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# Propagation of periodic wave trains along the magnetic field in a collision-free plasma

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## Abstract

In this work, a systematic study, examining the propagation of periodic and solitary waves along the magnetic field in a cold collision-free plasma, is presented. Employing the quasi-neutral approximation and the conservation of momentum flux and energy flux in the frame co-traveling with the wave, the exact analytical solution of the stationary solitary pulse is found analytically in terms of particle densities, parallel and transverse velocities, as well as transverse magnetic fields. Subsequently, this solution is generalized in the form of periodic waveforms represented by cnoidal-type waves. These considerations are fully analytical in the case where the total angular momentum flux  $L$ , due to the ion and electron motion together with the contribution due to the Maxwell stresses, vanishes. A graphical representation of all associated fields is also provided.

**Keywords:** Adlam–Allen model, wave propagation in plasmas, solitons, cnoidal waves, periodic solutions

(Some figures may appear in colour only in the online journal)

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## 1. Introduction

The study of solitary waves, their dynamics and interactions, as well as the associated notions of integrability and soliton theory have been long thought to originate with the famous Fermi-Pasta-Ulam-Tsingou problem [1]. Arguably, though, the explosion of interest in the field came a decade later and is chiefly credited to the numerical study of the FPU problem, and associated remarkable properties of the Korteweg–de Vries (KdV) equation, reported by Kruskal and Zabusky [2]. What is perhaps far less well-known is that a major part of the associated motivation (even cited as such in the work of [2]) stemmed from work at around the same time in the study of traveling waves in plasmas. More well-known in this regard is the seminal work of Washimi and Taniuti [3] connecting the dynamics of ion-acoustic waves in plasmas to the KdV equation. Yet, a ‘well-kept secret’ since its inception already many years earlier (i.e., in 1958–1960) has been the work of Adlam and Allen [4, 5]. These authors developed a model for the propagation of a solitary wave in a collisionless plasma along the  $x$ – and  $y$ –directions coupled with dynamics in the  $x$ – and  $y$ –directions for the electric field and a transverse magnetic field  $B_z$ .

Recently, the propagation of finite amplitude waves across a magnetic field in cold plasmas has been revisited for the demonstration of the  $\mathbf{j} \times \mathbf{B}$  force in a collisionless plasma [6]. On the other hand, some time ago, Montgomery [7] and Saffman [8] (see also Tidman and Krall [9]) discussed the large amplitude waves propagating along the magnetic field. Such parallel propagating solitary waves have been further examined with the core filled by oscillatory structures in more recent works. For example, Sauer *et al* [10] and Dubinin *et al* [11] discussed stationary nonlinear solutions, including localized waveforms with oscillating phase, looking like envelope solitons in the wave frame. Keeping in view this framework, Cattaert and Verheest [12] analyzed large amplitude weakly nonlinear (KdV type) oscillatory solitary structures in the plasma frame. Such a treatment has further been employed in solitary waves in dusty plasmas [13, 14]. In all the above cases, the core purpose is to obtain a one-dimensional solution which describes the motion of a pulse or solitary wave in the direction perpendicular or parallel to the magnetic field. Over the past year, both of these directions have been further considered. More specifically, in reference [15], analytical expressions in the co-traveling frame, for the form of the longitudinally propagating solitary wave, were obtained. Concurrently, in the transverse field case, the work of [16] examined various generalizations; these include the interaction of two solitary waves (identifying their repulsion), as well as periodic (cnoidal) solutions, that were identified numerically, which generalize the solitary waveform (and possess the latter as a special limit).

The present work studies analytically—and complements the analytics with numerical identification when needed—the system of equations for fully nonlinear electromagnetic waves propagating along a uniform magnetic field in a cold collisionless plasma. The oscillatory solutions of transverse particle velocities and transverse magnetic fields are derived without linearization. We shall solve the present problem using the concept of quasineutrality. In particular, it is assumed that an electrostatic field is produced by an infinitesimal difference between the electron and ion densities; the Gauss’ law for the electric field is not employed. The concept is valid in the case where the electron plasma frequency is much greater than the electron gyrofrequency. However, in a strongly nonlinear case, the possible deviation from quasineutrality condition in the longitudinal direction can be significant. In reference [19], the properties of ion- and electron-acoustic (longitudinal) solitons in a plasma without a magnetic field are considered with the understanding that solitons carry out one-way transfer of charged particles at a distance of several Debye radii. In particular, the electric currents with a DC component were induced. Subsequently [20], it was shown that these currents can be significant in the case

of large amplitudes. These considerations were further analyzed in the experimental study of dust-acoustic soliton currents [21]. This is, as can be observed from the above works, a particularly active direction of research, yet here we will restrict considerations to the regime where quasineutrality is a valid approximation.

The problem of interest herein is exposed in its full generality by involving nine space-time dependent fields, namely five velocity fields, two (transverse) magnetic fields, an electric field and a charge density field. Expressions for all of them are provided. The starting point is a reduction of the problem to an effective two- and eventually (through a polar decomposition) one degree-of-freedom problem for the transverse magnetic field components. In these variables, the solitary wave case of [15] is initially retrieved and is subsequently generalized to periodic solutions in the form of cnoidal waves. The derivation of the solitary wave is found to be analytically possible in the case of a vanishing angular momentum flux  $L$  (which is due to the ion and electron motion and the contribution due to the Maxwell stresses) in the plane of the transverse magnetic field. In the appendix A, we discuss the case of non-zero  $L$  which is of mathematical interest as it yields solely a possibility for periodic waves which we numerically reconstruct. In the astrophysical context, the present study may have applications in the Earth foreshock region and Jupiter's bow shock region where oscillatory solitary structures are expected naturally under many different conditions [22]. Nevertheless, we are not presently aware of an experiment that has observed the relevant structures.

Our presentation is structured as follows. In section 2, we revisit the analytical formulation of the problem in dimensionless units and reduce it to the two degree-of-freedom system for the transverse magnetic fields. In section 3, we discuss the analytical solutions and connect the problem to a Duffing oscillator, obtaining also the cnoidal wave solutions for the case of vanishing  $L$ . In section 4, we summarize our findings and present a number of conclusions, as well as directions for future research. The appendix A contains the case of non-vanishing  $L$  and a mathematical discussion of the reconstruction of the (solely) periodic orbits of the latter.

