The Parameter Dependence of Eddy Heat Flux in a Homogeneous Quasigeostrophic Two-Layer Model on a β Plane with Quadratic Friction

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ABSTRACT: This study investigates the parameter dependence of eddy heat flux in a homogeneous quasigeostrophic two-layer model on a β plane with imposed environmental vertical wind shear and quadratic frictional drag. We examine the extent to which the results can be explained by a recently proposed diffusivity theory for passive tracers in two-dimensional turbulence. To account for the differences between two-layer and two-dimensional models, we modify the two-dimensional theory according to our two-layer *f*-plane analyses reported in an earlier study. Specifically, we replace the classic Kolmogorovian spectral slope, −5/3, assumed to predict eddy kinetic energy spectrum in the former with a larger slope, −7/3, suggested by a heuristic argument and fit to the model results in the latter. It is found that the modified theory provides a reasonable estimate within the regime where both $\tilde{\beta} = \beta k_d^{-2} U^{-1}$ and the strength of the frictional drag, $\tilde{c}_D = c_D k_d^{-1}$, are much smaller than unity (here, c_D is the nondimensional drag coefficient divided by the depth of the layer, k_d is the wavenumber of deformation radius, and U is the imposed background vertical wind shear). For values of $\tilde{\beta}$ and $\tilde{c}_D$ that are closer to one, the theory works only if the full spectrum shape of the eddy kinetic energy is given. Despite the qualitative, fitting nature of this approach and its failure to explain the full parameter range, we believe its documentation here remains useful as a reference for the future attempt in pursuing a better theory.

KEYWORDS: Eddies; Fluxes; Turbulence; Quasigeostrophic models

1. Introduction

The maintenance of global climate is in large part determined by the baroclinic eddy equilibration of the extratropical troposphere. While these eddies have been well resolved in global climate models, an understanding of them that could be useful for the construction of conceptual models for climate change research remains missing. For the meridional heat fluxes of these eddies in particular, the pursuit of a quantitative understanding has indeed had a long history (Held 1999). An important theoretical framework for approaching this problem is founded on the seminal work by Rhines (1975, 1977) and Salmon (1978, 1980), where they studied homogeneous two-layer quasigeostrophic (QG) turbulence as a prototype of more complex atmospheric flows.

Numerical model simulations for this type of turbulence have been compared with theoretical scaling arguments in a series of papers in the intervening years (Larichev and Held 1995; Held and Larichev 1996; Lapeyre and Held 2003; Thompson and Young 2006, 2007; Chang and Held 2019, hereafter CH19; Gallet and Ferrari 2020). In the models investigated by these papers, an environmental mean vertical shear in the zonal flows is prescribed such that the environmental temperature and potential vorticity (PV) gradients in each layer are uniform. The deviations from these mean environmental gradients are assumed to be doubly periodic, so that all eddy statistics, including the heat (or mass) and PV fluxes are horizontally uniform, while the eddy momentum fluxes vanish on average. The heat flux, in particular, can be described by a diffusivity, defined as the flux divided by the prescribed gradient. The parameter dependence of this diffusivity, the thermal diffusivity, is then the central concern of the theoretical scaling arguments discussed therein.

However, some of these existing scaling arguments lack quantitative agreement with the numerical simulations. Moreover, all of them consider the limits where the thermal diffusivity depends only on β or the strength of the friction individually. As a step to improve and generalize the existing arguments, our goal in this paper is to better understand the thermal diffusivity's dual dependence on β and the strength of the quadratic friction. In this work, we effectively try to extend the work of CH19 on the $\beta = 0$ limit, which studied the dependence on the strength of the quadratic friction with $\beta = 0$, to the cases of nonzero β. We do so by combining it with the work of Kong and Jansen (2017, hereafter KJ17) on the diffusivity of passive tracers in externally stirred two-dimensional turbulence. Specifically, KJ17 proposed a passive tracer diffusivity theory for two-dimensional turbulence that describes both the dependence on β and the strength of the quadratic friction. We modify their theory to account for the nature of thermal diffusivity in the two-layer turbulence. We then investigate the ability of this modified theory to explain the numerical simulations, conducted using the model documented in CH19 except for the inclusion of the β effect.

The rest of the paper is organized as follows. In section 2, we selectively review the relevant theoretical scaling arguments from previous literature, both in the case of advection of passive tracers in externally stirred two-dimensional turbulence and of temperature in the two-layer self-stirred model. In section 3, we describe the two-dimensional turbulence theory

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of KJ17 and our modifications to make it applicable for the two-layer turbulence, following the heuristic argument described in CH19. Section 4 presents a comparison between the numerical simulations and the theoretical prediction obtained by the arguments proposed in section 3, with their discrepancy examined in further detail. In section 5, we summarize our results and discuss some additional, unsettled questions that prevent us from developing a quantitative theory for the full parameter range.

2. A review on some existing scaling theories

To setup the background of the derivation of the generalized diffusivity theory in section 3, this section surveys some of the relevant scaling arguments that have been proposed in the literature for two-dimensional turbulence and two-layer QG turbulence. The connection between two-dimensional turbulence and two-layer QG turbulence is first noticed by Rhines and Salmon. They postulated that the energy in two-layer turbulence is typically extracted from the mean available potential energy by the eddy heat flux into the baroclinic mode, transferred to the barotropic mode near the radius of deformation, and cascaded inversely within the barotropic mode to larger horizontal scales. If the cascade is sufficiently extensive, this results in most of the energy being barotropic and at large enough scale that the baroclinic streamfunction (temperature in the two-layer model) acts essentially like a passive tracer mixed by the barotropic flow. For this reason, it is common for existing studies to develop the scaling theories for the eddy heat flux in the self-stirred two-layer model based on the theories for the flux of a passive tracer in the externally stirred two-dimensional (barotropic) model, as discussed in more details in the following.

