# Excitation Conditions for Uniform Exponential Stability of the Cooperative Gradient Algorithm Over Weakly Connected Digraphs

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***Abstract*—In this letter, we study the problem of robust adaptive parameter estimation over networks with persistently exciting (PE) nodes and cooperative estimation dynamics. For this problem, it is well known that for networks characterized by undirected connected graphs, the property of *uniform exponential stability* (UES) can be established under a cooperative PE condition that relaxes the standard individual PE assumptions traditionally used in adaptive control. However, it is an open question whether similar cooperative PE conditions can also be used in *general directed graphs*. We provide an answer to this question by characterizing a *generalized cooperative PE condition* that is proved to be necessary and sufficient for UES in cooperative gradient dynamics evolving over arbitrary weakly connected digraphs. We also derive a similar generalized cooperative *data-based* condition for distributed learning dynamics that use recorded data instead of persistently exciting signals. We further present numerical experiments that study the rates of convergence of the dynamics.**

***Index Terms*—Adaptive control, networked control systems, Lyapunov methods.**

## I. Introduction

One of the cornerstones of many adaptive and learning-based controllers is their ability to achieve robust real-time parameter estimation under sufficient excitation in the system [1]. For single-agent systems, this task can be achieved by using the well-known *gradient algorithm* [2], [3], which can be analyzed by studying the dynamics

$$\dot{x} = -k_1\phi(t)\phi(t)^\top x, \qquad k_1 > 0, \tag{1}$$

where $\phi$ is a regressor function, and $x$ is the parameter estimation error. Systems of the form (1) also emerge frequently in the context of model-reference adaptive control [4], neuro-adaptive optimal control [5], and extremum seeking control [6]. The convergence and stability properties of system (1) under different assumptions on the mapping $t \mapsto \phi(t)$ have been extensively investigated over the last three decades, see for instance [2], [4], [7]. Among these properties, the notions of *uniform asymptotic and exponential stability* (UAS and UES, respectively) are of particular interest given that they additionally confer suitable robustness properties via Lyapunov converse theorems [8]. Indeed, one of the fundamental results in adaptive control states that system (1) renders the origin UES if and only if $\phi$ is persistently exciting (PE), i.e.,

$$\int_t^{t+T} \phi(\tau)\phi(\tau)^\top d\tau \succeq \alpha_1 I_n, \tag{2}$$

for all $t \geq t_0 \geq 0$, and for some $\alpha_1, T > 0$ [3, Th. 2.5.1].

When the adaptive or estimation problem is defined on a *networked multi-agent system* instead of a single-agent system, the gradient dynamics (1) can be suitably modified to incorporate cooperation between neighboring nodes (or agents) of the network. In this case, one can consider the *cooperative gradient algorithm* [9], that can be analyzed by studying the following dynamics for each agent $i$:

$$\dot{x}_i = F_i(x, t) := -k_1\phi_i(t)\phi_i(t)^\top x_i - k_2 \sum_{j\in\mathcal{N}_i} a_{ji}(x_i - x_j), \tag{3}$$

where $\mathcal{N}_i$ is the set of in-neighbors of agent $i$, $x_i$ and $\phi_i \in \mathbb{R}^n$ are the local state and regressor of the $i^{th}$ agent, $a_{ji} \geq 0$ is the weight associated with the $(j, i)$th edge in the digraph, and $k_2 > 0$ is a tunable gain. If $k_2 = 0$, it is easy to see that every agent will render the point $x_i^* = 0$ UES if and only if the local regressor $\phi_i$ satisfies the PE condition (2). However, as shown in [9] and [10], for networks characterized by *undirected graphs*, the incorporation of the cooperative term in (3) guarantees that the origin of the system is UES under the following weaker *cooperative PE* (C-PE) condition:

$$\int_t^{t+T} \sum_{i=1}^{N} \phi_i(\tau)\phi_i(\tau)^\top d\tau \succeq \alpha_1 I_n, \tag{4}$$

where the summation is taken over *all* agents of the network. This relaxation is conceptually and practically relevant for *large-scale* networked adaptive systems, where it is generally

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difficult to guarantee that *every* node of the network satisfies an individual PE condition of the form (2). However, by considering only undirected graphs, the class of problems that can be addressed by using dynamics of the form (3) is significantly narrowed down, compared to other networked control problems that are usually defined on general directed graphs. It thus leaves as open questions whether cooperative adaptive systems of the form (3) can also be extended to general directed graphs, and what classes of excitation conditions should be considered in this scenario. Some recent efforts in this direction have been made in [11]–[13] by using PE conditions to characterize the connectivity of the network, and in [5] by considering digraphs that are weight-balanced and strongly connected. However, to the knowledge of the authors, whether or not system (3) renders the origin UES under generalized cooperative PE conditions for arbitrary digraphs, as well as the necessity properties of these conditions, remain as open questions in the literature.

