SIAM J. MATH. ANAL.
Vol. 52, No. 4, pp. 3149–3191

RESPONSE SOLUTIONS TO QUASI-PERIODICALLY FORCED SYSTEMS, EVEN TO POSSIBLY ILL-POSED PDES, WITH STRONG DISSIPATION AND ANY FREQUENCY VECTORS*

FENFEN WANG† AND RAFAEL DE LA LLAVE‡

Abstract. We consider several models (including both multidimensional ordinary differential equations (ODEs) and partial differential equations (PDEs), possibly ill-posed), subject to very strong damping and quasi-periodic external forcing. We are interested in studying response solutions (i.e., quasi-periodic solutions with the same frequency as the forcing). Under some regularity assumptions on the nonlinearity and forcing, without any arithmetic condition on the forcing frequency ω, we show that the response solutions indeed exist. Moreover, the solutions we obtained possess optimal regularity in ε (where ε is the inverse of the coefficients multiplying the damping) when we consider ε in a domain that does not include the origin $\varepsilon = 0$ but has the origin on its boundary. We also show that response solutions are continuous in ε at 0. However, in general, the solutions may fail to be differentiable with respect to ε at $\varepsilon = 0$. In this paper, we allow multidimensional systems and we do not require that the unperturbed equations under consideration are Hamiltonian. One advantage of the method in the present paper is that it gives results for analytic, finitely differentiable and low regularity forcing, and nonlinearity, respectively. As a matter of fact, we do not even need that the forcing is continuous. Notably, we obtain results when the forcing is in L^2 space and the nonlinearity is just Lipschitz as well as in the case that the forcing is in H^1 space and the nonlinearity is $C^{1+\mathrm{Lip}}$. In the proof of our results, we reformulate the existence of response solutions as a fixed point problem in appropriate spaces of smooth functions.

Key words. strong dissipation, response solutions, singular perturbations

AMS subject classifications. 34D15, 35L05, 35B65, 42B30, 47H10, 70k75

DOI. 10.1137/19M1272159

1. Introduction.

In recent times, there has been much interest in the study of response solutions (i.e., solutions which have the same frequency as the forcing term) for nonlinear mechanical systems subject to strong damping (i.e., systems in which the term describing the damping contains a factor ε^{-1} with ε being a small parameter) and quasi-periodic external forcing. The mechanical systems under consideration are second order equations with respect to the time derivative and the damping is the term which corresponds to the time derivative of first order (see (1.1) and (1.2) below); this is a singular perturbation in ε. For more information in this field, we refer to [Bal94, Gen10a, CCCdlL17] and references therein.

We are interested in finding response solutions for two classes of equations. We first consider an ODE model of the form

$$x_{tt} + \frac{1}{\varepsilon}x_t + g(x) = f(\omega t),\ x \in \mathbb{R}^n. \tag{1.1}$$

*Received by the editors July 3, 2019; accepted for publication (in revised form) March 31, 2020; published electronically July 9, 2020.
https://doi.org/10.1137/19M1272159
Funding: The work of the first author was supported by CSC by the National Natural Science Foundation of China (NSFC) grants 11571201, 11971261. The work of the second author was partially supported by the National Science Foundation grant DMS-1800241. The work of the authors was supported by the National Science Foundation grant DMS-1440140 at MSRI during Fall 2018.
†School of Mathematics, Sichuan Normal University, Chengdu, Sichuan 610066, People's Republic of China (ffenwang@hotmail.com).
‡School of Mathematics, Georgia Institute of Technology, Atlanta GA, 30332 (rafael.delallave@math.gatech.edu).

Equation (1.1) is referred to as a *varactor* equation in the literature [Gen10a, CCdlL13, CFG14, GMV17, GV17].

We also consider PDE models. One particular example is obtained from the Boussinesq equation (derived in the paper [Bou72]) by adding a singular friction proportional to the velocity:

$$u_{tt} + \frac{1}{\varepsilon}u_t - \beta u_{xxxx} - u_{xx} = (u^2)_{xx} + f(\omega t, x),\ x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \tag{1.2}$$

where $\beta > 0$ is a parameter. Of course, (1.2) will be supplemented with periodic boundary conditions. We note that the positive sign of β makes (1.2) ill-posed. That is, there are many initial conditions that do not lead to solutions. It is, however, possible that there is a systematic way to construct many special solutions, for some ill-posed Boussinesq equations, which are physically observed (we refer to the papers [dlL09, dlLS19, CdlL19a, CdlL19b]).

In both (1.1) and (1.2), $\varepsilon > 0$ is a small parameter and $\omega \in \mathbb{R}^d \setminus \{0\}$ with $d \in \mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. The forcing f is quasi-periodic with respect to time t. Note that in the PDE (1.2), the forcing may depend on the space variable. At this moment, we think of the forcing as a quasi-periodic function taking values in a space of functions.

In (1.1), one considers the nonlinearity g as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $n \in \mathbb{N}_+$ and the forcing f as a function from $\mathbb{T}^d$ to $\mathbb{R}^n$. We will obtain several results depending on the regularity assumed for f and g. First, we will consider that the functions f and g are real analytic in the sense that they take real values for real arguments, which are what appear in physical applications, with $\varepsilon \in \mathbb{R}_+$. We will also consider highly differentiable functions f and g, such as $f \in H^m$ $(m > \frac{d}{2})$ and g is C^{m+l} $(l = 1, 2, \ldots)$. In addition, we will obtain results for rather irregular functions f and g. For example, the forcing f is in the L^2 space, the nonlinearity g is just Lipschitz or f is in the H^1 space, g is $C^{1+\mathrm{Lip}}$.

In (1.2), we consider the function $f : \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$. Analogously to the case of (1.1), we will present results for f being real analytic and finitely differentiable with high regularity. Note that in the study of the PDE model (1.2), we will just focus on the physically relevant case of a specific nonlinearity $(u^2)_{xx}$. It is possible to discuss general nonlinearities in a regularity class, but being unaware of a physical motivation, we leave these generalizations to the readers. We emphasize that the nonlinearity $(u^2)_{xx}$ in (1.2) is unbounded from one space to itself, but the fixed point problem we consider overcomes this problem since there will be smoothing factors.

From the physical point of view, the parameter ε is real. However, it is natural to consider ε in a complex domain when we consider our problem in an analytic setting. It is important to notice that the complex domain we choose does not include the origin but accumulates on it. Indeed, the solutions fail to be differentiable at $\varepsilon = 0$ in the generality considered in the present paper (see Remark 20). However, we will show that the response solutions depend continuously on ε at $\varepsilon = 0$.

1.1. Some remarks on the literature. The problem of the response solutions for dissipative systems has been studied by several methods. One method is based on developing asymptotic series and then show that they can be resummed using combinatorial arguments, which are established using the so-called *tree formalism.* This can be found in the literature [GBD05, GBD06, Gen10a, Gen10b]. Recent papers developing this method are [GMV17, GV17]. We point out that one important novelty of the papers [GMV17, GV17] is that no arithmetic condition is required in the frequency of the forcing. A later method is to reduce the existence of response solutions to a fixed point problem, which is analyzed in a ball in an appropriate Banach space, centered in the solution predicted by the asymptotic expansion. In this direction, we refer to [CCdlL13, CCCdlL17] and references therein. Note that the papers [CCdlL13, CCCdlL17] considered the perturbative expansion to low orders on ε and obtained a reasonably approximate solution in a neighborhood of $\varepsilon = 0$. Nevertheless, to obtain the asymptotic expansions, one needs to solve equations involving small divisors and assume some nondegeneracy conditions. Note that the small divisors assumed in [CCdlL13, CCCdlL17] are weaker than the Diophantine conditions in KAM theory. In this paper, we will not assume any small divisors conditions since we do not attempt to get the approximate solution through an asymptotic expansion.

Since the literature is growing, it is interesting to systematically compare results. There are several figures of merit for results on the existence of response solutions:

(1) the arithmetic properties required in the external forcing frequency, such as Diophantine condition, Bryuno condition, or even weaker conditions, etc.;
(2) the analyticity domain in ε established. Since we do not expect that the asymptotic series converges, this domain does not include a ball centered at the origin. We emphasize that the shape of this analyticity domain is very important to study properties of the asymptotic series. Having a parabolic domain shows that the asymptotic series is unique and it is also related to Borel summability. In this paper, we only establish wedge domains, but under extra Diophantine assumptions, parabolic domains established in [CFG13, CCdlL13, CCCdlL17] and Borel summability methods are considered in [GBD05, GBD06]. It seems that the optimal regularity domain is related to the Diophantine properties assumed. In this paper, we do not assume any Diophantine conditions and the solutions may fail to be differentiable at $\varepsilon = 0$. See Remark 20;
(3) whether the method gives some asymptotic expansions for the solutions;
(4) whether the method can deal with the forcing function f which has low regularity (e.g., $f \in L^2$ or $f \in H^1$) and the nonlinearity function g of low regularity (the case of piecewise differentiable functions appears in some applications);
(5) the generality of the models considered (e.g., whether the method requires that the system be Hamiltonian, reversible, etc.);
(6) smallness conditions imposed on functions f and g;
(7) the conditions required for the linear part of g, such as its eigenvalues are nonzero. We do not know whether our method can deal with the case of noninvertible A in the generality considered here. Research on extending the method to the case of degenerate fixed points is studied in [CdlLW19].

Notice that all these figures of merit cannot be accomplished at the same time. Obtaining more conclusions on the solutions (e.g., the existence of asymptotic expansions) will require more regularity and some arithmetic conditions on the frequency. Especially, allowing zero eigenvalues of the linear part of g is another issue which would deserve further investigation.

1.2. The method in the present paper. From the strictly logical point of view, our paper and [GMV17, GV17] are completely different even if they are motivated by the same physical problem for the model (1.1). More precisely, the present paper deals with not only the analytic setting but also the finitely differentiable case and even just the Lipschitz problem by the method of the fixed point theorem. In contrast, the papers [GMV17, GV17] apply resummation methods to establish the existence of response solutions under analytic conditions.

In the multidimensional case of (1.1), compared with [GMV17], the methods presented in this paper do not need that the oscillators without dissipation are Hamiltonian or that the linearization of g at the origin (denoted by A, which is an $n \times n$ matrix) is positive definite. Further, we do not assume that the matrix A is diagonalizable or symmetric. We allow Jordan blocks that appear naturally in the problems at resonance case appearing in [BG15, Gaz15].

However, we note that our method for the analytic case involves smallness assumptions in the forcing f but not in the nonlinear part (denoted by $\hat{g}$) of g. In the setting of L^2 and H^1, we involve just smallness assumptions on $\hat{g}$ but not f. For the highly differentiable case (i.e., H^m, $m > \frac{d}{2}$), we choose either smallness assumptions for f or $\hat{g}$, (see section 2.4 for more details). Explicitly, the smallness conditions imposed in f or $\hat{g}$ are determined by the eigenvalues of A and properties of the nonlinearity. See section 5.3 and section 6.1.2 for a concrete presentation.

As a further application, we consider adding dissipative terms to the Boussinesq equation of water waves in (1.2). Equation (1.2) is ill-posed in the sense that not all initial conditions lead to solutions. Nevertheless, we construct response solutions.

We note that the approach followed in [CCdlL13, CCCdlL17] has two steps. In the first step, one constructed series expansions in ε that produced approximate solutions. In a second step, one used a contraction mapping principle for an operator defined in a small ball near the approximate solutions obtained in the first step. Of course, this approach requires a very careful choice of the spaces in which the approximate solutions lie and the fixed point problems are formulated. One important consideration is that the spaces are chosen such that the operators involved map the spaces into themselves. Since some of the operators involved are diagonal in Fourier series, it is important that the norms can be read off from the Fourier coefficients. It will also be convenient that there are Banach algebra properties and properties of composition operators in the chosen space. This allows us to control the nonlinear terms easily. We have to say that it is the idea in [CCdlL13, CCCdlL17] that inspires our present treatment for (1.1) and (1.2).

To motivate the procedure adopted in this paper, we note that in the method of [CCdlL13, CCCdlL17], the fixed point part does not depend on any arithmetic condition on the forcing frequency. We will modify slightly the fixed point part to get response solutions analytic with respect to the parameter ε, for the analytic models (1.1) and (1.2), when ε ranges over a complex domain without any circle centered at the origin $\varepsilon = 0$. Our method (very different from resumming expansions) consists in transforming the original equations (1.1) and (1.2) into the fixed point equations (see (2.9) and (7.7), respectively). The main observation that allows us to solve the fixed point equations is that we are allowed to use the strong dissipation in the contraction mapping principle.

Our method also works for finitely differentiable problems. In such a case, we will introduce Sobolev spaces, in which the norms of functions are measured by the size of the Fourier coefficients. The solutions obtained, for (1.1) and (1.2) in the finitely differentiable setting, still possess the corresponding regularity in ε when ε ranges over a real domain without any circle centered at the origin $\varepsilon = 0$.

We think that the regularity results obtained in this paper are close to optimal. As for the optimality for the domain, we find that there exist arbitrarily small values of ε for which the operator we constructed is not a contraction and the method of the proof breaks down. Therefore, we conjecture that this is optimal and that indeed, regular solutions do not exist for these small parameter values and general forcing and nonlinearity. We also show, in Remark 20, that, in both analytic and finitely differentiable cases, there are examples in which the solution is not differentiable at $\varepsilon = 0$ when we remove the Diophantine condition on ω.

The lack of differentiability at $\varepsilon = 0$ is a reflection of the problem being a singular perturbation. In the case considered here that there are no nonresonance conditions on the frequency, the problem is more severe than in previously considered cases.

1.3. Some possible generalization. Our method could deal easily with the general case with the form

$$\mathbf{p}x_{tt} + \frac{1}{\varepsilon}\mathbf{q}x_t + g(x, \omega t) = f(\omega t), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $\mathbf{p}$, $\mathbf{q}$ are diagonal constant matrices and $g(x, \omega t) = Ax + \hat{g}(x, \omega t)$, where A is a matrix in Jordan block form and $\hat{g} : \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ is sufficiently regular. We leave the easy details to the interested readers. See Remark 17, which gives some simplified calculations after we have carried out the case in (1.1).

1.4. Organization of this paper. Our paper is organized as follows. In section 2, we present the idea of reformulating the existence of response solutions for (1.1) as a fixed point problem. To solve this fixed point equation, in section 3, we give the precise function spaces that we work in and we list their important properties, such as Banach algebra properties and the regularity of the composition operators. We state our three main results in the analytic case, the highly differentiable case, and with low regularity, respectively, in section 4. Section 5 is mainly devoted to the proof of our analytic result by the contraction mapping principle. In the process, we need to pay more attention to the invertibility of operators and regularity of composition operators. In section 6, we prove our regular result in the finitely differentiable case by combining the contraction argument with the implicit function theorem. Section 7 is an application to the ill-posed PDE model (1.2) by the similar idea used for the ODE model (1.1).

2. The formulation for (1.1). In this section, we give an overview of our treatment for ODE model (1.1), which can be rewritten as

$$\varepsilon x_{tt} + x_t + \varepsilon g(x) = \varepsilon f(\omega t),\ x \in \mathbb{R}^n, \tag{2.1}$$

where, as indicated before, the mappings are $g : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{T}^d \to \mathbb{R}^n$. We will reduce the existence of response solutions of (2.1) to an equivalent fixed point problem. To this end, it is crucial to make some assumptions for (2.1).

2.1. Preliminaries. For the analytic or highly differentiable functions f and g defining (2.1), we make the following assumptions.

Assumption 1. The average of f is 0 and $g(0) = 0$. Denote $A = Dg(0)$, which is an $n \times n$ matrix, the spectrum λ_j ($j = 1, \dots, n$) of A is real, and $\lambda_j \neq 0$.

Actually, we could weaken the assumptions on the regularity of the function g when considering low regularity results (e.g, L^2 or H^1). As we will see in section 6.2, instead of assuming g is differentiable, we just assume the following.

Assumption 2. g is Lipschitz in $\mathbb{R}^n$ and it can be expressed in the form

$$g(x) = Ax + \hat{g}(x),$$

where A is an $n \times n$ matrix and its spectra are real and nonzero. Moreover, the nonlinear part $\hat{g}$ satisfies that $Lip(\hat{g})$ is sufficiently small, depending on the spectral properties of A (the eigenvalues and the Jordan normal form) and the number theoretic properties of ω. We make explicit the Lipschitz constant in all steps. See (6.7) for more details.

Note that in the assumptions stated above we are not including that the matrix A is diagonalizable. Nondiagonalizable matrices appear naturally when considering oscillators at resonance, which is often a design goal in several applications in electronics or appear in mechanical systems with several nodes.

We emphasize that Assumption 2 involves an assumption on $\hat{g}$ for all values of its argument. This is needed when we consider solutions in L^2 which may be unbounded.

It is important to note also that once we have established the conclusion for g under Assumption 2, we can accommodate several physical situations such as piecewise linear nonlinearity with small breaks.

Without loss of generality, we assume that

$$k \cdot \omega \neq 0 \text{ for } k \in \mathbb{Z}^d \setminus \{0\}. \tag{2.2}$$

Indeed, if there is a $k_0 \in \mathbb{Z}^d \setminus \{0\}$ such that $k_0 \cdot \omega = 0$, we could reformulate the forcing with only $(d-1)$-dimensional variables which are orthogonal to k_0.

The condition (2.2) is called the *nonresonance* condition. If the nonresonance condition (2.2) is satisfied, then the set $\{\omega t\}_{t\in\mathbb{R}}$ is dense on the torus $\mathbb{T}^d$.

2.2. Quasi-periodic solutions, hull functions. In this paper, we are interested in finding quasi-periodic solutions with frequency $\omega \in \mathbb{R}^d \setminus \{0\}$. They are functions of time t with the form

$$x_\varepsilon(t) = U_\varepsilon(\omega t) \tag{2.3}$$

for a suitable function $U_\varepsilon : \mathbb{T}^d \to \mathbb{R}^n$, indexed by the small parameter ε. The function U_ε is often called the *hull function*. Substituting (2.3) into (2.1) and using that $\{\omega t\}_{t\in\mathbb{R}}$ is dense in $\mathbb{T}^d$, we obtain that (2.1) holds for a continuous function x if and only if the hull function U_ε satisfies

$$\varepsilon\,(\omega\cdot\partial_\theta)^2\,U_\varepsilon(\theta) + (\omega\cdot\partial_\theta)\,U_\varepsilon(\theta) + \varepsilon g(U_\varepsilon(\theta)) = \varepsilon f(\theta). \tag{2.4}$$

Hence, our treatment for (2.1) will be based on finding U_ε which solves (2.4). We will manipulate (2.4) to reformulate it as a fixed point equation that can be solved by the contraction argument.

The equation we will solve (2.4) involves a parameter ε (the inverse of the coefficient multiplying the damping). We will obtain solutions with delicate regularity in ε, which are objects in a space of functions. Precisely, in the analytic case (see section 5), we will get a solution U_ε of (2.4) depending analytically on ε when ε ranges on a complex domain Ω which does not include the origin $\varepsilon = 0$ but so that the origin is in the closure of Ω. In the finitely differentiable case (see section 6), the solution U_ε is differentiable in ε when ε is in a real domain $\widetilde{\Omega}$ which does not also include zero but includes it in its closure.

However, when we consider the regularity for the solution U_ε of (2.4) as ε goes to 0 along the set Ω, we get that U_ε is continuous in ε at 0 in the topologies used in the fixed point problem (see Lemma (21)). Moreover, we will show that, in the generality considered in this paper, there are cases in which the solution is not differentiable at $\varepsilon = 0$ (see Remark 20).

Later, we will develop analogous procedures for the PDE model (1.2) (see section 7). We anticipate that the treatment is inspired by this section presenting the formulation for the ODE model (1.1). The unknowns will not take values in $\mathbb{R}^n$, but rather will take values in a Banach space of functions. In addition, the PDF (1.2) is ill-posed and its nonlinearity is unbound, which make us do some more drastic rearrangement for its fixed point equation.

2.3. Formulation of the fixed point problem. In this part, we just present the formal manipulations. The precise setup will follow, but it is natural to present first the formal manipulations since the rigorous setting is chosen to make them precise.

Our goal is to transform (2.4) into an equivalent fixed point problem. We rewrite (2.4) as

$$\varepsilon\,(\omega\cdot\partial_\theta)^2\,U_\varepsilon(\theta)+(\omega\cdot\partial_\theta)\,U_\varepsilon(\theta)+\varepsilon AU_\varepsilon(\theta)=\varepsilon f(\theta)-\varepsilon\hat{g}(U_\varepsilon(\theta)), \tag{2.5}$$

where $A=Dg(0)$ and

$$\hat{g}(x)=g(x)-Ax.$$

Note that, in both the analytic case and the highly differentiable case, we use Assumption 1. It is obvious that

$$\hat{g}(0)=0,\ \ D\hat{g}(0)=0, \tag{2.6}$$

namely,

$$\hat{g}(x)=O(x^2),\ \ D\hat{g}(x)=O(x),$$

where $O(x)$ denotes the same order as x. As a consequence, $D\hat{g}$ is small (in many senses) in a small neighborhood of the origin $x=0$. We could also assume that $D\hat{g}$ is globally small in the whole of $\mathbb{R}^n$. This is trivial in the sense of complex analyticity by Liouville's theorem. When g is just Lipschitz, we need that $\mathrm{Lip}(\hat{g})$ is globally small, namely, Assumption 2.

