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Response solutions to the quasi-periodically forced systems with degenerate equilibrium: a simple proof of a result of W Si and J Si and extensions

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Abstract

We give a simple proof of the existence of response solutions in some quasi-periodically forced systems with degenerate fixed points. The same questions were answered by Si and Si (2018 *Nonlinearity* **31** 2361–18) using two versions of Kolmogorov–Arnold–Moser (KAM) theory. Our method is based on reformulating the existence of response solutions as a fixed point problem in appropriate spaces of smooth functions. By algebraic manipulations, the fixed point problem is transformed into a problem dealt with contraction mapping principle. Compared to the KAM method, the present method does not incur a loss of regularity. That is, the solutions we obtain have the same regularity as the forcing. Moreover, the method here applies when problems are only finitely differentiable. It also weakens slightly the non-resonance conditions on the forcing frequencies. Since the method is based on the contraction mapping principle, we also

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obtain automatically smooth dependence on parameters and, when studying complex versions of the problem we discover the new phenomenon of monodromy. We also present results for higher dimensional systems, but for higher dimensional systems, the concept of degenerate fixed points is much more subtle than in one dimensional systems. To illustrate the power of the method, we also consider two problems not studied by Si and Si (2018 *Nonlinearity* **31** 2361–18): the forcing with zero average and second order oscillators. We show that in the zero average forcing case, the solutions are qualitatively different, but for the second order oscillators are remarkably similar.

Keywords: degenerate fixed points, response solutions, fixed point theorem, second order oscillators

Mathematics Subject Classification numbers: 34D10, 34G20, 42B30, 47H10.

1. Introduction

The goal of this paper is to find response solutions to quasi-periodically forced systems with degenerate fixed points. The main technique we use is the contraction mapping theorem in carefully chosen Banach spaces.

1.1. The one-dimensional model

The 1-D version of the problem (the higher dimensional version of the problem will be formulated in section 6) is the following:

$$\dot{x} = x^l + h(\omega t, x) + \varepsilon f(\omega t, x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $l \in \mathbb{N}$ with $l \geqslant 2$, $0 < |\varepsilon| \ll 1$ is a small real parameter (the small adaptations needed for considering ε complex will be discussed in section 5), and ω is a vector in $\mathbb{R}^d$ with $d \in \mathbb{N}$. The function h is assumed to vanish in x to order higher than l. In the analytic case, vanishing to high order just means that $h(\theta, x) = x^{l+1}H(\theta, x)$ with H an analytic function. In the finitely differentiable case, we will just need that $\partial_x^j h(\theta, 0) = 0$ for $j = 0, 1, 2, \ldots, l$ (we will also need that all the derivatives up to a sufficiently high order are bounded for all x in a neighbourhood of the origin).

The functions will be assumed to have some regularity properties, which we will detail once we have detailed the spaces in which we will formulate the problem.

In our method the lower order terms do not play any important role and can get incorporated in f by scaling. We will keep it in the model to facilitate the comparison with the paper [SS18] but we advice the reader that all the terms that come from it will be subdominant.

The model (1.1) represents physically the forcing of a (one dimensional) fixed point which is degenerate. We recall that '*response solutions*' means solutions that have the same frequency as the forcing. The standard definition of quasi-periodic functions are functions of time of the form (2.2). Hence, the problem we are considering is to produce solutions of (1.1) of the form (2.2).

1.2. Assumptions in the frequency

Without loss of generality, we assume that, for $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$,

$$k \cdot \omega \neq 0, \quad \text{for } k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \setminus \{0\}, \tag{1.2}$$

where $k \cdot \omega = \sum_{i=1}^{d} k_i \omega_i$. Indeed, if there is a $k_0 \in \mathbb{Z}^d \setminus \{0\}$ such that $k_0 \cdot \omega = 0$, we could reformulate the forcing with only $(d-1)$-dimensional variables which are orthogonal to k_0.

In many related problems, one needs to assume not only (1.2) but also lower bounds on $|k \cdot \omega|$. It is remarkable that for the main results of this paper (and in [SS18]) the only requirement on ω is (1.2). Hence, the results hold without any assumption in the frequencies. In the study of some very degenerate results (not considered in [SS18]), we will impose some non-resonance conditions (some rather weak Diophantine properties (7.10) for the analytic case and the generally Diophantine properties for the finitely differentiable case). (See the section 7.)

1.3. The results in this paper

We will produce two main results for model (1.1), one assuming analytic regularity in the problem (see theorem 8) and another one for finite regularity (see theorem 12). These two results are aimed at the real parameter ε.

We will also consider the case of complex parameter ε and establish monodromy. (See section 5 for more details.) Moreover, we will consider analogues of (1.1) in higher dimensions and establish results in analytic and finite regularity. (See theorem 14 in section 6.)

We will also present results on the case of zero average forcing and on oscillators, which are second order problems and, in principle a singular perturbation. Remarkably, we obtain that in the case of zero average, the solutions are qualitatively different (see section 7), but in the oscillator case, the solutions are similar to the solutions in the first order (see section 8).

1.4. Relation to other papers

The same problem was studied in many other papers. In particular, it was studied in [SS18], using two versions of Kolmogorov–Arnold–Moser (KAM) theory. We refer to the comprehensive introduction of [SS18] for a review of related literature on the problem and other methods used to study it.

The method of this paper is very different from the method of [SS18] and the methods in other papers referred in [SS18]. The basic idea of our method is that we formulate the existence of response solutions as functional equation, which we manipulate till it becomes a fixed point in an appropriate space of functions. Algebraic manipulations transform the fixed point problem into a fixed point for contractions.

We anticipate that, perhaps, the most delicate step on our argument is the choice of spaces since we want that they satisfy several properties (see section 3). Similar methods had also been used in other response solution problems [CCdlL13, CCCdlL17, WdlL20]. In particular, we will follow the notation of [WdlL20] and refer to that paper for standard technical details (for example, well known properties of Sobolev spaces).

Eliminating the sophisticated KAM iteration allows us to deal straightforwardly with cases in which the problem is only finitely differentiable, and obtain automatically smooth dependence on parameters. Also the solutions produced have the same regularity as the forcing and we do not incur the loss of regularity that appears in KAM iteration.

The assumptions on the order of vanishing we use is slightly weaker than in [SS18]. We also weaken the non-resonance assumptions in the case that l is even. We do not need to assume a sign for the average, but in the even case, we need to restrict the values of ε. See the discussion of (2.7). In section 6 we obtain analogues of the results in higher dimensions. Since the proofs we present are based on soft methods, they also work for infinitely dimensional problems. The method allows to discuss complex values of the parameters. The use of the complex values for ε leads to the new phenomenon of '*monodromy*', which we study in section 5. We also consider some problems not considered in [SS18], namely, the case of zero average forcing (section 7) and second order degenerate oscillators (section 8).

1.5. Organization of this paper

This paper is organized as follows: in section 2, we present the main idea of reformulating the existence of response solutions for equation (1.1) as a fixed point problem. To solve this fixed point equation, in section 3, we give the precise function spaces that we work in and we list their important properties, such as Banach algebra property and composition operator. We state our main results and present the concrete proof in section 4. In section 5 we study the case of complex parameters and the monodromy phenomenon. In section 6, we deal with the generally high-dimensional system. In section 7, we generalize the system (1.1) to the one whose forcing is zero average. In section 8, we study the degenerate second oscillators. For the oscillators model, we just make some changes of variable to reduce this model to the one like (2.9) for model (1.1).

2. Overview of the method in one-dimensional system

In this section, we discuss heuristically the main ideas of our treatment. We will present in this section only the formal manipulations ignoring questions of domains etc. Those will be discussed later but indeed, the formal manipulations of this section, will be the motivations for the precise definitions later.

2.1. A guide

The manipulations we perform are rather systematic and very common in nonlinear analysis. We firstly identify what we expect to be the main part of the solution (in our case a constant). If we write the unknown as the guess plus an unknown correction, we see that the original equation is equivalent to an equation for the correction. We furthermore observe that the equation for the correction has a main part that can be inverted, then, we are left with a fixed point problem that has a good chance of being a contraction. Of course, identifying what are the main parts of the solution requires some experimentation (and some luck), but checking that a guess is the correct one, can be done systematically.

