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Universal reductions and solitary waves of weakly nonlocal defocusing nonlinear Schrödinger equations

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Abstract

We study asymptotic reductions and solitary waves of a weakly nonlocal defocusing nonlinear Schrödinger (NLS) model. The hydrodynamic form of the latter is analyzed by means of multiscale expansion methods. To the leading-order of approximation (where only the first of the moments of the response function is present), we show that solitary waves, in the form of dark solitons, are governed by an effective Boussinesq/Benney–Luke (BBL) equation, which describes bidirectional waves in shallow water. Then, for long times, we reduce the BBL equation to a pair of Korteweg–de Vries (KdV) equations for right- and left-going waves, and show that the BBL solitary wave transforms into a KdV soliton. In addition, to the next order of approximation (where both the first and second moment of the response function are present), we find that dark solitons are governed by a higher-order perturbed KdV (pKdV) equation, which has been used to describe ion-acoustic solitons in plasmas and water waves in the presence of higher-order effects. The pKdV equation is approximated by a higher-order integrable system and, as a result, only insubstantial changes in the soliton shape and velocity are found, while no radiation tails (in this effective KdV picture) are produced.

Keywords: nonlocal NLS, higher order KdV, multiple scales, solitary waves, integrable reductions

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1. Introduction

In many physical contexts, there appear systems featuring a spatially nonlocal nonlinearity. For instance, in nonlinear optics, the nonlinear correction to the refractive index at a particular point in space, depends on the light intensity in a certain spatial domain around this point. Such nonlocal nonlinear systems include thermal nonlinear media, e.g., vapors [1, 2] and liquid solutions [3, 4] (see also the reviews [5, 6]), plasmas [7–9], nematic liquid crystals [10, 11], dipolar Bose–Einstein condensates (BECs) [12], and so on. Nonlocality has been shown to be of paramount importance on the stability and dynamics of nonlinear waves and solitons. For instance, if the nonlocal nonlinearity is of the focusing type, collapse can be arrested in higher dimensions [13] (see also reference [5]) and, as a result, stable solitons exist in such settings [1, 2, 6, 14]. In the case of defocusing nonlocal nonlinearities, dark solitons do exist [15–18] and may feature an attractive interaction [15, 19] rather than a repulsive one, as in the case of a local nonlinearity (see the reviews [20, 21] and references therein). Furthermore, nonlocality can suppress the transverse (‘snaking’) instability of dark solitons and the associated dispersive shock waves [22].

An important class of nonlocal models, relevant to the physical settings mentioned above, is of the NLS type; a one-dimensional (1D) such model, expressed in dimensionless form, is of the form:

$$iu_t + \frac{1}{2}u_{xx} + \sigma \left[\int_{-\infty}^{\infty} R(x' - x) |u(x', t)|^2 dx' \right] u = 0, \quad (1)$$

where $u(x, t)$ is a complex field, $R(x)$ is a positive definite function describing the nonlocal response of the medium, and $\sigma = \pm 1$ corresponds to a focusing or a defocusing nonlinearity (for $\sigma = +1$ and $\sigma = -1$, respectively). Obviously, if the response function is singular, i.e., $R(x) = \delta(x)$ (where $\delta(x)$ is the Dirac δ function), then equation (1) reduces to the usual NLS equation, which is completely integrable in the $(1 + 1)$ -dimensional setting. Here, we are interested in the case where the spatial width of the response kernel $R(x)$ is small compared to the width of the density $|u|^2$. In such a case, nonlocality is weak and, following the analysis of reference [23], it is possible to reduce equation (1) to an effective local NLS model. There, nonlocality is effectively described by means of a local perturbation, in the form of a nonlinear potential term (see details below).

There exists a rich variety of soliton structures that have been predicted to occur in nonlocal NLS models, depending on the form of nonlocality. Particularly, weakly nonlocal media support bright and dark solitons [23], while in fully nonlocal models, apart from bright solitons [10, 11] and dark solitons [16, 17], antidark solitons are also possible [18]. In higher-dimensional settings, there also exist more complex structures, such as vortex solitons [9, 14], stable rotating dipole solitons [24], spiraling solitons and multipole localized modes [25], ring dark and antidark solitons [26, 27], as well as dark lump solitons and X-, Y-, or H-shaped waves, and other more complicated structures, composed by antidark soliton stripes [28, 29]. In addition, more recently, other new solutions (such as N th rational solutions) of nonlocal defocusing NLS equations have been constructed [30], while the Cauchy problem with step initial data and the long-time behavior of the pertinent solutions were studied [31].

In this work, our scope is to study analytically the above mentioned weakly nonlocal setting, stemming from equation (1), with a defocusing nonlinearity ($\sigma = -1$). Our scope is twofold. First, by means of our analytical approach, which relies on a multiscale analysis of the hydrodynamic form of the pertinent NLS model, we reduce the weakly nonlocal NLS to a number of important equations appearing in a variety of physical contexts. Importantly, all these effective

nonlinear evolution equations occur at various stages of the asymptotic analysis, revealing the asymptotic behavior of the model at different scales. Second, having derived these equations, we present their solitary wave solutions, which are then used for the construction of approximate dark soliton solutions of the weakly nonlocal NLS. We are thus able to find a wealth of asymptotic reductions of the original NLS model, determine corresponding dark solitary waves that may occur at various different scales, and also establish interesting connections with other physical contexts.

To be more specific, at an intermediate stage (in terms of proper slow temporal and spatial scales), first we derive an equation of the form of the Boussinesq [32] or the Benney–Luke [33] equation (hereafter, this model will be referred to as the BBL equation). The Boussinesq and BL models have been used to describe the propagation of bidirectional waves in shallow water [32–34], while similar Boussinesq-type equations appear in studies of waves in plasmas [35, 36], electrical and mechanical lattices [37], and so on. We then use a traveling wave ansatz, and derive exact solitary wave solutions of the BBL equation, which correspond to a weak dark soliton solution of the original NLS equation. Next, we study the long-time behavior of the BBL equation and, similarly to the water wave problem [32, 34], we reduce the BBL model to a pair of KdV equations that govern right- and left-propagating waves. We also show that if the formal perturbation parameter is sufficiently small, then the BBL solitary wave reduces to a KdV soliton.

