Deterministic ill-posedness and probabilistic well-posedness of the viscous nonlinear wave equation describing fluid-structure interaction

Jeffrey Kuan and Sunčica Čanić
Department of Mathematics
University of California Berkeley

April 9, 2021

Abstract

We study low regularity behavior of the nonlinear wave equation in $\mathbb{R}^2$ augmented by the viscous dissipative effects described by the Dirichlet-Neumann operator. Problems of this type arise in fluid-structure interaction where the Dirichlet-Neumann operator models the coupling between a viscous, incompressible fluid and an elastic structure. We show that despite the viscous regularization, the Cauchy problem with initial data (u, u_t) in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, is ill-posed whenever $0 < s < s_{cr}$, where the critical exponent s_{cr} depends on the degree of nonlinearity. In particular, for the quintic nonlinearity u^5, the critical exponent in $\mathbb{R}^2$ is $s_{cr} = 1/2$, which is the same as the critical exponent for the associated nonlinear wave equation without the viscous term. We then show that if the initial data is perturbed using a Wiener randomization, which perturbs initial data in the frequency space, then the Cauchy problem for the quintic nonlinear viscous wave equation is well-posed almost surely for the supercritical exponents s such that $-1/6 < s \leq s_{cr} = 1/2$. To the best of our knowledge, this is the first result showing ill-posedness and probabilistic well-posedness for the nonlinear viscous wave equation arising in fluid-structure interaction.

1 Introduction

We study low regularity behavior of the nonlinear wave equation augmented by the viscous effects described by the Dirichlet-Neumann operator typically arising in fluid-structure interaction problems:

$$\partial_{tt} u - \triangle u + u^p + 2\mu\sqrt{-\triangle}\partial_t u = 0, \quad (x, y) \in \mathbb{R}^2, t \in \mathbb{R}, \tag{1}$$

where $p > 0$ is an odd integer, and $\mu > 0$. The model above can be thought of as a mathematical prototype for the interaction between a prestressed, stretched membrane and a viscous, incompressible fluid. The membrane (an infinitely large drum surface) is modeled by the linear wave equation:

$$\partial_{tt} u - \triangle u = f, \quad (x, y) \in \mathbb{R}^2, t \in \mathbb{R},$$

where $u = u(x, y)$ is a scalar function describing transverse membrane displacement. We assume for simplicity that the structure experiences displacement only in the transverse, z direction and hence experiences no tangential displacements from its reference configuration. The incompressible,

viscous Newtonian fluid is located under the membrane in the half space $z < 0$, modeled by the Stokes equations:

$$\left.\begin{array}{rcl} \nabla\pi &=& \mu\triangle \boldsymbol{v}, \\ \nabla\cdot\boldsymbol{v} &=& 0, \end{array}\right\} \quad \text{in } \Omega = \{(x,y,z)\in\mathbb{R}^3 : z<0\}, \tag{2}$$

where π is the fluid pressure, $\boldsymbol{v}$ is the fluid velocity, and μ is the kinematic viscosity coefficient. The first equation in (2) describing the second Newton's law of motion (balance of forces), can be written as

$$\nabla\cdot\boldsymbol{\sigma}(\pi,\boldsymbol{v}) = 0,$$

where the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = -\pi\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{v}) \tag{3}$$

for Newtonian fluids, $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{D}(\boldsymbol{v})$ denotes the symmetrized gradient of velocity.

The fluid and structure are coupled through two coupling conditions: the kinematic and dynamic coupling conditions. For simplicity, we will be assuming that the coupling is at the fixed fluid-structure interface, which we denote by $\Gamma = \{(x,y,z)\in\mathbb{R}^3 : z = 0\}$. The fixed interface corresponds to the reference configuration of the stretched (prestressed) membrane. The coupling conditions read:

- The kinematic coupling condition (describing continuity of velocities):

$$\boldsymbol{v}|_\Gamma = u_t\boldsymbol{e}_z, \quad (x,y,z)\in\Gamma, \tag{4}$$

  where $\boldsymbol{e}_z$ is the unit vector pointing in the positive z direction. Notice that the kinematic coupling condition (4) states that on the boundary Γ, the tangential components of the trace of the fluid velocity are equal to zero. Only the normal component of the fluid velocity is assumed to be nonzero on Γ.

- The dynamic coupling condition (describing balance of forces, i.e., the second Newton's law of motion):

$$\partial_{tt}u - \triangle u = -\boldsymbol{\sigma}\boldsymbol{e}_z\cdot\boldsymbol{e}_z + F_{\text{ext}}(u). \tag{5}$$

  The right hand-side of (5) describes the jump in the normal stress (traction) across the fluid-structure interface, where $F_{\text{ext}}(u)$ denotes external force, which in general may depend on u, acting on the membrane in the normal direction $-\boldsymbol{e}_z$.

In our model we will be assuming that the external force is a nonlinear function of u, and that it is given by

$$F_{\text{ext}}(u) = -u^p,$$

where the nonlinear term u^p models, for example, the nonlinear spring-type effects, distributed across the membrane surface by, say, a surrounding medium (e.g., structure) sitting on top of the membrane. These types of external source terms have been used in modeling blood flow in compliant arteries, see [28], to describe the tissue surrounding pulsating arteries. In cylindrical geometry, the nonlinear term in the cylindrical nonlinear membrane/shell equations appears due to the contribution of the circumferential stress/strain, as was done in [39].

Next, we compute the contribution of the term arising from the Cauchy stress tensor, $-\boldsymbol{\sigma}\boldsymbol{e_z}\cdot\boldsymbol{e_z}$, on the right hand side of (5), which is evaluated on the fixed fluid-structure interface Γ. Using (3),

$$-\boldsymbol{\sigma}\boldsymbol{e_z}\cdot\boldsymbol{e_z} = \pi - 2\mu\frac{\partial v_z}{\partial z}.$$

Because we are evaluating this quantity on Γ, we note that

$$\frac{\partial v_z}{\partial z} = 0 \qquad \text{on } \Gamma, \tag{6}$$

by the divergence-free condition $\nabla \cdot \boldsymbol{v} = 0$ and the fact that $v_x = v_y = 0$ on Γ, since we are assuming that the structure experiences displacement only in the transverse z direction. Therefore,

$$-\boldsymbol{\sigma e_z} \cdot \boldsymbol{e_z} = \pi, \tag{7}$$

where π is the fluid pressure given as a solution to the Stokes equations (2).

So it remains to find an appropriate expression for π in terms of the structure displacement u as follows. We will derive the formula

$$\pi = -2\mu\sqrt{-\Delta}u_t \quad \text{on } \Gamma, \tag{8}$$

under the assumption that u and u_t are smooth functions, along with their spatial derivatives, that are rapidly decreasing at infinity. We will also impose the boundary conditions on (2), stating that the fluid velocity is bounded on the lower half space, and the pressure π has a limit equal to zero as $|x| \to \infty$ in the lower half space.

To derive the formula (8), we note that by taking the inner product of the first equation in (2) with $\boldsymbol{e_z}$, we obtain

$$\frac{\partial \pi}{\partial z} = \mu \Delta_{x,y} v_z + \mu \frac{\partial^2 v_z}{\partial z^2}, \tag{9}$$

where $\Delta_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Furthermore, by taking the divergence of the first equation in (2), and by using the divergence-free condition, we get that the pressure π is harmonic. Thus, if we can compute the right hand side of (9) on Γ, we can recover π as the solution to a Neumann boundary value problem for Laplace's equation in the lower half space, with the boundary condition requiring that π goes to zero at infinity.

To compute the right hand side of (9), we need to compute v_z. We use a Fourier analysis argument. From (9) we see that v_z satisfies

$$\Delta v_z = \frac{\partial \pi}{\partial z}.$$

Taking the Laplacian on both sides, and using the fact that π is harmonic, we obtain

$$\Delta^2 v_z = 0, \qquad \text{on } \Omega = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}. \tag{10}$$

Thus, v_z satisfies the biharmonic equation with the following two boundary conditions: from the kinematic coupling condition we get

$$v_z(x, y, 0) = u_t(x, y, 0), \qquad \text{on } \Gamma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}, \tag{11}$$

and by (6), we have

$$\frac{\partial v_z}{\partial z}(x, y, 0) = 0, \qquad \text{on } \Gamma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}. \tag{12}$$

We remark that the biharmonic equation to analyze the Stokes problem has been previously used in many works, see for example [35].

We solve (10) with boundary conditions (11) and (12) by taking a Fourier transform in the variables x and y, but not in z. We will denote the Fourier variables associated with x and y by ξ_1 and ξ_2, and we will denote $\boldsymbol{\xi} = (\xi_1, \xi_2)$, $|\boldsymbol{\xi}|^2 = \xi_1^2 + \xi_2^2$. The Fourier transform equation then reads:

$$|\xi|^4 \widehat{v_z}(\boldsymbol{\xi}, z) - 2|\boldsymbol{\xi}|^2 \frac{\partial^2}{\partial z^2} \widehat{v_z}(\boldsymbol{\xi}, z) + \frac{\partial^4}{\partial z^4} \widehat{v_z}(\boldsymbol{\xi}, z) = 0. \tag{13}$$

The general solution to (13) is

$$\widehat{v_z}(\boldsymbol{\xi}, z) = C_1(\boldsymbol{\xi}) e^{|\boldsymbol{\xi}|z} + C_2(\boldsymbol{\xi}) z e^{|\boldsymbol{\xi}|z} + C_3(\boldsymbol{\xi}) e^{-|\boldsymbol{\xi}|z} + C_4(\boldsymbol{\xi}) z e^{-|\boldsymbol{\xi}|z}.$$

Because $e^{-|\boldsymbol{\xi}|z}$ and $ze^{-|\boldsymbol{\xi}|z}$ are unbounded in the lower half plane where $z < 0$, we exclude these two terms and are left with

$$\widehat{v_z}(\boldsymbol{\xi}, z) = C_1(\boldsymbol{\xi}) e^{|\boldsymbol{\xi}|z} + C_2(\boldsymbol{\xi}) z e^{|\boldsymbol{\xi}|z}. \tag{14}$$

In Fourier variables, the two boundary conditions (11) and (12) translate to

$$\widehat{v_z}(\boldsymbol{\xi}, 0) = \widehat{u_t}(\boldsymbol{\xi}), \qquad \frac{\partial \widehat{v_z}}{\partial z}(\boldsymbol{\xi}, 0) = 0,$$

which allow us to solve for the general functions $C_1(\boldsymbol{\xi})$ and $C_2(\boldsymbol{\xi})$ in (14), giving the result

$$\widehat{v_z}(\boldsymbol{\xi}, z) = \widehat{u_t}(\boldsymbol{\xi}) e^{|\boldsymbol{\xi}|z} - |\xi| \widehat{u_t}(\boldsymbol{\xi}) z e^{|\boldsymbol{\xi}|z}. \tag{15}$$

We can now compute the right hand side of (9). Taking the Fourier transform of (9) in the x and y variables, and evaluating the equation on Γ by using the kinematic coupling condition (4), we get

$$\frac{\partial \widehat{\pi}}{\partial z}(\boldsymbol{\xi}, 0) = -\mu |\xi|^2 \widehat{u_t}(\boldsymbol{\xi}) + \mu \frac{\partial^2 \widehat{v_z}}{\partial z^2}(\boldsymbol{\xi}, 0).$$

From the explicit formula for $\widehat{v_z}(\boldsymbol{\xi}, z)$ in (15), we conclude that

$$\frac{\partial \widehat{\pi}}{\partial z}(\boldsymbol{\xi}, 0) = -2\mu |\xi|^2 \widehat{u_t}(\boldsymbol{\xi}). \tag{16}$$

We have now obtained that the pressure π is a harmonic function in the lower half space, satisfying a Neumann boundary condition, posed in Fourier space as (16). To obtain π on Γ and recover formula (8), we can now employ the Neumann to Dirichlet operator.