In this letter, we address these questions by introducing a novel generalized cooperative PE condition for multi-agent systems with topologies characterized by *weakly connected digraphs*. We show that this excitation condition is *necessary* and *sufficient* for UES in systems of the form (3) with bounded regressors. Moreover, we present extensions to systems where recorded data is available to some of the nodes of the network, in the spirit of concurrent learning. In summary, the following are the main contributions of this letter:

1) We introduce a generalized cooperative PE condition for arbitrary weakly connected digraphs. We show that, for systems of the form (3), this generalized condition is *sufficient* and *necessary* for UES of the origin. Unlike the sufficient results presented in [9] for undirected graphs, our analysis for sufficiency makes use of the notion of uniform complete observability (UCO) [3], [14], under a judicious choice of the output injection matrix. This technique allows us to overcome the restrictions imposed by the lack of symmetry in the Laplacian matrix $\mathcal{L}$ that emerges in undirected graphs. It also allows us to dispense with the assumption that $(\mathcal{L}+\mathcal{B})+(\mathcal{L}+\mathcal{B})^\top \succ 0$, for an arbitrary non-negative, nonzero diagonal matrix $\mathcal{B}$, as considered in [5], which only holds when the digraph is strongly connected and weight-balanced (see [15, Lemma 13]).

2) As a by-product of the proof of our previous result, we derive a novel weak cooperative "richness" condition for data-enabled parameter estimation dynamics in the spirit of concurrent learning, see [16], [17]. Previous results in the literature established similar conditions for *undirected* graphs, see [6]. However, to the best knowledge of the authors, the characterization of the "richness" condition for general digraphs was absent in the literature.

3) Finally, we investigate through numerical examples the rate of convergence of system (3), for different classes of digraphs of different sizes.

The results of this letter are instrumental to the design of distributed adaptive controllers for large-scale network systems that go beyond those considered in this letter. Indeed, our results can be used to characterize which nodes of the network need to satisfy PE conditions (individual or cooperative) in order to guarantee uniform exponential parameter identification in distributed adaptive control problems. In the data-enabled case, our results characterize the nodes of the network that require "jointly sufficiently" rich data, and also uncover the existence of nodes that have a marginal effect in the estimation dynamics.

The rest of this letter is organized as follows: Section II presents some preliminaries. Sections III and IV presents our main results and the proofs. Section V presents numerical results, and finally Section VI ends with conclusions.

## II. Preliminaries

We denote as $E_{ij}$ the matrix with all entries equal to zero except at the $(i,j)$th entry, which is equal to one. We use $\mathbf{1}_M \in \mathbb{R}^M$ to denote a vector of ones. An identity matrix of dimension $n$ is denoted by $I_n$. We use $e_i$ to denote the standard basis in $\mathbb{R}^n$. A block matrix $\mathbf{F}$ is represented in terms of its $(i,j)$th matrix block as $\mathbf{F} = [\mathbf{F}_{ij}]$. We use $\mathbf{diag}\{\cdot\}$ to build a block diagonal matrix from given matrices and $\text{diag}(\cdot)$ to build a diagonal matrix from a given vector. We use $|\cdot|$ to denote the standard Euclidean norm, and $\|\cdot\|$ to denote the Frobenius norm. We also use $\|\phi\|_{[t,t+T]} := (\int_t^{t+T} \|\phi(t)\|^2 dt)^{\frac{1}{2}}$. Given a matrix $C$, we use $\rho(C)$ to denote the spectral radius of $C$. An *M-matrix* is a square matrix $A = sI - C$ with non-positive off-diagonal entries, where $C$ is a non-negative matrix, and $s \geq \rho(C)$. This matrix is non-singular when $s > \rho(C)$.

A directed graph or digraph $G(V, E)$ is characterized by the set of nodes $V = \{1, 2, 3, \ldots, N\}$, and the set of directed edges $E$. The edge set $E$ consists of ordered pairs of the form $(j, i)$, which indicates a directed link from node $j$ to node $i$. We assume that the digraphs are simple, i.e., there are no self-arcs. If there exists a directed edge $(j, i)$ in $E$, then node $j$ is called an in-neighbor of node $i$. We assign a positive weight $a_{ji}$ to each edge $(j, i)$. By default, $a_{ji} = 0$ if $(j, i)$ is not an edge. Then, we define a weighted *Laplacian* matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ associated with the digraph by setting $l_{ij} := -a_{ji}$ for $j \neq i$ and $l_{ij} := \sum_{j=1}^N a_{ji}$ for $i = j$.

A digraph is *strongly connected* if for any two distinct nodes $i$ and $j$ in the digraph, there is a path from $i$ to $j$. A digraph $G$ is *weakly connected* if the undirected graph, obtained by ignoring the orientations of the edges, is connected. A weakly connected digraph $G$ can have multiple strong connected components (SCC). Following [18], the *skeleton* of the digraph $G$ is the digraph $G_c$ obtained by condensing each SCC to a single node. The nodes in the skeleton digraph with no in-neighbors are called leading nodes, and the corresponding SCCs in $G$ are called the *leading SCCs*. By relabeling the nodes of $G$, the Laplacian matrix $L$ of the weakly connected digraph $G$ can be written as a lower block triangular form $L = [L_s, 0; L_{sf}, L_f]$. The matrix $L_s$ is block diagonal $L_s = \mathbf{diag}\{L_{s1}, \ldots, L_{sK}\}$, where each diagonal block is an irreducible, singular M-matrix, corresponding to a certain leading SCC. The lower-left block $-L_{sf}$ of $L$ is a non-negative matrix and the lower-right block $L_f$ is a non-singular M-matrix.

We also recall the following two definitions, see [3], [19]. The first one concerns the stability properties of systems with inputs. The second one is common in adaptive control.