a. Diffusivity of passive tracers in two-dimensional flow with $\beta = 0$

For homogeneous two-dimensional turbulence to maintain a statistically steady state, energy is typically input at a small forcing scale with a prescribed energy injection rate ε. To the extent that most energy cascades to large scales, ε can also be thought of as the strength of the inverse energy cascade. In the absence of a β effect, large-scale friction stops the cascade by dissipating energy. When the friction is in the form of quadratic drag, the frictional damping strength is an inverse length scale or a wavenumber (c_D). If the forcing scale and domain size are irrelevant in the parameter regime of interest (and if no other properties of the forcing other than ε are relevant) the system is then characterized by only two parameters: ε and c_D. By dimensional analysis, the eddy diffusivity (D) for a passive tracer with an imposed environmental tracer gradient should scale as

$$D \sim \varepsilon^{1/3} c_D^{-4/3}, \tag{1}$$

which has been shown to be supported by simulations of passive tracer transport in barotropic models (e.g., Grianik et al. 2004; KJ17). Additionally, the relevant halting length scale, again by dimensional analysis, is then

$$L \sim c_D^{-1}. \tag{2}$$

b. Diffusivity of passive tracers in two-dimensional flow with β

The presence of β breaks the isotropy of the equations and channels the barotropic inverse energy cascade into Rossby waves and zonal jets that do not participate directly in meridional mixing. When β alone is large enough to achieve this redirection before the cascade is halted by friction, one can hope that diffusivity depends on β rather than friction. The physical picture is that the frictional energy loss occurs in the jets, allowing the eddies responsible for the meridional mixing of passive tracers to be independent of the friction strength. If this regime is achievable, then by dimensional analysis the diffusivity should scale as

$$D \sim \varepsilon^{3/5} \beta^{-4/5}. \tag{3}$$

This limit has also been confirmed in simulations of passive tracer transport in two-dimensional turbulence (e.g., Smith et al. 2002; KJ17). In addition, the halting scale in this β dominated limit, which marks the peak in the spectrum of the meridional component of the velocity, can also be derived by dimensional analysis as

$$L \sim \varepsilon^{1/5} \beta^{-3/5} \tag{4}$$

(Vallis and Maltrud 1993).

c. Thermal diffusivity in two-layer model with $\beta = 0$

As mentioned earlier, given the connection between the two-dimensional and two-layer turbulence, it is tempted to apply Eq. (1) to the thermal diffusivity in two-layer model with $\beta = 0$ (so long as the damping is sufficiently small that the kinetic energy is predominately barotropic). In the two-layer model, ε is however not prescribed. Instead, it is the mean vertical shear in the zonal flows between the two layers, U, or equivalently the mean gradient of baroclinic streamfunction, that is prescribed. The energy is generated through the eddy mixing that taps the available potential energy associated with this mean vertical zonal shear,

$$\varepsilon_p \equiv D_\tau U^2 k_d^2 = D_\tau T^{-2}, \tag{5}$$

where D_τ is the thermal diffusivity and k_d is the wavenumber of the deformation radius, also externally imposed in QG theory, and the time scale $T \equiv (U k_d)$. This equation is exact in the homogeneous model. Setting $\varepsilon = \varepsilon_p$, combining this exact energetic relation with Eq. (1) and taking $D = D_\tau$, the resulted scaling is

$$D_\tau \sim c_D^{-2} T^{-1} \tag{6}$$

(Held 1999). This scaling is however found by CH19 to poorly predict the thermal diffusivity in $\beta = 0$ two-layer simulations. It explained the decrease of thermal diffusivity with increasing strength of the quadratic friction but overestimates this sensitivity.

d. Thermal diffusivity in two-layer model with β

Again, to apply Eq. (3) to the two-layer thermal diffusivity, one has to consider that ε is internally computed in the two-layer model. Letting $\varepsilon = \varepsilon_p$ and solving Eqs. (5) and (3) simultaneously, Held and Larichev (1996) obtained

$$D_\tau \sim \beta^{-2} T^{-3}. \tag{7}$$

Strictly speaking, a complication in applying Eq. (3) to the two-layer turbulence is that the latter has three distinct diffusivities: the thermal diffusivity, and the diffusivity of the upper- and lower-layer PV, respectively. These three diffusivities are simply related due to the vanishing of domain mean momentum and PV fluxes (e.g., Vallis 1988) and they approach each other only when criticality $\xi \equiv Uk_d^2/\beta$ approaches to infinity. Thus, without further justification for why it is the temperature (baroclinic streamfunction) that behaves most like a passive tracer, Eq. (7) is better regarded as only applicable to the strongly unstable flows where ξ is large. Lapeyre and Held (2003) indeed found in numerical simulations that Eq. (3) actually fits best for the diffusivity of lower-layer PV and described how to modify Eq. (7) appropriately.

In fact, the state of knowledge regarding the validity of Eq. (7) is somewhat confusing. On one hand, Eq. (7) and its variants have been shown capable of at least qualitatively explaining the diffusivity defined by poleward eddy heat transport in more complex systems, including an inhomogeneous two-layer QG channel model (Pavan and Held 1996; Zurita-Gotor 2007), a multilayer Boussinesq channel model (Jansen and Ferrari 2013), and a comprehensive atmospheric-only general circulation model (Barry et al. 2002). On the other hand, Eq. (7) has also been shown to be problematic even for explaining the two-layer β-plane simulations. Specifically, Thompson and Young (2007) found that Eq. (7) is not the asymptotic limit for large β as it is assumed. For large β, the thermal diffusivity is not merely a function of β but still depends on the strength of friction. This complicated dual dependence is therefore the subject of study we like to address in section 3.

3. Toward a theory for the two-layer model incorporating c_D and β

The review in section 2 suggests that the scaling arguments derived from the two-dimensional turbulence theory, both for $\beta = 0$ and nonzero β, cannot be directly applied to explain the thermal diffusivity in two-layer turbulence. Rather than to completely give up on this theoretical framework, CH19 set in to understand more exactly why it breaks down. At the limit of $\beta = 0$, they found the main reason being that the barotropic eddy kinetic energy spectrum of the two-layer flows develops a slope that is steeper than obtained by the classic Kolmogorovian argument. The Kolmogorovian argument is fundamentally based on the idea that ε alone can fully characterize the stirring, which is evidently not the case in the two-dimensional turbulence simulations. Taking a heuristic approach to factor in this difference, CH19 described a scaling that agrees better with the simulations. In this section, we explore how these findings in the $\beta = 0$ limit can be further extended to understand the case when β is nonzero. Specifically, we accordingly modify KJ17's recently proposed theory that explains both the β and c_D dependence of passive tracer diffusivity in two-dimensional turbulence. We start with introducing their theory and next describe the modification we made based on CH19's heuristic argument.

a. KJ17's theory for two-dimensional turbulence

For passive tracers in two-dimensional turbulence, considering the full parameter range where both β and c_D may play a role, dimensional analysis suggests that the diffusivity should be a function of

$$\mu \equiv \varepsilon^{-1/5} \beta^{3/5} c_D^{-1}, \tag{8}$$

the only dimensionless form among all combinations of ε, β, and c_D. The parameter μ can be thought of as the zonostrophic index (e.g., Galperin et al. 2010) modified for quadratic drag. It can also be thought of as the ratio of the halting wavenumbers in the frictionally dominated and β dominated regimes, i.e., the ratio of the two length scales defined in Eqs. (2) and (4). KJ17 then proposed that the diffusivity in two-dimensional turbulence takes the form

$$D = D|_{\beta=0} \mathscr{F}(\mu), \tag{9}$$

where the function $\mathscr{F}$ describes the roles of β in causing the differences between the diffusivity (D) and its corresponding value at the $\beta = 0$ limit ($D|_{\beta=0}$).