It is well-known that the Dirichlet to Neumann operator for Laplace's equation in the lower half space (with the solution to Laplace's equation having a limit of zero at infinity) is given by $\sqrt{-\Delta}$, see for example [6]. Therefore, the Neumann to Dirichlet operator for Laplace's equation in the lower half space (with the solution to Laplace's equation having a limit of zero at infinity) is a Fourier multiplier of the form $\frac{1}{|\xi|}$. Since π is a harmonic function satisfying the Neumann boundary condition (16), by applying the Neumann to Dirichlet operator we get

$$\widehat{\pi}(\xi) = -2\mu |\xi| \widehat{u_t}(\xi) \qquad \text{on } \Gamma,$$

which establishes the desired formula (8).

The result in (8), along with (7), implies the following form of the dynamic coupling condition:

$$\partial_{tt} u - \triangle u + u^p + 2\mu \sqrt{-\triangle} u_t = 0,$$

which accounts for the influence of the fluid viscosity within the fluid domain Ω and its trace on the domain boundary Γ. For simplicity, we will set $2\mu = 1$, and study the equation

$$\partial_{tt}u - \triangle u + u^p + \sqrt{-\triangle}u_t = 0.$$

We will refer to equation (1) as the *viscous nonlinear wave equation* (vNLW).

We are interested in the Cauchy problem for equation (1), where $p > 0$ is an odd integer, and $\mu > 0$, supplemented with initial data:

$$u(0,\cdot) = f \quad \text{and} \quad u_t(0,\cdot) = g, \tag{17}$$

where $(f,g) \in \mathcal{H}^s(\mathbb{R}^2) = H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$. Here H^s denotes the usual (inhomogeneous) Sobolev space.

The analysis of fluid-structure interaction problems involving incompressible, viscous fluids and elastic structures started in the early 2000's with works in which the coupling between the fluid and structure was assumed across a fixed fluid-structure interface (*linear coupling*) as in [1,2,14,22], and was then extended to problems with *nonlinear coupling* in the works [3,8–10,12,13,15–19,21,23,24, 29–34]. In all these studies, a major underlying reason for the well-posedness is the regularization by the fluid viscosity and the dispersive nature of the wave-like operators in more than one spatial dimension. One of the main questions is "by how much" does the fluid viscosity regularize the coupled problem? How does the viscous regularization "compete" with the nonlinearities in the problem? In the present work, we study the influence of fluid viscosity and nonlinearity on the well-posedness of the Cauchy problem for the nonlinear viscous wave equation by studying the following two problems:

(P1) For a given exponent $p > 0$, which is an odd integer describing the nonlinearity in the problem, is there a critical exponent s_{cr} such that equation (1) with initial data $(f,g) \in \mathcal{H}^s(\mathbb{R}^2) = H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, with $0 < s < s_{cr}$, is *ill-posed* in the sense that the solution mapping

$$(f,g) \mapsto u,$$

which takes the initial data $(f,g) \in \mathcal{H}^s(\mathbb{R}^2)$ and maps it to a solution $u \in C^0([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2))$, fails to be continuous?

(P2) If well-posedness fails for some critical exponent s_{cr}, how "generic" is that behavior? Is there a random perturbation of the initial data that would provide well-posedness, even for "supercritical" initial data, namely for $s < s_{cr}$, and how generic is that random perturbation?

As we shall see, the answer to problem (P1) is yes. Namely, despite the regularization by fluid viscosity, there is a critical exponent s_{cr} depending on the nonlinearity p, below which the viscous nonlinear wave equation (1) is not well-posed for the initial data $u \in C^0([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2))$ such that $0 < s < s_{cr}$. Namely, as $t \to 0$, the energy of the low frequency Fourier modes gets transferred to the high frequencies so that the H^s-norm of the solution becomes arbitrarily large as $t \to 0$, even though the H^s-norm of the initial data is arbitrarily small.

The answer to problem (P2) is that the ill-posedness addressed in problem (P1) is *not generic* in a certain sense. Namely, we show for the quintic nonlinearity $p = 5$, for example, that if we randomize the initial data using a Wiener randomization, which perturbs initial data in frequency space via independent random variables with bounded six moments (associated with $p = 5$), then the Cauchy problem for the quintic nonlinear viscous wave equation (1) will be well-posed almost surely.

To obtain answers to problems (P1) and (P2), we start by looking for symmetries of equation (1). Symmetries can provide insight into the questions of if and when well-posedness may be expected. As in the theory of nonlinear dispersive equations [7, 11, 25], we look for scaling symmetries. As we will see later, the scaling symmetry

$$u(t,x) \to \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x)$$

preserves solutions of equation (1). Furthermore, it preserves the $\dot{H}^s$ norm of the solution when $s = s_{cr}$, where

$$s_{cr} = \frac{n}{2} - \frac{2}{p-1} = 1 - \frac{2}{p-1},$$

and n is the dimension of $\mathbb{R}^n$ ($n = 2$ in the present work). Here, $\dot{H}^s(\mathbb{R}^n)$ denotes the homogeneous Sobolev space, equipped with the norm

$$||u||^2_{\dot{H}^s(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi.$$

Note that this is the same critical exponent as for the defocusing nonlinear wave equation [11, 25]:

$$\partial_{tt} u - \Delta u + u^p = 0, \tag{18}$$

where the defocusing case corresponds to the choice of the positive sign in front of the nonlinear term u^p. We will see that in terms of energy inequalities and ill-posedness behavior, the viscous nonlinear wave equation (1) shares similar properties with the nonlinear wave equation (18), but the viscous nonlinear wave equation also exhibits novel behavior that arises from dissipation of energy due to viscosity.

The presence of a critical exponent is crucial for the analysis of nonlinear partial differential equations, in particular for dispersive equations. Above the critical exponent, dispersive equations usually exhibit **well-posedness**, which can be established from fixed point arguments in combination with dispersive estimates for the linear problem [7, 25]. For example, such a result was established first for the nonlinear Schrödinger equation above the critical exponent by Cazenave and Weissler in [7], and a well-posedness result for the nonlinear wave equation above the critical exponent was established by Lindblad and Sogge in [25].

In the case of the viscous nonlinear wave equation, we expect to have well-posedness for the exponents s above the critical exponent s_{cr} using similar approaches as in [7,25], since the presence of viscous regularization can only improve solution behavior. *Well-posedness* of the Cauchy problem for equation (1) (in the Hadamard sense) is defined by requiring that there exists a unique solution $u \in C^0([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2))$ such that the solution mapping

$$(f,g) \mapsto u,$$

which takes the initial data

$$(f,g) \in \mathcal{H}^s(\mathbb{R}^2) = H^s \times H^{s-1}$$

and maps it to a solution $u \in C^0([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2))$, is continuous.

In this manuscript, however, we are interested in the supercritical ($s < s_{cr}$) behavior of equation (1). Below the critical exponent which preserves the $\dot{H}^{s_{cr}}$ norm of the scaling symmetry $u(t,x) \to \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x)$, we would heuristically expect **ill-posedness**.

Indeed, such ill-posedness behavior was most famously considered for certain ranges of s of initial data in $\mathcal{H}^s$ for the nonlinear wave equation and the nonlinear Schrödinger equation by Christ, Colliander, and Tao in [11].

It is not clear *a priori* that the viscous nonlinear wave equation would embody the same property because of the regularizing effects by the fluid viscosity. However, in Sec. 2, we show that a similar ill-posedness result can be obtained using a similar procedure for the viscous nonlinear wave equation with initial data in $\mathcal{H}^s$ with $0 < s < s_{cr}$. In particular, we will show lack of continuity of the solution map for the viscous nonlinear wave equation when $0 < s < s_{cr}$.

As a remark, we note that for the nonlinear wave equation, because solutions can satisfy finite speed of propagation, it is possible to construct initial data for which there is instantaneous blowup [11]. In particular, it can be shown that for a certain choice of initial data, there exist no $T > 0$ for which there is a weak solution to the nonlinear wave equation in $C([0,T]; H^s)$. For more details, see the discussion in [11]. However, we would not expect instantaneous blowup for the viscous nonlinear wave equation, since the solutions no longer obey finite speed of propagation due to the viscous effects in (1). We will show, instead, that the solution map, which associates the solution u to the initial data $u(0)$, is not continuous in the sense that the H^s norm of the solution can grow without bound as $t \to 0$ even as the initial data $u(0)$ has infinitesimally small H^s norm. More precisely, we will show that if $0 < s < s_{cr}$, then for every $\epsilon > 0$, there exists a solution u of the viscous nonlinear wave equation (19) and a positive time t such that

$$||u(0)||_{H^s} < \epsilon, \quad u_t(0) = 0, \quad 0 < t < \epsilon, \quad ||u(t)||_{H^s} > \epsilon^{-1},$$

for some $u(0) \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz class. Thus, the solution map for the equation (19) is not continuous at $(0,0) \in H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ for $0 < s < s_{cr}$. The main mechanism that allows this behavior is transfer of energy from low to high Fourier modes. The proof is based on studying the visco-dispersive limit $\nu \to 0$ in:

$$\partial_{tt} u - \nu^2 \Delta u + \nu \sqrt{-\Delta} \partial_t u + u^p = 0,$$

and understanding how the unbounded growth of the H^s norm in time of the solution for the visco-dispersive limit equation $\nu = 0$ translates into the solution behavior of the perturbed equation with $\nu > 0$ small, as $t \to 0$. By utilizing the various symmetries of the solutions to the viscous nonlinear wave equation, one can show that the unbounded growth of the H^s norm in time for the visco-dispersive limit $\nu = 0$, translates into similar growth of the H^s norm of the solution to the perturbed equation with $\nu > 0$ small. Using scaling symmetries of the viscous nonlinear wave (vNLW) equation, we can then allow this unbounded H^s norm growth to occur at arbitrarily small times $t \to 0$, even when the initial data has infinitesimally small H^s norm. The ill-posedness result is presented in Sec. 2.

In Sec. 3, we start preparing for the probabilistic well-posedness result, by deriving Strichartz estimates, which will be crucial in proving probabilistic well-posedness for the supercritical nonlinear viscous wave equation with initial data belonging to $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, where s is below the critical exponent ($-1/6 < s \leq 1/2 = s_{cr}$). The Strichartz estimates, which are *linear* estimates derived for the linear homogeneous and linear inhomogeneous equations, will be used to prove probabilistic well-posedness of the *nonlinear* viscous wave equation by considering the nonlinear term as an inhomogeneous (source) term for the corresponding linear problem, and then using a fixed point argument to show the existence of a unique solution, which we will then show also depends continuously on the initial data.

Strichartz estimates, which are estimates that control the $L_t^q L_x^r$ norms of solutions to linear dispersive equations in terms of the initial data and source terms, are crucial for establishing well-posedness results for dispersive equations. These estimates were first determined by Strichartz in [37] in the context of a Fourier restriction problem, and were found to be equivalent to estimates for the linear wave equation. They were extended to a more general abstract context by Keel and

Tao in [20]. In Sec. 3, we use the theorem by Keel and Tao in [20] for general Strichartz estimates to derive the corresponding estimates for the linear viscous wave equation (homogeneous and inhomogeneous). We show that the presence of viscous regularization influences Strichartz estimates in two ways: the viscous Strichartz estimates hold for a larger range of admissible exponents q and r, and the homogeneous viscous Strichartz estimates hold even in 1D. This is interesting because it is well-known that the linear wave equation in one dimension does not possess such Strichartz estimates. See Sogge [36] for a detailed exposition of Strichartz estimates and generalized Strichartz estimates.