## 2. Basic equations of motion

The framework of physical interest, similarly to the case of [15], is as follows. We wish to describe ions and electrons propagating under a *common* velocity field and electric field along the  $x$ -direction. In the quasi-neutral setting of interest (where the charge carriers satisfy  $n_e \approx n_i = n$ ), the opposite charges bear unequal velocities along the  $y$ - and  $z$ -directions. In these directions, there exists a nontrivial (non-constant) magnetic field  $B_y$  and  $B_z$ . For a cold collisionless plasma, first we write the equations in SI units governing the motion of a cold gas consisting of electrons and one type of positive ion, as follows:

$$\left( \frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla \right) \mathbf{v}_s = \frac{q_s}{m_s} [\mathbf{E} + \mathbf{v}_s \times \mathbf{B}], \quad (1)$$

and

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0. \quad (2)$$

The index  $s$  can either stand for e, i.e., electrons or for i, i.e., ions. In response, the electromagnetic fields generated by particle motion are:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \mu_0 \sum_s q_s n_s \mathbf{v}_s + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \end{aligned} \quad (3)$$

where  $m_s = m_{e,i}$  are the respective masses,  $\epsilon_0$  and  $\mu_0$  denote vacuum's electric permittivity and magnetic permeability, respectively,  $q_s = \pm e$  is the charge, and  $n_s$  is the density of the charges. We add equation (1) for ions and electrons and obtain:

$$nD(m_i \mathbf{v}_i + m_e \mathbf{v}_e) = \frac{1}{\mu_0} \left[ (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (\mathbf{B}^2) \right]. \quad (4)$$

Here, we have used the total derivative notation  $D = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}$  for compactness. The electric stresses are negligible, since quasi-neutrality has been assumed. The above vector equation only refers to change in momentum flux. From equation (1), the energy equation can also be written down immediately as follows:

$$\frac{m_i n}{2} Dv_x^2 + \frac{m_i n}{2} Dv_{i\perp}^2 + \frac{m_e n}{2} Dv_{e\perp}^2 = 0. \quad (5)$$

### 2.1. Normalized equations

In what follows, and although it is possible to work with equations (1)–(5) as expressed in dimensional units, it is convenient—for notational simplicity and more straightforward connection with the numerical results—to use dimensionless equations. Before that, recall that the full set of fields involves, in addition to the density, the magnetic and electric fields,  $\mathbf{B} = (B_1, B_y(x), B_z(x))$  and  $\mathbf{E} = (E(x), 0, 0)$  respectively, as well as the velocity field  $(v_x, v_{ey,iy}, v_{ez,yz})$ . Notice the common velocity along the  $x$ -axis  $v_x$  of the electrons and the ions. More specifically, we start by rewriting the equations in normalized form. The unit for the distance is  $d = \sqrt{m_e m_i / e^2 \mu_0 n_1 (m_e + m_i)}$ . The corresponding characteristic speed  $v_A = B_1 / \sqrt{\mu_0 N_1 (m_e + m_i)}$  is the Alfvén velocity,  $\Omega_e = e B_1 / m_e$  is the electron angular frequency and  $\alpha = m_e / m_i$  is the mass ratio. The unit for the velocity employed in this paper was obtained by multiplying the distance  $d$  by  $\Omega_e$  (i.e.,  $v^* = \frac{v}{\Omega_e}$ ). We then define the normalized variables:  $t \mapsto \Omega_e t$ ,  $E_x \mapsto E_x / v^* B_1$ ,  $x \mapsto x/d$ ,  $\mathbf{v} \mapsto \mathbf{v}/v^*$ ,  $\mathbf{B} \mapsto \mathbf{B}/B_1$ . In these units, the equations become dimensionless in the form that we now discuss.

(a). Dimensionless equations for electrons and ions: by implementing the above mentioned rescaling, we acquire the dimensionless form of the Newtonian equations of motion of the particles (electrons and ions with respective subscripts), involving the force from the electric and the magnetic field ( $\mathbf{v} \times \mathbf{B}$ ).

$$\left( \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} \right) \begin{pmatrix} v_x \\ v_{ey} \\ v_{ez} \end{pmatrix} = - \left[ \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_{ey} B_z - v_{ez} B_y \\ -v_x B_z + v_{ez} \\ v_x B_y - v_{ey} \end{pmatrix} \right], \quad (6)$$

$$\left( \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} \right) \begin{pmatrix} v_x \\ v_{iy} \\ v_{iz} \end{pmatrix} = \alpha \left[ \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_{iy} B_z - v_{iz} B_y \\ -v_x B_z + v_{iz} \\ v_x B_y - v_{iy} \end{pmatrix} \right]. \quad (7)$$

Notice once again that in the above equations the electric field lies along the  $x$ -direction, while the magnetic field  $B_1$  is constant along the same direction and has now been normalized to unity. In equations (6) and (7), the derivatives are the total derivatives involving traveling along the  $x$ -direction, hence a potential traveling configuration will simply mean that we set the partial derivatives with respect to the (dimensionless) time to zero, allowing the waves to move along the  $x$  direction without changing shape.

(b). Additional field equations. In addition to equations (6) and (7), we have the continuity equation for the density (of both ions and electrons in our quasi-neutral setting):

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}\right) n + n \frac{\partial v_x}{\partial x} = 0, \quad (8)$$

while the full set of 9 equations for the 9 fields is completed by the two components of Ampère's Law along the  $y$ - and  $z$ -direction:

$$\frac{\partial B_y}{\partial x} = \frac{n}{\alpha + 1} (v_{iz} - v_{ez}), \quad (9)$$

$$\frac{\partial B_z}{\partial x} = -\frac{n}{\alpha + 1} (v_{iy} - v_{ey}). \quad (10)$$

It is worthwhile to note that algebraic manipulations of the 6 Newtonian equations can lead to a reformulation of 3 of them as momentum flux equations [15] in the following form:

$$\begin{aligned} Dv_x + \frac{\alpha}{n} \frac{\partial B_{\perp}^2}{\partial x} &= 0, \\ D\left(\frac{1}{\alpha} v_{iy} + v_{ey}\right) - \frac{\alpha + 1}{n} \frac{\partial B_y}{\partial x} &= 0, \\ D\left(\frac{1}{\alpha} v_{iz} + v_{ez}\right) - \frac{\alpha + 1}{n} \frac{\partial B_z}{\partial x} &= 0, \end{aligned} \quad (11)$$

where again we use  $D = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}$ , and  $B_{\perp}^2 = B_x^2 + B_y^2$ . Here, it is interesting to note that the electric stresses are negligible compared to the magnetic stresses when the concept of quasineutrality is employed.