Specifically, to determine $\mathscr{F}(\mu)$, KJ17 considered the effects of β in generating Rossby waves. They developed a theory for $\mathscr{F}(\mu)$ based on the physical picture of the suppression of meridional mixing by zonal wave propagation. This picture is first proposed by Ferrari and Nikurashin (2010), where the concern is a passive tracer mixed meridionally by the eddies dominated by a single zonal wavenumber K. Assuming these eddies propagate at a phase speed c with respect to a zonal flow $\overline{u}$, Ferrari and Nikurashin (2010) argued that the diffusivity can be determined as

$$D = \frac{D_{\mathrm{ml}}}{1 + \dfrac{(\overline{u} - c)^2}{2\nu_1^2 E}}, \tag{10}$$

where $D_{\mathrm{ml}} \equiv \nu_1^{-1} E^{1/2} K^{-1}$ is obtained by traditional mixing length theory assuming the eddy decorrelation time scale $T_e \equiv (\nu_1 E^{1/2} K)^{-1}$ with ν_1 being the dimensionless proportionality constant and E the total eddy kinetic energy. The denominator in Eq. (10) is then a suppression factor depending on the time scale ratio of eddy decorrelation and zonal flow advection.

Following Ferrari and Nikurashin (2010), KJ17 extended the single wavenumber picture to consider the full diffusivity spectrum,

$$D = \int_0^\infty \mathbb{D}(k)\, dk, \tag{11}$$

with $\mathbb{D}(k)$ determined in the same manner as Eq. (10). In particular, they treated each wavenumber k independently and assumed that the relevant Doppler shifted phase speed in $\mathbb{D}(k)$ is simply β/k^2 and the relevant eddy kinetic energy is simply the energy at that wavenumber, $\mathbb{E}(k)$. In terms of $\mathbb{E}(k)$, one can also write $\mathbb{D}_{\mathrm{ml}}(k) = \nu_1^{-1}[\mathbb{E}(k)k]^{1/2}k^{-1}$. Putting together, $\mathbb{D}(k)$ is therefore obtained as

$$\mathbb{D}(k) = \frac{\nu_1^{-1}\mathbb{E}(k)^{1/2}k^{-3/2}}{1 + \dfrac{\nu_2\beta^2}{2\nu_1^2\mathbb{E}(k)k^5}}, \tag{12}$$

where the constant ν_2 is additionally added to include some freedom in the strength of the suppression.

It is noted that Eq. (12) is still not a closure as $\mathbb{E}(k)$ has to be internally computed by numerical models. To further express $\mathbb{E}(k)$ in terms of the external parameters, KJ17 also assumed a Kolmogorov spectrum (with Kolmogorov constant C) for the wavenumbers above the wavenumber of the halting scale for $\beta = 0$, $\lambda_0 c_D$, i.e., the inverse of Eq. (2) with λ_0 being the dimensionless proportionality constant,

$$\mathbb{E}(k) = \begin{cases} C\varepsilon^{2/3}k^{-5/3}, & k \geq \lambda_0 c_D \\ 0, & k < \lambda_0 c_D \end{cases}. \tag{13}$$

Here, the simple choice of Eq. (2) appears to ignore the importance of the β-halting scale [i.e., Eq. (4)] for determining $\mathbb{E}(k)$, but the hope is that the strength of the suppression factor for scales between the two halting scale makes this choice less significant. Combining Eqs. (11)–(13), closed expressions for D, $D|_{\beta=0}$, and $\mathscr{F}(\mu) = D/D|_{\beta=0}$ are then readily obtained. To simplify the expression for $\mathscr{F}(\mu)$, an extra step can be taken to change the variable of integration from k to $\kappa \equiv kc_D^{-1}$. With this help, the functional form $\mathscr{F}(\mu)$ can then be written as

$$\frac{D}{D|_{\beta=0}} = \mathscr{F}(\mu) = \frac{4\lambda_0^{4/3}}{3}\int_{\lambda_0}^{\infty} \frac{\kappa^{-7/3}}{1 + \lambda_1\mu^{10/3}\kappa^{-10/3}}\,d\kappa. \tag{14}$$

Again for simplicity, the newly introduced constant $\lambda_1 \equiv \nu_2/(2\nu_1^2 C)$ now subsumes all the previously defined constants, while the prefactor assures that $\mathscr{F} = 1$ for $\mu = \beta = 0$. Also, its associated prediction for $D|_{\beta=0}$ is simply Eq. (1). This expression is the final product of KJ17's theory.

b. *Modifications for two-layer turbulence based on CH19's heuristic argument*

Despite the number of assumptions, KJ17 concluded that Eq. (14) can indeed fit their simulations for the diffusivity of a passive tracer in two-dimensional turbulence reasonably well. Yet there are reasons to expect that numerical simulations with two-layer models do not follow Eq. (14). The most obvious one is that, at $\beta = 0$ limit, $D|_{\beta=0}$ already does not scale as Eq. (6), as shown in CH19. It therefore makes sense to first discuss why Eq. (6) fails and to modify KJ17's theory accordingly.

In particular, CH19 found that the failure of Eq. (6) has to do with the absence of a well-defined Kolmogorov inertial range in the simulated two-layer barotropic eddy kinetic energy spectrum. Recall that Eq. (6) can be derived by dimensional analysis in section 2, assuming the nontrivial parameters determining the relevant eddy scales being c_D and ε only. This is equivalently to assume Eq. (13), i.e., $\mathbb{E}(k)$ to follow a Kolmogorov spectrum with an inverse cascade rate, $\varepsilon(k)$, that is independent of wavenumber, k, and equals to the energy input from the stirring at a single large wavenumber. CH19 argued that in the two-layer models, associated with the self-stirring due to baroclinic instability, the energy input from the baroclinic into the barotropic mode is not localized at the radius of deformation, but spreads to lower wavenumbers as well. This leads them to heuristically assume $\varepsilon(k) \sim \varepsilon_p(k/k_d)^{-x}$, with x an undetermined parameter representing the nonlocality of energy transfer (and $x = 0$ reducing to the classic assumption). Along with the fact that, $\mathbb{E}(k)$ locally is still determined by local k and local inverse cascade rate $\varepsilon(k)$, this results in a modified barotropic eddy kinetic energy spectrum in the form

