

On the CLT for additive functionals of Markov chains

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Abstract

In this paper we study the additive functionals of Markov chains via conditioning with respect to both past and future of the chain. We shall point out new sufficient projective conditions, which assure that the variance of partial sums of n consecutive random variables of a stationary Markov chain is linear in n . The paper also addresses the central limit theorem problem and is listing several open questions.

1 Introduction

Throughout the paper assume that $(\xi_n)_{n \in \mathbb{Z}}$ is a stationary Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable space $(S, \mathcal{A})$. We suppose that there is a regular conditional distribution for ξ_1 given ξ_0 denoted by $Q(x, A) = \mathbb{P}(\xi_1 \in A | \xi_0 = x)$. In addition Q denotes the Markov operator acting via $(Qf)(x) = \int_S f(s)Q(x, ds)$. Denote by $\mathcal{F}_n = \sigma(\xi_k, k \leq n)$ and $\mathcal{F}^n = \sigma(\xi_k, k \geq n)$. The invariant distribution is denoted by $\pi(A) = \mathbb{P}(\xi_0 \in A)$ and Q^* denotes the adjoint of Q . Next, let $\mathbb{L}_0^2(\pi)$ be the set of measurable functions on S such that $\int f^2 d\pi < \infty$ and $\int f d\pi = 0$. For a function $f \in \mathbb{L}_0^2(\pi)$ let

$$X_i = f(\xi_i), \quad S_n = \sum_{i=1}^n X_i.$$

Note that for every $k \in \mathbb{Z}$, $Q^k f(\xi_0) = \mathbb{E}(X_k | \xi_0)$, while $(Q^*)^k f(\xi_0) = \mathbb{E}(X_{-k} | \xi_0)$. We denote by $\|X\|$ the norm in $\mathbb{L}^2(\mathbb{P})$ and by $\|f\|_\pi$ the norm in $\mathbb{L}_0^2(\pi)$. Sometimes, we shall also use the notations

$$V_n = I + Q + \dots + Q^n \quad \text{and} \quad V_n^* = I + Q^* + \dots + (Q^*)^n.$$

In some statements, we assume that the stationary Markov chain is ergodic, i.e. the only invariant functions $Qf = f$ are the constant functions. Concerning the central limit theorem for additive functionals of a stationary and ergodic Markov

chain, many of the results in the literature are given under sufficient conditions either in terms of $Q^k f$ or $V_n f$. Among them we mention the pioneering works by Gordin (1969), Gordin and Lifshitz (1978), Heyde (1974), McLeish (1975) and Volný (1993) among others. For a survey see Peligrad (2010) and the book by Merlevède et al. (2019).

Maxwell and Woodroffe (2000) introduced a more general condition than in the papers mentioned above, namely

$$\sum_{n \geq 1} \frac{\|E(S_n|\xi_0)\|}{n^{3/2}} < \infty. \quad (1)$$

In the same paper, they showed that (1) implies that

$$\frac{E(S_n^2)}{n} \rightarrow \sigma^2 \quad (2)$$

and

$$\frac{S_n}{\sqrt{n}} \Rightarrow N(0, \sigma^2), \quad (3)$$

where $\Rightarrow$ denotes the convergence in distribution and $N(0, \sigma^2)$ is a normally distributed random variable. Later on, Peligrad and Utev (2005) established the functional form of the CLT under (1).

There are examples of Markov chains pointing out that, in general, condition (1) is as sharp as possible in some sense. Peligrad and Utev (2005) constructed an example showing that for any sequence of positive constants (a_n) , $a_n \rightarrow 0$, there exists a stationary Markov chain such that

$$\sum_{n \geq 1} a_n \frac{\|E(S_n|\xi_0)\|}{n^{3/2}} < \infty$$

but $S_n/\sqrt{n}$ is not stochastically bounded. This example and other counterexamples provided by Volný (2010), Dedecker (2015) and Cuny and Lin (2016), show that, in general, condition

$$\sum_{n \geq 1} \frac{\|E(S_n|\xi_0)\|^2}{n^2} < \infty \quad (4)$$

does not assure that (2) holds and also does not assure (3).

However, by using Proposition 2.2 in Cuny (2011), which connects (4) with a spectral condition, we know that (4) is sufficient for the CLT given in (3), in case when the Markov chain is normal ($QQ^* = Q^*Q$), as announced by Gordin and Lifshitz (1981) and proven in Section IV.7 in Borodin and Ibragimov (1995) and also, independently, in Derriennic and Lin (1996).