2.2. Some elementary notations

For a function $f : \mathbb{T}^d \times \mathbb{R}^n \to \mathbb{R}^n$, we denote:

$$\begin{aligned} \overline{f}(x) &:= \int_{\mathbb{T}^d} f(\theta, x) \mathrm{d}\theta, \\ \widetilde{f}(\theta, x) &:= f(\theta, x) - \overline{f}(x). \end{aligned} \tag{2.1}$$

We refer to $\overline{f}$ as the average of f with respect to θ and $\widetilde{f}$ as the oscillatory part of f.

We look for quasi-periodic solutions with forcing frequency $\omega \in \mathbb{R}^d$. They are functions of time t with the form

$$x(t) = a + V(\omega t), \tag{2.2}$$

where $a \in \mathbb{R}$ is a number and $V : \mathbb{T}^d \to \mathbb{R}$ is a function to be determined. Note that representation of the function $x(t)$ is not unique. From $a_1 + V_1(\omega t) = a_2 + V_2(\omega t)$, we can only conclude that $\widetilde{V_1} = \widetilde{V_2}$, $a_1 - a_2 = \overline{V_2} - \overline{V_1}$.

Remark 1. A good heuristic guide to guess that the dominative term in the response function (2.2) is a constant is the '*averaging principle*' (presented and partially justified in

[Min62, BM61, Hal80]) which suggests that one substitutes the forcing terms by their averages to obtain the leasing approximations. Of course, the present paper can be considered as another justification of the method.

In our case, the averaged equations of the system (1.1) are:

$$\dot{x} = x^l + \varepsilon \overline{f}(x)$$

and the equilibrium is obtained by solving $x^l + \varepsilon \overline{f}(x) = 0$, which we can further approximate by $x^l + \varepsilon \overline{f}(0) = 0$.

Note that the case $\overline{f}(0) = 0$ is a situation where the averaging principle does not provide any guidance and indeed, we will see that the leading part has a different form and, hence, the solutions in this case are qualitatively different from those with non-zero average forcing. (See section 7.)

Remark 2. Note that we depart slightly from the notation of [SS18]. We write the forcing as $\varepsilon f(\omega t, x)$. The paper [SS18] writes the forcing as $f(\omega t, x; \varepsilon)$.

The paper [SS18] presents two main theorems about analytic functions.

Theorem 3.1 in [SS18] assumes Diophantine condition

$$|k \cdot \omega| \geqslant \gamma / \Omega(|k|), \qquad \ln(\Omega(t))/t \to 0 \tag{2.3}$$

and a sign on the average. We do not need any conditions in ω.

In theorem 3.2 of [SS18], the Diophantine conditions (2.3) are eliminated, but there are two new assumptions:

- That the function agrees with the average to order ε^2, see (3.6) in [SS18]. In our notation, this amounts to $\widetilde{f}(\theta, 0) = 0$ (we only need it is small enough).
- $h = \mathcal{O}(x^{2l})$, we assume $h = \mathcal{O}(x^{l+1})$.

2.3. The invariance equations

Substituting (2.2) into equation (1.1) and using that $\{\omega t\}_{t\in\mathbb{R}}$ is dense in $\mathbb{T}^d$, we obtain that (1.1) holds for a continuous function x if and only if a and V satisfy

$$\begin{aligned}(\omega \cdot \partial_\theta)\, V(\theta) &= (a + V(\theta))^l + h(\theta, a + V(\theta)) + \varepsilon f(\theta, a + V(\theta)) \\ &= a^l + l a^{l-1} V(\theta) + S(a, V(\theta)) + h(\theta, a + V(\theta)) \\ &\quad + \varepsilon \overline{f}(0) + \varepsilon \widetilde{f}(\theta, 0) + \varepsilon g(\theta, a + V(\theta)),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned} S(a, V) &= (a + V)^l - a^l - l a^{l-1} V, \\ g(\theta, x) &= f(\theta, x) - f(\theta, 0). \end{aligned} \tag{2.5}$$

Note that the equation (2.4) is slightly undetermined because of the lack of uniqueness in the representation (2.2). This undetermination will be useful for us.

2.4. An important assumption

A crucial assumption in our treatment (as well as that in [SS18]) is:

$$\overline{f}(0) \neq 0. \tag{2.6}$$

The importance of the assumption (2.6) is that the leading term in the response solution will be a constant. Moreover, we will modify the method for the case that $\overline{f}(0) \neq 0$ to study the situation when $\overline{f}(0) = 0$ but the results (i.e. the form of the solutions) are qualitatively different. (See section 7.)

2.5. The leading term of the solution

Our first step is to choose a in (2.2) such that:

$$a^l + \varepsilon\overline{f}(0) = 0. \tag{2.7}$$

Note that this choice is possible in several cases. If l is odd, we can find such an a solving (2.7) for all ε real. If l is even, we can find a solving (2.7) for all ε such that $\varepsilon\overline{f}(0)$ has negative sign. Depending on the sign of $\overline{f}(0)$ we obtain solutions in the positive real interval or in the negative real interval.

In the even l case, we obtain two solutions in the appropriate interval of ε. Each of them could be taken as the basis to find the corrections V so that we get two response solutions. As we vary ε, we obtain two branches of solutions.

We note that finding a as above makes sense even for values of ε which are complex, provided, of course, that we allow for complex valued solutions. In section 5, we will take up the issue of complex values of ε. The use of complex values allows for much more topology and we discover the phenomenon of '*monodromy*'.

Once we have accomplished finding an a which eliminates several terms in (2.4), we study the remaining equation. We find it convenient to introduce the linear operator:

$$\mathcal{L}_a := \omega \cdot \partial_\theta - la^{l-1} \tag{2.8}$$

defined on one-dimensional periodic functions of $\theta \in \mathbb{T}^d$.

2.6. The equation for the corrections

Using the choice of a in (2.7) and the notation (2.8), we see that the equation (2.4) is equivalent to the following equation for V:

$$\mathcal{L}_a(V(\theta)) = S(a, V(\theta)) + h(\theta, a + V(\theta)) + \varepsilon\widetilde{f}(\theta, 0) + \varepsilon g(\theta, a + V(\theta)). \tag{2.9}$$

If we select spaces in which $\mathcal{L}_a$ is boundedly invertible, then the equation (2.9) can be transformed into:

$$\begin{aligned} V(\theta) &= \mathcal{L}_a^{-1}\left(S(a, V(\theta)) + h(\theta, a + V(\theta)) + \varepsilon\widetilde{f}(\theta, 0) + \varepsilon g(\theta, a + V(\theta))\right) \\ &\equiv \mathcal{T}_a(V)(\theta). \end{aligned} \tag{2.10}$$

We will show that we can apply the contraction mapping principle to the equation (2.10) once we identify appropriate Banach spaces and a ball in them mapped to itself by the operator $\mathcal{T}_a$ defined in (2.10). In the following section, we will make the choice of spaces explicitly.

3. Choice of spaces and some preliminary results on them

To make precise the calculations in section 2, we just need to choose appropriate function spaces and check that we can carry the steps indicated formally there and indeed obtain that $\mathcal{T}_a$ is a contraction in a ball.

3.1. Some preliminary considerations

There are a few guiding principles in the choice of spaces:

- The norms of the functions in the spaces can be read off from the size of the Fourier coefficients. In such a way, the norm of the operator $\mathcal{L}_a$ defined in (2.8), which is diagonal in Fourier series, can be estimated very precisely from one space in the class to itself.
- The spaces have to possess good Banach algebra properties for multiplication so that one can perform nonlinear analysis.
- The operator of composition in the left can be estimated.

With the above considerations, it is reasonable to consider the following well known spaces which have been found useful in many nonlinear problems (in particular, they were used in problems similar to ours in [CCdlL13, CCCdlL17, WdlL20]).

3.2. Some standard spaces we will use

For $\rho \geqslant 0$, we denote by

$$\mathbb{T}^d_\rho = \left\{ \theta \in \mathbb{C}^d/(2\pi\mathbb{Z})^d \, : \, \mathrm{Re}(\theta_j) \in \mathbb{T}, \quad \sup_{j=1,\ldots,d} |\mathrm{Im}(\theta_j)| \leqslant \rho \right\}.$$

We denote the Fourier expansion of a periodic function $f(\theta)$ on $\mathbb{T}^d_\rho$ by

$$f(\theta) = \sum_{k\in\mathbb{Z}^d} \widehat{f}_k \, \mathrm{e}^{\mathrm{i}k\cdot\theta},$$

where $k\cdot\theta = \sum_{j=1}^d k_j\theta_j$ represents the Euclidean product in $\mathbb{C}^d$ and $\widehat{f}_k$ are the Fourier coefficients of f.