$$\mathbb{E}(k) \sim \varepsilon_p^{2/3}k_d^{2x/3}k^{-(5+2x)/3}. \tag{15}$$

Consistently, the revised scaling for diffusivity and mixing length become [cf. their Eq. (16)]

$$\tilde{D} \sim \tilde{c}_D^{-[(4+x)/(1+x)]/2}, \tag{16}$$

$$\tilde{L} \sim \tilde{c}_D^{-1/(1+x)}, \tag{17}$$

where the tilde represents the nondimensionalization by the mean wind shear (U) and the wavenumber of deformation radius (k_d), i.e., $\tilde{D} = Dk_dU^{-1}$, $\tilde{L} = Lk_d$, and $\tilde{c}_D = c_Dk_d^{-1}$. With a positive x, these revised scaling then suggest a c_D dependence weaker than predicted by Eqs. (6) and (2) as seen qualitatively in CH19's two-layer simulations.

To be more quantitative, the parameter x needs to be specified. CH19 obtained $x \approx 1$ from the fit to D_τ in their $\beta = 0$ numerical simulations. Assuming $x \approx 1$ for simplicity, we can here write Eqs. (15)–(17) as

$$\mathbb{E}(k) \sim \varepsilon_p^{2/3}k_d^{2/3}k^{-7/3}, \tag{18a}$$

$$\tilde{D} \sim \tilde{c}_D^{-5/4}, \tag{18b}$$

$$\tilde{L} \sim \tilde{c}_D^{-1/2}. \tag{18c}$$

According to Eq. (18), we can then modify Eq. (13) and the frictional halting scale to get a revised diffusivity to apply to two-layer thermal diffusivity,

$$D_\tau = \int_{\lambda_0 c_D^{1/2}k_d^{1/2}}^{\infty} \frac{\nu_1^{-1}C^{1/2}\varepsilon_p^{1/3}k_d^{1/3}k^{-8/3}}{1 + \dfrac{\nu_2\beta^2}{2\nu_1^2 C\varepsilon_p^{2/3}k_d^{2/3}k^{8/3}}}\,dk. \tag{19}$$

To obtain an expression for $D_\tau/D_\tau|_{\beta=0}$, one has to again take into account the complication not present in the two-dimensional case that, for fixed external parameters in the two-layer model, ε_p depends on D_τ. In the $\beta = 0$ limit, it is only the factor of $\varepsilon_p^{1/3}$ in the numerator that survives. After changing the integration variable by setting $\kappa \equiv kc_D^{-1/2}k_d^{-1/2}$, we have

$$\frac{D_\tau}{D_\tau|_{\beta=0}} = \left(\frac{\varepsilon_p}{\varepsilon_p|_{\beta=0}}\right)^{1/3}\mathscr{F}, \tag{20}$$

where

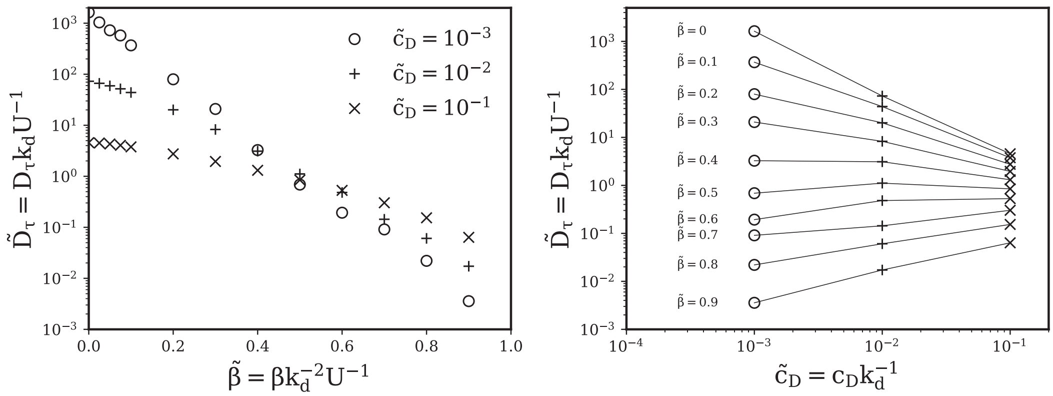

FIG. 1. The parameter dependence of the simulated thermal diffusivity with symmetric drag: (left) $\tilde{D}_\tau = D_\tau k_d U^{-1}$ plotted against $\tilde{\beta} = \beta k_d^{-2} U^{-1}$ with different $\tilde{c}_D = c_D k_d^{-1}$ values indicated by different symbols, and (right) $\tilde{D}_\tau$ plotted against $\tilde{c}_D$ with results sharing the same value of $\tilde{\beta}$ connected by lines.

$$\mathscr{T} = \frac{5\lambda_0^{5/3}}{3} \int_{\lambda_0}^{\infty} \frac{\kappa^{-8/3}}{1 + \lambda_1 \mu_2^{10/3} \kappa^{-8/3}} d\kappa \qquad (21)$$

and

$$\mu_2 \equiv \varepsilon_p^{-1/5} \beta^{3/5} c_D^{-2/5} k_d^{-3/5}. \qquad (22)$$

The prefactor in Eq. (21) once again ensures that $\mathscr{T} = 1$ when $\beta = 0$. The definition of the dimensionless parameter μ_2 in Eq. (22) is analogous to μ defined in Eq. (8). We can further use the energetic constraint, i.e., Eq. (5), which requires $\varepsilon_p \propto D_\tau$, to obtain

$$\frac{D_\tau}{D_\tau|_{\beta=0}} = \mathscr{T}^{3/2}. \qquad (23)$$

While the final expression is complex, we consider this as the natural extension of the $\beta = 0$ results in CH19 to nonzero β following the approach in KJ17 for the passive two-dimensional case. In fact, Eq. (23) is an implicit equation for D_τ. It indicates that, in the two-layer case, the overall parameter dependence of thermal diffusivity is characterized by μ_2. Yet μ_2 by definition depends on ε_p, which in turn depends on D_τ as stated in Eq. (5). We therefore have to solve Eqs. (23) and (5) together to further get the explicit prediction for D_τ itself. The extent to which this prediction is supported by numerical simulations will be examined in the next section.