In Sec. 5 we use the "global" $L_t^q L_x^r$ Strichartz estimates and combine them with the "local" $C^0([0,T], H^s(\mathbb{R}^n))$ estimates, presented in Sec. 4, to study probabilistic well-posedness for the supercritical nonlinear viscous wave equation. We show that under Wiener randomization of initial data, we can get a local probabilistic well-posedness result for the nonlinear viscous wave equation in $\mathbb{R}^2$ that brings the threshold exponent from $s_{cr} = 1/2$ down to $s > -1/6$.

This result is in the spirit of Burq and Tzvetkov [5], who studied probabilistic well-posedness for the supercritical cubic *wave equation*. Ever since Christ, Colliander, and Tao in [11] showed an ill-posedness result below the critical exponent for the nonlinear wave equation, there has been considerable interest in the behavior of such supercritical wave equations under randomization of the initial data. Burq and Tzvetkov showed in [5] that under appropriate randomization, one has local existence to the supercritical cubic wave equation almost surely for initial data in $H^s(M) \times H^{s-1}(M)$ appropriately randomized, where $s \geq 1/4$, and M is a compact, three-dimensional manifold. The probabilistic randomization decreases the threshold for such local existence from the critical exponent $s_{cr} = 1/2$ for the cubic nonlinear wave equation to $s = 1/4$.

The randomization used by Burq and Tzvetkov in [5] relies heavily on the fact that there is a discrete spectrum for $-\Delta$ on a compact Riemannian manifold M. The randomization of initial data is then performed by randomizing each of the eigenfunction components of the initial data, using a sequence of independent random variables. See [5] for more details.

Our problem, however, is posed on $\mathbb{R}^2$, which means that we do not have a discrete spectrum for $-\Delta$, and so the randomization from Burq and Tzvetkov in [5] does not carry over to our case. However, we use an analogue of this discrete randomization, known as the Wiener randomization. Wiener randomization was developed by Bényi, Oh, and Pocovnicu in [4]. Such a randomization was used in recent years to produce existence results for randomized initial data on Euclidean domains for the nonlinear wave equation by Lührmann and Mendelson in [26], and for the incompressible Euler equations by Wang and Wang in [38], for example. We describe the Wiener randomization and its properties in Sec. 5, and show how it can be used to produce a probabilistic local existence result for the supercritical nonlinear viscous wave equation. We prove that for a random perturbation of initial data in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ based on Wiener randomization, the supercritical quintic nonlinear viscous wave equation is well-posed almost surely, even for $-1/6 < s \leq s_{cr} = 1/2$. In contrast with the cubic *wave equation* for which supercritical probabilistic well-posedness holds for the exponents $s \geq 1/4$ in the case of three-dimensional compact manifolds without boundaries, see [5], we show that the viscous dissipation allows us to bring the threshold exponent from $s_{cr} = 1/2$ all the way down to negative Sobolev space exponents s for which $s > -1/6$.

2 Ill-posedness for viscous nonlinear wave equation

We study the Cauchy problem for the viscous nonlinear wave equation

$$\begin{aligned} \partial_{tt} u - \Delta u + \sqrt{-\Delta}\partial_t u + u^p &= 0 \text{ on } \mathbb{R}^n, \\ u(0,x) = f(x), \quad u_t(0,x) &= g(x), \end{aligned} \tag{19}$$

where $p > 1$ is a positive odd integer. The case of $n = 2$ corresponds to our given fluid-structure interaction model, but we will use general n, as all arguments here hold for general n. To study ill-posedness as specified in (P1) of Sec. 1, we begin by investigating the scaling symmetries of this equation and determine the critical exponent s_{cr} that preserves the homogeneous Sobolev $\dot{H}^s(\mathbb{R}^n)$ norm of solutions of (19) under this scaling symmetry. We recall that the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ is defined as a completion of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|f\|_{\dot{H}^s} := (2\pi)^{-n/2} \||\xi|^s \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}.$$

We start by first noticing that for positive λ, we have $\sqrt{-\Delta}\,[u(\lambda x)] = \lambda(\sqrt{-\Delta}u)(\lambda x)$. Indeed,

$$\sqrt{-\Delta}\,[u(\lambda x)] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \lambda|\xi|\widehat{u}(\xi) e^{i\lambda x\cdot\xi} d\xi = \lambda(\sqrt{-\Delta}u)(\lambda x).$$

Therefore, the following scaling map:

$$u(t,x) \to \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x) \tag{20}$$

preserves solutions to the partial differential equation above. A calculation shows that the critical exponent for equation (19) that preserves the homogeneous Sobolev $\dot{H}^s(\mathbb{R}^n)$ norm of this scaling, is given by:

$$s_{cr} = \frac{n}{2} - \frac{2}{p-1}. \tag{21}$$

We note that this is exactly the critical exponent for the defocusing nonlinear wave equation [11]

$$\partial_{tt} u - \Delta u + u^p = 0 \text{ on } \mathbb{R}^n. \tag{22}$$

Christ, Colliander, and Tao have shown in [11] that this defocusing nonlinear wave equation for odd integers $p > 1$ exhibits ill-posedness for initial data $(f, g) \in H^s \times H^{s-1}$ where $0 < s < s_{cr}$. We ask whether equation (19) exhibits similar behavior. Intuitively, one can show through an energy estimate that even though both the nonlinear wave equation (18) and viscous nonlinear wave equation (19) share the same critical exponent, the presence of the viscous term in (19) dissipates energy, which might yield better estimates and a different ill-posedness result. We will show that this is, in fact, not the case, although there are some differences. More precisely, we will show that the viscous nonlinear wave equation (19) has the same ill-posedness property as the nonlinear wave equation (18), indicating that *nonlinear effects are dominant over viscous regularization* associated with the fluid-structure coupling. However, as we will see below, the viscous contribution has the potential to *slow down* the speed of how "fast" the H^s norm of the solution grows.

A way to show this is to use a variation of the dispersive limit argument in [11]. More precisely, we consider

$$\partial_{tt} u - \nu^2 \Delta u + \nu\sqrt{-\Delta}\partial_t u + u^p = 0 \tag{23}$$

for $\nu > 0$, and study the solution in the *visco-dispersive limit* when $\nu \to 0$. In the visco-dispersive limit, one formally gets a limiting equation

$$\partial_{tt} u + u^p = 0, \tag{24}$$

whose solution can be written out explicitly, and it can be shown that the solution exhibits rapid growth of the H^s norm in time. The goal is to then show that for small ν's, solutions to (23) with $\nu > 0$ small are close to the solution with $\nu = 0$. Note that equation (23) can be considered as a "perturbation" of the visco-dispersive limit equation (24).

To show that for small ν's, solutions of (23) are close to the solution with $\nu = 0$, let ϕ_0 be any smooth compactly supported function on $\mathbb{R}^n$. The initial value problem we consider is

$$\begin{aligned} \partial_{tt}u - \nu^2\Delta u + \nu\sqrt{-\Delta}\partial_t u + u^p = 0 \text{ on } \mathbb{R}^n, \\ u(0,x) = \phi_0(x), \qquad \partial_t u(0,x) = 0, \end{aligned} \tag{25}$$

and the visco-dispersive limit of this initial value problem is

$$\begin{aligned} \partial_{tt}u + u^p = 0, \\ u(0,x) = \phi_0(x), \qquad \partial_t u(0,x) = 0. \end{aligned} \tag{26}$$

The solution of (26) is given by

$$\phi^{(0)}(t,x) = \phi_0(x)V\left(t|\phi_0(x)|^{\frac{p-1}{2}}\right) = \phi_0(x)V\left(t\left(\phi_0(x)\right)^{\frac{p-1}{2}}\right), \tag{27}$$

where V is the smooth periodic solution to

$$V'' + V^p = 0, \qquad V(0) = 1, \qquad V'(0) = 0.$$

Note that V is even, which is why we can remove the absolute values in (27).

Crucial for the proof is the following scaling property of the visco-dispersive limit equation (25): if $u(t,x)$ is a solution to (25), then the entire one-parameter family of functions

$$\lambda^{-\frac{2}{p-1}}u(\lambda^{-1}t, \lambda^{-1}\nu x), \quad \lambda > 0,$$

obtained via the scaling map (20), is a solution to the viscous nonlinear wave equation.

We want to argue that for small values of ν, solutions to both initial value problems (25) and (26) are close for a bounded set of times, which increases as $\nu \to 0$. We make this statement precise in the following proposition.

Note on notation. In what follows, we use k for the exponent in H^k whenever we want to emphasize that the Sobolev exponent is an integer, while we use s for the Sobolev exponent in H^s when the Sobolev exponent can be a general real number, possibly fractional.

Proposition 2.1. Let $p > 1$ be a positive odd integer, and let $k \geq n+1$ be an integer. Suppose ϕ_0 is a compactly supported smooth function, and let $\phi^{(0)}$ be the solution to (26). Given any $\delta > 0$, there exist $C, c > 0$ depending on p, k, δ, and $\phi_0 \in C_0^\infty(\mathbb{R}^n)$, such that for all $0 < \nu \leq c$, there exists a solution $\phi(t,x)$ of (25) such that

$$||\phi(t) - \phi^{(0)}(t)||_{H^k(\mathbb{R}^n)} + ||\phi_t(t) - \phi_t^{(0)}(t)||_{H^k(\mathbb{R}^n)} \leq C|\nu|^{1-\delta}, \quad \text{for all } 0 \leq t \leq c|\log\nu|^c.$$

Proof. The proof is based on energy methods. We begin by defining

$$w(t,x) = \phi(t,x) - \phi^{(0)}(t,x),$$

where $\phi(t,x)$ is the solution to (25) and $\phi^{(0)}(t,x)$ is the solution to (26). Then, w satisfies the following initial value problem, where we use G to denote the nonlinearity $G(z) = z^p$:

$$\begin{aligned} \partial_{tt}w - \nu^2\Delta w + \nu\sqrt{-\Delta}\partial_t w = \nu^2\Delta\phi^{(0)} - G(\phi^{(0)} + w) + G(\phi^{(0)}) - \nu\sqrt{-\Delta}\partial_t\phi^{(0)}, \\ w(0,x) = 0, \qquad \partial_t w(0,x) = 0. \end{aligned} \tag{28}$$

One can use energy estimates to show that this initial value problem has a solution in $H^{k+1} \times H^k$, as long as the $H^{k+1} \times H^k$ norm of the solution is bounded. The main idea to prove this is to first obtain the existence of a unique *local* solution, and then extend the solution locally whenever the energy norm is bounded. The proof of this is by a Picard iteration argument. For a full proof of the existence of a solution to (28) as long as the $H^{k+1} \times H^k$ norm of the solution is bounded, we refer the reader to the Appendix.

