Editors' Suggestion

Noncommutative field theory and composite Fermi liquids in some quantum Hall systems

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Composite Fermi liquid metals arise at certain special filling fractions in the quantum Hall regime and play an important role as parents of gapped states with quantized Hall response. They have been successfully described by the Halperin-Lee-Read (HLR) theory of a Fermi surface of composite fermions coupled to a $U(1)$ gauge field with a Chern-Simons term. However, the validity of the HLR description when the microscopic system is restricted to a single Landau level has not been clear. Here for the specific case of bosons at filling $\nu = 1$, we build on earlier work from the 1990s to formulate a low-energy description that takes the form of a *noncommutative* field theory. This theory has a Fermi surface of composite fermions coupled to a $U(1)$ gauge field with no Chern-Simons term but with the feature that all fields are defined in a noncommutative space-time. An approximate mapping of the long-wavelength, small-amplitude gauge fluctuations yields a commutative effective-field theory which, remarkably, takes the HLR form but with microscopic parameters correctly determined by the interaction strength. Extensions to some other composite Fermi liquids, and to other related states of matter are discussed.

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I. INTRODUCTION

The celebrated quantum Hall effects occur when electrons move in two space dimensions in a large perpendicular magnetic field such that the number of magnetic flux quanta is comparable to the number of particles. Our interest in this paper is in metallic phases of matter—dubbed composite Fermi liquids (see Refs. [1,2] for reviews)—that play a foundational role in our overall understanding of phenomena in the quantum Hall regime. In electronic systems, these occur at specific even-denominator filling fractions $\nu = \frac{1}{2}, \frac{1}{4}, \ldots$ They are striking experimental examples of metals that are not standard Landau Fermi liquids. Further, they act as parent phases for an entire prominent sequence (the Jain states [3]) of topological ordered states with quantized Hall conductivity that are observed experimentally.

Theoretically, the earliest and phenomenologically successful description of the experimentally observed metallic phase at $\nu = \frac{1}{2}$ was provided by the seminal work [4] of Halperin, Lee, and Read (HLR). The HLR theory employed a construction [5,6]—known as flux attachment—where the original interacting electron problem is formally rewritten in terms of a fermionic degree of freedom (dubbed the composite fermion) together with a dynamical $U(1)$ gauge field with a Chern-Simons term. In a mean-field description, the composite fermion sees a reduced effective magnetic field B^* compared to the physical magnetic field B. At filling $\nu = \frac{1}{2}$, the effective field $B^* = 0$, and the composite fermions form a Fermi surface. The HLR theory explained a number of essential experimental observations at $\nu = \frac{1}{2}$ and made further predictions that were confirmed in subsequent experiments. The HLR theory also has received considerable numerical support. This same general procedure was also extended to $\nu = \frac{1}{4}$ and captured the observed physics.

Despite its striking phenomenological success, there were a number of fundamental theoretical questions and difficulties raised by the HLR theory that led to many further developments in subsequent years [7]. The most crucial difficulty comes from considering the limit in which the Coulomb interaction energy is smaller than the Landau-level spacing. This limit is routinely used in numerical calculations used to confirm the HLR theory and is not an unreasonable approximation for experiments. Then it is appropriate to define the quantum Hall problem by projecting the Coulomb interaction to the highest occupied Landau level and ignoring all other levels. [At $\nu = \frac{1}{2}$, this means that we define the problem entirely within the lowest Landau level (LLL).] In this limit, there is only interaction energy as the electron kinetic energy is completely quenched by the Landau-level structure. Correspondingly, there is a single energy scale that is set by the interaction strength; for the Coulomb interaction, this energy scale is $\frac{e^2}{l_B}$ where e is the electron charge and l_B is the magnetic length. Formally, the projection to the LLL can be thought of as the limit where the bare electron mass $m \to 0$. The HLR theory, however, is not faithful to this projection. Indeed, in the mean-field approximation of HLR the composite fermion effective mass is the same as the bare electron mass m. Thus it is important to understand how to implement the physics of flux attachment while working purely within a single Landau level. A second crucial shortcoming—specific to electrons at $\nu = 1/2$—has to do with a particle-hole symmetry present at that filling [8] when the electron motion is restricted to a single Landau level. This symmetry is known—through numerical calculations—to be preserved by the composite Fermi liquid ground state [9,10] but yet is not manifest in the HLR description and is possibly even absent. This issue has attracted tremendous attention in recent years, resulting in a proposal for a modified theory [11] of the half-filled Landau level in terms of a Dirac composite fermion. This proposal has been substantiated through field theoretic duality transformations and other physical arguments [12–17] and by numerical calculations [10]. There have also been discussions of the emergence of particle hole as an approximate symmetry within the HLR theory [18,19]. However, all this progress on particle-hole symmetry did not address the basic issue of projecting to a single Landau level but rather sidestepped it (see, e.g., the discussion in the review, Ref. [20]). Hence we will not focus on it in this paper.

Thus an outstanding question in the theory of composite Fermi liquids is to provide a microscopic derivation of an effective-field theory by working within the LLL. Despite considerable theoretical attention [21–25] in the 1990s, this old question (for a review, see Ref. [7]) remained unanswered.

Theoretically, composite Fermi liquids are also expected to arise when the charged particles are bosons rather than fermions. For bosons at filling factor $\nu = 1$, attaching one flux quantum converts the bosons to composite fermions moving in an effective field $B^* = 0$. Then a composite Fermi liquid state can arise. Numerical calculations [26] show that the true ground state (with, say, a repulsive contact interaction) is an incompressible fractional quantum hall state—the bosonic Pfaffian—obtained by condensing pairs of composite fermions. Nevertheless, the unpaired metallic composite fermion state is interesting to consider as a possible ground state (for some interaction), and as a parent state for understanding the bosonic Pfaffian. In pioneering work in the 1990s, Read developed [23] a LLL theory of the composite Fermi liquid state of bosons at filling $\nu = 1$. This work was based on an exact though redundant representation [22], introduced by Pasquier and Haldane, of the Hilbert space of these bosons in terms of composite fermions. The redundancy in the representation leads to constraints in the theory. A Hartree-Fock solution [22,23] leads to a compressible state with a composite Fermi surface. Fluctuations beyond Hartree-Fock were treated diagrammatically in Ref. [23] within a conserving approximation and lead to physically sensible results for response functions similar to, but not identical to, those in the HLR theory.

In this paper, we will revisit the theory of Ref. [23] to pose and answer several questions that follow from it. What is a low-energy effective-field theory for the composite Fermi liquid that results from this theory? How exactly is it related to the HLR action? Can one understand the emergence of the paired state in numerics within this microscopic analytic framework? Is it possible to generalize these results to other composite Fermi liquids? An answer to the first question was, in fact, suggested in Sec. II D of Read's paper. Specifically, the suggestion was that the low-energy theory consists of a Fermi surface of composite fermions coupled to a dynamical $U(1)$ gauge field a without a Chern-Simons term. The external background $U(1)$ gauge field A was then proposed to couple linearly to $\frac{da}{2\pi}$. This suggestion, which we review in Appendix A, was, however, not explicitly obtained based on the diagrammatic calculations in the bulk of Ref. [23]. A derivation of this suggested effective theory using field theoretic duality transformations was also provided much later in Ref. [27], however, this derivation is not faithful to the LLL limit. Even later, this effective action was also proposed [28] to arise when HLR is projected to the LLL limit through an emergent Berry phase of the composite fermions. Assuming its correctness, the Lagrangian of the suggested effective theory is clearly distinct from HLR; do the two Lagrangians describe the same universal aspects of the physics or do they describe distinct phases of matter?

In this paper, we obtain a low-energy effective theory that captures the microscopic formulation and results of Ref. [23]. We show that this takes the form of a *noncommutative field theory*, i.e., a theory defined in terms of fields that move in a space with noncommutative coordinates. The theory is expressed in terms of a single composite fermion field coupled to a noncommutative dynamical $U(1)$ gauge field a without a Chern-Simons term. This is a precise formulation of the suggestion made in Ref. [23] but with many crucial differences. Specifically, we will find that it does not have the form suggested there and which we review in Appendix A. We then use an approximate mapping—due to Seiberg and Witten [29]—for the long wavelength, low-amplitude gauge fluctuations between noncommutative and commutative field theories. We show that the resulting approximate commutative field theory action is precisely of the HLR form but with parameters (like the composite fermion effective mass) faithful to the energetics of the LLL. We also examine the energetic stability of the paired state in a Hartree-Fock calculation within the Pasquier-Haldane-Read framework, and find that the paired state indeed wins in agreement with the numerics. Finally, we will show that these methods can be readily generalized to describe composite Fermi liquids formed by a system of spin-1/2 bosons at a total filling $\nu_T = 1$. Such a spinful bosonic composite Fermi liquid has been found in numerical calculations.

Noncommutative field theories were first contemplated in physics a long time ago [30]. Interest in them re-emerged in the late 1990s (and faded in the early 2000s) as they appeared in certain limits of string theory; for reviews, see Refs. [31,32]. It has long been recognized that a single Landau level provides a wonderful physical example of noncommutative geometry (as the guiding center coordinates do not commute within a Landau level). A rigorous proof of the quantization of the Hall conductivity in the integer quantum Hall effect used methods of noncommutative geometry [33]. For incompressible quantum Hall states, a noncommutative effective field theory was proposed in Refs. [34,35]. This is a possible alternate to the successful standard commutative Chern-Simons topological quantum field theory description of topologically ordered quantum Hall states, and has the potential to be obtained from a LLL description. However, we are not aware of such a microscopic derivation. The noncommutative geometry of the Landau level also plays an important role in the work of Haldane [36] on the geometrical description of the fractional quantum Hall effect. For the composite Fermi liquids of interest in this paper, the relevance of noncommutative geometry was pointed out in Read's original paper [23] but the full formulation in terms of noncommutative field theory was not developed.

II. PRELIMINARIES

A. The basic problem

We consider bosons of charge-1 in a magnetic field B in two space dimensions at filling $\nu = 1$. We take the bosons to occupy states in the LLL whose degeneracy we denote N. Given a basis set $|m\rangle$ ($m = 1, \ldots.., N$) of one-particle states for the Landau level, the many-particle Hilbert space is defined by the states

$$|\psi\rangle = \Sigma_{\{m_i\}} a_{m_1,\ldots\ldots,m_N} |m_1, \ldots\ldots.., m_N\rangle, \tag{1}$$

with the $a_{m_1,\ldots,m_N}$ symmetric under permutations. Clearly, the number of particles equals the Landau-level degeneracy reflecting the filling $\nu = 1$. The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\int \frac{d^2\mathbf{q}}{(2\pi)^2} U(\mathbf{q})\rho_L(\mathbf{q})\rho_L(-\mathbf{q}). \tag{2}$$

Here $\mathbf{q}$ is the momentum and the Hermitian operators $\rho_L(\mathbf{q})$ satisfy the algebra [37]:

$$[\rho_L(\mathbf{q}), \rho_L(\mathbf{q}')] = 2i \sin\left(\frac{(\mathbf{q}\times\mathbf{q}')l_B^2}{2}\right)\rho_L(\mathbf{q}+\mathbf{q}'). \tag{3}$$

This is known as the Girvin-MacDonald-Platzman (GMP) algebra. $l_B^2 = \frac{1}{B}$ is the magnetic length. The $\rho_L(\mathbf{q})$ are (up to an overall $\mathbf{q}$-dependent factor that we absorb into the interaction) the physical density operators projected to the LLL. Then we have

$$U(\mathbf{q}) = e^{-\frac{q^2 l_B^2}{2}} U_0(\mathbf{q}), \tag{4}$$

where U_0 is the Fourier transform of the real-space microscopic two-body repulsion between the bosons. We will work with a delta-function repulsion so $U_0(\mathbf{q}) = U_0$ independent of $\mathbf{q}$. Note that the only length scale is l_B and the only energy scale in the problem is $\frac{U_0}{l_B^2}$. Unless specified, we will henceforth work in units where $l_B = 1$. (How this effective Hamiltonian is obtained by projecting a microscopic Hamiltonian with an infinite number of Landau levels to the LLL is explained well in the literature, see, e.g., Ref. [7].)

We will begin with a representation of the GMP algebra in terms of canonical fermion operators $c_\mathbf{k}$ found by Pasquier and Haldane [22], and developed extensively by Read [23]. We write

$$\rho_L(\mathbf{q}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} c^\dagger_{\mathbf{k}-\mathbf{q}} c_\mathbf{k} e^{i\frac{\mathbf{k}\times\mathbf{q}}{2}}. \tag{5}$$

The fermion operators satisfy the usual anticommutation relations:

$$\{c_\mathbf{k}, c^\dagger_{\mathbf{k}'}\} = (2\pi)^2\delta^{(2)}(\mathbf{k}-\mathbf{k}'). \tag{6}$$

This is a redundant description and requires dealing with a constraint:

$$\rho_R(\mathbf{q}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} c^\dagger_{\mathbf{k}-\mathbf{q}} c_\mathbf{k} e^{-i\frac{\mathbf{k}\times\mathbf{q}}{2}} = (2\pi)^2 \underline{\rho}\,\delta^{(2)}(\mathbf{q}). \tag{7}$$

Here $\underline{\rho} = \frac{B}{2\pi} = \frac{1}{2\pi l_B^2}$ is the mean density. It is readily seen that ρ_R satisfies a GMP algebra but with a sign opposite to Eq. (3). Furthermore, ρ_R commutes with ρ_L at all momenta and hence with the Hamiltonian itself. The constraint operators may (for large but finite N) be thought of as generators of $U(N)$ gauge transformations (corresponding to a large redundancy in representing the physical Hilbert space in terms of the fermion operators). The fermions $c_\mathbf{k}$ are interpreted as (the LLL version of) the composite fermions. For a discussion of a physical picture in terms of vortex attachment to the particles, we refer to Ref. [23]. Note that the $\mathbf{q} \to 0$ limit of Eq. (5) implies that the total number of composite fermions equals the number of physical bosons.

Substituting Eq. (5) into Eq. (2) gives a four-fermion Hamiltonian which must be solved together with the constraint Eq. (7) imposed. A simple Hartree-Fock approximation that respects translation symmetry seeks a solution where

$$\langle c^\dagger_\mathbf{k} c_\mathbf{k}\rangle \neq 0. \tag{8}$$

The resulting Hartree-Fock Hamiltonian takes the form

$$\mathcal{H}_{\mathrm{HF}} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \epsilon_\mathbf{k} c^\dagger_\mathbf{k} c_\mathbf{k}. \tag{9}$$

The composite fermions then form a Fermi sea and we get a mean-field description of a composite Fermi liquid. To treat fluctuations beyond Hartree-Fock, we note that the Hartree-Fock "order parameter" $c^\dagger_\mathbf{k} c_\mathbf{k}$ does not commute with $\rho_R(\mathbf{q})$ except at $\mathbf{q} = 0$. Thus the huge group of gauge transformations generated by ρ_R is broken spontaneously (Higgsed). The important fluctuations are those generated by $\mathbf{q} \approx 0$—these can be thought of as a $U(1)$ gauge field. Thus we should expect to end up with an effective description in terms of a Fermi surface of composite fermions coupled to an emergent dynamical $U(1)$ gauge field. The precise form of this effective theory and its relationship with HLR will be discussed in the bulk of this paper. We will show that the effective theory is conveniently formulated as a noncommutative field theory and that HLR emerges in a long-wavelength approximation.

B. Noncommutative field theory

To set the stage, we provide a lightning review of the basic formalism of noncommutative field theory. A detailed exposition may be found in Refs. [31,32]. Consider 2 + 1-dimensional space-time where the two spatial coordinates X and Y do not commute:

$$[X, Y] = i\Theta. \tag{10}$$

Here Θ—known as the noncommutative parameter—is a constant. We can think of X, Y as operators in a space of states. In the specific context of the LLL, X, Y will be the components of the guiding center coordinate. They are operators in the space of single-particle eigenstates of LLL. We are interested in fields that live in this noncommutative space. To that end, first let us define scalar functions $f(\mathbf{R})$ [where $\mathbf{R} = (X, Y)$]. As X, Y do not commute, we need to specify what we mean by $f(\mathbf{R})$. A standard choice is known as the Weyl ordering which defines functions in terms of their Fourier transform:

$$f(\mathbf{R}) = \int \frac{d^2\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{R}}\tilde{f}(\mathbf{k}). \tag{11}$$

Here $\tilde{f}(\mathbf{k})$ is an ordinary function of the ordinary momentum $\mathbf{k}$ whose components commute with each other. The plane-wave factor

$$\tau_\mathbf{k} \equiv e^{i\mathbf{k}\cdot\mathbf{R}} \tag{12}$$

may be defined through its power series expansion and fixes the ordering of X and Y. The inverse Fourier transform is readily obtained:

$$\tilde{f}(\mathbf{k}) = (2\pi)^{\frac{1}{2}} \int d^2\mathbf{R}\mathrm{Tr}\big(f(\mathbf{R})e^{-i\mathbf{k}\cdot\mathbf{R}}\big). \quad (13)$$

Here $\int$Tr is over the space spanned by $\mathbf{R}$. For notational convenience, we will drop the Tr symbol in the subsequent equations and simply write $\int d^2\mathbf{R}$.