Definition 3. For $\rho \geqslant 0$, $m \in \mathbb{N}$, we denote by $H^{\rho,m}$ the space of analytic functions V in $\mathbb{T}^d_\rho$ with finite norm:

$$\begin{aligned} H^{\rho,m} &:= H^{\rho,m}(\mathbb{T}^d, \mathbb{C}^n) \\ &= \left\{ V : \, \mathbb{T}^d_\rho \to \mathbb{C}^n \mid \|V\|^2_{H^{\rho,m}} = \sum_{k\in\mathbb{Z}^d} |\hat{V}_k|^2 \, \mathrm{e}^{2\rho|k|}(|k|^2+1)^m < +\infty \right\}. \end{aligned}$$

It is obvious that the space $\left(H^{\rho,m}, \ \|\cdot\|_{H^{\rho,m}}\right)$ is a Banach space and indeed a Hilbert space. From the real analytic point of view, we consider the Banach space $H^{\rho,m}$ of the functions that take real values for real arguments. This is Banach space over the reals.

For $\rho = 0$, $H^m := H^{0,m}(\mathbb{T}^d, \mathbb{R}^n)$ is the standard Sobolev space, we refer to the reference [Tay97] for more details. Moreover, when $m > \frac{d}{2}$, by the Sobolev embedding theorem, we obtain that $H^{m+p}(\mathbb{T}^d, \mathbb{R}^n)$ $(p = 1, 2, \ldots)$ embeds continuously into $C^p(\mathbb{T}^d, \mathbb{R}^n)$.

For $\rho > 0$, functions in the space $H^{\rho,m}$ are analytic in the interior of $\mathbb{T}^d_\rho$ and extend to Sobolev functions of order m on the boundary of $\mathbb{T}^d_\rho$. For $m > d$, the space $H^{\rho,m}$ can also be considered as closed subspace of the Sobolev space in the $2d$-dimensional real manifold with boundary $\mathbb{T}^d_\rho$.

3.3. Some standard properties of the Sobolev spaces $H^{\rho,m}$

It is well known that the Sobolev spaces $H^{\rho,m}$ defined above satisfy the Banach algebra property for large enough m (we refer to [Tay97] for more details).

Lemma 4 (Banach algebra properties). *We have the following properties in two cases:*

(*a*) *Sobolev case: for $\rho = 0$, $m > \frac{d}{2}$, there exists a constant $C_{m,d} > 0$ depending only on m, d such that for $V_1, V_2 \in H^m$, the product $V_1 \cdot V_2 \in H^m$ and*

$$\|V_1 V_2\|_{H^m} \leqslant C_{m,d}\|V_1\|_{H^m}\|V_2\|_{H^m}.$$

(*b*) *Analytic case: for $\rho > 0, m > d$, there exists a constant $C_{\rho,m,d} > 0$ depending on ρ, m, d such that for $V_1, V_2 \in H^{\rho,m}$, the product $V_1 \cdot V_2 \in H^{\rho,m}$ and*

$$\|V_1 V_2\|_{H^{\rho,m}} \leqslant C_{\rho,m,d}\|V_1\|_{H^{\rho,m}}\|V_2\|_{H^{\rho,m}}.$$

In particular, $H^{\rho,m}$ is a Banach algebra when ρ, m, d are as above.

It is interesting to remark that the value of m is what controls the Banach algebra properties (which are crucial for us). On the other hand, for regularity, the parameter ρ is much more relevant. For a KAM argument, one could use many different sets of spaces since the Newton method would overcome all these difficulties. The present method of using only a contraction argument is much more restrictive on the spaces we use since we cannot lose any regularity in the iterative step and we also need some Banach algebra properties.

The Banach spaces $H^{\rho,m}$ seem a good compromise between having norms given by Fourier coefficients (which makes the linear estimates efficient) and having Banach algebra properties. They are also Hilbert spaces which makes spectral theory particularly powerful. These properties have been found useful in several areas such as quantum field theory.

The following results on composition are also rather standard.

Lemma 5 (composition properties). *Assume that $\rho > 0$.*

Let $g : \mathbb{T}^d_\rho \times B \to \mathbb{C}^n$ with B being an open ball around the origin in $\mathbb{C}^n$ and assume that g is analytic in $\mathbb{T}^d_\rho \times B$.

Then, for $V \in H^{\rho,m}(\mathbb{T}^d_\rho, \mathbb{C}^n) \cap L^\infty(\mathbb{T}^d_\rho, \mathbb{C}^n)$ with $V(\mathbb{T}^d_\rho) \subset B$, we have

$$\|g(\theta, V)\|_{H^{\rho,m}} \leqslant C_{\rho,m}\left(\|V\|_{L^\infty}\right)\left(1 + \|V\|_{H^{\rho,m}}\right) \tag{3.1}$$

for some $C_{m,d} > 0$ depending on the norm of V. Moreover, when $m > d$,

$$\begin{aligned}&\|g(\theta, V + W) - g(\theta, V) - D_V g(\theta, V) \cdot W\|_{H^{\rho,m}}\\ &\quad\leqslant C_{\rho,m,d}\left(\|V\|_{L^\infty}\right)\left(1 + \|V\|_{H^{\rho,m}}\right)\|W\|^2_{H^{\rho,m}}.\end{aligned} \tag{3.2}$$

In the case that $\rho = 0$, it suffices to assume that $g \in C^{m+2}$ in real neighbourhood and that $m > d/2$. Then, we have (3.1) and (3.2).

The results in lemma 5 are somewhat standard. For the sake of completeness, we give some sketch. Many details, counterexamples for related statements, etc are in [AZ90, IKT13, Tay97] or in [WdlL20].

The reason for the inequality (3.2) is that, by the fundamental theorem of calculus

$$g(\theta, V(\theta) + W(\theta)) - g(\theta, V(\theta)) - D_V g(\theta, V(\theta)) \cdot W(\theta) = \int_0^1 \int_0^1 t D_V^2 g(\theta, V(\theta) + stW(\theta)) \cdot W^2(\theta) \mathrm{d}s \ \mathrm{d}t.$$

Then, we get the desired result by the facts that $D_V^2 g(\theta, V(\theta) + stW(\theta)) \in H^{\rho,m}$ and its $H^{\rho,m}$ norm is bounded uniformly in t, s and that $H^{\rho,m}$ is a Banach algebra under multiplication by lemma 4 and using (3.1) for the second derivative.

To establish the standard inequality (3.1), it suffices to use the Faa di Bruno formula for derivatives and then, the Moser–Nirenberg inequalities for products of derivatives.

Remark 6. We call attention that we are considering only the cases when the Sobolev embedding theorem applies and the functions we are considering are bounded. This allows the consequence that the bounds in (3.2) are the bounds of the derivatives of g in the range of the functions considered.

Remark 7. Note that, in the case of analytic regularity, (3.2) establishes that the left composition operator $\mathcal{C}_g(V)(\theta) := g(\theta, V(\theta))$, considered as a function from the space $H^{\rho,m}$ to itself, is differentiable. This shows that the composition operator $\mathcal{C}_g$ is analytic.

Note also that (3.2) establishes that the derivative is the multiplication by the another left composition. Hence, we can apply the same result to obtain higher differentiability properties (under appropriate hypothesis). This shows that if $g \in C^{p+m+2}$ with $p = 0, 1, \ldots$ and $m > d/2$, the left composition operator $\mathcal{C}_g$ is C^{p+1} acting on the space H^m. We refer to [AZ90].

4. Existence of response solutions for one-dimensional system

In this section, we implement the strategy discussed at the beginning of section 2 using the spaces discussed in section 3.

4.1. Analytic case

In this section, we state the main result and the corresponding proof for the model (1.1) in which the forcing is analytic.

4.1.1. The main result. **Theorem 8.** *We study the equation (1.1) with h vanishing to order $(l+1)$ at zero.*

Assume that f, h are analytic in $\mathbb{T}^d_\rho \times B$ with B being an open ball around the origin in the space $\mathbb{C}$ and $\widetilde{f}(\theta, 0) \in H^{\rho,m}(\mathbb{T}^d, \mathbb{C})$ for some $\rho > 0, m > d$.