We will use the energy method to estimate the size of w. We note that we can derive an energy estimate for the inhomogeneous linear problem

$$u_{tt} - \nu^2 \Delta u + \nu \sqrt{-\Delta} \partial_t u = F(t,x) \tag{29}$$

as follows. Define the ν-wave energy of a solution u (depending on ν) by

$$E_\nu(u(t)) := \int \frac{1}{2}|u_t(t,x)|^2 + \frac{\nu^2}{2}|\nabla u(t,x)|^2 dx.$$

By multiplying the equation (29) by u_t and integrating in space, we get

$$\int \partial_{tt} u \cdot \partial_t u - \nu^2 \Delta u (\partial_t u) dx = \int F(t,x) \partial_t u(x) dx - \nu \int \sqrt{-\Delta} \partial_t u \cdot \partial_t u dx,$$

or equivalently (if u decays rapidly at infinity)

$$\frac{d}{dt}(E_\nu(u(t))) = \int F(t,x) \partial_t u(x) dx - \nu \int \sqrt{-\Delta} \partial_t u \cdot \partial_t u dx.$$

Using Plancherel's theorem,

$$\begin{aligned} \frac{d}{dt}(E_\nu(u(t))) &= \int F(t,x) \partial_t u(x) dx - \nu ||u_t||^2_{\dot{H}^{1/2}} \leq \int F(t,x) \partial_t u(x) dx \\ &\leq ||F(t,\cdot)||_{L^2} \cdot \sqrt{2} (E_\nu(u(t)))^{1/2} \leq ||F(t,\cdot)||_{L^2} \cdot 2 (E_\nu(u(t)))^{1/2}, \end{aligned}$$

where we applied the Cauchy-Schwarz inequality. Therefore,

$$\frac{d}{dt}\left(E_\nu^{1/2}(w(t))\right) \leq ||F(t,\cdot)||_{L^2},$$

which gives the desired energy inequality. Using the fact that derivatives commute with $(\partial_{tt} - \nu^2 \Delta + \nu \sqrt{-\Delta} \partial_t)$ (since $\sqrt{-\Delta}$ is a Fourier multiplier and hence commutes with ordinary derivatives which are also Fourier multipliers), we can get an estimate on the derivatives too. In particular, if we define

$$E_{\nu,k}(w(t)) := \sum_{|\alpha| \leq k} E_\nu(\partial_x^\alpha w(t)),$$

we have the energy inequality

$$\frac{d}{dt}\left(E_{\nu,k}^{1/2}(w(t))\right) \leq C ||F(t,\cdot)||_{H^k},$$

for a constant C depending only on k. Applying this energy inequality to (28) gives

$$\frac{d}{dt}\left(E_{\nu,k}^{1/2}(w(t))\right) \leq C \left(||\nu^2 \Delta \phi^{(0)}||_{H^k} + ||\nu \sqrt{-\Delta} \partial_t \phi^{(0)}||_{H^k} + ||G(\phi^{(0)} + w)(t) - G(\phi^{(0)})(t)||_{H^k} \right). \tag{30}$$

In addition, we note that we can get estimates on spatial derivatives as follows:

$$||w(t)||_{H^k} \leq \int_0^t ||w_t(t')||_{H^k} dt' \leq C \int_0^t E_{\nu,k}^{1/2}(w(t'))dt' \leq Cte(t), \tag{31}$$

where

$$e(t) := \sup_{0 \leq t' \leq t} E_{\nu,k}^{1/2}(w(t')).$$

Note that $\frac{d}{dt}e(t) \leq \max\left(\frac{d}{dt}\left(E_{\nu,k}^{1/2}(w(t))\right), 0\right)$, and so, since the right hand side of (30) is nonnegative, we have

$$\frac{d}{dt}e(t) \leq C\left(||\nu^2 \Delta \phi^{(0)}||_{H^k} + ||\nu\sqrt{-\Delta}\partial_t \phi^{(0)}||_{H^k} + ||G(\phi^{(0)} + w)(t) - G(\phi^{(0)})(t)||_{H^k}\right). \tag{32}$$

Recall the form of

$$\phi^{(0)}(t,x) = \phi_0(x) V\left(t\left(\phi_0(x)\right)^{\frac{p-1}{2}}\right),$$

and note that $|\phi_0(x)|^{\frac{p-1}{2}}$ is smooth since $p > 1$ is odd, so $\frac{p-1}{2}$ is a positive integer. We can deduce (recalling that ϕ_0 is fixed) that

$$||\nu^2 \Delta \phi^{(0)}||_{H^k} \leq C\nu^2(1+|t|)^{k+2} \leq C\nu^2(1+|t|)^C. \tag{33}$$

Using Hölder and Sobolev inequalities as in [11] at the bottom of pg. 11, one obtains

$$||G(\phi^{(0)} + w)(t) - G(\phi^{(0)})(t)||_{H^k} \leq C(1+|t|)^C ||w(t)||_{H^k}(1 + ||w(t)||_{H^k})^{p-1}. \tag{34}$$

Estimate (31) now implies

$$||G(\phi^{(0)} + w)(t) - G(\phi^{(0)})(t)||_{H^k} \leq C(1+|t|)^C (e(t) + e(t)^p). \tag{35}$$

Finally, we have to estimate $||\nu\sqrt{-\Delta}\partial_t \phi^{(0)}||_{H^k}$. Note that

$$\begin{aligned} ||\nu\sqrt{-\Delta}\partial_t \phi^{(0)}||_{H^k} &= \nu \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (1+|\xi|^2)^k |\xi|^2 |\partial_t \phi^{(0)}(t,\xi)|^2 d\xi \\ &\leq \nu \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (1+|\xi|^2)^{k+1} |\partial_t \phi^{(0)}(t,\xi)|^2 d\xi \\ &\leq \nu ||\partial_t \phi^{(0)}||_{H^{k+1}} \leq C\nu(1+|t|)^{k+1} \leq C\nu(1+|t|)^C, \end{aligned} \tag{36}$$

and so by using (32), (33), (35), and (36), we get

$$\frac{d}{dt}e(t) \leq C(1+|t|)^C(\nu^2 + \nu + e(t) + e(t)^p).$$

Taking c (and hence ν, since $0 \leq \nu < c$) sufficiently small, and using an a priori estimate $e(t) \leq 1$ (which we will recover later by bootstrap, and which is valid by continuity for small t since $e(t) = 0$), we get

$$\frac{d}{dt}e(t) \leq C(1+|t|)^C(\nu + e(t)).$$

Gronwall's inequality gives that

$$e(t) \leq C\nu \cdot \exp(C(1+|t|)^C).$$

Now we choose c sufficiently small such that for all $0 \leq t < c|\log\nu|^c$ and $0 < \nu \leq c$, we have

$$C\nu \cdot \exp(C(1+|t|)^C) \leq C'\nu^{1-\delta},$$

$$C\nu \cdot \exp(C(1+|t|)^C) \leq 1/2.$$

To see that this is possible, note that we can choose c sufficiently small so that $cC << 1$, $c << 1$, and $Cc^C << \frac{\delta}{2}$. Then, for all $0 \leq t < c|\log\nu|^c$ and $0 < \nu \leq c$,

$$\begin{aligned} C\nu \cdot \exp(C(1+|t|)^C) &\leq C\nu \cdot \exp(C + C|t|^C) \leq Ce^C\nu \cdot \exp(Cc^C|\log\nu|^{cC}) \\ &\leq Ce^C\nu \cdot \exp(Cc^C|\log\nu|) = Ce^C\nu \cdot \exp(-Cc^C\log\nu) = Ce^C\nu \cdot \nu^{(-Cc^C)} \\ &\leq Ce^C\nu^{1-\frac{\delta}{2}} \leq Ce^C\nu^{1-\delta} = C'\nu^{1-\delta}. \end{aligned}$$

Note that C' is independent of the sufficiently small c we choose. Then, by making c even smaller if necessary, we can also get $C'\nu^{1-\delta} << 1/2$ for all $0 < \nu \leq c$ to get the second inequality above. Then a bootstrap continuity argument (based on the second inequality above, which implies $e(t) \leq 1/2$) can be used to justify the a priori assumption that $e(t) \leq 1$ for the t we are considering in $0 \leq t < c|\log\nu|^c$, $0 < \nu \leq c$. Combining the definition of $e(t)$ and estimate (31) gives the desired result. (For estimate (31), we note that with the choices above, $te(t) \leq C''\nu^{1-\delta}$ still for all $0 \leq t < c|\log\nu|^c$, since $|\log\nu|^c \leq |\log\nu| \leq C_r\nu^{-r}$ for any $r > 0$ and since we showed earlier that with our choice of C, we actually had $e(t) \leq C'\nu^{1-\frac{\delta}{2}}$. In particular, we can set $r = \delta/2$.) □

Remark 2.1. Christ, Colliander, and Tao showed an analogous result for the nonlinear wave equation (18), but the resulting exponent in the analogue of the lemma above is $\nu^{2-\delta}$. This shows that the dissipative effect of the $\nu\sqrt{-\Delta}\partial_t u$ term makes the solution of the perturbed viscous nonlinear wave equation (25) "less close" than the solution of the analogously perturbed initial value problem for the nonlinear wave equation (18) (which is just (25) without the $\nu\sqrt{-\Delta}\partial_t u$ term) to the dispersive limit solution. Thus, there is the potential for the solution of the perturbed initial value problem (25) to have H^s norm growing less fast than for the corresponding perturbed initial value problem for the nonlinear wave equation.

The important feature of the above lemma is that for $\delta > 0$ sufficiently small, $\nu^{1-\delta}$ still goes to 0 as ν goes to 0. Therefore, the proofs of ill-posedness in Christ, Colliander, and Tao [11] still apply to this equation. In particular, we have the following result, which holds for general n, although $n = 2$ will correspond to the specific case for our given fluid-structure interaction model.

Theorem 2.1. Let $p > 1$ be a positive odd integer. If $0 < s < s_{cr}$, where s_{cr} is given by (21), then for every $\epsilon > 0$, there exists a solution u of the viscous nonlinear wave equation (19) and a positive time t such that

$$||u(0)||_{H^s} < \epsilon, \quad u_t(0) = 0, \quad 0 < t < \epsilon, \quad ||u(t)||_{H^s} > \epsilon^{-1},$$

for some $u(0) \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class. Thus, the solution map for the equation (19) is not continuous at $(u(0), u_t(0)) = (0,0) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ for $0 < s < s_{cr}$.

For completeness, we present the main ideas of the proof here, and refer the reader to [11] for more details. The main idea is to utilize the various symmetries of the vNLW equation (19) to create a family of solutions depending on ν, that "translates" H^s growth of the solution to the visco-dispersive limit equation ($\nu = 0$) into the unbounded growth of the H^s norm of solutions to the vNLW equation (19) at progressively smaller times $t \to 0$.

In particular, we have two symmetries:

- *The visco-dispersive scaling symmetry*: If $u(t,x)$ solves $\partial_{tt}u - \nu^2\Delta u + \nu\sqrt{-\Delta}\partial_t u + u^p = 0$, then $u(t,\nu x)$ solves the original vNLW equation (19); and
- *The vNLW scaling symmetry*: If $u(t,x)$ solves the original vNLW equation (19), then so does $\lambda^{\frac{2}{p-1}}u(\lambda t, \lambda x)$.