*Definition 1:* A smooth system $\dot{x} = f(t, x, u)$ is said to render the origin *input-to-state stable* (ISS) if $\exists\ \beta \in \mathcal{KL}$ and $\zeta \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u$, the solution $x$ exists for all $t \geq t_0$, and satisfies:

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \zeta(\sup_{t_0 \leq \tau \leq t} |u(\tau)|).$$

When $u = 0$, and $\exists\ c_1, c_2 > 0$ such that $\beta(r, s) = c_1 r e^{-c_2 s}$, the origin is *uniformly globally exponential stable.*

*Definition 2:* For $\dot{x} = A(t)x$, $y = Cx$, the pair $(A, C)$ is said to be *uniformly completely observable* (UCO) if $\exists\ \alpha_1, \alpha_2, T > 0$ such that $\forall\ t \geq t_0$, the following holds:

$$\alpha_1 I \preceq \int_t^{t+T} \Psi(\tau, t)^\top C(\tau)^\top C(\tau)\Psi(\tau, t)d\tau \preceq \alpha_2 I,$$

where $\Psi(\cdot, \cdot)$ is the state transition matrix of the system.

## III. Generalized Cooperative PE Condition and Main Results

We consider a multi-agent system composed of $N$ agents, whose information flow topology is given by a digraph $G$, and where the dynamics of each agent $i$ are characterized by system (3). This type of dynamics emerge frequently in parameter identification problems where the agents of the network aim to cooperatively learn a global parameter $\theta$ by using individual real-time measurements of a signal of the form $y_i(t) = \theta^\top \phi_i(t)$, where $\phi$ is a known regressor, see for instance [2], [3]. Indeed, by defining the local estimate $\hat{y}_i(t) := \hat{\theta}_i(t)^\top \phi_i(t)$, and the quadratic cost $\mathcal{J}_i := e_i^2/2$, with $e_i := \hat{y}_i - y_i$, each agent can update its own parameter estimate $\hat{\theta}_i$ via the following cooperative gradient descent rule:

$$\dot{\hat{\theta}}_i = -k_1 \nabla_{\hat{\theta}_i} \mathcal{J}_i - k_2 \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\theta}_i - \hat{\theta}_j).$$

By defining $x_i := \hat{\theta}_i - \theta$, and noticing that $\nabla_{\hat{\theta}_i} \mathcal{J}_i = e_i \nabla_{\hat{\theta}_i} e_i = \phi_i(t)\phi_i(t)^\top x_i$, the error dynamics $\dot{x}_i$ are precisely given by (3). For these dynamics, we are interested in the properties of the regressors $\phi_i$ that provide *sufficient* and *necessary* conditions for UES of the origin in an *arbitrary digraph G*. To do this, we first take a common assumption in the literature [2], [7], [9], namely, that each $\phi_i$ is uniformly bounded above. Next, we have the following definition:

*Definition 3:* Let $G$ be a weakly connected digraph with $N$ nodes. The regressors $\{\phi_i\}_{i=1}^N$ are said to satisfy the *generalized cooperative PE* (gC-PE) condition for $G$ if for each leading SCC $k$, with total agents $M_k$, we have

$$\int_t^{t+T_k} \sum_{i=1}^{M_k} \phi_i(\tau)\phi_i^\top(\tau)d\tau \succeq \alpha_{k,1} I_n \tag{5}$$

for some $T_k, \alpha_{k,1} > 0$, and for all $t \geq t_0 \geq 0$.

*Remark 1:* Since each $\phi_i$ is assumed to be uniformly bounded above, the integral in (5) is also bounded above by $\alpha_{k,2} I_n$ for some positive $\alpha_{k,2}$.

The gC-PE condition generalizes existing cooperative PE conditions for multi-agent systems, such as those presented in [9] for undirected graphs (bidirectional), and in [5] for weight-balanced and strongly connected digraphs. In particular, when the digraph is strongly connected, it follows that $M_k = N$, and in this case we recover the C-PE condition (4). Similarly, when the digraph is fully disconnected, (5) reduces to the standard PE condition (2) applied to *every* agent of the network.

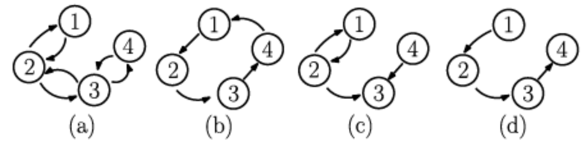


Fig. 1. Examples of digraphs. (a-b) Strongly Connected (c-d) Weakly Connected but not Strongly Connected.

We are now ready to present the main result of this letter.

*Theorem 1:* Consider a multi-agent system with individual agent dynamics given by (3). Suppose that the digraph $G$ is weakly connected; then, the origin of the system is UES if and only if the gC-PE condition (5) holds.

*Example 1:* Consider the digraphs shown in Figure 1. The gC-PE condition (5) reduces to the C-PE condition (4) for networks (a) and (b), which guarantees UES under the dynamics (3). However for networks (c) and (d), the gC-PE condition is stronger than the C-PE condition. For example: in order to satisfy gC-PE condition in network (c), two sets of agents, i.e., $\{1,2\}$ and $\{4\}$, must satisfy the C-PE condition separately.

The next corollary is an immediate consequence of the proof of Theorem 1, which will be provided in the next section. To the best knowledge of the authors, this result is also novel in the literature of concurrent learning in multi-agent systems.