4. Comparison with the simulations

In this section, we present the numerical results of thermal diffusivity to test the modified theory discussed in section 3. Following CH19, we use the same numerical model to conduct a series of simulations in the parameter range: $10^{-3} \le \tilde{c}_D \equiv c_D k_d^{-1} \le 10^{-1}$ and $0 \le \tilde{\beta} \equiv \beta k_d^{-2} U^{-1} < 1$. The value $\tilde{c}_D \approx 10^{-1}$ corresponds to a drag coefficient in marine boundary layers (CH19). This $\tilde{c}_D$ range is also chosen to ensure the frictional damping is relatively weak so that eddy kinetic energy is predominately barotropic, and the $\tilde{\beta}$ range covers the regime with larger-than-one criticality (i.e., $\xi = \tilde{\beta}^{-1}$). For each combination of $\tilde{c}_D$ and $\tilde{\beta}$, we have prepared two different simulations: one with symmetric drag (quadratic drag with the same $\tilde{c}_D$ value in the two layers) and another with asymmetric drag (quadratic drag only in the bottom layer). When there is equal drag in both layers, β is the only source of asymmetry between the two layers. When drag is present in the lower layer only, it is an additional source of asymmetry between the two layers. The main focus will be on the simulations with symmetric drag, as in CH19.

All the other model parameters and the numerics are also kept the same as in CH19, except for the maximum resolved wavenumber ($k_{\max}$). For most of the cases in CH19, $k_{\max}/k_d = 127/50$ is chosen following Larichev and Held (1995) and Held and Larichev (1996). We have confirmed that this resolution is adequate to obtain robust results for small to moderate $\tilde{\beta}$. However, as $\tilde{\beta}$ gets closer to one, a higher resolution is generally required. Hence, for all the simulations examined in this study, we have chosen $k_{\max}/k_d = 255/50$. The convergence of simulations with the largest few $\tilde{\beta}$ at this resolution is still not guaranteed. As will be discussed later on, this is also the parameter regime where the modified theory mostly disagrees with the simulations. While it is unclear what the role of resolution may be in this discrepancy, we wish to point it out as a potential caveat of our analysis.

a. *Overview of the simulations*

Figure 1 provides an overview of the simulated thermal diffusivity (D_τ) with symmetric drag, diagnosed from the statistically steady state as $D_\tau = \langle \tau \partial_x \psi \rangle$, where τ and ψ are the baroclinic and barotropic streamfunctions. On the left, we plot the nondimensional thermal diffusivity ($\tilde{D}_\tau = D_\tau k_d U^{-1}$) against $\tilde{\beta}$ and use different symbols to indicate the different values of $\tilde{c}_D$. This figure can be compared with Fig. 3 of Thompson and Young (2007), whose simulations are conducted with linear drag in the lower layer. In spite of the difference in the frictional form, the $\tilde{D}_\tau$ dependence in our simulations very much resembles theirs. The value of $\tilde{D}_\tau$ is always reduced with increasing β for a given $\tilde{c}_D$. A characteristic $\tilde{\beta}$ where $\tilde{D}_\tau$ is insensitive to $\tilde{c}_D$ is identified around $\tilde{\beta} \approx 0.4$ and $\tilde{D}_\tau \approx 1$: smaller than this $\tilde{\beta}$, $\tilde{D}_\tau$ decreases as $\tilde{c}_D$ increases; larger than this $\tilde{\beta}$, $\tilde{D}_\tau$ increases as $\tilde{c}_D$ increases. The right-hand

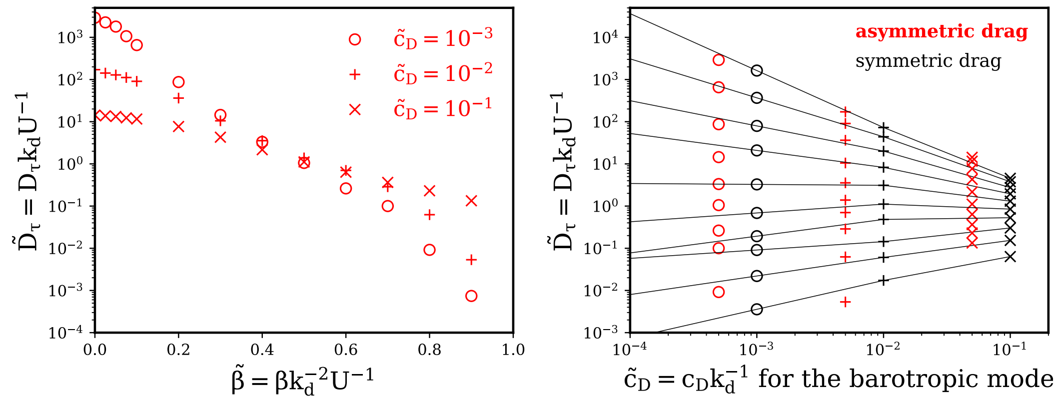

FIG. 2. As in Fig. 1, but for asymmetric drag (i.e., drag in the lower layer only). (right) Results from the symmetric drag case are superimposed (black). As in CH19, these are plotted so that they will agree if the diffusivities are only dependent on the strength of the frictional damping acting on barotropic flows (see text).

panel in Fig. 1, a plot of the variation of the diffusivity as a function of $\tilde{c}_D$ for fixed $\tilde{\beta}$, provides an alternative perspective on this behavior.

As mentioned earlier, for each simulation with symmetric drag, a simulation is also generated with asymmetric drag with the same $\tilde{c}_D$ in the lower layer for comparison. The results for $\tilde{D}_\tau$ with asymmetric drag are shown in Fig. 2. On the left-hand panel in Fig. 2, we see a qualitatively similar parameter dependence as compared to its symmetric counterpart over the explored parameter range. On the right-hand panel in Fig. 2, a more explicit comparison of simulations with symmetric and asymmetric drag is provided. Specifically, we plot the asymmetric drag results at half the $\tilde{c}_D$ value, since that is the effective damping strength for its barotropic mode when compared to the symmetric drag results. As discussed in CH19, this way of plotting can help us assess whether the two would agree if the thermal diffusivity were a function of the damping of the barotropic mode only. CH19 found that when $\tilde{\beta} = 0$ this simple adjustment works well for small $\tilde{c}_D$ but underestimates the symmetric diffusivity when $\tilde{c}_D$ gets closer to unity. This is owing to the fact that the flows becomes less barotropic, with maximum amplitude in the upper layer, resulting in weaker effective damping of the barotropic mode. The same qualitative behavior is seen here for nonzero β, indicating an additional complexity to consider when trying to understand the more realistic asymmetric drag simulations. Hence, to avoid this extra issue, the following discussion will be limited to the symmetric drag simulations, a cleaner setup for the comparison with the theory discussed in section 3.

b. *Theoretical fit*

To compare with the theory discussed in section 3, as in KJ17, our first step is to verify that $\tilde{D}_\tau$ normalized by $\tilde{D}_\tau|_{\beta=0}$ (the thermal diffusivity at $\beta = 0$ limit) can be in large part explained by one nondimensional parameter. In KJ17's original argument, this parameter is μ [Eq. (8)]; with our modification, it becomes μ_2 [Eq. (22)]. We begin by showing the data according to the former and then gradually changing the ways of presentation to demonstrate how different it is before and after this modification.