Using these symmetries, one can construct a family of solutions of the vNLW equation as follows. Fix an arbitrary compactly supported function $\phi_0 \in C_0^\infty(\mathbb{R}^n)$, and let $\phi^{(\nu)}(t,x)$, $\nu > 0$, denote the solution to the initial value problem:

$$\partial_{tt}u - \nu^2\Delta u + \nu\sqrt{-\Delta}\partial_t u + u^p = 0,$$

$$u(0,x) = \phi_0(x), \qquad \partial_t u(0,x) = 0,$$

with $(\phi_0, 0)$ as the initial data. Using the symmetries above, and the solution $\phi^{(\nu)}(t,x)$, we can construct an entire family of solutions

$$u^{(\nu,\lambda)}(t,x) = \lambda^{-\frac{2}{p-1}}\phi^{(\nu)}(\lambda^{-1}t, \nu\lambda^{-1}x) \tag{37}$$

to the original vNLW equation (19) corresponding to the following family of initial data for displacement:

$$u^{(\nu,\lambda)}(0,x) = \lambda^{-\frac{2}{p-1}}\phi_0(\nu\lambda^{-1}x).$$

One can show, following the bounds in [11], that for $0 < s < s_{cr} := \frac{n}{2} - \frac{2}{p-1}$, the H^s norm of the initial displacement $||u^{(\nu,\lambda)}(0,\cdot)||_{H^s(\mathbb{R}^n)}$ is bounded by

$$||u^{(\nu,\lambda)}(0,\cdot)||_{H^s(\mathbb{R}^n)} \leq C\lambda^{s_{cr}-s}\nu^{s-\frac{n}{2}},$$

whenever $0 < \lambda \leq \nu$, for a constant C independent of $0 < \lambda \leq \nu$. So given $\epsilon > 0$, we can choose λ and ν so that

$$\epsilon = C\lambda^{s_{cr}-s}\nu^{s-\frac{n}{2}} = C\frac{\lambda^{s_{cr}-s}}{\nu^{\frac{n}{2}-s}}, \quad \text{where } s_{cr} - s > 0 \text{ and } \frac{n}{2} - s > 0, \text{ since } 0 < s < s_{cr} < n/2. \tag{38}$$

This gives a whole family of possibilities for λ and ν, where $0 < \lambda \leq \nu$ for all $\nu > 0$ sufficiently small, since $\frac{n/2-s}{s_{cr}-s} > 1$. In particular, for any choice of λ, ν with ν sufficiently small satisfying (38), we will get that the norm of the initial displacement corresponding to the solution $u^{(\nu,\lambda)}$ in H^s is indeed less than ϵ.

To show that the H^s norm of the solution $u^{(\nu,\lambda)}(t,x)$ is greater than ϵ^{-1} for some $0 < t < \epsilon$, recall that we had an explicit form for the solution of the visco-dispersive limit initial value problem:

$$u_{tt} + u^p = 0,$$

$$u(0,x) = \phi_0(x), \qquad \partial_t u(0,x) = 0,$$

which was given by $\phi^{(0)}(t,x) = \phi_0(x)V\left(t(\phi_0(x))^{\frac{p-1}{2}})\right)$ for some smooth, even, periodic function V. Analyzing this $\phi^{(0)}$, one can see that $||\phi^{(0)}(t,\cdot)||_{H^k(\mathbb{R}^n)} \sim t^k$ for all nonnegative integers $k \geq 0$ and sufficiently large t, and hence the result also holds for all H^s for $s \geq 0$ not necessarily an integer, by interpolation. By Proposition 2.1, we see that for all $\nu > 0$ sufficiently small, $||\phi^{(\nu)}(t,\cdot)||_{H^s(\mathbb{R}^n)}$ has the same behavior:

$$||\phi^{(\nu)}(t,\cdot)||_{H^s(\mathbb{R}^n)} \sim t^s, \tag{39}$$

for $s \geq 0$ and for times $1 << t \leq c|\log\nu|^c$ (which can become increasingly large as $\nu \to 0$). Thus, the function $\phi^{(\nu)}$ transfers its energy to increasingly higher frequencies as time progresses.

We now want to show that this translates into increasingly high H^s norm of $u^{(\nu,\lambda)}(t,x)$, for some time between 0 and ϵ. Indeed, by recalling that $\phi^{(\nu)}$ appears in the definition of $u^{(\nu,\lambda)}(t,x)$ in the following way, see (2):

$$u^{(\nu,\lambda)}(t,x) = \lambda^{-\frac{2}{p-1}}\phi^{(\nu)}(\lambda^{-1}t, \nu\lambda^{-1}x),$$

and by using Fourier transform to examine the H^s norm of $u^{(\nu,\lambda)}$ for all ν and λ satisfying (38), one obtains, following the steps in the proof of Theorem 2, pg. 17, Sec. 4 of [11], the following estimate

$$\|u^{(\nu,\lambda)}(\lambda t)\|_{H^s} \geq c'\epsilon t^s,$$

which holds for a constant $c' > 0$ independent of λ and ν. By choosing t large enough (depending on ϵ) and λ small enough (depending on t and ϵ), one gets the desired estimate. Full details can be found in [11].

While the proof in [11] uses times that can be both negative and positive (since the wave equation is reversible in time), the argument from [11] carries over to the vNLW equation case, where $t > 0$.

In conclusion, we have shown by Theorem 2.1, that the viscous nonlinear wave equation is ill-posed for $0 < s < s_{cr} = n/2 - 2/(p-1)$. In the second half of the manuscript we will show that this ill-posedness associated with the lack of continuity in the solution map, is in some sense a non-generic phenomenon, and that using probabilistic arguments we can still get "probabilistic" well-posedness even below the critical exponent.

Crucial for this argument will be Strichartz estimates, which we address next.

3 Strichartz estimates for the linear viscous wave equation

In this section we show that the linear viscous wave operator has strong decay properties that admit a large collection of Strichartz estimates. Before beginning this analysis, following the abstract Strichartz estimates for the linear wave equation and the Schrödinger equation of Keel and Tao [20], we first consider the Fourier representation of the solution to the initial value problem for the linear viscous wave equation. We emphasize that Strichartz estimates are estimates for the *linear* equation, which we will later use to prove probabilistic well-posedness results for the nonlinear problem studied in Sec. 5.

3.1 Fourier representation of solution to the homogeneous and the inhomogeneous linear viscous wave equation

The homogeneous linear viscous wave equation. Consider the Cauchy problem for the linear homogeneous viscous wave equation with initial conditions, $f, g \in \mathcal{S}(\mathbb{R}^n)$:

$$\partial_{tt}u - \Delta u + \sqrt{-\Delta}\partial_t u = 0,$$

$$u(0,x) = f(x), \qquad \partial_t u(0,x) = g(x).$$

Taking spatial Fourier transforms (assuming that u decays rapidly at infinity), we get

$$\partial_{tt}\widehat{u}(t,\xi) + |\xi|\partial_t\widehat{u}(t,\xi) + |\xi|^2\widehat{u}(t,\xi) = 0,$$

$$\widehat{u}(0,\xi)=\widehat{f}(\xi),\qquad \partial_t\widehat{u}(0,\xi)=\widehat{g}(\xi).$$

One can easily see that the solution is given by:

$$\widehat{u}(t,\xi)=\widehat{f}(\xi)e^{-\frac{|\xi|}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right)+\widehat{g}(\xi)e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}. \tag{40}$$

This formula makes sense when $f,g\in\mathcal{S}(\mathbb{R}^n)$, but it can also be extended to the case when $f,g\in L^2(\mathbb{R}^n)$ by density arguments.

An important thing to notice is that, in contrast with the wave equation, solution (40) has a damping term of the form $e^{-\frac{|\xi|}{2}t}$ associated with the viscous effects.

The inhomogeneous linear viscous wave equation. Consider the Cauchy problem for the inhomogeneous linear viscous wave equation:

$$\begin{gathered}(\partial_{tt}-\Delta+\sqrt{-\Delta}\partial_t)u(t,x)=F(t,x),\\ u(0,x)=f(x)\qquad \partial_t u(0,x)=g(x)\end{gathered} \tag{41}$$

To get the representation formula for the inhomogeneous viscous wave equation, we use Duhamel's principle. We follow the notation in Sogge [36]: for $\tau>0$, let $v(\tau;t,x)$ be the solution to the Cauchy problem:

$$\begin{gathered}(\partial_{tt}-\Delta+\sqrt{-\Delta}\partial_t)u=0,\\ u(\tau;0,x)=0,\qquad \partial_t u(\tau;0,x)=F(\tau,x).\end{gathered}$$

One can easily show that the solution to the inhomogeneous problem with source term $F(t,x)$ and zero initial data is then given by

$$u(t,x)=\int_0^t u(\tau;t-\tau,x)d\tau.$$

Therefore, for the inhomogeneous Cauchy problem (41), we get the final formula for the Fourier representation of its solution:

$$\begin{aligned}\widehat{u}(t,\xi)\quad &=\widehat{f}(\xi)e^{-\frac{|\xi|}{2}t}\left(\cos\left(\tfrac{\sqrt{3}}{2}|\xi|t\right)+\tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}|\xi|t\right)\right)+\widehat{g}(\xi)e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\\ &\quad+\int_0^t\widehat{F}(\tau,\xi)e^{-\frac{|\xi|}{2}(t-\tau)}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|}d\tau.\end{aligned} \tag{42}$$

We will write this formula alternatively as:

$$\begin{aligned}u(t,\cdot)\quad &=e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\tfrac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)+\tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)f+e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}g\\ &\quad+\int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}F(\tau,\cdot)d\tau.\end{aligned} \tag{43}$$

3.2 Statement of Strichartz estimates

Next, we present Strichartz estimates, which will be useful when studying the nonlinear viscous wave equation using fixed point arguments. The estimates provide information about how the $L_t^q L_x^r$ norm of the solution to the linear problem is controlled in terms of data, for both the homogeneous case, and the inhomogeneous case. While the homogeneous estimates follow using techniques similar to those used in the case of the linear wave equation and the linear Schorödinger equation, the inhomogeneous estimates will require different approaches because the associated evolution operator is now self-adjoint due to the viscous contribution, as we explain below. In both cases, the estimates will be given in terms of the $L_t^q L_x^r$ norms of the solution, obtained via an evolution operator $U(t)$ associated with each problem separately. Crucial for the proof is the following important abstract result about Strichartz estimates due to Keel and Tao [20], which uses the following definition of σ-admissible exponents (q, r) (see Sogge [36] or Keel and Tao [20]):

Definition 3.1. Let $\sigma > 0$. The exponent pair (q, r) is said to be σ-admissible if $q, r \geq 2$, $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{2}{q} + \frac{2\sigma}{r} \leq \sigma.$$

Theorem 3.1 (General Strichartz estimates, Keel and Tao [20])**.** Let $U(t), t \in \mathbb{R}$ be a one-parameter family of operators

$$U(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

such that the following two estimates hold:

1. The energy estimate holding uniformly in t:

$$||U(t)f||_{L_x^2} \leq C||f||_{L_x^2}, \qquad f \in L^2(\mathbb{R}^n), \tag{44}$$

2. The truncated dispersive decay estimate holding for some $\sigma > 0$, uniformly in τ and t:

$$||U(\tau)U^*(t)f||_{L_x^\infty} \leq C(1 + |t - \tau|)^{-\sigma}||f||_{L_x^1}, \tag{45}$$

   where $U^*(t)$ is the adjoint operator.

Then, for all σ-admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$, the following estimates hold:

$$||U(t)f||_{L_t^q L_x^r} \leq C||f||_{L_x^2}, \tag{46}$$

$$\left|\left|\int_{-\infty}^{\infty} U^*(\tilde{\tau})F(\tilde{\tau}, \cdot)d\tilde{\tau}\right|\right|_{L_x^2} \leq C||F||_{L_t^{q'} L_x^{r'}}, \tag{47}$$

$$\left|\left|\int_{-\infty}^{t} U(t)U^*(\tilde{\tau})F(\tilde{\tau}, \cdot)d\tilde{\tau}\right|\right|_{L_t^q L_x^r} \leq C||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{48}$$

where $\tilde{q}'$ and $\tilde{r}'$ are Hölder conjugates of $\tilde{q}$ and $\tilde{r}$, respectively.