*Corollary 1:* Let $F_i$ be defined as in (3). Consider a multi-agent system with individual agent dynamics given by

$$\dot{x}_i = F_i(x, t) - k_3 \sum_{l=1}^{\mathbf{L}} \phi_i(t_{l_i})\phi_i(t_{l_i})^\top x_i, \tag{6}$$

for all $i \in V$, where $k_3 > 0$. Suppose that the digraph $G$ is weakly connected and that there exist $\epsilon_k > 0$ and sequences $\{t_{l_i}\}_{l_i=1}^{\mathbf{L}}$ such that for each $k$th leading SCC of $G$, the collection $\{\phi_i(t_{l_i})\}_{l_i=1}^{\mathbf{L}}$ satisfies $\sum_{i=1}^{M_k} \sum_{l=1}^{\mathbf{L}} \phi_i(t_{l_i})\phi_i(t_{l_i})^\top \succeq \epsilon_k I_n$; then, the origin of (6) is UES.

A consequence of Corollary 1 is that an appropriate *data* allocation to the regressor functions of the leading SCCs of the graph is *sufficient* to guarantee UES under the dynamics (6). Note that this allocation is not unique. In particular, it can be satisfied by allocating enough *rich* data to one single node of each SCC, by uniformly distributing the data among all the nodes of the SCC, or by using any other allocation strategy that balances the two previous approaches.

## IV. Proof of Theorem 1

We devote the rest of this letter to prove Theorem 1. Some lengthy computations are omitted and presented in the extended manuscript [15].

### A. Cascaded Form and Change of Coordinates

Let $x := [x_s^\top, x_f^\top]^\top$ where $x_s$ is the state associated with the agents in leading SCCs, and $x_f$ is the state associated with the remaining agents. Similarly, let $\Phi := \mathbf{diag}\{\Phi_s, \Phi_f\}$ where the matrices $\Phi_s$ and $\Phi_f$ are block diagonal and defined as $\Phi_s := \mathbf{diag}\{\phi_{s1}, \dots, \phi_{sM}\}$, and $\Phi_f := \mathbf{diag}\{\phi_{f1}, \dots, \phi_{fM'}\}$. With the above annotations, system (3) can be written as:

$$\dot{x}_s = -\left(k_1 \Phi_s \Phi_s^\top + k_2 \mathbf{L}_s\right) x_s, \tag{7a}$$

$$\dot{x}_f = -\left(k_1 \Phi_f \Phi_f^\top + k_2 \mathbf{L}_f\right) x_f - k_2 \mathbf{L}_{sf} x_s, \tag{7b}$$

where $\mathbf{L}_s = L_s \otimes I_n$, $\mathbf{L}_{sf} = L_{sf} \otimes I_n$ and $\mathbf{L}_f = L_f \otimes I_n$. One can also obtain (7) by appealing to the Frobenius normal form (see, e.g., [20]).

*Remark 2:* Recall from Section II that $L_s$ is a block diagonal matrix with $K$ blocks in total, one for each leading SCC. This, together with the block diagonal structure of $\Phi_s \Phi_s^\top$, indicates that the dynamics of the leading SCCs (7a) are also decoupled from each other. Thus, without loss of generality, for our analysis we can assume that the weakly connected digraph $G$ contains only one leading SCC.

Next, to facilitate the analysis of (7), we define permutation matrices $\Pi_s$, $\Pi_f$ of dimensions $Mn \times Mn$ and $M'n \times M'n$ (with total blocks $M \times n$ and $M' \times n$) respectively,

$$\Pi_s = [\Pi_{s,ij}] \ , \quad \Pi_f = [\Pi_{f,ij}], \tag{8}$$

such that each block $\Pi_{s,ij} := E_{ji}$ and $\Pi_{f,ij} := E_{ji}$ is of dimensions $n \times M$ and $n \times M'$ respectively. The next result follows directly by computation. For completeness, step-by-step deduction is presented in the extended manuscript [15].

*Lemma 1:* Under permutation matrices $\Pi_s$ and $\Pi_f$, as defined in (8), the transformation of coordinates $x_c := \Pi_s^\top x_s$ and $x_w := \Pi_f^\top x_f$, results in the dynamics

$$\dot{x}_c = -(k_1 \mathbf{H}_c(t) + k_2 \mathbf{L}_c) x_c, \tag{9a}$$

$$\dot{x}_w = -(k_1 \mathbf{H}_w(t) + k_2 \mathbf{L}_w) x_w - k_2 \mathbf{L}_{cw} x_c, \tag{9b}$$

where $\mathbf{L}_c := \Pi_s^\top \mathbf{L}_s \Pi_s = I_n \otimes L_s$, $\mathbf{L}_w := \Pi_f^\top \mathbf{L}_f \Pi_f = I_n \otimes L_f$, $\mathbf{L}_{cw} := \Pi_f^\top \mathbf{L}_{sf} \Pi_s$, $\mathbf{H}_c := \Pi_s^\top \Phi_s \Phi_s^\top \Pi_s$, $\mathbf{H}_w := \Pi_f^\top \Phi_f \Phi_f^\top \Pi_f$. In particular, if we divide $\mathbf{H}_c$ (and $\mathbf{H}_w$) into $n \times n$ blocks, then each block is an $M \times M$ (and $M' \times M'$) diagonal matrix.