Based on (2.5) and denoting by Id the $n\times n$ identity matrix, we introduce the linear operator $\mathcal{L}_\varepsilon$ as

$$\mathcal{L}_\varepsilon=\varepsilon\,(\omega\cdot\partial_\theta)^2\,Id+(\omega\cdot\partial_\theta)\,Id+\varepsilon A \tag{2.7}$$

defined on n-dimensional periodic functions of $\theta\in\mathbb{T}^d$. Then, (2.5) can be rewritten as

$$\mathcal{L}_\varepsilon(U_\varepsilon(\theta))=\varepsilon f(\theta)-\varepsilon\hat{g}(U_\varepsilon(\theta)). \tag{2.8}$$

As shown in section 5.1, the linear operator $\mathcal{L}_\varepsilon$ is boundedly invertible in the special space $H^{\rho,m}$ defined in section 3 when ε ranges in a suitable complex domain. This allows (2.8) to be transformed into a fixed point problem as

$$U_\varepsilon(\theta)=\varepsilon\mathcal{L}_\varepsilon^{-1}\left[f(\theta)-\hat{g}(U_\varepsilon(\theta))\right]\equiv\mathcal{T}_\varepsilon(U_\varepsilon)(\theta), \tag{2.9}$$

where we have introduced the operator $\mathcal{T}_\varepsilon$. For a fixed ε, we can obtain a solution U_ε for (2.9) by the contraction mapping principle. Further, we want to get a solution U_ε possessing optimal regularity in ε. This can be achieved by considering operator $\mathcal{T}_\varepsilon$ above in a function space consisting of functions regular in ε (see section 5.2 for the analytic case and section 6.1.1 for the highly differentiable case). Specially, in the highly differentiable case, we will use the classical implicit function theorem (we refer to the references [Die69, LS90, KP13]) to get the regular results.

Two subtle points appear in the above strategy used in (2.9). One is the invertibility of the linear operator $\mathcal{L}_\varepsilon$ and the bound of its inverse. Another is the regularity of the composition operator $\hat{g} \circ U_\varepsilon$.

Fortunately, we observe that the linear operator $\mathcal{L}_\varepsilon$ is diagonal in the basis of Fourier functions. This suggests that we use some variants of Sobolev (or Bergman) spaces which provide analyticity—or in the low regularity case L^2 or H^1. Hence, it will be useful that the spaces we consider have norms that can be estimated very easily by estimating the Fourier coefficients. The estimates of the Fourier coefficients involve the assumptions that the eigenvalues of A are real and nonzero and that the range of ε is restricted to a domain not including the origin $\varepsilon = 0$ but accumulating at this origin. (See section 5.1.2 for details.)

For the estimates of nonlinear terms, we need that the composition operator $\hat{g} \circ U_\varepsilon$ is smooth and considered as a mapping acting on the spaces under consideration. The regularity of the composition on the left by a smooth function acting on variants of Sobolev spaces have been widely studied [Mar74, AZ90, IKT13]. In section 3, we will present the precise spaces and some properties in these spaces used to implement our program.

2.4. Some heuristic considerations on the smallness conditions required for the present method. Recall the fixed point equation (2.9); the operator we consider has the structure

$$U = \varepsilon \mathcal{L}_\varepsilon^{-1} f - \varepsilon \mathcal{L}_\varepsilon^{-1} \hat{g}(U) \equiv \mathcal{T}_\varepsilon(U).$$

To solve it by iteration, roughly, we need that the map $U \mapsto \varepsilon \mathcal{L}_\varepsilon^{-1} \hat{g} \circ U$ is a contraction in a domain that contains a ball around $\varepsilon \mathcal{L}_\varepsilon^{-1} f$. Of course, the notions of contraction and smallness depend on the spaces we choose. The results of existence of solutions U are sharper if we consider spaces of more regular functions and the results of local uniqueness are sharper if we consider spaces of less regular functions.

Both the contraction properties of $\varepsilon \mathcal{L}_\varepsilon^{-1} \hat{g} \circ U$ and the smallness properties of $\varepsilon \mathcal{L}_\varepsilon^{-1} f$ are formulated in appropriate norms (which change with the regularity considered). As we will see in section 5.1, the operator $\varepsilon \mathcal{L}_\varepsilon^{-1}$ can be bounded in appropriate norms, which allows us to just consider the smallness of f and the properties of the composition $\hat{g} \circ U$.

It is clear that we can trade off some of the smallness assumptions in $\hat{g}$ and f. If we are willing to make global assumptions of smallness on $\hat{g}$, we do not need any smallness assumption on f. If, on the other hand, we assume that $\hat{g}$ is smooth and $\hat{g}(0) = D\hat{g}(0) = 0$, we have that $\hat{g}$ is small (in many senses) in a small neighborhood at the origin $x = 0$. From this point of view, it is necessary to impose a smallness condition on f in this small neighborhood.

There are some caveats to these arguments.

In the analytic case, assuming that $D\hat{g}$ is small globally (even bounded) in the whole complex space $\mathbb{C}^n$, Liouville's theorem shows that it is constant, namely, $\hat{g}$ is linear. This makes our result true, but it is trivial and we will not state it. Of course, Liouville's theorem is only a concern for analytic results.

In the low regularity cases (e.g., L^2 or H^1 when $d \geq 2$), the range of f may be the whole of $\mathbb{R}^n$, hence we need to make global assumptions on smallness in $\hat{g}$. In the highly differentiable case (e.g., $H^m,\ m > \frac{d}{2}$), we prove our results under two types of smallness assumptions (See section 6.1.)

We also advance that in the case of H^1 regularity, the contraction argument we use will be somewhat more sophisticated. (See section 6.2.)

3. Function spaces.

3.1. Choice of spaces. To implement the fixed point problem outlined in section 2, we need to precisely define function spaces with appropriate norms. The discussion in section 5 will make clear, it is very convenient that the norms can be expressed in terms of the Fourier coefficients of functions. In such a case, the inverse of the linear operator $\mathcal{L}_\varepsilon$ can be easily estimated just by estimating its Fourier coefficients. We are allowed to choose a special base in such a way that the Fourier coefficients of the multiplier operator $\mathcal{L}_\varepsilon$ have the Jordan standard form. (See section 5.1.1.)

We also need the spaces to possess other properties allowing us to control the composition $\hat{g} \circ U$ in (2.9) with ease, such as Banach algebras properties under multiplication and the properties of the composition operators.

In this section, we use the same notations for Banach spaces as in [dlL09, CCdlL13, dlLS19].

For $\rho \geq 0$, we denote

$$\mathbb{T}^d_\rho = \left\{\theta \in \mathbb{C}^d/(2\pi\mathbb{Z})^d \,:\, \mathrm{Re}(\theta_j) \in \mathbb{T},\ |\mathrm{Im}(\theta_j)| \leqslant \rho,\ j = 1, \ldots, d\right\}.$$

Then, we denote the Fourier expansion of a periodic function $f(\theta)$ on $\mathbb{T}^d_\rho$ by

$$f(\theta) = \sum_{k \in \mathbb{Z}^d} \widehat{f}_k e^{\mathrm{i}k\cdot\theta},$$

where $k\cdot\theta = \sum_{j=1}^d k_j\theta_j$ represents the Euclidean product in $\mathbb{C}^d$ and $\widehat{f}_k$ are the Fourier coefficients of f. If f is analytic and bounded on $\mathbb{T}^d_\rho$, then the Fourier coefficients satisfy the Cauchy bounds

$$|\widehat{f}_k| \leq \max_{\theta \in \mathbb{T}^d_\rho} |f(\theta)| \cdot e^{-|k|\rho}$$

with $|k| = \sum_{j=1}^d |k_j|$.

Definition 3. *For $\rho \geq 0$, $m \in \mathbb{N}_+$, we denote by $H^{\rho,m}$ the space of analytic functions U in $\mathbb{T}^d_\rho$ with finite norm*

$$\begin{aligned} H^{\rho,m} &:= H^{\rho,m}(\mathbb{T}^d) \\ &= \left\{U : \mathbb{T}^d_\rho \to \mathbb{C}^n \ \Big|\ \|U\|^2_{H^{\rho,m}} = \sum_{k \in \mathbb{Z}^d} \left|\widehat{U}_k\right|^2 e^{2\rho|k|}(|k|^2+1)^m < +\infty\right\}. \end{aligned}$$

It is obvious that the space $(H^{\rho,m},\ \|\cdot\|_{H^{\rho,m}})$ is a Banach space and indeed a Hilbert space. From the real analytic point of view, we consider the Banach space $H^{\rho,m}$ of the functions that take real values for real arguments.

For $\rho = 0$, $H^m(\mathbb{T}^d) := H^{0,m}(\mathbb{T}^d)$ is the standard Sobolev space; we refer to the references [Tay97, AF03] for more details. Moreover, when $m > \frac{d}{2}$, by the Sobolev embedding theorem (see Chapters 2 and 6 in [Tay97]), we obtain that $H^{m+l}(\mathbb{T}^d)$ $(l = 1, 2, \ldots)$ embeds continuously into $C^l(\mathbb{T}^d)$.

For $\rho > 0$, functions in the space $H^{\rho,m}$ are analytic in the interior of $\mathbb{T}^d_\rho$ and extend to Sobolev functions on the boundary of $\mathbb{T}^d_\rho$.

Remark 4. As a matter of fact, when $\rho > 0$ and $m > d$, the space $H^{\rho,m}$ can be identified with a closed space of the standard Sobolev space $H^m(\mathbb{T}^d_\rho)$ consisting of functions which are complex differentiable. The manifold $\mathbb{T}^d_\rho$ has $2d$ real dimensions so that, when $m > d$, the standard Sobolev embedding theorem shows that $H^{\rho,m+l}$ $(l = 1, 2, \ldots)$ embeds continuously into $C^l(\mathbb{T}^d_\rho)$. Since the uniform limit of complex differentiable functions is also complex differentiable, we conclude that our space is a closed space of the standard Sobolev space of $\mathbb{T}^d_\rho$ considered as a $2d$-dimensional real manifold. Several variants of this idea appear already in Bergman spaces in [RS75, RS80].

We also point out that the set of functions in $H^{\rho,m}$ which take real values for real arguments is a closed set in $H^{\rho,m}$ (this set is also a linear space over the reals). Since we will show that our operators map this set into itself, we get that the fixed point we produce will be such that they give real values for real arguments.

3.2. Properties of the chosen spaces $H^{\rho,m}$ above. We note several well-known properties of the space $H^{\rho,m}$ defined in section 3.1, which will play a crucial role in what follows.

LEMMA 5 (interpolation inequalities). *For any $0 \le i \le m$, $0 \le \nu \le 1$, denote $s = (1-\nu)i + \nu m$, then we have the following inequalities:*

(1) Sobolev case: *For $f \in H^m$, there exists a constant $C_{i,m} > 0$ depending only on i, m such that*

$$\|f\|_{H^s} \le C_{i,m} \cdot \|f\|_{H^i}^{1-\nu} \cdot \|f\|_{H^m}^{\nu}. \tag{3.1}$$

(2) Analytic case: *For $\rho > 0$, $g \in H^{\rho,m}$, there exists a constant $C_{i,\rho,m} > 0$ depending only on i, ρ, m such that*

$$\|g\|_{H^{\rho,s}} \le C_{i,\rho,m} \cdot \|g\|_{H^{\rho,i}}^{1-\nu} \cdot \|g\|_{H^{\rho,m}}^{\nu}. \tag{3.2}$$

The inequality (3.1) is the very standard Sobolev interpolation inequality in the literature [Tay97, Zeh75]. Since, as mentioned before, the spaces $H^{\rho,m}(\mathbb{T}^d)$ can be considered as a subspace of the standard Sobolev space in $\mathbb{T}^d_\rho$, we also have (3.2).

LEMMA 6 (Banach algebra properties). *We have the following properties in two cases:*

(1) Sobolev case (see [AF03, Tay97]): *Let $m > \frac{d}{2}$; there exists a constant $C_{m,d} > 0$ depending only on m, d such that for u_1, $u_2 \in H^m$, the product $u_1 \cdot u_2 \in H^m$ and*

$$\|u_1 u_2\|_{H^m} \le C_{m,d} \|u_1\|_{H^m} \|u_2\|_{H^m}.$$

(2) Analytic case: *For $\rho > 0$, $m > d$, there exists a constant $C_{\rho,m,d} > 0$ depending only on ρ, m, d such that for u_1, $u_2 \in H^{\rho,m}$, the product $u_1 \cdot u_2 \in H^{\rho,m}$ and*

$$\|u_1 u_2\|_{H^{\rho,m}} \le C_{\rho,m,d} \|u_1\|_{H^{\rho,m}} \|u_2\|_{H^{\rho,m}}.$$

In particular, $H^{\rho,m}$ is a Banach algebra when ρ, m, d are as above.

To analyze the operator $\mathcal{T}_\varepsilon$ defined in (2.9), we need to estimate the properties of the composition operator $\hat{g} \circ U$. The following properties are well-known consequences of Gagliardo–Nirenberg inequalities.

LEMMA 7 (composition properties). *We have the following properties in the two cases:*

(1) Sobolev case (see [Mar74, Tay97, IKT13]): *Let* $g \in C^m(\mathbb{R}^n, \mathbb{R}^n)$ *and assume that* $g(0) = 0$. *Then, for* $u \in H^m(\mathbb{T}^d, \mathbb{R}^n) \cap L^\infty(\mathbb{T}^d, \mathbb{R}^n)$, *we have*

$$\|g(u)\|_{H^m} \le C_m\left(\|u\|_{L^\infty}\right)\left(1 + \|u\|_{H^m}\right),$$

where $C_m(\eta) = \sup_{|x| \le \eta,\, \alpha \le m} |D^\alpha g(x)|$. *Particularly, when* $m > \frac{d}{2}$ *(so that, by the Sobolev embedding theorem* $H^m \subset L^\infty$*), if* $g \in C^{m+2}$ *and* $u, v, u+v \in H^m$, *then*

$$\begin{aligned} &\|g \circ (u+v) - g \circ u - Dg \circ u \cdot v\|_{H^m} \\ &\qquad \le C_{m,d}\left(\|u\|_{L^\infty}\right)\left(1 + \|u\|_{H^m}\right)\|g\|_{C^{m+2}}\|v\|_{H^m}^2 \end{aligned} \tag{3.3}$$

for some $C_{m,d} > 0$ *depending on the norm of* u.

(2) Analytic case: *Let* $g : B \to \mathbb{C}^n$ *with* B *being an open ball around the origin in* $\mathbb{C}^n$ *and assume that* g *is analytic in* B. *Then, for* $u \in H^{\rho,m}(\mathbb{T}^d_\rho, \mathbb{C}^n) \cap L^\infty(\mathbb{T}^d_\rho, \mathbb{C}^n)$ *with* $u(\mathbb{T}^d_\rho) \subset B$, *we have*

$$\|g(u)\|_{H^{\rho,m}} \le C_{\rho,m}\left(\|u\|_{L^\infty}\right)\left(1 + \|u\|_{H^{\rho,m}}\right).$$

Moreover, when $m > d$,

$$\begin{aligned} &\|g \circ (u+v) - g \circ u - Dg \circ u \cdot v\|_{H^{\rho,m}} \\ &\qquad \le C_{\rho,m,d}\left(\|u\|_{L^\infty}\right)\left(1 + \|u\|_{H^{\rho,m}}\right)\|v\|_{H^{\rho,m}}^2. \end{aligned}$$

The complete proof of Lemma 7 can be found in Proposition 1 in [Mar74], Proposition 3.9 in [Tay97], or Proposition 2.20 in [IKT13]. To make our paper self-contained, we just give a sketch of the ideas for the inequality (3.3), but refer the interested readers to the references above.

Since

$$\begin{aligned} &g \circ (u+v)(\theta) - g \circ u(\theta) - Dg \circ u(\theta) \cdot v(\theta) \\ &\qquad = \int_0^1 \int_0^t D^2 g \circ (u + sv)(\theta) \cdot v^2(\theta)\, ds dt, \end{aligned}$$

we get the desired result by the facts that $D^2 g \circ (u + tsv) \in H^m$ and its H^m norm is bounded uniformly in (t, s) and that H^m is a Banach algebra under multiplication by Lemma 6. The range of the derivative Dg is an $n \times n$ matrix, which can be identified with $\mathbb{R}^{n^2}$. Note that the dimension of the range of g does not play any role in our arguments.

The proof of Lemma 7 is rather elementary in the analytic case.

As a matter of fact, Lemma 7 gives not only the composition operator is differentiable but also presents a formula for the derivative. It is easy to check that the same argument leads to higher derivatives of the composition operator if we assume more regularity for the function g. More precisely, we have the following proposition.

Proposition 8 (regularity of composition operators). *We have the results in two cases:*

(1) Sobolev case: *Let* $m > \frac{d}{2}$. *Then, the left composition operator*

$$\mathcal{C}_g : H^m(\mathbb{T}^d, \mathbb{R}^n) \to H^m(\mathbb{T}^d, \mathbb{R}^n)$$

defined by

$$\mathcal{C}_g[u](\theta) = g(u(\theta)),$$

has the following properties:
If $g \in C^{m+1}(\mathbb{R}^n, \mathbb{R}^n)$*, then* $\mathcal{C}_g$ *is Lipschitz.*
If $g \in C^{m+l+1}(\mathbb{R}^n, \mathbb{R}^n)$ $(l = 1, 2, \ldots)$*, then* $\mathcal{C}_g$ *is* C^l*. Moreover, the derivative of the operator* $\mathcal{C}_g$ *is given by*

$$(D\mathcal{C}_g[u]v)(\theta) = Dg(u)v(\theta).$$

(2) Analytic case: *Let* $\rho > 0$*. Assume that* $m > d$ *and* $g : B \to \mathbb{C}^n$*, where* B *is an open ball around the origin in* $\mathbb{C}^n$ *and is analytic in* B*.*
Let $u_0 \in H^{\rho,m}$ *be such that* $u_0(\mathbb{T}^d_\rho) \subset B$*. Then for all* u *in a neighborhood* $\mathcal{U}$ *of* u_0 *in* $H^{\rho,m}$*, the operator* $\mathcal{C}_g : \mathcal{U} \to H^{\rho,m}$ *is analytic. Moreover, for* $v \in H^{\rho,m}$*, the derivative of the operator* $\mathcal{C}_g$ *is given by*

$$(D\mathcal{C}_g[u]v)(\theta) = Dg(u)v(\theta).$$

Proof. In fact, Lemma 7 shows that the operator $\mathcal{C}_g$ is C^1 when $g \in C^{m+2}$. For $g \in C^{m+l+1}$, we can proceed by induction. If we have proved the result for l and the formula for the derivative, we obtain the case for $l + 1$. Indeed, if $g \in C^{m+l+1}$, we have $\mathcal{C}_g$ is C^l. Then, for $g \in C^{m+l+2}$, $Dg \in C^{m+l+1}$, we get $D\mathcal{C}_g$ is C^l by induction. Namely, $\mathcal{C}_g$ is C^{l+1}.

In the analytic case, we start by observing that $u(\mathbb{T}^d_\rho) \subset B$ is a compact set by the Sobolev embedding theorem. Hence, it is at a bounded distance from the boundary of B. If the neighborhood of u is sufficiently small, the range of all the functions will also be contained in B. Then, we obtain our result by Lemma 7. We can also refer to [CCCdlL17] for more details. ∎

Note that, for the Sobolev case in Proposition 8, the regularity of $\mathcal{C}_g$ is not optimal; we refer to [RS96, AZ90, IKT13] for more results. Note also that, for the analytic case in Proposition 8, the result is not the most general result. There are results in the case of regularity where the Sobolev embedding theorem does not give continuity. In these cases, we need to pay more attention to the ranges of the functions. Since the functions are differentiable in the complex sense, we obtain that the composition operator $\mathcal{C}_g$ is differentiable in the complex sense by the chain rule to obtain the derivative. Further, to get that the operator $\mathcal{C}_g$ is analytic, we just recall the Cauchy result that also holds for functions whose arguments range over a complex Banach space. See [HP74].

4. Statement of the main results.

In this section, we state several results for the model (2.1). These results are aimed at different regularities of the forcing f: analyticity (Theorem 9), finite (but high enough) number of derivatives (Theorem 12), and low regularity (Theorem 14).

THEOREM 9. *Consider the model* (1.1)*. Suppose that* $f \in H^{\rho,m}(\mathbb{T}^d)$ *for some* $\rho > 0$, $m > d$ *and* g *is analytic in an open ball around the origin in the space* $\mathbb{C}^n$*.*

If Assumption 1 *holds and* $\|f\|_{H^{\rho,m}}$ *is small enough, then, for* $\varepsilon \in \Omega$*, where*

$$\Omega := \Omega(\sigma, \mu) = \{\varepsilon \in \mathbb{C} : \mathrm{Re}(\varepsilon) \geq \mu\,|\mathrm{Im}(\varepsilon)|,\ \sigma \leq |\varepsilon| \leq 2\sigma\} \tag{4.1}$$

with $\mu > \mu_0$ *for* $\mu_0 > 0$ *sufficiently large and* $\sigma > 0$ *sufficiently small, there is a unique solution* $U_\varepsilon \in H^{\rho,m}(\mathbb{T}^d)$ *for* (2.4)*.*

Furthermore, considering U_ε *as a function of* ε*, the mapping* $\varepsilon \mapsto U_\varepsilon : \Omega \to H^{\rho,m}(\mathbb{T}^d)$ *is analytic.*

In addition, when $\varepsilon \in \Omega$ *and* $\varepsilon \to 0$ *along the set* Ω*, the solution* $U_\varepsilon \to 0$ *and the mapping* $\varepsilon \mapsto U_\varepsilon$ *is continuous at* $\varepsilon = 0$*.*

Remark 10. The statement of Theorem 9 does not impose any Diophantine condition on the forcing frequency ω. Since we do not expand the solution as a power series in ε, there is no equation involving the small divisor appearing. We will, however, not get that the solution is differentiable with respect to ε at the origin $\varepsilon = 0$ and this may indeed be false in the generality considered in this paper. (See Remark 20.)