## 2.2. Equations in co-travelling frame

Remarkably, and despite their complexity for a multitude of fields, it is possible to tackle the above 9 equations when looking for a traveling wave. We thus now turn to the setting of solutions that do not depend on time explicitly. There are two potential approaches towards reducing the problem. One is to attempt to solve the equations involving the transverse magnetic components as a function of the speeds, and formulate ordinary differential equations (ODEs) for the latter. The second is to express the velocity fields in terms of  $B_y$  and  $B_z$ ; here, we follow this latter approach. As explained before, in the co-traveling frame, the partial derivative with respect to time vanishes. This way, the relevant Newtonian ODEs read:

(a). For electrons:

$$v_x \frac{\partial}{\partial x} \begin{pmatrix} v_x \\ v_{ey} \\ v_{ez} \end{pmatrix} = - \left[ \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_{ey} B_z - v_{ez} B_y \\ -v_x B_z + v_{ez} \\ v_x B_y - v_{ey} \end{pmatrix} \right], \quad (12)$$

(b). For ions:

$$v_x \frac{\partial}{\partial x} \begin{pmatrix} v_x \\ v_{iy} \\ v_{iz} \end{pmatrix} = \alpha \left[ \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_{iy} B_z - v_{iz} B_y \\ -v_x B_z + v_{iz} \\ v_x B_y - v_{iy} \end{pmatrix} \right]. \quad (13)$$

The continuity equation is now reduced to a simple algebraic equation:

$$n v_x = C \quad (14)$$

The momentum flux equation (11) can now be integrated to yield:

$$\begin{aligned} \frac{C v_x}{\alpha} + \frac{1}{2} B_{\perp}^2 &= E_1, \\ \frac{C}{\alpha + 1} \left( \frac{1}{\alpha} v_{iy} + v_{ey} \right) - B_y &= E_2, \\ \frac{C}{\alpha + 1} \left( \frac{1}{\alpha} v_{iz} + v_{ez} \right) - B_z &= E_3, \end{aligned} \quad (15)$$

where  $E_{1,2,3}$  are the corresponding integration constants. Equation (15) describe the momentum flow in the three directions  $x$ ,  $y$  and  $z$ . The first term on the left-hand side (in all three equations) is clearly the rate of flow of momentum of the particles in the directions  $x$ ,  $y$  and  $z$ . The second term is the rate of flow of momentum associated with the magnetic field.

(c). Equations for the two-dimensional potential well. One can now combine the momentum flux equations for the  $y$ - and  $z$ - components of the velocity field with the Ampère's law components, and express  $v_{iy,ey}$  and  $v_{iz,ez}$  in terms of the transverse magnetic field components  $B_y$  and  $B_z$ . The resulting equations read:

$$v_{iy,e} = \frac{\alpha B_y}{C} + \frac{(-\alpha, 1)}{n} \frac{\partial B_z}{\partial x} + \frac{\alpha}{C} E_2, \quad (16)$$

$$v_{iz,ez} = \frac{\alpha B_z}{C} + \frac{(\alpha, -1)}{n} \frac{\partial B_y}{\partial x} + \frac{\alpha}{C} E_3. \quad (17)$$

We now differentiate the components of Ampère's law with respect to  $x$ , multiply by  $v_x$ , and substitute the resulting velocity derivatives from Newton's equations, as well as the velocity components in terms of the magnetic fields  $B_y, B_z$  [see equations (16) and (17)]. This way, we can obtain the following pair of equations involving *solely* the transverse magnetic field components:

$$v_x \frac{\partial}{\partial x} \left( v_x \frac{\partial B_y}{\partial x} \right) = - \frac{\partial \Phi}{\partial B_y} - \frac{(1 - \alpha)}{n} C \frac{\partial B_z}{\partial x}, \quad (18)$$

$$v_x \frac{\partial}{\partial x} \left( v_x \frac{\partial B_z}{\partial x} \right) = - \frac{\partial \Phi}{\partial B_z} + \frac{(1 - \alpha)}{n} C \frac{\partial B_y}{\partial x}, \quad (19)$$

where the effective potential  $\Phi$  can be defined as:

$$\Phi \equiv \Phi(B_y, B_z) = \frac{1}{8} \alpha B_{\perp}^4 - \frac{1}{2} \alpha (E_1 - 1) B_{\perp}^2 + \alpha (E_2 B_y + E_3 B_z). \quad (20)$$

**2.2.1. Conservation laws for the two-dimensional system.** By multiplying the first of equation (18) by  $\frac{\partial B_y}{\partial x}$  and similarly equation (19) by  $\frac{\partial B_z}{\partial x}$ , and adding the two we get

$$\frac{1}{2} \left[ \left( v_x \frac{\partial B_y}{\partial x} \right)^2 + \left( v_x \frac{\partial B_z}{\partial x} \right)^2 \right] + \Phi(B_y, B_z) = W = \text{const.} \quad (21)$$

The above equation implies that  $W$  is a conserved quantity for the system. Here,  $W$  stands for the difference between the flux of total kinetic energy and the kinetic energy carried by the plasma particles along  $x$ -direction.

One can similarly manipulate the two equations (18) and (19) (multiplying the first by  $B_z$  and the second by  $B_y$  and subtracting) to obtain another conserved quantity, namely:

$$B_z \left( v_x \frac{\partial B_y}{\partial x} \right) - B_y \left( v_x \frac{\partial B_z}{\partial x} \right) + \frac{(1-\alpha)}{2} B_\perp^2 - \alpha \int (E_2 B_z - E_3 B_y) dx = L = \text{const}, \quad (22)$$

where  $L$  represents the total angular momentum flux. The explicit presence of an integral in equation (22) suggests that it is far more amenable to analytical (or semi-analytical) manipulations in the case of  $E_2 = E_3 = 0$ . We will return to this point in what follows.

(a) Polar coordinate analysis of the ODEs. Using a polar decomposition in the form:

$$B_y = B_\perp \cos \theta, \quad B_z = B_\perp \sin \theta,$$

we obtain

$$\left( \frac{\partial B_y}{\partial x} \right)^2 + \left( \frac{\partial B_z}{\partial x} \right)^2 = \left( \frac{\partial B_\perp}{\partial x} \right)^2 + \left( B_\perp \frac{\partial \theta}{\partial x} \right)^2. \quad (23)$$

Upon substituting these in equation (21), we obtain:

$$\frac{1}{2} \left[ \left( v_x \frac{\partial B_\perp}{\partial x} \right)^2 + B_\perp^2 \left( v_x \frac{\partial \theta}{\partial x} \right)^2 \right] + \Phi(B_\perp) = W. \quad (24)$$

Furthermore, the left-hand side of equation (24) is reshaped as:

$$B_z \left( \frac{\partial B_y}{\partial x} \right) - B_y \left( \frac{\partial B_z}{\partial x} \right) = -B_\perp^2 \frac{\partial \theta}{\partial x}. \quad (25)$$

Then, reformulation of the corresponding conservation law of equation (22) leads to:

$$\left[ \frac{(1-\alpha)}{2} - v_x \frac{\partial \theta}{\partial x} \right] B_\perp^2 + \alpha \int B_\perp (E_2 \sin \theta - E_3 \cos \theta) dx = L. \quad (26)$$

