GLOBAL BIFURCATION OF ANTI-PLANE SHEAR FRONTS

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Abstract. We consider anti-plane shear deformations of an incompressible elastic solid whose reference configuration is an infinite cylinder with a cross section that is unbounded in one direction. For a class of generalized neo-Hookean strain energy densities and live body forces, we construct unbounded curves of front-type solutions using global bifurcation theory. Some of these curves contain solutions with deformations of arbitrarily large magnitude.

1. Introduction

Consider an elastic solid whose undisturbed state is an infinite cylinder $\Omega \times \mathbb{R}$ where the coordinates are chosen so that the cross-section $\Omega := \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ lies in the xy-plane and the generator parallels the z-axis. For simplicity, suppose that the lateral boundaries at $\{y = \pm\pi/2\}$ are held fixed. *Anti-plane shear* occurs when the solid is displaced out of the xy-plane and the deformation is independent of z. This leads to considerable analytical simplification since the full three-dimensional field equations can be reduced to a two-dimensional scalar elastostatic model. Anti-plane shear is studied in connection to contact mechanics [31], rectilinear steady flow of incompressible non-Newtonian fluids [9], structures with cracks [26], and phase transitions in solids [30], among many other areas. From a purely mathematical perspective, this model is interesting as it is known to support a rich variety of nontrivial equilibria [21]. The existence of such solutions on bounded domains has been established by several authors [31, 34] through variational methods.

The present work concerns large anti-plane shear *fronts*, by which we mean static equilibria where the displacement has distinct limits as $x \to -\infty$ and $x \to +\infty$. While investigations of fronts are ubiquitous in the literature of reaction-diffusion equations and mathematical biology, for instance, they are largely unexamined in the context of elastostatics. Recently, the existence of local curves of fronts lying in a neighborhood of the undisturbed state was proved via center manifold reduction techniques [3]. We now use analytic global bifurcation theory to extend these families into the non-perturbative regime. Ultimately, this furnishes equilibria exhibiting deformation gradients of arbitrary magnitude, sometimes referred to as "solutions in the large" [14].

Global bifurcation has proven to be successful in treating a host of elasticity problems posed on bounded domains [15, 18, 19, 17, 16, 14]. In order to study fronts, though, one must naturally take Ω to be unbounded in the x-direction. Classical bifurcation theory is ill-adapted to this setting. For example, standard hypotheses for degree theoretic global bifurcation are that the nonlinear operator is Fredholm index 0 and locally proper [29]. However, the linearized anti-plane shear equation at the undisturbed state in fact fails to be Fredholm. Conditions guaranteeing local properness for quasilinear equations set on the whole space have been obtained by several authors [28, 11, 12]. However, these impose hypotheses on the limiting problems at infinity that can be difficult to verify in practice, and their functional analytic framework does not allow for fronts. With that in mind, the authors developed a new, general global bifurcation theoretic approach [2, 4] specifically to analyze (fully nonlinear) PDEs set on non-compact domains without imposing global compactness assumptions. This machinery was then used to construct large-amplitude solitary water waves and hydrodynamics bores. In part, our objective here is to demonstrate that these results have the potential to address questions of physical relevance in nonlinear elasticity.

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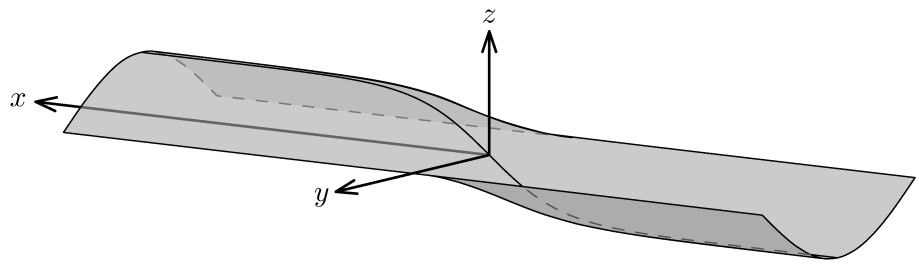

FIGURE 1. Sketch of a front solution satisfying the symmetry and monotonicity conditions in Theorem 1.1(b). The graph $z = u(x, y)$ is the image of the strip $\{z = 0, |y| < \frac{\pi}{2}\}$ under the anti-plane displacement $\mathrm{id} + u(x, y)e_z$.

1.1. **The model.** Let us now set down the governing equations. For anti-plane shear, the deformation must have the form $\mathrm{id} + u(x, y)e_z$, where e_z is the standard basis vector in the z-direction; see Figure 1. Further supposing the material is isotropic, hyperelastic, and incompressible, the principal invariants of the Cauchy–Green tensor are given by $I_1 = I_2 = 3 + |\nabla u|^2$ and $I_3 = 1$, and the strain energy density becomes $W = W(I_1, I_2)$.

A characteristic feature of the the anti-plane shear problem is that it is in general overdetermined: the solution to the reduced scalar PDE, called the out-of-plane displacement, must also satisfy the other two in-plane equations. Within the context of both linear and nonlinear elasticity theory, necessary and sufficient admissibility conditions on the strain energy of various elastic materials have been derived which ensure that nontrivial anti-plane shear deformation can be sustained [24, 25, 23, 33, 22, 27]. Specifically, it is proved in [24] that, under the so-called "ellipticity condition",

$$\frac{d}{dR}\left[R\left(\frac{\partial W}{\partial I_1}(I_1, I_2) + \frac{\partial W}{\partial I_2}(I_1, I_2)\right)\Big|_{I_1=I_2=3+R^2}\right] > 0 \quad \text{for all} \quad R \geq 0, \tag{1.1}$$

the energy function W is admissible if and only if[1] there exists some constant $k \in \mathbb{R}$ such that

$$k\frac{\partial W}{\partial I_1}(I_1, I_2) + (k-1)\frac{\partial W}{\partial I_2}(I_1, I_2) = 0. \tag{1.2}$$

It is also pointed out in [34] that (1.1) is equivalent to the convexity of the mapping $\xi \in \mathbb{R}^2 \mapsto W(3 + |\xi|^2, 3 + |\xi|^2)$, which is assumed in most variational treatments of the problem.

We restrict attention to generalized neo-Hookean materials [21] with

$$W = \overline{W}(I_1),$$

in which case (1.2) is automatically satisfied. Writing the strain energy density as a function of $|\nabla u|^2$ alone,

$$\mathcal{W}(|\nabla u|^2) := \overline{W}(3 + |\nabla u|^2), \tag{1.3}$$

the ellipticity condition (1.1) translates to

$$\mathcal{W}'(q) + 2\mathcal{W}''(q)q > 0 \quad \text{for} \quad q \geq 0. \tag{1.4}$$

Now we consider the system subject to an applied force which depends on the displacement u, that is, a "live" body force, or a live load [6, Section 2.7]. It is easy to see that in order to sustain nontrivial anti-plane shear, the body force can only be applied in the axial direction and needs to be independent of z. Following [19], we will consider the body force to be parameter dependent and denote the force density by $-b(u, \lambda)e_z$ for some parameter λ.

[1]It is further remarked in [23] that (1.2) remains sufficient for admissibility even in the absence of ellipticity (1.1).

A static equilibrium then corresponds to a solution of the quasilinear PDE

$$\begin{cases} \nabla \cdot \big(\mathcal{W}'(|\nabla u|^2)\nabla u\big) - b(u,\lambda) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Here the homogeneous Dirichlet boundary condition simply means that the cylinder is clamped along its boundary. Condition (1.1) ensures that the PDE (1.5) is elliptic. The case where (1.4) fails is quite interesting, particularly as it relates crack formation (see [21, Section 6] and the references therein), but is beyond the scope of the present paper.

1.2. **Structural assumptions.** We are motivated by the example of a quadratic neo-Hookean material subjected to simple harmonic forcing:

$$\mathcal{W}(q) = q + w_1 q^2, \qquad b(\varkappa,\lambda) = -(1+\lambda)\varkappa, \tag{1.6}$$

where here the constant $w_1 > 0$ and the parameter value $\lambda = 0$ is "critical" in a sense which will be made precise later. Our results, however, apply to a much wider class of materials and forcings which satisfy the following structural conditions.

First we make the symmetry assumption that

$$b(\,\cdot\,,\lambda) \text{ is odd}, \tag{1.7}$$

and hence that (1.5) is invariant under the reflection $u \mapsto -u$. This greatly simplifies the analysis as we are then able to restrict attention to solutions u which are odd in the unbounded variable x.

Next, we assume that both $\mathcal{W}$ and b are analytic in their arguments. This allows us to use analytic global bifurcation theory, and also to make an expansion near the reference configuration at $\lambda = 0$. We require this expansion to have the form

$$\begin{aligned} \mathcal{W}(q) &= q + w_1 q^2 + O(|q|^3), \\ b(\varkappa,\lambda) &= -(1+\lambda)\varkappa + b_2\varkappa^3 + O((|\varkappa| + |\lambda|^{1/2})^4), \end{aligned} \tag{1.8}$$

where the constants w_1 and b_2 satisfy the strict inequality

$$b_2 + 2w_1 > 0. \tag{1.9}$$

This allows us to use the existence theory in [3, Section 3] for small solutions with $0 < \lambda \ll 1$. An expanded version of that result is given below in Section 4.

Finally, we require several global sign conditions. For the strain energy density, we impose the "enhanced" ellipticity condition

$$3\mathcal{W}''(q) + 2\mathcal{W}'''(q)q = \big(\mathcal{W}'(q) + 2\mathcal{W}''(q)q\big)' > 0 \quad \text{for} \quad q > 0, \tag{1.10}$$

which implies (1.4) since $\mathcal{W}'(0) = 1 > 0$ by (1.8). For the body force, we suppose that

$$\begin{cases} \text{for all } \lambda \geq 0,\ b(\,\cdot\,,\lambda) \text{ is a strictly decreasing convex function on } (0,\infty), \\ \text{for all } \varkappa > 0,\ b(\varkappa,\,\cdot\,) \text{ is strictly decreasing and unbounded on } [0,\infty), \\ b_\varkappa(0,\lambda) < -1 \text{ for } \lambda > 0. \end{cases} \tag{1.11}$$

The first two of these conditions encode the physically intuitive assumption that the magnitude of the body forcing increases as either the displacement or loading parameter is increased. Together with the previous hypotheses, the inequalities (1.10) and (1.11) guarantee that the set of x-independent solutions of (1.5) has a particularly simple structure, and (1.10) is further used to establish an important a priori bound. The conditions in (1.11) only concern $\lambda \geq 0$ because, as we will see, the solutions we construct will all satisfy this inequality. Finally, let us reiterate that the motivational choice of strain energy and body force (1.6) satisfy all of the above requirements.

1.3. **Main results.** Our main result is the following.