Remark 11. The smallness conditions on $\|f\|_{H^{\rho,m}}$ in Theorem 9 (as well as Theorems 12, 14 in what follows) will be made explicit during the proof. We anticipate that they depend on the spectral properties of A (the eigenvalues and the size of the projection), the number theoretic properties of ω, and the size of nonlinearity $\hat{g}$. We make explicit the small conditions in all steps.

THEOREM 12. *Consider the model* (1.1). *Suppose that* $f \in H^m(\mathbb{T}^d)$ *with* $m > \frac{d}{2}$ *and* $g \in C^{m+l}(\mathbb{R}^n, \mathbb{R}^n)$ $(l = 1, 2, \ldots)$.

If Assumption 1 *is satisfied and* $\|f\|_{H^m}$ *is small enough (or* $Lip(\hat{g})$ *is sufficiently small in the whole of* $\mathbb{R}^n$ *in the sense of Assumption* 2*), then, for* $\varepsilon \in \widetilde{\Omega}$, *where*

$$\widetilde{\Omega} := \widetilde{\Omega}(\sigma) = \{\varepsilon \in \mathbb{R}_+ : \sigma \leq |\varepsilon| \leq 2\sigma\} \tag{4.2}$$

with sufficiently small $\sigma > 0$, *there exists a unique solution* $U_\varepsilon \in H^m(\mathbb{T}^d)$ *for* (2.4).

Moreover, the solution U_ε *obtained above has the following regularity in* ε:

- *If* $g \in C^{m+1}(\mathbb{R}^n, \mathbb{R}^n)$, *then the mapping* $\varepsilon \mapsto U_\varepsilon : \widetilde{\Omega} \to H^m(\mathbb{T}^d)$ *is Lipschitz.*
- *If* $g \in C^{m+l+1}(\mathbb{R}^n, \mathbb{R}^n)$, *then the mapping* $\varepsilon \mapsto U_\varepsilon : \widetilde{\Omega} \to H^m(\mathbb{T}^d)$ *is* C^l.

In addition, when $\varepsilon \in \widetilde{\Omega}$ *and* $\varepsilon \to 0$ *along the set* $\widetilde{\Omega}$, *the solution* $U_\varepsilon \to 0$ *and the mapping* $\varepsilon \mapsto U_\varepsilon$ *are continuous at* $\varepsilon = 0$.

Remark 13. Note that if $Lip(\hat{g})$ is sufficiently small in the whole of $\mathbb{R}^n$, we do not impose any small condition on f. Otherwise, it is necessary to give the small condition on f. (Recall the analysis in section 2.4.)

We emphasize that the regularity of the solution U_ε in ε stated in Theorem 12 depends on the regularity of the composition operator presented in Proposition 8. Even if we show that the derivatives with respect to ε exist for all $\varepsilon > 0$, we do not make any claim about the limit of the derivatives as ε goes to 0.

The following Theorem 14 is for the situation when the forcing and the nonlinearity are rather irregular.

THEOREM 14. *We study* (1.1). *Suppose that* $f \in L^2(\mathbb{T}^d)$ *and* g *is globally Lipschitz continuous on* $\mathbb{R}^n$ *satisfying Assumption* 2. *Then, for* $\varepsilon \in \widetilde{\Omega}$, *there is a unique solution* $U_\varepsilon \in L^2(\mathbb{T}^d)$ *for* (2.4).

Under the above assumptions if $f \in H^1(\mathbb{T}^d)$ *and* $g \in C^{1+\mathrm{Lip}}$, *then the unique solution* U_ε *constructed above is in* $\cap_{0 \leq s < 1} H^s$.

Note that Theorem 14 applies to some piecewise linear models (the Lipschitz constant of the derivatives has to be sufficiently small). Such models appear naturally in many areas.

We also stress that in Theorem 14, for $f \in H^1(\mathbb{T}^d)$, we cannot claim that the solution is in H^1, but only that it belongs to the intersection $\cap_{0 \leq s < 1} H^s$. We do not have a contraction argument in this case, but we can estimate the speed of convergence of the iterative procedure in the space H^s for $0 \leq s < 1$.

In the analytic case (Theorem 9) and in the highly differentiable regularity (Theorem 12), when $m > (\frac{d}{2} + 2)$, we have that the solution U_ε is C^2 with respect to the argument θ. Hence, the quasi-periodic solutions $x(t)$ obtained through (2.3) is also a twice differentiable function of time. As a consequence, the solutions we have produced satisfy the differential equation (1.1) in the classical sense. In the lower regularity case, the solutions we produce solve the equation in the sense that the Fourier coefficients of (2.4) are the same in both sides. This is equivalent to solving (1.1) in the weak sense since the trigonometric polynomials are dense in the space of C^∞ test functions.

In this paper, we also present some results for PDE model (1.2). Since the formulation requires new definitions and auxiliary lemmas, we postpone the formulation of the results until section 7.

5. Analytic case: Proof of Theorem 9.

We prove Theorem 9 in the analytic sense by considering the fixed point equation (2.9) in the Banach space $H^{\rho,m}$ for any $\varepsilon \in \Omega(\sigma,\mu)$. Recall (2.9),

$$U_\varepsilon(\theta) = \mathcal{L}_\varepsilon^{-1}\left[\varepsilon f(\theta) - \varepsilon\hat{g}(U_\varepsilon(\theta))\right] \equiv \mathcal{T}_\varepsilon(U_\varepsilon)(\theta). \tag{5.1}$$

The first concern is the invertibility of the linear operator $\mathcal{L}_\varepsilon$ and the quantitative bounds on its inverse when ε ranges over the complex domain $\Omega(\sigma,\mu)$ defined in (4.1). We remark that it is impossible to obtain the same bounds if ε belongs to the imaginary axis. In fact, we conjecture that the optimal domain of ε, when the solution U_ε of (5.1) is considered as a function of ε, do not extend to the imaginary axis.

Second, since we want to obtain a solution U_ε analytic in ε, we will define a space consisting of functions analytic in ε. (See section 5.2.) By reinterpreting the fixed point problem in the space $H^{\rho,m,\Omega}$, we obtain rather directly the analytic dependence on ε of the solutions U_ε. The delicate steps are to show that, for each $\varepsilon \in \Omega(\sigma,\mu)$, the operator $\mathcal{T}_\varepsilon$ maps a ball centered at the origin in the space $H^{\rho,m}$ to itself and it is a contraction in this ball.

5.1. Estimates on the inverse operator $\mathcal{L}_\varepsilon^{-1}$.

For the analytic nonlinearity g, the linear part A is dominant with respect to the nonlinear part $\hat{g}$. Moreover, the Lipschitz constant of $\hat{g}$ can be small enough in a sufficiently small domain.

We now study the linear operator defined by

$$\mathcal{L}_\varepsilon = \varepsilon\,(\omega\cdot\partial_\theta)^2\,Id + (\omega\cdot\partial_\theta)\,Id + \varepsilon A.$$

Our main result in this section includes that $\mathcal{L}_\varepsilon$ is boundedly invertible from the analytic function space $H^{\rho,m}$ to itself when ε ranges over a complex conical domain $\Omega(\sigma,\mu)$, which is away from the imaginary axis. Of course, this result requires the condition on A in Assumption 1.

A key ingredient for the result is that the norms of the functions can be read off from the sizes of the Fourier series and that the operator $\mathcal{L}_\varepsilon$ acts in a very simple matter in Fourier series. Indeed, if the matrix A was diagonal, the operator $\mathcal{L}_\varepsilon$ will be just a Fourier multiplier in each component (this case is worth keeping in mind as a heuristic guide).

5.1.1. Some elementary manipulations.

A consequence of Assumption 1 is that there exists a basis of generalized eigenvectors $\Phi_i \in \mathbb{C}^n$ $(i = 1,2,\ldots,n)$ such that

$$A\Phi = J\Phi,\ \ \Phi = (\Phi_1,\ldots,\Phi_n)^\top, \tag{5.2}$$

where J is the standard Jordan normal form. That is, for $1 \le p \le n$, $1 \le j \le p$,

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_p \end{pmatrix}_{n\times n}, \quad J_j = \begin{pmatrix} \lambda_j & & & 0 \\ 1 & \lambda_j & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_j \end{pmatrix}_{\mathbf{d}\times\mathbf{d}}.$$

Note that the subscript $n \times n$ denotes the dimension of the matrix J. The symbol $\mathbf{d}$ $(1 \le \mathbf{d} \le n)$ represents the multiplicity of the eigenvalue λ_j. More precisely, the Jordan matrix J depends on the spectra of the matrix A.

When we write a function $U_\varepsilon \in H^{\rho,m}$ in the Fourier expansion as

$$U_\varepsilon(\theta) = \sum_{k\in\mathbb{Z}^d} \widehat{U}_{k,\,\varepsilon} e^{\mathrm{i}k\cdot\theta} = \sum_{k\in\mathbb{Z}^d} \widetilde{\widehat{U}}_{k,\,\varepsilon} \Phi e^{\mathrm{i}k\cdot\theta}$$

with $\widehat{U}_{k,\,\varepsilon}$, $\widetilde{\widehat{U}}_{k,\,\varepsilon} \in \mathbb{C}^n$ and $\Phi \in \mathbb{C}^{n^2}$ being the one specified in (5.2), the operator $\mathcal{L}_\varepsilon$ acting on the Fourier basis becomes

$$\mathcal{L}_\varepsilon(\Phi e^{\mathrm{i}k\cdot\theta}) = \left(-\varepsilon(k\cdot\omega)^2 Id + \mathrm{i}(k\cdot\omega)Id + \varepsilon J\right)\Phi e^{\mathrm{i}k\cdot\theta} =: L_\varepsilon(k\cdot\omega)\Phi e^{\mathrm{i}k\cdot\theta},$$

where

$$\begin{aligned} L_\varepsilon(a) &= -\varepsilon a^2 Id + \mathrm{i}aId + \varepsilon J \\ &= \begin{pmatrix} L_{\varepsilon,1}(a) & & & 0 \\ & L_{\varepsilon,2}(a) & & \\ & & \ddots & \\ 0 & & & L_{\varepsilon,p}(a) \end{pmatrix}_{n\times n} \end{aligned} \tag{5.3}$$

with, for $1 \le j \le p$,

$$L_{\varepsilon,j}(a) = \begin{pmatrix} l_{\varepsilon,j}(a) & & 0 & \\ \varepsilon & l_{\varepsilon,j}(a) & & \\ & \ddots & \ddots & \\ 0 & & \varepsilon & l_{\varepsilon,j}(a) \end{pmatrix}_{\mathbf{d}\times\mathbf{d}}$$

with

$$l_{\varepsilon,j}(a) = -\varepsilon a^2 + \mathrm{i}a + \varepsilon\lambda_j. \tag{5.4}$$

The formula (5.3) gives that

$$L_\varepsilon^{-1}(a) = \begin{pmatrix} L_{\varepsilon,1}^{-1}(a) & & & 0 \\ & L_{\varepsilon,2}^{-1}(a) & & \\ & & \ddots & \\ 0 & & & L_{\varepsilon,p}^{-1}(a) \end{pmatrix}_{n\times n} \tag{5.5}$$

with, for $1 \le j \le p$,

(5.6)
$$L_{\varepsilon,j}^{-1}(a) = \begin{pmatrix} l_{\varepsilon,j}^{-1}(a) & & & & 0 \\ -\varepsilon l_{\varepsilon,j}^{-2}(a) & l_{\varepsilon,j}^{-1}(a) & & & \\ \varepsilon^2 l_{\varepsilon,j}^{-3}(a) & -\varepsilon l_{\varepsilon,j}^{-2}(a) & l_{\varepsilon,j}^{-1}(a) & & \\ \vdots & \ddots & \ddots & \ddots & \\ (-1)^{\mathbf{d}-1}\varepsilon^{\mathbf{d}-1} l_{\varepsilon,j}^{-\mathbf{d}}(a) & \cdots & \varepsilon^2 l_{\varepsilon,j}^{-3}(a) & -\varepsilon l_{\varepsilon,j}^{-2}(a) & l_{\varepsilon,j}^{-1}(a) \end{pmatrix}_{\mathbf{d}\times\mathbf{d}}.$$

Consequently, to estimate the inverse of $\mathcal{L}_\varepsilon$, it suffices to estimate

$$\Gamma_\varepsilon := \sup_{a\in\mathbb{R}} |L_\varepsilon^{-1}(a)| \ge \sup_{k\in\mathbb{Z}^d} |L_\varepsilon^{-1}(k\cdot\omega)|, \tag{5.7}$$

where the matrix norm is defined by

$$|L| = \max_{i,j} |L_{ij}| \tag{5.8}$$

with L_{ij} being the (i,j)th variable of the matrix L. Of course, any other norm will work just as well since the nonlinear operators we need to estimate will have a small norm for any precise definition of the metric.

5.1.2. Bounds on L_ε^{-1} given in (5.5). For the matrix $L_\varepsilon(a)$ defined in (5.3), once we obtain the infimum of $|l_{\varepsilon,j}(a)|$ in (5.4) for $a \in \mathbb{R}$, we get the estimates of Γ_ε defined in (5.7). The following estimates are similar to those in [CCdlL13], which considered only the 1-dimensional case. We now present the details for the n-dimensional case.

Note that the estimates we obtain also apply to the standard Sobolev space H^m, which allows us to conclude very quickly the results for the finitely differentiable case presented in section 6. We first deal with two special cases, which throw some light on the general case. Of course, from the purely logical point of view, these special cases can be omitted since they can be covered in the general discussion. We note that Case 1 with $\varepsilon \in \mathbb{R}_+$ is the special case needed in the finite differentiability result. So it is worth dealing with it explicitly.

Case 1. When $\varepsilon \in \mathbb{R}_+$, we have, for $a \in \mathbb{R}$,

$$\begin{aligned} |l_{\varepsilon,j}(a)|^2 &= |-\varepsilon a^2 + \mathrm{i}a + \varepsilon\lambda_j|^2 \\ &= (-\varepsilon a^2 + \varepsilon\lambda_j)^2 + a^2 \\ &= \varepsilon^2 a^4 + (1 - 2\varepsilon^2\lambda_j)a^2 + \varepsilon^2\lambda_j^2. \end{aligned}$$

Take $G(v) = \varepsilon^2 v^2 + (1 - 2\varepsilon^2\lambda_j)v + \varepsilon^2\lambda_j^2$ with $v = a^2 \ge 0$. It is obvious that $G(v) \ge G(0) = \varepsilon^2\lambda_j^2$ since $DG(v) = 2\varepsilon^2 v + (1 - 2\varepsilon^2\lambda_j) > 0$ due to the smallness of ε. Therefore, we have

$$\inf_{a\in\mathbb{R}} |l_{\varepsilon,j}(a)| \ge |\varepsilon\lambda_j|. \tag{5.9}$$

Equivalently,

$$\sup_{a\in\mathbb{R}} |l_{\varepsilon,j}(a)|^{-1} \le |\varepsilon\lambda_j|^{-1}.$$

Together with (5.6) and the matrix norm defined in (5.8), we have that

$$\begin{aligned}\Gamma_\varepsilon &= \sup_{a\in\mathbb{R}} |L_\varepsilon^{-1}(a)| = \sup_{a\in\mathbb{R}} \max_{1\le j\le p} |L_{\varepsilon,j}^{-1}(a)|\\ &= \sup_{a\in\mathbb{R}} \max_{1\le j\le p} \left(\max_{1\le m\le \mathbf{d}} |\varepsilon^{m-1} l_{\varepsilon,j}^{-m}(a)| \right)\\ &\le \max_{1\le j\le p} \left(\max_{1\le m\le \mathbf{d}} |\varepsilon|^{m-1}\cdot|\varepsilon|^{-m}|\lambda_j|^{-m} \right)\\ &\le |\varepsilon|^{-1} \max_{1\le j\le p} \left(\max_{1\le m\le \mathbf{d}} |\lambda_j|^{-m} \right).\end{aligned}$$

Case 2. When ε is pure imaginary, i.e., $\varepsilon = \mathrm{i}s$ with s small enough. In this case, there exists a real root a such that $|l_{\varepsilon,j}(a)| = 0$ since the discriminant $1 + 4s^2\lambda_j > 0$ (by the smallness of s) for $-sa^2 + a + s\lambda_j = 0$. Hence, the operator $\mathcal{L}_\varepsilon$ is unbounded if the small parameter ε locates in the imaginary axis, which makes the contraction mapping principle inapplicable.

We conjecture that no solutions for (2.5) exist when ε is purely imaginary because zero divisors can be considered as resonance.

To study the analyticity in ε of the function U_ε satisfying (5.1), it will be interesting to study the inverse of $\mathcal{L}_\varepsilon$ when ε ranges over the complex domain $\Omega(\sigma,\mu)$ defined in (4.1).

In what follows, in order to avoid having many constants, we will follow standard practice and denote by C_λ any constant depending only on the eigenvalues λ_j ($j = 1,\dots,n$) of the matrix A, but not ε.

PROPOSITION 15. *For Γ_ε defined in* (5.7)*, when $\varepsilon \in \Omega(\sigma,\mu)$, we have*

$$\Gamma_\varepsilon \le \sigma^{-1} C_\lambda.$$

Proof. Fix

$$\varepsilon = s_1 + \mathrm{i}s_2$$

for ε lining on a conical domain $\Omega(\sigma,\mu)$; we have $s_1 \ge \mu|s_2|$, where $\mu > \mu_0$ with some sufficiently large positive constant μ_0 (e.g., $\mu_0 > 10^3$), and $\sigma^2 \le s_1^2 + s_2^2 \le 4\sigma^2$. Namely,

$$\frac{1}{\sqrt{1+10^{-6}}}\cdot\sigma < \frac{1}{\sqrt{1+\mu^{-2}}}\cdot\sigma \le s_1 \le 2\sigma. \tag{5.10}$$

Then, one obtains that

$$\begin{aligned}|l_{\varepsilon,j}(a)|^2 &= |-\varepsilon a^2 + \mathrm{i}a + \varepsilon\lambda_j|^2\\ &= \left|-s_1(a^2-\lambda_j) - \mathrm{i}(s_2a^2 - a - s_2\lambda_j)\right|^2\\ &= s_1^2(a^2-\lambda_j)^2 + \left[s_2(a^2-\lambda_j) - a\right]^2.\end{aligned} \tag{5.11}$$

If $\lambda_j < 0$, it is obvious that

$$|l_{\varepsilon,j}(a)|^2 \ge s_1^2(a^2-\lambda_j)^2 \ge s_1^2\lambda_j^2. \tag{5.12}$$

The remaining task is to estimate $|l_{\varepsilon,j}(a)|^2$ in (5.11) in the case of $\lambda_j > 0$. The first term vanishes at the point $a = \pm\sqrt{\lambda_j}$. We define two regions in a as the following:

$$I_1 = [(1-10^{-3})\sqrt{\lambda_j},\ (1+10^{-3})\sqrt{\lambda_j}] \cup [(-1-10^{-3})\sqrt{\lambda_j},\ (-1+10^{-3})\sqrt{\lambda_j}],$$
$$I_2 = \mathbb{R}\setminus I_1.$$

When $a \in I_2$, we obtain the estimate

$$|l_{\varepsilon,j}(a)|^2 \geq s_1^2(a^2 - \lambda_j)^2 \geq s_1^2 \cdot O(|\lambda_j|^2). \tag{5.13}$$

In the case of $a \in I_1$, it is clear that $[s_2(a^2 - \lambda_j) - a] = O(s_2) - a$. Therefore,

$$\begin{aligned} |l_{\varepsilon,j}(a)|^2 &\geq \left[s_2(a^2 - \lambda_j) - a\right]^2 \\ &= [O(s_2) - a]^2 \geq \frac{a^2}{2} = O(|\lambda_j|) \geq O(|\lambda_j|) \cdot s_1^2 \end{aligned} \tag{5.14}$$

by the smallness of s_1 and s_2 (depending on ε). Note that the last inequality in the above estimate is very wasteful but we want to get estimates comparable to the ones we have in the other pieces. The inequalities (5.10), (5.12), (5.13), and (5.14) allow that

$$\inf_{a\in\mathbb{R}} |l_{\varepsilon,j}(a)| \geq \sigma \cdot O\left(\min\left\{|\lambda_j|,\, |\lambda_j|^{\frac{1}{2}}\right\}\right).$$

Equivalently,

$$\sup_{a\in\mathbb{R}} |l_{\varepsilon,j}(a)|^{-1} \leq \sigma^{-1} C_{\lambda_j}, \tag{5.15}$$

where

$$C_{\lambda_j} = O\left(\max\left\{|\lambda_j|^{-1},\, |\lambda_j|^{-\frac{1}{2}}\right\}\right). \tag{5.16}$$

Combining with the formulas in (5.5), (5.6), the matrix norm defined in (5.8), and (5.15) we obtain that

$$\begin{aligned} \Gamma_\varepsilon &= \sup_{a\in\mathbb{R}} |L_\varepsilon^{-1}(a)| = \sup_{a\in\mathbb{R}} \max_{1\leq j\leq p} |L_{\varepsilon,j}^{-1}(a)| \\ &= \sup_{a\in\mathbb{R}} \max_{1\leq j\leq p} \left(\max_{1\leq m\leq \mathbf{d}} |\varepsilon^{m-1} l_{\varepsilon,j}^{-m}(a)|\right) \\ &\leq \max_{1\leq j\leq p} \left(\max_{1\leq m\leq \mathbf{d}} \sigma^{m-1} \cdot \sigma^{-m} (C_{\lambda_j})^m\right) \\ &\leq \sigma^{-1} C_\lambda \end{aligned} \tag{5.17}$$

with

$$C_\lambda = \max_{\substack{1\leq j\leq p \\ 1\leq m\leq \mathbf{d}}} \left(C_{\lambda_j}\right)^m = O\left(\max_{\substack{1\leq j\leq p \\ 1\leq m\leq \mathbf{d}}} \left\{|\lambda_j|^{-m},\, |\lambda_j|^{-\frac{m}{2}}\right\}\right). \tag{5.18}$$

□

It follows from Proposition 15 that, for each $\varepsilon \in \Omega(\sigma, \mu)$,

$$\left|\varepsilon L_\varepsilon^{-1}(a)\right| \leq \sigma \cdot \sigma^{-1} C_\lambda \leq C_\lambda. \tag{5.19}$$

Due to the fact that the norm in the space $H^{\rho,m}$ is characterized by the Fourier coefficients, we have

$$\left\|\varepsilon \mathcal{L}_\varepsilon^{-1}\right\|_{H^{\rho,m}\to H^{\rho,m}} = \sup_{k\in\mathbb{Z}^d} \left|\varepsilon L_\varepsilon^{-1}(k\cdot\omega)\right| \leq C_\lambda. \tag{5.20}$$

This inequality is crucial in the contraction mapping argument used in section 5.3.