We now proceed to study the stability properties of system (7) by studying the stability properties of system (9). In Section IV-B, we show that the subsystem (9a) is UES if and only if the gC-PE condition holds, and in Section IV-C, we show that the subsystem (9b) is ISS. We conclude the proof of Theorem 1 in Section IV-D.

### B. UES of Subsystem (9a)

This subsection establishes the following proposition.

*Proposition 1:* The dynamics (9a) render the origin UES if and only if the gC-PE condition (5) holds.

To prove Proposition 1, we need a few preliminary results. Let $\mathbf{w} \in \mathbb{R}^M$, with $|\mathbf{w}| = 1$, be a left eigenvector of the irreducible, singular M-matrix $L_s$ corresponding to the zero eigenvalue, i.e., $L_s^\top \mathbf{w} = 0$, then by the Perron Frobenius Theorem, we can choose $\mathbf{w}$ such that all of its entries are positive. Next, we define the matrices $P_s := \mathrm{diag}(\mathbf{w})$, $\mathbf{P}_c := I_n \otimes P_s$ and $\mathbf{M}_c := \mathbf{L}_c^\top \mathbf{P}_c + \mathbf{P}_c \mathbf{L}_c$, where $\mathbf{L}_c$ is defined in Lemma 1. We have the following results for these matrices:

*Lemma 2:* The following holds: (i) $\mathbf{P}_c$ is diagonal and positive definite, (ii) $\mathbf{M}_c \succeq 0$, (iii) $\dim(\ker(\mathbf{M}_c)) = n$.

*Proof:* (i) It follows directly from the definition of $\mathbf{P}_c$. (ii) Using the properties of Kronecker product, we can write $\mathbf{M}_c$ as, $\mathbf{M}_c = I_n \otimes (L_s^\top P_s + P_s L_s)$. Since $L_s$ is an irreducible, singular M-matrix, by [21, Th. 4.31], we have that $L_s^\top P_s + P_s L_s \succeq 0$. This establishes that $\mathbf{M}_c$ is positive semi-definite. (iii) Let $(\lambda_j, v_j)$, $j = 1, \ldots, M$, be the eigenvalue and eigenvector pairs of positive semi-definite matrix $L_s^\top P_s + P_s L_s$. The eigenvalues can be arranged as,

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M. \tag{10}$$

Since $L_s \mathbf{1}_M = 0$, $P_s \mathbf{1}_M = \mathbf{w}$, and $L_s^\top \mathbf{w} = 0$, it then follows that: $\ker(L_s^\top P_s + P_s L_s) = \mathrm{Span}(\mathbf{1}_M)$. This implies that $\lambda_1 = 0$ and $\lambda_j > 0$ for all $j \in \{2, 3, \ldots, M\}$. Each eigenvalue $\lambda_j$ in (10) is repeated $n$ times to form the spectrum of $\mathbf{M}_c$, with the corresponding eigenvectors given by $e_i \otimes v_j$ for all $i \in \{1, 2, \ldots, n\}$. Hence, $\ker(\mathbf{M}_c) = \mathrm{Span}(e_i \otimes \mathbf{1}_M)$ and $\dim(\ker(\mathbf{M}_c)) = n$. This concludes the proof. ■

*Corollary 2:* Let $\mathbf{H}_c$ and $\mathbf{P}_c$ be defined in Lemmas 1 and 2, respectively. Then, $\mathbf{P}_c \mathbf{H}_c$ is positive semi-definite.

*Proof:* Recall from Lemma 1 that $\mathbf{H}_c$ is a positive semi-definite block matrix, with each of its blocks being a diagonal matrix. Also, recall from Lemma 2 that $\mathbf{P}_c = I_n \otimes P_s$ is a block diagonal matrix, with all the blocks being the same diagonal matrix. It then follows that $\mathbf{P}_c$ and $\mathbf{H}_c$ commute. Corollary 2 then follows from the fact that the product of two commuting positive semi-definite matrices is again positive semi-definite [22, Th. 3]. ■

Next, we let the matrices $\mathbf{A}$ and $\mathbf{C}$ be defined as follows:

$$\mathbf{A} := -(k_1 \mathbf{H}_c + k_2 \mathbf{L}_c), \ \ \mathbf{C} := \begin{bmatrix} \sqrt{2k_1} (\mathbf{P}_c \mathbf{H}_c)^{\frac{1}{2}} \\ \sqrt{k_2} \mathbf{M}_c^{\frac{1}{2}} \end{bmatrix}. \tag{11}$$

We show below that the pair $(\mathbf{A}, \mathbf{C})$ is UCO under the gC-PE condition. This fact is key to prove Proposition 1. It takes several steps to establish the fact, and we will use the output injection theorem [3, Lemma 2.5.2] applied to the pair $(0, \mathbf{C})$. To proceed, we first have the following lemma.

*Lemma 3:* The pair $(0, \mathbf{C})$ in (11) is UCO if and only if the gC-PE condition (5) holds.