Note that as $\mathbf{R}$ is an operator, we should regard $f(\mathbf{R})$ also as an operator. It will be convenient to associate to this operator an ordinary function $f(\mathbf{x})$ of commuting coordinates $\mathbf{x}$ by taking the ordinary inverse transformation of $\tilde{f}(\mathbf{k})$.

Next consider the product of two operator-valued functions $f(\mathbf{R})$ and $g(\mathbf{R})$. It is easy to see from the definition through the Fourier transforms that

$$f(\mathbf{R})g(\mathbf{R}) = \int \frac{d^2\mathbf{k}d^2\mathbf{k}'}{(2\pi)^3}\Big(e^{-i\frac{\Theta\mathbf{k}\times\mathbf{k}'}{2}}\tilde{f}(\mathbf{k})\tilde{g}(\mathbf{k}')\Big)e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}}. \quad (14)$$

Taking the ordinary inverse Fourier transform, we find that the ordinary function $f(\mathbf{x}) * g(\mathbf{x})$ that corresponds to this product is not the ordinary product of functions but rather to a modification known as the star product. Thus the Fourier transform of $f(\mathbf{x}) * g(\mathbf{x})$ at momentum $\mathbf{q}$ is

$$\int \frac{d^2\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\tilde{f}(\mathbf{q}-\mathbf{k})\tilde{g}(\mathbf{k})e^{-i\frac{\Theta\mathbf{k}\times\mathbf{q}}{2}}. \quad (15)$$

The star product is associative but not commutative. We will henceforth remove the tilde from the momentum space variables and simply use the argument (coordinate versus momentum space) as an identification of which object we are talking about. Derivatives of operators can also be readily defined and correspond to ordinary derivatives of the functions $f(\mathbf{x})$.

Given an operator-valued field $\phi(\mathbf{R})$, consider a term in a putative Lagrangian such as $\phi(\mathbf{R})\phi(\mathbf{R})$. Its integral $\int d^2\mathbf{R}\phi(\mathbf{R})\phi(\mathbf{R})$ can be written as an ordinary Fourier space integral and hence is precisely defined. Thus we can build field theories defined in noncommutative space. Equivalently, we can also work with the corresponding ordinary fields $\phi(\mathbf{x})$ and write the action with all products being star products. For instance, the noncommutative ϕ^4 theory has an action that can be written

$$S = \int d\tau d^2\mathbf{x}[\partial_\mu\phi * \partial_\mu\phi + r\phi * \phi + u\phi * \phi * \phi * \phi]. \quad (16)$$

From the definitions above, we see that the noncommutativity shows up only in the product that defines the quartic term.

Noncommutative gauge theories can be similarly defined. We will only need to work with $U(1)$ gauge fields $a_\mu(\mathbf{R}, \tau)$ which will again be defined in terms of their Fourier transform or the corresponding $a_\mu(\mathbf{x}, \tau)$. We use the latter below. The corresponding field strength is

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + i[a_\nu, a_\nu]_*. \quad (17)$$

Here we introduced the $*$ commutator:

$$[A, B]_* = A * B - B * A. \quad (18)$$

Gauge transformations correspond to

$$a_\mu \to a_\mu + \partial_\mu\lambda + i[a_\mu, \lambda]_*. \quad (19)$$

These leave the field strength $f_{\mu\nu}$ invariant.

C. Summary of results

We can now state the main results of this paper. We begin by showing that an effective low-energy theory that describes the results of Ref. [23] is a noncommutative gauge theory with the imaginary time Lagrangian in terms of a composite fermion field c:

$$\mathcal{L} = \bar{c} * D_0 c + ia_0\underline{\rho} + \frac{1}{2m^*}\overline{D_i c}D_i c. \quad (20)$$

Here the covariant derivatives are defined through

$$D_\mu c = \partial_\mu c - ic * a_\mu - iA_\mu * c, \quad (21)$$

where a_μ ($\mu = 0, 1, 2$) is a dynamical $U(1)$ gauge field and A_μ is an external background $U(1)$ gauge field. The noncommutative parameter $\Theta = -l_B^2$. The composite fermion effective mass m^* is determined by the interaction strength. The composite fermions have a density $\underline{\rho}$. The theory thus takes the expected form of a Fermi surface of composite fermions coupled to a dynamical $U(1)$ gauge field. Furthermore, in agreement with what is claimed in Ref. [23], this theory has no Chern-Simons term for the dynamical gauge field.

However, these results come with the added feature not mentioned in Ref. [23], namely, this is a noncommutative field theory. In contrast, the standard descriptions (such as HLR) of composite Fermi liquids obtained without paying restricting to the LLL is in terms of commutative field theories. We will show that there is an approximate mapping of the theory in Eq. (20) to a commutative field theory which remarkably is the same as the effective action of HLR (with some calculable subleading corrections) but with an effective mass set by the interaction strength.

The key technical tool we use is known as the Seiberg-Witten map. This map enables trading a noncommutative field theory for a commutative one in a systematic series expansion[1] in powers of the noncommutative parameter $\Theta = -l_B^2$. As Θ is dimensionful this should really be regarded as an expansion in $(l_B q)^2$, $l_B^2\delta\rho_L$ where q is the momentum of the gauge field, and $\delta\rho_L$ is the deviation of the real space density from it's mean, i.e., as a long wavelength, low-amplitude expansion.

Thus we conclude that though the noncommutative field theory Eq. (20) is a more microscopically faithful effective theory, its approximate equivalence to HLR in the long wavelength limit vindicates the use of HLR for addressing many universal physical properties.

We also examine the energetics of pairing of composite fermions within the Hartree-Fock theory. Within this treatment we find that the composite Fermi liquid is unstable to pairing. However the pairing gap is numerically small compared to the Fermi energy of the composite fermions. Thus fluctuations beyond mean field may affect the relative stability of the paired state as compared to the composite Fermi liquid.

[1]For pure noncommutative $U(1)$ gauge theory, there is an exact nonperturbative version [38–41] of the Seiberg-Witten map relating it to a commutative $U(1)$ gauge theory. A physically appealing understanding [42] (see also Ref. [34]) of this result relates it to the map in fluid dynamics between the Lagrangian and Eulerian frameworks.

In particular it is known that gauge fluctuations oppose pairing. Nevertheless the mean field stability of the paired state is encouraging and agrees with existing numerical results.

We generalize these methods to study the problem of two-component bosons with full $U(2)$ symmetry at a total filling fraction $\nu_T = 1$. Numerical work [43,44] has indicated the presence of a spin unpolarized composite Fermi liquid at this filling. We develop an effective non-commutative field theory of this composite Fermi liquid, and show again that it reduces to a HLR form in the long wavelength, low amplitude approximation. At the mean-field level, we find a pairing instability which is somewhat weaker than for the spinless case.

III. PASQUIER-HALDANE-READ CONSTRUCTION FOR COMPOSITE FERMION

A. Parton construction and gauge structure

We begin with a brief discussion of some physical pictures [45] that motivate the formal parton construction of the composite fermion. In the LLL, the process of flux attachment to the particles to produce the composite fermion is replaced by the concept of vortex attachment. The vortex comes with a depletion of charge density at its core. A heuristic argument shows that the depletion is precisely equal to the boson charge. Thus we may view the composite fermion as a bound state of the boson with electric charge $+1$ and a vortex with electric charge -1. Such a bound state is electrically neutral but has a dipole moment $\mathbf{d}$ that is determined[2] by the composite fermion momentum $\mathbf{k}$:

$$\mathbf{d} = l_B^2 \mathbf{k} \times \hat{z}. \tag{22}$$

This dipole moment gives an appealing physical picture for how the composite fermion gets a dispersion. It is simply the polarization energy $\sim \mathbf{d}^2$ (for small $|\mathbf{d}|$) which leads to a $\mathbf{k}$-dependent energy.

The Pasquier-Haldane formalism begins with a redundant description of the Hilbert space of bosons at $\nu = 1$ in terms of the composite fermion Hilbert space. We will follow the presentation in Ref. [23]. Introduce the fermion operators $c_{nm}, c_{mn}^\dagger$ satisfying

$$\{c_{mn}, c_{m'n'}^\dagger\} = \delta_{mm'}\delta_{nn'}. \tag{23}$$

Here m, n are integers that range from 1 to N. Below we will identify N to be the total number of orbitals in a single Landau level, and will eventually take $N \to \infty$. The basis states of the physical Hilbert space [see Eq. (1)] of the bosons is then constructed as

$$|m_1, ..., m_N\rangle = \epsilon^{n_1 n_2 .... n_N} c_{n_1 m_1}^\dagger c_{n_2 m_2}^\dagger ...... c_{n_N m_N}^\dagger |0\rangle. \tag{24}$$

Here $|0\rangle$ is the Fock vacuum annihilated by all the c operators, and ϵ is fully antisymmetric with $\epsilon^{12....N} = 1$. A sum over repeated indices is assumed. Clearly, the states thus constructed are fully symmetric (and hence describe bosons) and correspond to a total number N of bosons in the N orbitals. Thus we describe bosons at filling $\nu = 1$. The antisymmetrization implied by the ϵ symbol implies that physical states are singlets under $SU(N)$ transformations of the n index of the c_{mn} operators. The generators of these transformations are

$$\rho_{nn'}^R = c_{nm}^\dagger c_{mn'} \tag{25}$$

Here we have included a global $U(1)$ generator given by $c_{nm}^\dagger c_{nm}$ so the ρ^R generate $U(N)$ transformations. The constraint that physical states are $SU(N)$ singlets can be restated in terms of these operators as

$$\rho_{nn'}^R |\psi_{\text{phys}}\rangle = \delta_{nn'} |\psi_{\text{phys}}\rangle. \tag{26}$$

These $SU(N)$ transformations express an $SU(N)$ gauge redundancy in the representation of the boson Hilbert space in terms of fermions. We can similarly define unitary $U(N)$ transformations on the physical left index m generated by

$$\rho_{mm'}^L = c_{nm'}^\dagger c_{mn.} \tag{27}$$

Unlike the transformations on the right index, the transformations generated by ρ^L are physical operations in the boson Hilbert space. Note that the overall $U(1)$ generator $N = \mathrm{Tr}\rho^L = \mathrm{Tr}\rho^R$ is shared by both the left and right generators (and in any case is fixed to be N as we work with a fixed total number of bosons). We will refer to ρ^L and ρ^R as the left and right densities, respectively.

We will now proceed slightly differently from the development in Ref. [23] in a manner suitable for our purposes. We will identify the m and n indices to denote orbitals in a single Landau level. Then we may regard the matrix c_{nm} as the matrix elements of an abstract operator c defined in such a Landau level. As is well known, a single Landau level is a noncommutative space. Specifically consider the guiding center coordinate $\mathbf{R}$ of a single particle moving in the LLL. Its components X, Y satisfy

$$[X, Y] = -il_B^2. \tag{28}$$

We can use $\mathbf{R}$ to define a magnetic translation operator:

$$\tau_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{R}}. \tag{29}$$

Any matrix in the LLL can be expanded in terms of the matrix elements of $\tau_{\mathbf{k}}$. Thus we write

$$c_{mn} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \langle m|\tau_{\mathbf{k}}|n\rangle c_{\mathbf{k}}. \tag{30}$$

The momentum space operator $c_{\mathbf{k}}$ is readily seen to satisfy the usual anticommutation relations:

$$\{c_{\mathbf{k}}, c_{\mathbf{k}'}^\dagger\} = (2\pi)^2 \delta^{(2)}(\mathbf{k} - \mathbf{k}'). \tag{31}$$

The densities $\rho^{L,R}$ may also be similarly expressed in momentum space. We define

$$\rho_{mm'}^L = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \langle m|\tau_k|m'\rangle \rho^L(\mathbf{k}), \tag{32}$$

$$\rho_{nn'}^R = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \langle n|\tau_k|n'\rangle \rho^R(\mathbf{k}). \tag{33}$$

It is readily seen that $\rho^L(\mathbf{k})$ satisfies the standard GMP algebra of Eq. (3) as expected for the physical boson density. The $\rho^R(\mathbf{k})$ also satisfies a GMP algebra but with the opposite

[2]This follows from the relation between position and momentum in the LLL.

sign:

$$[\rho^R(\mathbf{q}), \rho^R(\mathbf{q}')] = -2i \sin\left(\frac{(\mathbf{q}\times\mathbf{q}')l_B^2}{2}\right)\rho^R(\mathbf{q}+\mathbf{q}'). \quad (34)$$

Furthermore, ρ^L and ρ^R commute with each other. In momentum space, the constraint in Eq. (25) then becomes exactly Eq. (7). Furthermore, the left and right densities—when expressed in terms of the momentum space fermion operators—take precisely the forms given in Eqs. (5) and (7). Thus, as advertised before, the momentum space version of the Pasquier-Haldane representation faithfully reproduces the Hilbert space of bosons at $\nu = 1$.

The $c(\mathbf{k})$ are identified with composite fermion destruction operators at momentum $\mathbf{k}$. To bolster this interpretation, let us establish that these fermions have the correct dipole moment given by Eq. (22).

B. Dipole moment of the composite fermion

Consider the deviation $\delta\rho_L(\mathbf{q})$ of the physical density operator from its mean:

$$\delta\rho_L(\mathbf{q}) = \rho_L(\mathbf{q}) - \rho_L(\mathbf{q}=0). \quad (35)$$

Expanding to $o(\mathbf{q})$, we get

$$\delta\rho_L(\mathbf{q}) \simeq \int \frac{d^2k}{(2\pi)^2}\left(\frac{i}{2}\mathbf{k}\times\mathbf{q}c^\dagger_{\mathbf{k}}c_{\mathbf{k}} - \mathbf{q}\cdot\frac{\partial c^\dagger_{\mathbf{k}}}{\partial\mathbf{k}}c_{\mathbf{k}}\right) + o(q^2). \quad (36)$$

The second term involving $\frac{\partial c^\dagger_{\mathbf{k}}}{\partial\mathbf{k}}c_{\mathbf{k}}$ can be simplified using the definition of $c_{\mathbf{k}}$ but we will use a different argument to obtain its form in the physical Hilbert space. To the same order in $\mathbf{q}$, the deviation $\delta\rho_R(\mathbf{q})$ of the constraint density is

$$\delta\rho_R(\mathbf{q}) \simeq \frac{d^2k}{(2\pi)^2}\left(-\frac{i}{2}\mathbf{k}\times\mathbf{q}c^\dagger_{\mathbf{k}}c_{\mathbf{k}} - \mathbf{q}\cdot\frac{\partial c^\dagger_{\mathbf{k}}}{\partial\mathbf{k}}c_{\mathbf{k}}\right) + o(q^2). \quad (37)$$

Acting on physical states in the Hilbert space, we must have

$$\delta\rho_R(\mathbf{q})|\psi_{\text{phys}}\rangle = 0. \quad (38)$$

It follows that

$$\int \frac{d^2k}{(2\pi)^2}\mathbf{q}\cdot\frac{\partial c^\dagger_{\mathbf{k}}}{\partial\mathbf{k}}c_{\mathbf{k}}|\psi_{\text{phys}}\rangle = -i\int\frac{d^2k}{(2\pi)^2}\frac{\mathbf{k}\times\mathbf{q}}{2}c^\dagger_{\mathbf{k}}c_{\mathbf{k}}|\psi_{\text{phys}}\rangle. \quad (39)$$

Substituting in Eq. (36), we get

$$\delta\rho_L(\mathbf{q})|\psi_{\text{phys}}\rangle \simeq i\int\frac{d^2k}{(2\pi)^2}\,\mathbf{k}\times\mathbf{q}c^\dagger_{\mathbf{k}}c_{\mathbf{k}}|\psi_{\text{phys}}\rangle. \quad (40)$$

In the long wavelength limit, in real space, we write

$$\delta\rho_L(\mathbf{x}) = -\boldsymbol{\nabla}\cdot\mathbf{P}, \quad (41)$$

where $\mathbf{P}$ is the net dipole moment per unit area. In Fourier space we then have $\delta\rho_L(\mathbf{q}) = -i\mathbf{q}\cdot\mathbf{P}_{\mathbf{q}} \simeq -i\mathbf{q}\cdot P_{\mathbf{q}=0}$ to leading order in $\mathbf{q}$. Thus we identify

$$\mathbf{P}_{\mathbf{q}=0} = \int\frac{d^2k}{(2\pi)^2}\,\mathbf{k}\times\hat{\mathbf{z}}c^\dagger_{\mathbf{k}}c_{\mathbf{k}}. \quad (42)$$

Thus, each fermion at momentum $\mathbf{k}$ can be assigned a dipole moment exactly as given by Eq. (22).