In the left-hand panel of Fig. 3, we present D_τ in the symmetric drag simulations in the same way as in Fig. 3 of KJ17.

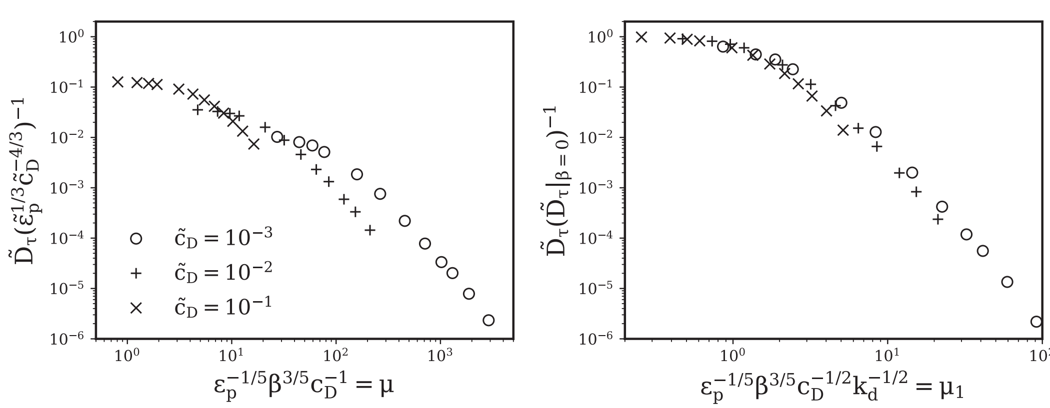

FIG. 3. The parameter dependence of the simulated thermal diffusivity with symmetric drag: (left) $\tilde{D}_\tau$ normalized by $\tilde{\varepsilon}_p^{1/3}\tilde{c}_D^{-4/3}$ plotted against the nondimensional parameter $\mu = \varepsilon_p^{-1/5}\beta^{3/5}c_D^{-1}$, and (right) $\tilde{D}_\tau$ normalized by $\tilde{D}_\tau|_{\beta=0}$ plotted against the nondimensional parameter $\mu_1 = \varepsilon_p^{-1/5}\beta^{3/5}c_D^{-1/2}k_d^{-1/2}$.

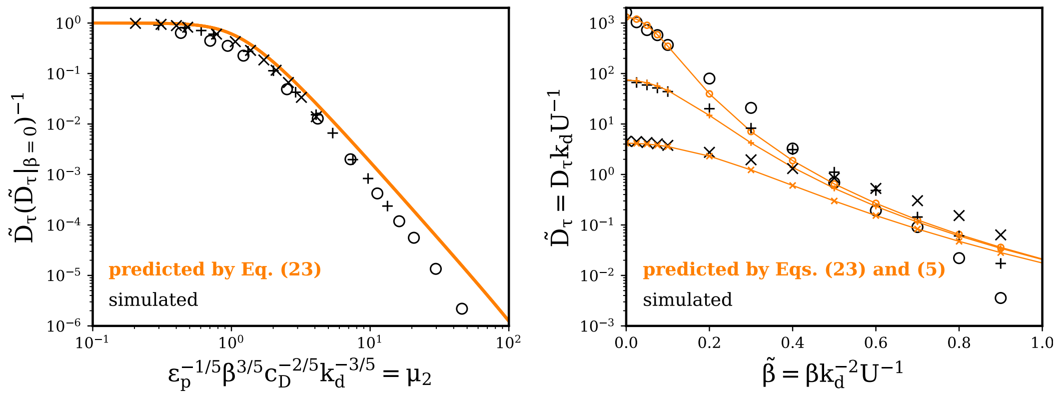

FIG. 4. Verifying the theoretical prediction, Eq. (23), for $\tilde{D}_\tau$ with symmetric drag: (left) $\tilde{D}_\tau$ normalized by $\tilde{D}_\tau|_{\beta=0}$ plotted against the nondimensional parameter $\mu_2 = \varepsilon_p^{-1/5}\beta^{3/5}c_D^{-2/5}k_d^{-3/5}$, and (right) $\tilde{D}_\tau$ plotted against $\tilde{\beta}$. The simulations are shown in black (as in Fig. 1), and the theoretical predictions are colored. (right) The $\tilde{D}_\tau$ prediction is obtained by solving Eqs. (23) and (5) simultaneously to eliminate ε_p dependence.

Specifically, this is to verify Eq. (9) with Eq. (1) as the $\beta = 0$ normalization. We plot D_τ normalized by Eq. (1) as a function of μ, where ε in Eq. (1) and μ is identified as the simulated energy generation, ε_p. Note that this log–log plot does not show the $\beta = 0$ limit. Unlike what is observed for two-dimensional passive tracer diffusivity in KJ17's Fig. 3, the two-layer simulations here clearly do not collapse into one curve.

This scatter of data is actually what to expect since we already know $\tilde{D}_\tau|_{\beta=0}$ does not follows Eq. (1) but rather Eq. (18) from the results in CH19. Also, as discussed in section 3, μ can be interpreted as the ratio of frictional and β halting scales, and CH19 showed that the frictional halting scale in the two-layer simulations does not follow Eq. (2) but rather Eq. (18). Guided by the $\beta = 0$ results in CH19, if we assume that the frictional halting scale is instead given by Eq. (18) and retain the use of β halting scale as Eq. (4), this halfway modification of the parameter μ becomes

$$\mu_1 \equiv \varepsilon^{-1/5}\beta^{3/5}c_D^{-1/2}k_d^{-1/2}. \tag{24}$$

The right-hand panel in Fig. 3 shows the simulated $\tilde{D}_\tau$ normalized by the simulated $\tilde{D}_\tau|_{\beta=0}$ as a function of μ_1. We continue to set ε in μ_1 equal to ε_p. Note that k_d dependence is explicitly introduced by this modification to the frictional halting scale and the $\beta = 0$ diffusivity in Eq. (18). This plot demonstrates a more compact description of the β and c_D dependence of $\tilde{D}_\tau$. Yet, according to the modified theory, we expect $\tilde{D}_\tau/\tilde{D}_\tau|_{\beta=0}$ is characterized not by μ_1 but μ_2 defined in Eq. (22).

