Subdivided Claws and the Clique-Stable Set Separation Property

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Abstract

Let \mathcal{C} be a class of graphs closed under taking induced subgraphs. We say that \mathcal{C} has the clique-stable set separation property if there exists $c \in \mathbb{N}$ such that for every graph $G \in \mathcal{C}$ there is a collection \mathcal{P} of partitions (X,Y) of the vertex set of G with $|\mathcal{P}| \leq |V(G)|^c$ and with the following property: if K is a clique of G, and G is a stable set of G, and $G \cap G = \emptyset$, then there is $(X,Y) \in \mathcal{P}$ with $K \subseteq X$ and $G \subseteq Y$. In 1991 M. Yannakakis conjectured that the class of all graphs has the clique-stable set separation property, but this conjecture was disproved by M. Göös in 2014. Therefore it is now of interest to understand for which classes of graphs such a constant C exists. In this paper we define two infinite families $C \cap C$, of graphs and show that for every $C \cap C$ and $C \cap C$ and $C \cap C$ has the clique-stable set separation property.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. A clique in G is a set of pairwise adjacent vertices, and a stable set is a set of pairwise non-adjacent vertices. Let \mathcal{C} be a class of graphs closed under taking induced subgraphs. We say that \mathcal{C} has the clique-stable set separation property if there exists $c \in \mathbb{N}$ such that for every graph $G \in \mathcal{C}$ there is a collection \mathcal{P} of partitions (X,Y) of the vertex set of G with $|\mathcal{P}| \leq |V(G)|^c$ and with the following property: if K is a clique of G, and S is a stable set of G, and $K \cap S = \emptyset$, then there is $(X,Y) \in \mathcal{P}$ with $K \subseteq X$ and $S \subseteq Y$. This property plays an important role in a large variety of fields: communication complexity, combinatorial optimization, constraint satisfaction and others (for a comprehensive survey of these connections see [3]).

In 1991 Mihalis Yannakakis conjectured that the class of all graphs has the clique-stable set separation property [5], but this conjecture was disproved by Mika Göös in 2014 [2]. Therefore it is now of interest to understand for which classes of graphs such a constant c exists; our main result falls into that category.

Let G be a graph and let X, Y be disjoint subsets of V(G). We denote by G[X] the subgraph of G induced by X, by N(X) the set of all vertices of $V(G) \setminus X$ with a neighbor in X, and by N[X] the set $N(X) \cup X$. We say that X is complete to Y if every vertex of X is adjacent to every vertex of Y, and that X is anticomplete to Y if every vertex of X is non-adjacent to every vertex of Y. We say

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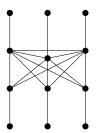
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that X and Y are matched if every vertex of X has exactly one neighbor in Y, and every vertex of Y has exactly one neighbor in X (and therefore |X| = |Y|). For a graph H, we say that G is H-free if no induced subgraph of G is isomorphic to H.

Next we define two types of graphs. Let $p, q \in \mathbb{N}$. We define the graph $F_S^{p,q}$ as follows:

- $V(F_S^{p,q}) = K \cup S_1 \cup S_2 \cup S_3$ where K is a clique, S_1, S_2, S_3 are stable sets, and the sets K, S_1, S_2, S_3 are pairwise disjoint;
- $|K| = |S_1| = p$, and K and S_1 are matched;
- $|S_2| = |S_3| = q$, and S_2 and S_3 are matched;
- K is complete to S_2 ;
- there are no other edges in $F_S^{p,q}$.

The graph $F_K^{p,q}$ is obtained from $F_S^{p,q}$ by making all pairs of vertices of S_3 adjacent.



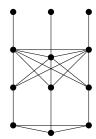


Figure 1: The graphs $F_S^{3,3}$ and $F_K^{3,3}$

Let $\mathcal{F}^{p,q}$ be the class of all graphs that are both $F_S^{p,q}$ -free and $F_K^{p,q}$ -free. We can now state our main result:

1.1 For all p, q > 0 the class $\mathcal{F}^{p,q}$ has the clique-stable set separation property.

Since the clique-stable set separation property is preserved under taking complements, we immediately deduce:

1.2 For all p,q > 0 the class of graphs whose complements are in $\mathcal{F}^{p,q}$ has the clique-stable set separation property.

2 The Proof

In this section we prove 1.1. The idea of the proof comes from [1]. Let $G \in \mathcal{F}^{p,q}$. Define \mathcal{P}_1 to be the set of all partitions $(N[X], V(G) \setminus N[X])$ and $(N(X), V(G) \setminus N(X))$ where X is a subset of V(G) with |X| < p. Clearly $|\mathcal{P}_1| \le 2|V(G)|^p$.

Write R = R(q, q) to mean the smallest positive integer R such that every 2-coloring of the edges of the complete graph on R vertices contains a monochromatic complete graph on q vertices. Ramsey's Theorem [4] implies:

2.1 $R(q,q) \leq 2^{2q}$.

For $a, b \in \mathbb{N}$ let the graph $F_{a,b}$ be defined as follows:

- $V(F_{a,b}) = K_1 \cup S_1 \cup S_2 \cup W$ where K_1 is a clique, S_1, S_2 are stable sets, and the sets K_1, S_1, S_2, W are pairwise disjoint;
- $|K_1| = |S_1| = a$, and K_1 and S_1 are matched;
- $|S_2| = |W| = b$, and S_2 and W are matched;
- K_1 is complete to S_2 ;
- there is no restriction on the adjacency of pairs of vertices of W;
- there are no other edges in $F_{a,b}$.