Theorem 1.1 (Global bifurcation of anti-plane shear fronts)**.** *Suppose that the body force and strain energy satisfy the structural conditions* (1.7)–(1.11)*. There exists a continuous curve* $\mathscr{C}$ *of solutions to* (1.5) *admitting the* C^0 *parameterization*

$$\mathscr{C} = \{(u(s), \lambda(s)) : 0 < s < \infty\} \subset C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times (0, \infty)$$

with $(u(s), \lambda(s)) \to (0, 0)$ *as* $s \to 0+$*, and satisfying the following.*

(a) (Symmetry and monotonicity) *Each* $(u(s), \lambda(s)) \in \mathscr{C}$ *is a strictly increasing monotone front with*

$$\begin{aligned} \partial_x u(s) &> 0 \qquad \text{in } \Omega, \\ \partial_y u(s) &< 0 \qquad \text{for } x, y > 0. \end{aligned} \tag{1.12}$$

Moreover, $u(s)$ *is odd in* x *and even in* y*.*

(b) (Unboundedness) *In the limit* $s \to \infty$ *we have blowup in that*

$$\|\partial_y u(s)\|_{C^0},\ \lambda(s) \longrightarrow \infty \qquad \text{as } s \to \infty. \tag{1.13}$$

(c) (Analyticity) *The curve* $\mathscr{C}$ *is locally real-analytic.*

Figure 1 shows a sketch of a front satisfying the symmetry and monotonicity conditions in (a).

Remark 1.2. Since (1.5) is invariant under reflections in x, one can reflect each of the solutions in $\mathscr{C}$ to obtain a curve of strictly *decreasing* monotone fronts with the inequalities in (1.12) reversed.

Remark 1.3. Compared to other global bifurcation results in nonlinear elasticity, the definitiveness in (1.13) is quite unusual. More typical is that one has several alternatives, such as reconnection to the reference configuration. On the other hand, our results are limited to the case of anti-plane shear.

Remark 1.4. One can relax many of these hypotheses at the cost of additional ambiguity regarding the limiting behavior along $\mathscr{C}$. For example, the assumption that $b(\varkappa, \cdot)$ is unbounded on $[0, \infty)$ is used only in Lemma 6.5. Without it, we would still have that $\lambda(s) \to \infty$, but not necessarily the blowup of $\partial_y u(s)$. On the other hand, if (1.10) does not hold globally, then it may happen that the system loses ellipticity in the limit. This scenario is of particular interest to crack formation but unfortunately is difficult to treat through our methodology. In particular, it could coincide with any of the alternatives in Section 6.1. We also rely on (1.10) in our study of the conjugate flow problem in Section 3. If it does not hold, then there may exist multiple x-independent solutions of (1.5) that are conjugate in the sense of (3.3). Were this to occur, then we must allow for the possibility that a heteroclinic degeneracy develops in the limit along $\mathscr{C}$; see Theorem 6.1(A2).

Under the opposite sign conditions on the coefficients in (1.9), there exist spatially localized anti-plane shear equilibria for which $u(x, \cdot)$ vanishes in the limits $x \to \pm\infty$. Curves of small solutions of this form were also obtained in [3]. In a forthcoming paper, Hogancamp [20] continues these families globally by means of the general theory in [2]. Unlike (1.13), he finds that the solutions broaden while remaining uniformly bounded in $C^{k+\alpha}$, for all $k \geq 0$.

Remark 1.5. Observe that while (1.13) implies the existence of a monotone front solution for all $\lambda > 0$, it does not give uniqueness. Indeed, there may be many turning points along the global bifurcation curve. There can also be secondary bifurcations resulting in branches of solutions not captured by Theorem 1.1. On the other hand, $\mathscr{C}$ is maximal among all locally analytic curves of increasing monotone front solutions containing the reference state $(u, \lambda) = (0, 0)$. Moreover, the curve comprises *all* such solutions in a neighborhood of $(u, \lambda) = (0, 0)$ in the sense that every sufficiently small front is an element of $\mathscr{C}$ up to translation and reflection.

Remark 1.6. Here we are taking the regularity of u to be slightly better than the classical one. This choice simplifies the maximum principle arguments used to establish (1.12). In fact, since $\mathcal{W}$ and b are real analytic, a simple bootstrapping argument using elliptic theory shows that u is $C^{k+\alpha}_{\mathrm{b}}$ for any $k \geq 0$ as soon as it is C^2_{b}.

The rest of the paper is organized as follows. In Section 2, we define the function spaces suited for the study of front-type solutions and the limiting linearized operators. In Section 3, we consider the so-called conjugate flow problem for the system. This is crucial to the global continuation argument as it will allow us to characterize (and then rule out) the loss of compactness. Next, in Section 4, we briefly recapitulate the small-amplitude existence result from [3, Section 3] and prove some additional facts. Section 5 is then devoted to establishing that certain monotonicity properties are preserved along closed sets of solutions extending these local curves. Finally, these components are assembled in Section 6 to give the proof of Theorem 1.1.

2. Preliminaries

2.1. **Function spaces.** We begin by setting down a functional analytic framework. First, define

$$\mathscr{X} := \left\{ u \in C^{3+\alpha}(\overline{\Omega}) : u|_{\partial\Omega} = 0 \right\}, \qquad \mathscr{Y} := C^{1+\alpha}(\overline{\Omega}).$$

Note that the elements of $\mathscr{X}$ and $\mathscr{Y}$ are only locally Hölder continuous; we denote the corresponding space of uniformly Hölder continuous functions by $\mathscr{X}_{\mathrm{b}}$ and $\mathscr{Y}_{\mathrm{b}}$, respectively. We also will append a subscript of "o" to indicate the subspace of functions that are odd in x and even in y. Finally we denote $\mathscr{X}'$ be the subspace of $\mathscr{X}$ consisting of functions independent of x, and likewise $\mathscr{Y}'$. The PDE (1.5) can then be recast as the abstract operator equation

$$\mathscr{F}(u, \lambda) = 0,$$

where $\mathscr{F}$ is a real-analytic mapping $\mathscr{X}_{\mathrm{b}} \times \mathbb{R} \to \mathscr{Y}_{\mathrm{b}}$ and $\mathscr{X}_{\mathrm{b,o}} \times \mathbb{R} \to \mathscr{Y}_{\mathrm{b,o}}$.

To study fronts, we follow the strategy of [4] and conduct most of the work in the spaces

$$\begin{aligned} \mathscr{X}_\infty &:= \Big\{ u \in \mathscr{X}_{\mathrm{b,o}} : \lim_{x\to\pm\infty} \partial^\beta u \text{ exists for all } |\beta| \leq 3 \Big\}, \\ \mathscr{Y}_\infty &:= \Big\{ f \in \mathscr{Y}_{\mathrm{b,o}} : \lim_{x\to\pm\infty} \partial^\beta f \text{ exists for all } |\beta| \leq 1 \Big\}, \end{aligned} \tag{2.1}$$

with all the above limits are uniform in y. Intuitively, $\mathscr{X}_\infty$ is the largest closed subspace of $\mathscr{X}_{\mathrm{b}}$ containing all front-type solutions with the desired symmetry properties (see, [4, Lemma 2.3]). Since the solutions we construct will all have $\lambda > 0$, it is convenient to introduce the open set

$$\mathcal{U}_\infty := \mathscr{X}_\infty \times (0, \infty). \tag{2.2}$$

One can easily confirm that $\mathscr{F} \colon \mathcal{U}_\infty \to \mathscr{Y}_\infty$ is real analytic.

2.2. **Limiting operators and their principal eigenvalues.** If $(u, \lambda) \in \mathscr{X}_\infty \times \mathbb{R}$ is a front-type solution of (1.5), then the Fréchet derivative $\mathscr{F}_u(u, \lambda)$ is a linear elliptic operator whose coefficients have well-defined point-wise limits as $x \to \pm\infty$. Taking those limits and restricting the domain to x-independent functions yields the so-called *transversal linearized operators* at $x = -\infty$ and $x = +\infty$:

$$\mathscr{L}'_\pm(u, \lambda) \colon \mathscr{X}' \to \mathscr{Y}' \qquad v \mapsto \lim_{x\to\pm\infty} \mathscr{F}_u(u, \lambda) v.$$

In our setting, these will be Sturm–Liouville-type ODE operators posed on the interval $\Omega' := (-\frac{\pi}{2}, \frac{\pi}{2})$, and hence they possess principal eigenvalues that we denote by $\sigma_0^\pm(u, \lambda)$. Recall that the principal eigenvalues bound the remainder of the spectrum from above and are characterized by the corresponding eigenfunction being strictly positive on Ω'. The problem here is self-adjoint, so the entirety of the spectrum lies on the real axis. It is also worth noting that, if u is odd in x, then the structure of the equation (1.5) implies that $\sigma_0^-(u, \lambda) = \sigma_0^+(u, \lambda)$.

3. Conjugate flows

If (u,λ) is a front, then its limiting states $\lim_{x\to\pm\infty} u(x,\,\cdot\,)$ are themselves x-independent solutions of the PDE (1.5). This leads us to study the ODE

$$\begin{cases} \partial_y\big(\mathcal{W}'(U_y^2)U_y\big) - b(U,\lambda) = 0 & \text{in } \Omega' = (-\frac{\pi}{2},\frac{\pi}{2}), \\ U = 0 & \text{on } \partial\Omega' = \big\{y = \pm\frac{\pi}{2}\big\}. \end{cases} \tag{3.1}$$

Given two solutions to the above ODE, a natural question is whether they can be connected by a front solution of the full anti-plane shear system. In this section we give a partial answer to this question, which will be needed at several key points in the global bifurcation theoretic argument leading to Theorem 1.1.

Our main tool is a conserved quantity of the full system. Naturally, the anti-plane shear model (1.5) carries a variational structure in that it is formally given by

$$\delta \int_\Omega \mathcal{L}(u,\nabla u,\lambda)\,dx\,dy = 0 \tag{3.2}$$

with suitable boundary conditions on the admissible variations, and where the Lagrangian density

$$\mathcal{L}(u,\nabla u,\lambda) := \frac{1}{2}\mathcal{W}(|\nabla u|^2) + \mathcal{B}(u,\lambda).$$

Here,

$$\mathcal{B}(\varkappa,\lambda) := \int_0^{\varkappa} b(\tilde{\varkappa},\lambda)\,d\tilde{\varkappa}$$

is the primitive of b. Note that (1.7) then implies that $\mathcal{B}(\,\cdot\,,\lambda)$ is even.

Now, for $(u,\lambda)\in\mathscr{X}_{\mathrm{b}}\times\mathbb{R}$, define the functional

$$\begin{aligned} \mathscr{H}(u,\lambda;x) &:= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big(\mathcal{L}(u,\nabla u,\lambda) - \mathcal{L}_\xi(u,\nabla u,\lambda)u_x\Big)\,dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\mathcal{W}(|\nabla u|^2) - u_x^2\mathcal{W}'(|\nabla u|^2) + \mathcal{B}(u,\lambda)\right)\,dy. \end{aligned}$$

By [4, Lemma 3.1], $\mathscr{H}(u,\lambda;x)$ is independent of x provided (u,λ) is a solution to (1.5). This can of course also be verified by a direct calculation, and should be understood as a consequence of the variational structure (3.2) and the equation's translation invariance in x. Following [4, Definition 3.3], we call two distinct functions $U_\pm\in C^2(\Omega')$ *conjugate flows* if

$$\mathscr{F}(U_-,\lambda) = \mathscr{F}(U_+,\lambda) = 0, \quad \text{and} \quad \mathscr{H}(U_+,\lambda) = \mathscr{H}(U_-,\lambda). \tag{3.3}$$

That is, $(U_\pm,\lambda)$ both solve (3.1) and are on the same level set of the conserved quantity $\mathscr{H}$. Naturally, being conjugate flows is a necessary condition for U_+ and U_- to represent the limiting states of a front solution to (1.5).