If (2.6) holds and $\|\widetilde{f}(\theta, 0)\|_{H^{\rho,m}}$ is small enough compared to $|\overline{f}(0)|$, then, there exists a $\varepsilon_0 > 0$ such that, defining $\mathcal{I} = (-\varepsilon_0, \varepsilon_0)$ for l odd and $\mathcal{I} = (-\varepsilon_0, 0)$ when l even and $\overline{f}(0) > 0$, and $\mathcal{I} = (0, \varepsilon_0)$ when l even and $\overline{f}(0) < 0$, we have that for all $\varepsilon \in \mathcal{I}$, there exists a solution of (1.1) of the form (2.2) in $H^{\rho,m}$.

Moreover, the solution for equation (1.1) is locally unique.

By reading the proof in the following part, we obtain explicit estimates of the domain where local uniqueness holds. Roughly, they are domains of size $\approx |\varepsilon|^{1/l}$. This is consistent with the fact that in the case that l is even we obtain several solutions at this distance (or when we consider complex valued solutions).

When we have a locally unique solution for all values of ε, we can discuss the regularity with respect to the parameter ε. It follows that one can get that it is analytic in ε.

Remark 9. Note that the above regularity statement, does not include regularity at $\varepsilon = 0$. This seems possible under some very weak Diophantine properties such as (7.10). One obtains an approximate solution as a polynomial in ε and starts a contraction mapping around it. We omit a precise formulation and a proof. See [WdlL20, CCCdlL17].

4.1.2. Proof of theorem 8. In this section, we prove theorem 8 by considering the fixed point equation (2.10) in Banach space $H^{\rho,m}$.

It is easy to obtain the quantitative bounds on the inverse of $\mathcal{L}_a$ defined in (2.8) with a being the one in (2.7), as an operator from the space $H^{\rho,m}$ to itself. Indeed, when we write a function $V(\theta) \in H^{\rho,m}$ in the Fourier expansion as

$$V(\theta) = \sum_{k\in\mathbb{Z}^d} \hat{V}_k \, \mathrm{e}^{\mathrm{i}k\cdot\theta},$$

the operator $\mathcal{L}_a$ acting on the Fourier basis becomes

$$\mathcal{L}_a(\mathrm{e}^{\mathrm{i}\,k\cdot\theta}) = \left(\mathrm{i}(k\cdot\omega) - la^{l-1}\right)\mathrm{e}^{\mathrm{i}\,k\cdot\theta} = :L_a(k\cdot\omega)\mathrm{e}^{\mathrm{i}k\cdot\theta}$$

with $L_a(k\cdot\omega) = \mathrm{i}(k\cdot\omega) - la^{l-1}$.

Due to the fact that the norm in the space $H^{\rho,m}$ is characterized by the Fourier coefficients, we obtain that

$$\begin{aligned}\|\mathcal{L}_a^{-1}\|_{H^{\rho,m}\to H^{\rho,m}} &= \sup_{k\in\mathbb{Z}^d}\left|L_a^{-1}(k\cdot\omega)\right| = \sup_{k\in\mathbb{Z}^d}\frac{1}{|\mathrm{i}(k\cdot\omega) - la^{l-1}|}\\ &\leqslant \frac{1}{|la^{l-1}|} = \frac{1}{l|\overline{f}(0)|^{1-\frac{1}{l}}}|\varepsilon|^{-1+\frac{1}{l}}.\end{aligned} \tag{4.1}$$

Remark 10. In section 5, we will consider the case that a is complex.

We remark that when a is complex, we have, by the same argument

$$\|\mathcal{L}_a^{-1}\|_{H^{\rho,m}\to H^{\rho,m}} \leqslant 1/Re(la^{l-1}) = 1/\mathrm{dist}(la^{l-1}, \mathrm{i}\mathbb{R}), \tag{4.2}$$

where $\mathrm{dist}(la^{l-1}, \mathrm{i}\mathbb{R})$ is the distance between la^{l-1} and $\mathrm{i}\mathbb{R}$, which is $\mathrm{Re}(la^{l-1})$.

For simplicity, we will omit the subscript of $\|\mathcal{L}_a^{-1}\|_{H^{\rho,m}\to H^{\rho,m}}$ and use the notation $\|\mathcal{L}_a^{-1}\|$ in the following. We also simplify the notation $\|\cdot\|_{H^{\rho,m}}$ as $\|\cdot\|_{\rho,m}$ when there is no confusing.

We now look for a fixed point for the operator $\mathcal{T}_a$ defined in (2.10). Consider a ball $\mathcal{B}_r(0)$ around the origin in $H^{\rho,m}$ with radius $r > 0$. We will show that one can obtain r such that $\mathcal{T}_a(\mathcal{B}_r(0)) \subset \mathcal{B}_r(0)$ and $\mathcal{T}_a$ is a contraction on $\mathcal{B}_r(0)$.

For $S(a, V)$, $g(\theta, a + V)$ defined in (2.5), by the fact that one has that the Lipschitz constant of the nonlinear terms over a ball with radius r small is

$$\begin{aligned}&\mathrm{Lip}_V(S) \leqslant C|a|^{l-2}r,\\ &\mathrm{Lip}_V(h) \leqslant C(|a| + r)^l,\\ &\mathrm{Lip}_V(g) \leqslant C,\end{aligned}$$

where $\mathrm{Lip}_V(h)$ denotes the Lipschitz constant of $h(\theta, V)$ with respect to the second argument V in the ball of radius r, and C is a positive constant depending on l and f.

Note that in the contraction arguments, there are two conditions (that the ball gets mapped into itself and that the map is a contraction in the ball). We obtain two results: existence and uniqueness. The uniqueness result is stronger taking large balls and the existence is stronger for smaller balls. Hence, it is good to have some flexibility.

For any V_1, $V_2 \in \mathcal{B}_r(0)$, we have, assuming that r is small (remember that a is given by (2.7))

$$\begin{aligned}
&\|\mathcal{T}_a(V_1) - \mathcal{T}_a(V_2)\|_{\rho,m} \\
&\quad \leqslant \|\mathcal{L}_a^{-1}\| \left(\mathrm{Lip}_V(S) + \mathrm{Lip}_V(h) + |\varepsilon|\mathrm{Lip}_V(g)\right) \|V_1 - V_2\|_{\rho,m} \\
&\quad \leqslant C\|\mathcal{L}_a^{-1}\| \left(|a|^{l-2}r + (|a| + r)^l + |\varepsilon|\right) \|V_1 - V_2\|_{\rho,m}.
\end{aligned} \tag{4.3}$$

Note that we have used remark 6 to take advantage of the fact that some functions appearing in $\mathcal{T}_a$ vanish to a high order. The most delicate term above is the derivative of S which takes advantage of S not only being second order in V but also a being small.

Taking $|a| \approx |\varepsilon|^{1/l}$, $\|\mathcal{L}_a^{-1}\| \approx |\varepsilon|^{-1+1/l}$ into account, we see that if we take $r = A|\varepsilon|^{1/l}$ with A sufficiently small, it follows from (4.3) that $\mathcal{T}_a$ is a contraction of a factor $1/10$ in the ball of radius r for $|\varepsilon|$ sufficiently small.

Now we try to identify the conditions that the ball $\mathcal{B}_r(0)$ with r chosen as above gets mapped into itself for small ε.

If r satisfies the conditions that make $\mathcal{T}_a$ a contraction in $\mathcal{B}_r(0)$, we have:

$$\begin{aligned}
&\|\mathcal{T}_a(V)\|_{\rho,m} \\
&\quad \leqslant \|\mathcal{T}_a(0)\|_{\rho,m} + \|\mathcal{T}_a(V) - \mathcal{T}_a(0)\|_{\rho,m} \\
&\quad \leqslant \|\mathcal{L}_a^{-1}\| \left(|\varepsilon|\|\widetilde{f}(\theta,0)\|_{\rho,m} + \|h(\theta,a)\|_{\rho,m} + |\varepsilon|\|g(\theta,a)\|_{\rho,m}\right) + r/10 \\
&\quad \leqslant C|\varepsilon|^{-1+1/l}\left(|\varepsilon|\|\widetilde{f}(\theta,0)\|_{\rho,m} + |\varepsilon|^{1+2/l} + |\varepsilon|^{1+1/l}\right) + r/10.
\end{aligned} \tag{4.4}$$

Therefore under the assumption that

$$\|\widetilde{f}(\theta,0)\|_{\rho,m} \tag{4.5}$$

is small enough we obtain that $\mathcal{T}_a(\mathcal{B}_r(0)) \subset \mathcal{B}_r(0)$ and we already had that $\mathcal{T}_a$ is a contraction in this ball.