Finally, we use the reductive perturbation method [38] to analyze higher-order effects arising from the consideration of moderate widths of the response kernel. In this case, we show that dark solitons are governed by a 5th-order pKdV equation, which stems naturally from the underlying Hamiltonian systems [39, 40], and is related to the first higher-order equation in the KdV hierarchy [41]. This pKdV model, which is known to describe ion-acoustic solitons [42, 43] and shallow water waves [44] under the influence of higher-order effects, has been studied in the context of asymptotic integrability of weakly dispersive nonlinear wave equations [45, 46]. An approximate soliton solution of the derived pKdV is presented, and it is shown that it is only slightly deformed as compared to the original KdV soliton obtained to the leading-order of approximation. In addition, the perturbation theory for solitons [47–49] is employed in order to determine the effect of the perturbation on the soliton characteristics under the action of the higher-order effects. We find that the soliton amplitude remains unchanged, while no radiation tails are produced during the evolution in the higher-order KdV approximation.

The manuscript is organized as follows. In section 2 we present the model equations, while section 3 is devoted to the asymptotic analysis and the presentation of the soliton solutions. Finally, in section 4 we present our conclusions and discuss possibilities for future work.

2. Model equations

We consider a physical system, which is governed by the following dimensionless 1D nonlocal defocusing NLS equation for the unknown complex field $u(x, t)$:

$$iu_t + \frac{1}{2}u_{xx} - n(I)u = 0, \quad (2)$$

where subscripts denote partial derivatives, and the real function $n(I)$, with $I = |u(x, t)|^2$, is given by the following convolution integral:

$$n(I) = \int_{-\infty}^{\infty} R(x' - x)I(x', t) dx', \quad (3)$$

with the kernel $R(x)$ describing the response function of the nonlocal medium. This nonlocal model describes, in the context of optics, beam propagation in thermal media [1–4]; in this case, $u(x, t)$ is the electric field envelope, n is the nonlinear change of the refractive index (that depends on the light intensity I), while t represents the propagation direction. A similar situation occurs in plasmas, but with n denoting the relative electron temperature perturbation [7–9], as well as in nematic liquid crystals, with n being the perturbation of the optical director angle from its static value due to the presence of the light field [10, 11].

In all the above cases, the kernel $R(x)$ may be considered to be a real, positive definite, localized and symmetric function [50], obeying the normalization condition $\int_{-\infty}^{+\infty} R(x)dx = 1$. A physically relevant form of the kernel, that finds applications in all the above mentioned contexts, is:

$$R(x) = \frac{1}{2d} \exp\left(-\frac{|x|}{d}\right), \quad (4)$$

where $d > 0$ is a spatial scale that measures the degree of nonlocality ($d = 0$ corresponds to the limit of local nonlinearity). Then, introducing the Fourier transform pair for a function $f(x)$ as:

$$\begin{aligned} \hat{f}(k) &= \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{ikx} dx, \\ f(x) &= \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx} dk, \end{aligned}$$

it can be seen that, using the convolution property, equation (3) can be expressed in the wavenumber domain as:

$$\hat{n} = \hat{R}(k)\hat{I}(k, t). \quad (5)$$

Taking now into regard that, for the kernel of equation (4), one has: $\hat{R}(k) = (1 + d^2k^2)^{-1}$, it can readily be found that equation (5) can be rewritten as: $(1 + d^2k^2)\hat{n} = \hat{I}(k, t)$. Then, applying the inverse Fourier transform to the latter equation, it is found that $n - d^2n_{xx} = I$. In other words, equations (2) and (3) are equivalent to the following system of coupled partial differential equations (PDEs):

$$iu_t + \frac{1}{2}u_{xx} - nu = 0, \quad (6)$$

$$d^2n_{xx} - n = -|u|^2. \quad (7)$$

In this work, we focus on the case where the response function is narrow as compared to the width of the beam's intensity (the so-called weakly nonlocal limit) [23]. Then, $n(x, t)$ in equation (3) may be approximated appropriately, so that (2) may be written in local form, so that such a weakly nonlocal medium may be described by a local PDE. To do this, we express $n(x, t)$ in Fourier space and, recalling that equation (3) is a convolution integral, we have:

$$\hat{n}(k, t) = \hat{R}(k)\hat{I}(k, t).$$

Next, we expand $\hat{R}(k)$ in a Taylor series:

$$\hat{R}(k) = \sum_{n=0}^{\infty} \frac{\hat{R}^{(n)}(0)}{n!} k^n,$$

where $\hat{R}^{(n)}(k) \equiv d^n \hat{R}(k)/dk^n$, and using Fourier transform properties we can find:

$$\begin{aligned} n(x, t) &= \mathcal{F}^{-1} \{ \hat{R}(k) \hat{I}(k, t) \} = \mathcal{F}^{-1} \left\{ \sum_{n=0}^{\infty} \frac{\hat{R}^{(n)}(0)}{n!} k^n \hat{I}(k, t) \right\} \\ &= \sum_{n=0}^{\infty} \frac{\hat{R}^{(n)}(0)}{n!} \mathcal{F}^{-1} \{ k^n \hat{I}(k, t) \} = \sum_{n=0}^{\infty} \frac{\hat{R}^{(n)}(0)}{n!} i^n \frac{\partial^n}{\partial x^n} I(x, t). \end{aligned} \quad (8)$$

To this end, using the expansion (8) and the properties of the Fourier transform for $\hat{R}^{(n)}(0)$, we obtain from equation (2) the following local PDE:

$$iu_t + \frac{1}{2}u_{xx} - \left[\sum_{n=0}^{\infty} a_n \frac{\partial^n}{\partial x^n} (|u|^2) \right] u = 0, \quad (9)$$

where

$$a_n = \frac{(-1)^n}{n!} \left[\int_{-\infty}^{\infty} x^n R(x) dx \right]. \quad (10)$$

Note that if $n = 0$ (corresponding to the singular δ -function kernel) then $\hat{R}^{(0)} = \hat{R}(0) = \int_{-\infty}^{\infty} R(x) dx = 1$, i.e., $a_0 = 1$, and equation (9) reduces to the cubic NLS equation, while for $n \neq 0$, given the symmetric nature of R , we have:

$$a_{2n} > 0, \quad \text{and} \quad a_{2n+1} = 0, \quad \forall n \in \mathbb{N}.$$

For instance, in the case of the kernel $R(x)$ given by equation (4), it is straightforward to find that:

$$\int_{-\infty}^{+\infty} x^n R(x) dx = \frac{1}{2d} \int_{-\infty}^{+\infty} x^n e^{-|x|/d} dx = \begin{cases} d^n n!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

and hence $a_{2n} = d^{2n}$.