Proposition 3.1 (Conjugate flows)**.** *Assume that* (1.7), (1.10), *and* (1.11) *hold.*

(a) *If $\lambda > 0$, there are exactly three solutions of the ODE* (3.1) *which do not change sign: the unique positive solution $U =: U_+(\lambda)$, its reflection $U_-(\lambda) := -U_+(\lambda)$, and the trivial solution $U\equiv 0$.*
(b) *$U_+(\lambda)$ and $U_-(\lambda)$ are conjugate in the sense of* (3.3)*, but neither of them is conjugate to $U\equiv 0$.*
(c) *If $\lambda = 0$, then the unique solution of* (3.1) *is the trivial solution $U\equiv 0$.*

In (a) there will also exist many other solutions which change sign, but these do not play a major role in our analysis. Before giving the proof of this result, let us state and prove a corollary for the original problem of finding monotone front solutions to the PDE (1.5).

Corollary 3.2 (Limiting states of monotone symmetric fronts)**.** *Suppose that* $(u,\lambda) \in \mathscr{X}_\infty \times [0,\infty)$ *is a front solution of* (1.5) *which is strictly monotone in that* $\partial_x u > 0$ *in* Ω*. Then necessarily* $\lambda > 0$ *and* $\lim_{x\to\pm\infty} u(x,\,\cdot\,) = U_\pm(\lambda)$*.*

Proof. Since u is odd in x, $u(0,y) = 0$ for $|y| \le \frac{\pi}{2}$. The monotonicity of u in x therefore implies that its limiting states $U_\pm := \lim_{x\to\pm\infty} u(x,\,\cdot\,)$ satisfy $U_- < 0 < U_+$ for $|y| < \frac{\pi}{2}$. In particular, $U_\pm$ are distinct and hence conjugate by the above discussion. If $\lambda = 0$, then Proposition 3.1(c) forces $U_- = U_+ = 0$, which is a contradiction. If $\lambda > 0$, then Proposition 3.1(a)–(b) imply that $U_\pm = U_\pm(\lambda)$ as desired. □

Proof of Proposition 3.1. First we rewrite the interior equation of (3.1) as a planar system

$$\begin{cases} U_y = V, \\ V_y = \dfrac{b(U,\lambda)}{\mathcal{W}'(V^2) + 2\mathcal{W}''(V^2)V^2}. \end{cases} \tag{3.4}$$

Here the denominator is strictly positive thanks to (1.4), which follows in turn from the stronger condition (1.10). Fix $\lambda > 0$. It is easy to confirm from (1.8) and (1.11) that the origin is the only rest point for this system, and the quantity

$$H(U,V,\lambda) := \mathcal{W}'(V^2)V^2 - \frac{1}{2}\mathcal{W}(V^2) - \mathcal{B}(U,\lambda) \tag{3.5}$$

is conserved. Note that $\mathcal{B} \le 0$ and from (1.4) we have

$$2q\mathcal{W}'(q) - \mathcal{W}(q) = \int_0^q \big(\mathcal{W}'(s) + 2\mathcal{W}''(s)s\big)\, ds > 0 \quad \text{for } q > 0.$$

Hence $H(U,V,\lambda) \ge 0$ and thus all orbits are periodic and centered on $(U,V) = (0,0)$.

Consider the level set

$$H(U,V,\lambda) = c,$$

for a fixed $c > 0$. It will intersect the V-axis at $(0, \pm V_0(c))$ for some unique $V_0(c) > 0$. To find a positive solution to the boundary value problem (3.1), we look for a value of c such that the orbit through $(0, V_0(c))$ arrives at $(0, -V_0(c))$ in time $\pi = |\partial\Omega'|$.

It is convenient to switch to polar coordinates for the dependent variables:

$$(U,V) \longmapsto (R,\Theta), \qquad U = R\cos\Theta, \quad V = R\sin\Theta.$$

This transforms the planar system (3.4) to

$$\begin{cases} R_y = \dfrac{1}{2}R\sin(2\Theta) + \dfrac{b(R\cos\Theta,\lambda)\sin\Theta}{f(R,\Theta)}, \\ \Theta_y = -\sin^2(\Theta) + \dfrac{b(R\cos\Theta,\lambda)\cos\Theta}{Rf(R,\Theta)}, \end{cases} \tag{3.6}$$

where, thanks to (1.4),

$$f(R,\Theta) := \mathcal{W}'(R^2\sin^2\Theta) + 2\mathcal{W}''(R^2\sin^2\Theta)R^2\sin^2\Theta > 0.$$

From (3.6) and the symmetry of the equation, we compute that the time required for the orbit to travel from $(0, V_0(c))$ to $(0, -V_0(c))$ is

$$\begin{aligned} P(c,\lambda) &:= 2\int_{\frac{\pi}{2}}^{0} \frac{d\theta}{\Theta_y(r(\theta,c,\lambda)\sin\theta)} \\ &= 2\int_0^{\frac{\pi}{2}} \frac{r(\theta,c,\lambda)f(r(\theta,c,\lambda),\theta)}{r(\theta,c,\lambda)f(r(\theta,c,\lambda),\theta)\sin^2\theta - b(r(\theta,c,\lambda)\cos\theta,\lambda)\cos\theta}\, d\theta, \end{aligned} \tag{3.7}$$

where $r = r(\theta, c, \lambda) > 0$ is defined to be the unique positive solution to

$$H(r\cos\theta, r\sin\theta, \lambda) = c. \tag{3.8}$$

We will often suppress the arguments of r in the interest of readability.

Differentiating (3.8) in c, we see that

$$1 = \frac{r_c}{r}\left(q(\mathcal{W}'(q) + 2q\mathcal{W}''(q)) - b(\varkappa, \lambda)\varkappa\right)\Big|_{\substack{q=r^2\sin^2\theta \\ \varkappa = r\cos\theta}}$$

and hence $r_c > 0$ by (1.4) and (1.11). Likewise, differentiating the period map P gives

$$P_c(c, \lambda) = 2\int_0^{\frac{\pi}{2}} \frac{\partial_c[\Theta_y(r\sin\theta)]}{\Theta_y^2}\, d\theta.$$

A direct computation shows that

$$\partial_c[\Theta_y(r\sin\theta)] = -\frac{r_c\cos\theta}{r^2 f^2(r,\theta)}\left(f(r,\theta)(b - \varkappa b_\varkappa) + 2bq(3\mathcal{W}'' + 2q\mathcal{W}''')\right)\Big|_{\substack{q=r^2\sin^2\theta \\ \varkappa = r\cos\theta}} > 0$$

for $\theta \in (0, \frac{\pi}{2})$ by (1.10) and (1.11). Therefore, we conclude that

$$P_c(c, \lambda) > 0 \quad \text{for} \quad \lambda > 0, \tag{3.9}$$

which confirms that P is strictly increasing in c.

On the other hand, sending $c \searrow 0$ in (3.7) we have for each $\lambda > 0$,

$$\begin{aligned}\lim_{c\searrow 0} P(c, \lambda) &= \lim_{c\searrow 0} 2\int_0^{\frac{\pi}{2}} \frac{f(r,\theta)}{f(r,\theta)\sin^2\theta - \frac{b(r\cos\theta,\lambda)\cos^2\theta}{r\cos\theta}}\, d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta - b_\varkappa(0,\lambda)\cos^2\theta} = \frac{\pi}{\sqrt{-b_\varkappa(0,\lambda)}} < \pi,\end{aligned}$$

where we used (1.11) in the last inequality. A solution to (3.1) will correspond to the energy levels c with

$$P(c, \lambda) = \frac{\pi}{k}, \quad k = 1, 2, \ldots. \tag{3.10}$$

Simply by the implicit function theorem there is, for each $k \geq 1$, a curve $c = c_k(\lambda)$ in the (c, λ)-plane satisfying (3.10). Since $P_c > 0$, it follows that

$$c_1(\lambda) < c_2(\lambda) < \cdots.$$

We are only interested in the first of these, which gives rise to the unique positive solution $U_+(\lambda)$ of (3.1) corresponding to $P(c, \lambda) = \pi$. Repeating the above calculation at $\lambda = 0$ we find that the period map is still monotone in c but since $b_\varkappa(0, 0) = -1$,

$$\lim_{c\searrow 0} P(c, 0) = \pi.$$

Thus, in this case the only solution to (3.1) is the trivial one $U = 0$. Since $b_{\varkappa\lambda}(0, 0) = 1$ by (1.8), this result extends to $-1 \ll \lambda < 0$.

It remains to show (b). That U_+ and U_- are conjugate is clear from the symmetry of the equations, and by the above arguments, the only other solution of (3.1) which does not change sign is the trivial one $U = U_\pm(0) \equiv 0$. Since $\mathscr{H}(0, \lambda) = 0$, it therefore suffices to show that $\mathscr{H}(U_\pm(\lambda), \lambda) \neq 0$ for $\lambda > 0$. To that end, denoting $\dot{U}_\pm := \partial_\lambda U_\pm$, we compute that

$$\begin{aligned}\frac{d}{d\lambda}\mathscr{H}(U_\pm(\lambda), \lambda) &= \frac{d}{d\lambda}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\mathcal{W}(U_{\pm y}^2) + \mathcal{B}(U_\pm, \lambda)\right) dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\mathcal{W}'(U_{\pm y}^2)U_{\pm y}\dot{U}_{\pm y} + b(U_\pm, \lambda)\dot{U}_\pm + \mathcal{B}_\lambda(U_\pm, \lambda)\right) dy\end{aligned}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{B}_\lambda(U_\pm, \lambda)\, dy,$$

where the last line follows from integrating by parts and the equation (3.1) satisfied by U, and we have suppressed some λ dependencies for readability. By the sign condition (1.11), it then follows that $\lambda \mapsto \mathscr{H}(U_\pm(\lambda), \lambda)$ is strictly decreasing. This completes the proof. □

Remark 3.3. The above proof gives us additional information about $U_+(\lambda)$. In particular, it is even and strictly decreasing for $y > 0$.

4. Small-amplitude theory

In this section, we recall the existence theory for anti-plane shear fronts in a neighborhood of the undisturbed state. This was first obtained in [3, Section 3], but we will establish some further properties in preparation for the global continuation. Without loss of generality we only consider increasing fronts, as the analogous local curve of decreasing fronts can be obtained by simply reflecting in x (cf. Remark 1.2).