C. Hartree-Fock theory

Substituting the expression for ρ_L in terms of the c fermions leads to a four-fermion Hamiltonian. In a Hartree-Fock treatment, Ref. [23] sought for and found a ground state with a filled Fermi sea. Here we will modify this treatment in two different ways. First, rather than write the Hamiltonian just in terms of $\delta\rho_L$ we will use $\rho_L - \rho_R$ for the (deviation from the mean of the) physical density. A second modification is that we will allow for mean-field states where the composite fermions are paired.

We start with the Hamiltonian

$$H = \frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\tilde{U}(\mathbf{q}) : \rho^L_{\mathbf{q}}\rho^L_{-\mathbf{q}} : \quad (43)$$

together with the gauge constraint

$$\rho^R_{\mathbf{q}}|\psi_{\text{phys}}\rangle = \underline{\rho}(2\pi)^2\delta^2(\mathbf{q})|\psi_{\text{phys}}\rangle. \quad (44)$$

The normal ordering in Eq. (43) is potentially important when we use the expression Eq. (5) for ρ_L in terms of the composite fermion operators. However, it is readily checked that

$$: \rho_L(\mathbf{q})\rho_L(-\mathbf{q}) : -\rho_L(\mathbf{q})\rho_L(-\mathbf{q}) = -\int\frac{d^2k}{(2\pi)^2}c^\dagger_{\mathbf{k}}c_{\mathbf{k}} = -\underline{\rho}. \quad (45)$$

As this is just a constant, we can remove the normal ordering from Eq. (43). We specialize to a δ-function repulsive interaction, with the projected two-body potential $\tilde{U}(\mathbf{q}) = U(\mathbf{q})e^{-q^2/2} = Ue^{-q^2/2}$.

It is straightforward to insert Eq. (5) and do a Hartree-Fock mean field of the resulting Hamiltonian, as was done by Read [23]. Restricting to unpaired translation-invariant states, the composite fermion acquires a dispersion. However, as observed already in Ref. [23], this mean-field treatment has some physically unsatisfactory features. To see this, note that the dispersion in this mean field is given by

$$\epsilon_k = \frac{1}{2}\tilde{U}(0)\int\frac{d^2k'}{2\pi}\langle c^\dagger_{\mathbf{k}'}c_{\mathbf{k}'}\rangle - \frac{1}{2}\int\frac{d^2k'}{2\pi}\tilde{U}(\mathbf{k}-\mathbf{k}')\langle c^\dagger_{\mathbf{k}}c_{\mathbf{k}'}\rangle. \quad (46)$$

The first is the Hartree term and the second is the Fock term. Thus the dispersion comes entirely from the Fock term. At first sight, this gives a sensible dispersion ϵ_k which increases monotonically as k increases. However, the origin of the dispersion through the Fock term is different from the physical picture for how the composite fermion gets a dispersion, namely, from the polarization energy due to its dipole moment. This polarization energy should have been a Hartree effect which, within this mean field, does not affect the dispersion.

This problem appears in a much more severe form if we we were to use the same formalism to treat a system of fermions at $\nu = 1$. Then, instead of composite fermions $c_{\mathbf{k}}$, we would introduce composite bosons $b_{\mathbf{k}}$. Proceeding as above would give a boson dispersion of the same general structure as Eq. (46) except that the Fock term now comes with a positive sign. This yields a problematic composite boson dispersion that monotonically *decreases* with k. Indeed, in this problem we simply expect to condense the composite boson at $\mathbf{k} = 0$ so as to recover the obvious answer (of a fully filled Landau level with an integer quantum Hall effect). The inability of this mean field to capture this physics suggests that it is not a good physical starting point.

Here we will rectify these problems by modifying the Hamiltonian (following Ref. [46]) by replacing $\delta\rho_L$ by $\rho_L - \rho_R$. Within the physical Hilbert space, both these expressions have the same matrix elements. However, as seen in the previous subsection, $\rho_L - \rho_R$ has the right long wavelength dipole moment and hence leads to a more physical Hartree-Fock mean field Hamiltonian.

Thus, we work with the modified Hamiltonian:

$$\tilde{H} = \frac{1}{2}\int \frac{d^2q}{(2\pi)^2}\tilde{U}(\mathbf{q})\big(\rho^L_{\mathbf{q}} - \rho^R_{\mathbf{q}}\big)\big(\rho^L_{-\mathbf{q}} - \rho^R_{-\mathbf{q}}\big). \tag{47}$$

We begin by writing this in normal ordered form

$$\begin{aligned}\tilde{H} &= \int \frac{d^2k}{2\pi}\int \frac{d^2q}{2\pi} 2\tilde{U}(\mathbf{q})\sin^2\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right)c^\dagger_{\mathbf{k}}c_{\mathbf{k}} \\ &\quad + \frac{1}{2}\int \frac{d^2q}{(2\pi)^2}\tilde{U}(\mathbf{q}) : (\rho^L_{\mathbf{q}} - \rho^R_{\mathbf{q}})(\rho^L_{-\mathbf{q}} - \rho^R_{-\mathbf{q}}) : \\ &= \int \frac{d^2k}{2\pi}(1 - e^{-\frac{k^2}{2}})U c^\dagger_{\mathbf{k}}c_{\mathbf{k}} \\ &\quad + \frac{1}{2}\int \frac{d^2q}{(2\pi)^2}\tilde{U}(\mathbf{q}) : (\rho^L_{\mathbf{q}} - \rho^R_{\mathbf{q}})(\rho^L_{-\mathbf{q}} - \rho^R_{-\mathbf{q}}) : . \end{aligned} \tag{48}$$

The two terms in the last line of Eq. (48) have clear physical meaning. The first term is a one-body term for the bare self-energy of a single dipole. This is precisely the polarization energy of the dipole which should contribute to the dispersion. Despite being interaction driven, it does not depend on the occupation n_k of other dipole states. The second term can be thought of as a dipole-dipole interaction, which is still a two-body term that needs a mean-field treatment.

The Hartree-Fock calculation procedure is standard. For the quartic term in Eq. (48), we adopt the translation and rotational symmetric mean-field ansatz,

$$\begin{aligned}\langle c^\dagger_{\mathbf{k}}c_{\mathbf{k}'}\rangle &= \delta^{(2)}(\mathbf{k} - \mathbf{k}')n_k, \\ \langle c_{\mathbf{k}}c_{\mathbf{k}'}\rangle &= \delta^{(2)}(\mathbf{k} + \mathbf{k}')d(k) = \delta(\mathbf{k} + \mathbf{k}')e^{il\theta_k}d(|k|),\end{aligned} \tag{49}$$

where the pairing angular momentum l is an odd integer. The effective mean-field Hamiltonian is reduced into a quadratic form

$$H_{\mathrm{MF}} = \int \frac{d^2k}{2\pi}\,\epsilon_k c^\dagger_{\mathbf{k}}c_{\mathbf{k}} + \Delta_{\mathbf{k}}c_{\mathbf{k}}c_{-\mathbf{k}} + \Delta^*_{\mathbf{k}}c^\dagger_{-\mathbf{k}}c^\dagger_{\mathbf{k}}. \tag{50}$$

We begin by ignoring the possibility of pairing. The best unpaired state has a filled circular Fermi surface (centered at $\mathbf{k} = 0$) of composite fermions with a Fermi momentum k_F determined in the usual way:

$$\frac{\pi k_F^2}{(2\pi)^2} = \underline{\rho}. \tag{51}$$

We identify this state with (the mean-field description of) the composite Fermi liquid. The composite fermion dispersion $\tilde{\epsilon}_k$, with occupation n_k given by the Fermi sea, is

$$\tilde{\epsilon}_{\mathbf{k}} = U\left(1 - e^{-\frac{k^2}{2}}\right) - 2Ue^{-\frac{k^2}{2}}\int_0^{k_F}\frac{dk'}{2\pi}k'e^{-\frac{k'^2}{2}}(I_0(kk') - 1), \tag{52}$$

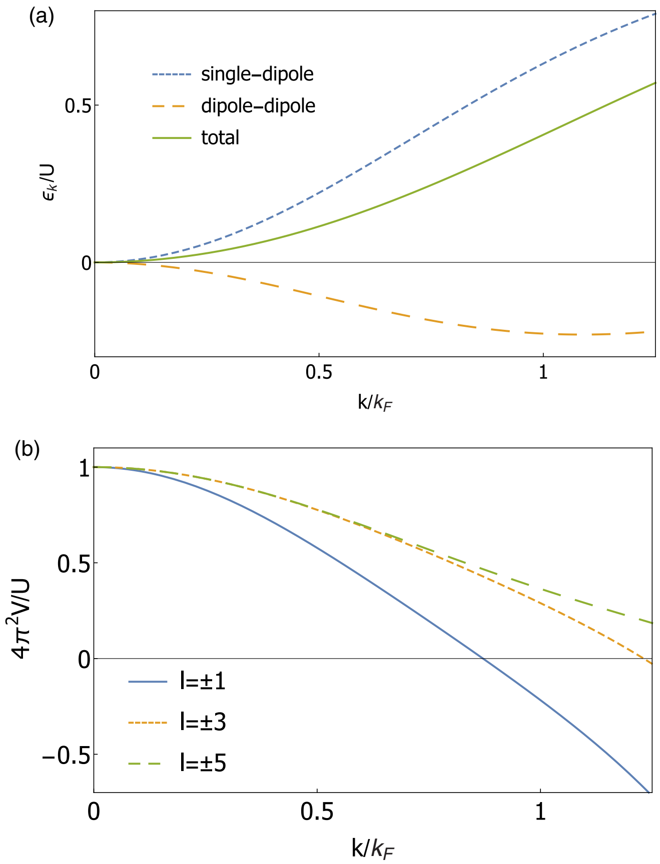

FIG. 1. Mean-field dispersion and partial wave components for different angular momentum. Horizontal axis show k/k_F. (a) Composite fermion dispersion at mean-field level without pairing included. The blue and orange curves denote contributions from normal ordering (marked as "single-dipole") and from normal-ordered term (marked as dipole-dipole), respectively. The green curve, their sum, is the total dispersion. (b) Partial wave components for different pairing channels, as in Eq. (54). The labels stand for corresponding angular momentum l. At the Fermi surface of composite fermion, only $l = \pm 1$ is attractive.

where $I_l(z)$ is the modified Bessel function. This modified dispersion replaces the one above in Eq. (46). The first term arises from the polarization energy of a single dipole and the second term from the dipole-dipole interaction. In Fig. 1, where the contributions from two terms are plotted separately, we show that the self-energy term dominates over dipole-dipole term for all k, which is sensible, since the interdipole contribution only screens and weakens the intradipole interaction. It can be checked that $\tilde{\epsilon}_k$ is a monotonically increasing function of k. The composite fermion effective mass m^* for states near the Fermi surface is given (within the Hartree-Fock approximation), as usual, by

$$\frac{K_F}{m^*} = \left[\frac{\partial\tilde{\epsilon}_{\mathbf{k}}}{\partial k}\right]_{k=k_F}. \tag{53}$$

Numerically, we find the value of effective mass to be $m^* = \frac{1.54}{U_0}$.

If instead we had solved the problem of fermions at $\nu = 1$ within the same framework using composite bosons, the coefficient of the first term in Eq. (52) is unaffected while the second term has the opposite sign. As the first term dominates, the composite boson dispersion has its minimum at $k = 0$ and increases with increasing k. Thus this modified mean field yields physically sensible answers for both bosons at $\nu = 1$ and for fermions at $\nu = 1$.

Returning to bosons at $\nu = 1$, the ground-state energy of the mea-field composite Fermi liquid is $0.2913UN$, where N is the Landau level degeneracy.

We now include the possibility of pairing of composite fermions. We will assume that any pairing that is found is weak in the sense that the pairing gap Δ is small compared to the Fermi energy E_F of the composite fermions. Note that as the only energy scale in the problem is U, there can be no parametric separation between Δ and E_F. Nevertheless this assumption is justified a posteriori as the solution we will find will have small $\frac{\Delta}{E_F}$.

First, we show that $l = \pm 1$ is the only possible pairing channel by examining the partial wave components of the pairing potential:

$$V_l(k) = \frac{U}{2} e^{-k^2} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(e^{i\frac{k^2}{2}\sin\theta} - e^{-i\frac{k^2}{2}\sin\theta} \right)^2 e^{k^2\cos\theta - il\theta}$$
$$= -U e^{-k^2} \left(I_l(2k^2) - 1 \right). \quad (54)$$

The behavior of $V_l(k)$ is plotted in Fig. 1. At the Fermi surface, only $l = \pm 1$ channels are attractive.

Now we carry out a numerical self-consistent calculation. As shown in Fig. 2, we find a stable solution in the $l = \pm 1$ pairing channel, whose energy is $0.2908U$, below that of the composite Fermi liquid solution. Furthermore the pairing is weak ($\Delta/E_F \ll 1$) thereby justifying our approximations.

In numerical exact diagonalization calculations [26] of the microscopic Hamiltonian for bosons at $\nu = 1$ it is found that the ground state is a gapped topologically ordered state that can be thought of as a paired state of composite fermions and not the composite Fermi liquid. It is interesting that the mean-field treatment described here captures this preference for a paired state. However, this mean-field treatment also has an artifact, namely, the degeneracy between $l = \pm 1$ pairing channels. This is a direct consequence of identifying the physical density operator with $\rho^L - \rho^R$, which makes the interaction term symmetric [46] under a discrete antiunitary operation that interchanges ρ_L and ρ_R. The constraint, however, does not have this symmetry. Thus we expect that fluctuations beyond the mean field will select between these two possibilities. The two mean-field paired ground states correspond to two topologically distinct states, both of which has a charged edge mode propagating along the direction fixed by the Landau level, and a neutral edge mode propagating in parallel or antiparallel directions.

Though the paired state wins over the composite Fermi liquid in the mean field, it is still interesting to consider the fate of the composite Fermi liquid beyond mean field. The energetic preference for the paired state may be altered by a different microscopic interaction; furthermore, it is conceivable, as the pairing is in any case weak, that there is a temperature window in which the physics of the composite Fermi liquid is relevant even if the true ground state is paired.

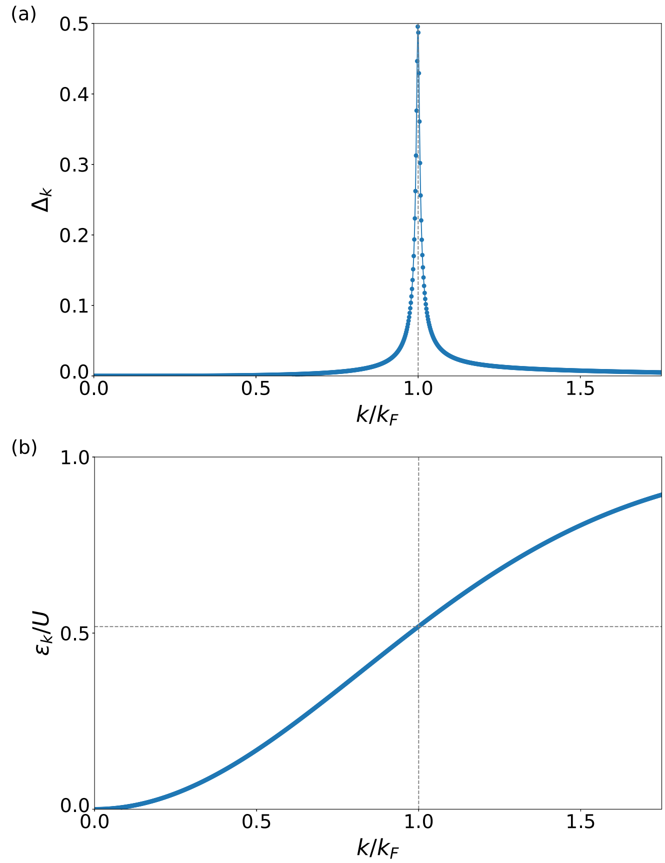

FIG. 2. The self-consistent Hatree-Fock mean-field solution without single-particle potential V, and with $l = +1$ pairing channel turned on. (a) $p + ip$ pairing order parameter. (b) Dispersion of composite fermion induced by interaction. The horizontal dotted line shows the chemical potential and the vertical dotted line marks $k = k_F$.