According to the above discussion, in the case of a response kernel of sufficiently small width compared to the width of the intensity $I(x, t) \equiv |u(x, t)|^2$, we may use, to a first approximation, $n(I) \approx I + a_2 \partial_x^2 I$, and find that equation (9) is reduced to the following modified NLS equation:

$$iu_t + \frac{1}{2}u_{xx} - (|u|^2 + a_2 \partial_x^2 |u|^2) u = 0, \quad (11)$$

with the parameter a_2 characterizing the nonlocality. The above equation has been studied in nonlinear optics [23], as well as (in the case of a focusing nonlinearity) in plasma physics, where the parameter a_2 may take both positive and negative values [51], and the continuum limit of discrete molecular structures [52]. It has also been shown that equation (11) possesses stable soliton solutions, bright or dark for a focusing or a defocusing nonlinearity (i.e., a plus or a minus sign in front of the parenthesis in equation (11)) respectively [23].

Here, we are interested in studying the role of higher-order effects on dark soliton dynamics, which may be accounted for by the inclusion of higher-order terms in the Taylor expansion of $n(x, t)$. Thus, below, we will use multiscale expansion methods to study dark solitons of

equation (9) up to the following level of approximation: $n(I) \approx I + a_2 \partial_x^2 I + a_4 \partial_x^4 I$. In this case, equation (9) is obviously reduced to the following higher-order modified NLS equation:

$$iu_t + \frac{1}{2}u_{xx} - (|u|^2 + a_2 \partial_x^2 |u|^2 + a_4 \partial_x^4 |u|^2) u = 0. \quad (12)$$

In the next section, we will use multiscale expansion methods and derive universal models the soliton solutions of which will then be used to obtain approximate dark soliton solutions of equation (12).

3. Asymptotic analysis

3.1. The cw solution and its stability

To start our analysis, first we introduce the Madelung transformation

$$u(x, t) = u_0 \rho^{1/2}(x, t) \exp[i\phi(x, t)],$$

(where u_0 is an arbitrary real constant), and derive from equation (9) the following system of two coupled PDEs for the amplitude ρ and phase ϕ ,

$$\phi_t + u_0^2 \sum_{n=0}^{\infty} a_{2n} \frac{\partial^{2n}}{\partial x^{2n}} \rho + \frac{1}{2} \phi_x^2 - \frac{1}{2} \rho^{-1/2} \left(\rho^{1/2} \right)_{xx} = 0, \quad (13)$$

$$\rho_t + (\rho \phi_x)_x = 0. \quad (14)$$

Obviously, the system (13) and (14) possesses a simple homogeneous solution, namely:

$$\rho = 1, \quad \phi = -u_0^2 t, \quad (15)$$

which corresponds to the continuous-wave (cw) solution $u = u_0 \exp(-iu_0^2 t)$ of equation (9). Since below we will seek dark soliton solutions of equation (9) on top of this cw background, it is necessary to investigate the stability of the cw solution. To do so, we assume that $\rho = 1 + \Delta\rho$, $\phi = -u_0^2 t + \Delta\phi$, where the perturbations $\Delta\rho$, $\Delta\phi$ (with $|\Delta\rho| \ll 1$, $|\Delta\phi| \ll 1$) behave like $\exp[i(kx - \omega t)]$; this way, equations (13) and (14) lead to the following dispersion relation for the perturbations' frequency ω and wavenumber k :

$$\omega^2 = k^2 \left[c^2 \sum_{n=0}^{\infty} a_{2n} (ik)^{2n} + \frac{1}{4} k^2 \right], \quad (16)$$

where

$$c^2 = u_0^2, \quad (17)$$

is the speed of the small-amplitude (linear) waves propagating on top of the cw background, the so-called 'speed of sound'. Equation (16) shows that the cw is modulationally stable, i.e., $\omega \in \mathbb{R} \forall k \in \mathbb{R}$, provided that $\sum_{n=0}^{\infty} a_{2n} (ik)^{2n} > 0$. Obviously this occurs for $n = 0$ (corresponding to the local NLS with the defocusing nonlinearity). On the other hand, for $n = 1$ the cw is modulationally stable as long as the parameter α , defined as:

$$\alpha = 1 - 4u_0^2 a_2, \quad (18)$$

is positive. This condition is fulfilled if, for a fixed nonlocality parameter a_2 , the cw background intensity u_0^2 does not exceed a critical value $I_0^{(cr)}$, i.e., $u_0^2 \leq I_0^{(cr)} \equiv (4a_2)^{-1}$, in accordance to the analysis of reference [53]; the same holds if, for a fixed cw intensity u_0^2 , the nonlocality parameter a_2 does not exceed the critical value $a_2^{(cr)}$, i.e., $a_2 \leq a_2^{(cr)} \equiv (4u_0^2)^{-1}$. In addition, for $n = 2$, the dispersion relation (16) becomes:

$$\omega^2 = c^2 k^2 \left(1 + \frac{\alpha}{4c^2} k^2 + a_4 k^4 \right). \quad (19)$$

Notice that in the case of the kernel (4), the series in equation (16) converges to $(1 + d^2 k^2)^{-1}$ and, as a result, equation (16) takes the form:

$$\omega^2 = \frac{c^2 k^2}{1 + d^2 k^2} + \frac{1}{4} k^4, \quad (20)$$

which is the dispersion relation of the nonlocal model (6) and (7). Obviously, in this fully nonlocal case, the cw background is always stable.