Theorem 1 *Gordin and Lifshitz (1981). Assume that the Markov chain is normal, stationary and ergodic and satisfies (4). Then (2) and (3) hold.*

For reversible Markov chains ($Q = Q^*$) condition (4) is equivalent to (2) and also with the convergence of $E_\pi(fV_nf)$ (see Kipnis and Varadhan (1986), Proposition 2.2 in Cuny (2011) and the remarks following Proposition 1 in Derriennic and Lin [9]). Furthermore in this case the functional CLT also holds.

A natural question is to ask what will be a natural generalization of Theorem 1 to Markov processes, which are not necessarily normal. In other words what will be a natural minimal condition to be added to (4), which will insure (2) and (3).

A possibility is to impose besides (4) a similar condition, but conditioning this time with respect to the future of the process

$$\sum_{n \geq 1} \frac{\|E(S_n|\xi_n)\|^2}{n^2} < \infty. \quad (5)$$

It is interesting to point out that for normal Markov chains conditions (4) and (5) coincide.

In the operator notation, conditions (4) and (5) could be written in the following alternative form:

$$\sum_{n \geq 1} \frac{\|V_n(f)\|_\pi^2}{n^2} < \infty \text{ and } \sum_{n \geq 1} \frac{\|V_n^*(f)\|_\pi^2}{n^2} < \infty. \quad (6)$$

This paper has double scope. First, in Section 2, we shall raise some open questions concerning the CLT for the additive functionals of a Markov chain under conditions related to (6). In the following section we support these conjectures by proving some partial results. We shall show, for instance, that (6) implies (2) and we shall comment that the CLT holds up to a random centering. The proofs are given in Sections 4 and 5.

2 Open problems

We shall list here several natural open problems.

Problem 2 *For a stationary and ergodic Markov chain is it true (or not) that condition (6) implies that the CLT in (3) holds?*

In terms of the individual random variables, let us note that, by the triangle inequality, stationarity and Lemma 14 in the last section, applied with $a_k = \|E(X_k|\xi_0)\|$ (and also with $a_k = \|E(X_{-k}|\xi_0)\|$), we obtain

$$\sum_{n \geq 1} \frac{\|E(S_n|\xi_0)\|^2}{n^2} \leq 4 \sum_{k \geq 1} \|E(X_k|\xi_0)\|^2$$

and also

$$\sum_{n \geq 1} \frac{\|E(S_n|\xi_n)\|^2}{n^2} \leq 4 \sum_{k \geq 1} \|E(X_{-k}|\xi_0)\|^2.$$

Then, clearly (4) is implied by

$$\sum_{k \geq 1} \|E(X_k | \xi_0)\|^2 < \infty \quad (7)$$

and (5) is implied by

$$\sum_{k \geq 1} \|E(X_{-k} | \xi_0)\|^2 < \infty. \quad (8)$$

These two last conditions can be reformulated as:

$$\sum_{k \geq 1} \|Q^k f\|_\pi^2 < \infty \text{ and } \sum_{k \geq 1} \|(Q^*)^k f\|_\pi^2 < \infty. \quad (9)$$

As a matter of fact there are summability conditions that interpolates between (6) and (9), which are known under the name of square root conditions. Following Derriennic and Lin [8] the operator $\sqrt{I-Q}$ is defined by

$$\sqrt{I-Q} := I - \sum_{n \geq 1} \delta_n Q^n,$$

where $\sqrt{1-x} = 1 - \sum_{n \geq 1} \delta_n x^n$, with $\delta_n > 0$, $n \geq 1$ and $\sum_{n \geq 1} \delta_n = 1$. By the equivalent definitions in Corollary 2.12 in Derriennic and Lin [8], $f \in \sqrt{1-Q}\mathbb{L}_2(\pi)$ is equivalent to

$$\sum_{k=1}^n \frac{1}{k^{1/2}} Q^k f \text{ converges in } \mathbb{L}_2(\pi).$$