IV. FLUCTUATIONS ABOUT THE MEAN-FIELD COMPOSITE FERMI LIQUID: NONCOMMUTATIVE FIELD THEORY

In this section, we will go beyond the mean-field treatment and incorporate fluctuations to obtain an effective action for the composite Fermi liquid. The mean-field CFL state is not invariant under the right gauge transformations generated by $\rho_R(\mathbf{q})$ (except in the trivial limit $\mathbf{q} = 0$, which does not correspond to a generator of a gauge transformation),

$$c_{mn} \to c_{mn'} U^R_{n'n}, \quad (55)$$

where U^R is an $SU(N)$ matrix. As the $q = 0$ U_R rotations are unbroken, the important fluctuations beyond mean field are long wavelength rotations by U_R. To capture these, we will introduce a dynamical $U(1)$ gauge field a_μ that couples to the right three currents of the c fermions. We will also include a coupling to a background noncommutative gauge field A_μ that corresponds to left gauge transformations generated by ρ_L,

$$c_{mn} \to U^L_{mm'} c_{m'n}, \quad (56)$$

with U^L another $SU(N)$ matrix.

We begin with the effective Hartree-Fock action for the composite fermions. In what follows, we will replace the

Hartree-Fock dispersion $\epsilon_{\mathbf{k}}$ by a simpler quadratic dispersion,

$$\tilde{\epsilon}_{\mathbf{k}} \to \frac{\mathbf{k}^2}{2m^*}, \tag{57}$$

with the m^* given in Eq. (53). As the low-energy physics is dominated by states near the Fermi surface anyway, this replacement is innocuous: It only modifies the dispersion away from the Fermi surface. The imaginary time Hartree-Fock action may then be written

$$\mathcal{S}_{\rm HF} = \int d\tau \frac{d^2\mathbf{k}}{(2\pi)^2}\ \bar{c}_{\mathbf{k}} \frac{dc_{\mathbf{k}}}{d\tau} - \left(\frac{\mathbf{k}^2}{2m^*}\right) c^\dagger_{\mathbf{k}} c_{\mathbf{k}}. \tag{58}$$

It is understood that the fermions are at a nonzero mean density $\underline{\rho}$. This could be implemented explicitly by including a chemical potential term but we will not do so.

To proceed, we could try to continue to work in k space; however, the action of the gauge fluctuations in k space is complicated and mixes fermion operators at different momenta. We could try going back to the matrix operators c_{mn} on which the gauge transformations act simply. However, then the mean-field action looks complicated.[3]

These problems are nicely circumvented by passing to a slightly abstract real space formulation in non-commutative space without choosing any basis for the Landau level. To that end, we *define* the composite fermion field $c(\mathbf{R}, \tau)$ as a function of the noncommutative guiding center coordinate $\mathbf{R}$, and imaginary time τ:

$$c(\mathbf{R}, \tau) = \int \frac{d^2\mathbf{k}}{(2\pi)^{\frac{1}{2}}} e^{i\mathbf{k}\cdot\mathbf{R}} c_{\mathbf{k},\tau}, \tag{59}$$

$$\bar{c}(\mathbf{R}, \tau) = \int \frac{d^2\mathbf{k}}{(2\pi)^{\frac{1}{2}}} e^{-i\mathbf{k}\cdot\mathbf{R}} \bar{c}_{\mathbf{k},\tau}. \tag{60}$$

As emphasized in Sec. II B, functions of $\mathbf{R}$ should be viewed as operators that act within the space of single-particle LLL states. In a basis $\{|m\rangle\}$ for these single-particle LLL states, the matrix elements of the (Grassmann-valued) function $c(\mathbf{R}, \tau)$ are

$$\langle m|c(\mathbf{R}, \tau)|n\rangle = \int \frac{d^2\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \langle m|e^{i\mathbf{k}\cdot\mathbf{R}}|n\rangle c_{\mathbf{k},\tau}. \tag{61}$$

Thus we see that these matrix elements are precisely the Grassmann fields c_{mn} corresponding to the matrix-values composite fermion operators we have been working with.

As explained in Sec. II B, such noncommutative fields can be traded for fields $c(\mathbf{x})$ defined in ordinary space $\mathbf{x}$ so long as we modify products to star products. We will use this formulation below.

In terms of these fields, the mean-field action becomes

$$\mathcal{S}_{\rm HF} = \int d\tau d^2\mathbf{x} \left(\bar{c}(\mathbf{x}, \tau) * \frac{dc_{\mathbf{x}}}{d\tau} + \frac{1}{2m}\nabla\bar{c}(\mathbf{x}, \tau) * \nabla c(\mathbf{x}, \tau)\right) \tag{62}$$

The right and left gauge transformations of Eqs. (55) and (56) act on $c(\mathbf{x}, \tau)$ through

$$c(\mathbf{x}, \tau) \to U^L(\mathbf{x}, \tau) * c(\mathbf{R}, \tau) * U^R(\mathbf{x}, \tau), \tag{63}$$

with $U^{L,R} = e^{i\theta_{L,R}(\mathbf{x},\tau)}$; the exponential is defined through its power series with all products being star products.

To obtain a gauge invariant effective action for the fluctuations, we introduce a dynamical noncommutative $U(1)$ gauge field $a_\mu(\mathbf{x}, \tau)$ and a background $U(1)$ gauge field $A_\mu(\mathbf{x}, \tau)$. Under the gauge transformation Eq. (63), these transform as

$$a_\mu \to U_R^\dagger * a_\mu * U_R + iU_R^\dagger * \partial_\mu U_R, \tag{64}$$

$$A_\mu \to U_L * A_\mu * U_L^\dagger + i\partial_\mu U_L * U_L^\dagger. \tag{65}$$

We will assume that a_μ and A_μ are both slowly varying on the scale of the magnetic length. They thus respond to long wavelength gauge transformations, which is what we are interested in. The important components of the fermion fields are, however, not at low momenta but rather at momenta close to the Fermi surface.

We now recall the covariant derivatives introduced in Eq. (21):

$$D_\mu c = \partial_\mu c - ic * a_\mu - iA_\mu * c. \tag{66}$$

Under the gauge transformations of Eq. (63), these derivatives transform as

$$D_\mu c \to U^L(\mathbf{x}, \tau) * D_\mu c(\mathbf{x}, \tau) * U^R(\mathbf{x}, \tau). \tag{67}$$

It will be useful below to also note the infinitesimal form of these gauge transformations. Under an infinitesimal right gauge transformation $U_R = 1 + i\theta_R$, we have

$$c \to c + ic * \theta_R, \tag{68}$$

$$a_\mu \to a_\mu + \partial_\mu\theta_R + i(a_\mu * \theta_R - \theta_R * a_\mu), \tag{69}$$

and under an infinitesimal left gauge transformation $U_L = 1 + i\theta_L$, we have

$$c \to c + i\theta_L * c, \tag{70}$$

$$A_\mu \to A_\mu + \partial_\mu\theta_L + i(\theta_L * A_\mu - A_\mu * \theta_L). \tag{71}$$

We can now construct an effective action that correctly captures the effect of gauge fluctuations about the mean-field state; we simply replace all the derivatives in Eq. (62) by covariant derivatives to get Eq. (20):

$$\mathcal{S} = \int d^2\mathbf{x} d\tau\ \ \bar{c} * D_0 c + ia_0\underline{\rho} + \frac{1}{2m^*}\overline{D_i c} D_i c. \tag{72}$$

Thus, the formulation as a noncommutative field theory readily allows us to identify and to formulate a theory of the important fluctuations beyond mean field. The fermions are at the nonzero density $\underline{\rho}$. As promised, this is a theory of a Fermi surface coupled to a $\overline{U}(1)$ gauge field without a Chern-Simons term; however, the theory is defined in noncommutative space.

V. APPROXIMATION AS A COMMUTATIVE FIELD THEORY

The noncommutative effective field theory is the natural result of describing the composite Fermi liquid in the LLL. It

[3]The same is true if we choose explicit wave functions for the Landau orbitals, for instance, in the symmetric gauge to define fermion operators [23] $c(z, \bar{w})$ as a function of a holomorphic coordinate z and an antiholomorphic coordinate $\bar{w}$.

is hard to directly compare it to other proposed field theories for this composite Fermi liquid which are defined in commutative space (corresponding to the absence of Landau-level projection). However, the noncommutativity of space occurs at the scale of the magnetic length l_B, and we might suppose that for fluctuations at a wavelength much bigger than l_B there is an approximate commutative effective field theory. Interestingly, precisely such an approximate mapping between noncommutative and commutative gauge theories was discovered in a well-known paper by Seiberg and Witten [29]. Here we extend the Seiberg-Witten map to include fermion fields and apply it to the field theory of the composite Fermi liquid. This will enable us to obtain an approximate commutative field theory for long-wavelength fluctuations of the composite Fermi liquid.

A. The Seiberg-Witten map

The Seiberg-Witten map is usually presented as an expansion of the noncommutative fields and gauge-transformation parameters in powers of the noncommutativity parameter $\Theta = -l_B^2$. The coefficients in this expansion are expressed in terms of commutative fields and corresponding gauge transformation parameters. We will follow this presentation here.

We formally seek a map from noncommutative fields and gauge-transformation parameters $(a_\mu, A_\mu, c, \theta_R, \theta_L)$ to commutative fields and gauge-transformation parameters denoted $(\widehat{a}_\mu, \widehat{A}_\mu, \psi, \widehat{\theta}_R, \widehat{\theta}_L)$ of the form

$$\begin{aligned} a_\mu &= a_\mu(\widehat{a}_\nu), \\ A_\mu &= A_\mu(\widehat{A}_\nu), \\ c &= c(\psi, \widehat{a}_\mu, \widehat{A}_\mu), \\ \theta_R &= \theta_R(\widehat{a}_\mu, \widehat{\theta}_R), \\ \theta_L &= \theta_L(\widehat{A}_\mu, \widehat{\theta}_L). \end{aligned} \tag{73}$$

We require that the hatted fields satisfy the standard commutative gauge transformations:

$$\begin{aligned} \widehat{a}_\mu &\to \widehat{a}_\mu + \partial_\mu \widehat{\theta}_R, \\ \widehat{A}_\mu &\to \widehat{A}_\mu + \partial_\mu \widehat{\theta}_L, \\ \psi &\to \psi + i\psi(\widehat{\theta}_L + \widehat{\theta}_R). \end{aligned} \tag{74}$$

It is *a priori* not clear that such a map will exist but we will find it explicitly to linear order in Θ. Furthermore, the map will determine the noncommutative fields at a space-time point in terms of the commutative fields (and their derivatives) at the same point. Finally note that the Seiberg-Witten map relates the gauge-transformation parameters in a manner that depends on the gauge-field configurations.

The assumption that the map is analytic around $\Theta = 0$ allows us to write it down as

$$\begin{aligned} A(\widehat{A}) &= \widehat{A} + \Delta A(\widehat{A}), \quad a(\widehat{a}) = \widehat{a} + \Delta a(\widehat{a}), \\ \theta_L(\widehat{\theta}_L, \widehat{A}) &= \widehat{\theta}_L + \Delta\theta_L(\widehat{\theta}_L, \widehat{A}), \\ \theta_R(\widehat{\theta}_R, \widehat{a}) &= \widehat{\theta}_R + \Delta\theta_R(\widehat{\theta}_R, \widehat{a}), \\ c(\psi, \widehat{A}, \widehat{a}) &= \psi + \Delta\psi(\psi, \widehat{A}, \widehat{a}). \end{aligned} \tag{75}$$

Here ΔA, Δa, $\Delta\theta_R$, $\Delta\theta_L$, and Δc are all of $o(\Theta)$.

To determine the map, we start with two sets of fields in commutative space $(\widehat{A}, \widehat{a}, \psi)$ and $(\widehat{A}', \widehat{a}', \psi')$, which differ by a standard infinitesimal gauge transform in Eq. (74). The map, if it exists, should send them, respectively, to two sets of fields in noncommutative space (A, a, c) and (A', a', ψ), which by assumption are also connected through a gauge transform in noncommutative space as described by Eqs. (68)–(71). Moreover, the two gauge-transformation parameters for the commutative fields and their noncommutative counterparts are related by the second line of Eq. (75). The constraints are written as

$$\begin{aligned} &A_\mu(\widehat{A} + \partial\widehat{\theta}_L) - A_\mu(\widehat{A}) = \partial_\mu\theta_L(\widehat{\theta}_L, \widehat{A}) + i\big[\theta_L(\widehat{\theta}_L, \widehat{A}), A_\mu(\widehat{A})\big]_*, \\ &a_\mu(\widehat{a} + \partial\widehat{\theta}_R) - a_\mu(\widehat{a}) = \partial_\mu\theta_R(\widehat{\theta}_R, \widehat{a}) + i\big[a_\mu(\widehat{a}), \theta_R(\widehat{\theta}_R, \widehat{a})\big]_*, \\ &c\big(\psi + i\psi(\widehat{\theta}_L + \widehat{\theta}_R), \widehat{A} + \partial\widehat{\theta}_L, \widehat{a} + \partial\widehat{\theta}_R\big) - c\big(\psi, \widehat{A}, \widehat{a}\big) \\ &\quad = i\big(\theta_L(\widehat{\theta}_L, \widehat{A}) * c(\psi, \widehat{A}, \widehat{a}) + c(\psi, \widehat{A}, \widehat{a}) * \theta_R(\widehat{\theta}_R, \widehat{a})\big), \end{aligned} \tag{76}$$

where we have used a shorthand notation $[A, B]_* = A * B - B * A$. Now, Eq. (76) can be expanded using Eq. (75). By comparing terms up to first order in θ_L, θ_R and Θ, one can show the map is

$$\begin{aligned} \Delta A_\mu(\widehat{A}) &= -\frac{\Theta}{2}\epsilon^{\nu\rho}\widehat{A}_\nu(\partial_\rho\widehat{A}_\mu + \widehat{F}_{\rho\mu}), \\ \Delta a_\mu(\widehat{a}) &= +\frac{\Theta}{2}\epsilon^{\nu\rho}\widehat{a}_\nu(\partial_\rho\widehat{a}_\mu + \widehat{f}_{\rho\mu}), \\ \Delta\theta_L(\widehat{\theta}_L, \widehat{A}) &= -\frac{\Theta}{2}\epsilon^{\mu\nu}\widehat{A}_\mu\partial_\nu\widehat{\theta}_L, \\ \Delta\theta_R(\widehat{\theta}_R, \widehat{a}) &= +\frac{\Theta}{2}\epsilon^{\mu\nu}\widehat{a}_\mu\partial_\nu\widehat{\theta}_R, \\ \Delta\psi(\psi, \widehat{A}, \widehat{a}) &= +\frac{\Theta}{2}\epsilon^{\mu\nu}\big[(\widehat{a}_\mu - \widehat{A}_\mu)\partial_\nu\psi - i\widehat{a}_\mu\widehat{A}_\nu\psi\big], \end{aligned} \tag{77}$$

where $\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{\mu\nu} = 0$ for all other entries. $\widehat{F}_{\mu\nu} = \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu$, $\widehat{f}_{\mu\nu} = \partial_\mu\widehat{a}_\nu - \partial_\nu\widehat{a}_\mu$ are the field strengths.

B. Emergence of HLR theory

Using the Seiberg-Witten map, we can rewrite the noncommutative Lagrangian in Eq. (72) in terms of the fields defined in commutative space to obtain a Lagrangian, formally to linear order in Θ. Let us write the resulting Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \tag{78}$$

where $\mathcal{L}_0$ is formally of order Θ^0 and $\mathcal{L}_1$ is formally of order Θ.