*Proof (Sufficiency):* The arguments used in the proof will be similar to the ones in [9, Th. 1]. The main difference is that, here we adopt a UCO based approach and perform the analysis for $\mathbf{M}_c$ instead of using the Laplacian, which is now not necessarily symmetric. First, note that the Observability Gramian of the pair $(0, \mathbf{C})$ is

$$\Omega(t) := \int_t^{t+T} (2k_1 \mathbf{P}_c \mathbf{H}_c(\tau) + k_2 \mathbf{M}_c) d\tau. \tag{12}$$

Because each $\phi_i$ is uniformly upper bounded, it follows that $\Omega(t)$ is also uniformly upper bounded (details of the arguments can be found in [15]). But, here, we will only need to compute a uniform lower bound. To do this, using eigenvectors from the proof of Lemma 2, we define the following matrices:

$$\mathbf{V} := [e_1 \otimes v_2, \ldots, e_n \otimes v_2, \ldots, e_1 \otimes v_M, \ldots, e_n \otimes v_M],$$

$$\mathbf{F} := \frac{1}{\sqrt{M}} [e_1 \otimes \mathbf{1}_M, e_2 \otimes \mathbf{1}_M, \ldots, e_n \otimes \mathbf{1}_M]. \tag{13}$$

Note that the matrix $[\mathbf{F}, \mathbf{V}]$ is orthogonal. For any vector $\mathbf{x} \in \mathbb{R}^{Mn}$, we write $\mathbf{x} = \mathbf{F}\mathbf{b} + \mathbf{V}\mathbf{c}$, where $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^{(M-1)n}$. We show that under the gC-PE condition (5), $\mathbf{x}^\top \Omega(t) \mathbf{x} > 0$ for all $t \geq t_0$ and all $\mathbf{x} \neq 0$. Indeed, for $\mathbf{c} \neq 0$, we have that $\mathbf{x}^\top \Omega(t) \mathbf{x} \geq k_2 \lambda_2 \mathbf{c}^\top \mathbf{c} > 0$, where $\lambda_2$ comes from (10). For $\mathbf{c} = 0$, we have that:

$$\mathbf{x}^\top \Omega(t) \mathbf{x} = 2k_1 \mathbf{b}^\top \sum_{i=1}^{M} w_i \int_t^{t+T} \phi_{si}(\tau) \phi_{si}(\tau)^\top d\tau \mathbf{b}. \tag{14}$$

Under the gC-PE condition (5) we obtain:

$$\mathbf{x}^\top \Omega(t)\mathbf{x} \geq 2k_1\underline{w}\alpha_1\mathbf{b}^\top\mathbf{b} > 0, \quad \text{for } \underline{w} := \min_i w_i. \tag{15}$$

Hence, $\Omega(t)$ is positive definite for all $t \geq t_0$. Since the graph is time-invariant, we can apply the same contradiction argument of [9, Proof of Theorem 1], to establish that under the gC-PE condition (5), $\Omega(t)$ is *uniformly* positive definite.

*Necessity:* Suppose, to the contrary, that the gC-PE condition does not hold, yet the pair $(0, \mathbf{C})$ is UCO. Then, there exists positive constants $T, \sigma > 0$ such that $\Omega(t)$ in (12) satisfies $\Omega(t) \succeq \sigma I_{Mn}$, for all $t \geq t_0$. Consider a vector $\mathbf{x} \in \text{span}(\mathbf{F})$. Then, (14) still holds and, moreover,

$$\mathbf{x}^\top \Omega(t)\mathbf{x} \geq \sigma\mathbf{b}^\top\mathbf{b}, \quad \forall t \geq t_0. \tag{16}$$

By defining $\bar{w} := \max_i w_i$, the right hand side of (14) is upper bounded by $2k_1\bar{w}\mathbf{b}^\top S(t)\mathbf{b}$ where $S(t)$ is given by $S(t) := \sum_{i=1}^M \int_t^{t+T} \phi_{si}(\tau)\phi_{si}(\tau)^\top d\tau$. Because the gC-PE condition does not hold, we have that for all $T, \alpha_1 > 0$ there exists $t^* \geq t_0$ such that $S(t^*) \prec \alpha_1 I_n$. By choosing $\alpha_1 = \sigma/(2k_1\bar{w})$, we have $\mathbf{x}^\top\Omega(t^*)\mathbf{x} < \sigma\mathbf{b}^\top\mathbf{b}$, which contradicts (16). ∎

Let $\Delta$ be the pseudo inverse of $\mathbf{M}_c$. Then, we define a gain matrix $\mathbf{K}$ as follows:

$$\mathbf{K} := \left[\sqrt{\tfrac{k_1}{2}}\mathbf{P}_c^{-\frac{1}{2}}\mathbf{H}_c^{\frac{1}{2}}, \ \sqrt{k_2}\ \mathbf{L}_c\Delta^{\frac{1}{2}}\right]. \tag{17}$$

Next, we use this gain matrix $\mathbf{K}$ to show that the pair $(\mathbf{A}, \mathbf{C})$ in (11) is UCO.

*Lemma 4:* The pair $(\mathbf{A}, \mathbf{C})$ in (11) is UCO if and only if the gC-PE condition (5) holds.