We will use estimate (46) of this theorem to get appropriate estimates on the solution of the homogeneous linear viscous wave equation, defined via certain evolutions operators $U(t)$ that we define below, applied to the given data. Estimates (47) and (48) are usually used to estimate the Duhamel contribution of the inhomogeneous term for the linear wave equation and the Schrödinger equation. In our case, however, since the contribution from the viscous regularization is self-adjoint, we will have to resort to different approaches to estimate the Duhamel contribution in the inhomogeneous case, as we explain below.

Our main results on Strichartz estimates are the following.

Theorem 3.2 (**Strichartz estimates for homogeneous linear viscous wave equation).** Let u be a solution to the Cauchy problem

$$(\partial_{tt} - \Delta + \sqrt{-\Delta}\partial_t)u = 0, \tag{49}$$

$$u(0,\cdot) = f, \qquad \partial_t u(0,\cdot) = g.$$

Then, for any $\sigma > 0$ and any σ-admissible pair (q,r), $r < \infty$, there exists a constant $C_\sigma > 0$ depending only on σ, such that for every time $0 < T < \infty$, the following estimate holds:

$$||u||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} + ||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \leq C_\sigma ||f||_{\dot{H}^s(\mathbb{R}^n)} + C_\sigma ||g||_{\dot{H}^{s-1}(\mathbb{R}^n)}, \tag{50}$$

provided that the gap condition

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s \tag{51}$$

holds.

Theorem 3.3 (**Strichartz estimates for inhomogeneous linear viscous wave equation).** Let $n \geq 2$ and let (q,r) and $(\tilde{q},\tilde{r})$ be any two pairs with $\tilde{q},\tilde{r} \geq 2$, $1 < \tilde{q}' < q < \infty$, $1 < \tilde{r}' < r \leq \infty$, where $\tilde{q}'$ and $\tilde{r}'$ are Hölder conjugates of $\tilde{q}$ and $\tilde{r}$, respectively. Let u be a solution to the Cauchy problem

$$(\partial_{tt} - \Delta + \sqrt{-\Delta}\partial_t)u = F,$$

$$u(0,\cdot) = 0, \qquad \partial_t u(0,\cdot) = 0.$$

Then, there exists a constant $C_{q,\tilde{q},r,\tilde{r}} > 0$ depending on $q,\tilde{q},r,\tilde{r}$, such that for every time $0 < T < \infty$, the following estimate holds:

$$||u||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} + ||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \leq C_{q,\tilde{q},r,\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}, \tag{52}$$

provided that the gap condition

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 \tag{53}$$

holds.

Remark 3.1 (**Dimension $n = 1$).** Note that the homogeneous Strichartz estimates in Theorem 3.2 hold for any dimension n, including $n = 1$. This is interesting because it is well-known that the linear wave equation in one dimension does not possess such Strichartz estimates. However, the viscous wave equation in one dimension has homogeneous Strichartz estimates due to the dissipating effects of the viscosity.

Remark 3.2 (**The gap condition).** The gap condition is a natural condition to impose in both cases, as it is the exact condition needed for the inequalities above to respect the scaling symmetry of solutions in time and space. This makes the inequality scale invariant, and this property will later play an important role, especially in the proof of Theorems 3.2 and 3.3.

Remark 3.3 (**Admissible exponents** (q,r)**).** The Strichartz estimates for the viscous wave equation are better than those of the wave equation in the sense that both the homogeneous and inhomogeneous estimates hold for a larger class of admissible exponents. Again, this is due to the dissipative effects of viscosity.

Classical Strichartz estimates for the linear Schrödinger equation and the linear wave equation usually follow immediately from the abstract Strichartz estimates by Keel and Tao [20] in Theorem 3.1 above. In the case of the viscous wave equation, this will indeed be true for the homogeneous estimates, but it will not be true for the inhomogeneous estimates, because the dissipative portion of the evolution operator is self-adjoint. This makes the proof of the inhomogeneous Strichartz estimates considerably more subtle, and the proof will employ techniques from harmonic analysis that are markedly different from the techniques used to prove the homogeneous estimates.

In the proof of Theorems 3.2 and 3.3, we will use the following well-known Littlewood-Paley theorem, which will help us reduce the problem to proving the estimates for the components in the Littlewood-Paley decomposition.

Lemma 3.1 (Littlewood-Paley lemma)**.** Let $\beta \in C_0^\infty(\mathbb{R}_+)$, $0 \leq \beta \leq 1$, with support in $[1/2, 2]$ give the Littlewood-Paley decomposition

$$\sum_{j=-\infty}^{\infty} \beta\left(\frac{\xi}{2^j}\right) = 1 \text{ for all } \xi > 0.$$

Define the Littlewood-Paley operators:

$$G_j(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \beta\left(\frac{|\xi|}{2^j}\right) \widehat{G}(t,\xi) d\xi,$$

which send G to its Littlewood-Paley decomposition $\{G_j\}_{j=-\infty}^{\infty}$. Then, the following estimates hold:

- If $2 \leq r < \infty$ and $2 \leq q \leq \infty$,

$$||G||^2_{L_t^q L_x^r} \leq C \sum_{j=-\infty}^{\infty} ||G_j||^2_{L_t^q L_x^r}. \tag{54}$$

- If $1 < r \leq 2$ and $1 \leq q \leq 2$,

$$\sum_{j=-\infty}^{\infty} ||G_j||^2_{L_t^q L_x^r} \leq C||G||^2_{L_t^q L_x^r}. \tag{55}$$

To prove Theorems 3.2 and 3.3 we introduce the Littlewood-Paley operators $U^{(j)}(t)$ and $V^{(j)}(t)$ that account for the contribution of the initial data f and g, separately, to the solution of (49). We recall (40):

$$\widehat{u}(t,\xi) = e^{-\frac{|\xi|}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right)\widehat{f}(\xi) + e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\widehat{g}(\xi),$$

and introduce:

$$U^{(j)}(t)f(x) = \chi_{[0,T]}(t)\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-\frac{|\xi|}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right)\widehat{f}(\xi)\beta\left(\frac{|\xi|}{2^j}\right)d\xi,$$

and

$$V^{(j)}(t)g(x) = \chi_{[0,T]}(t)\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\widehat{g}(\xi)\beta\left(\frac{|\xi|}{2^j}\right)d\xi.$$

As we shall see later, only the operators $U^{(j)}(t)$ and $V^{(j)}(t)$ with $-2 \le j \le 2$ will be relevant for the proof. Operators $U^{(j)}(t)$ and $V^{(j)}(t)$, $-2 \le j \le 2$, satisfy the following estimates, which follow from Keel and Tao [20]:

Lemma 3.2 (Estimates on $U^{(j)}(t)f(x)$ and $V^{(j)}(t)g(x)$)**.** Given $\sigma > 0$, there exists a constant C_σ independent of $T > 0$ such that for all σ-admissible pairs (q, r), $(\tilde{q}, \tilde{r})$ and for all integers $-2 \le j \le 2$,

$$||U_\pm^{(j)}(t)f||_{L_t^q L_x^r} \le C_\sigma ||f||_{L^2},$$

$$\left|\left|\int_{-\infty}^{\infty} (U_\pm^{(j)})^*(\tau)F(\tau,\cdot)d\tau\right|\right|_{L_x^2} \le C_\sigma ||F||_{L_t^{q'} L_x^{r'}},$$

$$\left|\left|\int_{-\infty}^{t} U_\pm^{(j)}(t)(U_\pm^{(j)})^*(\tau)F(\tau,\cdot)d\tau\right|\right|_{L_t^q L_x^r} \le C_\sigma ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

$$||V^{(j)}(t)g||_{L_t^q L_x^r} \le C_\sigma ||g||_{L^2},$$

$$\left|\left|\int_{-\infty}^{\infty} (V^{(j)})^*(\tau)F(\tau,\cdot)d\tau\right|\right|_{L_x^2} \le C_\sigma ||F||_{L_t^{q'} L_x^{r'}},$$

$$\left|\left|\int_{-\infty}^{t} V^{(j)}(t)(V^{(j)})^*(\tau)F(\tau,\cdot)d\tau\right|\right|_{L_t^q L_x^r} \le C_\sigma ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Proof. We verify the necessary conditions in Theorem 3.1 to get the desired estimates. We start with inequality (44). It is clear from Plancherel's theorem that

$$||U^{(j)}(t)f||_{L_x^2} \le C||f||_{L^2},$$

where C is independent of t and j, since $\left|e^{-\frac{|\xi|}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right)\beta\left(\frac{|\xi|}{2^j}\right)\right| \le 2$.

Similarly, by Plancherel's formula, we can deduce that

$$||V^{(j)}(t)g||_{L_x^2} \le C||g||_{L^2}$$

for some C that is independent of t and j. To see this, we simply note that there exists a constant C such that

$$\left|e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\beta\left(\frac{|\xi|}{2^j}\right)\right| \le C \tag{56}$$

for all $t > 0$ and for all $\xi \in \mathbb{R}^n$. This is due to the support properties of $\beta\left(\frac{|\xi|}{2^j}\right)$ and the fact that $-2 \le j \le 2$, so that the quantity on the left hand side of (56) is potentially nonzero only for $1/8 \le |\xi| \le 8$.

To verify (45), we fix an arbitrary $\sigma > 0$ and verify the estimate

$$||U_\pm^{(j)}(t)(U_\pm^{(j)})^*(\tau)f||_{L_x^\infty} \le C_\sigma(1 + |t-\tau|)^{-\sigma}||f||_{L_x^1}$$

for some constant C_σ. It suffices to prove this inequality for positive *integers* σ. We calculate

$$U^{(j)}(t)(U^{(j)})^*(\tau)f(x) = \chi_{[0,T]}(t)\chi_{[0,T]}(\tau)\int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}(t+\tau)}e^{ix\cdot\xi}a(\xi,t)a(\xi,\tau)\widehat{f}(\xi)\beta^2\left(\frac{|\xi|}{2^j}\right)d\xi,$$

where

$$a(\xi,t) := \cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right).$$

Assume t and τ are such that $0 \le \tau, t \le T$ where $T > 0$ is arbitrary. Fix an integer $k > n/2$. Then,

$$||U^{(j)}(t)(U^{(j)})^*(\tau)f||^2_{H^k_x} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^k\left|e^{-\frac{|\xi|}{2}(t+\tau)}a(\xi,t)a(\xi,\tau)\widehat{f}(\xi)\beta^2\left(\frac{|\xi|}{2^j}\right)\right|^2 d\xi.$$