Remark 11. Note that the smallness assumption (4.5) depends on $|\overline{f}(0)|$. Indeed a more detailed analysis shows that we could write (4.5) as $\|\widetilde{f}(\theta,0)\|_{\rho,m}/|\overline{f}(0)|$ sufficiently small.

It follows from the fixed point theorem in the Banach space $H^{\rho,m}$ that there exists a unique solution $V \in H^{\rho,m}$ for equation (2.4). This produces a solution $x(t) = a + V(\theta)$ for equation (1.1). Notice that, once we fix a, the V is unique in the chosen ball. This shows that the solution $x(t) = a + V(\theta)$ of (1.1) is locally unique.

From the contraction mapping properties, we obtain easily regularity with respect to parameters, since the regularity of solutions of contraction mappings with parameters is standard. In particular, we note that the contraction mapping for analytic families is very standard.

4.2. The finitely differentiable case

Theorem 12. *We study the equation* (1.1) *with* h *vanishing to order* $(l+1)$ *at zero.*

Suppose that f, $h \in C^{m+p}(\mathbb{T}^d \times \mathbb{R}, \mathbb{R})$ $(p = 1, 2, \ldots)$ *and* $\widetilde{f}(\theta, 0) \in H^m(\mathbb{T}^d, \mathbb{R})$ *with* $m > \frac{d}{2}$.

If $\overline{f}(0) \neq 0$ *and* $\|\widetilde{f}(\theta, 0)\|_m$ *is sufficiently small compared to* $|\overline{f}(0)|$*, then, there exists a* $\varepsilon_0 > 0$ *such that, defining* $\mathcal{I} = (-\varepsilon_0, \varepsilon_0)$ *for l odd and* $\mathcal{I} = (-\varepsilon_0, 0)$ *when l even and* $\overline{f}(0) > 0$*, and* $\mathcal{I} = (0, \varepsilon_0)$ *when l even and* $\overline{f}(0) < 0$*, we have that for all* $\varepsilon \in \mathcal{I}$*, there exists a solution of* (1.1) *of the form of* (2.2) *in* $H^m(\mathbb{T}^d, \mathbb{R})$.

Moreover, the solution of equation (1.1) *is locally unique.*

The same strategy presented for theorem 8 applies also to the case that f is finitely differentiable (but with sufficiently high derivatives). Therefore, similar to the way in section 4.1 and together with lemma 5 in Sobolev case, we can easily prove theorem 12.

Remark 13. Using remark 7 we see that, in the analytic case (resp. when g is sufficiently differentiable), the operator $\mathcal{T}_a$ is analytic from $H^{\rho,m}$ to itself (resp. several times differentiable from H^m to itself) with m as in the main theorems.

Since the operator is differentiable with respect to ε we obtain that the solution produced depends analytically (resp. differentially) on parameters.

5. The case of complex ε. The phenomenon of monodromy

The previous analysis has shown that the leading term in the solution (2.2) is a constant. Note that we have shown that $\|V\|_{\rho,m}$ is much smaller than $|a|$.

The leading effect is the equation (2.7), which is an algebraic equation. The study of the algebraic equation is much more natural when all the variables are complex. Allowing complex values for a makes superfluous to distinguish between odd and even l, but it emphasizes that we can get more solutions.

Note that all the other arguments that we have developed to compute the correction V work just as well when they are complex valued.

An elementary remark is that if we consider a closed path in the ε plane $\varepsilon = \alpha \exp 2\pi \mathrm{i}t$, $t \in [0, 1]$, $\alpha \in \mathbb{R}$, we see that the solutions move only in a segment

$$a = (-\overline{f}(0)\alpha)^{1/l} \exp(2\pi \mathrm{i}s), \quad s \in [0, 1/l].$$

Hence, if we continue a while we vary ε along a circle, the a does not come to the same value. If we repeat the path above l times ($\varepsilon = \alpha \exp 2\pi \mathrm{i}t$, $t \in [0, l]$), then a gets back to the original value. This is the phenomenon of monodromy.

When we consider the nonlinear problem, we observe that we can not apply the contraction argument if a is close to the imaginary axis. On the other hand, in a region of the form $|\mathrm{Im}(\varepsilon)| \leqslant C|\mathrm{Re}(\varepsilon)|$, we obtain that $\mathrm{dist}(a, \mathrm{i}\mathbb{R})$ is comparable with $|\varepsilon|^{1/l}$. These regions in complex ε are geometrically a ball with $2l$ cones removed.

In these regions, the argument developed in this paper applies and we get the results. The solutions depend differentially and they are a small deformation of the solutions. Hence the space of the solutions contains a branch surface (minus some cuts).

Monodromy has appeared in other problems in degenerate perturbation theory [dlLT94, JdlLZ99], but the regions excluded are a more elaborate since the analysis is more elaborate.

We note that the fact that for complex ε we get several solutions at a distance $\mathcal{O}(|\varepsilon|^{1/l})$. This shows that one cannot hope to obtain contraction in larger balls by methods that work also for complex valued functions such as the soft methods employed here.

6. Higher dimensional phase space

In this section, we consider the existence of response solutions for the n-dimensional quasi-periodically forced system:

$$\dot{x} = \phi(x) + h(\omega t, x) + \varepsilon f(\omega t, x), \tag{6.1}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous function of degree l, i.e.

$$\phi(\lambda x) = \lambda^l \phi(x), \quad \lambda \in \mathbb{R}_+, \ x \in \mathbb{R}^n, \tag{6.2}$$

and h vanishes to order $(l + 1)$ in x. Of course, one important example of homogeneous functions is the polynomials all of whose terms have degree l, but there are other functions. The polynomials are precisely those that are $(l + 1)$ times differentiable at the origin, but it is natural to consider functions which are not differentiable at the origin. We note that the form (6.1) appears naturally when we are considering functions and expanding them in Taylor polynomials. We keep the lowest degree.

Note that the range of ϕ will be always a cone. We note also that for a homogeneous function, taking derivatives of (6.2), we have Euler's formula:

$$(D\phi)(\lambda x) = \lambda^{l-1} D\phi(x). \tag{6.3}$$

The strategy is very similar to the one used when $n = 1$. We assume (2.6) (in the sense that $\overline{f}(0) = \overline{f}_j(0)$ with $\overline{f}_j(0) \neq 0\,(j = 1, 2, \ldots, n)$) and that

$$\begin{aligned} &\overline{f}(0) \in \text{interior}(\text{range}(\phi)), \\ &\text{or} \\ &-\overline{f}(0) \in \text{interior}(\text{range}(\phi)). \end{aligned} \tag{6.4}$$

In the first case of (6.4), we will obtain results for all $0 < -\varepsilon \ll 1$ and in the second case, we will obtain results for $0 < \varepsilon \ll 1$. Of course, both cases can happen at the same time. We will introduce the following notation, for a positive constant ε_0,

$$\mathcal{I} = \begin{cases} [0, \varepsilon_0) \\ (-\varepsilon_0, 0] \\ (-\varepsilon_0, \varepsilon_0) \end{cases}$$

depending on whether only the first of (6.4) is true, only the second of (6.4) is true or both of (6.4) are true.

We indicate that the assumption (6.4) is an analogue in higher dimensions of the assumption (2.6) in the one dimensional phase space case.

Using (6.4) in the second case, we will be able to find $a_0 \in \mathbb{R}^n$ such that $\phi(a_0) = -\overline{f}(0)$ and hence, $a = \varepsilon^{1/l} a_0$ satisfies $\phi(a) = -\varepsilon \overline{f}(0)$ for positive ε. Analogously, in the first case of (6.4), we get a defined for negative ε. For simplicity of notation, we will only discuss the second case from now on. One can obtain the other case by changing ε to $-\varepsilon$.