In fact, to get μ_2, we have to also consider the modification of β halting scale. If we assume that the slope of the barotropic eddy kinetic energy spectrum follows Eq. (18), we can estimate the characteristic velocity in the barotropic mode in eddies of scale k, $V(k) \sim [\mathbb{E}(k)k]^{1/2}$. Solving for the length scale k^{-1} at which V matches the characteristic phase speed of a Rossby wave, β/k^2, the resulted length scale with $\varepsilon = \varepsilon_p$ is

$$L \sim \varepsilon^{1/4}k_d^{1/4}\beta^{-3/4}. \tag{25}$$

The ratio of the frictional halting scale in Eq. (18) to the modified β halting scale identified as Eq. (25) is $\varepsilon^{-1/4}\beta^{3/4}c_D^{-1/2}k_d^{-3/4}$. If we equivalently raise this expression to the 4/5 power to make the similarity with μ and μ_1, we then recover μ_2 defined in Eq. (22). The left-hand panel in Fig. 4 is an analogous plot to those in Fig. 3, but now using μ_2 as the abscissa. The change is a modest but noticeable improvement.

After confirming the tightness of the relationship when plotting against μ_2, we can next try to test the theoretical prediction by the modified theory, i.e., Eq. (23). This analytical expression is used to fit the simulations in the left-hand panel in Fig. 4, after choosing $\lambda_0 = 1$ and $\lambda_1 = 1.25$ (corresponding to the visually estimated $\nu_1 = \sqrt{2}$, $\nu_2 = 4$, and $C = 0.8$). The parameter μ_2 still involves ε_p in its definition. As discussed in section 3, we can further eliminate ε_p using the exact relationship between D_τ and ε_p in Eq. (5). Iterating to solve Eq. (23) simultaneously with Eq. (5) then yields the explicit prediction for $\tilde{D}_\tau$. The dependence of this $\tilde{D}_\tau$ prediction on $\tilde{\beta}$ and $\tilde{c}_D$ is shown in the right-hand panel of Fig. 4. Comparing with the simulations, the result is encouraging for $\tilde{\beta} \lesssim 0.4$, especially given the numerous assumptions in the modified theory. Yet it misses the transition to a regime of increasing $\tilde{D}_\tau$ with increasing $\tilde{c}_D$ for larger $\tilde{\beta}$.

c. *Sensitivity to the energy spectrum*

The above results point out the value of Eq. (23) in predicting $\tilde{D}_\tau$ when $\tilde{\beta}$ is small or moderate but also its limitation when $\tilde{\beta}$ is large. A natural question is then to ask which assumptions in the theory are inconsistent with the simulations. For this purpose, we take advantage of the intermediate result of the theory, Eq. (12), a diagnostic relation between D_τ and the barotropic eddy kinetic energy spectrum $\mathbb{E}(k)$.

Taking $\mathbb{E}(k)$ directly from numerical simulations, we plot the diffusivity ratio and diffusivity itself estimated using Eq. (12) in Fig. 5. While there are still some discrepancies, Eq. (12)'s estimation nicely explains the simulated $\tilde{c}_D$ dependence for large values of $\tilde{\beta}$.

This comparison informs us that Eq. (23)'s breakdown in the large $\tilde{\beta}$ regime has to do with the series of assumptions we introduced to express $\mathbb{E}(k)$ in terms of the external parameters. We have plotted the simulated $\mathbb{E}(k)$ in Fig. 6, with each panel

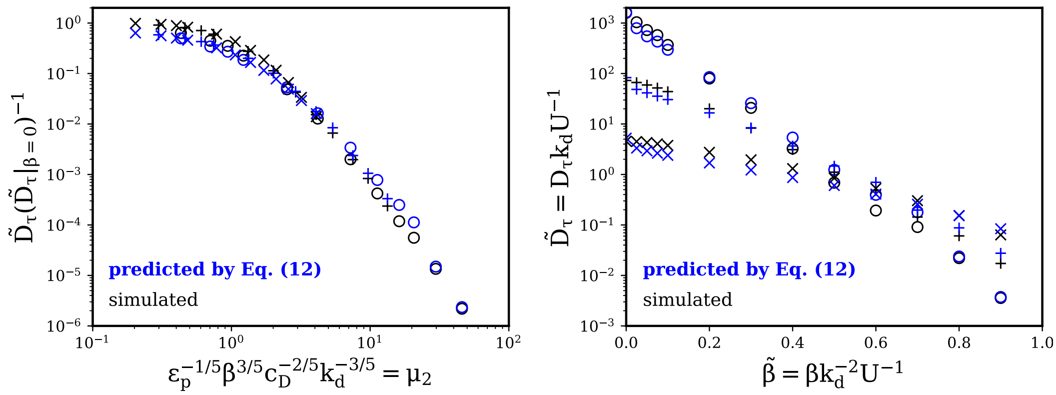

FIG. 5. As in Fig. 4, but with the prediction obtained by Eq. (12) and the simulated barotropic eddy kinetic energy spectrum [$\mathbb{E}(k)$].

containing a series of simulations using the same $\tilde{c}_D$ but varying $\tilde{\beta}$, the spectrum with the largest amplitude corresponding to the $\tilde{\beta}=0$ case. These $\tilde{\beta}=0$ cases all have a wide range of wavenumbers with a fairly well defined spectral slope close to $k^{-7/3}$. The spectra for the nonzero $\tilde{\beta}$ cases suggest that this assumption continues to give a fair description for the energy spectra over a limited range, presumably marking an inverse cascade, qualitatively justifying holding the value of x in Eq. (15) fixed with varying β.

The source of the discrepancy is presumably the crude way that the energy spectrum is cutoff at large scales in Eq. (13), which assumes that the suppression effect for large β is adequate for estimating the diffusivity without altering the spectral shape from the $\beta = 0$ case. In addition, for $\tilde{\beta}$ approaching unity, the issue of whether to identify the diffusivity emerging from these approximations with the diffusivity of lower-layer PV rather than the thermal diffusivity is presumably relevant (Lapeyre and Held 2003). But we have not been able to improve the fit to the simulations along these lines.

5. Summary and discussion

In this study, we examine the parameter dependence of eddy heat flux on the frictional wavenumber of quadratic drag c_D, and β in a homogeneous quasigeostrophic two-layer model. Seeking a theoretical understanding of the parameter control on this eddy heat flux, or the thermal diffusivity defined accordingly, has been a long-lasting problem. The goal of this study is to connect the results and arguments in previous two-layer studies to the recent proposal made by KJ17 on understanding the same parameter dependence of passive tracer diffusion in a two-dimensional externally stirred model. Particularly, building on our findings reported in CH19, we propose a heuristic modification on KJ17's two-dimensional theory and discuss its potential to explain the two-layer thermal diffusivity D_τ.