Before proceeding with the asymptotic analysis of the weakly nonlocal NLS model, it is worth mentioning the following. For right-going waves, the dispersion relation (19) becomes $\omega = ck[1 + (\alpha/4c^2)k^2 + a_4 k^4]^{1/2}$, and in the long-wavelength limit ($k \ll 1$), it can be reduced to the form:

$$\omega \approx ck + \frac{\alpha}{8c} k^3 + \left(\frac{1}{2} ca_4 - \frac{\alpha^2}{128c^3} \right) k^5.$$

Then, using $\omega \mapsto i\partial_t$ and $k \mapsto -i\partial_x$ to revert to the corresponding (linear) PDE, and introducing a reference frame moving with velocity c , i.e., $x \mapsto x - ct$, it can be found that the linear PDE for a field $Q(x, t)$ associated to the above dispersion relation is:

$$Q_t - \frac{\alpha}{8c} Q_{xxx} + \left(\frac{1}{2} ca_4 - \frac{\alpha^2}{128c^3} \right) Q_{xxxxx} = 0. \quad (21)$$

The above PDE has the form of a linearized 5th-order KdV equation. The full nonlinear version will be derived below in section 3.4.

3.2. The Boussinesq/Benney–Luke equation and the solitary wave solution

We first consider an intermediate stage of the asymptotic analysis, and seek solutions of equations (13) and (14) in the form of the following asymptotic expansions:

$$\phi = -u_0^2 t + \varepsilon^{1/2} \Phi(X, T), \quad \rho = 1 + \sum_{j=1}^{\infty} \varepsilon^j \rho_j(X, T), \quad (22)$$

where $0 < \varepsilon \ll 1$ is a formal small parameter that sets the soliton's amplitude [i.e., below, we will find soliton solutions valid up to order $\mathcal{O}(\varepsilon)$]. Here, it is assumed that the phase Φ and amplitudes ρ_j are unknown real functions of the slow variables:

$$X = \varepsilon^{1/2} x, \quad T = \varepsilon^{1/2} t. \quad (23)$$

Substituting the expansions (22) into equations (13) and (14), and equating terms of the same order in ε , we obtain the following results. First, equation (13) reads:

$$\Phi_T + u_0^2 \rho_1 + \varepsilon \left(u_0^2 \rho_2 + \frac{1}{2} \Phi_X^2 - \frac{1}{4} \alpha \rho_{1XX} \right) = \mathcal{O}(\varepsilon^2), \quad (24)$$

while equation (14) yields, at $\mathcal{O}(\varepsilon^{3/2})$ and $\mathcal{O}(\varepsilon^{5/2})$, the following equations respectively:

$$\rho_{1T} + \Phi_{XX} = 0, \quad (25)$$

$$\rho_{2T} + (\rho_1 \Phi_X)_X = 0. \quad (26)$$

Differentiating equation (24) once with respect to T , and using $\rho_{1T} = -(1/u_0^2)\Phi_{TT}$ (from the leading-order part of equation (24)) as well as equations (25) and (26), we eliminate the functions ρ_1 and ρ_2 from the resulting equation, and arrive at the following equation for Φ :

$$\Phi_{TT} - c^2 \Phi_{XX} + \varepsilon \left[\frac{1}{4} \alpha \Phi_{XXXX} + \frac{1}{2} (\Phi_X^2)_T + (\Phi_T \Phi_X)_X \right] = \mathcal{O}(\varepsilon^2). \quad (27)$$

At the leading-order, equation (27) is a 2nd-order linear wave equation, with the wave velocity c given by equation (17), as found above. On the other hand, at order $\mathcal{O}(\varepsilon)$, equation (27) features a fourth-order dispersion and quadratic nonlinear terms, thus resembling the (1 + 1)-dimensional variants of the Boussinesq [32] or Benney–Luke [33] equations.

It is now possible to derive an exact solitary wave solution to the above BBL model, equation (27). This can be done upon seeking traveling wave solutions of the form:

$$\Phi = \Phi(s), \quad s = X - vT, \quad (28)$$

where v is the velocity of the traveling wave. Assuming vanishing boundary conditions for $\Phi' \equiv d\Phi/ds$, i.e., $\Phi' \rightarrow 0$ as $s \rightarrow \pm\infty$, we substitute in equation (27) and obtain the following 3d-order ordinary differential equation (ODE):

$$\frac{1}{4} \varepsilon \alpha \Phi''' + (v^2 - c^2) \Phi' - \frac{3}{2} \varepsilon v \Phi'^2 = 0, \quad (29)$$

where primes denote derivatives with respect to s . Next, we assume that the unknown field ρ_1 also depends on s , i.e., $\rho_1 = \rho_1(s)$, with $\rho_1(s) \rightarrow 0$ as $s \rightarrow \pm\infty$. Then, we may use $\Phi_T = -u_0^2 \rho_1$ from the leading-order part of equation (24) and obtain the auxiliary equation $\Phi' = (c^2/v) \rho_1$. Substituting the latter equation into equation (29), we derive the following 2nd-order ODE for ρ_1 :

$$\rho_1'' + \frac{4}{\varepsilon \alpha} (v^2 - c^2) \rho_1 - \frac{6c^2}{\alpha} \rho_1^2 = 0, \quad (30)$$