By Proposition 4.6 of Cohen et al. (2017), applied with $b_n = (1+n)^{-1/2}$, we know that $f \in \sqrt{1-Q}\mathbb{L}_2(\pi)$ implies (4) and also $f \in \sqrt{1-Q^*}\mathbb{L}_2(\pi)$ implies (5). On the other hand, by Proposition 9.2 in Cuny and Lin (2016), condition (7) implies $f \in \sqrt{1-Q}\mathbb{L}_2(\pi)$ and condition (8) implies $f \in \sqrt{1-Q^*}\mathbb{L}_2(\pi)$. Furthermore, according to Corollary 4.7 in Cohen et al. (2017), for normal contractions $f \in \sqrt{1-Q}\mathbb{L}_2(\pi)$ is equivalent to (4). We mention that Volný (2010) constructed an example of a (non-normal) Markov operator Q and $f \in \sqrt{1-Q}\mathbb{L}_2(\pi)$ for which the asymptotic variance of $\|S_n\|^2/n$ does not exist. Note however that if $f \in \sqrt{1-Q}\mathbb{L}_2(\pi) \cap \sqrt{1-Q^*}\mathbb{L}_2(\pi)$ then (2) holds (see Proposition 1 in [9] and the remarks following this proposition).

These considerations suggest that the following conjecture deserves to be studied, of course, in case the answer to Problem 2 is negative.

Problem 3 *If the Markov chain is stationary and ergodic is it true (or not) that $f \in \sqrt{1-Q}\mathbb{L}_2(\pi) \cap \sqrt{1-Q^*}\mathbb{L}_2(\pi)$ implies that the CLT in (3) holds?*

Finally, if the answer to Problem 3 is negative we could ask the following question:

Problem 4 *If the stationary Markov chain is ergodic in the ergodic theoretical sense is it true (or not) that condition (9) implies that (3) holds?*

3 Results

We give here a few results in support of the open problems which have been raised in the previous section. Point (a) of the next theorem deals with the variance of partial sums, which plays a very important role in the CLT. In the next theorem by totally ergodic chain we understand that Q^k is ergodic for any positive integer k .

Theorem 5 *Assume that conditions (4) and (5) hold. Then:*

(a) *The limit in (2) holds, namely*

$$\lim_{n \rightarrow \infty} \frac{E(S_n^2)}{n} = \sigma^2.$$

(b) *If the chain is totally ergodic then the following limit exists*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|E(S_n | \xi_0, \xi_n)\|^2 = \eta^2 \quad (10)$$

and

(c)

$$\frac{S_n - E(S_n | \xi_0, \xi_n)}{\sqrt{n}} \Rightarrow N(0, \sigma^2 - \eta^2). \quad (11)$$

As an immediate consequence, by the discussion in the previous section, we also have the following corollaries:

Corollary 6 *The conclusion of Theorem 5 also holds for $f \in \sqrt{1-Q}\mathbb{L}_2(\pi) \cap \sqrt{1-Q^*}\mathbb{L}_2(\pi)$.*

Corollary 7 *The conclusion of Theorem 5 also holds under the couple of conditions (7) and (8).*

Remark 8 *Let us mention that the conclusion of Corollary 6 does not hold if we assume only that $f \in \sqrt{1-Q}\mathbb{L}_2(\pi)$ as shown in Volný (2010). Also, the conclusion of Corollary 7 does not hold under just (7). Dedecker (2015) constructed a relevant example, which has been reformulated in Proposition 9.5. in Cuny and Lin (2016). This example shows that there exists a Markov operator Q on some $\mathbb{L}_2(\pi)$ and a function $f \in \mathbb{L}_0^2(\pi)$ satisfying*

$$\sum_{k \geq 1} (\log k) \|Q^k f\|_\pi^2 < \infty$$

and such that $\|S_n\|^2/n \rightarrow \infty$ as $n \rightarrow \infty$.

An interesting question asked in Problem 2 is whether the random centering in the point (c) of Theorem (5) can be avoided altogether. We can prove this fact under the condition

$$\sum_{n \geq 1} \frac{\|E(S_n | \xi_0, \xi_n)\|^2}{n^2} < \infty, \quad (12)$$

namely:

Corollary 9 *Assume (12) holds. Then (2) and (3) hold.*

By Lemma 14, relation (12) is implied by

$$\sum_{k \geq 1} \|E(X_0 | \xi_{-k}, \xi_k)\|^2 < \infty. \quad (13)$$

Therefore we obtain the following corollary which was also pointed out in Peligrad (2020):

Corollary 10 *Assume that (13) holds. Then (2) and (3) hold.*

4 Proofs

Let us comment first about conditions (4) and (5). We are going to establish two lemmas (Lemma 12 and Lemma 13) showing that condition (4) implies that

$$\sum_{i \geq 0} \frac{\|E(S_{2^i} | \xi_0)\|^2}{2^i} < \infty, \quad (14)$$

while condition (5) implies that

$$\sum_{i \geq 0} \frac{\|E(S_{2^i} | \xi_{2^i})\|^2}{2^i} < \infty. \quad (15)$$

As a matter of fact, we can replace conditions (4) and (5) in Theorem 5 by conditions (14) and (15).