Note that for $-2 \le j \le 2$, $\beta(|\xi|/2^j)$ is supported in $1/8 \le |\xi| \le 8$ and $0 \le \beta(\xi) \le 1$. Therefore, since $|a(\xi,t)| \le 2$,

$$\begin{aligned}||U^{(j)}(t)(U^{(j)})^*(\tau)f||^2_{H^k_x} &\le \frac{1}{(2\pi)^n}\int_{1/8\le|\xi|\le 8} 4\cdot 65^k e^{-\frac{1}{8}(t+\tau)}|\widehat{f}(\xi)|^2 d\xi \\ &= C_k e^{-\frac{1}{8}(t+\tau)}\int_{1/8\le|\xi|\le 8}|\widehat{f}(\xi)|^2 d\xi \le C'_k e^{-\frac{1}{8}(t+\tau)}||\widehat{f}||^2_{L^\infty_\xi} \le C'_k e^{-\frac{1}{8}(t+\tau)}||f||^2_{L^1_x},\end{aligned}$$

implying

$$||U^{(j)}(t)(U^{(j)})^*(\tau)f||_{H^k_x} \le C_k e^{-\frac{1}{16}(t+\tau)}||f||_{L^1_x} \le C_k e^{-\frac{1}{16}|t-\tau|}||f||_{L^1_x},$$

for $0 \le \tau, t \le T$, since $|t-\tau| \le t+\tau$. Because $e^{-\frac{1}{16}|t-\tau|}$ decays exponentially, it decays faster than $(1+|t-\tau|)^{-\sigma}$ for any positive integer σ. In particular,

$$||U^{(j)}_\pm(t)(U^{(j)}_\pm)^*(\tau)f||_{H^k_x} \le C_\sigma(1+|t-\tau|)^{-\sigma}||f||_{L^1_x},$$

for $0 \le t, \tau \le T$ and for all $-2 \le j \le 2$. For all other t, τ, the left hand side is zero by the characteristic functions $\chi_{[0,T]}$. Since $k > n/2$, $H^k(\mathbb{R}^n)$ embeds into $L^\infty(\mathbb{R}^n)$, and so we have:

$$||U^{(j)}_\pm(t)(U^{(j)}_\pm)^*(\tau)f||_{L^\infty_x} \le C||U^{(j)}_\pm(t)(U^{(j)}_\pm)^*(\tau)f||_{H^k_x} \le C_\sigma(1+|t-\tau|)^{-\sigma}||f||_{L^1_x},$$

which shows that assumption (45) holds for the operator $U^{(j)}$, $-2 \le j \le 2$.

To show that the operator $V^{(j)}$ with $-2 \le j \le 2$ satisfies assumption (45), we proceed in a similar way. We calculate

$$V^{(j)}(t)(V^{(j)})^*(\tau)g(x) = \chi_{[0,T]}(t)\chi_{[0,T]}(\tau)\int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}(t+\tau)}e^{ix\cdot\xi}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\cdot\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\tau\right)}{\frac{\sqrt{3}}{2}|\xi|}\widehat{g}(\xi)\beta^2\left(\frac{|\xi|}{2^j}\right)d\xi.$$

Again, we only need to consider $0 \le \tau, t \le T$ where $T > 0$ is arbitrary. Fix an integer $k > n/2$, and note that

$$\begin{aligned}||V^{(j)}(t)(V^{(j)})^*(\tau)g||^2_{H^k_x} &\le \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^k\left|e^{-\frac{|\xi|}{2}(t+\tau)}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\cdot\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\tau\right)}{\frac{\sqrt{3}}{2}|\xi|}\widehat{g}(\xi)\beta^2\left(\frac{|\xi|}{2^j}\right)\right|^2 d\xi \\ &\le C\int_{1/8\le|\xi|\le 8}(1+|\xi|^2)^k\left|e^{-\frac{|\xi|}{2}(t+\tau)}\widehat{g}(\xi)\beta^2\left(\frac{|\xi|}{2^j}\right)\right|^2 d\xi,\end{aligned}$$

where we used

$$\left|\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\tau\right)}{\frac{\sqrt{3}}{2}|\xi|}\beta\left(\frac{|\xi|}{2^j}\right)\right| \le C$$

uniformly on the support of $\beta\left(\frac{|\xi|}{2^j}\right)$ for $-2 \leq j \leq 2$. Since $-2 \leq j \leq 2$, we get

$$||V^{(j)}(t)(V^{(j)})^*(\tau)g||^2_{H^k_x} \leq C \int_{1/8 \leq |\xi| \leq 8} (1+|\xi|^2)^k e^{-|\xi|(t+\tau)} |\widehat{g}(\xi)|^2 d\xi$$
$$\leq C \int_{1/8 \leq |\xi| \leq 8} 65^k e^{-\frac{1}{8}(t+\tau)} |\widehat{g}(\xi)|^2 d\xi \leq C_k e^{-\frac{1}{8}(t+\tau)} ||\widehat{g}||^2_{L^\infty_\xi} \leq C_k e^{-\frac{1}{8}(t+\tau)} ||g||^2_{L^1_x}.$$

Therefore, recalling that $k > n/2$, for all $-2 \leq j \leq 2$ we have

$$||V^{(j)}(t)(V^{(j)})^*(\tau)g||_{L^\infty_x} \leq C||V^{(j)}(t)(V^{(j)})^*(\tau)g||_{H^k_x} \leq C_k e^{-\frac{1}{16}(t+\tau)} ||g||_{L^1_x}$$
$$\leq C_k e^{-\frac{1}{16}|t-\tau|} ||g||_{L^1_x} \leq C_\sigma (1+|t-\tau|)^{-\sigma} ||g||_{L^1_x},$$

where $|t-\tau| \leq t+\tau$ since we are considering $0 \leq \tau, t \leq T$. The estimates from the statement of Lemma 3.2 now follow directly from Keel and Tao [20]. □

3.2.1 Proof of Theorem 3.2: Strichartz estimates for the homogeneous problem

To prove the Strichartz estimates for the homogeneous linear viscous wave equation stated in Theorem 3.2 we define $u_j(t,x)$, $f_j(x)$, $g_j(x)$ for $j \in \mathbb{Z}$ by

$$u_j(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \beta\left(\frac{|\xi|}{2^j}\right) \widehat{u}(t,\xi) d\xi, \tag{57}$$

$$f_j(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \beta\left(\frac{|\xi|}{2^j}\right) \widehat{f}(\xi) d\xi, \qquad g_j(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \beta\left(\frac{|\xi|}{2^j}\right) \widehat{g}(\xi) d\xi. \tag{58}$$

It is easy to see that u_j solves the corresponding linear viscous wave equation with initial data f_j, g_j. Notice that restricted on the time interval $[0,T]$, u_j can be written in terms of $U^{(j)}(t)$ and $V^{(j)}(t)$ as follows:

$$u_j(t,x) = \frac{1}{(2\pi)^n} \left(U^{(j)}(t)f(x) + V^{(j)}(t)g(x)\right) = \frac{1}{(2\pi)^n} \sum_{k=j-2}^{j+2} U^{(k)}(t)f_j(x) + V^{(k)}(t)g_j(x), \ t \in [0,T], \tag{59}$$

where the second equality follows from the fact that $\text{supp}(\beta) \subset [1/2, 2]$. Obtaining estimates for u_j will be based on using the results from Lemma 3.2, and the Littlewood-Paley decomposition estimates from Lemma 3.1. More precisely, we first show in Step 1 below that given a solution u for any initial data (f,g), it suffices to obtain the corresponding estimate for each u_j, uniform in $j \in \mathbb{Z}$, for the initial data f_j, g_j defined via (58). In Step 2, we then simplify this even further by showing that, in fact, it suffices to simply consider initial data whose spatial Fourier transforms are supported in the annulus $1/2 \leq |\xi| \leq 2$. Thus, what we show in Steps 1 and 2 below is that it suffices to obtain the estimate (50) from Theorem 3.2 for initial data f, g that have spatial Fourier transforms that are all supported in the annulus $1/2 \leq |\xi| \leq 2$. The proof will then follow from estimates presented in Lemma 3.2.

Step 1. Since β is supported in $[1/2, 2]$, we claim that it suffices to show that u_j satisfies the estimate (50) from Theorem 3.2 for the data f_j, g_j defined by (58), where the constant C in the estimate (50) is *independent* of j. Since $q, r \geq 2$, we can use (54) to get the results for general f, g.

In particular, given a general solution u to the linear viscous wave equation with initial data f, g, we can construct u_j, f_j, g_j as in (57) and (58). Suppose that for all such u, f, g, the functions u_j for $j \in \mathbb{Z}$ satisfy the estimate

$$\begin{aligned} ||u_j||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} + ||u_j(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u_j(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \\ \leq C||f_j||_{\dot{H}^s(\mathbb{R}^n)} + C||g_j||_{\dot{H}^{s-1}(\mathbb{R}^n)}, \end{aligned} \tag{60}$$

where the constant C is independent of $j \in \mathbb{Z}$.

Estimate (60) implies that for a constant C' independent of $j \in \mathbb{Z}$,

$$\begin{aligned} \max\left\{||u_j||^2_{L_t^q([0,T];L_x^r(\mathbb{R}^n))}, ||u_j(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)}, ||\partial_t u_j(T,\cdot)||^2_{\dot{H}^{s-1}(\mathbb{R}^n)}\right\} \\ \leq C'\left(||f_j||^2_{\dot{H}^s(\mathbb{R}^n)} + ||g_j||^2_{\dot{H}^{s-1}(\mathbb{R}^n)}\right). \end{aligned} \tag{61}$$

Because $q, r \geq 2$ and $r \neq \infty$, we can apply estimate (54) in Lemma 3.1 to get

$$||u||^2_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} \leq C\sum_{j=-\infty}^{\infty} ||u_j||^2_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} \leq CC'\sum_{j=-\infty}^{\infty}\left(||f_j||^2_{\dot{H}^s(\mathbb{R}^n)} + ||g_j||^2_{\dot{H}^{s-1}(\mathbb{R}^n)}\right),$$

where the last inequality follows from (61). Because $\sum_{j=-\infty}^{\infty} \beta^2\left(\frac{|\xi|}{2^j}\right) \leq 1$, we have that

$$\sum_{j=-\infty}^{\infty} ||f_j||^2_{\dot{H}^s(\mathbb{R}^n)} \leq ||f||^2_{\dot{H}^s(\mathbb{R}^n)} \text{ and } \sum_{j=-\infty}^{\infty} ||g_j||^2_{\dot{H}^{s-1}(\mathbb{R}^n)} \leq ||g||^2_{\dot{H}^{s-1}(\mathbb{R}^n)}. \tag{62}$$

Combining the last three estimates, we obtain that there exists a $C > 0$ independent of f, g, such that

$$||u||^2_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} \leq C\left(||f||^2_{\dot{H}^s(\mathbb{R}^n)} + ||g||^2_{\dot{H}^{s-1}(\mathbb{R}^n)}\right) \leq C\left(||f||_{\dot{H}^s(\mathbb{R}^n)} + ||g||_{\dot{H}^{s-1}(\mathbb{R}^n)}\right)^2. \tag{63}$$

Similarly, using the fact that for some constant $c > 0$, $\sum_{j=-\infty}^{\infty} \beta^2\left(\frac{|\xi|}{2^j}\right) \geq c$, we conclude that

$$||u(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)} \leq c^{-1}\sum_{j=-\infty}^{\infty} ||u_j(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)}, \text{ and } ||\partial_t u(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)} \leq c^{-1}\sum_{j=-\infty}^{\infty} ||\partial_t u_j(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)},$$

where in the second inequality, we used the fact that $\widehat{\partial_t u_j}(t,\xi) = \beta(|\xi|/2^j)\widehat{\partial_t u}(t,\xi)$. Then by using (61) and the inequalities in (62), we can obtain the analogous inequalities

$$||u(T,\cdot)||^2_{\dot{H}^s} \leq C\left(||f||_{\dot{H}^s} + ||g||_{\dot{H}^{s-1}}\right)^2 \text{ and } ||\partial_t u(T,\cdot)||^2_{\dot{H}^{s-1}} \leq C\left(||f||_{\dot{H}^s} + ||g||_{\dot{H}^{s-1}}\right)^2. \tag{64}$$

Taking square roots in (63), (64), and adding the resulting equations gives the result in Theorem 3.2.