Equation (30) can be seen as the equation of motion of a unit mass particle in the presence of the effective potential $V(\rho_1) = (2/\varepsilon \alpha)(v^2 - c^2)\rho_1^2 - (2c^2/\alpha)\rho_1^3$. We assume that $v^2 - c^2 < 0$, i.e., we focus on traveling waves moving with a velocity smaller than the speed of sound (subsonic waves). Then, a simple analysis shows that, in this case, there exists a hyperbolic fixed point, at $\rho_1 = 0$ [corresponding to the global maximum of $V(\rho_1)$], and an elliptic fixed point, at $\rho_1 = (2/3\varepsilon c^2)(v^2 - c^2)$ [corresponding to the global minimum of $V(\rho_1)$]. In the phase plane of the system, associated to the hyperbolic fixed point that corresponds to zero energy E , i.e., for $(1/2)\rho_1^2 + V(\rho_1) = E = 0$, there exists a homoclinic orbit (separatrix). The latter is a trajectory of infinite period, which corresponds to a solution decaying at infinity, i.e., a solitary wave with vanishing asymptotics (note that if $v^2 - c^2 > 0$ the hyperbolic fixed point would become an elliptic one, and vice versa, and as a result the corresponding solitary wave would not asymptote to zero, as per our assumption above). It is then straightforward to find that this solitary wave solution can be expressed in the following explicit form:

$$\rho_1 = -\frac{1}{\varepsilon} \left(1 - \frac{v^2}{c^2} \right) \operatorname{sech}^2 \left[\frac{c}{\sqrt{\varepsilon \alpha}} \left(1 - \frac{v^2}{c^2} \right)^{1/2} (X - vT - X_0) \right], \quad (31)$$

where X_0 is an arbitrary constant that sets the initial soliton location. It can now readily be seen that the amplitude of the soliton is $\mathcal{O}(\varepsilon^{-1})$, while, generally, the condition $|\max(\rho_j)| = \mathcal{O}(1) \forall j \in \mathbb{N}$ should hold, as implied by the asymptotic expansion of ρ in equation (22). Hence, in order for the solution to be meaningful, i.e., the soliton amplitude is $\mathcal{O}(1)$, we assume that the (arbitrary so far) velocity v is sufficiently close to c , namely the following condition holds:

$$v^2/c^2 = 1 - \varepsilon\mu^2, \quad (32)$$

where μ is a $\mathcal{O}(1)$ parameter (recall that $v^2 - c^2 < 0$). Notice that, under this assumption, we detune from the sonic limit and explicitly consider solely solitary waves (which are gray solitons in the original model—see below) within this region. Then, the solution (31) becomes:

$$\rho_1 = -\mu^2 \text{sech}^2 \left[\frac{c\mu}{\sqrt{\alpha}} (X - c\sqrt{1 - \varepsilon\mu^2}T - X_0) \right], \quad (33)$$

and thus, an approximate solution [valid up to $\mathcal{O}(\varepsilon)$] of equation (9) is of the form:

$$u(x, t) \approx u_0 \left[1 - \varepsilon\mu^2 \text{sech}^2(\varepsilon^{1/2}\theta) \right]^{1/2} \exp \left[-iu_0^2 t + \frac{i\varepsilon^{1/2}c}{\sqrt{1 - \varepsilon\mu^2}} \tanh(\varepsilon^{1/2}\theta) \right], \quad (34)$$

$$\theta = \frac{c\mu}{\sqrt{\alpha}} \left(x - c\sqrt{1 - \varepsilon\mu^2}t - x_0 \right), \quad (35)$$

where we have used the equation $\Phi' = (c^2/v)\rho_1$ to derive the phase of the solution. It is clear that equation (34) represents a density dip on top of the cw background, with a phase jump across the density minimum, and hence it is a dark soliton.

Notice that, in our analysis above, the velocity v may be either positive or negative and, therefore, propagation of either right- or left-going waves is allowed, as should be expected from the bidirectional BBL equation. Below we will show that, indeed, and similarly to the water wave problem [32], the far-field of equation (27) is a pair of two KdV equations, for right- and left-going waves, and that the solitary wave (33) is accordingly reduced to right-going KdV soliton.

3.3. The KdV equation and the soliton solution

We now proceed to obtain the far-field equations stemming from the BBL model (27) by means of a multiscale asymptotic expansion method. We seek solutions of equation (27) in the form of the asymptotic expansion:

$$\Phi = \sum_{j=0}^{\infty} \varepsilon^j \Phi_j, \quad (36)$$

where the unknown functions Φ_j ($j = 0, 1, 2, \dots$) depend on the new variables:

$$\xi = X - cT, \quad \eta = X + cT, \quad \tau = \varepsilon T. \quad (37)$$

Substituting equation (36) into equation (27), we obtain the following results. First, at the leading-order, $\mathcal{O}(1)$:

$$4c^2 \Phi_{0\xi\eta} = 0, \quad (38)$$

which implies that Φ_0 is a superposition of a right-going wave, $\Phi_0^{(R)}$, depending on ξ , and a left-going one, $\Phi_0^{(L)}$, depending on η , namely:

$$\Phi_0 = \Phi_0^{(R)}(\xi) + \Phi_0^{(L)}(\eta). \quad (39)$$

Second, at order $\mathcal{O}(\varepsilon)$:

$$\begin{aligned} 4c^2\Phi_{1\xi\eta} = & -c \left(\Phi_{0\xi\xi}^{(R)}\Phi_{0\eta}^{(L)} - \Phi_{0\xi}^{(R)}\Phi_{0\eta\eta}^{(L)} \right) + \left(-2c\Phi_{0\tau}^{(R)} + \frac{\alpha}{4}\Phi_{0\xi\xi\xi}^{(R)} - \frac{3c}{2}\Phi_{0\xi}^{(R)2} \right)_{\xi} \\ & + \left(2c\Phi_{0\tau}^{(L)} + \frac{\alpha}{4}\Phi_{0\eta\eta\eta}^{(L)} + \frac{3c}{2}\Phi_{0\eta}^{(L)2} \right)_{\eta}. \end{aligned} \quad (40)$$

It is clear that upon integrating equation (40) in ξ or η , the terms in parentheses in the right-hand side of this equation are secular, because they are functions of ξ or η alone, and not both. Removal of these terms leads to two uncoupled nonlinear evolution equations for $\Phi_0^{(R)}$ and $\Phi_0^{(L)}$. Furthermore, employing $\Phi_{TT} = -u_0^2\rho_{1T} = -c^2\rho_{1T}$ from the leading-order part of equation (24) it can readily be seen that the amplitude ρ_1 can also be decomposed to a left- and a right-going wave, i.e., $\rho_1 = \rho_1^{(R)} + \rho_1^{(L)}$, with

$$\Phi_{0\xi}^{(R)} = c\rho_1^{(R)}, \quad \Phi_{0\eta}^{(L)} = -c\rho_1^{(L)}. \quad (41)$$