Theorem 4.1 (Small-amplitude fronts)**.** *Let conditions* (1.7)–(1.11) *hold. There exists* $\varepsilon_0 > 0$ *and a local* C^0 *curve*

$$\mathscr{C}_{\mathrm{loc}} = \big\{(u^\varepsilon, \varepsilon^2) : 0 < \varepsilon < \varepsilon_0\big\} \subset \mathscr{X}_\infty \times (0, \infty)$$

of solutions to (1.5) *with the following properties.*

(a) (Asymptotics) *The solutions on* $\mathscr{C}_{\mathrm{loc}}$ *have the leading-order form*

$$u^\varepsilon(x, y) = a_1 \varepsilon \tanh\left(\frac{\varepsilon x}{\sqrt{2}}\right) \cos(y) + O(\varepsilon^2) \qquad \text{in } C^{3+\alpha}_{\mathrm{b}}(\overline{\Omega}), \tag{4.1}$$

with $a_1 = 2/\sqrt{3(b_2 + 2w_1)}$*. In particular,* $\mathscr{C}_{\mathrm{loc}}$ *bifurcates from the trivial solution* $(u, \lambda) = (0, 0)$*.*

(b) (Uniqueness) *In a neighborhood of* $(0, 0)$*, all front solutions of* (1.5) *are, up to reflection and translation in* x*, contained in* $\mathscr{C}_{\mathrm{loc}}$*.*

(c) (Kernel) *The kernel of the linearized problem at* $(u^\varepsilon, \varepsilon^2)$ *is generated by* $\partial_x u^\varepsilon$*.*

Proof. The existence of $\mathscr{C}_{\mathrm{loc}}$, as well as the asymptotic information in (4.1), is proved in [3, Theorem 3.1] using a center manifold reduction approach. This theory constructs a sufficiently smooth coordinate map[2]

$$\Psi \colon \mathbb{R}^3 \to C^{3+\alpha}(\overline{\Omega_1}), \qquad \text{where } \Omega_1 := (-1, 1) \times \Omega',$$

with the following properties. First, any solution along $\mathscr{C}_{\mathrm{loc}}$ can be recovered from its trace $v := u(\,\cdot\,, 0)$ via

$$u(x, y) = v(x)\varphi_0(y) + \Psi(v(x), v'(x), \varepsilon)(0, y), \tag{4.2}$$

where here

$$\varphi_0(y) := \cos y$$

is an element of the kernel of $\mathscr{L}'(0, 0)$. Second, the trace v of any sufficiently small solution solves the second order ODE

$$v'' = f(v, v', \varepsilon), \qquad \text{where} \quad f(A, B, \varepsilon) := \frac{d^2}{dx^2}\Big|_{x=0} \Psi(A, B, \varepsilon)(x, 0). \tag{4.3}$$

By its construction and because of the symmetries of the problem, Ψ satisfies

$$\begin{aligned} &\Psi(0, 0, \varepsilon) = 0 \text{ for all } \varepsilon, && \Psi(-A, B, \varepsilon) = -\Psi(A, B, \varepsilon), \\ &\Psi_A(0, 0, 0) = \Psi_B(0, 0, 0) = 0, && \Psi(A, -B, \varepsilon)(-x, y) = \Psi(A, B, \varepsilon)(x, y). \end{aligned} \tag{4.4}$$

[2]The function Ψ in [3] in fact takes values in a certain weighted Hölder space on Ω, but for our present purposes it is sufficient to restrict its output to the truncated domain Ω_1.

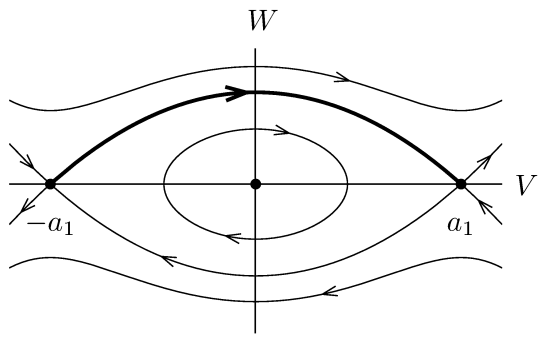

FIGURE 2. Phase portrait for the rescaled reduced ODE (4.7) at $\varepsilon = 0$. The thick curve is the heteroclinic orbit connecting the rest points $(-a_1, 0)$ and $(a_1, 0)$.

Finally, Ψ has the expansion

$$\begin{aligned}\Psi(A,B,\varepsilon)(x,y) = -\frac{1}{2}x^2\cos(y)\varepsilon^2 A + \left[\frac{3b_2+6w_1}{8}x^2\cos y + \frac{b_2-6w_1}{32}(\cos y - \cos(3y))\right]A^3 \\ + O\Big((|A|+|B|^{1/2}+|\varepsilon|)^4\Big).\end{aligned} \tag{4.5}$$

Inserting (4.5) into (4.3) and switching to the rescaled variables

$$x =: X/\varepsilon, \qquad v(x) =: \varepsilon V(X), \qquad v_x(x) =: \varepsilon^2 W(X), \tag{4.6}$$

we obtain the first order system

$$\begin{cases} V_X = W, \\ W_X = -V + a_1^{-2}V^3 + R(V,W,\varepsilon), \end{cases} \tag{4.7}$$

with the rescaled error term $R(V,W,\varepsilon) = O\left(|\varepsilon|(|V|+|W|)\right)$. When $\varepsilon = 0$, (4.7) has the explicit heteroclinic solution

$$V^0(X) = a_1 \tanh\frac{X}{\sqrt{2}}, \qquad W^0(X) = \frac{a_1}{\sqrt{2}}\operatorname{sech}^2\frac{X}{\sqrt{2}}.$$

shown in Figure 2. Since (4.7) enjoys the symmetries $(V(X),W(X)) \mapsto (V(-X),-W(-X))$ and $(V(X),W(X)) \mapsto (V(X),-W(X))$, one can show that this heteroclinic connection persists for small $\varepsilon > 0$. Denote this solution by $(V^\varepsilon, W^\varepsilon)$, and the corresponding limits as $x \to \pm\infty$ by $(V^\varepsilon_\pm, 0)$. Undoing the scaling and using (4.2), one obtains both the existence of $\mathscr{C}_{\mathrm{loc}}$ and the asymptotics (4.1).

By the proof of Proposition 3.1, all small solutions (U,λ) of the transverse ODE (3.1) besides the trivial solution $U = 0$ have $\lambda = \varepsilon^2 > 0$ and $U = U_\pm(\lambda)$. Moreover, U_+ and U_- are conjugate to one another but not to $U = 0$. Thus all small fronts which are not constant in x must have $U_\pm(\varepsilon^2)$ as their asymptotic states. In particular, $V^\varepsilon_\pm$ above is nothing other than $U_\pm(\varepsilon^2)(0)$.

We can now prove the uniqueness claimed in (b). Suppose that (u,λ) is a nontrivial front which is small in the sense that $\|u\|_{C^{3+\alpha}(\Omega)} + |\lambda| \le \delta \ll 1$. By the above discussion we necessarily have $\lambda = \varepsilon^2 > 0$, and moreover $u \to U_\pm(\varepsilon^2)$ or $u \to U_\mp(\varepsilon^2)$ as $x \to \pm\infty$. Reflecting in x if necessary, we can assume that we are in the first case. Choosing δ small enough, we can also guarantee that the trace $v = u(\,\cdot\,,0)$ solves (4.3). Rescaling, we obtain a solution (V,W) of (4.7), with the same limiting states $(V^\varepsilon_\pm, 0)$ as the orbit $(V^\varepsilon, W^\varepsilon)$ corresponding to u^ε. We claim that in fact (V,W) is a translate of $(V^\varepsilon, W^\varepsilon)$. To prove this claim, consider the phase portrait of (4.7), which is qualitatively the same as the one shown in Figure 2. The unstable manifold $\mathfrak{W}^{\mathrm{u}}(V^\varepsilon_-, 0)$ has two portions, one in the upper half plane and one in the lower half plane. If (V,W) is not a translate of $(V^\varepsilon, W^\varepsilon)$, then its orbit must coincide with the portion of $\mathfrak{W}^{\mathrm{u}}(V^\varepsilon_-, 0)$ in the lower half plane. In particular, $V < V^\varepsilon_- < 0$

for $X \ll -1$. Since $V \to V_+^\varepsilon > 0$ as $X \to +\infty$, there must be some X_0 where $V_X(X_0) = W(X_0) = 0$. Shifting X_0 to the origin, reversibility implies that $(V(X), W(X)) = (V(-X), -W(-X))$. But then (V, W) is a homoclinic orbit with $(V, W) \to (V_-^\varepsilon, 0)$ as $|X| \to \infty$, which is a contradiction.

Combining this uniqueness with the symmetry of (1.5) under reflections in x, y, and u, we immediately obtain that solutions along $\mathscr{C}_{\rm loc}$ are odd in x and even in y. Thus $\mathscr{C}_{\rm loc} \subset \mathscr{X}_\infty \times (0, \infty)$.

To prove (c) we adopt the idea of [3, Theorem 1.6] which, in our setting, says that $\dot u$ is in the kernel of the linearized operator at (u, λ) only if $\dot v := \dot u(\,\cdot\,, 0)$ solves the linearized reduced equation

$$\dot v'' = \nabla_{(A,B)} f(v, v', \lambda) \cdot (\dot v, \dot v').$$

Equivalently, the corresponding rescaled quantities $(\dot V, \dot W)$ solve a nonautonomous planar system

$$\begin{pmatrix} \dot V_X \\ \dot W_X \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 + 3a_1^{-2}V^2 + R_V(V, W, \varepsilon) & R_W(V, W, \varepsilon) \end{pmatrix} \begin{pmatrix} \dot V \\ \dot W \end{pmatrix} =: \mathcal{M}(X) \begin{pmatrix} \dot V \\ \dot W \end{pmatrix}.$$

Taking limits, we find that

$$\lim_{X \to \pm\infty} \mathcal{M}(X) = \begin{pmatrix} 0 & 1 \\ -1 + 3a_1^{-2}V_\pm^2 + R_V(V_\pm, 0, \varepsilon) & R_W(V_\pm, 0, \varepsilon) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 + O(\varepsilon) & O(\varepsilon) \end{pmatrix},$$

and hence that $\mathcal{M}(X)$ is strictly hyperbolic for $|X| \gg 1$ with one negative and one positive eigenvalue. A standard dynamical systems argument implies that there cannot be two linearly independent solutions of the reduced linearized problem that are uniformly bounded. We may then conclude that the kernel of the linearized operator is indeed generated by $\partial_x u^\varepsilon$. □

Let us next consider the principal eigenvalues for the transversal linearized problems at infinity along $\mathscr{C}_{\rm loc}$.