We then have

$$\begin{aligned} \mathcal{L}_0 = {} & \bar{\psi}\partial_0\psi - i(\widehat{a}_0 + \widehat{A}_0)\bar{\psi}\psi + i\widehat{a}_0\underline{\rho} \\ & + \frac{1}{2m^*}\big|\big(\partial_i - i(\widehat{a}_i + \widehat{A}_i)\big)\psi\big|^2. \end{aligned} \tag{79}$$

$\mathcal{L}_1$ has contributions coming from several pieces of the noncommutative Lagrangian. We begin with the term $ia_0\underline{\rho}$. Using the Seiberg-Witten map we see that the $o(\Theta)$ piece from this is

$$i\frac{\underline{\rho}\Theta}{2}\epsilon^{\alpha\beta}\widehat{a}_\alpha(\partial_\beta\widehat{a}_0+\widehat{f}_{\beta 0}). \tag{80}$$

It is readily seen to be the Chern-Simons term (after an integration by parts):

$$i\frac{\underline{\rho}\Theta}{2}\epsilon^{\alpha\beta\gamma}\widehat{a}_\alpha\partial_\beta\widehat{a}_\gamma. \tag{81}$$

Note that as $\Theta=-l_B^2$ and $\underline{\rho}=\frac{1}{2\pi l_B^2}$, the coefficient of the Chern-Simons term is precisely [4] $-\frac{1}{4\pi}$.

Next consider the contribution at $o(\Theta)$ from the term involving the covariant derivative. We split this into two parts coming from the two terms in the Seiberg-Witten map for the fermion fields:

$$\begin{aligned}\Delta\psi(\psi,\widehat{A},\widehat{a})&=\Delta^{(1)}\psi+\Delta^{(2)}(\psi),\\ \Delta^{(1)}\psi&=+\frac{\Theta}{2}\epsilon^{\mu\nu}(\widehat{a}_\mu-\widehat{A}_\mu)\partial_\nu\psi,\\ \Delta^{(2)}\psi&=-i\frac{\Theta}{2}\epsilon^{\nu\mu}\widehat{a}_\mu\widehat{A}_\nu\psi.\end{aligned} \tag{82}$$

Thus we write

$$\bar{c}*D_0c=\bar{\psi}\widehat{D}_0\psi+\mathcal{L}_{\tau,1}+\mathcal{L}_{\tau,2}, \tag{83}$$

where the $\mathcal{L}_{\tau,2}$ term comes from $\Delta\psi^{(2)}$ and $\mathcal{L}_{\tau,1}$ represents the remaining contributions. We also define $\widehat{D}_\mu=\partial_\mu-i(\widehat{a}_\mu+\widehat{A}_\mu)$ as the standard covariant derivative for the commutative fields. We then have

$$\begin{aligned}\mathcal{L}_{\tau,1}=-\frac{\Theta}{2}\epsilon^{\alpha\beta}\big\{&\partial_0(\widehat{a}_\beta-\widehat{A}_\beta)\bar{\psi}\partial_\alpha\psi+\partial_\beta(\widehat{a}_0-\widehat{A}_0)(\partial_\alpha\bar{\psi}\psi)\\ &+(\widehat{a}_\beta-\widehat{A}_\beta)\big(\partial_\alpha(\bar{\psi}\partial_0\psi)-i(\widehat{a}_0+\widehat{A}_0)\partial_\alpha(\bar{\psi}\psi)\big)\\ &+\big[i\widehat{a}_\alpha(\partial_\beta\widehat{a}_0+\widehat{f}_{\beta 0})-i\widehat{A}_\alpha(\partial_\beta\widehat{A}_0+\widehat{F}_{\beta 0})\big]\bar{\psi}\psi\big\}.\end{aligned} \tag{84}$$

These terms can be simplified, as we now explain. To that end, we consider the equation of motion obtained from $\mathcal{L}_0$ by varying the dynamical gauge fields. This gives

$$j^\mu=\frac{\delta\mathcal{L}}{\delta a_\mu}=0. \tag{85}$$

The spatial components yield the equation

$$\bar{\psi}\big(\widehat{D}_i\psi\big)-\big(\overline{\widehat{D}_i\psi}\big)\psi=0. \tag{86}$$

It follows that

$$\bar{\psi}\ \partial_i\psi=\tfrac{1}{2}\partial_i(\bar{\psi}\psi)+i(\widehat{a}_i+\widehat{A}_i)\bar{\psi}\psi, \tag{87}$$

$$\partial_i\bar{\psi}\ \psi=\tfrac{1}{2}\partial_i(\bar{\psi}\psi)-i(\widehat{a}_i+\widehat{A}_i)\bar{\psi}\psi. \tag{88}$$

[4] Note that we have obtained a coefficient of $o(1)$ from a term that is formally of order Θ. This is because we are at a density of composite fermions that is order $\frac{1}{|\Theta|}$. We will return to this point at the end of this section.

We use this to reduce the first line of $\mathcal{L}_{\tau,1}$ in Eq. (84) to

$$\begin{aligned}-\frac{\Theta}{2}\epsilon^{\alpha\beta}\bigg[&\big(\partial_0(\widehat{a}_\beta-\widehat{A}_\beta)+\partial_\beta(\widehat{a}_0-\widehat{A}_0)\big)\frac{1}{2}\partial_\alpha(\bar{\psi}\psi)\\ &-i(\widehat{a}_\alpha+\widehat{A}_\alpha)\big(\partial_\beta(\widehat{a}_0-\widehat{A}_0)-\partial_0(\widehat{a}_\beta-\widehat{A}_\beta)\big)\bar{\psi}\psi\bigg]\end{aligned} \tag{89}$$

or more compactly as

$$\begin{aligned}\frac{\Theta}{2}\epsilon^{\alpha\beta}\bigg[&\frac{1}{2}\partial_0\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\bar{\psi}\psi+i(\widehat{a}_\alpha+\widehat{A}_\alpha)\\ &\times\big(\partial_\beta(\widehat{a}_0-\widehat{A}_0)-\partial_0(\widehat{a}_\beta-\widehat{A}_\beta)\big)\bar{\psi}\psi\bigg].\end{aligned} \tag{90}$$

The second line of Eq. (84) can be written

$$\begin{aligned}-\frac{\Theta}{2}\epsilon^{\alpha\beta}[&(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\alpha(\bar{\psi}\widehat{D}_0\psi)\\ &+i(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\alpha(\widehat{a}_0+\widehat{A}_0)\bar{\psi}\psi].\end{aligned} \tag{91}$$

We may now sum together the first, second, and third lines of Eq. (84). Expanding out the resulting products of gauge fields, and using the antisymmetry of $\epsilon^{\alpha\beta}$, we find that the third line is exactly canceled by contributions from the other two lines. The remaining terms lead to

$$\begin{aligned}\mathcal{L}_{\tau,1}=-\frac{\Theta}{2}\epsilon^{\alpha\beta}\big(&-\tfrac{1}{2}\partial_0\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\bar{\psi}\psi\\ &+(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\alpha(\bar{\psi}\widehat{D}_0\psi)+i\partial_0(\widehat{a}_\alpha\widehat{A}_\beta)\bar{\psi}\psi\big).\end{aligned} \tag{92}$$

The last term is not gauge invariant under combined gauge transformations of $\widehat{a}$ and $\widehat{A}$. However, we show in Appendix B that it is exactly canceled by $\mathcal{L}_{\tau,2}$. Thus we have

$$\begin{aligned}\mathcal{L}_{\tau,1}+\mathcal{L}_{\tau,2}=\frac{\Theta}{2}\epsilon^{\alpha\beta}\big(&\tfrac{1}{2}\partial_0\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\bar{\psi}\psi\\ &-(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\alpha(\bar{\psi}\widehat{D}_0\psi)\big).\end{aligned} \tag{93}$$

In the corresponding action, we integrate the last term by parts (and throw away total derivative terms) to obtain

$$\begin{aligned}\mathcal{L}_{\tau,1}+\mathcal{L}_{\tau,2}=\frac{\Theta}{2}\epsilon^{\alpha\beta}\big(&\tfrac{1}{2}\partial_0\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\bar{\psi}\psi\\ &+\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\bar{\psi}\widehat{D}_0\psi\big).\end{aligned} \tag{94}$$

The spatial gradient term can be similarly handled. Details may be found in Appendix B. We show there that the leading order in Θ term is

$$\mathcal{L}_x=-\frac{\Theta}{2}\epsilon^{\alpha\beta}\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\frac{1}{2m^*}|\widehat{D}_i\psi|^2. \tag{95}$$

Combining all these contributions, we thus obtain (to linear order in Θ) the effective commutative Lagrangian:

$$\begin{aligned}\mathcal{L}&=\mathcal{L}_{\rm HLR}+\mathcal{L}_{\rm corr}\\ \mathcal{L}_{\rm HLR}&=\bar{\psi}\partial_0\psi-i(\widehat{a}_0+\widehat{A}_0)\bar{\psi}\psi+i\widehat{a}_0\underline{\rho}\\ &\quad+\frac{1}{2m^*}|\big(\partial_i-i(\widehat{a}_i+\widehat{A}_i)\big)\psi|^2-i\frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\widehat{a}_\alpha\partial_\beta\widehat{a}_\gamma,\\ \mathcal{L}_{\rm corr}&=\frac{\Theta}{2}\epsilon^{\alpha\beta}\bigg(-\frac{1}{2}\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\partial_0(\bar{\psi}\psi)\\ &\quad+\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)(\bar{\psi}\widehat{D}_0\psi-\frac{1}{2m^*}|\widehat{D}_i\psi|^2)\bigg).\end{aligned} \tag{96}$$

Remarkably, the first term $\mathcal{L}_{HLR}$ this is precisely the HLR action for the composite Fermi liquid, while the second term $\mathcal{L}_{corr}$ is a subleading correction. To make this identification, first note that $\widehat{A}$ represents an additional probe gauge field on top of the basic magnetic field B that defines the Landau level. If we introduce a vector potential $\underline{A}_\mu = (0, \underline{A}_x, \underline{A}_y)$ such that

$$\nabla \times \underline{\mathbf{A}} = B, \tag{97}$$

then the total external gauge field is

$$A_{\text{tot},\mu} = \widehat{A}_\mu + \underline{A}_\mu. \tag{98}$$

We then have

$$\mathcal{L}_{\text{HLR}} = \mathcal{L}[\psi, \widehat{a} - \underline{A} + A_{\text{tot}}] + i\widehat{a}_0\underline{\rho} - i\frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\widehat{a}_\alpha\partial_\beta\widehat{a}_\gamma. \tag{99}$$

We similarly define a new dynamical gauge field a_{tot} through

$$a_{\text{tot},\mu} = \widehat{a}_\mu - \underline{A}_\mu. \tag{100}$$

We then get

$$\mathcal{L}_{\text{HLR}} = \mathcal{L}[\psi, a_{\text{tot}} + A_{\text{tot}}] + ia_{\text{tot},0}\underline{\rho} - \frac{i}{4\pi}\epsilon^{\alpha\beta\gamma}(a_{\text{tot},\alpha} + \underline{A}_\alpha)\partial_\beta(a_{\text{tot},\gamma} + \underline{A}_\gamma). \tag{101}$$

Expanding out the Chern-Simons term and using $A_{\text{tot},0} = 0$, $\underline{\rho} = \frac{B}{2\pi}$, we get the Lagrangian

$$\mathcal{L}_{\text{HLR}} = \mathcal{L}[\psi, a_{\text{tot}} + A_{\text{tot}}] - \frac{i}{4\pi}\epsilon^{\alpha\beta\gamma}a_{\text{tot},\alpha}\partial_\beta a_{\text{tot},\gamma}, \tag{102}$$

which is the standard form of HLR. However, the HLR Lagrangian is usually derived through the flux attachment procedure without invoking the projection to the LLL. The composite fermion mass appearing in the usual HLR action is the bare electron mass. Here we have derived the HLR action within the LLL. It appears as an approximation to the more microscopically correct noncommutative field theory Eq. (20). The composite fermion mass that appears in the HLR action thus obtained is determined by the interactions.

Let us now examine the terms in $\mathcal{L}_{\text{corr}}$. In the absence of the probe background gauge field ($\widehat{A} = 0$), the first term is a coupling between the internal electric field and the density gradient. This is small so long as we limit ourselves to long wavelength density fluctuations. The second term involves corrections, of order $\delta\rho l_B^2 \ll 1$, to terms already present in $\mathcal{L}_{\text{HLR}}$. Here $\delta\rho$ is the fluctuation of the density in real space. (We used the relationship $\epsilon^{ij}\partial_i\widehat{a}_j = 2\pi\delta\rho$ implied by $\mathcal{L}_{\text{HLR}}$). Thus this is small, with the further assumption that we limit ourselves to small amplitude fluctuations of the density.

Thus the HLR Lagrangian emerges as an approximate description of the full noncommutative field theory for long-wavelength, low-amplitude gauge fluctuations. The crucial Chern-Simons term arises with a properly quantized coefficient $-\frac{1}{4\pi}$. Does the presence of a mean density of order $\frac{1}{|\Theta|}$ invalidate the expansion in powers of Θ? The mean density sets the Fermi momentum $k_F \sim \frac{1}{\sqrt{|\Theta|}}$. Clearly, we can not assume that the fermions are at long wavelength though the important gauge fluctuations are at long wavelength. It is thus reassuring that the smallness of the correction terms in $\mathcal{L}_{\text{corr}}$ only invoked the long-wavelength, low-amplitude limit for the gauge fluctuations. This then is a justification of the use of HLR theory for many physical properties (e.g., the compressibility or transport in the presence of a smooth impurity potential) even when restricted to the LLL. If, however, we are interested in universal short-distance properties, such as $2K_F$ singularities in density correlations, it may be safer to go back to the full noncommutative field theory.

Another consequence of the emergence of the Chern-Simons term with an $o(1)$ coefficient is that we must re-examine Eq. (86) for the current that we used to obtain the $o(\Theta)$ correction to the action. The Chern-Simons term will lead to an extra Hall current contribution to this equation which will lead to an additional correction to the HLR action. We do this in the Appendix and show that this extra correction is of the form

$$-\frac{\Theta}{4\pi}m^*\big((\widehat{f}_{01} - \widehat{F}_{01})\widehat{f}_{01} + (\widehat{f}_{02} - \widehat{F}_{02})\widehat{f}_{02}\big). \tag{103}$$

This is an innocuous correction for long-wavelength gauge fluctuations.

Can we understand why at the end of the day we only obtain a self Chern-Simons term for $\widehat{a}$? In particular, based on the interpretation of the composite fermion as a vortex, one might have expected a mutual Chern-Simons term of the form $\frac{i}{2\pi}\widehat{A} \wedge d\widehat{a}$ which is not found in our derivation. To understand this, we note that the noncommutative Lagrangian, apart from the term $-ia_0\underline{\rho}$ that comes from Lagrangian multiplier for the gauge constraint Eq. (7), has a symmetry $\widehat{A} \leftrightarrow \widehat{a}$, $\Theta \to -\Theta$. This symmetry of part of the Lagrangian precludes any mutual Chern-Simons term between $\widehat{a}$ and $\widehat{A}$. The term $-ia_0\underline{\rho}$ contributes only an internal Chern-Simons term $\sim a \wedge da$.

A few qualitative (and somewhat heuristic) remarks on the results of this section may be useful. A well-known way to understand the usual HLR construction (without the LLL restriction) for bosons at $\nu = 1$ is in terms of a traditional parton representation where we write the microscopic boson operator b as a product of two fermions:

$$b = \psi f. \tag{104}$$

This comes with a $U(1)$ gauge constraint $\psi^\dagger\psi = f^\dagger f$. This introduces a $U(1)$ gauge field. Further the total number of f (or ψ particles) equals the total number of bosons. We assume that the ψ fermions carry the global $U(1)$ charge of the boson and see the external magnetic field. The f fermions then are neutral under the global $U(1)$. In a mean-field description of the composite Fermi liquid, there is a mean internal gauge flux that cancels the external gauge flux. Then the ψ fermions see net effective zero magnetic field and form a Fermi surface while the f fermions are in an integer quantum Hall state with $\sigma_{xy} = 1$. Integrating out the f fermions, we get the standard HLR action with a Chern-Simons term for the fluctuations of the internal gauge field. Now, the c fermions occuring in the Pasquier-Haldane-Read formulation may roughly be thought of as the LLL version of the ψ fermions in the standard parton construction. The constraint that the right density does not fluctuate may be represented formally by introducing a filled Landau level of f fermions and writing

$$\rho^R_{nn'} = f_n^\dagger f_{n'}, \tag{105}$$

where f_n destroys an f fermion in the Landau orbital n. It is natural then that the contribution of the background density (which technically is the origin of the Chern-Simons term) gives a Chern-Simons term.