We note that, because of (6.3),

$$(D\phi)(a) = \varepsilon^{1-1/l} D\phi(a_0). \tag{6.5}$$

We will make the assumption that

$$\mathrm{Spec}(D\phi(a_0)) \cap \mathrm{i}\mathbb{R} = \emptyset. \tag{6.6}$$

Hence,

$$\sup_{t\in\mathbb{R}} \|(\mathrm{i}t - D\phi(a_0))^{-1}\| < \infty.$$

And, using (6.5) we have

$$\sup_{t\in\mathbb{R}} \|(\mathrm{i}t - D\phi(a))^{-1}\| = \sup_{t\in\mathbb{R}} \|(\mathrm{i}t - \varepsilon^{1-1/l} D\phi(a_0))^{-1}\| \leqslant C\varepsilon^{-1+1/l}.$$

If we define, as before $\mathcal{L}_a$, we have

$$\|\mathcal{L}_a^{-1}\| \leqslant C\varepsilon^{-1+1/l}.$$

Since the composition estimates are the same for higher dimensional vectors as in the case of one dimensional vectors, we follow exactly the proof of theorems 8 and 12 and obtain:

Theorem 14. *Consider the equation* (6.1) *with h vanishing to order* $(l + 1)$ *and f satisfying* (2.6) *and* (6.4).

Assume that ϕ is homogeneous of degree l, i.e. (6.2).

If f, h are analytic, $\widetilde{f}(\theta, 0) \in H^{\rho,m}$ with $\rho > 0, m > d$, and $\|\widetilde{f}(\theta, 0)\|_{\rho,m}$ is small enough, then for all $\varepsilon \in \mathcal{I}$, we obtain a solution of (6.1) *of the form* (2.2) *in $H^{\rho,m}$.*

If f, h are $C^{m+p} (p = 1, 2, \ldots)$, $\widetilde{f}(\theta, 0) \in H^m$ with $m > d/2$, and $\|\widetilde{f}(\theta, 0)\|_m$ is small enough, then for all $\varepsilon \in \mathcal{I}$, we obtain a solution of (6.1) *of the form* (2.2) *in H^m.*

Moreover, the solution of (6.1) *is locally unique.*

Remark 15. We note that the method can be generalized to the case that $\phi(x)$ is not a homogeneous function. The key is that we can solve $\phi(a) = -\varepsilon\overline{f}(0)$ and that we can get bounds of $\|(\mathrm{i}t - D\phi(a))^{-1}\|$.

This is possible under several sets of conditions, such as ϕ being the sum of homogeneous functions, etc. We will not explore these possibilities.

7. Results when the average forcing vanishes

Both in our previous treatment and in [SS18], the assumption (2.6) plays an important role. In this section, we present some results without this assumption. We will, however need other assumptions, such as Diophantine condition.

According to the heuristic principles we described in section 2.1, the constant a from solving $a^l + \varepsilon\overline{f}(0) = 0$ is the dominant part in (2.4). This is based on the condition $\overline{f}(0) \neq 0$. In this part, we remove this condition. Therefore, we need to take the function V from solving the homological equation $\partial_\omega V = \widetilde{f}(\theta, 0)$ as the dominant part in (7.3). To deal with this equation we need some non-resonance assumptions, see (7.10), which are much weaker than Brjuno assumptions, for analytic case, and (7.16), which is the standard Diophantine assumptions, for finitely differentiable case.

As we will see, our results have different assumptions depending on whether $l = 2$ or $l > 2$. The difference between the two ranges of l is real and not an artifact of the methods since the solutions are somewhat different.

Since the method is mainly algebraic manipulations and contractions, it also leads easily to results when the average is not zero but it is small compared with other quantities that appear.

This goes in the opposite direction of the results in the previous sections where we assumed that other quantities are small compared with the average. We note that the solutions we produce in both cases are qualitatively different in the two regimes so that it seems clear that there is some bifurcation, but we do not know how to formulate this precisely, much less to develop a theory.

7.1. Formulation of results in the zero average forcing case

7.1.1. Description of the method for $l > 2$. In this section we will describe the method we propose in an informal way. We will ignore for the moment, precise definitions of spaces and formulating precisely the hypotheses. This will be done immediately afterward, after the steps to be taken are clarified. The informal assumption will clarify the reasons for our choices.

We assume that in (1.1), we have $\overline{f}(0) = 0$. We will try to find solutions of the form

$$x(t) = \varepsilon V(\omega t) + U(\omega t). \tag{7.1}$$

We choose V to solve the (dominant) equation

$$\omega \partial_\theta V = \widetilde{f}(\theta, 0). \tag{7.2}$$

In the analytic case, to solve (7.2), whose small divisor is $\mathrm{i}(k \cdot \omega)$, we need to impose some non-resonance conditions. As a matter of fact, we take

$$|k \cdot \omega| \geqslant \gamma \ \exp(-\eta |k|).$$

See (7.10) in the following part for more details. With the estimate (7.10), by shrinking the complex domain ρ to $(\rho - \eta)$ we can guarantee the solution to this equation is controllable. As for finitely differentiable case, since there is no complex domain, we have to lose the regularity m. In this case the condition (7.10) is not enough, we need the standard Diophantine condition (7.16).

Notice that the Diophantine conditions (7.10) are much weaker than the assumptions in KAM theory. The reason is that in our case, we only need to solve small divisor equations twice. Hence, we can afford that they have a more drastic effect than in KAM theory where one needs to solve infinitely many small divisor equations as part of an iterative process. In our case, we solve small divisor equations only to set up a contraction argument.

Also, we note that the solutions of (7.2) will never be unique since we can add a constant. In what follows, we will assume that we have chosen the V and transform the equation for the fluctuation accordingly. We will not revisit the choice of V (except at the end of the discussion in section 7.1.2, where we will find that there is an advantage in choosing the constant so that $\overline{g_1} + 2\overline{V} \neq 0$).

In this section, we will assume that $l > 2$. As we will see in section 7.1.2, the case $l = 2$ leads to a different answer with different non-resonance conditions.

We will find it convenient to introduce some notation for the expansions of g in the second variable (of course, this is just continuing the expansion of the forcing f, but we will keep the notation g we used before)

$$g(\theta, x) = g_1(\theta)x + g_>(\theta, x),$$

where, of course, $g_>(\theta, 0) = 0$, $D_x g_>(\theta, 0) = 0$.

Once we have chosen the function V solving (7.2), $x(t)$ given by (7.1) solves (1.1) if and only if U solves

$$\omega\partial_\theta U = (U + \varepsilon V)^l + h(\theta, U + \varepsilon V) + \varepsilon g_1(\theta)(U + \varepsilon V) + \varepsilon g_>(\theta, U + \varepsilon V). \tag{7.3}$$

We will see that the main part of the equation (7.3) is the following:

$$\mathcal{M}U \equiv \omega\partial_\theta U - \varepsilon g_1(\theta)U.$$

As indicated in the sketch of the strategy, we will try to invert $\mathcal{M}$ to formulate (7.3) as a fixed point equation.

As we will see more precisely in lemma 20, the operator $\mathcal{M}$ can be inverted provided that

$$\overline{g_1} \neq 0 \tag{7.4}$$

as well as some very weak Diophantine equations and we can obtain bounds in the Sobolev spaces we have used in the previous sections. Then, the equation (7.3) is equivalent to

$$U = \mathcal{M}^{-1}\left((U + \varepsilon V)^l + h(\theta, U + \varepsilon V) + \varepsilon^2 g_1(\theta)V + \varepsilon g_>(\theta, U + \varepsilon V)\right), \tag{7.5}$$

which is of a form very similar to (2.10).

Once we have the estimates for $\mathcal{M}$, the Lipschitz properties of the nonlinear terms can be estimated rather easily when $l > 2$. As it turns out, the term $(U + \varepsilon V)^l$ has very small Lipschitz constant when l is larger. When $l = 2$, we will have to rearrange the equations a bit more. See section 7.1.2.

Remark 16. It is a natural question to ask what will happen if the average forcing is zero and (7.4) fails. It seems plausible that one can make progress identifying other leading terms which will have to vanish and solve the auxiliary equation. Eliminating the assumption (2.6) seems to bring in the qualitatively different assumption (7.10), but higher order non-resonance seems to bring no new phenomenon.

7.1.2. Description of the method for $l = 2$. As before, we start by a heuristic description of the method. We will keep as much of the notation introduced in section 7.1.1.