Next, using equation (26), we find

$$v_x \frac{\partial \theta}{\partial x} = \frac{(1-\alpha)}{2} - \frac{L}{B_\perp^2} + \frac{\alpha}{B_\perp^2} \int B_\perp (E_2 \sin \theta - E_3 \cos \theta) dx, \quad (27)$$

which is substituted into equation (24) to give:

$$\begin{aligned} & \frac{1}{2} \left( v_x \frac{\partial B_\perp}{\partial x} \right)^2 + \frac{B_\perp^2}{2} \left( \frac{(1-\alpha)}{2} - \frac{L}{B_\perp^2} + \frac{\alpha}{B_\perp^2} \int B_\perp (E_2 \sin \theta - E_3 \cos \theta) dx \right)^2 \\ & + \frac{\alpha B_\perp^4}{8} - \alpha (E_1 - 1) \frac{B_\perp^2}{2} + \alpha B_\perp (E_2 \cos \theta + E_3 \sin \theta) - W = 0 \end{aligned} \quad (28)$$

It is now evident that the integral involving  $E_{2,3}$  precludes us from further continuing the analysis. Hence, we will hereafter assume that  $E_2 = E_3 = 0$ . We now distinguish the following two possibilities. Either the constant  $L$  can be selected to vanish (in line with what was also done in reference [15]), or it can be selected to be  $L \neq 0$ . The former scenario is of direct physical relevance given the vanishing of the angular momentum. The latter ( $L \neq 0$ ) case is a topic of mathematical interest even though it does not appear to us to have a direct physical interpretation in the setting at hand. For this reason, here we focus our analytical considerations to the case of  $L = 0$ , while we relegate the topic of  $L \neq 0$  to an appendix A given the mathematical interest in the latter case in its own right. Then, the energy conservation of equation (28) reads:

$$\frac{1}{2} \left( v_x \frac{\partial B_{\perp}}{\partial x} \right)^2 + \frac{B_{\perp}^2}{8} (1 - \alpha)^2 + \frac{\alpha B_{\perp}^4}{8} - \alpha (E_1 - 1) \frac{B_{\perp}^2}{2} - W = 0, \quad (29)$$

where the first of the momentum flux equations allows us to express:

$$v_x = \frac{\alpha}{C} \left( E_1 - \frac{1}{2} B_{\perp}^2 \right).$$

In line with the discussion of [15], we can express the dimensionless constants of equation (29) as  $E_1 = M^{*2}/\alpha$  and  $C = M^*$ , where  $M^* = v_1/v_{\text{Ae}}$  is the electron Alfvén Mach number  $0.5 < M^* < 0.707$ . Substituting into equation (29), and introducing the dimensionless parameter  $\lambda^2 = M^{*2} - \frac{(1+\alpha)^2}{4}$ , we obtain:

$$\frac{1}{2} \left( v_x \frac{\partial B_{\perp}}{\partial x} \right)^2 + \frac{B_{\perp}^2}{8} [\alpha B_{\perp}^2 - 4\lambda^2] - W = \frac{1}{2} \left( v_x \frac{\partial B_{\perp}}{\partial x} \right)^2 + V_{\text{eff}} - W = 0. \quad (30)$$

The effective potential now naturally assumes the form of the quartic one arising in a Duffing oscillator. Hence it is amenable to analytical considerations as discussed in the following section.

### 3. Analytical and numerical solutions

We now tackle the solutions of equation (28), focusing as discussed above on the scenario where  $E_2 = E_3 = 0$ . We separate the analytically tractable and physically relevant case of  $L = 0$  which we present here from the numerically examined one of  $L \neq 0$  that is relegated to appendix A. We notice that in all the relevant quantities that we have examined so far, we do not simply find derivatives such as  $\frac{\partial}{\partial x}$ , but rather these derivatives always appear multiplied by  $v_x$ . This renders it rather natural to consider a transformation of coordinates from  $x$  to a new spatial variable  $x'$  that ‘absorbs’ this factor of  $v_x$ . This is done by considering the  $x \mapsto x'$  transformation defined through  $\frac{\partial}{\partial x'} = v_x \frac{\partial}{\partial x}$ . Then equation (29) becomes:

$$\frac{1}{2} \left( \frac{\partial B_{\perp}}{\partial x'} \right)^2 + V_{\text{eff}} = W, \quad (31)$$

with the effective potential  $V_{\text{eff}}$  given by:

$$V_{\text{eff}} = -p B_{\perp}^2 + q B_{\perp}^4, \quad (32)$$

where  $p = \lambda^2/2$  and  $q = \alpha/8$ . Since  $V_{\text{eff}}$  is of the form of a well-known double-well potential or Duffing oscillator, we can seek solutions in the form of Jacobi elliptic functions [17]. A key advantage of this approach is that then the special solitonic solutions of the earlier work of [15] merely become special cases of the more general elliptic function waveforms. In particular, we choose:

$$B_{\perp}(x') = A \operatorname{dn}(bx', k), \quad (33)$$

where  $\operatorname{dn}$  is a Jacobi elliptic function and  $k$  is the elliptic modulus. Bearing in mind that

$$\frac{\partial B_{\perp}}{\partial x'} = -bk^2 A \operatorname{cn}^2(bx', k) \operatorname{sn}^2(bx', k),$$

and by also using the identities involving the Jacobi elliptic functions:

$$k^2 \operatorname{cn}^2(bx', k) = k^2 - 1 + \operatorname{dn}^2(bx', k),$$

$$k^2 \operatorname{sn}^2(bx', k) = 1 - \operatorname{dn}^2(bx', k),$$

we get:

$$\frac{1}{2} \left( \frac{\partial B_{\perp}}{\partial x'} \right)^2 = \frac{A^2 b^2 (k^2 - 1)}{2} + \frac{(2 - k^2) b^2}{2} B_{\perp}^2 - \frac{b^2}{2A^2} B_{\perp}^4. \quad (34)$$

We substitute (34) to (31) and, by comparing with (32), we infer the solvability conditions:

$$W = \frac{A^2 b^2 (k^2 - 1)}{2}, \quad A = \sqrt{\frac{p}{q(2 - k^2)}} = \frac{2\lambda}{\sqrt{\alpha(2 - k^2)}} \quad \text{and} \quad b = \sqrt{\frac{2p}{2 - k^2}} = \frac{\lambda}{\sqrt{2 - k^2}}. \quad (35)$$