Finally, we briefly comment on the relationship to the ideas of Ref. [28] on the emergent Berry phase of composite fermions in the LLL. That paper proposed that as the LLL limit was taken, the composite fermions of the HLR theory will develop a Fermi-surface Berry phase of -2π (for bosons at $\nu = 1$). This Berry phase will then give an anomalous Hall effect for the internal gauge field that exactly cancels the Chern-Simons term of the original HLR theory. This then was suggested to be a way to reconcile the two effective Lagrangians discussed in Appendix A. The detailed analysis presented here partially supports this proposal but also shows its limitation. The correct effective noncommutative field theory in the LLL has no Chern-Simons term, but the right density operator expressed in terms of the composite fermions has a form factor $e^{-\frac{i}{2}\mathbf{k}\times\mathbf{q}}$. Considering this for small $|\mathbf{q}|$, we can think of this form factor as describing a Berry connection $\mathbf{A}(\mathbf{k})$ in momentum space:

$$\mathcal{A}(\mathbf{k}) = -\tfrac{1}{2}\hat{z}\times\mathbf{k}. \tag{106}$$

The corresponding Berry curvature is

$$\mathcal{B} = -1. \tag{107}$$

Thus we could say that the Chern-Simons term of the HLR theory has been accommodated instead by an anomalous Hall effect that will result from the form factor associated with the composite fermion density in the LLL theory. However, the full structure that results in the LLL is the noncommutative field theory and not the commutative effective-field theory of Eq. (A1). In the commutative approximation to the full noncommutative field theory, the density operator has no nontrivial form factor (and hence no Berry phase). The Seiberg-Witten map trades the theory of fermions with a gauge field coupling to densities with a nontrivial form factor to a theory of different fermions with a gauge field coupling to densities without such a form factor but with a Chern-Simons term.

VI. DOPING THE COMPOSITE FERMI LIQUID: THE JAIN STATES

Apart from their intrinsic interest, composite Fermi liquids also play a crucial role as parent states of the Jain series of gapped quantum Hall states at nearby fillings. For the bosonic composite Fermi liquid at $\nu = 1$, the nearby Jain states occur at a filling $\frac{p}{p+1}$ with p a large integer of either sign. Topological aspects of the Jain states are described by multicomponent abelian Chern-Simons gauge theories. These topological quantum field theories of course do not capture dynamical aspects of the state, for instance, the quasiparticle gaps or details of the magnetoroton mode, etc. However, for large $|p|$, both topological and some dynamical properties are universally determined by properties of the composite Fermi liquid at $\nu = 1$. Thus, armed as we are, with a LLL theory of the composite Fermi liquid we can obtain a LLL description of the large $|p|$ Jain states. This is not straightforward directly in the original Pasquier-Haldane-Read framework: Moving away from $\nu = 1$ requires using rectangular matrices c_{mn} which leads to technical complications. However, the effective field theory description readily allows us to dope away from $\nu = 1$.

To that end, it is simplest to just use the approximate mapping to the commutative theory described in the last section. If we initially ignore the extra $\mathcal{L}_{\rm corr}$ term, then there is no difference with the usual HLR theory. Moving away from $\nu = 1$ by changing the external magnetic field at fixed boson density, we have

$$\epsilon_{ij}\partial_i\widehat{A}_j = \delta B. \tag{108}$$

The internal magnetic field $\widehat{b} = \epsilon_{ij}\partial_i\widehat{a}_j$ has an average value

$$\langle\mathbf{b}\rangle = 2\pi\left(\langle\bar{\psi}\psi\rangle - \underline{\rho}\right) \;= 0. \tag{109}$$

As usual, the *net* average magnetic field seen by the composite fermions is

$$B^* = \delta B + \langle\widehat{b}\rangle \;= \delta B. \tag{110}$$

Jain states form when the composite fermions fill p Landau levels which happens when $\underline{\rho} = \frac{p\delta B}{2\pi} = \frac{p(B_{\rm tot}-B)}{2\pi}$, which gives a filling $\frac{2\pi\rho}{B_{\rm tot}} = \frac{p}{p+1}$.

Next consider the $\mathcal{L}_{\rm corr}$ term. The potentially important effect comes from the second term. Replacing $\widehat{b} - \delta B$ by its average $-\delta B$, we find the approximate Lagrangian:

$$\mathcal{L} = \left(1 + \frac{\Theta\delta B}{2}\right)\bar{\psi}\widehat{D}_0\psi + i\widehat{a}_0\underline{\rho} + \frac{1}{2m^*}\left(1 - \frac{\Theta\delta B}{2}\right)|\widehat{D}_i\psi|^2 - i\frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\widehat{a}_\alpha\partial_\beta\widehat{a}_\gamma. \tag{111}$$

We can now redefine the ψ field (and using $|\Theta\delta B| = |\frac{\delta B}{B}| \ll 1$ to set the coefficient of the time derivative to 1):

$$\tilde{\psi} \approx \left(1 + \frac{\Theta\delta B}{4}\right)\psi. \tag{112}$$

The Lagrangian then becomes

$$\tilde{\mathcal{L}} = \bar{\tilde{\psi}}\widehat{D}_0\tilde{\psi} + i\widehat{a}_0\underline{\rho} + \frac{1}{2\tilde{m}^*}|\widehat{D}_i\tilde{\psi}|^2 - i\frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\widehat{a}_\alpha\partial_\beta\widehat{a}_\gamma. \tag{113}$$

Thus the effect of $\mathcal{L}_{\rm corr}$ is to change the bare mass m^* to $\tilde{m}^*$ given by

$$\tilde{m}^* = m^*\left(1 - \frac{\delta B}{B}\right). \tag{114}$$

The Landau-level spacing of the composite fermions $\frac{\delta B}{\tilde{m}^*}$ gives a rough estimate of the gap of the Jain state.[5] Using the mean-field estimate for m^* from Eq. (53), we thus get an approximate gap for the large $|p|$ Jain states:

$$\Delta \approx 0.65U_0|\delta B|\left(1 + \frac{\delta B}{B}\right). \tag{115}$$

[5]This will be renormalized by gauge fluctuations which lead for small δB to a singular correction to the effective mass. So the effective mass given in Eq. (114) may be expected to capture the correct gap in a window of small but not too small $|\delta B|$.

In the future, it should be interesting to correctly obtain the coupling of the Jain states to geometry (and calculate the shift/Hall viscosity) within this framework.

VII. SPINFUL BOSONS IN LLL AT TOTAL FILLING $\nu_T = 1$

In this section, we generalize our results to a system of two-component bosons with global $U(2)$ symmetry in a magnetic field at a total filling factor $\nu_T = 1$. The physical Hilbert space is spanned by states fully symmetric under exchange of two particles:

$$|(m_1,\sigma_1),\ldots,(m_n,\sigma_n)\rangle\,, \tag{116}$$

where m_i label orbitals (in some basis) in the LLL, and σ_i is the $SU(2)$ spin of the ith particle. There is a total density operator $\rho^L_{\mathbf{q}}$ that satisfies the GMP algebra. In addition, there is a spin density operator $S^{L,\alpha}(\mathbf{q})$ ($\alpha = 1, 2, 3$) that satisfies the following commutation relations:

$$\begin{aligned}\left[S^{L,\alpha}_{\mathbf{q}},\rho^L_{\mathbf{q}'}\right] &= 2i\sin\left(\frac{\mathbf{q}\times\mathbf{q}'}{2}\right)S^{L,\alpha}_{\mathbf{q}+\mathbf{q}'},\\ \left[S^{L,\alpha}_{\mathbf{q}},S^{L,\beta}_{\mathbf{q}'}\right] &= 2i\epsilon^{\alpha\beta\gamma}\cos\left(\frac{\mathbf{q}\times\mathbf{q}'}{2}\right)S^{L,\gamma}_{\mathbf{q}+\mathbf{q}'}\\ &\quad+2i\delta^{\alpha\beta}\sin\left(\frac{\mathbf{q}\times\mathbf{q}'}{2}\right)\rho^L_{\mathbf{q}+\mathbf{q}'}.\end{aligned} \tag{117}$$

We will consider a Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}U(\mathbf{q})\rho^L_{\mathbf{q}}\rho^L_{-\mathbf{q}}. \tag{118}$$

The treatment can be readily generalized to a more general $U(2)$ symmetric Hamiltonian that includes, for example, an interaction between the spin densities.

This system has been studied numerically in Refs. [43,44] for a contact interaction and there is evidence for a spin-unpolarized composite Fermi liquid. Below we will provide an analytic microscopic theory.[6]

A. Pasquier-Haldane construction

The Pasquier-Haldane construction introduced in Sec. III A can be naturally generalized to include spin by introducing spinful composite fermion $c_{\sigma,mn}$ that satisfy anticommutation relations:

$$\{c_{\sigma,mn},c^\dagger_{n'm',\sigma'}\} = \delta_{\sigma,\sigma'}\delta_{mm'}\delta_{nn'}. \tag{119}$$

Many-body states in the physical Hilbert space are then represented by

$$|(m_1,\sigma_1),\ldots,(m_n,\sigma_n)\rangle = \epsilon^{n_1,\ldots,n_N}c^\dagger_{n_1;\sigma_1,m_1}\ldots c^\dagger_{n_Nm_N,\sigma_N}\,|0\rangle\,. \tag{120}$$

The antisymmetrization over internal indices n_i means that physical states are singlets under the $SU(N)$ right transformations generated by $\rho^R_{nn'} - \delta_{nn'}$ where $\rho^R_{nn'}\,|\psi\rangle = \delta_{nn'}\,|\psi\rangle$, where the right density is now

$$\rho^R_{nn'} = \sum_{m\sigma}c^\dagger_{nm,\sigma}c_{\sigma,mn'}. \tag{121}$$

Thus we have the constraint

$$\rho^R_{nn'}\,|\psi_{\text{phys}}\rangle = \delta_{nn'}\,|\psi_{\text{phys}}\rangle\,. \tag{122}$$

We can now go to momentum space using the plane-wave operators $e^{i\mathbf{q}\cdot\mathbf{R}}$. It is readily checked that the $\rho^L_{\mathbf{q}}, S^{L,\alpha}_{\mathbf{q}}$ satisfy the commutation algebra of Eqs. (117). Furthermore, just as before, the right density operator ρ^R satisfies the GMP algebra but with the opposite sign from ρ^L. The ρ^R also commute with $\rho^L, S^{L,\alpha}$.

We note that we can define a right spin density operator

$$S^{R,\alpha}_{nn'} = \frac{1}{2}\sum_{ss'm}c^\dagger_{nm,s}\sigma^\alpha_{ss'}c_{s,mn'} \tag{123}$$

which also has vanishing matrix elements between physical states. To show this, consider a matrix element of the commutator of right spin density and right density. By virtue of the gauge constraint Eq. (38),

$$\langle\psi_1|\,[S^{R,\alpha}_{\mathbf{q}},\rho^R_{\mathbf{q}'}]\,|\psi_2\rangle = 0 \text{ for } \forall\mathbf{q}'\neq 0, \tag{124}$$

where $|\psi_1\rangle\,, |\psi_2\rangle$ are physical states that satisfy $\rho^R_{\mathbf{q}}\,|\psi_{1,2}\rangle = 0$ for $\forall q\neq 0$. It follows therefore from the commutation algebra in Eq. (117) that

$$\langle\psi_1|\,S^{R,\alpha}_{\mathbf{q}}\,|\psi_2\rangle = 0 \text{ for } \forall\psi_1,\psi_2, q\neq 0. \tag{125}$$

However, unlike $\delta\rho^R$, the operator $S^{R,\alpha}$ (for $\mathbf{q}\neq 0$) does not simply annihilate physical states. Rather it takes physical states to unphysical states. We illustrate this with an explicit example in Appendix D. So, the right spin density is not a generator for gauge fluctuations and the gauge structure of our spinful composite fermion construction is still $SU(N)$.

B. Hartree-Fock theory

We can now proceed completely similarly to our previous discussion. The Hamiltonian is expressed in terms of the c fermions, and the resulting four-fermion term can be solved within a Hartree-Fock approximation. We first describe a spin-unpolarized composite Fermi liquid solution (no pairing terms) with

$$\langle c^\dagger_{\mathbf{k}s}c_{\mathbf{k}s'}\rangle = n_{\mathbf{k}}\delta^{(2)}(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \tag{126}$$

with $n_{\mathbf{k}} = 1$ for $\mathbf{k}$ inside a circular Fermi surface of radius k_F, and zero otherwise. The Fermi momentum k^s_F is the one appropriate for spinful fermions, i.e., it satisfies

$$\frac{2\pi(k^s_F)^2}{(2\pi)^2} = \underline{\rho}. \tag{127}$$

We then get the dispersion for the spinful composite fermion:

$$\tilde{\epsilon}_{\mathbf{k}} = U(1-e^{-\frac{k^2}{2}}) - 2Ue^{-\frac{k^2}{2}}\int_0^{k^s_F}dk'\,k'e^{-\frac{k'^2}{2}}(I_0(kk')-1). \tag{128}$$

The dispersion incorporates two terms as described in the spinless case before. The first term is the intradipole interaction, which is unchanged compared to Eq. (52). The second

[6]It is also easy to treat N-component bosons with global $U(N)$ symmetry at a total filling $\nu_T = 1$ for general N but we will not do so here.

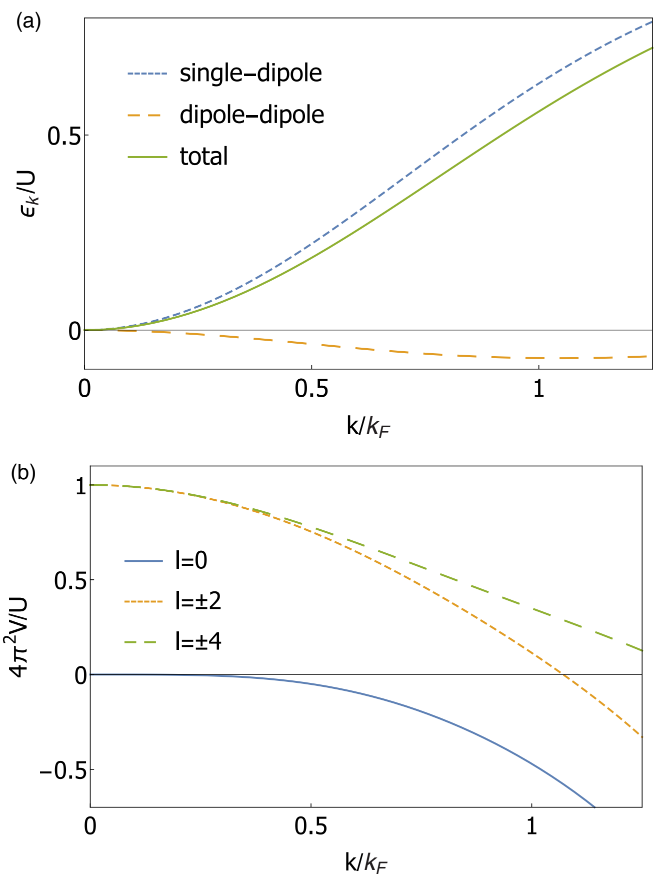

FIG. 3. Mean-field dispersion and partial wave components for different angular momentum. Horizontal axis shows k/k_F. (a) Composite fermion dispersion at mean-field level without pairing included. The dashed curves in blue and orange denote contributions from the single-dipole and from the dipole-dipole terms, respectively. The green curve is their sum, and hence the total dispersion. (b) Partial wave components for even parity pairing channels, as in Eq. (54). The labels stand for corresponding angular momentum l. At the Fermi surface of spinful composite fermion, s-wave channel is attractive.

term is an interdipole interaction, which is different from that of Eq. (52) due to the different Fermi surface structure. We plot the mean-field dispersion in Fig. 3. We note that the dipole-dipole term is significantly weaker than that of spinless case, since the reduced size of the Fermi surfaces lead to smaller dipole moments, which provide weaker screening.

Next we include the possibility of pairing to discuss the stability of the composite Fermi liquid. Note that compared to the spinless problem, the spin degrees of freedom allows for both even and odd angular momentum pairing.