Then, using the above expressions, the equations for $\Phi_0^{(R)}$ and $\Phi_0^{(L)}$ stemming from equation (40) yield the following two uncoupled KdV equations for $\rho_1^{(R)}$ and $\rho_1^{(L)}$:

$$\rho_{1\tau}^{(R)} - \frac{\alpha}{8c}\rho_{1\xi\xi\xi}^{(R)} + \frac{3c}{2}\rho_1^{(R)}\rho_{1\xi}^{(R)} = 0, \quad (42)$$

$$\rho_{1\tau}^{(L)} + \frac{\alpha}{8c}\rho_{1\eta\eta\eta}^{(L)} - \frac{3c}{2}\rho_1^{(L)}\rho_{1\eta}^{(L)} = 0. \quad (43)$$

We have thus shown that, indeed, the far field of equation (27) is a pair of KdV equations for a left- and a right-going wave. We will also show that the KdV soliton is directly connected with the solitary wave solution (33) of the BBL equation. To do this, let us consider the right-going wave, and write down the soliton solution of the KdV equation (42):

$$\rho_1(\xi, \tau) = -\frac{\kappa^2\alpha}{c^2}\text{sech}^2 Z, \quad Z = \kappa \left(\xi + \frac{\kappa^2\alpha}{2c}\tau - \xi_0 \right), \quad (44)$$

where κ is an arbitrary $\mathcal{O}(1)$ parameter. When expressed in terms of the original variables, the above KdV soliton reads:

$$\rho_1(x, t) = -\frac{\kappa^2\alpha}{c^2}\text{sech}^2 \left\{ \varepsilon^{1/2}\kappa \left[x - c \left(1 - \varepsilon \frac{\kappa^2\alpha}{2c^2} \right) t - x_0 \right] \right\}, \quad (45)$$

showing that the velocity of the KdV soliton is $v_s = c[1 - \varepsilon(\kappa^2\alpha)/(2c^2)]$. On the other hand, returning to the solitary wave (33), it is observed that if the free parameter μ is taken to be such that:

$$\mu^2 = \kappa^2\alpha/c^2, \quad (46)$$

the solitary wave (33) can be expressed in terms of the original variables as:

$$\rho_1 = -\frac{\kappa^2 \alpha}{c^2} \operatorname{sech}^2 \left[\varepsilon^{1/2} \kappa \left(x - c \sqrt{1 - \varepsilon \frac{\kappa^2 \alpha}{c^2}} t - x_0 \right) \right], \quad (47)$$

and, hence, the solitary wave velocity is $V_s = c \sqrt{1 - \varepsilon(\kappa^2 \alpha)/c^2}$. It is now clear that for sufficiently small ε we may use the approximation $V_s \approx v_s = c[1 - \varepsilon(\kappa^2 \alpha)/(2c^2)]$, showing that the solitary wave (33) transforms into the KdV soliton (45). The latter, gives rise to an approximate dark soliton solution of equation (9), similar to that in equation (34), but with μ^2 given by equation (46) and with the substitution $\sqrt{1 - \varepsilon \mu^2} \mapsto 1 - \varepsilon \mu^2/2$.

At this point, we should mention that, in the case of the weakly nonlocal system we consider here, the presented approximate solutions are always dark solitons, due to the condition $\alpha > 0$ following from the requirement of the stability of the cw background state. Nevertheless, in the fully nonlocal system characterized by the kernel (4) (see equations (6) and (7)), the cw is always modulationally stable as mentioned above. As a result, the parameter α may also take negative values and, thus, the nonlocal system, also possesses approximate *antidark* soliton solutions (i.e., density humps rather than dips on top of the cw background), as predicted in references [26, 28, 29]. Obviously, these are supersonic structures (here, $V_s > c$) which can only be found in the strongly nonlocal regime, and cannot be supported in the weakly nonlocal case under consideration.

3.4. Reductive perturbation method and higher-order effects

The formal derivation of the KdV equation (42) from the BBL model (27) for long times, and particularly at the scales of equation (37), suggest that the KdV equation (e.g., for the right-going wave) can also be obtained directly from the hydrodynamic equations (13) and (14). This can be done upon employing the reductive perturbation method (RPM) [38]. In the framework of the RPM, as we will see, it is straightforward to take into regard higher-order effects (namely, the term $a_4 \rho_{xxxx}$ in equation (13)) and thus derive an effective higher-order KdV equation describing dark solitons in weakly nonlocal media.

We start by seeking solutions of equations (13) and (14) in the form of the asymptotic expansions

$$\phi = -u_0^2 t + \sum_{j=0}^{\infty} \varepsilon^{j+1/2} \phi_j(\xi, \tau), \quad \rho = 1 + \sum_{j=1}^{\infty} \varepsilon^j \rho_j(\xi, \tau), \quad (48)$$

where the unknown functions ρ_j and ϕ_j ($j \in \mathbb{N}$) depend on the stretched coordinates:

$$\xi = \varepsilon^{1/2} (x - ct), \quad \tau = \varepsilon^{3/2} t, \quad (49)$$

where c is speed of sound (see equation (17)). Substituting the expansions (48) into equations (13) and (14), and using the variables (49), we obtain a hierarchy of coupled equations, which are to be solved order by order in ε [note that, as before, both parameters a_2 and a_4 are assumed to be of order $\mathcal{O}(1)$]. Particularly, to the leading order, i.e., to orders $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^{3/2})$, equations (13) and (14) respectively lead to the following linear equations,

$$c\phi_{1\xi} - u_0^2 \rho_1 = 0, \quad (50)$$

$$c\rho_{1\xi} - \phi_{1\xi\xi} = 0. \quad (51)$$

The compatibility condition of equation (51) is the algebraic equation (17). To the next order, namely to order $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon^{5/2})$, equations (13) and (14) respectively read

$$c\phi_{2\xi} - u_0^2\rho_2 = \phi_{1\tau} - \frac{\alpha}{4}\rho_{1\xi\xi} + \frac{1}{2}\phi_{1\xi}^2, \quad (52)$$

$$c\rho_{2\xi} - \phi_{2\xi\xi} = \rho_{1\tau} + (\rho_1\phi_{1\xi})_\xi, \quad (53)$$

with α given in equation (18). Using equation (50), the unknown function ϕ_1 is expressed by means of ρ_1 , i.e.,

$$\phi_{1\xi} = c\rho_1. \quad (54)$$

Then, the compatibility conditions of equations (52) and (53) are determined, once equation (52) is differentiated with respect to ξ , equation (53) is multiplied by c , and the resulting equations are added. This way, it is found that the compatibility condition at this order is the KdV equation (42) for the unknown amplitude function ρ_1 , in accordance with the analysis of the previous section.