The central point of our modification is on the assumed slope of the barotropic eddy kinetic energy spectrum $\mathbb{E}(k)$. Existing theories that are based on two-dimensional theory for passive tracer diffusion have been known to struggle in predicting D_τ in numerical simulations. In CH19, we revisited the problem and found that, at the $\beta = 0$ limit, this is owing to the inaccuracy of Kolmogorovian argument adopted in these theories. We also found that heuristically assuming a $-7/3$ spectral slope of $\mathbb{E}(k)$ that is steeper than Kolmogorovian slope, $-5/3$, to qualitatively account for the input of barotropic energy as the cascade moves energy to larger scales, provides us a reasonable estimate for $D_\tau|_{\beta=0}$ (i.e., D_τ at $\beta = 0$). Since KJ17's two-dimensional theory for nonzero β has utilized a Kolmogorovian argument, we here incorporate the same heuristic modification into their theory, assuming this spectral slope does not vary with β.

With this modification, the result is a prediction that the dependence of $D_\tau/D_\tau|_{\beta=0}$ on c_D and β collapses to a dependence on a nondimensional parameter, $\mu_2 = \varepsilon_p^{-1/5}\beta^{3/5}c_D^{-2/5}k_d^{-3/5}$. Physically, μ_2 can also be interpreted as the ratio of the frictional halting scale and β halting scale, with both of them being modified in the same way to consider the steeper slope of energy spectrum and introducing the explicit dependence on the wavenumber of deformation radius k_d. This prediction for $D_\tau/D_\tau|_{\beta=0}$ is confirmed by simulations in the parameter range $10^{-3} \le \tilde{c}_D \equiv c_D k_d^{-1} \le 10^{-1}$ and $0 \le \tilde{\beta} \equiv \beta k_d^{-2} U^{-1} < 1$.

Since the expression for μ_2 also contains ε_p the energy production rate, while ε_p is itself closely related to D_τ, we try to further close the theory by eliminating this ε_p dependence to solve for D_τ from external parameters only. The result is again of value in this limited parameter range, but is inadequate for large β. (While we have not highlighted it here, it is also inadequate for large c_D.) Since an alternative prediction obtained by returning to the simulated $\mathbb{E}(k)$ shows no sign of the same discrepancy, this is likely due to the approximation concerning the shape of the energy spectrum.

These results do not lead us to a better physical understanding of the large β regime in which diffusivity increases with increasing drag. There are several perspectives that potentially explain this kind of behavior. One is that in these relatively weakly unstable cases, one can think of the flow as approximated by a steady wave, for which the flux vanishes when there is no dissipation, by the familiar nonacceleration theorem. The dissipation then produces a flux that increases with the damping strength as it breaks this nonacceleration limit. An alternative perspective involves the barotropic governor (James 1987), in which strong damping, acting

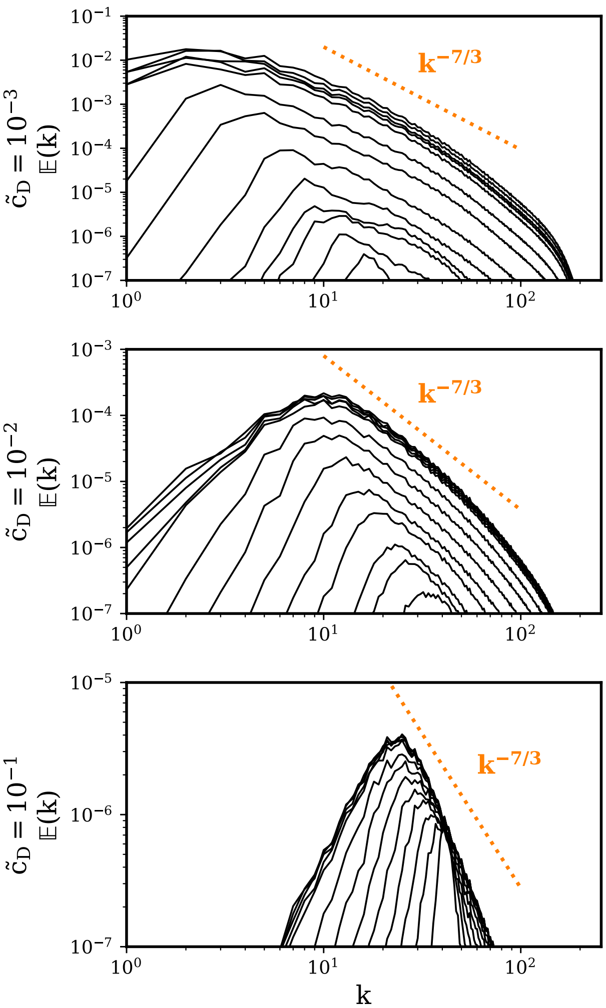

FIG. 6. Barotropic eddy kinetic energy spectra [$\mathbb{E}(k)$] as a function of the total wavenumber (k) in symmetric drag simulations: each panel shows a set of simulations with the same $\tilde{c}_D$ but varying $\tilde{\beta}$. The dashed line indicates the spectral slope $k^{-7/3}$.

particularly on the zonal mean winds, weakens jets that otherwise interfere with the baroclinic eddy production. Neither of these perspectives are incorporated into our theory.

Neither have we been able to take into account the results of Lapeyre and Held (2003) in which the diffusivity of lower-layer PV is identified with the diffusivity obtained from the scaling theory. This modification alters the theory when β approaches unity, but our attempt to incorporate it into the theory described here does not result in any improvements to the fit with the simulations. In addition, the theories of Gallet and Ferrari (2020) and Thompson and Young (2006, 2007) focus on the dynamics of coherent structures, either as barotropic vortices that advect the baroclinic streamfunction as a passive tracer, or as baroclinic vortices and jets that participate more directly in the heat flux. We do not pretend to be able to unify these various perspectives on this intriguing problem.

In summary, we have made an attempt to present a theory for the eddy heat flux in the homogeneous two-layer model, staying close to ideas developed for passive tracer transport in two-dimensional turbulence by accepting the picture in which the barotropic mode of the two-layer model plays a critical role, agreeing qualitatively with the classic picture conceptualized in Rhines and Salmon's work and formulated more formally as a scaling theory by Held and Larichev (1996). The results presented here may provide some guidance on how to approach a more complete theory.

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