*Proof:* Because each $\phi_i$ is uniformly upper bounded, it is shown in [15] that the gain matrix $\mathbf{K}$ is locally integrable, i.e., there exists $c_3 > 0$ such that $\int_t^{t+T} \|\mathbf{K}(\tau)\|^2 d\tau \leq Tc_3$, for all $t \geq t_0$. Moreover, by computation, $\mathbf{KC} = -\mathbf{A}$. By Lemma 3, $(0, \mathbf{C})$ is UCO if and only if the gC-PE condition holds. Thus, by the output injection theorem [3, Lemma 2.5.2], we conclude that the lemma holds. ∎

We will now prove Proposition 1:

*Proof of Proposition 1 (Sufficiency):* Let $\mathbf{P}_c$ be the diagonal, positive definite matrix introduced in Lemma 2. We consider the following Lyapunov function $V_c = x_c^\top\mathbf{P}_c x_c$, which satisfies

$$\dot{V}_c(t) = -x_c(t)^\top[2k_1\mathbf{P}_c\mathbf{H}_c(t) + k_2\mathbf{M}_c]x_c(t) \leq 0,$$

where we used Lemma 2 and Corollary 2 to have $\mathbf{M}_c \succeq 0$ and $\mathbf{P}_c\mathbf{H}_c \succeq 0$. Now, we show that $\dot{V}_c(t)$ decreases exponentially fast on average. In particular, we have

$$\int_t^{t+T} \dot{V}_c(\tau)d\tau = -\int_t^{t+T} x_c(\tau)^\top\mathbf{C}^\top\mathbf{C}x_c(\tau)d\tau, \tag{18}$$

where $\mathbf{C}$ is defined in (11). Denote by $\Psi(\tau, t)$, the state transition matrix of the system (9a). Then, the right hand side of (18) can be written as

$$-x_c(t)^\top\int_t^{t+T} \Psi(\tau, t)^\top\mathbf{C}^\top\mathbf{C}\Psi(\tau, t)d\tau x_c(t), \tag{19}$$

where the integral defines the observability gramian of the pair $(\mathbf{A}, \mathbf{C})$, defined in (11). Using the gC-PE condition, we apply Lemma 4 to establish that $(\mathbf{A}, \mathbf{C})$ is UCO. This implies the existence of $\sigma > 0$ such that $\int_t^{t+T} \dot{V}_c(\tau)d\tau \leq -\sigma|x_c(t)|^2$. Thus, by [3, Th. 1.5.2], the dynamics (9a) render the origin UES.

*(Necessity):* Suppose, to the contrary, that the gC-PE condition does not hold, yet the origin of system (9a) is UES. Consider the Lyapunov function $V_c$. Using $\underline{w}$ from (15) and $\bar{w} := \max_i w_i$, it follows that $\underline{w}|x_c(t)|^2 \leq V_c(t) \leq \bar{w}|x_c(t)|^2$. Since the gC-PE condition does not hold, by Lemma 4 the pair $(\mathbf{A}, \mathbf{C})$ is not UCO. Hence, for all $T, \sigma' > 0$ there exists $t^* > t_0$ such that the observability gramian $\Omega'(t^*)$ of the pair $(\mathbf{A}, \mathbf{C})$ satisfies $\Omega'(t^*) < \sigma' I_{Mn}$. By substituting this in (19), we get $\int_{t^*}^{t^*+T} \dot{V}_c(\tau)d\tau > -\sigma'|x_c(t^*)|^2$. After further computation, we choose $\sigma' = \underline{w}/2$ to get

$$V_c(t^*)/2 < V_c(t^* + T). \tag{20}$$

Next, consider a solution $x_c(t)$ of subsystem (9a) starting at $(t^*, x_{c,0})$ and let $x_{c,0} \neq 0$. Since the origin of subsystem (9a) is UES, there exists constant $c_1, c_2 > 0$ such that $|x_c(t)| \leq c_1|x_{c,0}|e^{-c_2(t-t^*)}$. Using $V_c(t^* + T) \leq \bar{w}c_1^2\ |x_{c,0}|^2e^{-2c_2T}$, we pick $T$ sufficiently large such that the inequality $\bar{w}c_1^2\ |x_{c,0}|^2e^{-2c_2T} \leq 0.5V_c(t^*)$ holds. Then, $V_c(t^* + T) \leq V_c(t^*)/2$, which contradicts (20). ∎

### C. ISS of Subsystem (9b)

This subsection is dedicated to the following result.

*Proposition 2:* The dynamics (9b) render the origin ISS, with respect to the input $x_c$, and UES when $x_c = 0$.

*Proof:* Recall from Section II that $L_f$ is the lower-right block of the weighted Laplacian matrix $L = [L_s, 0; L_{sf}, L_f]$ and it is a non-singular M-matrix. By [21, Th. 4.25], there exists a positive diagonal matrix $P_f := \text{diag}(L_f^{-1}\mathbf{1}_{M'})$ such that $L_f^\top P_f + P_f L_f \succeq \eta I$ for some $\eta > 0$. Now we define $\mathbf{P}_w := I_n \otimes P_f$ and similar to Lemma 2, it can be shown that: $\mathbf{L}_w^\top\mathbf{P}_w + \mathbf{P}_w\mathbf{L}_w \succeq \eta I$. Using the Lyapunov function $V_w = x_w^\top\mathbf{P}_w x_w$, we obtain

$$\begin{aligned}\dot{V}_w(t) = &-x_w^\top\Big[2k_1\mathbf{P}_w\mathbf{H}_w(t) + k_2(\mathbf{L}_w^\top\mathbf{P}_w + \mathbf{P}_w\mathbf{L}_w)\Big]x_w \\ &- 2k_2x_w^\top\mathbf{P}_w\mathbf{L}_{cw}x_c \leq -k_2\eta|x_w|^2 - 2k_2x_w^\top\mathbf{P}_w\mathbf{L}_{cw}x_c,\end{aligned}$$

where we used the fact that $\mathbf{P}_w\mathbf{H}_w \succeq 0$ (the proof of which is similar to the proof of Corollary 2), and $\mathbf{L}_w^\top\mathbf{P}_w + \mathbf{P}_w\mathbf{L}_w \succeq \eta I$. Using Cauchy-Schwartz inequality we get

$$\dot{V}_w(t) \leq -k_2\eta|x_w|^2 + 2k_2|x_w|\ \|\mathbf{P}_w\mathbf{L}_{cw}\|\ |x_c|.$$