Proof of point (a) of Theorem 5

The proof of point (a) of Theorem 5 is related to the proof of Proposition 2.1 in Peligrad and Utev (2005), but it takes advantage of the Markov property. It includes several steps.

1. Upper bound on a subsequence

We shall establish first the following recurrence formula which has interest in itself.

Denote

$$\Delta_{2^r} = \sum_{i=0}^{r-1} \frac{\|E(S_{2^i} | \xi_0)\| \cdot \|E(S_{2^i} | \xi_{2^i})\|}{2^i}.$$

Then, for $2^{r-1} \leq n < 2^r$, we have the following bound:

$$E(S_{2^r}^2) \leq 2^r [E(X_0^2) + \Delta_{2^r}]. \quad (16)$$

To establish it, denote by $\bar{S}_n = \sum_{k=n+1}^{2n} \xi_k$. So, by stationarity

$$E(S_{2n}^2) = 2E(S_n^2) + 2E(S_n \bar{S}_n).$$

Note that, by the properties of conditional expectation and by the Markov property,

$$E(S_n \bar{S}_n) = E[E(S_n E(\bar{S}_n | \mathcal{F}_n))] = E[E(S_n E(\bar{S}_n | \xi_n))] = E[E(S_n | \xi_n) E(\bar{S}_n | \xi_n)].$$

We see that, by recurrence we have

$$E(S_{2^r}^2) = 2^r \left[E(X_0^2) + \sum_{j=0}^{r-1} \frac{1}{2^j} E[E(S_{2^j} | \xi_{2^j}) E(\bar{S}_{2^j} | \xi_{2^j})] \right]. \quad (17)$$

Now, by Hölder's inequality and stationarity,

$$|E[E(S_n | \xi_n) E(\bar{S}_n | \xi_n)]| \leq \|E(S_n | \xi_n)\| \cdot \|E(\bar{S}_n | \xi_n)\|,$$

and (16) follows.

2. Limit on a subsequence

Note that, if $\sup_r \Delta_{2^r} < \infty$, then $\sum_{j=0}^{r-1} 2^{-j} E[E(S_{2^j} | \xi_0) E(\bar{S}_{2^j} | \xi_{2^j})]$ converges as $r \rightarrow \infty$, say to L . Then, by (17), we have that

$$\frac{1}{2^r} E(S_{2^r}^2) \rightarrow \sigma^2 \text{ as } r \rightarrow \infty, \quad (18)$$

where $\sigma^2 = E(X_0^2) + L$.

3. Limiting variance for $S_n / \sqrt{n}$

We show here that if conditions (4) and (5) hold then (2) holds.

For $2^{r-1} \leq n < 2^r$, we use the binary expansion

$$n = \sum_{k=0}^{r-1} 2^k a_k \quad \text{where} \quad a_{r-1} = 1 \quad \text{and} \quad a_k \in \{0, 1\}. \quad (19)$$

Then, we apply the following representation

$$S_n = \sum_{j=0}^{r-1} U_{2^j} a_j \quad \text{where} \quad U_{2^j} = \sum_{i=n_{j-1}+1}^{n_j} X_i, \quad n_j = \sum_{k=0}^j 2^k a_k, \quad n_{-1} = 0. \quad (20)$$

Clearly, for $a_j = 0$, $U_{2^j} = 0$.

Then we use the representation

$$E(S_n^2) = \sum_{i=0}^{r-1} a_i E(U_{2^i}^2) + \sum_{i \neq j=0}^{r-1} a_i a_j E(U_{2^i} U_{2^j}) \equiv I_n + J_n. \quad (21)$$

Now, by stationarity, the representation of n in (19) and the convergence in (18) we obtain

$$\frac{I_n}{n} = \frac{1}{n} \sum_{i=0}^{r-1} a_i 2^i \left(\frac{E(S_{2^i}^2)}{2^i} - \sigma^2 \right) + \sigma^2 \rightarrow \sigma^2 \text{ as } n \rightarrow \infty.$$