Lemma 4.2 (Local spectral non-degeneracy)**.** *In the setting of Theorem 4.1, every solution $(u, \lambda) \in \mathscr{C}_{\rm loc}$ with $0 < \lambda \ll 1$ is spectrally non-degenerate in that*

$$\sigma_0^-(u, \lambda) = \sigma_0^+(u, \lambda) < 0. \tag{4.8}$$

Proof. Note that $\sigma_0^\pm(0, 0) = 0$. Therefore, we must show that the principal eigenvalues perturb to the left as λ moves away from 0 along $\mathscr{C}_{\rm loc}$. As observed before, we need only consider σ_0^+ as the spectrum at $x = -\infty$ is the same.

For simplicity we will drop the $\pm$ and further write the limiting transversal linearized operator as

$$\mathscr{L}'(u, \lambda)\psi = \partial_y \left(\left(\mathcal{W}'(U_y^2) + 2U_y^2 \mathcal{W}''(U_y^2) \right) \partial_y \psi \right) - b_\varkappa(U, \lambda)\psi,$$

where recall that (U, λ) solves the problem

$$\mathscr{F}'(U, \lambda) := \partial_y \big(\mathcal{W}'(U_y^2) U_y \big) - b(U, \lambda) = 0. \tag{4.9}$$

Clearly $\mathscr{F}'_U(U, \lambda) = \mathscr{L}'(u, \lambda)$. The proof of Theorem 4.1 (or indeed the arguments in [3]) shows that the solution U of (4.9) depends smoothly on $\varepsilon = \sqrt{\lambda}$. Moreover, sending $x \to \infty$ in (4.1) yields the expansion

$$U(\varepsilon)(y) = \lim_{x \to \infty} u^\varepsilon(x, y) = a_1 \varepsilon \cos y + O(\varepsilon^2) \qquad \text{in } C^{3+\alpha}([-\tfrac{\pi}{2}, \tfrac{\pi}{2}]).$$

Using dots to denote derivatives in ε, we therefore have

$$\dot U(0)(y) = a_1 \varphi_0 = a_1 \cos y. \tag{4.10}$$

Note that we are abusing notation somewhat by writing U as a function of ε rather than λ.

With the asymptotics for U in hand, we now turn to the eigenvalue problem, which in our notation is

$$\big(\mathscr{L}'(U(\varepsilon), \varepsilon^2) - \sigma_0 \big) \varphi = 0. \tag{4.11}$$

We know that $(\varphi, \sigma_0, \varepsilon) = (\varphi_0, 0, 0)$ solves (4.9)–(4.11). By a familiar implicit function theorem argument, we deduce that there is a unique curve of nearby solutions, with φ and σ_0 depending smoothly on ε.

Differentiating (4.11) in ε, we find

$$\mathscr{L}'(U, \varepsilon^2)\dot{\varphi} + \dot{\mathscr{L}}'(U, \varepsilon^2)\varphi = \dot{\sigma}_0\varphi + \sigma_0\dot{\varphi}, \tag{4.12}$$

where

$$\dot{\mathscr{L}}'(U, \varepsilon^2)\psi := \partial_y \left(\left(6\mathcal{W}'' + 4U_y^2\mathcal{W}'''\right) U_y\dot{U}_y\partial_y\psi \right) - \left(b_{\varkappa\varkappa}\dot{U} + 2\varepsilon b_{\lambda\varkappa} \right)\psi$$

and we are suppressing the arguments of $\mathcal{W}(U_y^2)$ and $b(U, \lambda)$ for readability. In particular, at $\varepsilon = 0$ where $U = 0$ and $\sigma_0 = 0$, this becomes

$$\dot{\mathscr{L}}'(0, 0)\psi = -b_{\varkappa\varkappa}(0, 0)\dot{U}(0)\psi = 0$$

in light of (1.8). Therefore multiplying (4.12) by φ, evaluating at $\varepsilon = 0$, and then integrating, we obtain

$$\dot{\sigma}_0(0) = 0,$$

which forces us to proceed to higher order derivatives.

Differentiating (4.12) with respect to ε we find that

$$\mathscr{L}'(U, \varepsilon^2)\ddot{\varphi} + 2\dot{\mathscr{L}}'(U, \varepsilon^2)\dot{\varphi} + \ddot{\mathscr{L}}'(U, \varepsilon^2)\varphi = \ddot{\sigma}_0\varphi + 2\dot{\sigma}_0\dot{\varphi} + \sigma_0\ddot{\varphi}, \tag{4.13}$$

where

$$\begin{aligned}\ddot{\mathscr{L}}'(U, \varepsilon^2)\psi := \partial_y\Big[\Big(\big(6\mathcal{W}'' + 4q\mathcal{W}'''\big)\big(\dot{U}_y^2 + U_y\ddot{U}_y\big) + \big(20\mathcal{W}''' + 8q\mathcal{W}^{(4)}\big)U_y^2\dot{U}_y^2\Big)\partial_y\psi\Big] \\ - \left(b_{\varkappa\varkappa\varkappa}\dot{U}^2 + b_{\varkappa\varkappa}\ddot{U} + 4\varepsilon b_{\lambda\varkappa\varkappa}\dot{U} + 4\varepsilon^2 b_{\lambda\lambda\varkappa} + 2b_{\lambda\varkappa} \right)\psi.\end{aligned}$$

At $\varepsilon = 0$, then,

$$\ddot{\mathscr{L}}'(0, 0)\psi = 12w_1\partial_y\big(\dot{U}_y^2\partial_y\psi\big) - \big(6b_2\dot{U}^2 - 2\big)\psi.$$

Multiplying (4.13) by φ and integrating, we find that, at $\varepsilon = 0$,

$$\begin{aligned}\ddot{\sigma}_0(0)\|\varphi_0\|_{L^2}^2 &= -12w_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{U}_y^2(\partial_y\varphi_0)^2\,dy - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(6b_2\dot{U}^2 - 2\right)\varphi_0^2\,dy \\ &= -12a_1^2w_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\partial_y\varphi_0)^2\,dy - 6a_1^2b_2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi_0^4\,dy + 2\|\varphi_0\|_{L^2}^2,\end{aligned}$$

where here we have made use of (4.10) and the fact that $\varphi|_{\varepsilon=0} = \varphi_0 = \cos y$. Calculating the explicit integrals yields

$$\ddot{\sigma}_0(0) = -2,$$

which in turn proves that (4.8) holds for $0 < \varepsilon \ll 1$. □

The final task of this section is to verify that the solutions on $\mathscr{C}_{\text{loc}}$ exhibit the monotonicity properties claimed in (1.12). For that purpose, we first state an elementary (but very useful) lemma.

Lemma 4.3 (Nodal cone)**.** *Let $\mathscr{W}$ and $\mathscr{Z}$ be Banach spaces, and suppose that $\mathcal{N} \subset \mathscr{Z}$ is an open cone (that is, invariant under multiplication by strictly positive scalars). If $G\colon \mathbb{R} \times \mathscr{W} \to \mathscr{Z}$ is continuous and*

$$\frac{G(s, w)}{s} \to \psi \in \mathcal{N} \qquad \text{as } (s, w) \to (0, 0),\ s \neq 0, \tag{4.14}$$

then, for all sufficiently small (s, w) with $\pm s > 0$, we have $\pm G(s, w) \in \mathcal{N}$.

Proof. Since $\mathcal{N}$ is open, (4.14) implies that $G(s, w)/s \in \mathcal{N}$ for all (s, w) sufficiently small and $s \neq 0$. But then $|s|G(s, w)/s$ is also in $\mathcal{N}$, and so the result holds. □

Using this result, the desired monotonicity properties follow quickly from the reduction formula (4.2) and our understanding of the behavior of $v = u^\varepsilon(\,\cdot\,,0)$ obtained in the proof of Theorem 4.1.

Lemma 4.4 (Local monotonicity)**.** *Each $(u,\lambda) \in \mathscr{C}_{\mathrm{loc}}$ is a strictly increasing monotone front that exhibits the nodal properties:*

$$u_x > 0 \qquad \text{in } \Omega, \tag{4.15a}$$

$$u_y < 0 \qquad \text{in } (0,\infty)\times(0,\tfrac{\pi}{2}]. \tag{4.15b}$$

Proof. Consider the x-directional monotonicity (4.15a). In view of the center manifold reduction (4.2) and scalings in (4.6), we have that

$$\partial_x u^\varepsilon(x,y) = v'(x)\varphi_0(y) + \Psi_A(v(x),v'(x),\varepsilon)(0,y)v'(x) + \Psi_B(v(x),v'(x),\varepsilon)(0,y)v''(x), \tag{4.16}$$

where recall that $\varphi_0(y) = \cos y$, $v = \varepsilon V^\varepsilon(\varepsilon x)$, and $(V^\varepsilon, W^\varepsilon)$ solves the rescaled ODE (4.7). From the phase portrait, we have seen that v is strictly increasing (as $W^\varepsilon > 0$) and odd with $v(x) > 0$ for $x > 0$.

Having (4.16) in mind, set

$$\begin{aligned}
\mathscr{W} &:= \mathbb{R}^3,\\
\mathscr{Z} &:= \left\{ z \in C^1([-\tfrac{\pi}{2},\tfrac{\pi}{2}]) : z(-\tfrac{\pi}{2}) = z(\tfrac{\pi}{2}) = 0 \right\},\\
\mathcal{N} &:= \left\{ z \in \mathscr{Z} : z > 0 \text{ on } (-\tfrac{\pi}{2},\tfrac{\pi}{2}),\ \mp z'(\pm\tfrac{\pi}{2}) > 0 \right\},
\end{aligned}$$

and consider the mapping $G\colon \mathbb{R}\times\mathscr{W} \to \mathscr{Z}$ defined by

$$G(B;A,C,\varepsilon) := B\varphi_0 + \Psi_A(A,B,\varepsilon)(0,\,\cdot\,)B + \Psi_B(A,B,\varepsilon)(0,\,\cdot\,)C.$$

Due to (4.4), $B \mapsto \Psi_B(A,B,\varepsilon)(0,\,\cdot\,)$ is odd. Since Ψ_B is smooth, there must therefore exist a continuous mapping $\Xi\colon \mathbb{R}^3 \to \mathscr{Z}$ satisfying

$$\Psi_B(A,B,\varepsilon)(0,\,\cdot\,) = \Xi(A,B,\varepsilon)B.$$

Thus,

$$\frac{G(B;A,C,\varepsilon)}{B} = \varphi_0 + \Psi_A(A,B,\varepsilon)(0,\,\cdot\,) + \Xi(A,B,\varepsilon)C.$$

Sending $(B,A,C,\varepsilon) \to 0$, we find that the left-hand side limits to $\varphi_0 \in \mathcal{N}$. Applying Lemma 4.3, we conclude that $\pm G(B;A,C,\varepsilon) \in \mathcal{N}$ for $\pm B > 0$ and (B,A,C,ε) sufficiently small. Comparing this to the asymptotics for $\partial_x u^\varepsilon$ in (4.16), we must then have that (4.15a) holds for $0 < \varepsilon \ll 1$.