Remark 16. By (5.17), we see that Γ_ε can be bounded by σ^{-1} when σ is the minimum distance to the origin in the domain $\Omega(\sigma,\mu)$. Then it follows from (5.19) that the bad factors σ^{-1} can be dominated by the good factor σ. This is the reason why we choose $\sigma \le |\varepsilon| \le 2\sigma$, whose maximum and minimum distances to the origin are comparable. Note, however, that the estimate for $\varepsilon\mathcal{L}_\varepsilon^{-1}$ in (5.20) is independent of σ, so we obtain uniqueness of solutions for different σ, i.e., the solutions obtained for different σ agree for the ε in the intersection.

Remark 17. We note that the method presented in this present paper can accommodate small modifications leading to several generalizations. For example, we have the general equation (1.3) with $\mathbf{p} = \operatorname{diag}(\mathbf{p}_1,\dots,\mathbf{p}_n)$, $\mathbf{q} = \operatorname{diag}(\mathbf{q}_1,\dots,\mathbf{q}_n)$ being a diagonal matrix satisfying $\mathbf{p}_j$, $\mathbf{q}_j \in \mathbb{R}\setminus\{0\}$, $j = 1,\dots,n$. In this general case, the only modification with the present exposition is that the calculation for $l_{\varepsilon,j}(a)$ in (5.11) becomes

$$\begin{aligned}|l_{\varepsilon,j}(a)|^2 &= |-\varepsilon\mathbf{p}_j a^2 + \mathrm{i}\mathbf{q}_j a + \varepsilon\lambda_j|^2\\ &= \left[-s_1(\mathbf{p}_j a^2-\lambda_j) - \mathrm{i}(s_2\mathbf{p}_j a^2 - \mathbf{q}_j a - s_2\lambda_j)\right]^2\\ &= s_1^2(\mathbf{p}_j a^2-\lambda_j)^2 + \left[s_2(\mathbf{p}_j a^2-\lambda_j) - \mathbf{q}_j a\right]^2,\end{aligned}$$

which makes no difference in our discussion in Proposition 15.

5.2. Analyticity in ε.

Recall (2.9),

$$U(\theta) = \varepsilon\mathcal{L}_\varepsilon^{-1}\left[f(\theta) - \hat{g}(U(\theta))\right] \tag{5.21}$$

with U being a function of ε defined by $U = U_\varepsilon$. In this way, we define the operator $\mathcal{T}$ acting on functions analytic in ε, taking values in $H^{\rho,m}$, given by

$$\mathcal{T}(U) \equiv \varepsilon\mathcal{L}_\varepsilon^{-1}\left[f - \hat{g}(U)\right]. \tag{5.22}$$

Since we want to obtain the solution U_ε depending analytically on ε, we reinterpret $\mathcal{T}$ above as an operator acting on the space $H^{\rho,m,\Omega}$ consisting of analytic functions of ε taking values in $H^{\rho,m}$ with ε ranging over the domain $\Omega(\sigma,\mu)$. We endow the space $H^{\rho,m,\Omega}$ with the supremum norm

$$\|U\|_{\rho,m,\Omega} = \sup_{\varepsilon\in\Omega}\|U_\varepsilon\|_{\rho,m}, \tag{5.23}$$

where we use the abbreviation $\|\cdot\|_{\rho,m} := \|\cdot\|_{H^{\rho,m}}$ defined in Definition 3. The supremum norm of ε in (5.23) makes $H^{\rho,m,\Omega}$ a Banach space. Moreover, it is also a Banach algebra under multiplication when $m > d$ by Lemma 6.

We now show that the operator $\mathcal{T}$ defined in (5.22) maps the space $H^{\rho,m,\Omega}$ into itself.

LEMMA 18. *Assume that $m > d$. If $U \in H^{\rho,m,\Omega}$, then $\mathcal{T}(U) \in H^{\rho,m,\Omega}$. Precisely, if the mapping $\varepsilon \mapsto U_\varepsilon : \Omega \to H^{\rho,m}$ is complex differentiable, then the mapping $\varepsilon \mapsto \mathcal{T}_\varepsilon(U_\varepsilon) : \Omega \to H^{\rho,m}$ is complex differentiable as well.*

Proof. From the definition (5.22), we know that the operator $\mathcal{T}$ is composed of operators $\varepsilon\mathcal{L}_\varepsilon^{-1}$ and $\hat{g}$. It is clear that the map $\varepsilon \mapsto \hat{g}(U_\varepsilon) : \Omega \to H^{\rho,m}$ is complex differentiable since $\hat{g}$ is analytic and it does not depend on ε explicitly. Therefore, it suffices to show that the map $\varepsilon \mapsto \varepsilon\mathcal{L}_\varepsilon^{-1}(V_\varepsilon) : \Omega \to H^{\rho,m}$ is complex differentiable when V_ε, considered as a function from Ω to $H^{\rho,m}$, is complex differentiable.

We prove that the derivatives of $\varepsilon\mathcal{L}_\varepsilon^{-1}(V_\varepsilon)$ with respect to ε exist in the space $H^{\rho,m-\tau}(d<\tau\le m)$ instead of $H^{\rho,m}$. Then, we apply, somewhat surprisingly, Lemma 36 in Appendix A to conclude that the derivatives we consider indeed exist in the space $H^{\rho,m}$.

For a fixed $\varepsilon\in\Omega$, we expand $V_\varepsilon(\theta)$ as

$$V_\varepsilon(\theta)=\sum_{k\in\mathbb{Z}^d}\widehat{V}_{k,\,\varepsilon}e^{\mathrm{i}k\cdot\theta}$$

with

$$\widehat{V}_{k,\,\varepsilon}=\int_{\mathbb{T}^d_\rho}V_\varepsilon(\theta)e^{-\mathrm{i}k\cdot\theta}d\theta \tag{5.24}$$

satisfying

$$\left|\widehat{V}_{k,\,\varepsilon}\right|\le\|V_\varepsilon\|_{\rho,m}\,e^{-\rho|k|}\left(|k|^2+1\right)^{-\frac{m}{2}}. \tag{5.25}$$

Taking the derivative with respect to ε for (5.24), we have that

$$\frac{d}{d\varepsilon}\widehat{V}_{k,\,\varepsilon}=\int_{\mathbb{T}^d_\rho}\left(\frac{d}{d\varepsilon}V_\varepsilon\right)(\theta)e^{-\mathrm{i}k\cdot\theta}d\theta \tag{5.26}$$

with

$$\left|\frac{d}{d\varepsilon}\widehat{V}_{k,\,\varepsilon}\right|\le\left\|\frac{d}{d\varepsilon}V_\varepsilon\right\|_{\rho,m}e^{-\rho|k|}(|k|^2+1)^{-\frac{m}{2}}. \tag{5.27}$$

It follows from section 5.1 that

$$\varepsilon\mathcal{L}_\varepsilon^{-1}(V_\varepsilon)=\sum_{k\in\mathbb{Z}^d}\varepsilon L_\varepsilon^{-1}(\omega\cdot k)\widehat{V}_{k,\,\varepsilon}e^{\mathrm{i}k\cdot\theta}$$

with L_ε^{-1} defined in (5.5). By (5.15) and $\sup_{a\in\mathbb{R}}|l_{\varepsilon,j}(a)|^{-1}a^2\le\sigma^{-1}C_\lambda$ (which can be easily obtained using the same technique as (5.15)), we have that

$$\begin{aligned}&\left|\frac{d}{d\varepsilon}\left[\varepsilon^{\mathbf{d}}l_{\varepsilon,j}^{-\mathbf{d}}(\omega\cdot k)\right]\right|\\&\quad=\left|\mathbf{d}\cdot\varepsilon^{\mathbf{d}-1}\cdot l_{\varepsilon,j}^{-\mathbf{d}}(\omega\cdot k)-\mathbf{d}\cdot\varepsilon^{\mathbf{d}}\cdot l_{\varepsilon,j}^{-\mathbf{d}-1}(\omega\cdot k)\cdot\left(-(\omega\cdot k)^2+\lambda_j\right)\right|\\&\quad\le C_\lambda\cdot\sigma^{-1}.\end{aligned}$$

Together with the formulas (5.5) and (5.6), one gets

$$\begin{aligned}&\left|\frac{d}{d\varepsilon}\left(\varepsilon L_\varepsilon^{-1}(\omega\cdot k)\widehat{V}_{k,\,\varepsilon}\right)\right|\\&\quad\le\left|\frac{d}{d\varepsilon}\left(\varepsilon L_\varepsilon^{-1}(\omega\cdot k)\right)\right|\left|\widehat{V}_{k,\,\varepsilon}\right|+\left|\varepsilon L_\varepsilon^{-1}(\omega\cdot k)\right|\left|\frac{d}{d\varepsilon}\widehat{V}_{k,\,\varepsilon}\right|\\&\quad\le C_\lambda\cdot\sigma^{-1}\left(\left|\widehat{V}_{k,\,\varepsilon}\right|+\left|\frac{d}{d\varepsilon}\widehat{V}_{k,\,\varepsilon}\right|\right).\end{aligned} \tag{5.28}$$

Hence, (5.25), (5.27), and (5.28) yield that

$$\begin{aligned}
&\left\| \frac{d}{d\varepsilon}\left(\varepsilon L_\varepsilon^{-1}(\omega\cdot k)\widehat{V}_{k,\,\varepsilon}\right)e^{\mathrm{i}k\cdot\theta}\right\|_{\rho,\,m-\tau} \\
&\quad\le C_\lambda\cdot\sigma^{-1}\left(\left|\widehat{V}_{k,\,\varepsilon}\right|+\left|\frac{d}{d\varepsilon}\widehat{V}_{k,\,\varepsilon}\right|\right)\left\|e^{\mathrm{i}k\cdot\theta}\right\|_{\rho,\,m-\tau} \\
&\quad\le C_\lambda\cdot\sigma^{-1}\left(\|V_\varepsilon\|_{\rho,m}+\left\|\frac{d}{d\varepsilon}V_\varepsilon\right\|_{\rho,m}\right)e^{-\rho|k|}\left(|k|^2+1\right)^{-\frac{m}{2}} \\
&\qquad\cdot e^{\rho|k|}\left(|k|^2+1\right)^{\frac{m-\tau}{2}} \\
&\quad\le C_\lambda\cdot\sigma^{-1}\left(\|V_\varepsilon\|_{\rho,m}+\left\|\frac{d}{d\varepsilon}V_\varepsilon\right\|_{\rho,m}\right)(|k|^2+1)^{-\frac{\tau}{2}}.
\end{aligned}$$

Due to $\sum_{|k|=\kappa}1\le 2^d\kappa^{d-1}$, $k\in\mathbb{Z}^d$ (see [Ad63]), and choosing $d<\tau\le m$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d}(|k|^2+1)^{-\frac{\tau}{2}} &\le \sum_{\kappa=0}^{\infty}\sum_{|k|=\kappa}(\kappa^2+1)^{-\frac{\tau}{2}}\le\sum_{\kappa=0}^{\infty}(\kappa^2+1)^{-\frac{\tau}{2}}\sum_{|k|=\kappa}1 \\
&\le 2^d\sum_{\kappa=0}^{\infty}(\kappa^2+1)^{-\frac{\tau}{2}}\kappa^{d-1} \\
&\le 2^d\sum_{\kappa=0}^{\infty}(\kappa^2+1)^{-\frac{\tau-d+1}{2}}<\infty.
\end{aligned}$$

As a consequence, it follows from the Weierstrass M-test that the series

$$\sum_{k\in\mathbb{Z}^d}\frac{d}{d\varepsilon}\left(\varepsilon L_\varepsilon^{-1}(\omega\cdot k)\widehat{V}_{k,\,\varepsilon}\right)e^{\mathrm{i}k\cdot\theta}$$

converges uniformly on $\varepsilon\in\Omega$ in the space $H^{\rho,m-\tau}$. The fact that these formal derivatives are uniformly convergent shows that they are the true derivatives. Namely,

$$\frac{d}{d\varepsilon}\left(\varepsilon\mathcal{L}_\varepsilon^{-1}(V_\varepsilon)\right)=\sum_{k\in\mathbb{Z}^d}\frac{d}{d\varepsilon}\left(\varepsilon L_\varepsilon^{-1}(\omega\cdot k)\widehat{V}_{k,\,\varepsilon}\right)e^{\mathrm{i}k\cdot\theta}.$$

Therefore, we have that the mapping $\varepsilon\mapsto\varepsilon\mathcal{L}_\varepsilon^{-1}(V_\varepsilon):\Omega\to H^{\rho,m-\tau}$ is complex differentiable. Since $H^{\rho,m}\subset H^{\rho,m-\tau}$, we conclude that the mapping $\varepsilon\mapsto\varepsilon\mathcal{L}_\varepsilon^{-1}(V_\varepsilon):\Omega\to H^{\rho,m}$ is complex differentiable with derivatives in $H^{\rho,m-\tau}$ by Lemma 36 in Appendix A. ∎

5.3. Existence of the fixed point. Recall that the fixed point equation is

$$U(\theta)=\varepsilon\mathcal{L}_\varepsilon^{-1}\left[f(\theta)-\hat{g}(U(\theta))\right]\equiv\mathcal{T}(U)(\theta). \tag{5.29}$$

The proof of the existence of the solutions for the above equation is based on the fixed point theorem in the Banach space $H^{\rho,m,\Omega}$. We consider a ball $\mathcal{B}_r(0)$ around the origin in $H^{\rho,m,\Omega}$ with radius $r>0$ (chosen later) and we prove that $\mathcal{T}(\mathcal{B}_r(0))\subset\mathcal{B}_r(0)$ so that $\mathcal{T}$ is a contraction in the ball $\mathcal{B}_r(0)$.

It follows from (2.6) ($\hat{g}(0)=D\hat{g}(0)=0$) and Proposition 8 that the Lipschitz constant of the composition operator $\hat{g}\circ U$ is bounded by a constant times the radius

r when $U \in \mathcal{B}_r(0)$. Combining this with (5.20), for any elements U_1, $U_2 \in \mathcal{B}_r(0)$, we get, assuming that r is sufficiently small such that $C_\lambda \cdot r < \frac{1}{2}$,

$$\begin{aligned}\|\mathcal{T}(U_1) - \mathcal{T}(U_2)\|_{\rho,m,\Omega} &= \sup_{\varepsilon\in\Omega} \left\|\varepsilon\mathcal{L}_\varepsilon^{-1}\hat{g}(U_1) - \varepsilon\mathcal{L}_\varepsilon^{-1}\hat{g}(U_2)\right\|_{\rho,m} \\ &\le C_\lambda \cdot r \cdot \|U_1 - U_2\|_{\rho,m,\Omega} \\ &\le \frac{1}{2}\|U_1 - U_2\|_{\rho,m,\Omega}.\end{aligned}$$

Therefore, $\mathcal{T}$ is a contraction in the ball $\mathcal{B}_r(0)$.

Now we try to identify the conditions that the ball $\mathcal{B}_r(0)$ with r chosen as above gets mapped into itself. Now that we have chosen a radius r so that $\mathcal{T}$ is a contraction in $\mathcal{B}_r(0)$, in the following, we show that for ε, $\|f\|$ satisfying suitable assumptions (these are the assumptions stated in Theorem 9) the operator $\mathcal{T}$ maps the ball into itself.

Indeed, for $U \in \mathcal{B}_r(0)$, one has

$$\begin{aligned}\|\mathcal{T}(U)\|_{\rho,m,\Omega} &\le \|\mathcal{T}(0)\|_{\rho,m,\Omega} + \|\mathcal{T}(U) - \mathcal{T}(0)\|_{\rho,m,\Omega} \\ &\le C_\lambda\|f\|_{\rho,m} + \frac{1}{2}r.\end{aligned}$$

Under the assumption that $\|f\|_{\rho,m}$ is small enough in such a way that

$$C_\lambda\|f\|_{\rho,m} \le \frac{1}{2}r,$$

we get $\mathcal{T}(\mathcal{B}_r(0)) \subset \mathcal{B}_r(0)$.

In conclusion, by the fixed point theorem in the Banach space $H^{\rho,m,\Omega}$, there exists a unique solution $U \in H^{\rho,m,\Omega}$ analytic in ε for (5.29).

Remark 19. When we consider the operator $\mathcal{T}$ defined in (5.22) in the Banach space $H^{\rho,m,\Omega}$, the solution U_ε obtained via fixed point theorem does not lose any regularity on ε. That is, the solution U_ε naturally depends analytically on the parameter ε. However, in the finitely differentiable case, when we take $\varepsilon \in \widetilde{\Omega} \subset \mathbb{R}$ instead of $\varepsilon \in \Omega \subset \mathbb{C}$, the contraction mapping principle is not enough to get a solution U_ε with optimal regularity in ε since when $\rho = 0$, the space $H^{\rho,m,\widetilde{\Omega}}$ is no longer a Banach space with supremum in ε. We will combine this with the implicit function theorem to get the optimal regularity. (See section 6.1 for more details.) It is worth pointing out that in the low regularity case, especially in H^1, we need a more sophisticated contraction argument in some sense since there is no Lipschitz property for the composition operator $\hat{g} \circ u$ in H^1. (See section 6.2.)

Remark 20. We emphasize that the general solution U_ε obtained above may be not differentiable in ε at the origin $\varepsilon = 0$ since we do not impose any Diophantine condition for the frequency ω. Indeed, if U_ε was differentiable, we denote the derivative $U^{(1)}(\theta) := \frac{dU_\varepsilon(\theta)}{d\varepsilon}\,|_{\varepsilon=0}$ and assume $U_\varepsilon = 0$ at point $\varepsilon = 0$. Then, taking the derivative in ε at $\varepsilon = 0$ for (2.4), $U^{(1)}$ would satisfy that

$$(\omega \cdot \partial_\theta)\, U^{(1)}(\theta) = f(\theta). \tag{5.30}$$

If ω is sufficiently Liouvillean (e.g., $|\omega \cdot k| \ge \exp(-|k|^2)$; such an ω can be easily constructed for infinitely many k), we can easily construct analytic function f so that $U^{(1)}(\theta)$ solving (5.30) cannot even be a distribution.

Note that this argument also excludes very weak notions of differentiability (such as the existence of limits of quotients through a subsequence). We just argue that if such a limit exists, the limit would have to satisfy an equation that does not have solution.

LEMMA 21. *For the solution U_ε constructed above, we have that, for $\varepsilon \in \Omega$ and ε going to 0 along the set Ω, U_ε tends to 0. Moreover, the mapping $\varepsilon \mapsto U_\varepsilon$ is continuous at $\varepsilon = 0$.*

Proof. We take $\rho_1 > \rho > 0$ so that both the space $H^{\rho_1,m}$ and the space $H^{\rho,m}$ satisfy the assumptions of Theorem 9. Denote by U^1_ε, U_ε the solutions obtained by applying Theorem 9 to $H^{\rho_1,m}$, $H^{\rho,m}$, respectively. Then, we observe that $U^1_\varepsilon = U_\varepsilon$ by $U^1_\varepsilon \in H^{\rho_1,m} \subseteq H^{\rho,m}$ and the uniqueness conclusion in $H^{\rho,m}$. Moreover, we note that the set $\{U^1_\varepsilon \,|\, \varepsilon \in \overline{\Omega}\}$, where $\overline{\Omega}$ denotes the closure of Ω, is bounded in $H^{\rho_1,m}$ and hence it is precompact in $H^{\rho,m}$ topology.

To show that U_ε is continuous in ε at $\varepsilon = 0$, it suffices to verify that the graph $\mathcal{G}$ of U, that is,

$$\mathcal{G} := \left\{(\varepsilon, U_\varepsilon) | \, \varepsilon \in \overline{\Omega}\right\},$$

is compact in the $H^{\rho,m}$ topology. Since a ball in $H^{\rho_1,m}$ is precompact in $H^{\rho,m}$, we just need to prove that $\mathcal{G}$ is closed. Indeed, the sequence $(\varepsilon_n, U_{\varepsilon_n}) \in \mathcal{G}$ if and only if (2.8) is satisfied, that is,

$$\mathcal{L}_{\varepsilon_n}(U_{\varepsilon_n}(\theta)) = \varepsilon_n f(\theta) - \varepsilon_n \hat{g}(U_{\varepsilon_n}(\theta)).$$

Taking the limits of $\varepsilon_n \to \varepsilon_*$, $U_{\varepsilon_n} \to U^*$ for $n \to \infty$, one can obtain that

$$\mathcal{L}_{\varepsilon_*}(U^*(\theta)) = \varepsilon_* f(\theta) - \varepsilon_* \hat{g}(U^*(\theta)).$$

Hence, we conclude that $(\varepsilon_*, U^*) \in \mathcal{G}$. ◻

6. Finitely differentiable case: Proofs of Theorems 12 and 14.

In this section we present the proof of Theorem 12, which concerns the highly differentiable forcing f. We also prove Theorem 14 in which the forcing is assumed to be L^2 or H^1. The method used for the finitely differentiable case, especially H^1, is different from that for the analytic case.