From the definition of R we immediately deduce:

2.2 G is $F_{p,R}$ -free.

For every triple $X = (K_1, S_1, S_2)$ of pairwise disjoint non-empty subsets of V(G) such that $|K_1| = |S_1| = p$ and $|S_2| < R$ we define the partition P_X of V(G) as follows. Let Z be the set of all vertices of G that are anticomplete to $K_1 \cup S_1$. Let A_X be the set of all vertices v of G such that

- either $v \in K_1$, or v is complete to K_1 , and
- either v has a neighbor in S_1 , or v has a neighbor in $Z \setminus N(S_2)$.

Note that A_X is disjoint from $S_1 \cup Z$. Define $P_X = (A_X, V(G) \setminus A_X)$, and let \mathcal{P}_2 be the set of all such partitions P_X . Since $|K_1 \cup S_1 \cup S_2| \leq 2p + R - 1$, and since by 2.1 $R \leq 2^{2q}$, we deduce that $|\mathcal{P}_2| < |V(G)|^{2p+2^{2q}}$.

In order to complete the proof of 1.1 we will prove the following:

2.3 For every clique K and stable set S of G such that $K \cap S = \emptyset$, there exists $(X,Y) \in \mathcal{P}_1 \cup \mathcal{P}_2$ with $K \subseteq X$ and $S \subseteq Y$.

Proof: Let K and S be as in the statement of 2.3.

(1) We may assume that K is a maximal clique of G, and S is a maximal stable set of G.

Let K' be a maximal clique of G with $K \subseteq K'$, and let S' be a maximal stable set of G with $S \subseteq S'$. If $K' \cap S' = \emptyset$, then the existence of the desired partition for K, S follows from the existence of such a partition for K', S'; thus we may assume that $K' \cap S' \neq \emptyset$. Since K' is a clique and S' is a stable set, it follows that $|K' \cap S'| = 1$, say $K' \cap S' = \{v\}$. But now the partitions $(N[\{v\}], V(G) \setminus N[\{v\}])$ and $(N(\{v\}), V(G) \setminus N(\{v\}])$ are both in \mathcal{P}_1 , and at least one of them has the desired property. This proves (1).

In view of (1) from now on we assume that K is a maximal clique of G, and S is a maximal stable set of G. Consequently every vertex of K has a neighbor in S. Let $S'_1 \subseteq S$ be a minimal subset of S such that every vertex of K has a neighbor in S'_1 . It follows from the minimality of S'_1 that there is a subset K'_1 of K such that S'_1 and K'_1 are matched. If $|S'_1| < p$, then the partition $(N(S'_1), V(G) \setminus N(S'_1)) \in \mathcal{P}_1$ has the desired property, so we may assume that $|S'_1| \ge p$.

Let S_1 be a subset of S_1' with $|S_1| = p$, and let $K_1 = N(S_1) \cap K_1'$. Then S_1 and K_1 are matched, and so $|K_1| = p$. Let Z be the set of vertices of G that are anticomplete to $S_1 \cup K_1$. Then $S_1' \setminus S_1 \subseteq Z \cap S$, and in particular every vertex of K has a neighbor either in S_1 or in $Z \cap S$. Let S' be the subset of vertices of $S \setminus S_1$ that are complete to K_1 . Note that $S' \cap Z = \emptyset$. Let S_2 be a minimal subset of S' such that $N(S_2) \cap Z = N(S') \cap Z$. It follows from the minimality of S_2 that there is a subset $W \subseteq Z \cap N(S')$ such that W and S_2 are matched. Observe that $G[K_1 \cup S_1 \cup S_2 \cup W]$ is isomorphic to $F_{p,|S_2|}$ (with K_1, S_1, S_2, W as in the definition of $F_{a,b}$). It follows from 2.2 that $|S_2| < R$.

Let $X = (K_1, S_1, S_2)$. We claim that the partition $P_X \in \mathcal{P}_2$ has the desired property for the pair K, S. Recall that $P_X = (A_X, V(G) \setminus A_X)$, where A_X is the set of all vertices v of G such that

- either $v \in K_1$, or v is complete to K_1 , and
- either v has a neighbor in S_1 , or v has a neighbor in $Z \setminus N(S_2)$.

We need to show that $K \subseteq A_X$, and $S \cap A_X = \emptyset$.

(2)
$$K \subseteq A_X$$
.

Let $k \in K$. Clearly either $k \in K_1$ or k is complete to K_1 . Moreover, k has a neighbor in S'_1 , and $S'_1 \subseteq S_1 \cup (Z \cap S)$. Since S is a stable set, it follows that $Z \cap S \subseteq Z \setminus N(S_2)$, and thus k has a neighbor either in S_1 , or in $Z \setminus N(S_2)$. This proves (2).

(3)
$$S \cap A_X = \emptyset$$
.

Suppose that $s \in S \cap A_X$. Then $s \notin K_1$; therefore s is complete to K_1 , and so $s \in S'$. Since S is a stable set, it follows that s is anticomplete to S_1 , and therefore s has a neighbor in $Z \setminus N(S_2)$. But $N(S') \cap Z = N(S_2) \cap Z$, a contradiction. This proves (3).

Now 2.3 follows from (2) and (3).

This completes the proof of 1.1.

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