The argument for the y-directional monotonicity is very similar. Again from the representation formula (4.2) one can compute that

$$\partial_y u^\varepsilon(x,y) = v(x)\varphi_0'(y) + \Psi_y(v(x),v'(x),\varepsilon)(0,y).$$

Let us redefine

$$\begin{aligned}
\mathscr{W} &:= \mathbb{R}^2,\\
\mathscr{Z} &:= \left\{ z \in C^1([-\tfrac{\pi}{2},\tfrac{\pi}{2}]) : z \text{ odd} \right\},\\
\mathcal{N} &:= \left\{ z \in \mathscr{Z} : z < 0 \text{ on } (0,\tfrac{\pi}{2}],\ z'(0) < 0 \right\},
\end{aligned}$$

and consider the map $G\colon \mathbb{R}\times\mathscr{W} \to \mathscr{Z}$ given by

$$G(A;B,\varepsilon) := A\varphi_0' + \Psi_y(A,B,\varepsilon)(0,\,\cdot\,).$$

By (4.4), $A \mapsto \Psi_y(A,B,\varepsilon)(0,\,\cdot\,)$ is odd, so we can find a smooth map $\Upsilon\colon \mathbb{R}^3 \to \mathscr{Z}$ such that

$$\Psi_y(A,B,\varepsilon)(0,\,\cdot\,) = \Upsilon(A,B,\varepsilon)A.$$

Proceeding as before, one can then easily check that G satisfies the hypothesis (4.14) of Lemma 4.3, and hence (4.15b) holds for all $0 < \varepsilon \ll 1$. □

5. Nodal pattern

While Lemma 4.4 ensures that the small anti-plane shear fronts obtained in [3] are strictly increasing, it is not obvious that this property should persist beyond the perturbative regime. In [4], it is nevertheless shown that $\partial_x u$ has a fixed sign on any connected set of solutions that are spectrally non-degenerate in the sense of (4.8), provided it has a fixed sign at a single solution in this set. In particular, this implies that the solutions obtained via global continuation will also be monotone in x. The argument relies on repeated use of the maximum principle, and it applies to a broad class of elliptic PDE that includes the anti-plane shear model (1.5). Our equation enjoys an additional symmetry: invariance under reflection in y. This will allow us to infer that y-directional monotonicity (4.15b) is likewise preserved on the global bifurcation curve. Consider the fundamental half-strip

$$R^+ := (0,\infty)\times(0,\tfrac{\pi}{2}),$$

whose boundary components we denote

$$B^+ := [0,\infty)\times\{0\},\qquad T^+ := [0,\infty)\times\{\tfrac{\pi}{2}\},\qquad L^+ := \{0\}\times[0,\tfrac{\pi}{2}].$$

The main result of this section is the following, where we recall that a front (u,λ) is called *strictly increasing* if $\partial_x u > 0$ in Ω.

Theorem 5.1 (Nodal properties)**.** *Suppose that $\mathcal{K}\subset\mathcal{U}_\infty$ is a connected set of strictly increasing monotone fronts and the spectral nondegeneracy condition* (4.8) *holds along it. If some $(u,\lambda)\in\mathcal{K}$ exhibits the nodal properties*

$$u_y < 0 \qquad \text{in } \overline{R^+}\setminus(L^+\cup B^+), \tag{5.1a}$$

$$u_{yy} < 0 \qquad \text{on } B^+\setminus\{(0,0)\}, \tag{5.1b}$$

$$u_{xy} < 0 \qquad \text{on } L^+\setminus\{(0,0)\}, \tag{5.1c}$$

$$u_{xyy}(0,0) < 0, \tag{5.1d}$$

then every element of $\mathcal{K}$ satisfies (5.1)*.*

We start in the next lemma by establishing some basic information about the boundary behavior of u and its derivatives on R^+.

Lemma 5.2 (Boundary behavior)**.** *Let $(u,\lambda)\in\mathcal{U}_\infty$ be a strictly increasing monotone front solution of* (1.5)*. Then*

$$u,\, u_{yy} = 0 \qquad \text{on } T^+, \tag{5.2a}$$

$$u_{xy} < 0 \qquad \text{on } T^+, \tag{5.2b}$$

$$u,\, u_{xx} = 0 \qquad \text{on } L^+, \tag{5.2c}$$

$$u_y = 0 \qquad \text{on } B^+. \tag{5.2d}$$

Proof. Most of these are immediate consequences of the boundary conditions or symmetry. Indeed, $T^+\subset\partial\Omega$, hence u vanishes identically there. This implies further that $\partial_x^k u = 0$ on T^+ for all $k\geq 0$, so evaluating the PDE (1.5) along the top yields

$$\left[\mathcal{W}'(u_y^2) + 2u_y^2\mathcal{W}''(u_y^2)\right] u_{yy} = 0 \qquad \text{on } T^+.$$

From (1.4), this gives $u_{yy}=0$ on T^+, proving (5.2a). The argument for (5.2c) is similar: u (and hence $\partial_y^k u$) vanishes identically along L^+ by oddness. Using this and the equation (1.5) gives

$$\left[\mathcal{W}'(u_x^2) + 2u_x^2\mathcal{W}''(u_x^2)\right] u_{xx} = 0 \qquad \text{on } L^+.$$

That u_y vanishes on B is likewise a consequence the evenness of u in y. Finally, to obtain (5.2b), we observe that u_x is in the kernel of the linearized operator at (u,λ). That is, $v := u_x$ satisfies

$$\nabla\cdot\big(\mathcal{W}'(|\nabla u|^2)\nabla v + 2\mathcal{W}''(|\nabla u|^2)(\nabla u\otimes\nabla u)\nabla v\big) - b_\varkappa(u,\lambda)v = 0 \qquad \text{in } R^+. \tag{5.3}$$

By assumption, $u_x = v > 0$, and thus it attains its minimum along T^+. Recalling (1.4) and (1.11), the sign of u_{xy} in (5.2b) then follows from the Hopf lemma (for positive solutions). □

Using the above identities and the maximum principle, we can show that the full set of nodal properties (5.1) can be collapsed to just transversal monotonicity (5.1a).

Lemma 5.3 (Simplified nodal properties)**.** *Suppose that* $(u, \lambda) \in \mathcal{U}_\infty$ *is a strictly monotone increasing front solution of* (1.5) *that satisfies* (5.1a)*. Then* u *also satisfies* (5.1b)–(5.1d)*.*

Proof. Differentiating the PDE (1.5) in y, we see that $v = u_y$ satisfies the linear elliptic equation (5.3). Moreover, $u_y < 0$ in R^+ according to (5.1a), so u_y attains its supremum on R^+ at $(0, 0)$ thanks to (5.2d). Combining (5.2c) and (5.2d), we infer that

$$u_{yx},\, u_{yy},\, u_{yxx},\, u_{yyy} = 0 \qquad \text{at } (0, 0). \tag{5.4}$$

But this is in violation of the Serrin edge-point lemma as ∇u_y and $D^2 u_y$ cannot simultaneously vanish at a maximum; see, for example, [10, Theorem E.9]. Having arrived at a contradiction, we must have that (5.1d) holds. Likewise, since u_y vanishes identically along $L^+ \cup B^+$ and is negative in R^+, we obtain (5.1b) directly from the Hopf boundary-point lemma. The same argument shows that $u_{xy} < 0$ on L^+ except at the lower corner $(0, 0)$ and possibly at the upper corner $(0, \frac{\pi}{2})$. But notice that u_y attains its minimum at $(0, \frac{\pi}{2})$, and moreover (5.2a) and (5.2c) give

$$u_{yy}, u_{yxx},\, u_{yyx},\, u_{yyy} = 0 \qquad \text{at } (0, \tfrac{\pi}{2}). \tag{5.5}$$

Thus the Serrin edge-point lemma ensures that $u_{yx}(0, \frac{\pi}{2}) < 0$, proving (5.1c). □

For later use, we observe that the above lemma together with Lemmas 4.2 and 4.4 has the following immediate corollary.

Corollary 5.4 (Local nodal pattern)**.** *Each* $(u, \lambda) \in \mathscr{C}_{\text{loc}}$ *is a strictly increasing monotone front that exhibits the full set of nodal properties* (5.1)*.*

The next two lemmas show that these nodal properties are (relatively) open and closed in a suitable topology. The main tools are [4, Lemmas 2.7 and 2.8], which require the spectral nondegeneracy condition (4.8).

Lemma 5.5 (Open property)**.** *Suppose that* $(\bar{u}, \bar{\lambda}) \in \mathcal{U}_\infty$ *is a strictly increase monotone front solution of* (1.5) *that satisfies* (4.8) *and* (5.1)*. There exists* $\delta = \delta(\bar{u}, \bar{\lambda}) > 0$ *such that, if* $(u, \lambda) \in \mathcal{U}_\infty$ *is a solution with*

$$\|u - \bar{u}\|_{C^3(R^+)} + |\lambda - \bar{\lambda}| < \delta, \qquad \lim_{x\to\infty} u = U_+(\lambda), \tag{5.6}$$

then u *is a strictly increasing monotone front and satisfies* (5.1)*.*

Proof. By Corollary 3.2, the limiting state as $x \to +\infty$ corresponding to $(\bar{u}, \bar{\lambda})$ is $\bar{U}_+ := U_+(\bar{\lambda})$. Now by Remark 3.3, for any $\bar{\lambda} \geq \lambda_0 > 0$, we have

$$\bar{U}_{+y} < 0 \quad \text{on } (0, \tfrac{\pi}{2}], \qquad \bar{U}_{+y}(\tfrac{\pi}{2}) < -\delta_1, \qquad \bar{U}_{+yy}(0) < -\delta_2,$$

for some $\delta_1, \delta_2 > 0$ depending only on λ_0. It follows from this, (5.1a), and (5.1b) that, for any $\varepsilon > 0$, there exists $\delta = \delta(\lambda_0, \varepsilon)$ so that

$$u_y < 0 \qquad \text{on } (\varepsilon, \infty) \times [0, \pi/2],$$

for all (u, λ) satisfying (5.6). Note that the principal eigenvalues depend continuously on the coefficients of the operator (see, for example, [4, Lemma A.2]), hence $\sigma_0^\pm(\bar{u}, \bar{\lambda}) < 0$ implies $\sigma_0^\pm(u, \lambda) < 0$ for δ small enough. Perhaps shrinking it even further, we can then ensure that (u, λ) is a strictly monotone front by applying [4, Lemma 2.7].

Consider next the sign of u_y in a neighborhood of L^+. As in the proof of Lemma 5.3, we see that (5.2c) and (5.2d) imply that its derivatives at $(0,0)$ satisfy (5.4). Expanding u_y at the origin gives

$$u_y(x,y) = \frac{1}{2}u_{yyx}(0,0)xy + O(|x|^3 + |y|^3) \qquad \text{in } R^+.$$

But $\bar{u}_{xyy}(0,0) < 0$ according to (5.1d), and so by perhaps further shrinking δ, we can ensure that $u_y < 0$ on $\mathcal{O} \cap R^+$, for some open ball $\mathcal{O} \ni (0,0)$. Let us now fix ε to be half the radius of $\mathcal{O}$.