It remains to prove that $|J_n|/n \rightarrow 0$. Let $0 \leq i < j < r$. Then, by the properties of Markov chains and Hölder's inequality

$$|E(U_{2^i} U_{2^j})| \leq |E(U_{2^i} | \xi_{n_i}) E(U_{2^j} | \xi_{n_{j-1}})| \leq \|E(S_{2^i} | \xi_{2^i})\| \cdot \|E(S_{2^j} | \xi_0)\|. \quad (22)$$

Hence,

$$\begin{aligned} |J_n| &\leq \frac{1}{2} \sum_{0 \leq i \neq j \leq r-1} 2^{i/2} 2^{j/2} \left(\frac{\|E(S_{2^i} | \xi_{2^i})\|^2}{2^i} + \frac{\|E(S_{2^j} | \xi_0)\|^2}{2^j} \right) \leq \\ &\frac{1}{2(\sqrt{2}-1)} 2^{r/2} \left(\sum_{i=0}^{r-1} 2^{i/2} \frac{\|E(S_{2^i} | \xi_{2^i})\|^2}{2^i} + \sum_{j=0}^{r-1} 2^{j/2} \frac{\|E(S_{2^j} | \xi_0)\|^2}{2^j} \right). \end{aligned}$$

We can easily see that $E|J_n|/n \rightarrow 0$ because of (14) and (15). $\square$

Proof of points (b) and (c) of Theorem 5

For obtaining the points (b) and (c) of Theorem 5, we shall combine the result in point (a) with a new CLT for additive functionals of Markov chains given by Peligrad (2020), namely:

Theorem 11 *Peligrad (2020). Assume that if the chain is totally ergodic and*

$$\sup_{n \geq 1} \frac{E(S_n^2)}{n} < \infty. \quad (23)$$

Then, the following limit exists

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|S_n - E(S_n | \xi_0, \xi_n)\|^2 = \theta^2 \quad (24)$$

and

$$\frac{S_n - E(S_n | \xi_0, \xi_n)}{\sqrt{n}} \Rightarrow N(0, \theta^2).$$

Note that, since (2) holds, then clearly (23) holds and in addition (24) implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|E(S_n | \xi_0, \xi_n)\|^2 = \sigma^2 - \theta^2.$$

It follows that points (b) and (c) of Theorem 5 follow with $\eta^2 = \sigma^2 - \theta^2$.

Proof of Corollary 9.

By the properties of the conditional expectation we see that condition (12) implies that (4) and (5) are satisfied and therefore, by point (b) of Theorem 5, the limit in (10) exists. If this limit is not 0, note that (12) cannot be satisfied. Therefore (12) implies that the limit in (10) is 0. We can apply now Theorem 3.1 in Billingsley (1999) to conclude that in (11) the random centering is not needed if we assume (12).

5 Auxiliary results

The following lemma holds for any subadditive sequence $(V_m)_{m \geq 1}$ of positive numbers. Its proof is inspired by Lemma 2.8. in Peligrad and Utev (2005). Because of the subtle differences we shall give it here. The main difference is that the sequence V_m^2 is not subadditive.

Lemma 12 *For any positive subadditive sequence $(V_m)_{m \geq 1}$ of positive numbers we have*

$$\sum_{i \geq 1} \frac{V_{2^i}^2}{2^i} \leq 65 \sum_{k \geq 1} \frac{V_k^2}{k^2}.$$

Proof. We recall first a property on the page 806 in Peligrad and Utev (2005). For a finite set of real numbers C , denote by $|C|$ its cardinal. Consider a positive integer N and the set

$$A_N = \{1 \leq i \leq N : V_i \geq V_N/2\}.$$

Property : $|A_N| \geq N/2$; that is A_N contains at least $N/2$ elements.

We start the proof of this lemma by adding the variables in blocks in the following way:

$$\sum_{n \geq 2} \frac{V_n^2}{n^2} = \sum_{r \geq 0} \sum_{n=4^r+1}^{4^{r+1}} \frac{V_n^2}{n^2} \geq \sum_{r \geq 0} \frac{1}{4^{2r+2}} \sum_{n=4^r+1}^{4^{r+1}} V_n^2. \quad (25)$$

Define

$$C_r = \{n \in \{4^r + 1, \dots, 4^{r+1}\} : V_n \geq V_{4^{r+1}}/2\} = A_{4^{r+1}} \cap \{4^r + 1, \dots, 4^{r+1}\}.$$