6.1. Proof of Theorem 12.

When the forcing term f and the nonlinear term g are finitely differentiable, we consider $\varepsilon \in \widetilde{\Omega}$ defined in (4.2) for (2.1).

6.1.1. Regularity in ε.

In order to get solutions U_ε with some regularity in ε, we need to consider the operator $\mathcal{T}$ defined in (5.22) acting on the space $H^{m,\widetilde{\Omega}}$ consisting of differentiable functions of ε, taking values in H^m, with ε ranging over the domain $\widetilde{\Omega}$. Moreover, we endow $H^{m,\widetilde{\Omega}}$ with the supremum norm

$$\|U\|_{H^{m,\widetilde{\Omega}}} = \sup_{\varepsilon \in \widetilde{\Omega}} \|U_\varepsilon\|_{H^m}, \tag{6.1}$$

which is similar to the analytic case in section 5.2. Note that H^m is a Banach space and it is a Banach algebra when $m > \frac{d}{2}$ by Lemma 6. However, $H^{m,\widetilde{\Omega}}$ (in contrast with the analytic version $H^{\rho,m,\widetilde{\Omega}}$) is not a Banach space with the supremum norm defined in (6.1). In this case, if we just apply the fixed point theorem to the proof of Theorem 12 in the space $H^{m,\widetilde{\Omega}}$, we may lose some regularity in the argument ε. To avoid this shortcoming, we will combine the contraction argument with the implicit function theorem such that the solution U_ε with optimal regularity in ε can be obtained.

For the convenience of the implicit function theorem, we introduce the operator $\mathbf{T}$ involving the arguments ε and U as the following:

$$\mathbf{T}(\varepsilon, U) := U - \mathcal{T}(U), \tag{6.2}$$

where $\mathcal{T}$ is given in (5.22). This makes it clear to obtain the solution U, as a function of ε, having the same regularity as $\mathbf{T}$ by the implicit function theorem.

More precisely, for some $\varepsilon_0 \in \widetilde{\Omega}$, we first produce a solution U_{ε_0} such that $\mathbf{T}(\varepsilon_0, U_{\varepsilon_0}) = 0$. To get the optimal regularity of the solution U, taking $\widetilde{\Omega}$ to H^m, we apply the classical implicit function theorem for the operator $\mathbf{T}$. In this process, it is crucial to study the differentiability of the operator $\mathbf{T}$, mapping $\widetilde{\Omega} \times H^m$ to H^m, with respect to the arguments (ε, U) as well as the invertibility of $D_U \mathbf{T}(\varepsilon_0, U_{\varepsilon_0})$.

As a matter of fact, we can easily get the differentiability of the operator $\mathcal{T}$ with respect to the argument $U \in H^m$ since the operator $\mathcal{L}_\varepsilon$ is linear and the differentiability properties of the left composition operator $\hat{g} \circ U$ are already studied carefully in [IKT13, AZ90].

The key to our results will be the differentiability of the operator $\mathcal{T}$ with respect to ε as given by the following.

PROPOSITION 22. *Fix any $m \in \mathbb{N}$ with $m > \frac{d}{2}$ and $\sigma > 0$. We consider the map that $\varepsilon\mathcal{L}_\varepsilon^{-1} \in B(H^m, H^m)$ for every $\varepsilon \in \widetilde{\Omega}$, where $B(H^m, H^m)$ denotes the set of bounded operators from the space H^m to itself.*

For any $l \in \mathbb{N}$, the map $\varepsilon \mapsto \varepsilon\mathcal{L}_\varepsilon^{-1}$ is C^l considered as a mapping from $\widetilde{\Omega}$ to $B(H^m, H^m)$. Moreover, for any $l \in \mathbb{N}$ and $\varepsilon \in \widetilde{\Omega}$, $\frac{d^l}{d\varepsilon^l}(\varepsilon\mathcal{L}_\varepsilon^{-1}) \in B(H^m, H^m)$.

As a matter of fact, something stronger is true. The map $\varepsilon \mapsto \varepsilon\mathcal{L}_\varepsilon^{-1}$ is real analytic for $\varepsilon \in \widetilde{\Omega}$ and the radius of analyticity can be bounded uniformly for all $\varepsilon \in \widetilde{\Omega}$.

Proof. The key to the proof is the observation that, as noted in (5.9) in section 5.1.2, $|l_{\varepsilon,j}(a)| \geq |\varepsilon||\lambda_j| \geq \sigma|\lambda_j|$ for $\varepsilon \in \widetilde{\Omega}$.

To study the expansion in powers of δ for $l^{-1}_{\varepsilon+\delta,\, j}(a)$, we rewrite

$$\begin{aligned} l^{-1}_{\varepsilon+\delta,\, j}(a) &= \big((\varepsilon+\delta)(\lambda_j - a^2) + \mathrm{i}a\big)^{-1} \\ &= \big(\varepsilon(\lambda_j - a^2) + \mathrm{i}a + \delta(\lambda_j - a^2)\big)^{-1} \\ &= \big(\varepsilon(\lambda_j - a^2) + \mathrm{i}a\big)^{-1} \left(1 + \delta \frac{\lambda_j - a^2}{\varepsilon(\lambda_j - a^2) + \mathrm{i}a}\right)^{-1}. \end{aligned} \tag{6.3}$$

It is easy to see that the factor $\frac{\lambda_j - a^2}{\varepsilon(\lambda_j - a^2) + \mathrm{i}a}$ is bounded uniformly in a (compute the limit as $|a|$ tends to infinity and observe that the function is continuous in a since the denominator does not vanish) and uniformly in ε when ε ranges in an interval bounded away from zero.

Therefore, we can expand $(1 + \delta\frac{\lambda_j - a^2}{\varepsilon(\lambda_j - a^2) + \mathrm{i}a})^{-1}$ in (6.3) in powers of δ using the geometric series formula. Moreover, the radii of convergence are bounded uniformly in $\varepsilon \in \widetilde{\Omega}$ and the values of the coefficients in the expansion are also bounded uniformly in $a \in \mathbb{R}, \varepsilon \in \widetilde{\Omega}$.

Using the formula (5.5) in section 5.1.1 for the inverse $\mathcal{L}_\varepsilon^{-1}$, we also obtain that the matrices $L^{-1}_{\varepsilon+\delta}$ can be expanded in powers of δ with coefficients that are bounded uniformly in $a \in \mathbb{R}$, $\varepsilon \in \widetilde{\Omega}$.

We note that the operators $\mathcal{L}_\varepsilon^{-1}$ are multiplier operators (in the sense used in Fourier series). That is, for $\widehat{f}_k$ being the Fourier coefficients of function f in the space

H^m, the Fourier coefficients $(\widehat{\mathcal{L}_\varepsilon^{-1} f})_k$ of function $(\mathcal{L}_\varepsilon^{-1} f)$ in the space H^m have the structure

$$(\widehat{\mathcal{L}_\varepsilon^{-1} f})_k = \left(L_\varepsilon^{-1}\right)_k \widehat{f}_k, \tag{6.4}$$

where each $(L_\varepsilon^{-1})_k$ is an $n \times n$ matrix (i.e., $(L_\varepsilon^{-1})_k = L_\varepsilon^{-1}(k \cdot \omega)$ as specified in (5.5)). From the discussion in the above paragraph, we know that, for each k, $(L_\varepsilon^{-1})_k$ is uniformly analytic in ε. Thus, we conclude that the operator $\mathcal{L}_\varepsilon^{-1}$ is analytic in ε by (6.4).

In addition, we know that the Fourier indices k only enter into the multipliers $(L_\varepsilon^{-1})_k$ through $k \cdot \omega$ and the supremum of $(L_\varepsilon^{-1})_k$ over the Fourier index is bounded by the supremum in a, which is studied in the previous section 5.1.2. Together with the fact that the norms of functions in Sobolev spaces are measured by the size of the Fourier coefficients, we have that, for all $m > \frac{d}{2}$, the norm of $\mathcal{L}_\varepsilon^{-1}$ considered as an operator from the Sobolev space H^m to itself is defined by

$$\left\|\mathcal{L}_\varepsilon^{-1}\right\|_{H^m \to H^m} = \sup_{k \in \mathbb{Z}^d} \left\|\left(L_\varepsilon^{-1}\right)_k\right\| = \sup_{k \in \mathbb{Z}^d} \left\|L_\varepsilon^{-1}(k \cdot \omega)\right\|. \tag{6.5}$$

Note that the norms of $\|L_\varepsilon^{-1}(k\cdot\omega)\|$ are just finite-dimensional matrix norms. As a consequence, by (5.20), we can bound $\|\mathcal{L}_\varepsilon^{-1}\|_{H^m \to H^m}$ by the supremum of the multipliers defined in (6.5). Therefore, when we write $\mathcal{L}_{\varepsilon+\delta}^{-1} = \sum_{n=0}^{\infty} (\mathcal{L}_\varepsilon^{-1})_n \delta^n$, $\|(\mathcal{L}_\varepsilon^{-1})_n\|_{H^m \to H^m}$ can be bounded by the way of (6.5). That means $\frac{d^l}{d\varepsilon^l}(\varepsilon \mathcal{L}_\varepsilon^{-1}) \in B(H^m, H^m)$ for every $\varepsilon \in \widetilde{\Omega}$. ∎

6.1.2. Existence of the solutions. With all the above preliminaries established, now we turn to finishing the proof of Theorem 12. We divide the proof into two steps. First, for a fixed $\varepsilon_0 \in \widetilde{\Omega}$, we find a fixed point U_{ε_0} of $\mathcal{T}_\varepsilon$ defined in (2.9) by considering a domain $\mathcal{P} \subset H^m$ with $\mathcal{T}_\varepsilon(\mathcal{P}) \subset \mathcal{P}$ on which $\mathcal{T}_\varepsilon$ is a contraction. Second, we use the classical implicit function theorem to verify that the solution U_ε we obtained in the first step possesses the optimal regularity in ε.

Step 1. As we state in section 2.4, there are two ways to prove that $\mathcal{T}_\varepsilon$ is a contraction. One is that we choose a small ball in H^m such that $\mathrm{Lip}(\hat{g})$ is small in this ball. Meanwhile, we impose a smallness condition on f in this ball. In this way, the operator $\mathcal{T}_\varepsilon$ maps this ball into itself and it is a contraction in this ball. (We omit the details here since it is similar to section 5.3.) We can also assume that $\mathrm{Lip}(\hat{g})$ (or $D\hat{g}$) is globally small (the assumption is stated is Assumption 2) in such a way that

$$C_\lambda \cdot \mathrm{Lip}(\hat{g}) < \frac{1}{2}. \tag{6.6}$$

This shows that

$$\mathrm{Lip}(\hat{g}) \le O\left(\min_{\substack{1 \le j \le p \\ 1 \le m \le \mathbf{d}}} \left\{|\lambda_j|^m, \, |\lambda_j|^{\frac{m}{2}}\right\}\right) \tag{6.7}$$

by (5.18).

In this case, for a fixed $\varepsilon \in \widetilde{\Omega}$ and $U_1, U_2 \in H^m$, it follows from (5.20) that

$$\begin{aligned} \|\mathcal{T}_\varepsilon(U_1) - \mathcal{T}_\varepsilon(U_2)\|_{H^m} &= \|\varepsilon \mathcal{L}_\varepsilon^{-1}(\hat{g}(U_1) - \hat{g}(U_2))\|_{H^m} \\ &\le C_\lambda \cdot \mathrm{Lip}(\hat{g}) \cdot \|U_1 - U_2\|_{H^m} \\ &\le \frac{1}{2}\|U_1 - U_2\|_{H^m}. \end{aligned}$$

This makes $\mathcal{T}_\varepsilon$ a contraction in the whole space H^m.

In summary, we get a fixed point $U_{\varepsilon_0} \in H^m$ of (2.9) for some $\varepsilon_0 \in \widetilde{\Omega}$.

Step 2. It follows from Propositions 8 and 22 that the operator $\mathbf{T}$ defined in (6.2) is C^l with respect to the argument $(\varepsilon, U) \subseteq \widetilde{\Omega} \times H^m$. Based on Step 1, we have $\mathbf{T}(\varepsilon_0, U_{\varepsilon_0}) = 0$. Moreover, $D_U\mathbf{T}(\varepsilon_0, U_{\varepsilon_0}) = Id - D_U\mathcal{T}(\varepsilon_0, U_{\varepsilon_0}) = Id - \varepsilon_0\mathcal{L}_{\varepsilon_0}^{-1}D\hat{g}(U_{\varepsilon_0})$ is invertible since $\varepsilon_0\mathcal{L}_{\varepsilon_0}^{-1}$ is bounded and $D\hat{g}(U_{\varepsilon_0})$ is sufficiently small. Therefore, by the implicit function theorem, there exist an open neighborhood, included in $\widetilde{\Omega} \times H^m$ of $(U_{\varepsilon_0}, \varepsilon_0)$ and a C^l function U_ε satisfying $\mathbf{T}(\varepsilon, U_\varepsilon) = 0$ on this neighborhood.

6.2. Proof of Theorem 14. In this section, we will prove Theorem 14 in a different way from the first two cases (analytic and highly differentiable cases). The key problem is the properties of the composition operator $\hat{g} \circ u$ in the space $H^1(\mathbb{T}^d)$ or space $L^2(\mathbb{T})$.

6.2.1. The properties of compositions.

PROPOSITION 23. *For the composition operator defined by*

$$\mathcal{C}_{\hat{g}}[u](\theta) = \hat{g}(u(\theta)), \tag{6.8}$$

we have the following properties:

If we consider $\mathcal{C}_{\hat{g}}$ acting on $L^2(\mathbb{T}^d, \mathbb{R}^n)$ and assume that $\hat{g}$ is globally Lipschitz continuous on $\mathbb{R}^n$, then

$$\mathcal{C}_{\hat{g}}:\, L^2(\mathbb{T}^d, \mathbb{R}^n) \to L^2(\mathbb{T}^d, \mathbb{R}^n)$$

is Lipschitz continuous.

If we consider $\mathcal{C}_{\hat{g}}$ acting on $H^1(\mathbb{T}^d, \mathbb{R}^n)$ and assume that $\hat{g} \in C^{1+\mathrm{Lip}}$, then

$$\mathcal{C}_{\hat{g}}:\, H^1(\mathbb{T}^d, \mathbb{R}^n) \to H^1(\mathbb{T}^d, \mathbb{R}^n)$$

is bounded and continuous. In particular, given $\epsilon > 0$, there exists a constant $\widetilde{\delta} := \widetilde{\delta}(\epsilon, \mathrm{Lip}(\hat{g}), \hat{g}(0)) > 0$ so that $\|u\|_{H^1} \le \widetilde{\delta}$ implies $\|\mathcal{C}_{\hat{g}}(u)\|_{H^1} \le \epsilon$.

Proof. Since $\hat{g}$ is globally Lipschitz continuous on $\mathbb{R}^n$, denote $M = \mathrm{Lip}(\hat{g})$ (for ease of notation, we will use M in the following part) and for $u, v \in L^2(\mathbb{T}^d, \mathbb{R}^n)$, we get

$$|\hat{g}(u(\theta)) - \hat{g}(v(\theta))| \le M|u(\theta) - v(\theta)|.$$

Therefore,

$$\|\hat{g} \circ u - \hat{g} \circ v\|_{L^2} \le M\|u - v\|_{L^2}.$$

We refer to [AZ90, KS00] for the properties of the operator $\mathcal{C}_{\hat{g}}$ mapping space $H^1(\mathbb{T}^d, \mathbb{R}^n)$ to itself. ∎

Remark 24. We emphasize that for our results in L^2 and H^m $(m > \frac{d}{2})$, it is needed to assume that $Lip(\hat{g})$ is globally arbitrary small. This allows us to obtain that the operator $\mathcal{T}_\varepsilon$ in (2.9) is a contraction in the whole space.

However, due to the lack of Lipschitz regularity for the operator $\mathcal{C}_{\hat{g}}$ acting on the space H^1 (see [AZ90]), we need to choose a ball in H^1 so that the operator $\mathcal{T}_\varepsilon$ maps this ball into itself. Note that the chosen ball does not need to be small. We also do not require that the forcing be small in H^1.

6.2.2. Existence of the solutions. Now, we give the proof of Theorem 14.

First we give the proof for the result in space L^2. By Parseval's identity, we know that the L^2-norm is also expressible in terms of the Fourier coefficients. Together with the bound of $\varepsilon\mathcal{L}_\varepsilon^{-1}$ in (5.20), we have that $\mathcal{T}_\varepsilon(L^2) \subset L^2$. Moreover, for $u, v \in L^2$, one has

$$\|\mathcal{T}_\varepsilon(u) - \mathcal{T}_\varepsilon(v)\|_{L^2} = \left\|\varepsilon\mathcal{L}_\varepsilon^{-1}\left(\hat{g}\circ u - \hat{g}\circ v\right)\right\|_{L^2} \leq C_\lambda M\|u - v\|_{L^2}. \tag{6.9}$$

It follows from Assumption 2 that M is small enough. Therefore, $\mathcal{T}_\varepsilon$ is a contraction in L^2. The result in L^2 space follows immediately.

Now, we present the proof for the result in H^1. Using the interpolation inequality in Lemma 5, we obtain, for $n \geq 1$, $0 \leq s < 1$, that

$$\begin{aligned}
&\left\|\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^{n}(u)\right\|_{H^s} \\
&\quad\leq C_{0,1}\left\|\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^{n}(u)\right\|_{L^2}^{1-s}\left\|\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^{n}(u)\right\|_{H^1}^{s} \\
&\quad\leq C_{0,1}\left[(C_\lambda M)^n\right]^{1-s}\|\mathcal{T}_\varepsilon(u) - u\|_{L^2}^{1-s}\left\|\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^{n}(u)\right\|_{H^1}^{s},
\end{aligned} \tag{6.10}$$

where the second inequality comes from (6.9) inductively. Note that the inequality (6.6) gives that $[(C_\lambda M)^n]^{1-s}$ is decreasing exponentially.

The remaining task is to show that $\|\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^{n}(u)\|_{H^1}$ in (6.10) can be bounded independently of the iteration step n. As a matter of fact, from Proposition 23, we know that $u \in H^1$ implies $\hat{g}\circ u \in H^1$. Moreover, it is easy to check that

$$\|\hat{g}\circ u\|_{H^1} \leq M\|u\|_{H^1}.$$

Therefore, we get

$$\|\mathcal{T}_\varepsilon(u)\|_{H^1} = \left\|\varepsilon\mathcal{L}_\varepsilon^{-1}\left(f - \hat{g}\circ u\right)\right\|_{H^1} \leq C_\lambda\|f\|_{H^1} + C_\lambda M\|u\|_{H^1}.$$

We now choose a ball $B_r(0)$ centered at the origin in H^1 such that $B_r(0)$ is mapped by $\mathcal{T}_\varepsilon$ into itself. This can be achieved whenever we take r such that

$$C_\lambda\|f\|_{H^1} \leq \frac{1}{2}r \tag{6.11}$$

since $C_\lambda M < \frac{1}{2}$ given in (6.6). Note that the radius r chosen by (6.11) depends on the function f, which can be any function in H^1. As a consequence, for every $u \in B_r(0)$ and $n \in \mathbb{N}$, we obtain that $\mathcal{T}_\varepsilon^n(u) \in B_r(0)$ and

$$\|\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^{n}(u)\|_{H^1} \leq 2r.$$

Thus, (6.10) becomes

$$\|\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^{n}(u)\|_{H^s} \leq C_{0,1}\left[(C_\lambda M)^n\right]^{1-s}(2r)^s\,\|\mathcal{T}_\varepsilon(u) - u\|_{L^2}^{1-s}. \tag{6.12}$$

This indicates that the sequence $\mathcal{T}_\varepsilon^n(u)$ has a limit $u^* \in H^s$ and the fixed point obtained by the contraction mapping in L^2 should be in H^s. Note that (6.12) allows one to bound the distance in H^s from an initial guess to the true solution. That is,

$$\begin{aligned}\|u^* - u\|_{H^s} &= \left\| \lim_{n\to 0} \mathcal{T}_\varepsilon^n(u) - u \right\|_{H^s} \\ &= \left\| \sum_{n=0}^{\infty} \left[\mathcal{T}_\varepsilon^{n+1}(u) - \mathcal{T}_\varepsilon^n(u) \right] \right\|_{H^s} \\ &\le C_{0,1}(2r)^s \|\mathcal{T}_\varepsilon(u) - u\|_{L^2}^{1-s} \sum_{n=0}^{\infty} \left[(C_\lambda M)^n \right]^{1-s} \\ &\le C_{0,1}(2r)^s \left[1 - (C_\lambda M)^{1-s} \right]^{-1} \|\mathcal{T}_\varepsilon(u) - u\|_{L^2}^{1-s}.\end{aligned}$$

Remark 25. As shown in [AZ90], the conditions for composition operators mapping $H^{1+\delta}$ to itself are very strict. There are many mapping results for the composition in $H^{1+\delta} \cap L^\infty$, but it is not clear how the L^∞ norm behaves under the Fourier multipliers.