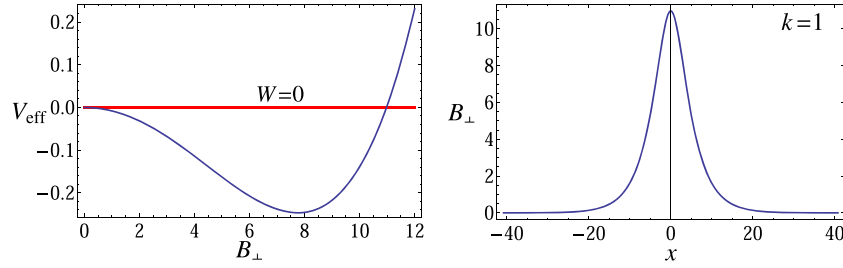
Thus, there exists a solution of equation (31) of the form of equation (33) with  $A$  and  $b$  given by (35). The inverse transformation  $x' \mapsto x$  is defined through the direct  $\frac{\partial}{\partial x'} = v_x \frac{\partial}{\partial x}$  transformation which implies  $\frac{dx}{dx'} = v_x$ . Thus, using equation (15), it holds that:

$$x = \int v_x(x') dx' = \frac{\alpha}{C} \int \left( E_1 - \frac{1}{2} B_{\perp}^2 \right) dx'. \quad (36)$$

For the specific form of  $B_{\perp}$ , the above becomes

$$x = \frac{\alpha E_1}{C} x' - \frac{\alpha A^2}{2bC} E(\operatorname{am}(bx', k), k) = Cx' - \frac{2\lambda}{C\sqrt{2 - k^2}} E(\operatorname{am}(bx', k), k), \quad (37)$$

where  $E$  stands for the Jacobi integral of the second kind, and  $\operatorname{am}$  for the Jacobi amplitude function. It is worth noting that the only free parameter in these solutions is the elliptic modulus  $k$ . As  $k$  varies between  $0 \leq k \leq 1$ , we switch from a constant (equilibrium) solution at the minimum of  $V_{\text{eff}}$  (for  $k = 0$ ) to a soliton solution of vanishing energy (per equation (35)) for  $k = 1$ . Any intermediate value gives rise to a periodic solution



**Figure 1.** Effective Potential (left panel) and soliton solution) for  $B_{\perp}$  (right panel), in the case  $L = 0$  and  $k = 1$  as given analytically by equations (32) and (33).

with periodicity  $T = 2K(k)$  where  $K(k)$  stands for the complete elliptic integral of the first kind.

### 3.1. The soliton solution for $k = 1$

In this case,  $W = 0$  as can be seen also in figure 1. For all the calculations in this work, the values of the parameters  $C = M^* = 0.5164 \Rightarrow \lambda = 0.128$ , have been used. Since  $W = 0$  corresponds to the local maximum of the effective potential  $V_{\text{eff}}$ , the corresponding solution is a homoclinic one. The form of the  $B_{\perp}$  solution will be a solitonic one as the one depicted in the right panel of figure 1. In particular, for  $k = 1$ , solution (33) becomes

$$B_{\perp}(x') = A \operatorname{sech}(bx') = \frac{2\lambda}{\sqrt{\alpha}} \operatorname{sech}(\lambda x'), \quad (38)$$

while the  $x' \mapsto x$  transformation becomes in this case

$$\begin{aligned} x &= \int_0^{x'} v_x(t) dt = \frac{1}{C} \int [\alpha E_1 - 2\lambda^2 \operatorname{sech}^2(\lambda x')] dx' \\ &= \frac{1}{C} [\alpha E_1 x' - 2\lambda \tanh(\lambda x')] = Cx' - \frac{2\lambda}{C} \tanh(\lambda x'). \end{aligned} \quad (39)$$

The combined result of (38) and (39) provides the exact form of  $B_{\perp} = B_{\perp}(x)$  shown in the right panel of figure 1.

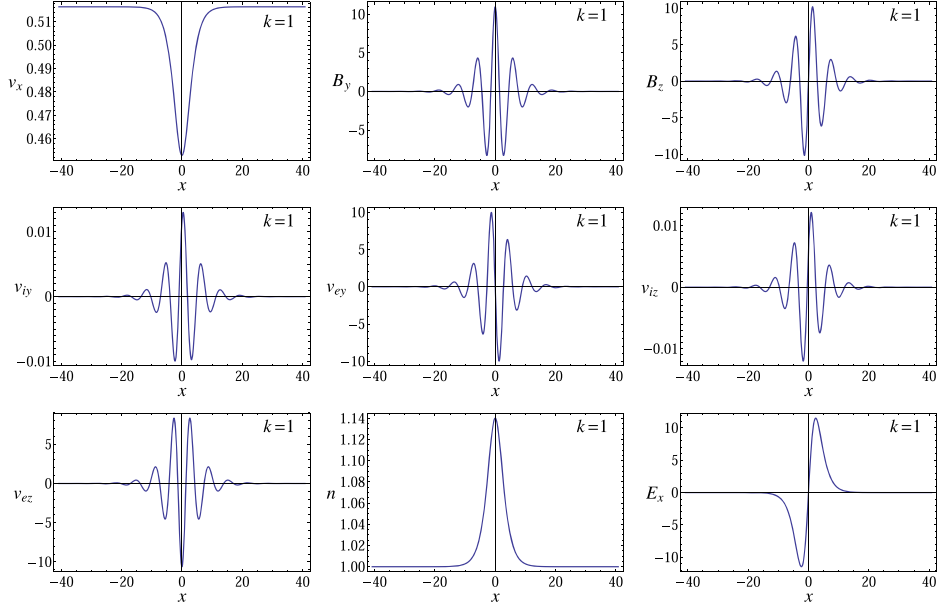
As mentioned above, the equation for  $v_x$  is calculated by the first one among equation (15), rewritten here

$$v_x = \frac{\alpha}{C} \left( E_1 - \frac{1}{2} B_{\perp}^2 \right).$$

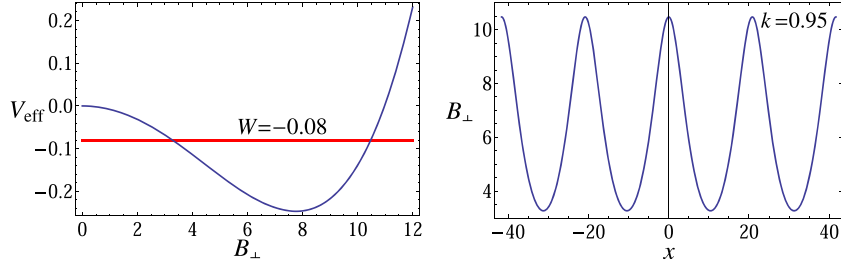
On the other hand,  $\theta$  can be calculated through (27), which reads for  $E_2 = E_3 = L = 0$

$$\frac{\partial \theta}{\partial x'} = \frac{(1 - \alpha)}{2} \Rightarrow \theta = \frac{(1 - \alpha)}{2} x', \quad (40)$$

by considering (without loss of generality) that the constant of integration  $\theta_0 = 0$ . Using (40) and (38), we can calculate  $B_y$  and  $B_z$  through  $B_y = B_{\perp} \cos \theta$ ,  $B_z = B_{\perp} \sin \theta$ , while  $n$  can be calculated through  $n = \frac{C}{v_x}$ . The fields  $v_{iy,ey}$  and  $v_{iz,ez}$  are calculated by equations (16) and (17), namely (for our case of  $E_2 = E_3 = 0$ ):



**Figure 2.** The nine fields (5 velocity components, including a common one, for ions and electrons, 2 transverse magnetic field components, a longitudinal electric field and a charge density) which describe the system for  $L = 0$  and  $k = 1$ .