Note that, by applying the above property with $N = 4^{r+1}$, it is easy to see that

$$|C_r| \geq |A_{4^{r+1}}| - |\{1, 2, \dots, 4^r\}| \geq 4^{r+1}/2 - 4^r = 4^r.$$

It follows that

$$\sum_{n \geq 2} \frac{V_n^2}{n^2} \geq \sum_{r \geq 0} \frac{4^r}{4^{2r+4}} V_{4^{r+1}}^2 \geq \frac{1}{64} \sum_{r \geq 1} \frac{1}{4^r} V_{4^r}^2,$$

which implies

$$\sum_{r \geq 1} \frac{1}{2^{2r}} V_{2^{2r}}^2 \leq 64 \sum_{n \geq 1} \frac{V_n^2}{n^2}.$$

Then, by the subadditivity property, we have $V_{2^{2r+1}} \leq 2V_{2^{2r}}$, so that

$$\sum_{r \geq 1} \frac{1}{2^{2r+1}} V_{2^{2r+1}}^2 \leq \sum_{r \geq 1} \frac{1}{2^{2r}} V_{2^{2r}}^2$$

and, as a consequence

$$\sum_{r \geq 1} \frac{1}{2^{2r}} V_{2^{2r}}^2 + \sum_{r \geq 1} \frac{1}{2^{2r+1}} V_{2^{2r+1}}^2 \leq 65 \sum_{n \geq 1} \frac{V_n^2}{n^2}$$

and the proof is complete. $\square$

Next lemma contains examples of subadditive sequences which are relevant for the proofs.

Lemma 13 *For any stationary Markov chain the sequences $(\|E(S_n|\xi_0)\|)_{n \geq 0}$, $(\|E(S_n|\xi_n)\|)_{n \geq 0}$ and $(\|E(S_n|\xi_0, \xi_n)\|)_{n \geq 0}$ are all subadditive.*

Proof. The proofs are similar. So it is enough to sketch the proof of only one of them. By the triangle inequality, Markov property and the properties of conditional expectation, for all positive integers m and n ,

$$\begin{aligned} \|E(S_{n+m}|\xi_0, \xi_{n+m})\| &\leq \|E(S_n|\xi_0, \xi_{n+m})\| + \|E(S_{n+m} - S_n|\xi_0, \xi_{n+m})\| \\ &= \|E(S_n|\mathcal{F}_0, \mathcal{F}^{n+m})\| + \|E(S_{n+m} - S_n|\mathcal{F}_0, \mathcal{F}^{n+m})\| \\ &\leq \|E(S_n|\mathcal{F}_0, \mathcal{F}^n)\| + \|E(S_{n+m} - S_n|\mathcal{F}_n, \mathcal{F}^{n+m})\| \end{aligned}$$

and the result follows. $\square$

Finally we give a lemma for sequences of real numbers.

Lemma 14 *Let $(a_i)_{i \geq 1}$ be a sequence of real numbers. Then, for any $m \geq 1$*

$$A_m := \sum_{k \geq 1}^m \frac{1}{k^2} \left(\sum_{i=1}^k a_i \right)^2 \leq 4 \sum_{i=1}^m a_i^2.$$

Proof. Note that

$$\begin{aligned} \left(\sum_{i=1}^k a_i \right)^2 &= \sum_{i=1}^k a_i^2 + 2 \sum_{i=1}^k a_i \sum_{j=1}^{i-1} a_j \leq \\ &2 \sum_{i=1}^k a_i \sum_{j=1}^i a_j. \end{aligned}$$

By changing the order of summation, taking into account that for $i \geq 1$ we have $\sum_{k \geq i} k^{-2} \leq 2i^{-1}$ and applying the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \sum_{k=1}^m \frac{1}{k^2} \sum_{i=1}^k a_i \sum_{j=1}^i a_j &= \sum_{i=1}^m a_i \sum_{j=1}^i a_j \sum_{k \geq i} \frac{1}{k^2} \\ &\leq 2 \sum_{i=1}^m |a_i| \frac{1}{i} \left| \sum_{j=1}^i a_j \right| \leq 2 \left(\sum_{i=1}^m a_i^2 \right)^{1/2} \left(\sum_{i=1}^m \frac{1}{i^2} \left| \sum_{j=1}^i a_j \right|^2 \right)^{1/2}. \end{aligned}$$

So

$$A_m \leq 2 \left(\sum_{i=1}^m a_i^2 \right)^{1/2} A_m^{1/2},$$

which completes the proof of the lemma. $\square$

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$\square$

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