As we will see, the conditions we need are different since the dominant terms that we need to consider are different.

In the case $l = 2$, we will rewrite (7.3) (which is equivalent for (1.1) with the notations introduced)

$$\begin{aligned}\omega\partial_\theta U = {} & U^2 + 2\varepsilon VU + \varepsilon^2 V^2 + h(\theta, U + \varepsilon V)\\ & + \varepsilon g_1(\theta)(U + \varepsilon V(\theta)) + \varepsilon g_>(\theta, U + \varepsilon V(\theta))\end{aligned} \tag{7.6}$$

which is equivalent to:

$$\begin{aligned}&(\omega\partial_\theta - (\varepsilon g_1(\theta) + 2\varepsilon V))\, U\\ &\quad = U^2 + \varepsilon^2 V^2 + h(\theta, U + \varepsilon V) + \varepsilon^2 g_1(\theta)V + \varepsilon g_>(\theta, U + \varepsilon V(\theta)).\end{aligned} \tag{7.7}$$

We proceed to invert the operator $\mathcal{N}$ defined by

$$\mathcal{N}U \equiv (\omega\partial_\theta - (\varepsilon g_1(\theta) + 2\varepsilon V))\, U$$

which can be done in the same way as we inverted $\mathcal{M}$ since they are operators of the same form. The difference of $\mathcal{M}$ and $\mathcal{N}$ is that $\mathcal{N}$ contains an extra multiplication term. Then, the equation (7.7) can be transformed into

$$U = \mathcal{N}^{-1}\left(U^2 + \varepsilon^2 V^2 + h(\theta, U + \varepsilon V) + \varepsilon^2 g_1(\theta)V + \varepsilon g_>(\theta, U + \varepsilon V(\theta))\right). \tag{7.8}$$

Following the procedure in lemma 20, the operator $\mathcal{N}$ can be inverted provided that we have that

$$\overline{g_1} + 2\overline{V} \neq 0. \tag{7.9}$$

The equation (7.9) appears for the same reasons as (7.4). We note however that (7.9) can always be arranged if we choose, from the beginning the V solving (7.2) taking advantage of the lack of uniqueness of solutions of (7.2). Adding an arbitrary constant to them is always possible, so that (7.9) can always be satisfied.

Of course, the choice of $\overline{V}$ will affect some of the details of subsequent calculations and it will affect the value of ε_0 which determines the maximum size of the perturbations allowed but will not affect the qualitative arguments.

Remark 17. The reason why the case $l = 2$ is special is because the linear approximation of U in $(U + \varepsilon V)^l$ for general l is $l\varepsilon^{l-1}V^{l-1}$. We see that in the case that $l = 2$ this is a term of order ε of the same order of magnitude as εg_1, When $l > 2$, the linear in U approximation of $(U + \varepsilon V)^l$ is much smaller than the εg_1.

7.2. Precise formulation of the main results in the zero average case

Theorem 18. *Consider the differential equation of the form* (1.1) *with h vanishing to order* $(l + 1)$ *and* $\overline{f}(0) = 0$.

Assume that:

- *f, h are analytic, $\widetilde{f}(\theta, 0) \in H^{\rho,m}$ with $\rho > 0, m > d$, and $\|\widetilde{f}(\theta, 0)\|_{\rho,m}$ is small enough.*
- *The frequency ω satisfies* (7.10) *with some $\eta > 0$ smaller than ρ.*
- *In the case that $l > 2$, the average of g_1 is not zero.*

Then, for all $\varepsilon \in \mathcal{I}$, we obtain a solution of (1.1) *of the form* (7.1) *in $H^{\rho-\eta,m}$.*

We also have that if f, h are $C^{m+p}(p = 1, 2, \ldots)$, $\widetilde{f}(\theta, 0) \in H^m$ with $m > d/2$ and ω satisfies (7.16) *with some τ satisfying $d - 1 < \tau < m$, then we obtain a solution in $H^{m-\tau}$.*

Since the proof is based on contraction mappings, we also obtain local uniqueness and smooth dependence on parameters. We leave the straightforward formulation to the reader.

7.3. Some auxiliary lemmas

In this section, we present some auxiliary lemmas motivated by the sketch of the arguments indicated in sections 7.1.1 and 7.1.2. They will allow to carry out all the estimates required in the sketch and make it rigorous.

Lemma 19. *For some $\rho, \eta > 0$, if the frequency vector ω satisfies*

$$|k \cdot \omega| \geqslant \gamma\ \exp(-\eta|k|), \quad \text{for} k \in \mathbb{Z}^d \setminus \{0\}, \tag{7.10}$$

with $0 < \gamma \ll 1$, *then, for* $\rho > \eta$, *we have that if* $f \in H^{\rho,m}$ *has zero average, then there is a unique solution* V *of zero average of the equation*

$$\omega \cdot \partial_\theta V = f. \tag{7.11}$$

Moreover, we have $V \in H^{\rho-\eta,m}$ *and*

$$\|V\|_{\rho-\eta,m} \leqslant \gamma^{-1}\|f\|_{\rho,m}. \tag{7.12}$$

Proof. The proof is obvious if we realize that the equation (7.11) is equivalent to the Fourier coefficients of V as the following:

$$\mathrm{i}(k\cdot\omega)\,\hat{V}_k = \hat{f}_k, \quad \text{for } k \in \mathbb{Z}^d.$$

This determines $\hat{V}_k$ when $k \neq 0$ and normalizing V to zero average gives $\hat{V}_0 = 0$. Then, (7.12) establishes since the norm of V in the space $H^{\rho-\eta,m}$ is read off the size of the Fourier coefficients of V. □

Lemma 20. *For* $\rho, \gamma, \eta > 0$ *with* $\rho > \eta$ *and* $m > d$, *let* ω *satisfy* (7.10) *and* $\beta \neq 0$ *be a real constant.*

If $\varphi \in H^{\rho,m}$ *have zero average, then, for any* $f \in H^{\rho-\eta,m}$, *there is a unique solution* $V \in H^{\rho-\eta,m}$ *solving*

$$(\omega\partial_\theta + \beta + \varphi)V = f. \tag{7.13}$$

Furthermore, we have

$$\|V\|_{\rho-\eta,m} \leqslant |\beta|^{-1}\|f\|_{\rho-\eta,m}\ \exp(2\gamma^{-1}\|\varphi\|_{\rho,m}). \tag{7.14}$$

Remark 21. Note that the lemma 20 would be immediate under the extra assumption that $|\gamma|^{-1}\|\varphi\|_{\rho+\eta,m}$ sufficiently small. In such a case we could invert the operator $(\omega\partial_\theta + \beta)$ using Fourier series and then use the Neumann series to invert $(\omega\partial_\theta + \beta + \varphi)$. For our applications, it is desirable not to make the extra assumption.

Remark 22. Equations of the form (7.13) are called '*twisted cohomology equations*' in [Her83], which also develops techniques to solve them.

There are several interesting variants of (7.13) estimates.

Proof. The proof is very similar to the integrating factor method in linear ordinary differential equations.

We find Γ solving $\omega\partial_\theta\Gamma = \varphi$ (as in the case of the integrating factor, we remark that such Γ is unique up to an additive constant).

By lemma 19, we have

$$\|\Gamma\|_{\rho-\eta,m} \leqslant \gamma^{-1}\|\varphi\|_{\rho,m}$$

and, by the Banach algebra properties of the Sobolev norm,

$$\|\exp(\Gamma)\|_{\rho-\eta,m} \leqslant \exp(\gamma^{-1}\|\varphi\|_{\rho,m}).$$

Then, multiplying (7.13) by $\exp(\Gamma)$, we obtain that it is equivalent to

$$\begin{aligned} f\ \exp(\Gamma) &= \exp(\Gamma)\omega\partial_\theta V + \beta V\ \exp(\Gamma) + \exp(\Gamma)(\omega\partial_\theta\Gamma)V \\ &= (\omega\cdot\partial_\theta + \beta)\,(\exp(\Gamma)V). \end{aligned}$$

Hence, using that the operator $\omega \cdot \partial_\theta + \beta$ is invertible and so are the operators of multiplication by $\exp(\pm\Gamma)$, one has

$$V = \exp(-\Gamma)(\omega \cdot \partial_\theta + \beta)^{-1} f \ \exp(\Gamma). \tag{7.15}$$

From (7.15), the estimates claimed in (7.14) follow immediately. □

Remark 23. In case that $\beta = 0$, to use formula (7.15) we need to assume that $\exp(\Gamma)f$ has average zero. This shows that in this case we will require different arguments.