**Figure 3.** Effective potential (left panel) and periodic solution of  $B_{\perp}$  (right panel) for  $L = 0$  and  $k = 0.95$ , as given analytically by equations (32) and (33).

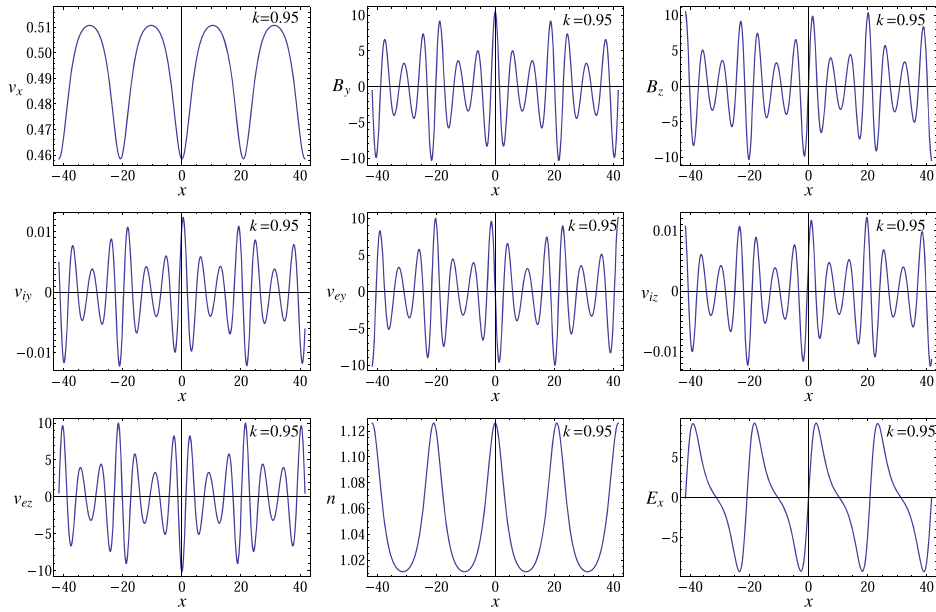
$$v_{iy,ey} = \frac{\alpha B_y}{C} + \frac{(-\alpha, 1)}{C} \frac{\partial B_z}{\partial x'}, \quad (41)$$

$$v_{iz,e_z} = \frac{\alpha B_z}{C} + \frac{(\alpha, -1)}{C} \frac{\partial B_y}{\partial x'}, \quad (42)$$

and finally  $E_x$  is given by the first of (12), which in the present case reads

$$E_x = \frac{1}{\alpha} \frac{\partial v_x}{\partial x'} - (v_{iy} B_z - v_{iz} B_y). \quad (43)$$

Thus, by recalling also (38) and (39), we can calculate all the fields that are needed for the full description of our system. The calculation for the present case is shown in figure 2. These



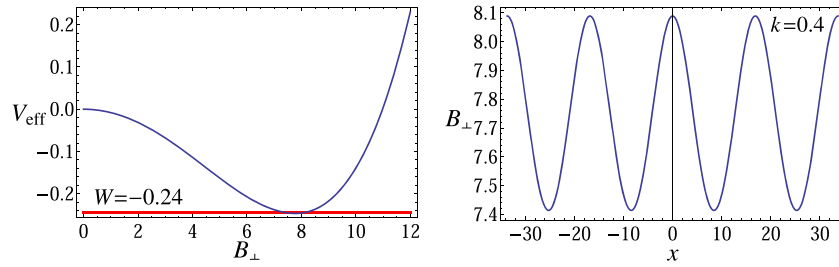
**Figure 4.** Reconstruction of the nine fields describing the system, as was done previously in figure 2, but now the solutions with  $L = 0$  and  $k = 0.95$ .

results are in line with the recent analysis of reference [15] and illustrate the ability of our formulation to not only retrieve these earlier findings (in an arguably more intuitive fashion from a dynamical systems point of view), but also the potential to extend them as will be done below. It is worthwhile to also note that the  $B_\perp$  field, as well as ones that depend directly on that such as  $v_x$  and  $n$ , have a ‘regular’ solitonic form. However, quantities involving the components of the magnetic field and the ones of the velocities (for both ions and electrons) in the transverse directions feature oscillations whose origin is now more transparent. These stem from the polar decomposition of the transverse magnetic field, endowing one of its components with a cosinusoidal and another with a sinusoidal variation, so that the relevant fields bear substantial resemblance to the notions of envelope solitons [18].

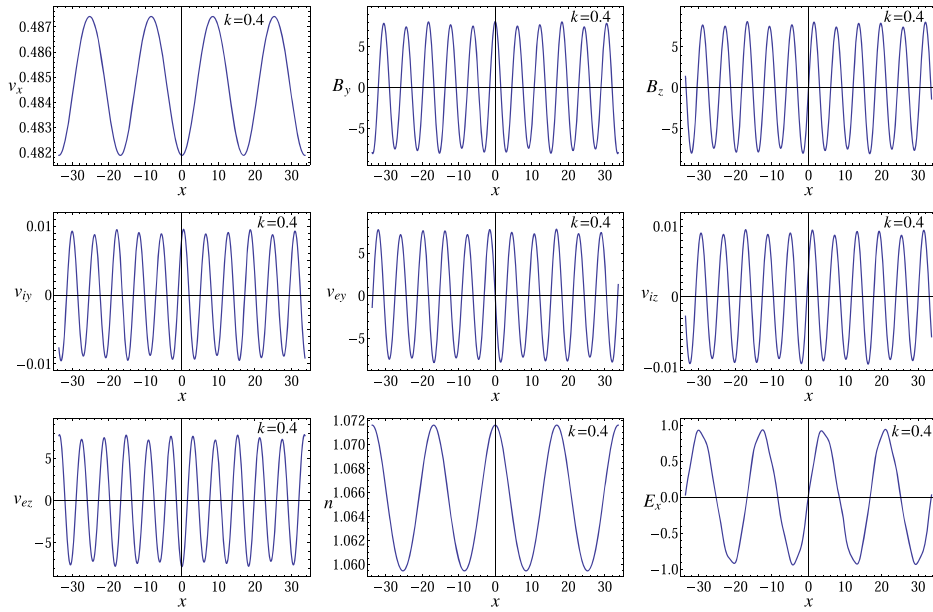
### 3.2. Periodic solutions for $k = 0.95$ and $k = 0.4$ ( $L = 0$ )

We now turn to the generalization involving the genuinely periodic state solutions. When  $k \neq 1$ , the total energy is  $W < 0$ . In particular, for  $k = 0.95$  we get  $W = -0.08$ , as can be seen in the left panel of figure 3. The corresponding solution of  $B_\perp$  is now indeed periodic and not solitonic; see the right panel of figure 3.