Similarly, choosing any point $(0,y_0) \in L^+ \setminus \mathcal{O}$, we have

$$u_y(x,y) = u_{xy}(0,y_0)x + O(x^2 + (y-y_0)^2) \qquad \text{in } R^+.$$

In view of (5.1c), we may then shrink δ further so that

$$u_y < 0 \qquad \text{on } ((0,\varepsilon) \times [0,\pi/2]) \setminus \mathcal{O}.$$

At last, then, we have shown that u satisfies (5.1a). Applying Lemma 5.3, we conclude that u exhibits all of the nodal properties (5.1). □

Lemma 5.6 (Closed property)**.** *Suppose that $\{(u_n,\lambda_n)\} \subset \mathcal{U}_\infty$ is a sequence of strictly monotone increasing front solutions to* (1.5) *that are each spectrally non-degenerate* (4.8) *and $(u_n,\lambda_n) \to (u,\lambda)$ in $\mathscr{X}_{\mathrm{b}} \times \mathbb{R}$ for some $(u,\lambda) \in \mathcal{U}_\infty$ also satisfying* (4.8)*. If each u_n exhibits the nodal properties* (5.1)*, then so does u.*

Proof. Since (4.8) holds by assumption, [4, Lemma 2.8] ensures that (u,λ) is also a strictly increasing monotone front. In view of Lemma 5.3, it therefore suffices to show that u satisfies (5.1a). By continuity, we have that $u_y \le 0$ in $\overline{R^+}$. We have already seen that u_y satisfies the linear elliptic PDE (5.3), and since $\sup_{R^+} u_y = 0$, we may apply the maximum principle to conclude that $u_y < 0$ in R^+.

Fix a point $(x_0, \frac{\pi}{2})$ with $x_0 > 0$. By monotonicity, u is a positive solution of the elliptic PDE (1.5), which can be viewed as linear. Since u attains its minimum value at $(x_0, \frac{\pi}{2})$, we have by the Hopf boundary-point lemma that $u_y < 0$ there. Together with the previous paragraph, this proves that u satisfies (5.1a) and hence (5.1). □

Combining these lemmas, the proof of Theorem 5.1 is now immediate.

6. Global continuation

6.1. **Abstract global bifurcation theory.** For the convenience of the reader, we record here the main tool for the proof of Theorem 1.1, which is the following general theorem on global bifurcation of monotone front solutions to elliptic PDE. To simplify the presentation, we will only discuss its application to the specific problem (1.5).

Suppose that we have a local curve $\mathscr{K}_{\mathrm{loc}}$ of strictly monotone front solutions to (1.5). Due to translation invariance, for any $(u,\lambda) \in \mathscr{K}_{\mathrm{loc}}$, the linearized operator $\mathscr{F}_u(u,\lambda)$ will have at least one kernel direction. As the first assumption, we require that the kernel is *exactly* one dimensional:

$$\ker \mathscr{F}_u(u,\lambda) = \operatorname{span}\{\partial_x u\}. \tag{H1}$$

We also impose a requirement on the spectrum of the transversal linearized operator:

$$\sigma_0^{\pm}(u,\lambda) < 0. \tag{H2}$$

Note that, as observed before, the spectrum of the transversal limiting problems at $x = -\infty$ and $x = +\infty$ are identical, so there is no ambiguity above. Lastly, assume that the local curve bifurcates from a singular point where the above spectral condition is violated. That is, suppose $\mathscr{K}_{\mathrm{loc}}$ admits the C^0 parameterization

$$\mathscr{K}_{\mathrm{loc}} = \{(u(\varepsilon),\lambda(\varepsilon)) : 0 < \varepsilon < \varepsilon_0\} \subset \mathcal{U},$$

where

$$(u(\varepsilon),\lambda(\varepsilon)) \to (u_0,\lambda_0) \in \mathcal{U} \quad \text{as } \varepsilon \to 0+, \quad \text{and} \quad \sigma_0^{\pm}(u_0,\lambda_0) = 0. \tag{H4}$$

Under the above hypotheses, we have the following global continuation result. It corresponds to [4, Theorem 1.2] combined with [4, Lemma 3.4].

Theorem 6.1. *Let $\mathscr{K}_{\mathrm{loc}}$ be a curve of strictly monotone front solutions to* (1.5) *bifurcating from a singular point as in* (H4)*. Assume that at each $(u,\lambda)\in\mathscr{K}_{\mathrm{loc}}$, the nondegeneracy* (H1) *and spectral* (H2) *conditions hold.*

Then, possibly after translation, $\mathscr{K}_{\mathrm{loc}}$ is contained in a global curve of strictly monotone front solutions $\mathscr{K}\subset\mathcal{U}_\infty$, parameterized as

$$\mathscr{K} := \{(u(s),\lambda(s)) : 0<s<\infty\}\subset\mathscr{F}^{-1}(0)$$

for some continuous $\mathbb{R}_+\ni s\longmapsto(u(s),\lambda(s))\in\mathcal{U}_\infty$ with the properties enumerated below.

(a) (Alternatives) *As $s\to\infty$, one of three alternatives must occur:*

(A1) (Blowup) *The quantity*

$$N(s) := \|u(s)\|_{C^{3+\alpha}} + |\lambda(s)| + \frac{1}{|\lambda(s)|} \longrightarrow \infty. \tag{6.1}$$

(A2) (Heteroclinic degeneracy) *There exist sequences $s_n\to\infty$ and $x_n\to\pm\infty$ with*

$$(u(s_n)(\,\cdot\,+x_n,\,\cdot\,),\,\lambda(s_n)) \longrightarrow (u_*,\lambda_*) \text{ in } C^3_{\mathrm{loc}}(\overline{\Omega})\times\mathbb{R}^2$$

for some monotone front solution $(u_*,\lambda_*)\in\mathcal{U}_\infty$, but the three limiting states

$$\lim_{x\to\mp\infty} u_*(x,\,\cdot\,),\quad \lim_{n\to\infty}\lim_{x\to+\infty} u(s_n)(x,\,\cdot\,),\quad \lim_{n\to\infty}\lim_{x\to-\infty} u(s_n)(x,\,\cdot\,),$$

are all distinct and pairwise conjugate in the sense of (3.3)*.*

(A3) (Spectral degeneracy) *There exists a sequence $s_n\to\infty$ with $\sup_n N(s_n)<\infty$ so that*

$$\sigma_0^{\pm}(u(s_n),\lambda(s_n))\to 0.$$

(b) *For all s sufficiently large, $(u(s),\lambda(s))\notin\mathscr{K}_{\mathrm{loc}}$. In particular, $\mathscr{K}$ is not a closed loop.*

(c) *At each parameter value $s\in(0,\infty)$, $\mathscr{K}$ admits a local real-analytic reparameterization.*

The above result is based on the foundational work of Dancer [8, 7] on analytic global bifurcation and its later refinement by Buffoni–Toland [1]. Because we wish to treat problems on unbounded domains, however, Theorem 6.1 is deliberately formulated so that no compactness hypotheses are imposed on $\mathscr{F}$ beyond the local curve. In particular, we do not ask that it is locally proper nor Fredholm index 0. Instead, the loss of compactness is viewed as an *alternative*. From this perspective, the novelty of Theorem 6.1 is that it completely characterizes how a failure of compactness can manifest in terms of the qualitative behavior of the solutions along the curve. Indeed, (A1), (A2), and (A3) each have analogues in global bifurcation of second-order ODEs. A discussion of these ideas can found in the recent announcement [5, Section 2].

Several authors [28, 11, 12] have developed degree theoretic global bifurcation machinery for quasilinear elliptic problems on unbounded domains. Their approach is more closely aligned to the classical results for bounded domains in that they make hypotheses implying that $\mathscr{F}$ is locally proper and Fredholm. Note that the local curve here bifurcates from a singular point where the latter assumption is violated. This hints at the fact that verifying these compactness criteria is actually quite subtle for our problem. Another barrier to applying [28, 11, 12] is that they are formulated in Sobolev spaces, whereas for fronts u is not integrable. While some adaptation of this theory may be possible, as we will see in this section, Theorem 6.1 offers a more elegant means of overcoming both difficulties.

It is also worth mentioning that the degree theoretic approach furnishes the entire connected component of $\mathscr{F}^{-1}(0)$ containing the trivial solution. In principle, this will be larger than $\mathscr{K}$, as it is possible that secondary bifurcations occur that result in additional solution branches. On the other hand, $\mathscr{K}$ is globally C^0 and locally real analytic, whereas with degree theory, the continuum one obtains is merely a closed set.

6.2. **Spectral non-degeneracy.** The next result will allow us to confirm that the spectral non-degeneracy persists not just for perturbative solutions, but globally.

Lemma 6.2 (Global non-degeneracy)**.** *Suppose that* $(u,\lambda)\in\mathcal{U}_\infty$ *is a strictly increasing monotone front. Then,* $\ker\mathscr{L}'_\pm(u,\lambda)$ *is trivial.*

Proof. As the following argument does not involve the parameter λ, for simplicity of the presentation, we will suppress all dependence on it. Also, as noted before, the transversal linearized problems at $x=\pm\infty$ are identical, so there is no need to append subscripts of $\pm$. In particular, we simply write U for the limiting state $\lim_{x\to\pm\infty}u(x,\,\cdot\,)=U_\pm(\lambda)$.

From (3.5) we know that U satisfies

$$H(U,U_y)=\mathcal{W}'(U_y^2)U_y^2-\frac{1}{2}\mathcal{W}(U_y^2)-\mathcal{B}(U)=\text{constant}. \tag{6.2}$$

Moreover by (1.11) and (1.10) we have

$$VH_V(U,V)>0\ \text{ for }\ V\neq 0. \tag{6.3}$$

Let $\Phi=\Phi(y;\mu)$ be the unique solution to the initial value problem

$$\begin{cases}\partial_y\big(\mathcal{W}'(\Phi_y^2)\Phi_y\big)-b(\Phi)=0 & \text{in }\Omega',\\ \Phi(-\frac{\pi}{2};\mu)=0,\quad \Phi_y(-\frac{\pi}{2};\mu)=\mu.\end{cases} \tag{6.4}$$

Clearly $\Phi(y;U_y(-\frac{\pi}{2}))=U(y)$. Differentiating (6.4) with respect to the parameter μ yields

$$\begin{cases}\partial_y\left[\big(\mathcal{W}'(U_y^2)+2U_y^2\mathcal{W}''(U_y^2)\big)\,\dot\Phi_y\right]-b_\varkappa(U)\dot\Phi=0 & \text{in }\Omega',\\ \dot\Phi(-\frac{\pi}{2})=0,\quad \dot\Phi_y(-\frac{\pi}{2})=1,\end{cases} \tag{6.5}$$

where $\dot\Phi(y):=\Phi_\mu(y;U_y(-\frac{\pi}{2}))$. If $\dot\Phi(\frac{\pi}{2})=0$, then (6.5) is precisely the statement that $\dot\Phi\in\ker\mathscr{L}'(u,\lambda)$. Since we are dealing with a boundary value problem for a linear second-order ODE, one can easily check that, conversely, $\ker\mathscr{L}'(u,\lambda)$ is trivial whenever $\dot\Phi(\frac{\pi}{2})\neq 0$.