For spin-triplet Cooper pairing, which has odd angular momentum, the pairing interaction is exactly the same as Fig. 1(b). In this case, the pairing in $l = \pm 1$ channel is no longer attractive at the reduced Fermi surface $k_F^s = k_F/\sqrt{2}$. For spin-singlet (even angular momentum) pairing, the pairing potential is shown in Fig. 3. We find the s-wave channel attractive. The attractive potential at the Fermi surface is around 25% weaker than that of the p-wave attraction for the spinless case. Thus, at the mean-field level the composite Fermi liquid will be unstable to pairing, and a topologically ordered ground state will result. Solving the Hartree-Fock equations numerically, when only triplet pairing channel is turned on, no pairing is observed. Allowing singlet pairing, the self-consistent mean-field calculation converge to the s-wave pairing state, with an energy gap $\frac{\Delta E}{UN} = 6 \times 10^{-6}$, an order of magnitude smaller than that of spinless case. This is consistent with our analytical results.

However, upon including fluctuations, the weaker pairing in the spinful problem may not be able to compete against the Amperean repulsion coming from the current-current interaction. In any case, we expect that the pairing is likely a weaker instability than in the spinless case. This is qualitatively consistent with what is seen in the numerics, where the CFL state seems to exist in the spinful model for currently accessible system sizes while the spinless case is in a paired state.

C. Effective field theory

Now we include fluctuations beyond Hartree-Fock to write down a low-energy effective field theory for the spinful composite Fermi liquid, completely parallel to what was done in Sec. IV. The Hartree-Fock composite Fermi liquid state breaks the right gauge transformations generated by ρ^R_{vq} for $\mathbf{q} \neq 0$,

$$c_{\sigma,mn} \to c_{\sigma,mn'} U^R_{n'n}, \tag{129}$$

where U^R is an $SU(N)$ matrix. Meanwhile, we also include left gauge transformations generated by left density ρ^L:

The important fluctuations, therefore, are gauge fluctuations at small $|\mathbf{q}|$. As before, we will include also a background gauge field that couples to left $SU(N)$ rotations,

$$c_{\sigma,mn} \to U^L_{mm'} c_{\sigma,m'n}, \tag{130}$$

with U^L another $SU(N)$ matrix. In principle, we could also include a background gauge field that couples to spin [or more precisely a $U(2)$ background gauge field that couples to both charge and spin] but we will not do so here. As before, these gauge fluctuations are readily incorporated in a path integral framework in terms of the noncommutative operator-valued fields,

$$c_s(\mathbf{R}, \tau) = \int \frac{d^2\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{R}} c_{\mathbf{k}s}(\tau), \tag{131}$$

or their corresponding ordinary fields $c_s(\mathbf{x}, \tau)$ which are multiplied by the star product. Following the development in Sec. IV, we find the noncommutative effective-field theory,

$$\mathcal{S} = \int d^2\mathbf{x} d\tau \;\; \bar{c}_s * D_0 c_s - i a_0 \underline{\rho} + \frac{1}{2m^*}|D_i c_s|^2, \tag{132}$$

where the spin index s is summed over.

Finally, our discussion on the Seiberg-Witten map still applies to this spinful case, only with a modification to include spin indices of composite fermion fields. Namely, we only substitute the last line of Eq. (77) with

$$\Delta\psi_\sigma(\psi_\sigma, \widehat{A}, \widehat{a}) = \frac{\Theta}{2}\epsilon^{\mu\nu}\big[(\widehat{a}_\mu - \widehat{A}_\mu)\partial_\nu\psi_\sigma - i\widehat{a}_\mu\widehat{A}_\nu\psi_\sigma\big]. \tag{133}$$

We then find that the noncommutative theory is mapped to a HLR theory for the spinful composite Fermi liquid (with subleading correction terms similar to Sec. V B):

$$\mathcal{L} = \bar{\psi}_\sigma \partial_0 \psi_\sigma - i(\widehat{a}_0 + \widehat{A}_0)\bar{\psi}_\sigma \psi_\sigma + i\widehat{a}_0 \underline{\rho} + \frac{1}{2m^*}\left|\left(\partial_i - i(\widehat{a}_i + \widehat{A}_i)\right)\psi_\sigma\right|^2 - i\frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\widehat{a}_\alpha \partial_\beta \widehat{a}_\gamma. \tag{134}$$

VIII. DISCUSSION

The noncommutative field theory formulation of the bosonic composite Fermi liquid within the LLL developed in this paper raises a number of other questions. The most important one is whether for fermions at $\nu = \frac{1}{2}$ in the LLL there is a similar formulation. Such a field theory will presumably automatically incorporate particle-hole symmetry and will reduce to the commutative field theory of the Dirac composite fermion (Ref. [11] or the more refined version in Ref. [17]). Unfortunately, a direct extension of the Pasquier-Haldane-Read representation (using, for instance, three-index fermionic partons (see the thesis [47]) is complicated and has not thus far led to progress [48].

Other problems that could be treated within the Pasquier-Haldane-Read formalism include mutlicomponent fermions or bosons in Landau levels at total integer filling. These have been of interest in various contexts. A further generalization is to introduce some dispersion to broaden the Landau level into a Chern band and study the competition between correlations and bandwidth. For bosons at $\nu = 1$, we will describe this competition and the evolution of the ground state elsewhere [49].

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APPENDIX A: FIELD THEORIES FOR THE BOSONIC COMPOSITE FERMI LIQUID

In this Appendix, we present the field theory suggested in Ref. [23] and further discussed in Refs. [27,28]. In this theory, the composite fermion field may be given an interpretation as a (fermionized) vortex of the physical bosons. Thus, we will refer to this as the vortex composite Fermi liquid (VCFL) and will denote the corresponding composite fermion field ψ_v. The Lagrangian takes the form

$$\mathcal{L}_{\rm VCFL} = \bar{\psi}_v(\partial_\tau - ia_0)\psi_v + \frac{1}{2m^*}|(\partial_i - ia_i)\psi_v|^2 + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda - \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \tag{A1}$$

Here a is the dynamical $U(1)$ gauge field and A_μ is the background $U(1)$ gauge field.

Though this form of the action was not explicitly written down in Ref. [23], the comments in Sec. II D of that paper suggested that this effective theory might describe the microscopic results in the bulk of the paper. This effective Lagrangian should be contrasted with that for the HLR theory:

$$\mathcal{L}_{\rm HLR} = \bar{\psi}(\partial_\tau - i(a_0 + A_0))\psi + \frac{1}{2m^*}|(\partial_i - i(a_i + A_i))\psi|^2 - \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{A2}$$

In the microscopic derivation of HLR, m^* is just the bare boson mass but if this Lagrangian emerges in a LLL theory we should regard m^* as a renormalized effective mass.

Both $\mathcal{L}_{\rm HLR}$ and $\mathcal{L}_{\rm VCFL}$ describe (possibly distinct) composite Fermi liquid phases of bosons at $\nu = 1$. In both theories, all local operators are bosonic; in particular, the operator with charge-1 under the background $U_A(1)$ gauge transformation is bosonic. The physical properties (deduced within, for instance, the random phase approximation) of both theories are similar and describe metallic compressible phases. Nevertheless, the two Lagrangians are different and it is not clear whether they describe the same infrared fixed point or not. Furthermore, it has also not been clear which, if any, of these two arises within a microscopic LLL treatment.

If we dispense with the LLL requirement, we can understand how to obtain either of these two theories. The HLR Lagrangian can of course be obtained by a flux attachment transformation of the original boson to composite fermion variables. The VCFL theory can be obtained as follows [27]. First, perform a standard charge-vortex duality transformation of the boson system to pass to a theory in terms of (bosonic) vortices coupled to a dynamical $U(1)$ gauge field. At boson filling $\nu = 1$, the vortices are at finite density and themselves see the boson density as an effective magnetic field; the vortices are then at a filling $\nu_{\rm vortex} = -1$. If we now do a flux attachment transformation to fermionize these vortices, we arrive at Eq. (A1) (up to corrections involving higher derivative terms).

A different possible relationship between the HLR theory and Eq. (A1) was described in Ref. [28]. These authors proposed that in the LLL limit the HLR composite fermions acquire a Fermi surface Berry phase -2π. Upon restricting to the vicinity of the Fermi surface, we should include an anomalous Hall effect contribution to the dynamics of the combined gauge field $a + A$. This then precisely yields the vortex composite Fermi liquid Lagrangian restricted to the modes near the Fermi surface.

APPENDIX B: DETAILS OF THE SEIBERG-WITTEN MAP

Here we provide some detail that was left out in the main text on the approximate mapping of the noncommutative effective-field theory to the commutative one. We will only discuss the spinless case.

The correction to covariant time derivative term is

$$\begin{aligned}\mathcal{L}_1^\tau &= \bar{c}D_0c - \bar{\psi}\widehat{D}_0\psi \\ &= \bar{\psi}\widehat{D}_0\Delta\psi + \Delta\bar{\psi}\widehat{D}_0\psi - i(\Delta a_0 + \Delta A_0)\bar{\psi}\psi \\ &\quad - i\bar{\psi}\left[(\psi * \widehat{a}_0 - \widehat{a}_0\psi) + (\widehat{A}_0 * \psi - \psi\widehat{A}_0)\right],\end{aligned} \tag{B1}$$

where $\widehat{D}_\mu\psi = (\partial_\mu - i\widehat{a}_\mu - i\widehat{A}_\mu)\psi$. The first two terms in Eq. (B1) give

$$\mathcal{L}^\tau_{1a} = \frac{\Theta}{2}\epsilon^{\alpha\beta}\{(\widehat{a}_\alpha - \widehat{A}_\alpha)\big[\bar{\psi}\widehat{D}_0(\partial_\beta\psi) + (\partial_\beta\bar{\psi})\widehat{D}_0\psi\big] - (i\widehat{a}_\alpha\widehat{A}_\beta\bar{\psi}\widehat{D}_0\psi + \text{H.c.})\partial_0(\widehat{a}_\alpha - \widehat{A}_\alpha)\bar{\psi}\partial_\beta\psi - i\partial_0(\widehat{a}_\alpha\widehat{A}_\beta)\bar{\psi}\psi\}, \tag{B2}$$

where the last two terms come from ∂_0 acting on $\Delta\psi$ in the first term of Eq. (B1). We separate out the gauge invariant part by organizing the term (B2) into

$$\mathcal{L}^\tau_{1a} = \frac{\Theta}{2}\epsilon^{\alpha\beta}\{(\widehat{a}_\alpha - \widehat{A}_\alpha)\partial_\beta(\bar{\psi}\widehat{D}_0\psi) + i(\widehat{a}_\alpha - \widehat{A}_\alpha)\partial_\beta(\widehat{a}_0 + \widehat{A}_0)(\bar{\psi}\psi)\partial_0(\widehat{a}_\alpha - \widehat{A}_\alpha)(\bar{\psi}\partial_\beta\psi) - i\partial_0(\widehat{a}_\alpha\widehat{A}_\beta)\bar{\psi}\psi\}. \tag{B3}$$

The first term is readily seen to be gauge invariant after integration by parts. The third term in Eq. (B1) becomes

$$\mathcal{L}^\tau_{1b} = \frac{\Theta}{2}\epsilon^{\alpha\beta}\big[-i\widehat{a}_\alpha(\partial_\beta\widehat{a}_0 + \widehat{f}_{\beta 0}) + i\widehat{A}_\alpha(\partial_\beta\widehat{A}_0 + \widehat{F}_{\beta 0})\big]\bar{\psi}\psi. \tag{B4}$$

The last term in Eq. (B1) is

$$\mathcal{L}^\tau_{1c} = \frac{\Theta}{2}\epsilon^{\alpha\beta}(\bar{\psi}\partial_\alpha\psi)\partial_\beta(\widehat{a}_0 - \widehat{A}_0). \tag{B5}$$

Now we sum up Eqs. (B3)–(B5) and get

$$\begin{aligned}\mathcal{L}^\tau_{1} = \frac{\Theta}{2}\epsilon^{\alpha\beta}\{&(\widehat{a}_\alpha - \widehat{A}_\alpha)\partial_\beta(\bar{\psi}\widehat{D}_0\psi) + i(\widehat{a}_\alpha - \widehat{A}_\alpha)\partial_\beta(\widehat{a}_0 + \widehat{A}_0)\bar{\psi}\psi + \partial_0(\widehat{a}_\alpha - \widehat{A}_\alpha)(\bar{\psi}\partial_\beta\psi) - (\bar{\psi}\partial_\beta\psi)\partial_\alpha(\widehat{a}_0 - \widehat{A}_0)\\ &- [i\widehat{a}_\alpha(\partial_\beta\widehat{a}_0 + \widehat{f}_{\beta 0}) - i\widehat{A}_\alpha(\partial_\beta\widehat{A}_0 + \widehat{F}_{\beta 0})]\bar{\psi}\psi\}.\end{aligned} \tag{B6}$$

Again we separate gauge invariant terms in the third line of Eq. (B6):

$$\begin{aligned}\mathcal{L}^\tau_{1} = \frac{\Theta}{2}\epsilon^{\alpha\beta}\{&(\widehat{a}_\alpha - \widehat{A}_\alpha)\partial_\beta(\bar{\psi}\widehat{D}_0\psi) + i(\widehat{a}_\alpha - \widehat{A}_\alpha)\partial_\beta(\widehat{a}_0 + \widehat{A}_0)\bar{\psi}\psi - i\big[\partial_\alpha(\widehat{a}_0 - \widehat{A}_0) - \partial_0(\widehat{a}_\alpha - \widehat{A}_\alpha)\big](\widehat{a}_\beta + \widehat{A}_\beta)\bar{\psi}\psi\\ &- \big[\partial_\alpha(\widehat{a}_0 - \widehat{A}_0) - \partial_0(\widehat{a}_\alpha - \widehat{A}_\alpha)\big](\bar{\psi}\widehat{D}_\beta\psi) - i[\widehat{a}_\alpha(\partial_\beta\widehat{a}_0 + \widehat{f}_{\beta 0}) - \widehat{A}_\alpha(\partial_\beta\widehat{A}_0 + \widehat{F}_{\beta 0})]\bar{\psi}\psi - i\partial_0(\widehat{a}_\alpha\widehat{A}_\beta)\bar{\psi}\psi\}.\end{aligned} \tag{B7}$$

After a step of integration by parts for the first two terms in Eq. (B7), one can organize the terms into the following form, which is almost symmetric:

$$\begin{aligned}\mathcal{L}^\tau_{1} = \frac{\Theta}{2}\epsilon^{\alpha\beta}\{&-\partial_\beta(\widehat{a}_\alpha - \widehat{A}_\alpha)(\bar{\psi}\widehat{D}_0\psi) - \partial_\alpha(\widehat{a}_0 - \widehat{A}_0)(\bar{\psi}\widehat{D}_\beta\psi) - \partial_0(\widehat{a}_\beta - \widehat{A}_\beta)(\bar{\psi}\widehat{D}_\alpha\psi)\\ &+ i\big[-\partial_\beta(\widehat{a}_\alpha - \widehat{A}_\alpha)(\widehat{a}_0 + \widehat{A}_0) - \partial_\alpha(\widehat{a}_0 - \widehat{A}_0)(\widehat{a}_\beta + \widehat{A}_\beta) + \partial_0(\widehat{a}_\alpha - \widehat{A}_\alpha)(\widehat{a}_\beta + \widehat{A}_\beta)\big]\bar{\psi}\psi\\ &+ i\big[-\widehat{a}_\alpha(\partial_\beta\widehat{a}_0 + \widehat{f}_{\beta 0}) + \widehat{A}_\alpha(\partial_\beta\widehat{A}_0 + \widehat{F}_{\beta 0}) + \partial_\beta\big((\widehat{a}_\alpha - \widehat{A}_\alpha)(\widehat{a}_0 + \widehat{A}_0)\big) - \partial_0(\widehat{a}_\alpha\widehat{A}_\beta)\big]\bar{\psi}\psi\}.\end{aligned} \tag{B8}$$