By a standard Lyapunov Theorem [19, Th. 4.19], we conclude that the proposition holds. ∎

### D. Proof of Theorem 1

*Sufficiency:* Since subsystems (9a) and (9b) (resp. (7a) and (7b)) are UES and ISS respectively when the gC-PE condition holds, therefore, their cascade is UES by [19, Lemma 4.7].

*Necessity:* Since subsystem (9a) (resp. (7a)) is not UES when the gC-PE condition does not hold, the original cascade system (7) is not UES.

## V. Simulation Results

Consider a network of 20 agents in Fig. 2. Each edge weight is chosen uniformly randomly in the interval $(0, 1)$ and each agent $i$ measures a target output of $y_i = (x_i - 1)^2$, where $x_i(t) := \exp(-0.01it)$. Agent $i$ chooses its regressor $\phi_i(t)$ as, $\phi_i(t) := \psi(x_i(t)) = [1, x_i, x_i^2]^\top$ and aims to estimate

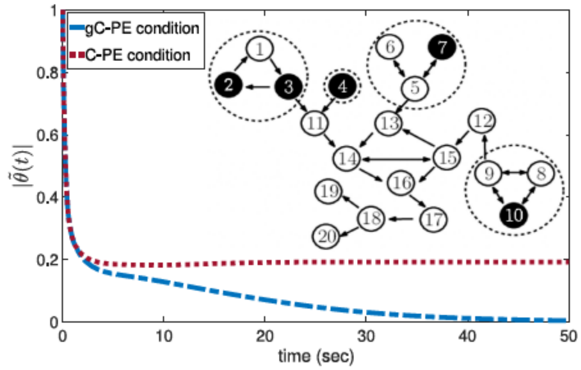


Fig. 2. Convergence of cooperative parameter estimation.

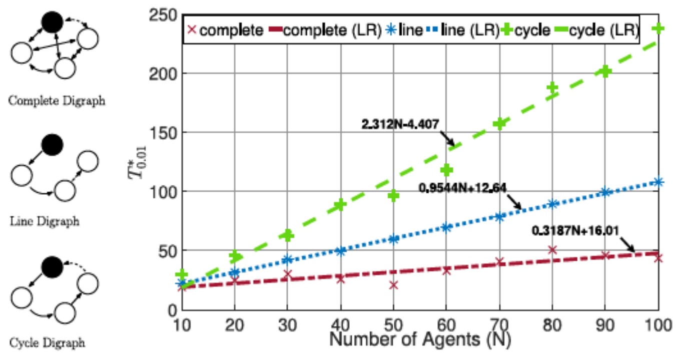


Fig. 3. Convergence time for different digraphs vs. number of nodes in the graph.

the parameter $\theta = [1, -2, 1]^\top$. We implement the individual dynamics (3) (with $k_1 = k_2 = 1$) for a normalized random initial condition (i.e., uniformly drawn from the unit sphere). In order to satisfy the gC-PE condition (5), agents 2, 3 are excited with $\sin(20t)$, $\cos(10t)$ respectively, and agents 4, 7, and 10 are excited with $\sin(20t) + \cos(10t) + \sin(10t)$. In this case, and as shown in Fig. 2, the norm of the error vector $x := [x_1^\top, \ldots, x_N^\top]^\top$, for $N = 20$, goes to zero. However, if we remove the excitation from agent 10, then the gC-PE condition no longer holds, and Fig. 2 shows that the norm of the estimation error does not converge to zero. This example highlights the need of the gC-PE condition for weakly connected digraphs. In Fig. 3, we investigate the convergence time of system (3) for complete, cycle, and line digraphs of different sizes. We approximate the convergence time by $T_{0.01} := \min\{t : |x(t)| \leq 0.01\}$. We add the excitation signal $\sin(20t) + \cos(10t) + \sin(10t)$ to agent 1, as highlighted black in Fig. 3. This makes the gC-PE condition (5) satisfied. For each class of digraphs and different sizes, we simulated (3) (with $k_1 = k_2 = 1$) for 20 random initial conditions uniformly drawn from the unit sphere. The mean convergence time $T^*_{0.01}$ and its linear regression is shown in Fig. 3.

## VI. Conclusion

In this letter, we have introduced a generalized cooperative PE (gC-PE) condition for the cooperative gradient dynamics (3) defined over weakly connected digraphs. Specifically, we have shown that system (3) renders the origin UES if and only if the gC-PE condition holds. A major technical challenge in establishing the result is the lack of symmetry of the Laplacian matrix of the digraph. To tackle the challenge, we propose a Lyapunov function to show in Section IV that the function decays exponentially fast by integrating the output injection theorem and results from graph theory.

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