Therefore, using the methods of this paper, it seems that there is a gap between the treatments possible for the forcing, either in H^s $(0 \le s < 1)$ or in H^m $(m > d/2)$.

7. Results for PDEs. An important observation is that, since the treatment of (1.1) did not use any properties of the dynamics of equation, we can treat even ill-posed PDEs. The ill-posed equation (1.2) is a showcase of the possibilities of our method for the model (1.1). The heuristic principle is that we can think of evolutionary PDEs as models similar to (1.1) in which the role of the phase space $\mathbb{R}^n$ is taken up by a function space (of functions of the spatial variable x). Note that the nonlinearities in PDE models can be not just compositions but more complicated operators (even unbounded). For example, the nonlinearity $(u^2)_{xx}$ in (1.2) is an unbounded operator from a function space to itself. However, the fixed point problem under consideration in the Banach space we choose overcomes this tricky problem. (See section 7.3 for more details.)

The solutions produced in this section point in the direction that ill-posed equations, even if they lack a general theory of the existence and uniqueness of solutions, may admit many solutions that have a good physical interpretation.

For convenience, we rewrite (1.2) as

$$\varepsilon u_{tt} + u_t - \varepsilon\beta u_{xxxx} - \varepsilon u_{xx} = \varepsilon(u^2)_{xx} + \varepsilon f(\omega t, x),\ x \in \mathbb{T},\ t \in \mathbb{R},\ \beta > 0, \tag{7.1}$$

with a periodic boundary condition.

We define the full Lebesgue measure set

$$\mathcal{O} = \left\{ \beta > 0 : \frac{1}{\sqrt{\beta}} \text{is not an integer} \right\}. \tag{7.2}$$

Note that we shall only work with values of β in $\mathcal{O}$ so that the eigenvalues of the linear operator $\varepsilon\beta\partial_{xxxx} + \varepsilon\partial_{xx}$ in (7.1) are different from zero in a such way that the linear operator $\mathcal{N}_\varepsilon$ defined in (7.5) is invertible. (See section 7.3 for the details.)

Remark 26. There are other models of friction besides the u_t term in (7.1) that one could consider. The treatment given in the present paper is a very general method and could be applied to several friction models, such as u_{txx}.

We note also that our method for the ill-posed equation (7.1) with positive parameter β also applies to well-posed equation (7.1) with negative parameter β. It is even easier for the well-posed case since the eigenvalues of the linear operator $\varepsilon\beta\partial_{xxxx}+\varepsilon\partial_{xx}$ in (7.1) are not zero such that we can invert the operator $\mathcal{N}_\varepsilon$ defined in (7.5).

However, we just consider the ill-posed model (7.1) that serves as motivation for the readers. This ill-posed case is what appears in water wave theory [Bou72].

7.1. Formulation of the fixed point problem. Similarly to section 2 for the ODE model (2.1), we need to reduce the equation (7.1) to a fixed point problem. In this section, we just present the formal manipulations omitting specification of spaces. Indeed, the precise spaces defined in section 7.2 will be motivated by the desire to justify the formal manipulations and that the operators considered are a contraction.

Our goal is to find response solutions of the form

$$u_\varepsilon(t,x) = U_\varepsilon(\omega t, x), \tag{7.3}$$

where, for each fixed ε, $U_\varepsilon : \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$. Inserting (7.3) into (7.1), we get the following functional equation for U_ε:

$$\begin{aligned}\varepsilon\left(\omega\cdot\partial_\theta\right)^2 U_\varepsilon(\theta,x) + \left(\omega\cdot\partial_\theta\right) U_\varepsilon(\theta,x) - \varepsilon\beta\partial_x^4 U_\varepsilon(\theta,x) - \varepsilon\partial_x^2 U_\varepsilon(\theta,x)\\ = \varepsilon\left(U_\varepsilon^2\right)_{xx} + \varepsilon f(\theta,x).\end{aligned} \tag{7.4}$$

The solution of (7.4) will be the centerpiece of our treatment.

Denote by $\mathcal{N}_\varepsilon$ the linear operator

$$\mathcal{N}_\varepsilon U_\varepsilon(\theta,x) = \left[\varepsilon\left(\omega\cdot\partial_\theta\right)^2 + \left(\omega\cdot\partial_\theta\right) - \varepsilon\beta\partial_x^4 - \varepsilon\partial_x^2\right] U_\varepsilon(\theta,x). \tag{7.5}$$

Then, (7.4) can be rewritten as

$$\mathcal{N}_\varepsilon U_\varepsilon(\theta,x) = \varepsilon\left(U_\varepsilon^2\right)_{xx} + \varepsilon f(\theta,x). \tag{7.6}$$

As we will see in section 7.3, the operator $\mathcal{N}_\varepsilon$ is boundedly invertible in some appropriate space for $\varepsilon \in \Omega(\sigma,\mu)$ defined in (4.1). In this case, (7.6) becomes

$$U_\varepsilon(\theta,x) = \varepsilon\mathcal{N}_\varepsilon^{-1}\left[\left(U_\varepsilon^2\right)_{xx} + f(\theta,x)\right] \equiv \mathcal{T}_\varepsilon\left(U_\varepsilon(\theta,x)\right), \tag{7.7}$$

where, for convenience, we introduce the operator $\mathcal{T}_\varepsilon$. In section 7.4 dealing with the analytic case, we will show that there exists a solution U_ε analytic in ε for (7.7) by the contraction mapping argument. Moreover, in section 7.5 carrying out the finitely differentiable case, we will combine the contraction mapping principle with the classical implicit function theorem to get the regular results.

From the formal manipulation above, we find that the first key point is to study the invertibility of the operator $\mathcal{N}_\varepsilon$ and give quantitative estimates on its inverse for ε in a complex domain. Note that the linear operator $\mathcal{N}_\varepsilon$ defined in (7.5) used to study PDE model (7.1) is much more complicated than the linear operator $\mathcal{L}_\varepsilon$ defined in (2.7) for ODE model (2.1) since $\mathcal{N}_\varepsilon$ involves not only the angle variable $\theta \in \mathbb{T}^d$ but also the space variable $x \in \mathbb{T}$. This leads to different calculations for the inverse of $\mathcal{N}_\varepsilon$.

The second crucial part is that the nonlinearity $(U_\varepsilon^2)_{xx}$ may be unbounded from one space to itself. However, it happens that $\varepsilon\mathcal{N}_\varepsilon^{-1}(U_\varepsilon^2)_{xx}$ is bounded. (See Lemmas 29 and 30 for more details.)

To get a fixed point for (7.7), analogously to the smallness arguments in section 2.4 for the ODE model (2.1), we also need to impose some smallness conditions for the PDE model (7.1). However, we only consider a specially nonlinear map $U \mapsto \varepsilon\mathcal{N}_\varepsilon^{-1}(U^2)_{xx}$, which is analytic, be a contraction in a domain that contains a ball around $\varepsilon\mathcal{N}_\varepsilon^{-1} f$. It is nontrivial to choose a sufficiently small ball and the forcing f is assumed to be small in this ball.

7.2. Choice of spaces and the statement of our results. In this section, we give the concrete spaces we work in. Again, we note that the main principle is that the norms of the functions needed to be expressed in terms of the Fourier coefficients associated with the Fourier basis in arguments θ and x. This permits us to estimate the inverse of the linear operator $\mathcal{N}_\varepsilon$ just by estimating its Fourier coefficients. We also need these spaces to possess the Banach algebra properties and the properties of composition operators so that the nonlinear terms can be controlled. From the point of view of analyticity in ε, it is necessary to define spaces consisting of analytic functions with respect to ε.

In a way analogous to the definition in section 3, for $\rho \geq 0$, $m, d \in \mathbb{N}_+$, we define the space of analytic functions U in $\mathbb{T}_\rho^{d+1}$ with finite norm

$$\begin{aligned}\mathcal{H}^{\rho,m} &:= \mathcal{H}^{\rho,m}(\mathbb{T}^{d+1})\\ &= \Bigg\{ U \,:\, \mathbb{T}_\rho^{d+1} \to \mathbb{C} \mid U(\theta,x) = \sum_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}} \widehat{U}_{k,j} e^{\mathrm{i}(k\cdot\theta + j\cdot x)},\\ &\qquad \|U\|_{\rho,m}^2 = \sum_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}} \left|\widehat{U}_{k,j}\right|^2 e^{2\rho(|k|+|j|)} \left(|k|^2+|j|^2+1\right)^m < +\infty \Bigg\}.\end{aligned}$$

It is obvious that the space $(\mathcal{H}^{\rho,m},\ \|\cdot\|_{\rho,m})$ is a Banach space as well as a Hilbert space.

We actually consider $\mathcal{H}_0^{\rho,m}$, which is a subspace of $\mathcal{H}^{\rho,m}$, consisting of functions $U \in \mathcal{H}^{\rho,m}$ with

$$\int_0^{2\pi} U(\theta,x)dx = 0. \tag{7.8}$$

In the physical applications, we also consider the closed subspace of $\mathcal{H}^{\rho,m}$ in which the functions take real values for real arguments.

Note that the choice of the normalization condition (7.8) is motivated by the assumption that

$$\int_0^{2\pi} f(\theta,x)dx = 0.$$

Here and after, we consider our fixed point problems in the space $\mathcal{H}_0^{\rho,m}$. To simplicity the notation, we still write $\mathcal{H}^{\rho,m}$ as $\mathcal{H}_0^{\rho,m}$.

For $\rho > 0$, $\mathcal{H}^{\rho,m}$ consists of functions which are analytic in the domain $\mathbb{T}_\rho^{d+1}$. For $\rho = 0$, $\mathcal{H}^m := \mathcal{H}^{0,m}$ is just the regular Sobolev space. In this case, we use the abbreviation $\|\cdot\|_m := \|\cdot\|_{0,m}$.

Similarly to Lemma 6, when $\rho > 0$, $m > (d+1)$ or $\rho = 0$, $m > \frac{d+1}{2}$, we still have the Banach algebra properties in the space $\mathcal{H}^{\rho,m}$.

Now we are ready to state our main results on the existence of quasi-periodic solutions for the PDE (7.1) in the cases of analyticity and finite differentiability.

THEOREM 27. *Consider the model* (7.1) *with the coefficient* β *working in the set* (7.2). *Assume that* $f \in \mathcal{H}^{\rho,m}(\mathbb{T}^{d+1})$ *with* $\rho > 0$, $m > (d+1)$.

If $\|f\|_{\rho,m}$ *is small enough (depending on the coefficient* β*, the number theoretic properties of* ω*, and the nonlinearity of* (7.1)*), then, for* $\varepsilon \in \Omega$ *defined in* (4.1)*, there exists a unique solution* $U_\varepsilon \in \mathcal{H}^{\rho,m}(\mathbb{T}^{d+1})$ *for* (7.4).

Furthermore, considering U_ε *as a function of* ε*, we have that* $\varepsilon \mapsto U_\varepsilon : \Omega \to \mathcal{H}^{\rho,m}$ *is analytic when* $m > (d+3)$*. In addition, when* $\varepsilon \in \Omega$ *and it goes to* 0 *along* Ω*, the solution* U_ε *tends to* 0 *and* $\varepsilon \mapsto U_\varepsilon$ *is continuous at* $\varepsilon = 0$.

Our method also applies to finitely differentiable forcing, but we omit the details.

THEOREM 28. *Consider* (7.1) *with the coefficient* β *working in the set* (7.2). *Assume that* $f \in \mathcal{H}^m(\mathbb{T}^{d+1})$ *with* $m > \frac{d+1}{2}$.

If $\|f\|_m$ *is small enough, then, for* $\varepsilon \in \widetilde{\Omega}$ *defined in* (4.2), *there exists a unique solution* $U_\varepsilon \in \mathcal{H}^m(\mathbb{T}^{d+1})$ *for* (7.4).

Furthermore, for any $l \in \mathbb{N}$, *the map* $\varepsilon \mapsto U_\varepsilon$ *is* C^l *(even real analytic) considered as a mapping from* $\widetilde{\Omega}$ *to* $\mathcal{H}^m$. *In addition, when* ε *goes to* 0 *along* $\widetilde{\Omega}$, *the solution* U_ε *tends to* 0 *and the map* $\varepsilon \mapsto U_\varepsilon$ *is continuous at* $\varepsilon = 0$.

7.3. The boundedness of the operator $\mathcal{T}_\varepsilon$ defined in (7.7) taking $\mathcal{H}^{\rho,m}$ into itself. For the PDE model (7.1), the nonlinear map $U \mapsto (U^2)_{xx}$ (which in the ODE case was a composition operator with $\hat{g} \circ U$) is an unbounded operator from a space to itself. We will show, however, that the map $U \mapsto \varepsilon\mathcal{N}_\varepsilon^{-1}(U^2)_{xx}$ is bounded from a space to itself. To this end, we give the following lemmas and propositions. Some of the results would generalize for a nonlinearity of the form $U \mapsto (g(U))_{xx}$. We will not pursue these specialized results in this paper, but we think it would be an interesting subject.

LEMMA 29. *Let* $U \in \mathcal{H}^{\rho,m}$. *Denote*

$$h(U) = (U^2)_{xx}. \tag{7.9}$$

Then, h *is analytic from the space* $\mathcal{H}^{\rho,m}$ *to the space* $\mathcal{H}^{\rho,m-2}$. *Moreover, for* $V \in \mathcal{H}^{\rho,m}$, *we have that*

$$\|Dh(U)V\|_{\rho,m-2} \le 2\|U\|_{\rho,m}\|V\|_{\rho,m}.$$

Proof. We rewrite $h = h_1 \circ h_2$ with

$$\begin{aligned} h_1 : \mathcal{H}^{\rho,m} &\to \mathcal{H}^{\rho,m-2}, \\ U &\mapsto U_{xx} \end{aligned}$$

and

$$\begin{aligned} h_2 : \mathcal{H}^{\rho,m} &\to \mathcal{H}^{\rho,m} \\ U &\mapsto U^2. \end{aligned}$$

It is obvious that both h_1 and h_2 are analytic. Therefore, the composition operator $h : \mathcal{H}^{\rho,m} \to \mathcal{H}^{\rho,m-2}$ is analytic. Moreover,

$$Dh(U)V = \frac{d}{d\xi}h(U+\xi V)\bigg|_{\xi=0} = \frac{d}{d\xi}\left((U+\xi V)^2\right)_{xx}\bigg|_{\xi=0} = 2(UV)_{xx}.$$

This shows that

$$\|Dh(U)V\|_{\rho,m-2} \le 2\|UV\|_{\rho,m} \le 2\|U\|_{\rho,m}\|V\|_{\rho,m}$$

by the Banach algebra property in the space $\mathcal{H}^{\rho,m}$. ∎

Lemma 29 allows that the map $U \mapsto (U^2)_{xx}$ is bounded from the space $\mathcal{H}^{\rho,m}$ to $\mathcal{H}^{\rho,m-2}$. To prove the boundedness of the operator $\mathcal{T}_\varepsilon$ defined in (7.7), the remaining task is to show that $\varepsilon\mathcal{N}_\varepsilon^{-1} : \mathcal{H}^{\rho,m-2} \to \mathcal{H}^{\rho,m}$ is bounded.

LEMMA 30. *For a fixed* $\varepsilon \in \Omega(\sigma,\mu)$, *the operator* $\varepsilon\mathcal{N}_\varepsilon^{-1}$ *taking the space* $\mathcal{H}^{\rho,m-2}$ *into* $\mathcal{H}^{\rho,m}$ *is bounded.*

Proof. We verify that $\|\varepsilon\mathcal{N}_\varepsilon^{-1}\|_{\mathcal{H}^{\rho,m-2}\to\mathcal{H}^{\rho,m}}$ can be bounded by the supremum of its multipliers, as we argued in the proof of Proposition 22.

Consider $V(\theta,x)=\sum_{\substack{k\in\mathbb{Z}^d\\ j\in\mathbb{Z}\setminus\{0\}}}\widehat{V}_{k,j}e^{\mathrm{i}(k\cdot\theta+j\cdot x)}\in\mathcal{H}^{\rho,m-2}$, in which $j\neq 0$ comes from the setting (7.8). Then, by the linear operator $\mathcal{N}_\varepsilon$ defined in (7.5), we have the following Fourier expansion

$$\mathcal{N}_\varepsilon^{-1}(V)(\theta,x)=\sum_{\substack{k\in\mathbb{Z}^d\\ j\in\mathbb{Z}\setminus\{0\}}}\frac{1}{-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)}\widehat{V}_{k,j}e^{\mathrm{i}(k\cdot\theta+j\cdot x)}.$$

Note that we consider β in the full measure set given by (7.2), i.e., $\beta j^4-j^2\neq 0$ for $j\in\mathbb{Z}\setminus\{0\}$.

Recall that our goal is to obtain the bound of the operator from $\mathcal{H}^{\rho,m-2}$ to $\mathcal{H}^{\rho,m}$. Since

$$\begin{aligned}\left\|\mathcal{N}_\varepsilon^{-1}(V)\right\|_{\rho,m}^2&=\sum_{\substack{k\in\mathbb{Z}^d\\ j\in\mathbb{Z}\setminus\{0\}}}\frac{1}{\left|-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)\right|^2}\left|\widehat{V}_{k,j}\right|^2\\&\quad\cdot e^{2\rho(|k|+|j|)}\left(|k|^2+|j|^2+1\right)^m\\&=\sum_{\substack{k\in\mathbb{Z}^d\\ j\in\mathbb{Z}\setminus\{0\}}}\frac{(|k|^2+|j|^2+1)^2}{\left|-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)\right|^2}\left|\widehat{V}_{k,j}\right|^2\\&\quad\cdot e^{2\rho(|k|+|j|)}\left(|k|^2+|j|^2+1\right)^{m-2}\\&\le\sup_{\substack{k\in\mathbb{Z}^d\\ j\in\mathbb{Z}\setminus\{0\}}}\frac{(|k|^2+|j|^2+1)^2}{\left|-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)\right|^2}\|V\|_{\rho,m-2}^2,\end{aligned}$$

therefore, it suffices to estimate the supremum of $\widetilde{\mathbf{N}}_\varepsilon^{-1}$ defined by

$$\begin{aligned}\widetilde{\mathbf{N}}_\varepsilon^{-1}(k,j)&:=\frac{k^2+j^2+1}{-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)}\\&=\frac{k^2}{-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)}\\&\quad+\frac{j^2}{-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)}\\&\quad+\frac{1}{-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)}\end{aligned}\tag{7.10}$$

for $k\in\mathbb{Z}^d$, $j\in\mathbb{Z}\setminus\{0\}$. It is clear that (7.10) includes three parts, in which the main difficulty is to estimate the second term. We now present the details for the second term in the case of $k\neq 0$ (it is easy for $k=0$). Equivalently, we just need to estimate the infimum of

$$N_\varepsilon(a,t):=\frac{-\varepsilon a^2+\mathrm{i}a-\varepsilon(\beta t^2-t)}{t},\ a=(k\cdot\omega)\in\mathbb{R}\setminus\{0\},\ t=j^2\in\mathbb{N}_+.\tag{7.11}$$

Taking $\varepsilon=s_1+\mathrm{i}s_2\in\Omega(\sigma,\mu)$, we have

$$|N_\varepsilon(a,t)|^2=s_1^2\left[\frac{a^2}{t}-(1-\beta t)\right]^2+\left[s_2\left(\frac{a^2}{t}-(1-\beta t)\right)-\frac{a}{t}\right]^2,\tag{7.12}$$

which has an infimum controlled by σ by an argument similar to Proposition 15. We will estimate (7.12) in the cases of $\beta > 1$ and $0 < \beta < 1$.

When $\beta > 1$, i.e., $1 - \beta t < 0$, which shows that

$$|N_\varepsilon(a,t)|^2 \geq s_1^2 \left[\frac{a^2}{t} - (1-\beta t)\right]^2 \geq (\beta t - 1)^2 s_1^2$$
$$\geq (\beta - 1)^2 s_1^2 \geq s_1^2 C_\beta$$

by $t \geq 1$. To simplify the notation, in what follows, the constant C_β denotes all constants that depend on β. For example, the constants $C_\beta^{\frac{1}{2}}$, C_β^{-1}, etc., are replaced by the same notation C_β.

We focus mainly on the case of $0 < \beta < 1$. To analyze (7.12), we divide $t \in \mathbb{N}_+$ into two regions.