Recalling the definition of the period map P in Section 3, from (6.4) it follows that

$$\Phi(P(0,\mu);\mu)=0.$$

Differentiating the above identity with respect to μ then gives

$$\Phi_y\big(P(H(0,\mu);\mu)\big)P_c(H(0,\mu))H_V(0,\mu)+\Phi_\mu\big(P(H(0,\mu));\mu\big)=0,$$

which at $y=\frac{\pi}{2}$ and $\mu=U_y(-\frac{\pi}{2})$ becomes

$$U_y(\tfrac{\pi}{2})P_c\big(H(0,U_y(-\tfrac{\pi}{2}))\big)H_V(0,U_y(-\tfrac{\pi}{2}))+\dot\Phi(\tfrac{\pi}{2})=0.$$

By (6.3) and (3.9), the first term above is negative. Thus, $\dot\Phi(\frac{\pi}{2})<0$, which proves that $\ker\mathscr{L}'(u,\lambda)$ is trivial. □

6.3. **Uniform regularity.** The purpose of this section is to derive a priori estimates on front solutions to the anti-plane shear equation that will eventually allow us to conclude that λ is unbounded along the global curve. Following [32, Chapter 7], we first show that the divergence structure of (1.5) and the uniform ellipticity condition (1.4) imply that there is an auxiliary function $\mathcal{P}(u,|\nabla u|^2,\lambda)$ that obeys a maximum principle. By exploiting certain coercivity properties of $\mathcal{P}$, this will allow us to control u in terms of λ.

More precisely, define

$$\mathcal{P}(\varkappa,q,\lambda):=2q\mathcal{W}'(q)-\mathcal{W}(q)-2\mathcal{B}(\varkappa,\lambda). \tag{6.6}$$

An application of [32, Theorem 7.1] gives

Lemma 6.3 (Maximum principle for $\mathcal{P}$)**.** *Assume that* (1.4) *holds and let* $(u,\lambda) \in \mathscr{X}_{\mathrm{b}} \times \mathbb{R}$ *be a solution to* (1.5)*. Then* $\mathcal{P}(u, |\nabla u|^2, \lambda)$ *cannot achieve an interior maximum or minimum except at critical points of* u*.*

Now, from (1.10) we see that

$$2q\mathcal{W}''(q) + \mathcal{W}'(q) > \mathcal{W}'(0) = 1 \qquad \text{for} \quad q > 0.$$

Integrating this leads to

$$2q\mathcal{W}'(q) - \mathcal{W}(q) > q \qquad \text{for} \quad q > 0.$$

From the definition of $\mathcal{P}$ (6.6) and negativity of $\mathcal{B}$ (1.11), we may then infer the bound

$$\mathcal{P}(u, |\nabla u|^2, \lambda) \geq |\nabla u|^2. \tag{6.7}$$

Supposing now that (u,λ) is a strictly monotone front, Lemma 6.3 implies that $\mathcal{P}(u, |\nabla u|^2, \lambda)$ is maximized at infinity. Combining this with the above inequality and a bootstrapping argument, we obtain the following.

Theorem 6.4 (Uniform regularity)**.** *Every monotone front solution* $(u,\lambda) \in \mathcal{U}_\infty$ *to* (1.5) *with* $|\lambda| < \Lambda$ *obeys the a priori bound*

$$\|u\|_{C^{3+\alpha}(\Omega)} < C$$

where the constant $C = C(\Lambda) > 0$*.*

Proof. Throughout the course of the proof, let C denote a generic positive constant depending only on Λ. First, notice that because u is a strictly monotone front, it is maximized and minimized only at infinity. Thus, by Corollary 3.2,

$$\|u\|_{C^0(\Omega)} = \sup_{\Omega'} |U_\pm(\lambda)| \leq C. \tag{6.8}$$

On the other hand, thanks to Lemma 6.3 and (6.7), we have

$$\|\nabla u\|_{C^0(\Omega)} \leq \|\mathcal{P}(u, |\nabla u|^2, \lambda)\|_{C^0(\Omega)} = \sup_{\Omega'} \mathcal{P}(U_\pm(\lambda), |\partial_y U_\pm(\lambda)|^2, \lambda) \leq C.$$

Together with (6.8), this gives control of $\|u\|_{C^1}$.

To upgrade this to the full $C^{3+\alpha}$ norm, we make use of a familiar elliptic regularity argument. Observe that $\partial_x u$ can be thought of as $W^{2,\infty}(\Omega)$ strong solution of (5.3). By the previous paragraph, the coefficients of this PDE are bounded uniformly in $L^\infty(\Omega)$ in terms of Λ. Using the De Giorgi–Nash-type estimate [13, Corollary 9.29], we find that

$$\|\partial_x u\|_{C^\alpha(\Omega_m)} \leq C\|\partial_x u\|_{C^0(\Omega)} < C,$$

for any $m \in \mathbb{Z}$, where $\Omega_m := (m, m+1) \times \Omega'$. Note that the constant above is independent of m, and so this implies further that $\|\partial_x u\|_{C^\alpha(\Omega)} < C$. The same reasoning gives an equivalent bound for $\partial_y u$, and hence $\|u\|_{C^{1+\alpha}} < C$.

Now, we can simply apply linear Schauder theory to (1.5) to see that

$$\|u\|_{C^{2+\alpha}(\Omega)} \leq C.$$

Bootstrapping the above argument once more leads to the desired $C^{3+\alpha}$ bound, completing the proof. □

Lemma 6.5 (Deformation gradient blowup)**.** *It holds that*

$$\|\partial_y U_\pm(\lambda)\|_{C^0} \longrightarrow \infty \qquad \text{as } \lambda \to \infty.$$

Proof. First, we observe that $\lambda \mapsto U_+(\lambda)|_{y=0}$ is strictly increasing. This simply follows from the fact that $\dot{U}_+ := \partial_\lambda U_+(\lambda)$ solves

$$\mathscr{L}'(U_+(\lambda),\lambda)\dot{U}_+ = -b_\lambda(U_+(\lambda),\lambda)$$

and vanishes on $\partial\Omega'$. In view of (1.11), the right-hand side above is strictly positive. On the other hand, Lemma 4.2 and Lemma 6.2 ensure that the principal eigenvalue of $\mathscr{L}'(U_+(\lambda),\lambda)$ is strictly negative for all $\lambda > 0$. We may therefore apply the maximum principle to conclude that $\dot{U}_+ > 0$.

Next, we recall that the equation satisfied by U_+ can be rewritten as the planar system (3.4) which has the first integral H given by (3.5). Since $U_+(\lambda)$ is odd, writing $V_+(\lambda) := U_{y+}(\lambda)$, we have that

$$c(\lambda) = H(U_+(\lambda), V_+(\lambda)) = -\mathcal{B}(U_+(\lambda),\lambda)|_{y=0} > -\mathcal{B}(U_+(\bar{\lambda}),\lambda)|_{y=0}$$

for $0 < \bar{\lambda} < \lambda$ by (1.11). Fixing a $\bar{\lambda} > 0$, the unboundedness of $b(\varkappa, \cdot\,)$ implies that $c(\lambda) \to \infty$ as $\lambda \to \infty$. On the other hand, evaluating H at the top of the domain reveals that

$$c(\lambda) = \left(\mathcal{W}'(V_+(\lambda)^2)V_+(\lambda)^2 - \frac{1}{2}\mathcal{W}(V_+(\lambda)^2)\right)\Big|_{y=\frac{\pi}{2}}.$$

Thus, $\|\partial_y U_+(\lambda)\|_{C^0} \to \infty$ as $\lambda \to \infty$, which completes the proof. □

6.4. **Proof of the main result.** Finally, we turn to Theorem 1.1. Recall that we have constructed a local curve $\mathscr{C}_{\text{loc}}$ of small-amplitude strictly increasing fronts in Section 4.

Proof of Theorem 1.1. We have already verified in Theorem 4.1(c) that the kernel condition (H1) holds along $\mathscr{C}_{\text{loc}}$. Lemma 4.2, moreover, implies that it satisfies the spectral assumption (H2). By construction, $\mathscr{C}_{\text{loc}}$ bifurcates from $(0,0)$, and hence (H4) holds. Lastly, in view of Corollary 5.4, the solutions on $\mathscr{C}_{\text{loc}}$ are strictly monotone increasing and exhibit the nodal properties (5.1).

We are therefore justified in applying Theorem 6.1 with $\mathscr{K}_{\text{loc}} = \mathscr{C}_{\text{loc}}$, furnishing a global bifurcation curve $\mathscr{C}$. The symmetry properties claimed in part (a) are encoded in the definition of the spaces (2.1), while the monotonicity (1.12) follows from the fact that the fronts in $\mathscr{C}_{\text{loc}}$ are strictly increasing and Theorem 5.1. The local real analyticity of $\mathscr{C}$ asserted in part (c) is likewise a direct consequence of Theorem 6.1(c).

Consider now the limiting behavior along $\mathscr{C}$. Suppose for the sake of contradiction that (A2) occurs, and without loss of generality assume that the sequence $x_n \to +\infty$. By the monotonicity properties in (1.12), the limiting front (u_*, λ_*) has $u_* \geq 0$ in Ω. Thus its limiting states

$$\lim_{x\to\pm\infty} u_*(x, \cdot\,) \geq 0.$$

By Proposition 3.1(a), these states must therefore be either $U_+(\lambda_*)$ or 0. However, the three limiting states

$$\lim_{x\to-\infty} u_*(x, \cdot\,) \quad \text{and} \quad \lim_{n\to\infty}\lim_{x\to\pm\infty} u(s_n)(x, \cdot\,) = U_\pm(\lambda_*),$$

are distinct and pairwise conjugate, and so this is impossible in view of Proposition 3.1(b).

Reconnection to the trivial solution $(0,0)$ is ruled out by Theorem 4.1(b) and Theorem 6.1(b). We claim further that the spectral degeneracy alternative (A3) can only happen in conjunction with blowup of λ as in (1.13). To see this, note that because the curve does not reconnect, if λ is uniformly bounded along it, then spectral degeneracy would imply there exists $0 < \lambda_* < \infty$ for which the kernel of $\mathscr{F}'_U(U_+(\lambda_*),\lambda_*)$ is nontrivial. But this is impossible due to Lemma 6.2.

Thus we are left only with blowup as in (6.1). On the other hand, $\liminf_{s\to\infty}\lambda(s) > 0$ by the previous paragraph, and so Theorem 6.4 ensures that $\lambda(s) \to \infty$. Finally, Lemma 6.5 implies that this also leads to blowup in the deformation gradient:

$$\|\partial_y u(s)\|_{C^0(\Omega)} \geq \sup|U_{y\pm}(\lambda(s))| \longrightarrow \infty \qquad \text{as } s \to \infty,$$

completing the proof of part (b). □

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