Importantly, the same reconstruction path of equations (39)–(43) can be utilized to obtain *all* 9 of the associated fields. In figure 4, the corresponding quantities and their spatial profiles (with respect to  $x$ ) are depicted. In the panels of  $B_\perp$  and  $v_x$  a clear elliptic function behavior can be recognized. The same periodic behavior is obvious for  $n$  and  $E_x$ ; notice the analogy of all of these features with the limiting case of  $k \rightarrow 1$ , where essentially the additional periods of the central wave are pushed to  $\infty$ . On the other hand, the behavior of the rest of the examined fields is quasi-periodic, due to the simultaneous action of two periodic quantities bearing different periodicities, namely  $B_\perp$  and  $\theta$ . Nevertheless, this quasi-periodic pattern can be analytically constructed through the decompositions and building blocks presented herein.



**Figure 5.** Effective potential (left panel) and periodic solution (right panel) of  $B_{\perp}$  for  $L = 0$  and  $k = 0.4$  as given analytically by equations (32) and (33).



**Figure 6.** Similar to figure 4, but now for  $L = 0$  and  $k = 0.4$ .

In the case  $k = 0.4$ , the value of the total energy is  $W = -0.24$  (see left panel of figure 5). Since the motion now occurs close to the local minimum of  $V_{\text{eff}}$ , the corresponding behavior is close to the harmonic one as it can be seen both in the right panel of figure 5 and in figure 6. This will be progressively more so as we approach the minimum of the potential, which is  $V_{\text{eff}_{\text{min}}} \simeq -0.2467$  and occurs for  $B_{\perp} \simeq 0.759$ . Nevertheless, the transverse velocity and magnetic fields retain their quasi-periodic functional form in this case too, as  $k \rightarrow 0$  and we tend to the near-linear, small amplitude (trigonometric) limit of the theory.

#### 4. Conclusions and future work

In the present work we have provided a full formulation, in the form of partial differential equations for nine fields, of the problem of longitudinal wave propagation in a cold, quasi-neutral, collision-free plasma. The guiding principles involved the Newtonian dynamics at the

level of the ions and the electrons, the equation of continuity and the Ampère law's components. This enabled the development of a system of nine equations for the five components of the velocity  $v_x, v_{iy,ey}, v_{iz,ez}$ , the two components of the transverse magnetic field ( $B_y, B_z$ ), the density  $n$  and the electric field  $E_x$ . While considering the full set of nine PDEs at the numerical level remains a particularly challenging task, we have been able to achieve a substantial simplification by considering the co-traveling frame. We have considered the reduction of the resulting nine ODEs into, arguably, the simplest formulation of a  $2 \times 2$  system for the transverse magnetic field components. This was eventually converted to a single degree of freedom setting via  $L$  considerations (with  $L$  being the total angular momentum flux). This allowed us to remarkably not only retrieve the solitary wave solutions of reference [15] for *all* the relevant fields, but also to provide a platform for generalizing these considerations to a mono-parametric family of periodic function solutions. We explained how/why some of the fields (like  $B_\perp, v_x, n$  or  $E_x$ ) feature solitonic or periodic character, while others (the transverse components of the velocity or the ones of the magnetic field) have, respectively, an envelope-soliton- or periodic-nature. In appendix A, we also consider the case of mathematical interest with non-vanishing  $L$ . There, the additional centrifugal contribution to the effective potential energy landscape precludes analytical solutions (and especially so solitary waves), yet still we can numerically identify periodic orbits in the system.

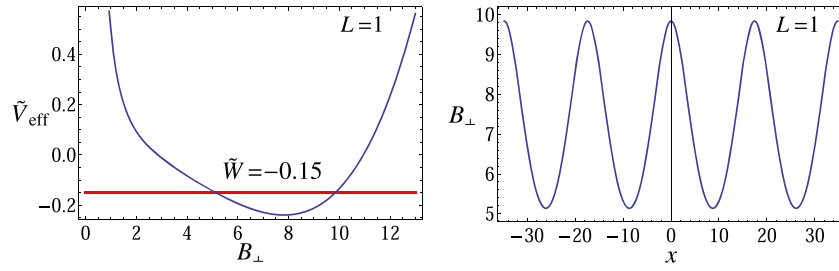
Naturally, there are numerous directions that are worthwhile to consider for future study in this vein. From a numerical point of view, it would be particularly interesting to explore the full set of partial differential equations describing this system, namely equations (6)–(11). In that framework, our solutions are traveling ones, or equivalently stationary ones in the co-traveling frame and hence it would also be natural to explore their dynamical stability. From the point of view of dynamical reductions, it would be particularly interesting (although quite challenging in its own right) to reduce the solutions via a reductive perturbation method to solutions of the Korteweg–de Vries or similar equations as was recently done for the transverse case in reference [16]. In a similar spirit as the latter work, exploring numerically at first the interactions between different solitary waves would be a topic of interest in its own right, especially given the fundamentally more complex form (reminiscent of an envelope soliton in the transverse components) of the solitary waves herein. Lastly, generalizing corresponding considerations to more complex scenarios, where the traveling wave may not be genuinely one dimensional, or where a dust granule may interact with the wave, are also of interest. This is essential because the dusty solitary currents [21] can influence the phenomena considered in the present analysis. Work along some of these directions is currently in progress and will be reported in future publications.

## Acknowledgments

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## Appendix A. The $L \neq 0$ case

In the case where  $L \neq 0$ , we still select  $E_2 = E_3 = 0$ , so as to have a tractable, effectively one degree-of-freedom scenario as concerns  $B_\perp$ . Then, the equation of energy, (28), becomes:



**Figure 7.** Effective potential (left panel) and periodic solution of  $B_{\perp}$  (right panel) for  $L = 1$  and  $\tilde{W} = -0.15$ .  $\tilde{V}_{\text{eff}}$  is given analytically by (A2) while  $B_{\perp}$  is calculated numerically.

$$\frac{1}{2} \left( \frac{\partial B_{\perp}}{\partial x'} \right)^2 + \tilde{V}_{\text{eff}} = \tilde{W}, \quad (\text{A1})$$

where

$$\tilde{V}_{\text{eff}} = V_{\text{eff}} + \frac{L^2}{2B_{\perp}^2} \quad \text{and} \quad \tilde{W} = W + \frac{(1-a)L}{2}. \quad (\text{A2})$$

Since the effective potential  $\tilde{V}_{\text{eff}}$  does not possess a local maximum at  $B_{\perp} = 0$  as in the  $L = 0$  case, but it tends to infinity as  $B_{\perp} \rightarrow 0$  (e.g. left panel of figure 7), we *cannot* achieve solitonic solutions with respect to  $B_{\perp}$ . That is to say, solutions with nontrivial  $L$  of the transverse magnetic field can only be periodic and not solitary wave states; see e.g. right panel of figure 7. On the other hand, since the potential is convex, there exists an infinite number of periodic solutions. We cannot acquire these solutions in closed form as before, due to the presence of the centrifugal potential term in equation (A2), but can study them numerically.