Case 1. When $t \geq [\frac{1}{\beta}] + 1$, we have that $1 - \beta t < 0$. Therefore

$$|N_\varepsilon(a,t)|^2 \geq s_1^2 \left[\frac{a^2}{t} - (1-\beta t)\right]^2 \geq s_1^2 C_\beta.$$

Case 2. When $1 \leq t \leq [\frac{1}{\beta}]$, we get that $t(1-\beta t) \in [C_\beta^1, C_\beta^2]$ with $C_\beta^2 \geq C_\beta^1 > 0$. It is clear that $\frac{a^2}{t} - (1-\beta t) = 0$ holds at $a^2 = t(1-\beta t) \in [C_\beta^1, C_\beta^2]$, namely, $a \in [-\sqrt{C_\beta^2}, -\sqrt{C_\beta^1}] \cup [\sqrt{C_\beta^1}, \sqrt{C_\beta^2}]$. Now, we divide the region in $a \in \mathbb{R}$ into two parts as follows:

$$I_1 = [(-1-10^{-3})\sqrt{C_\beta^2}, (-1+10^{-3})\sqrt{C_\beta^1}] \cup [(1-10^{-3})\sqrt{C_\beta^1}, (1+10^{-3})\sqrt{C_\beta^2}],$$
$$I_2 = \mathbb{R} \setminus I_1.$$

The case of $a \in I_2$ yields that

$$|N_\varepsilon(a,t)|^2 \geq s_1^2 \left[\frac{a^2}{t} - (1-\beta t)\right]^2 \geq s_1^2 C_\beta.$$

In the interval of $a \in I_1$, the term $\frac{a^2}{t} - (1-\beta t)$ can be bounded so that we can bound the second term in $|N_\varepsilon(a,t)|^2$, that is,

$$|N_\varepsilon(a,t)|^2 \geq \left[s_2\left(\frac{a^2}{t} - (1-\beta t)\right) - \frac{a}{t}\right]^2$$
$$= \left[O(s_2) - \frac{a}{t}\right]^2 \geq s_1^2 C_\beta$$

by the smallness of s_1, s_2 (due to $|\varepsilon|$ is sufficiently small). The above estimates for $|N_\varepsilon(a,t)|$ give that

$$|N_\varepsilon(a,t)| \geq s_1 C_\beta.$$

Therefore,

$$\inf_{a\in\mathbb{R},\, t\in\mathbb{N}_+} |N_\varepsilon(a,t)| \geq s_1 C_\beta \geq \sigma C_\beta \tag{7.13}$$

by the domain of $\varepsilon \in \Omega(\sigma,\mu)$. Consequently, for $\widetilde{\mathbf{N}}_\varepsilon^{-1}(k,j)$ defined in (7.10), we obtain

$$\sup_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}\setminus\{0\}} \left|\widetilde{\mathbf{N}}_\varepsilon^{-1}(k,j)\right| \leq \sup_{a\in\mathbb{R},\, t\in\mathbb{N}_+} \left|\widetilde{\mathbf{N}}_\varepsilon^{-1}(a,t)\right| \leq \sigma^{-1} C_\beta. \tag{7.14}$$

It follows that

$$\left\|\mathcal{N}_\varepsilon^{-1}(V)\right\|_{\rho,m} \leq \sigma^{-1} C_\beta \|V\|_{\rho,m-2}.$$

Since the norm in the space $\mathcal{H}^{\rho,m}$ can be characterized by the Fourier coefficients, we define

$$\left\|\mathcal{N}_\varepsilon^{-1}\right\|_{\mathcal{H}^{\rho,m-2}\to\mathcal{H}^{\rho,m}} = \sup_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}\setminus\{0\}} \left|\widetilde{\mathbf{N}}_\varepsilon^{-1}(k,j)\right|.$$

This allows us to conclude that

$$\left\|\varepsilon\mathcal{N}_\varepsilon^{-1}\right\|_{\mathcal{H}^{\rho,m-2}\to\mathcal{H}^{\rho,m}} \leq \sigma\cdot\sigma^{-1} C_\beta \leq C_\beta. \tag{7.15}$$ ◻

As a matter of fact, Lemmas 29 and 30 give that the operator $\mathcal{T}_\varepsilon$ defined in (7.7) is analytic from the space $\mathcal{H}^{\rho,m}$ to itself.

Remark 31. Note that the previous Lemma 30 includes the case of $\varepsilon \in \mathbb{R}$, which will be used later in the finitely differentiable case (see Lemma 33).

Note also that for (7.1), the nonlinearity will always be regular. Therefore, we just consider the finitely differentiable version with $m > \frac{d+1}{2}$. The analogue of the low regularity results for the ODE case would be easier to consider.

7.4. Proof of Theorem 27. In this section, we give the proof of Theorem 27.

7.4.1. Regularity in ε. Since we want to obtain solutions depending analytically on ε, proceeding as in section 5.2, we consider $\mathcal{T}_\varepsilon$ defined in (7.7) as a function of ε, namely, the operator $\mathcal{T} : \varepsilon \mapsto \mathcal{T}_\varepsilon$ acting on the space $\mathcal{H}^{\rho,m,\Omega}$ consisting of analytic functions of ε, taking values in $\mathcal{H}^{\rho,m}$ with ε ranging over the domain $\Omega(\sigma,\mu)$. We endow $\mathcal{H}^{\rho,m,\Omega}$ with supremum norm

$$\|U\|_{\rho,m,\Omega} = \sup_{\varepsilon\in\Omega(\sigma,\mu)} \|U_\varepsilon\|_{\rho,m},$$

which makes $\mathcal{H}^{\rho,m,\Omega}$ a Banach space. Moreover, it is also a Banach algebra when $m > (d+1)$. Based on Lemma 30, we show that the operator $\mathcal{T}$ maps the space $\mathcal{H}^{\rho,m,\Omega}$ into itself. The idea of the proof is similar to Lemma 18, but the details are different since the PDE model (7.1) involves a space variable x.

PROPOSITION 32. *If $m > (d+3)$, then the operator $\mathcal{T}$ defined in* (7.7) *maps the analytic Banach space $\mathcal{H}^{\rho,m,\Omega}$ into itself. Precisely, if the mapping $\varepsilon \mapsto U_\varepsilon : \Omega \to \mathcal{H}^{\rho,m}$ is complex differentiable, then, $\varepsilon \mapsto \mathcal{T}_\varepsilon(U_\varepsilon) : \Omega \to \mathcal{H}^{\rho,m}$ is also complex differentiable.*

Proof. From the fixed point equation (7.7), we know that $\mathcal{T}_\varepsilon$ is composed of $\varepsilon\mathcal{N}_\varepsilon^{-1}$ and h defined in Lemma 29. Lemma 29 gives that $h(\mathcal{H}^{\rho,m,\Omega}) \subset \mathcal{H}^{\rho,m-2,\Omega}$. Hence, it suffices to verify that $\varepsilon\mathcal{N}_\varepsilon^{-1}(\mathcal{H}^{\rho,m-2,\Omega}) \subset \mathcal{H}^{\rho,m,\Omega}$. In the following step, we use a similar method as that used in the proof of Proposition 18.

For a fixed $\varepsilon \in \Omega$, we expand $V_\varepsilon \in \mathcal{H}^{\rho,m-2}$ as

$$V_\varepsilon(\theta,x) = \sum_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}\setminus\{0\}} \widehat{V}_{k,j,\varepsilon} e^{\mathrm{i}(k\cdot\theta + j\cdot x)}$$

with

$$\left|\widehat{V}_{k,j,\varepsilon}\right| \leq \|V_\varepsilon\|_{\rho,m-2}\, e^{-\rho(|k|+|j|)}(|k|^2+|j|^2+1)^{-\frac{m-2}{2}}, \tag{7.16}$$

and

$$\left|\frac{d}{d\varepsilon}\widehat{V}_{k,j,\varepsilon}\right| \le \left\|\frac{d}{d\varepsilon}V_\varepsilon\right\|_{\rho,m-2} e^{-\rho(|k|+|j|)}(|k|^2+|j|^2+1)^{-\frac{m-2}{2}}. \tag{7.17}$$

It follows from (7.5) that

$$\varepsilon\mathcal{N}_\varepsilon^{-1}(V_\varepsilon)(\theta,x) = \sum_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}\setminus\{0\}} \varepsilon\mathbf{N}_\varepsilon^{-1}(k\cdot\omega,\, j)\widehat{V}_{k,j,\varepsilon}e^{\mathrm{i}(k\cdot\theta+j\cdot x)},$$

where

$$\mathbf{N}_\varepsilon^{-1}(k\cdot\omega,\, j) = \frac{1}{-\varepsilon(k\cdot\omega)^2+\mathrm{i}(k\cdot\omega)-\varepsilon(\beta j^4-j^2)} =: \mathbf{N}_\varepsilon^{-1}.$$

By (7.14), one has

$$\begin{aligned}
&\left|\frac{d}{d\varepsilon}\left(\varepsilon\mathbf{N}_\varepsilon^{-1}\widehat{V}_{k,j,\varepsilon}\right)\right| \\
&\quad \le |\mathbf{N}_\varepsilon^{-1}|\left|\widehat{V}_{k,j,\varepsilon}\right| + \left|\varepsilon\frac{d}{d\varepsilon}\mathbf{N}_\varepsilon^{-1}\right|\left|\widehat{V}_{k,j,\varepsilon}\right| + \left|\varepsilon\mathbf{N}_\varepsilon^{-1}\right|\left|\frac{d}{d\varepsilon}\widehat{V}_{k,j,\varepsilon}\right| \\
&\quad \le C_\beta\cdot\sigma^{-1}\left(\left|\widehat{V}_{k,j,\varepsilon}\right| + \left|\frac{d}{d\varepsilon}\widehat{V}_{k,j,\varepsilon}\right|\right).
\end{aligned}$$

Together with (7.16) and (7.17), we get

$$\begin{aligned}
&\left\|\frac{d}{d\varepsilon}\left(\varepsilon\mathbf{N}_\varepsilon^{-1}\widehat{V}_{k,j,\varepsilon}\right)e^{\mathrm{i}(k\cdot\theta+j\cdot x)}\right\|_{\rho,\, m-\tau} \\
&\quad \le C_\beta\cdot\sigma^{-1}\left(\left|\widehat{V}_{k,j,\varepsilon}\right| + \left|\frac{d}{d\varepsilon}\widehat{V}_{k,j,\varepsilon}\right|\right)\left\|e^{\mathrm{i}(k\cdot\theta+j\cdot x)}\right\|_{\rho,\, m-\tau} \\
&\quad \le C_\beta\cdot\sigma^{-1}\left(\|V_\varepsilon\|_{\rho,m-2} + \left\|\frac{d}{d\varepsilon}V_\varepsilon\right\|_{\rho,m-2}\right)e^{-\rho(|k|+|j|)}(|k|^2+|j|^2+1)^{-\frac{m-2}{2}} \\
&\qquad \cdot e^{\rho(|k|+|j|)}\left(|k|^2+|j|^2+1\right)^{\frac{m-\tau}{2}} \\
&\quad \le C_\beta\cdot\sigma^{-1}\left(\|V_\varepsilon\|_{\rho,m-2} + \left\|\frac{d}{d\varepsilon}V_\varepsilon\right\|_{\rho,m-2}\right)\left(|k|^2+|j|^2+1\right)^{-\frac{\tau-2}{2}}.
\end{aligned}$$

By choosing $d+3<\tau\le m$, we obtain that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}\setminus\{0\}}(|k|^2+|j|^2+1)^{-\frac{\tau-2}{2}} &\le \sum_{\widetilde{k}\in\mathbb{Z}^{d+1}}(|\widetilde{k}|^2+1)^{-\frac{\tau-2}{2}} \\
&\le \sum_{\kappa=0}^{\infty}(\kappa^2+1)^{-\frac{\tau-2}{2}}\sum_{|\widetilde{k}|=\kappa}1 \\
&\le 2^{d+1}\sum_{\kappa=0}^{\infty}(\kappa^2+1)^{-\frac{\tau-2}{2}}\kappa^d \\
&\le 2^{d+1}\sum_{\kappa=0}^{\infty}(\kappa^2+1)^{-\frac{\tau-d-2}{2}} < \infty.
\end{aligned}$$

Therefore, by the Weierstrass M-test, we conclude that the series

$$\sum_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}\setminus\{0\}} \frac{d}{d\varepsilon}\left(\varepsilon \mathbf{N}_\varepsilon^{-1}\widehat{V}_{k,j,\varepsilon}\right) e^{\mathrm{i}(k\cdot\theta+j\cdot x)}$$

converges uniformly on $\varepsilon \in \Omega$ in the space $\mathcal{H}^{\rho,m-\tau}$. Therefore, the map $\varepsilon \mapsto \varepsilon\mathcal{N}_\varepsilon^{-1}(V_\varepsilon) : \Omega \to \mathcal{H}^{\rho,m}$ is complex differentiable with derivatives in $\mathcal{H}^{\rho,m-\tau}$ by $\mathcal{H}^{\rho,m} \subset \mathcal{H}^{\rho,m-\tau}$ and Lemma 36 in the appendix. ◻

7.4.2. Proof of Theorem 27. We now start to deal with the fixed point equation

$$U(\theta,x) = \mathcal{N}_\varepsilon^{-1}\left[\varepsilon(U^2)_{xx} + \varepsilon f(\theta,x)\right] \equiv \mathcal{T}(U)(\theta,x) \tag{7.18}$$

in the space $\mathcal{H}^{\rho,m,\Omega}$. We will find a fixed point of $\mathcal{T}$ by considering a ball $\mathbb{B}_{\mathbf{r}}(0) \subset \mathcal{H}^{\rho,m,\Omega}$ such that $\mathcal{T}(\mathbb{B}_{\mathbf{r}}(0)) \subset \mathbb{B}_{\mathbf{r}}(0)$and $\mathcal{T}$ is a contraction in this ball.

By (7.15) and Lemma 29, for any $U_1, U_2 \in \mathbb{B}_{\mathbf{r}}(0)$, we have, assuming that $\mathbf{r}$ is small satisfying $C_\beta \cdot \mathbf{r} < \frac{1}{2}$,

$$\begin{aligned}\|\mathcal{T}(U_1) - \mathcal{T}(U_2)\|_{\rho,m,\Omega} &= \left\|\varepsilon\mathcal{N}_\varepsilon^{-1}h(U_1) - \varepsilon\mathcal{N}_\varepsilon^{-1}h(U_2)\right\|_{\rho,m,\Omega} \\ &= \sup_{\varepsilon\in\Omega}\left\|\varepsilon\mathcal{N}_\varepsilon^{-1}h(U_1) - \varepsilon\mathcal{N}_\varepsilon^{-1}h(U_2)\right\|_{\rho,m} \\ &\le C_\beta \cdot \mathbf{r}\|U_1 - U_2\|_{\rho,m,\Omega} \\ &< \frac{1}{2}\|U_1 - U_2\|_{\rho,m,\Omega}.\end{aligned}$$

This shows that $\mathcal{T}$ is a contraction in the ball $\mathbb{B}_{\mathbf{r}}(0)$.

Now we try to identify the conditions that the ball $\mathbb{B}_{\mathbf{r}}(0)$ with $\mathbf{r}$ chosen as above gets mapped into itself.

If $\mathbf{r}$ satisfies the conditions that make $\mathcal{T}$ a contraction in $\mathbb{B}_{\mathbf{r}}(0)$, we have that, for $U \in \mathbb{B}_{\mathbf{r}}(0)$,

$$\begin{aligned}\|\mathcal{T}(U)\|_{\rho,m,\Omega} &\le \|\mathcal{T}(0)\|_{\rho,m,\Omega} + \|\mathcal{T}(U) - \mathcal{T}(0)\|_{\rho,m,\Omega} \\ &\le C_\beta\|f\|_{\rho,m} + \frac{1}{2}\mathbf{r}.\end{aligned}$$

Therefore, under the assumption that $\|f\|_{\rho,m}$ is small enough such that $C_\beta\|f\|_{\rho,m} \le \frac{1}{2}\mathbf{r}$, we obtain that $\mathcal{T}(\mathbb{B}_{\mathbf{r}}(0)) \subset \mathbb{B}_{\mathbf{r}}(0)$.

In conclusion, there is a unique fixed point U in the space $\mathcal{H}^{\rho,m,\Omega}$ for (7.7). Namely, we obtain a solution U_ε analytic in ε for (7.4). For $\varepsilon \to 0$ along the set Ω, we conclude that the mapping $\varepsilon \mapsto U_\varepsilon$ is continuous at $\varepsilon = 0$. This can be proved in a similar way to that used to Lemma 21.

7.5. Proof of Theorem 28. In this section, we consider $\mathcal{T}$ defined in (7.18) acting on space $\mathcal{H}^{m,\widetilde{\Omega}}$ consisting of differentiable functions of ε taking values in $\mathcal{H}^m$ with ε ranging over the domain $\widetilde{\Omega}(\sigma,\mu)$ defined in (4.2). We endow $\mathcal{H}^{m,\widetilde{\Omega}}$ with the supremum norm

$$\|U\|_{m,\widetilde{\Omega}} = \sup_{\varepsilon\in\widetilde{\Omega}(\sigma,\mu)} \|U_\varepsilon\|_m. \tag{7.19}$$

We only have the result that the space $\mathcal{H}^m$ is a Banach space and it is also a Banach algebra when $m > \frac{d+1}{2}$ but not the space $\mathcal{H}^{m,\widetilde{\Omega}}$ with the supremum norm with respect to ε defined in (7.19). Consequently, the contraction mapping principle is not enough to get the solution U_ε with optimal regularity in ε. We will combine this with the implicit function theorem to obtain the regular solutions.

In order to use the implicit function theorem, analogously to section 6.1.1, the main issue is to study the differentiability of the operator $\mathcal{T}$ in (7.18) considered as an operator from $\widetilde{\Omega} \times \mathcal{H}^m$ to $\mathcal{H}^m$ as well as the invertibility of $D_U\mathcal{T}(\varepsilon, U)$.

We first present the result with respect to the argument U. Since Lemmas 29 and 30 also hold in the finitely differentiable setting, we have the following result when we work in the space $\mathcal{H}^m$.

LEMMA 33. *For a fixed $\varepsilon \in \widetilde{\Omega}(\sigma,\mu)$, the operator $\mathcal{T}_\varepsilon$ defined in* (7.7) *is analytic from the space $\mathcal{H}^m$ to itself.*

Now, we give the following proposition with the result that the operator $\mathcal{T}$ in (7.18) is differentiable in the argument ε. Note that $\mathcal{T}$ is composed by $\varepsilon\mathcal{N}_\varepsilon^{-1}$ and h defined in (7.9). Since $h(\mathcal{H}^m) \subset \mathcal{H}^{m-2}$, we need to verify that the derivatives of $\varepsilon\mathcal{N}_\varepsilon^{-1}$ with respect to ε is bounded from the space $\mathcal{H}^{m-2}$ to the space $\mathcal{H}^m$. Similarly to Proposition 22, we have the following.

PROPOSITION 34. *Fix any $m \in \mathbb{N}$ with $m > \frac{d+1}{2}$ and $\sigma > 0$. We consider the map that to every $\varepsilon \in \widetilde{\Omega}$, $\varepsilon\mathcal{N}_\varepsilon^{-1} \in B(\mathcal{H}^{m-2},\mathcal{H}^m)$. Moreover, for any $l \in \mathbb{N}$ and $\varepsilon \in \widetilde{\Omega}$, the map $\varepsilon \mapsto \varepsilon\mathcal{N}_\varepsilon^{-1}$ is C^l considered as a mapping from $\widetilde{\Omega}$ to $B(\mathcal{H}^{m-2},\mathcal{H}^m)$. Namely, $\frac{d^l}{d\varepsilon^l}(\varepsilon\mathcal{N}_\varepsilon^{-1}) \in B(\mathcal{H}^{m-2},\mathcal{H}^m)$.*

As a matter of fact, something stronger is true. The mapping $\varepsilon \mapsto \varepsilon\mathcal{N}_\varepsilon^{-1}$ is real analytic for $\varepsilon \in \widetilde{\Omega}$ and the radius of analyticity can be bounded uniformly for all $\varepsilon \in \widetilde{\Omega}$.

Proof. The idea of the proof is similar to Proposition 22. For $N_\varepsilon(a,t)$ defined in (7.11), we have $|N_\varepsilon(a,t)| \geq \sigma C_\beta$ by (7.13) in Lemma 30; we now expand $N_{\varepsilon+\delta}^{-1}(a,t)$ in powers of δ as

$$
\begin{aligned}
&N_{\varepsilon+\delta}^{-1}(a,t)\\
&= \left(-(\varepsilon+\delta)\left[\frac{a^2}{t} - (1-\beta t)\right] + \mathrm{i}\frac{a}{t}\right)^{-1}\\
&= \left(-\varepsilon\left[\frac{a^2}{t} - (1-\beta t)\right] + \mathrm{i}\frac{a}{t} - \delta\left[\frac{a^2}{t} - (1-\beta t)\right]\right)^{-1}\\
&= \left(-\varepsilon\left[\frac{a^2}{t} - (1-\beta t)\right] + \mathrm{i}\frac{a}{t}\right)^{-1}\left(1 - \delta\frac{\left[\frac{a^2}{t} - (1-\beta t)\right]}{-\varepsilon\left[\frac{a^2}{t} - (1-\beta t)\right] + \mathrm{i}\frac{a}{t}}\right)^{-1}.
\end{aligned} \tag{7.20}
$$

By the estimates in Lemma 30, we observe that the factor $\frac{[\frac{a^2}{t} - (1-\beta t)]}{-\varepsilon[\frac{a^2}{t} - (1-\beta t)] + \mathrm{i}\frac{a}{t}}$ is bounded uniformly in $a \in \mathbb{R}$, $t \in \mathbb{N}_+$, and $\varepsilon \in \widetilde{\Omega}$.

Therefore, we can expand $(1 - \delta\frac{[\frac{a^2}{t} - (1-\beta t)]}{-\varepsilon[\frac{a^2}{t} - (1-\beta t)] + \mathrm{i}\frac{a}{t}})^{-1}$ in (7.20) in powers of δ using the geometric series formula and the radii of convergence are bounded uniformly and the values of the function are also bounded in a ball which is uniform in $a \in \mathbb{R}, t \in \mathbb{Z}_+$, and $\varepsilon \in \widetilde{\Omega}$. That means N_ε^{-1} is uniformly analytic in ε for each $a \in \mathbb{R}, t \in \mathbb{Z}_+$, namely, $\widetilde{\mathbf{N}}_\varepsilon^{-1}$ given by (7.10) is analytic in ε.