Remark 24. Note that lemmas 19 and 20 are aimed at the analytic functions. When we consider our problem in finitely differentiable setting, we need to assume that the frequency ω satisfies

$$|k \cdot \omega| \geqslant \gamma |k|^{-\tau}, \quad \text{for } k \in \mathbb{Z}^d \setminus \{0\} \tag{7.16}$$

with $d - 1 < \tau < m$ and $0 < \gamma \ll 1$ (the condition $\tau > d - 1$ guarantees that the set whose elements are the frequencies satisfying (7.16) is of positive Lebesgue measure). Then, for $f \in H^m$, $m > \frac{d}{2}$ has zero average, there is a unique solution $V \in H^{m-\tau}$ of zero average of the equation

$$\omega \cdot \partial_\theta V = f$$

satisfying

$$\|V\|_{m-\tau} \leqslant \gamma^{-1} \|f\|_m. \tag{7.17}$$

Moreover, for $\psi \in H^{m-\tau}$ have zero average, then, for any $f \in H^{m-\tau}$, there is a unique solution $V \in H^{m-\tau}$ solving

$$(\omega\partial_\theta + \beta + \varphi)V = f$$

with

$$\|V\|_{m-\tau} \leqslant |\beta|^{-1} \|f\|_{m-\tau} \ \exp(2\gamma^{-1}\|\varphi\|_m).$$

7.4. Proof of the results in the zero average forcing case

We only present the detailed proof of theorem 18 in analytic case. The finitely differentiable case is similar.

7.4.1. The case $l > 2$. In the case $l > 2$, we will consider the equation (7.5) and check the hypotheses of the contraction mapping principle for the operator on the right.

The operator $\mathcal{M}$ fits into lemma 20 by taking $\beta = -\varepsilon\overline{g_1}$, $\varphi = -\varepsilon\widetilde{g}_1$. Therefore we obtain $\|\mathcal{M}\|^{-1} \leqslant C_1|\varepsilon|^{-1}$, where C_1 depends on g_1, γ. To simplify the notation, we still use C_1 to represent all constants (may depend on l, γ, f, h but not depend on ε).

Recall that (7.5) is an equation for U and that V has already been picked.

If we consider a ball of radius r with $r \leqslant A|\varepsilon|$, (we henceforth fix A, so that all the constants may depend on it), we can estimate the Lipschitz constants of the nonlinear terms in the right-hand side of (7.5) with respect to the U variable as the following:

$$\mathrm{Lip}_U\left((U + \varepsilon V)^l\right) \leqslant C_1|\varepsilon|^{l-1},$$

$$\mathrm{Lip}_U(\varepsilon h(\theta, U + \varepsilon V)) \leqslant C_1|\varepsilon|^l,$$

$$\mathrm{Lip}_U(\varepsilon g_>(U + \varepsilon V)) \leqslant C_1|\varepsilon|^2.$$

Note that the distance is measured in $H^{\rho-\eta,m}$.

Hence, we obtain that, the right-hand side of (7.5) has a Lipschitz constant bounded by $C_1|\varepsilon|$. We choose $|\varepsilon|$ small enough so that we get a contraction by $1/10$.

We also observe that for $U = 0$, the norm of the right-hand side of (7.5) is bounded from above by $C_1|\varepsilon|^{-1}(|\varepsilon|^l + |\varepsilon|^{l+1} + |\varepsilon|^2|\widetilde{f}(\theta, 0)| + |\varepsilon|^3)$. Since we assume that $|\widetilde{f}(\theta, 0)|$ is small enough, we get the ball to map into itself.

7.4.2. The case $l = 2$. The case $l = 2$ is based on the analysis of the operator in the right-hand side of (7.8).

This is actually easier than the case of $l > 2$. By lemma 20, we have that $\|\mathcal{N}\|^{-1} \leqslant C_1|\varepsilon|^{-1}$.

The Lipschitz constant of most nonlinear terms in a ball or radius $r = A|\varepsilon|$ are estimated the same. The only difference is that we have

$$\mathrm{Lip}_U(U^2) \leqslant C_1 A|\varepsilon|.$$

Hence, we have that the Lipschitz constant of the right-hand side of (7.5) in the ball of radius $A|\varepsilon|$ can be made smaller than $1/10$ by taking A small enough.

We also have that the $\|\cdot\|_{\rho-\eta,m}$ norm of the right-hand side of (7.5) at $U = 0$ can be estimated by $C_1\,(|\varepsilon|^{-1}(|\varepsilon|^2|\widetilde{f}(\theta, 0)| + |\varepsilon|^3)$. Thus, by taking $|\varepsilon|$ small enough, we can get that the operator maps the ball into itself.

8. Application to degenerate oscillators (second order equations)

Remarkably similar methods can be applied to the study of degenerate oscillators (second order equations).

$$\ddot{x} + \delta\dot{x} = x^l + h(\omega t, x) + \varepsilon f(\omega t, x) \tag{8.1}$$

where h, f are as in (1.1). Again, we aim to find solutions of the form (2.2).

Note that the equation (8.1) has two small parameters δ, ε. Depending on the relation among them, we will have that the dominant solution has different forms.

In this paper, we only aim to demonstrate the possibilities of the method and will only do one of the cases. We hope to come back to a more complete study.

A sample result is the following:

Theorem 25. *Consider the equation (8.1) with h, f as in (1.1).*

Assume that there exist a solving

$$a^l + \varepsilon\overline{f}(0) = 0$$

and choose one of them.

Assume:

- (2.6).
- $\|\widetilde{f}(\theta, 0)\|_{\rho,m}$ *is small enough compared to* $|\overline{f}(0)|$.
- $$\delta^2 + 2la^{l-1} \geqslant 0.$$

Then, the same conclusions hold as in theorems 8 and 12.

The proof is extremely similar to the study of (1.1). After we substitute (2.2) in (8.1) and cancel $a^l + \varepsilon \overline{f}(0)$ we see that (2.2) is a solution of (8.1) if and only if V satisfies:

$$\widetilde{\mathcal{L}}V = S(a, V) + \varepsilon \widetilde{f}(0, 0) + h(\theta, a + V(\theta)) + \varepsilon g(\theta, a + V(\theta)), \tag{8.2}$$

where

$$\widetilde{\mathcal{L}}V \equiv \left[(\omega \cdot \partial_\theta)^2 + \delta(\omega \cdot \partial_\theta) - la^{l-1}\right] V.$$

If the operator $\widetilde{\mathcal{L}}$ was invertible, (8.2) would be equivalent to

$$V = \widetilde{\mathcal{L}}^{-1} \left(S(a, V) + \varepsilon \widetilde{f}(0, 0) + h(\theta, a + V(\theta)) + \varepsilon g(\theta, a + V(\theta)) \right). \tag{8.3}$$

Note the similitude between (8.3) and (2.10). The only difference is the linear operator to be inverted.

Hence, we will need to study the invertibility of the operator $\widetilde{\mathcal{L}}$ and the norm of its inverse. We note that the operator $\widetilde{\mathcal{L}}$ is diagonal in Fourier series and it amounts to multiplying the k Fourier coefficient by

$$\widetilde{L}_k \equiv -(k \cdot \omega)^2 + \mathrm{i}\delta(k \cdot \omega) - la^{l-1}.$$

Hence, to estimate $\|\widetilde{\mathcal{L}}^{-1}\|$, it suffices to estimate from below the minimum of $|\widetilde{L}_k|$. Denoting $t = k \cdot \omega$, we have

$$\begin{aligned} |\widetilde{L}_k|^2 &= (-t^2 - la^{l-1})^2 + \delta^2 t^2 \\ &= t^4 + t^2(\delta^2 + 2la^{l-1}) + l^2 a^{2(l-1)} \\ &\geqslant |la^{l-1}|^2, \end{aligned}$$

where the last inequality comes from the assumption that $(\delta^2 + 2la^{l-1}) \geqslant 0$.

Once we have that, we see that the operator in (8.3) satisfies exactly the same bounds as the operator in (2.10) and the rest of the proof does not need any modification from the estimates in the proof of theorem 8 (see (4.3) and (4.4)).

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