Notice that in the present order of approximation, there is no contribution from the term $a_4u\partial_x^4|u|^2$ in equation (9), since the corresponding 4th-order derivative term $a_4\rho_{xxxx}$ in equation (13) is of higher-order. However, this term contributes in the next order of approximation. Indeed, proceeding to the next order, namely to $\mathcal{O}(\varepsilon^3)$ and $\mathcal{O}(\varepsilon^{7/2})$, equations (13)–(14) respectively lead to the following equations,

$$c\phi_{3\xi} - u_0^2\rho_3 = \phi_{2\tau} - \frac{\alpha}{4}\rho_{2\xi\xi} + c^2a_4\rho_{1\xi\xi\xi\xi} + \phi_{1\xi}\phi_{2\xi} + \frac{1}{8}\rho_{1\xi}^2 + \frac{1}{4}\rho_1\rho_{1\xi\xi}, \quad (55)$$

$$c\rho_{3\xi} - \phi_{3\xi\xi} = \rho_{2\tau} + (\rho_1\phi_{2\xi})_\xi + (\rho_2\phi_{1\xi})_\xi. \quad (56)$$

The compatibility conditions of equations (55) and (56), can also be obtained upon following the procedure described above. In particular, first, equation (55) is differentiated with respect to ξ , equation (56) is multiplied by c , and the resulting equations are added. Second, we use equation (54) to express $\phi_{1\xi}$ in terms of ρ_1 , as well as equation (53) to express $\phi_{2\xi}$ in terms of ρ_2 and ρ_1 , i.e.,

$$\phi_{2\xi} = c\rho_2 - c\rho_1^2 - \int \rho_{1\tau} d\xi, \quad (57)$$

where integration constants are equal to zero due to the boundary conditions. This way, we obtain from equations (55) and (56) the following equation, which involves solely the fields ρ_1 and ρ_2 :

$$\begin{aligned} \rho_{2\tau} - \frac{\alpha}{8c}\rho_{2\xi\xi\xi} + \frac{3c}{2}(\rho_1\rho_2)_\xi + \frac{1}{2c} \int \rho_{1\tau\tau} d\xi + \rho_{1\xi} \int \rho_{1\tau} d\xi + \frac{3}{4}(\rho_1^2)_\tau + \frac{c}{2}(\rho_1^3)_\xi \\ + \frac{a_2c}{2}(\rho_{1\xi}^2)_\xi - \frac{\alpha}{8c^2}\rho_{1\xi\xi\tau} - \frac{1-8a_2c^2}{8c}\rho_1\rho_{\xi\xi\xi} + \frac{a_4c}{2}\rho_{1\xi\xi\xi\xi} = 0. \end{aligned} \quad (58)$$

To this end, we multiply equation (58) by ε , and add it to the KdV of equation (42). Then, introducing the combined amplitude function

$$q = \rho_1 + \varepsilon\rho_2, \quad (59)$$

we obtain the following nonlinear evolution equation for the field $q(\xi, \tau)$:

$$q_\tau - \frac{\alpha}{8c} q_{\xi\xi\xi} + \frac{3c}{2} qq_\xi + \varepsilon P(q) = O(\varepsilon^2), \quad (60)$$

$$P(q) \equiv c_1 q^2 q_\xi + c_2 q_\xi q_{\xi\xi} + c_3 qq_{\xi\xi\xi} + c_4 q_{\xi\xi\xi\xi\xi}. \quad (61)$$

Notice that the terms involving $\rho_{1\tau}$ in equation (58) have been evaluated by substituting $\rho_{1\tau}$ from the KdV, equation (42). The coefficients c_j ($j = 1, 2, 3, 4$) in equation (61) are given by:

$$c_1 = -\frac{3}{8}c, \quad c_2 = \frac{1}{4c} \left(1 + \frac{5}{8}\alpha\right), \quad c_3 = \frac{1}{8c} \left(1 - \frac{1}{2}\alpha\right), \quad c_4 = \frac{1}{2}ca_4 - \frac{\alpha^2}{128c^3}. \quad (62)$$

It is readily seen that equation (60) has the form of a 5th-order pKdV equation. It is worth observing that equation (60) is reduced to the unperturbed KdV equation (42) in the limit of $\varepsilon = 0$, while its linearized version is identical to equation (21).

The 5th-order pKdV equation (60) has attracted attention, as a model describing the evolution of steeper waves, with shorter wavelengths than in the KdV model. As such, this equation has been used to describe solitons in plasmas [42, 43] and shallow water waves [44] in the presence of higher-order effects, and as a generic model that can be used to explain soliton emergence in experiments from arbitrary initial data, even when the Hamiltonian perturbations are quite large [39, 40].