To study the operator $\mathcal{N}_\varepsilon^{-1}$ as a mapping from the space $\mathcal{H}^{m-2}$ to the space $\mathcal{H}^m$, we need to consider its multiplier in the Fourier space. Precisely, for $\widehat{f}_{k,j}$ being the

Fourier coefficients of function f in the space $\mathcal{H}^{m-2}$, the Fourier coefficients $(\widehat{\mathcal{N}_\varepsilon^{-1} f})_{k,j}$ of function $(\mathcal{N}_\varepsilon^{-1} f)$ in the space $\mathcal{H}^m$ have the structure:

$$(\widehat{\mathcal{N}_\varepsilon^{-1} f})_{k,j} = \left(\widetilde{\mathbf{N}}_\varepsilon^{-1}\right)_{k,j} \widehat{f}_{k,j}$$

with $(\widetilde{\mathbf{N}}_\varepsilon^{-1})_{k,j} = \widetilde{\mathbf{N}}_\varepsilon^{-1}(k,\, j)$ given in (7.10). Hence, we get that $\mathcal{N}_\varepsilon^{-1}$ is analytic in ε.

Moreover, due to the fact that the norm in the space H^m is characterized by the Fourier coefficients, we can bound

$$\left\|\mathcal{N}_\varepsilon^{-1}\right\|_{H^{m-2}\to\mathcal{H}^m} = \sup_{k\in\mathbb{Z}^d,\, j\in\mathbb{Z}\setminus\{0\}} \left\|\widetilde{\mathbf{N}}_\varepsilon^{-1}(k,j)\right\| \tag{7.21}$$

due to the uniform boundedness of $\widetilde{\mathbf{N}}_\varepsilon^{-1}(k,j)$ in $k \in \mathbb{Z}^d$, $j \in \mathbb{Z} \setminus \{0\}$ by (7.14). Therefore, when we write $\mathcal{N}_{\varepsilon+\delta}^{-1} = \sum_{n=0}^{\infty}(\mathcal{N}_\varepsilon^{-1})_n \delta^n$, each $\|(\mathcal{N}_\varepsilon^{-1})_n\|_{\mathcal{H}^{m-2}\to\mathcal{H}^m}$ can be bounded in the sense of (7.21). That means $\frac{d^l}{d\varepsilon^l}(\varepsilon\mathcal{N}_\varepsilon^{-1}) \in B(\mathcal{H}^{m-2}, \mathcal{H}^m)$ for every $\varepsilon \in \widetilde{\Omega}$. ∎

Now, we start to prove Theorem 28 by constructing a fixed point U_{ε_0} for $\varepsilon_0 \in \widetilde{\Omega}$ first and then using the implicit function theorem to obtain the optimal regularity of U_ε in ε. It is similar to the proof in section 6.1.2. We omit some details here.

Proof. First, when we choose a small ball $\mathbb{B}_{\mathbf{r}}(0) \subset \mathcal{H}^{m,\widetilde{\Omega}}$, a similar process to section 7.4.2 allows us to obtain a fixed point $U_{\varepsilon_0} \in \mathcal{H}^m$ for some $\varepsilon_0 \in \widetilde{\Omega}$ by the contraction argument in this ball.

Then, according to Lemma 33 and Proposition 34, we obtain that the operator $\mathcal{T}$, defined in (7.18), acting on $\widetilde{\Omega} \times \mathcal{H}^m$ is C^l in arguments ε and U, respectively. Namely, $\mathbf{T}(\varepsilon, U) = U - \mathcal{T}(\varepsilon, U)$ is C^l in $\widetilde{\Omega} \times \mathcal{H}^m$. Based on the first step, we have $\mathbf{T}(\varepsilon_0, U_{\varepsilon_0}) = 0$. Moreover, $D_U\mathbf{T}(\varepsilon_0, U_{\varepsilon_0}) = Id - D_U\mathcal{T}(\varepsilon_0, U_{\varepsilon_0}) = Id - \varepsilon_0\mathcal{N}_{\varepsilon_0}^{-1}Dh(U_{\varepsilon_0})$ is invertible since $\varepsilon_0\mathcal{N}_{\varepsilon_0}^{-1}Dh(U_{\varepsilon_0})$ is sufficiently small in a small domain of the origin in $\mathcal{H}^m$. Therefore, by the implicit function theorem, there exist an open neighborhood included in $\widetilde{\Omega} \times \mathcal{H}^m$ of $(\varepsilon_0,\, U_{\varepsilon_0})$ and a C^l function U_ε satisfying $\mathbf{T}(\varepsilon, U_\varepsilon) = 0$ on this neighborhood. ∎

Appendix A. Some properties in analytic and finitely differentiable Banach spaces.

A.1. Analytic functions in Banach space.

DEFINITION 35. *Let X, Y be complex Banach spaces and $O \subset X$ is open. We say that $f : O \to Y$ is analytic if it is differentiable at all points of O and there exists a function $\gamma := \gamma_x(\|z\|)$ with $\frac{\gamma_x(\|z\|)}{\|z\|} \to 0$ as $\|z\| \to 0$, such that*

$$\|f(x+z) - f(x) - Df(x)\cdot z\| \le \gamma_x(\|z\|)$$

for all $x \in O$ and $z \in X$ such that $(x+z) \in O$.

Note that Definition 35 is a rather weak version of differentiability, but it is enough for this paper. For more analyticity of nonlinear functions in Banach spaces, we refer to [HP74, Muj86].

The main result of this appendix is the theory of complex analytic functions in Banach space, bootstrapping the meaning of derivatives of analytic functions. The result could be deduced from stronger results in [HP74, RS80], but we thought it would be useful to present a self-contained proof since this lemma could be useful in other applications.

LEMMA 36. *Let $U \subseteq \mathbb{C}$ be open and X, Y be complex Banach spaces, $X \subseteq Y$ with continuous embedding. Let $f : U \to X$, which is differentiable in Y for all $x \in U$, and*

$$\lim_{h\to 0} \left\| \frac{f(x+h) - f(x)}{h} - f'(x) \right\|_Y = 0. \tag{A.1}$$

Then, $f'(x) \in X$ and

$$\lim_{h\to 0} \left\| \frac{f(x+h) - f(x)}{h} - f'(x) \right\|_X = 0. \tag{A.2}$$

We start by proving the Cauchy–Goursat theorem for functions satisfying (A.1). The proof is rather straightforward. This will lead to a Cauchy formula, from which we can deduce (A.2).

PROPOSITION 37. *Let $g : U \to X \subseteq Y$, be differentiable everywhere in the sense of Y differentiable. Let γ be a triangle contour contained in U. Then*

$$\int_\gamma g(z)dz = 0.$$

Of course, by the usual approximation procedures, one can get the result for more general paths. This will not be needed for our purposes. Note that, by the fact that g is continuous as a function from U to Y, we know that the integrals over paths involved can be understood as Riemann integrals.

Proof. Suppose γ is a triangular contour with positive orientation, we construct four positively oriented contours that are triangles obtained by joining the midpoints of the sides of γ. Then, we have

$$\int_\gamma g(z)dz = \sum_{i=1}^{4} \int_{\gamma_i} g(z)dz.$$

Let γ_1 be selected such that

$$\left| \int_\gamma g(z)dz \right| \le \sum_{i=1}^{4} \left| \int_{\gamma_i} g(z)dz \right| \le 4 \left| \int_{\gamma_1} g(z)dz \right|.$$

If $\int_\gamma g(z)dz = b \neq 0$, we get

$$\left| \int_{\gamma_1} g(z)dz \right| \ge \frac{1}{4}|b|.$$

Proceeding by induction, we get a sequence of triangular contours $\{\gamma_n\}$, whose length equals $2^{-n}|\gamma|$, where $|\gamma|$ denotes the length of γ, such that

$$\left| \int_{\gamma_n} g(z)dz \right| \ge \frac{1}{4^n}|b|. \tag{A.3}$$

By the choice of γ_n, we have

$$\overline{\mathit{Interior\ of\ } \gamma_{n+1}} \subset \overline{\mathit{Interior\ of\ } \gamma_n}$$

and the length of the sides of γ_n goes to 0 as $n \to \infty$. Therefore there exists a unique point $z_0 \in \bigcap_n \overline{\mathit{Interior\ of\ } \gamma_n} \in U$.

Since g is differentiable at z_0, there is a function R such that

$$g(z) = g(z_0) + g'(z_0)(z - z_0) + R(z, z_0),$$

where

$$\|R(z, z_0)\|_Y \leq |z - z_0| w(|z - z_0|)$$

with $w(|z - z_0|) \to 0$ when $|z - z_0| \to 0$. Integrating g along γ_n, we find that

$$\begin{aligned}\int_{\gamma_n} g(z)dz &= \int_{\gamma_n} g(z_0)dz + \int_{\gamma_n} g'(z_0)(z - z_0)dz + \int_{\gamma_n} R(z, z_0)dz \\ &= [g(z_0) - g'(z_0)z_0] \int_{\gamma_n} 1dz + g'(z_0) \int_{\gamma_n} zdz + \int_{\gamma_n} R(z, z_0)dz \\ &= \int_{\gamma_n} R(z, z_0)dz.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\| \int_{\gamma_n} g(z)dz \right\|_Y &\leq |\gamma_n| \cdot \sup_{z \in \gamma_n} \|R(z, z_0)\|_Y \\ &\leq |\gamma_n| \cdot \frac{|\gamma_n|}{2} \cdot w\left(\frac{|\gamma_n|}{2}\right) \\ &\leq \frac{|\gamma|^2}{2 \cdot 4^n} w\left(\frac{|\gamma_n|}{2}\right)\end{aligned} \tag{A.4}$$

by $|z - z_0| < \frac{1}{2}|\gamma_n|$ for $z \in \gamma_n$. Comparing (A.3) and (A.4), we get $b = 0$. □

As a corollary, we obtain the same conclusion, but assuming only that g is differentiable at all points inside of the triangle except for the center of the small triangles.

Now we begin to prove Lemma 36. As is standard, for the function f in Lemma 36, fixing ϵ belonging to the interior of γ, we define

$$g_\epsilon(z) = \begin{cases} \frac{f(z) - f(\epsilon)}{z - \epsilon}, & z \neq \epsilon, \\ f'(z), & z = \epsilon, \end{cases}$$

which satisfies the hypothesis of Proposition 37 or its corollary. If γ is triangle centered at ϵ, then

$$0 = \int_\gamma g_\epsilon(z)dz = \int_\gamma \frac{f(z)}{z - \epsilon}dz - f(\epsilon) \int_\gamma \frac{1}{z - \epsilon}dz.$$

Hence we satisfy the formula

$$f(\epsilon) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z - \epsilon}dz.$$

Now, we can compute the derivative with respect to ϵ in space X and obtain

$$f'(\epsilon) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{(z - \epsilon)^2}dz.$$

Of course, since the derivative is obtained as limits of quotients, if the limit exists in X, it has to agree with the limit in Y.

A.2. Finitely differentiable functions in Banach space. For arbitrary Banach spaces $X_1, \ldots, X_i, Y, i \geq 1$, we denote by $A(X^{\otimes i}, Y)$ the space of symmetric continuous i-linear forms on $X^{\otimes i} := X_1 \times \cdots \times X_i$ taking values in Y. Now we present the converse to Taylor's theorem (see page 6 in the book [AR67]).

DEFINITION 38. *Let $O \subset X$ be a convex set and $F : O \to Y$, $f_i : O \to A(X^{\otimes i}, Y)$, $i = 0, \ldots, r$. For any $x \in O$ and $h \in X$ such that $(x+h) \in O$, we define $R(x,h)$ by*

$$F(x+h) = F(x) + \sum_{i=1}^{r} \frac{f_i(x)(h, \ldots, h)}{i!} + R(x,h).$$

If for any $0 \leq i \leq r$, f_i is continuous and for any $x \in O$, $\frac{\|R(x,h)\|_Y}{\|h\|_X^r} \to 0$ as $\|h\|_X^r \to 0$, then we say F is of class C^r on O and $D^i F = f_i$ for any $0 \leq i \leq r$.

DEFINITION 39. *We denote by $C^r(O,Y)$ the space of functions $f : O \to Y$ with continuous derivatives up to order r. We endow $C^r(O,Y)$ with the norm of the supremum of all the derivatives. Namely,*

$$\|f\|_{C^r} = \max_{0 \leq i \leq r} \sup_{x \in O} |[D^i f](x)|_{X^{\otimes i}, Y} \tag{A.5}$$

with

$$|\cdot|_{X^{\otimes i}, Y} \equiv \sup_{\|x_1\|_{X_1}=1, \ldots \|x_i\|_{X_i}=1} \|A(x_1, \ldots, x_i)\|_Y.$$

As is well known, the norm (A.5) *makes $C^r(O,Y)$ a Banach space.*

DEFINITION 40. *We denote by $C^{r+\mathrm{Lip}}(O,Y)$ the space of functions in $C^r(O,Y)$ whose rth derivative is Lipschitz. The Lipschitz constant is*

$$\mathrm{Lip}_{O,Y} D^r f = \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{|D^r f(x_1) - D^r f(x_2)|_{X^{\otimes r}, Y}}{\|x_1 - x_2\|_X}.$$

We note that since O may not be compact, this definition is different from the Whitney definition in which the topology is given by seminorms of the supremum in compact sets. We will not use the Whitney definition of C^r in this paper.

DEFINITION 41. *An open set O is called a compensated domain if there is a constant C such that given $x, y \in O$ there is a C^1 path γ contained in O joining x, y satisfying $|\gamma| \leq C\|x - y\|$.*

For O a compensated domain, we have the mean value theorem

$$\|f(x) - f(y)\|_Y \leq C\|f\|_{C^1(O,Y)}\|x - y\|_X.$$

In particular, C^1 functions in a compensated domain are Lipschitz. It is not difficult to construct noncompensated domains with C^1 functions which are not Lipschitz.

Of course a convex set is compensated and the compensation constant is 1. In our paper, we will just be considering domains which are balls or full spaces. See [dlLO99] for the effects of the compensation constants in many problems of the function theory.

A.3. The standard Sobolev space. As a matter of fact, we define

$$H^m(\mathbb{T}^d) := H^m(\mathbb{T}^d, \mathbb{R}^n) := \{U = (U_1, \ldots, U_n) | U_i \in H^m(\mathbb{T}^d, \mathbb{R}),\ i = 1, \ldots, n\}$$

equipped with the norm

$$\|U\|_{H^m} = \sum_{0\le i\le n} \|U_i\|_{H^m} \tag{A.6}$$

and

$$H^m(\mathbb{T}^d,\mathbb{R}) = \{U \in L^2(\mathbb{T}^d,\mathbb{R}) : D^{|\alpha|}U \in L^2(\mathbb{T}^d,\mathbb{R}),\ \ 0 \le |\alpha| \le m\},$$

where we use multi-index notation $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$, $|\alpha| = \sum_{i=1}^d \alpha_i$, and $x = (x_1,\dots,x_d) \in \mathbb{T}^d$, $D^\alpha := D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_d}^{\alpha_d}$. We define

$$\|U\|_{H^m(\mathbb{T}^d,\mathbb{R})} = \sum_{0\le|\alpha|\le m} \|D^\alpha U\|_{L^2}$$

with

$$\|U\|_{L^2} = \left(\int_{\mathbb{T}^d} |U(\theta)|^2 d\theta\right)^{\frac{1}{2}}.$$

Indeed, by Fourier transformation, the norm defined in (A.6) is equivalent to the norm defined by Definition 3 based on the Fourier coefficients. We refer to the books [AF03, Tay97] for more details.

Acknowledgments. We thank Profs. G. Gentile and L. Corsi, Dr. H. Cheng, and Ms. X. Xu for discussions and encouragement. We also thank two anonymous referees for very generous comments that improved the exposition. F. Wang is grateful to Georgia Institute of Technology for hospitality.

REFERENCES

[Ad63] V. I. Arnol'd, *Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian*, Uspekhi Mat. Nauk, 18 (1963), pp. 13–40.

[AF03] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Pure Appl. Math. 140, Elsevier, Amsterdam, 2003.

[AR67] R. Abraham and J. Robbin, *Transversal Mappings and Flows*, W. A. Benjamin, New York, 1967.

[AZ90] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Tracts in Math. 95, Cambridge University Press, Cambridge, 1990.

[Bal94] W. Balser, *From Divergent Power Series to Analytic Functions*, Theory and Application of Multisummable Power Series, Lecture Notes in Math. 1582, Springer, Berlin, 1994.

[BG15] E. Berchio and F. Gazzola, *The role of aerodynamic forces in a mathematical model for suspension bridges*, Discrete Contin. Dyn. Syst., 2015 (Special) (2015), pp. 112–121.

[Bou72] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pures Appl. (2), 17 (1872), pp. 55–108.

[CCCdlL17] R. C. Calleja, A. Celletti, L. Corsi, and R. de la Llave, *Response solutions for quasi-periodically forced, dissipative wave equations*, SIAM J. Math. Anal., 49 (2017), pp. 3161–3207.

[CCdlL13] R. C. Calleja, A. Celletti, and R. de la Llave, *Construction of response functions in forced strongly dissipative systems*, Discrete Contin. Dyn. Syst., 33 (2013), pp. 4411–4433.

[CdlL19a] H. Cheng and R. de la Llave, *Stable manifolds to bounded solutions in possibly ill-posed PDEs*, J. Differential Equations, 268 (2020), pp. 4830–4899.

[CdlL19b] H. Cheng and R. de la Llave, *Time dependent center manifold in PDEs*, Discrete Contin. Dyn. Syst., https://doi.org/10.3934/dcds.2020213.

[CdlLW19] H. Cheng, R. de la Llave, and F. Wang, *Response Solutions to the Quasi-periodically Forced Systems with Degenerate Equilibrium: A Simple Proof of a Result of W. Si and J. Si. and Extensions*, preprint, arXiv:1909.09915, 2019.

[CFG13] L. Corsi, R. Feola, and G. Gentile, *Domains of analyticity for response solutions in strongly dissipative forced systems*, J. Math. Phys., 54 (2013), 122701.

[CFG14] L. Corsi, R. Feola, and G. Gentile, *Convergent series for quasi-periodically forced strongly dissipative systems*, Commun. Contemp. Math., 16 (2014), 1350022.

[Die69] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York, 1969.

[dlL09] R. de la Llave, *A smooth center manifold theorem which applies to some ill-posed partial differential equations with unbounded nonlinearities*, J. Dynam. Differential Equations, 21 (2009), pp. 371–415.

[dlLO99] R. de la Llave and R. Obaya, *Regularity of the composition operator in spaces of Hölder functions*, Discrete Contin. Dyn. Syst., 5 (1999), pp. 157–184.

[dlLS19] R. de la Llave and Y. Sire, *An a posteriori KAM theorem for whiskered tori in Hamiltonian partial differential equations with applications to some ill-posed equations*, Arch. Ration. Mech. Anal., 231 (2019), pp. 971–1044.

[Gaz15] F. Gazzola, *Mathematical Models for Suspension Bridges*, MS&A. Model. Simul. Appl., Springer, Cham, Switzerland, 2015.

[GBD05] G. Gentile, M. V. Bartuccelli, and J. H. B. Deane, *Summation of divergent series and Borel summability for strongly dissipative differential equations with periodic or quasiperiodic forcing terms*, J. Math. Phys., 46 (2005), 062704.

[GBD06] G. Gentile, M. V. Bartuccelli, and J. H. B. Deane, *Quasiperiodic attractors, Borel summability and the Bryuno condition for strongly dissipative systems*, J. Math. Phys., 47 (2006), 072702.

[Gen10a] G. Gentile, *Quasi-periodic motions in strongly dissipative forced systems*, Ergodic Theory Dynam. Systems, 30 (2010), pp. 1457–1469.

[Gen10b] G. Gentile, *Quasiperiodic motions in dynamical systems: Review of a renormalization group approach*, J. Math. Phys., 51 (2010), 015207.

[GMV17] G. Gentile, A. Mazzoccoli, and F. Vaia, *Forced quasi-periodic oscillations in strongly dissipative systems of any finite dimension*, Commun. Contemp. Math., 21 (2019), 1850064.

[GV17] G. Gentile and F. Vaia, *Response solutions for forced systems with large dissipation and arbitrary frequency vectors*, J. Math. Phys., 58 (2017), 022703.

[HP74] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, rev. ed., Amer. Math. Soc. Colloq. Publ. XXXI, American Mathematical Society, Providence, RI, 1974.

[IKT13] H. Inci, T. Kappeler, and P. Topalov, *On the Regularity of the Composition of Diffeomorphisms*, Mem. Amer. Math. Soc. 226, American Mathematical Society, Providence, RI, 2013.

[KP13] S. G. Krantz and H. R. Parks, *The implicit function theorem*, Mod. Birkhäuser Class., Birkhäuser, New York, 2003.

[KS00] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Classics Appl. Math. 31, SIAM, Philadelphia, 2000.

[LS90] L. H. Loomis and S. Sternberg, *Advanced Calculus*, Jones and Bartlett, Boston, 1990.

[Mar74] J. Marsden, *Applications of Global Analysis in Mathematical Physics*, Math. Lect. Ser. 2, Publish or Perish, Boston, MA, 1974.

[Muj86] J. Mujica, *Complex Analysis in Banach Spaces*, Holomorphic Functions and Domains of Holomorphy in Finite and Infinite Dimensions, Notas Mat. 107, North-Holland Math. Stud. 120, North-Holland, Amsterdam, 1986.

[RS75] M. Reed and B. Simon, *Methods of Modern Mathematical Physics.* II. *Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[RS80] M. Reed and B. Simon, *Methods of Modern Mathematical Physics.* I. *Functional Analysis*, 2nd ed., Academic Press, New York, 1980.

[RS96] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Ser. Nonlinear Anal. Appl. 3, Walter de Gruyter, Berlin, 1996.

[Tay97] M. E. Taylor, *Partial Differential Equations.* III, Appl. Math. Sci. 117, Springer, New York, 1997.

[Zeh75] E. Zehnder, *Generalized implicit function theorems with applications to some small divisor problems.* I, Comm. Pure Appl. Math., 28 (1975), pp. 91–140.