The inverse transformation  $x' \mapsto x$  is calculated as in the  $L = 0$  case. In addition, since it is now true (per equation (27)) that

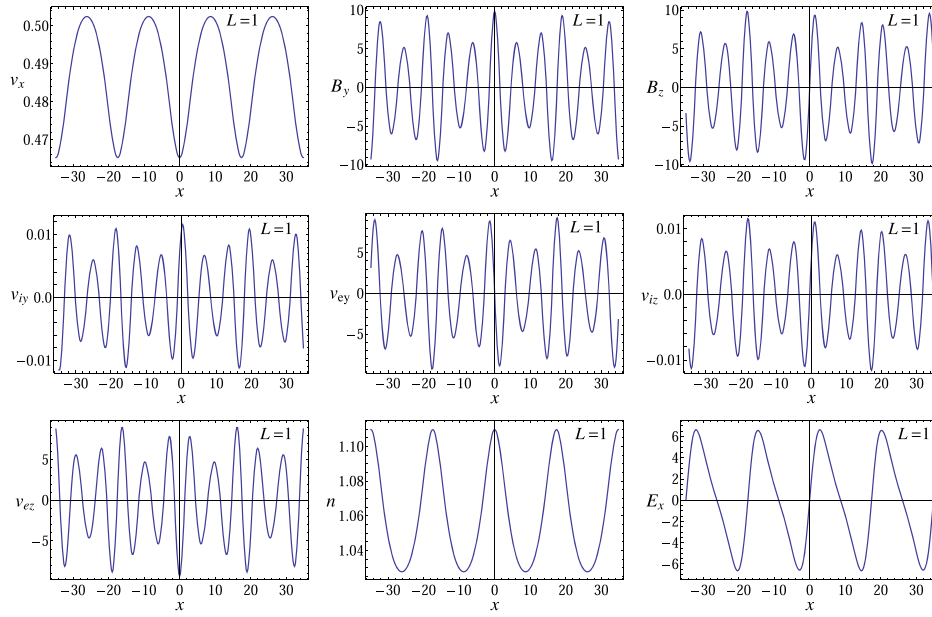
$$\frac{\partial \theta}{\partial x'} = \frac{1-\alpha}{2} - \frac{L}{B_{\perp}^2},$$

the angle  $\theta$  is calculated as

$$\theta = \frac{1-\alpha}{2} x' - \int \frac{L}{B_{\perp}^2(x')} dx'. \quad (\text{A3})$$

The various fields which describe our system are calculated as in the  $L = 0$  case.

Let us consider a situation where the total energy of the system is negative  $\tilde{W} = -0.15$  and  $L$  is non-zero, i.e.,  $L = 1$ . This is the case actually depicted in figure 7, where the form of the effective potential, as well as the corresponding periodic orbit are shown. We can clearly observe how the divergence of the centrifugal potential as  $B_{\perp}$  tends to smaller values leads to the sole existence of periodic orbits for the allowable range of  $B_{\perp}$ , such that  $V_{\text{eff}}(B_{\perp}) < \tilde{W}$ . The various fields are shown in figure 8. The central core of the solution retains a form reminiscent from before, yet the periodic character of the solution is also evident. We have also explored a variety of other cases, including ones with the same  $L$  but higher  $\tilde{W}$ , as well as ones with larger  $L$  and the same  $\tilde{W}$ . While the specifics of the solution (e.g., its period or the specifics of the



**Figure 8.** Reconstruction of the nine fields describing the system, in this case for  $L = 1$  and effective energy  $\tilde{W} = -0.15$ .

quasi-periodicity of the transverse velocity or magnetic components) may change, the overarching character of the waveforms does not. Hence, we do not show further such examples here.

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## References

- [1] Fermi E, Pasta J and Ulam S 1955 *Technical Report LA1940* Los Alamos National Laboratory
- [2] Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240
- [3] Washimi H and Taniuti T 1966 *Phys. Rev. Lett.* **17** 996
- [4] Adlam J H and Allen J E 1958 *Phil. Mag.* **3** 448
- [5] Adlam J H and Allen J E 1960 *Proc. Phys. Soc.* **75** 640
- [6] Allen J E and Gibson J 2017 *Phys. Plasmas* **24** 042106
- [7] Montgomery D C 1959 *Phys. Fluids* **2** 585
- [8] Saffman P G 1961 *J. Fluid Mech.* **11** 16
- [9] Tidman D A and Krall N A 1971 *Shock Waves in Collisionless Plasmas* (New York: Wiley)

- [10] Sauer K, Dubinin E and McKenzie J F 2002 *Geophys. Res. Lett.* **29** 2226
- [11] Dubinin E, Sauer K and McKenzie J F 2003 *J. Plasma Phys.* **69** 305
- [12] Cattaert T and Verheest F 2005 *Phys. Plasmas* **12** 012307
- [13] Gibson J 2018 Theoretical and computational studies of magnetized dusty plasmas *PhD Thesis* Imperial College London
- [14] Cramer N F and Verheest F 2018 *Phys. Plasmas* **25** 123702
- [15] Abbas G, Allen J E, Coppins M, Simons L and James L 2020 *Phys. Plasmas* **27** 042102
- [16] Allen J E, Frantzeskakis D J, Karachalios N I, Kevrekidis P G and Koukoulouyannis V 2020 *Phys. Rev. E* **102** 013209
- [17] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
- [18] Remoissenet M 1999 *Waves Called Solitons* (Berlin: Springer)
- [19] Trukhachev F M and Tomov A V 2016 *Cosmic Res.* **54** 351
- [20] Trukhachev F M, Tomov A V, Mogilevsky M M and Chugunin D V 2018 *Tech. Phys. Lett.* **44** 494
- [21] Trukhachev F M, Vasiliev M M, Petrov O F and Vasilieva E V 2019 *Phys. Rev. E* **100** 063202
- [22] Sauer K, Dubinin E and McKenzie J 2001 *Geophys. Res. Lett.* **28** 3589