The first line of Eq. (B8) is gauge invariant. The second line becomes

$$\begin{aligned}\frac{i\Theta}{2}\epsilon^{\mu\nu\rho}(\widehat{a}_\mu + \widehat{A}_\mu)\partial_\nu(\widehat{a}_\rho - \widehat{A}_\rho)\bar{\psi}\psi &= \frac{i\Theta}{2}\epsilon^{\mu\nu\rho}(\widehat{a}_\mu\partial_\nu\widehat{a}_\rho - \widehat{A}_\mu\partial_\nu\widehat{A}_\rho - \widehat{a}_\mu\partial_\nu\widehat{A}_\rho + \widehat{A}_\mu\partial_\nu\widehat{a}_\rho)\bar{\psi}\psi\\ &= \frac{i\Theta}{2}\epsilon^{\mu\nu\rho}\big[\widehat{a}_\mu\partial_\nu\widehat{a}_\rho - \widehat{A}_\mu\partial_\nu\widehat{A}_\rho + \partial_\nu(\widehat{a}_\rho\widehat{A}_\mu)\big]\bar{\psi}\psi.\end{aligned} \tag{B9}$$

The third line of Eq. (B8) becomes

$$\begin{aligned}&\frac{i\Theta}{2}\epsilon^{\mu\nu\rho}(-\widehat{a}_\mu\partial_\nu\widehat{a}_\rho + \widehat{A}_\mu\partial_\nu\widehat{A}_\rho)\bar{\psi}\psi + \frac{i\Theta}{2}\epsilon^{\alpha\beta}\big[-\partial_\beta(\widehat{a}_\alpha\widehat{a}_0) + \partial_\beta(\widehat{A}_\alpha\widehat{A}_0) + \partial_\beta\big((\widehat{a}_\alpha - \widehat{A}_\alpha)(\widehat{a}_0 + \widehat{A}_0)\big) - \partial_0(\widehat{a}_\alpha\widehat{A}_\beta)\big]\bar{\psi}\psi\\ &= \frac{i\Theta}{2}\epsilon^{\mu\nu\rho}(-\widehat{a}_\mu\partial_\nu\widehat{a}_\rho + \widehat{A}_\mu\partial_\nu\widehat{A}_\rho)\bar{\psi}\psi + \frac{i\Theta}{2}\epsilon^{\alpha\beta}\big[\partial_\beta(\widehat{a}_\alpha\widehat{A}_0 - \widehat{A}_\alpha\widehat{a}_0) - \partial_0(\widehat{a}_\alpha\widehat{A}_\beta)\big]\bar{\psi}\psi\\ &= \frac{i\Theta}{2}\epsilon^{\mu\nu\rho}(-\widehat{a}_\mu\partial_\nu\widehat{a}_\rho + \widehat{A}_\mu\partial_\nu\widehat{A}_\rho)\bar{\psi}\psi - \frac{i\Theta}{2}\epsilon^{\mu\nu\rho}\partial_\mu(\widehat{a}_\nu\widehat{A}_\rho)\bar{\psi}\psi.\end{aligned} \tag{B10}$$

Equations (B9) and (B10) cancel exactly. The remaining correction to the covariant time derivative is

$$\mathcal{L}^\tau_{1} = \frac{\Theta}{2}\epsilon^{\alpha\beta}\partial_\alpha(\widehat{a}_\beta - \widehat{A}_\beta)(\bar{\psi}\widehat{D}_0\psi) + \frac{\Theta}{4}\partial_0(\widehat{f}_{12} - \widehat{F}_{12})\bar{\psi}\psi. \tag{B11}$$

Next we turn to the part of the action in Eq. (72) involving the spatial covariant derivatives. To first order in the noncommutativity parameter Θ, we get

$$\mathcal{L}^s_{1} = \frac{1}{2m^*}\big(|D_\alpha c|^2 - |\widehat{D}_\alpha\psi|^2\big) \sim \frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\{\widehat{D}_\alpha\Delta\psi - i(\Delta a_\alpha + \Delta A_\alpha)\psi - \frac{i}{2}[\psi, \widehat{a}_\alpha - \widehat{A}_\alpha]_*\} + \text{H.c.} \tag{B12}$$

Using Seiberg-Witten map in Eq. (77), the first term in Eq. (B12) becomes

$$\begin{aligned}\mathcal{L}^s_{1a} &= \frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\widehat{D}_\alpha\Delta\psi + h.c.\\ &= \frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\frac{\Theta}{2}\epsilon^{\beta\gamma}\widehat{D}_\alpha\big[(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\gamma\psi-i\widehat{a}_\beta\widehat{A}_\gamma\psi\big]+\text{H.c.}\\ &= \frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\frac{\Theta}{2}\epsilon^{\beta\gamma}\{(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\gamma(\widehat{D}_\alpha\psi)\\ &\quad+\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\gamma\psi+i(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\gamma(\widehat{a}_\alpha+\widehat{A}_\alpha)\psi\\ &\quad-i\widehat{a}_\beta\widehat{A}_\gamma\widehat{D}_\alpha\psi-i\partial_\alpha(\widehat{a}_\beta\widehat{A}_\gamma)\psi\}+\text{H.c.}\end{aligned}\tag{B13}$$

The second term in Eq. (B12) is

$$\mathcal{L}^s_{1b} = -\frac{1}{2m^*}(\widehat{D}_\alpha\psi)^* i(\Delta a_\alpha+\Delta A_\alpha)\psi+\text{H.c.},\tag{B14}$$

where $\Delta\widehat{A}_\alpha$, $\Delta\widehat{a}_\alpha$ are Hermitian, which will become important later. The last term in Eq. (B12) gives

$$\begin{aligned}\mathcal{L}^s_{1c} &= -\frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\frac{i}{2}[\psi,\widehat{a}_\alpha-\widehat{A}_\alpha]_*+\text{H.c.}\\ &= \frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\frac{\Theta}{2}\epsilon^{\beta\gamma}\partial_\gamma(\widehat{a}_\alpha-\widehat{A}_\alpha)\partial_\beta\psi+\text{H.c.}\end{aligned}\tag{B15}$$

Equation (C2) guarantees that $(\widehat{D}_\alpha\psi)^*\psi = \frac{1}{2}\partial_\alpha(\bar{\psi}\psi)$ is real. So $\mathcal{L}^s_{1b}$ and the third and fourth terms of Eq. (B13) are purely imaginary and get canceled by their Hermitian conjugate. Now summing up Eqs. (B13)–(B15), we are left with

$$\begin{aligned}\mathcal{L}^s_1 =& \frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\frac{\Theta}{2}\epsilon^{\beta\gamma}\big\{(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\gamma(\widehat{D}_\alpha\psi)\\ &+\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\gamma\psi+\partial_\gamma(\widehat{a}_\alpha-\widehat{A}_\alpha)\partial_\beta\psi\\ &-i\partial_\alpha(\widehat{a}_\beta\widehat{A}_\gamma)\psi\big\}+\text{H.c.}\end{aligned}\tag{B16}$$

Thanks to Eq. (86), the last term of Eq. (B16) gets canceled by its Hermitian conjugate. Upon integration by part, the first term of Eq. (B16) (+H.c.) becomes

$$\begin{aligned}\mathcal{L}^s_{1a'} &= \frac{1}{2m^*}\frac{\Theta}{2}\epsilon^{\beta\gamma}(\widehat{a}_\beta-\widehat{A}_\beta)\partial_\gamma\big[(\widehat{D}_\alpha\psi)^*(\widehat{D}_\alpha\psi)\big]\\ &= \frac{1}{2m^*}\frac{\Theta}{2}\epsilon^{\beta\gamma}\partial_\beta(\widehat{a}_\gamma-\widehat{A}_\gamma)|\widehat{D}_\alpha\psi|^2\\ &= \frac{1}{2m^*}\frac{\Theta}{2}(\widehat{f}_{12}-\widehat{F}_{12})|\widehat{D}_\alpha\psi|^2\end{aligned}\tag{B17}$$

and the other term is

$$\begin{aligned}\mathcal{L}^s_{1b'} &= -\frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\frac{\Theta}{2}\epsilon^{\beta\gamma}[\partial_\alpha(\widehat{a}_\gamma-\widehat{A}_\gamma)\\ &\quad-\partial_\gamma(\widehat{a}_\alpha-\widehat{A}_\alpha)]\partial_\beta\psi+\text{H.c.}\\ &= -\frac{1}{2m^*}(\widehat{D}_\alpha\psi)^*\frac{\Theta}{2}\epsilon^{\beta\gamma}[\partial_\alpha(\widehat{a}_\gamma-\widehat{A}_\gamma)\\ &\quad-\partial_\gamma(\widehat{a}_\alpha-\widehat{A}_\alpha)]\widehat{D}_\beta\psi+\text{H.c.}\\ &= -\frac{1}{2m^*}\frac{\Theta}{2}\epsilon^{\beta\gamma}\big(\widehat{f}_{\alpha\gamma}-\widehat{F}_{\alpha\gamma}\big)(\widehat{D}_\alpha\psi)^*\widehat{D}_\beta\psi+\text{H.c.},\end{aligned}\tag{B18}$$

where in the second line we have added a vanishing term $\sim(\widehat{D}_\alpha\psi)^* i(\widehat{a}+\widehat{A})\psi+\text{H.c.}$ to get the covariant derivative. It is easy to check that

$$\epsilon_{\beta\gamma}\big(\widehat{f}_{\alpha\gamma}-\widehat{F}_{\alpha\gamma}\big) = \big(\widehat{f}_{12}-\widehat{F}_{12}\big)\delta_{\alpha\beta}.\tag{B19}$$

Consequently,

$$\mathcal{L}^s_1 = -\frac{1}{2m^*}\frac{\Theta}{2}(\widehat{f}_{12}-\widehat{F}_{12})|\widehat{D}_\alpha\psi|^2.\tag{B20}$$

APPENDIX C: CORRECTIONS FROM THE HALL CURRENT

In Eq. (86), it's assumed that the current is vanishing. However, strictly speaking, as we discussed in the main text, we should include an additional Hall current coming from the Chern-Simons term in the HLR action Eq. (96). To be precise, the current is

$$\begin{aligned}J^\alpha = \frac{\delta\mathcal{L}}{\delta\widehat{a}_\alpha} &= \frac{i}{2m^*}\big(\bar{\psi}D_\alpha\psi-\overline{(D_\alpha\psi)}\psi\big)\\ &= -i\Theta\underline{\rho}\epsilon^{\alpha\mu\nu}\partial_\mu a_\nu,\end{aligned}\tag{C1}$$

which is the Hall response to the internal gauge field. Therefore,

$$\bar{\psi}\;\widehat{D}_\alpha\psi = \frac{1}{2}\partial_\alpha(\bar{\psi}\psi)-m^*\Theta\underline{\rho}\epsilon^{\alpha\mu\nu}\partial_\mu a_\nu,\tag{C2}$$

$$\overline{(\widehat{D}_\alpha\psi)}\;\psi = \frac{1}{2}\partial_\alpha(\bar{\psi}\psi)+m^*\Theta\underline{\rho}\epsilon^{\alpha\gamma\delta}\partial_\gamma a_\delta.\tag{C3}$$

As a consequence, the correction to covariant time derivative term now becomes

$$\begin{aligned}\mathcal{L}^\tau_1 &= \frac{\Theta}{2}\epsilon^{\mu\nu\rho}\partial_\mu(\widehat{a}_\nu-\widehat{A}_\nu)(\bar{\psi}\widehat{D}_\rho\psi)\\ &= \frac{\Theta}{2}\epsilon^{\alpha\beta 0}\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)(\bar{\psi}\widehat{D}_0\psi)+\frac{\Theta}{2}\epsilon^{\mu\nu\alpha}\\ &\qquad\partial_\mu(\widehat{a}_\nu-\widehat{A}_\nu)\left(\frac{1}{2}\partial_\alpha(\bar{\psi}\psi)-m^*\Theta\underline{\rho}\epsilon^{\alpha\rho\sigma}\partial_\rho a_\sigma\right)\\ &= \frac{\Theta}{2}\epsilon^{\alpha\beta}\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)(\bar{\psi}\widehat{D}_0\psi)+\frac{\Theta}{4}\epsilon^{\alpha\beta}\partial_0\partial_\alpha(\widehat{a}_\beta-\widehat{A}_\beta)\\ &\qquad-\frac{\Theta^2}{2}m^*\underline{\rho}\epsilon^{\alpha\mu\nu}\partial_\mu(\widehat{a}_\nu-\widehat{A}_\nu)\epsilon^{\alpha\rho\sigma}\partial_\rho a_\sigma.\end{aligned}\tag{C4}$$

The additional term is as stated in Eq. (103),

$$-\frac{\Theta^2}{2}m^*\underline{\rho}((\widehat{f}_{01}-\widehat{F}_{01})\widehat{f}_{01}+(\widehat{f}_{02}-\widehat{F}_{02})\widehat{f}_{02}),\tag{C5}$$

which formally is of second order in Θ. However, as $\underline{\rho} = \frac{1}{2\pi|\Theta|}$, it really is of order Θ. For the spatial covariant derivative terms, no correction shows up at this order since in Appendix B, we have only used the fact that $\bar{\psi}D_\alpha\psi$ is real, which is still the case. Note that Eq. (103) is not the full correction for the action to $o(\Theta^2)$ since we have only kept $o(\Theta)$ terms in Seiberg-Witten map as well as later in the expansion of the action. It is, however, the only term of order $\Theta^2\underline{\rho}$.

APPENDIX D: RIGHT SPIN DENSITY

In this Appendix, we show that the right spin density defined in Eq. (123) does not annihilate all physical states. We write the generalized density operator as

$$\rho^{R,\alpha}_{nn'} = \sum_{m,ss'} c^{\dagger}_{n,ms}\sigma^{\alpha}_{s,s'}c_{s'm,n'}, \tag{D1}$$

where $\sigma^{\alpha} = (1, \Sigma^x/2, \Sigma^y/2, \Sigma^z/2)$, and Σ^i are the Pauli matrices. Then $\alpha = 0$ corresponds to the right density operator and $\alpha = 1, 2, 3$ correspond to the right spin-density operators defined in the main text. The physical Hilbert space is spanned by states

$$|\psi_{phys,m_i,s_i}\rangle = \epsilon^{n_1n_2\dots n_N} c^{\dagger}_{n_1,m_1s_1}c^{\dagger}_{n_2,m_2s_2}\dots c^{\dagger}_{n_N,m_Ns_N}|0\rangle . \tag{D2}$$

Applying right density on a physical state, we get

$$\begin{aligned}
&\rho^{R,\alpha}_{nn'}|\psi_{phys,m_i,s_i}\rangle \\
&= \epsilon^{n_1n_2\dots n_N} c^{\dagger}_{n,ms}\sigma^{\alpha}_{s,s'}c_{s'm,n'}c^{\dagger}_{n_1,m_1s_1}c^{\dagger}_{n_2,m_2s_2}\dots c^{\dagger}_{n_N,m_Ns_N}|0\rangle \\
&= \sum_j (-1)^{(j-1)}\epsilon^{n_1n_2\dots n_N}\delta_{n'n_j}\delta_{s's_j}\delta_{m,m_j}c^{\dagger}_{n,ms}\sigma^{\alpha}_{s,s'}\prod_{i\neq j}c^{\dagger}_{n_i,m_is_i}|0\rangle \\
&= \sum_j (-1)^{(j-1)}\epsilon^{n_1n_2\dots n_N}\delta_{n'n_j}\sigma^{\alpha}_{s,s_j}c^{\dagger}_{n,m_js}\prod_{i\neq j}c^{\dagger}_{n_i,m_is_i}|0\rangle ,
\end{aligned} \tag{D3}$$

where repeated indices are summed over.

It is sufficient to illustrate our point by considering a finite system and explicitly showing that the right spin-density operator takes a physical state to an nonphysical state. To that end, consider $N = 2$, i.e., a system with just two single-particle orbitals. The many-body Hilbert space is spanned by states with two c-fermions filling eight basis states. Consider the state

$$|\psi_{phys,1\uparrow 2\downarrow}\rangle = b^{\dagger}_{1\uparrow}b^{\dagger}_{2\downarrow}|0\rangle = (c^{\dagger}_{1,1\uparrow}c^{\dagger}_{2,2\downarrow} - c^{\dagger}_{2,1\uparrow}c^{\dagger}_{1,2\downarrow})|0\rangle \tag{D4}$$

where $|0\rangle$ is the vacuum state of composite fermion. This state is in the physical Hilbert space since the internal index is antisymmetrized. Applying the right spin operator S^z_{12}, one gets

$$S^z_{12}|\psi_{phys,1\uparrow 2\downarrow}\rangle = -2c^{\dagger}_{11\uparrow}c^{\dagger}_{12\downarrow}|0\rangle , \tag{D5}$$

which is a nonphysical state that does not get annihilated by right density $\rho^{R}_{nn'}$.

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