Additionally, an extended KdV equation, similar in form to equation (60), is related to the first higher-order equation in the KdV hierarchy [41]. In particular, using the transformations $\tau \mapsto -8c/\alpha\tau$ and $q \mapsto -(\alpha/2c^2)q$, equation (60) reduces [up to $O(\varepsilon)$] to the form:

$$q_\tau + q_{\xi\xi\xi} + 6qq_\xi + \varepsilon (\tilde{c}_1 q^2 q_\xi + \tilde{c}_2 q_\xi q_{\xi\xi} + \tilde{c}_3 qq_{\xi\xi\xi} + \tilde{c}_4 q_{\xi\xi\xi\xi\xi}) = 0,$$

where

$$\tilde{c}_1 = -(2\alpha/c^3)c_1, \quad \tilde{c}_{2,3} = (4/c)c_{2,3}, \quad \tilde{c}_4 = -(8c/\alpha)c_4,$$

where c_j are given by equation (62). In this case, the first higher-order equation in the KdV hierarchy is characterized by the following values of the coefficients:

$$\tilde{c}_1 = 1, \quad \tilde{c}_2 = \frac{2}{3}, \quad \tilde{c}_3 = \frac{1}{3}, \quad \tilde{c}_4 = \frac{1}{30}.$$

Obviously, in our case, equation (60) never falls in that integrable limit. In the more general nonintegrable case, the pKdV equation with arbitrary c_j has been studied in the context of asymptotic integrability of weakly dispersive and nonlinear wave equations [45, 46]. In this context, it was shown that there exist asymptotic transformations [45, 46, 54, 55] that reduce the pKdV to the KdV equation. Thus, an approximate [valid up to $O(\varepsilon)$] soliton solution of equation (60) can be found, which has the form of the traditional KdV soliton with a velocity-shift and a bounded shape correction. In particular, this approximate soliton solution of equation (60) reads (see, e.g., reference [55]):

$$q(\xi, \tau) = -\frac{\alpha}{2c^2} (A \operatorname{sech}^2 \theta + \varepsilon A^2 \lambda_1 \operatorname{sech}^2 \theta + \varepsilon A^2 \lambda_2 \operatorname{sech}^4 \theta) + O(\varepsilon^2), \quad (63)$$

$$\theta = \kappa \left(\xi + \frac{\alpha}{8c} V \tau - \xi_0 \right), \quad V = 2A - \frac{32c}{\alpha} \varepsilon c_4 A^2, \quad (64)$$

where $A = 2\kappa^2$, with $\kappa \in \mathbb{R}$ being the free $O(1)$ parameter of the KdV soliton in equation (44), while the constants λ_1 and λ_2 are given by:

$$\lambda_1 = \frac{120c^4c_4 + 2c^2\alpha c_2 + 8c^2\alpha c_3 + \alpha^2}{3c^3\alpha},$$

$$\lambda_2 = \frac{-360c^4c_4 - 6c^2\alpha c_2 - 12c^2\alpha c_3 - \alpha^2}{6c^3\alpha}.$$

Obviously, in the limit $\varepsilon \rightarrow 0$ the soliton of equation (63) reduces to the KdV soliton of equation (44).

Furthermore, employing the perturbation theory for solitons [47–49], we may obtain the following results. First, due to the presence of the perturbation $P(q)$ (see equation (61)), the parameter κ becomes time-dependent, namely $\kappa \mapsto \kappa(\tau)$, featuring an evolution determined by:

$$\frac{d\kappa}{d\tau} \propto \frac{1}{4\kappa} \int_{-\infty}^{+\infty} P(q_s) \text{sech}^2 Z \, dZ, \quad (65)$$

where q_s is the soliton of the unperturbed KdV equation (see equation (44)). Second, the amplitude of the radiation tails q_R produced by the perturbation $P(q)$ is proportional to the factor R given by:

$$q_R \propto R = \frac{1}{4\kappa^5} \int_{-\infty}^{+\infty} P(q_s) \tanh^2 Z \, dZ. \quad (66)$$

Evidently, $P(q_s)$ is an odd function of Z : indeed, since q_s is an even function of Z , its odd-order derivatives are odd functions, while its even-order ones are even functions. It is thus straightforward to conclude that the overall function $P(q_s)$ is odd. On the other hand, $\text{sech}^2 Z$ and $\tanh^2 Z$ are even functions of Z and thus both integrals in equations (65) and (66) vanish. Thus, the parameter κ which characterizes the soliton amplitude remains time-independent and, at the same time, no radiation tails are produced in this higher-order KdV approximation.

4. Conclusions

In conclusion, we have used multiscale expansion methods to study the dynamics of dark solitons in weakly nonlocal media, governed by a nonlinear Schrödinger model. In particular, we have analyzed the hydrodynamic form of the model and considered at first the leading-order of approximation, where only the first moment of the medium's response function is present. At an intermediate stage of the asymptotic analysis, we derived a BBL equation. Using a traveling wave ansatz, we derived exact solitary wave solutions of this equation, in the limiting case where the velocity of the solitary wave is sufficiently close to (and below) the speed of sound. Then, we considered the long-time behavior of the BBL equation and, upon introducing relevant scales and asymptotic expansions, we reduced the BBL model to a pair of KdV equations that govern right- and left-propagating waves. We have also shown that if the formal perturbation parameter becomes sufficiently small then the BBL solitary wave transforms into the KdV soliton.

We also used the reductive perturbation method to analyze higher-order effects. We thus considered the model at the next order of approximation, where the second moment of the response function comes into play. In this case, we found that dark solitons are governed by

a pKdV equation which, as it has been shown in the past, can be approximated by a higher-order integrable system. We have presented the exact soliton solution of the pKdV equation, and employed the perturbation theory for solitons to show that the soliton amplitude remains unchanged, while no radiation tails are produced during the evolution. Thus, it can be concluded that, in the presence of the higher-order effects, the dark soliton's shape and velocity are only insubstantially changed.

Our analysis and results suggest interesting directions for future investigations. First, it would be relevant to study analytically the dynamics of the derived soliton solutions in a higher-dimensional setting, and investigate the role of weak nonlocality on the transverse modulational instability of dark solitons (see, e.g., reference [22] for a relevant study). It would also be relevant to study dispersive shock waves, and particularly the role of higher-order effects, in weakly nonlocal media. Naturally, exploring numerically the quantitative aspects of the predictions herein both on the original dynamical system of the weak nonlocality, as well as in the full original setting of the model featuring the nonlocal kernel, would be of particular relevance and interest. Various predictions including the existence of the antidark, supersonic solitons or the range over which the gray subsonic solitons may exist are of particular interest within such an investigation. Finally, the derivation of higher-order nonlinear evolution equations that describe effectively soliton dynamics in higher-dimensional, fully nonlocal media is another quite interesting theme. Pertinent studies are in progress and relevant results will be reported elsewhere.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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