In the next step, we make a further simplification as follows.

Step 2. We have shown in the previous step that it suffices to show the uniform estimate (60) for all $j \in \mathbb{Z}$ and for all initial data (f, g) with corresponding solution u. In this step, we show that because of the gap condition (51), it suffices to show (60) for just $j = 0$. In particular, showing the estimate (60) for $j = 0$ with a constant C automatically gives the same estimate for all $j \in \mathbb{Z}$ with the same constant C, by the scaling symmetries of the viscous linear wave equation.

To see this, recall that

$$\widehat{h(\lambda x)} = \lambda^{-n}\widehat{h}(\xi/\lambda).$$

So it suffices to show that an estimate for a given f, g, u also holds for the corresponding functions $f(\lambda x), \lambda g(\lambda x), u(\lambda t, \lambda x)$. To verify this, we calculate

$$||u(\lambda t, \lambda x)||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} = \lambda^{-\frac{n}{r}-\frac{1}{q}}||u(t,x)||_{L_t^q([0,\lambda T];L_x^r(\mathbb{R}^n))},$$

$$||u(\lambda T, \lambda x)||_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{-\frac{n}{2}+s}||u(\lambda T, x)||_{\dot{H}^s(\mathbb{R}^n)}, \quad ||\lambda \partial_t u(\lambda T, \lambda x)||_{\dot{H}^{s-1}(\mathbb{R}^n)} = \lambda^{-\frac{n}{2}+s}||\partial_t u(\lambda T, x)||_{\dot{H}^{s-1}(\mathbb{R}^n)},$$

$$||f(\lambda x)||_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{-\frac{n}{2}+s}||f(x)||_{\dot{H}^s(\mathbb{R}^n)}, \quad ||\lambda g(\lambda x)||_{\dot{H}^{s-1}(\mathbb{R}^n)} = \lambda^{-\frac{n}{2}+s}||g(x)||_{\dot{H}^{s-1}(\mathbb{R}^n)}.$$

From here we see that we get the desired result since, by the gap condition,

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s.$$

Conclusion. From Steps 1 and 2 we conclude that we just need to show the estimate (60) for $j = 0$, for any initial data (f, g) and corresponding solution u. Since f_0, g_0 have spatial Fourier transforms supported in $1/2 \leq |\xi| \leq 2$, estimate (60) for $j = 0$ would be established if we more generally proved Theorem 3.2 for all initial data (f, g) that have $\widehat{f}, \widehat{g}$ supported in $1/2 \leq |\xi| \leq 2$. Thus, without loss of generality, we can assume that f, g have spatial Fourier transforms that are all supported in the annulus $1/2 \leq |\xi| \leq 2$. Note that, in this case, all homogeneous Sobolev norms $\dot{H}^s(\mathbb{R}^n)$ are equivalent to the $L^2(\mathbb{R}^n)$ norm.

Step 3. The proof of Theorem 3.2 now follows by combining Steps 1 and 2, the expression (59) for u_j in terms of the operators $U^{(j)}$ and $V^{(j)}$, and the estimates from Lemma 3.2.

More precisely, recall that the homogeneous solution can be written as

$$u(t,\cdot) = e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right) f + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} g,$$

and we want to prove

$$||u||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} + ||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \leq C||f||_{\dot{H}^s(\mathbb{R}^n)} + C||g||_{\dot{H}^{s-1}(\mathbb{R}^n)}.$$

This follows from the estimates on the operators $U^{(j)}$ and $V^{(j)}$, $-2 \leq j \leq 2$, in Lemma 3.2, and by using Steps 1 and 2 above to work with f, g which have Fourier transform supported in $1/2 \leq |\xi| \leq 2$ so that:

$$\begin{aligned}\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right)\widehat{f}(\xi)d\xi \\ = \sum_{j=-2}^{2}\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right)\widehat{f}(\xi)\beta\left(\frac{|\xi|}{2^j}\right)d\xi.\end{aligned}$$

Notice that we need $-2 \leq j \leq 2$ to cover all the j's for which the support of $\beta\left(\frac{|\xi|}{2^j}\right)$ intersects $1/2 \leq |\xi| \leq 2$. For g, we use the estimates on $V^{(j)}$, $-2 \leq j \leq 2$, in Lemma 3.2, and the same sum decomposition where we sum from $j = -2$ to $j = 2$.

For the terms $||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)}$ and $||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)}$, we note that these norms are equivalent to L^2 norms since u and $\partial_t u$ have spatial Fourier transforms that are also supported in $1/2 \leq |\xi| \leq 2$, in which case the inequality follows from Plancherel's theorem and the support properties of f and g. Note again that Plancherel's theorem works here since $||f||_{\dot{H}^s(\mathbb{R}^n)}$ and $||g||_{\dot{H}^{s-1}(\mathbb{R}^n)}$ are equivalent to $||f||_{L^2(\mathbb{R}^n)}$ and $||g||_{L^2(\mathbb{R}^n)}$ respectively, given the fact that the Fourier transforms of f and g are supported in $1/2 \leq |\xi| \leq 2$.

□

3.2.2 Proof of Strichartz estimates for the inhomogeneous problem

To prove Theorem 3.3 we first recall from (43) that for the inhomogeneous viscous wave equation with zero initial data, the solution can be represented as

$$u(t,\cdot) := \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau. \tag{65}$$

The goal is to estimate this inhomogeneous contribution. In particular, we must show that

$$||u||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} + ||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \leq C_{q,\tilde{q},r,\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}. \tag{66}$$

Unfortunately, since the dissipative portion of the evolution operator involving $e^{-\frac{\sqrt{-\Delta}}{2}}$ is self-adjoint, the results from Keel and Tao's theorem cannot be used here, since $U(t)U^*(\tau)$ has $e^{-\frac{\sqrt{-\Delta}}{2}(t+\tau)}$ instead of $e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)}$. Instead, we adopt ideas from fractional heat equations, see Miao, Yuan, and Zhang [27].

We start by first proving that

$$||u||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} \leq C_{q,\tilde{q},r,\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}. \tag{67}$$

For this purpose, we introduce the following family of operators

$$S(t) := e^{-\frac{\sqrt{-\Delta}}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}, \quad t > 0.$$

The operators $S(t)$ define a family of convolution kernels $K_t(x)$ via

$$S(t)\phi(x) = (K_t * \phi)(x),$$

where

$$K_t(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|} e^{ix\cdot\xi} d\xi = \frac{t^{1-n}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} e^{ix\cdot\frac{\xi}{t}} d\xi. \tag{68}$$

Using these operators, the solution u in (65) can be written as:

$$u(t,x) = \int_0^t \left[K_{t-\tau}(x) * F(\tau,x)\right] d\tau,$$

where the convolution $*$ is with respect to x. To obtain the desired estimates on u, we investigate the properties of the convolution kernels $K_t(x)$. In particular, we begin by defining the "unit-scale" kernel by

$$K(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} e^{ix\cdot\xi} d\xi, \tag{69}$$

and notice that it has the following important scaling property:

$$K_t(x) = t^{1-n} K\left(\frac{x}{t}\right). \tag{70}$$

Lemma 3.3. There exists a constant $C > 0$ such that the convolution kernel $K(x)$ satisfies the following pointwise estimate:

$$|K(x)| \leq C(1+|x|)^{-n-\alpha}, \ x \in \mathbb{R}^n,$$

for some $\alpha \geq 1/2$. Therefore, $K \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

Proof. The proof is similar to an argument used in Miao, Yuan, and Zhang [27] for a different kernel, where we use repeated integration by parts until we get the necessary decay.

More precisely, to perform integration by parts and to obtain the desired estimates, it is useful to introduce the following operator:

$$L(x,D) = \frac{x\cdot\nabla_\xi}{i|x|^2}, \quad \text{and its adjoint} \quad L^*(x,D) = -\frac{x\cdot\nabla_\xi}{i|x|^2}.$$

Operator $L(x,D)$ has the following crucial property

$$L(x,D)e^{ix\cdot\xi} = e^{ix\cdot\xi}.$$

We now integrate by parts to obtain:

$$\begin{aligned} K(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} e^{ix\cdot\xi} d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} L(e^{ix\cdot\xi}) e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) d\xi. \end{aligned} \tag{71}$$

Integration by parts is justified because $e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|}$ and all of its ξ derivatives are rapidly decreasing at infinity, and $e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|}$ is bounded near the origin.

To estimate the resulting integral, we will do a high and a low frequency estimate. For this purpose, define a radially symmetric smooth compactly supported function ρ such that:

$$\rho(\xi) = 1 \ \text{ if } |\xi| \leq 1, \qquad \rho(\xi) = 0 \ \text{ if } |\xi| \geq 2, \quad \text{and } \rho \text{ is decreasing radially.}$$

Then, we can split the integral (71) above into two integrals,

$$\begin{aligned} K(x) = &\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \rho\left(\frac{\xi}{\delta}\right) L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) d\xi \\ &+ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(1 - \rho\left(\frac{\xi}{\delta}\right)\right) L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) d\xi := I + II, \end{aligned} \tag{72}$$

where $\delta > 0$ will be chosen later.

Estimate of integral I. We start by estimating the factor involving the operator L^*:

$$\left| L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) \right| = \left| \frac{x \cdot \nabla_\xi}{i|x|^2} \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) \right| \leq \frac{1}{|x|} \left| \nabla_\xi \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) \right|$$
$$= \frac{1}{|x|} \left| -\frac{1}{2} e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} + e^{-\frac{|\xi|}{2}} \left(\frac{\cos\left(\frac{\sqrt{3}}{2}|\xi|\right)}{|\xi|} - \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|^2} \right) \right| \cdot |\nabla_\xi |\xi|| \,.$$

Now, from the Taylor expansion of sin and cos, and the boundedness of $|\nabla_\xi |\xi||$, there exists a constant $C > 0$ such that

$$|I| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \rho\left(\frac{\xi}{\delta}\right) L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) d\xi \right| \leq C|x|^{-1} \int_{|\xi|\leq 2\delta} d\xi = C|x|^{-1}\delta^n. \tag{73}$$

Estimate of integral II. Let N be an arbitrary positive integer such that $N \geq n+1$. Using integration by parts $N-1$ times, we get:

$$|II| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(1 - \rho\left(\frac{\xi}{\delta}\right)\right) L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) d\xi \right|$$
$$\leq C \int_{\mathbb{R}^n} \left| e^{ix\cdot\xi} (L^*)^{N-1} \left(\left(1 - \rho\left(\frac{\xi}{\delta}\right)\right) L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) \right) \right| d\xi$$
$$= C \int_{|\xi|\geq\delta} \left| e^{ix\cdot\xi} (L^*)^{N-1} \left(\left(1 - \rho\left(\frac{\xi}{\delta}\right)\right) L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) \right) \right| d\xi.$$

By the triangle inequality and the support properties of ρ, we get:

$$|II| \leq C \int_{|\xi|\geq\delta} \left| (L^*)^N \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) \right| d\xi$$
$$+ C \int_{\delta\leq|\xi|\leq 2\delta} \left| (L^*)^{N-1} \left(\rho\left(\frac{\xi}{\delta}\right) L^* \left(e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} \right) \right) \right| d\xi := II_A + II_B. \tag{74}$$

We estimate II_A as follows:

$$II_A \leq C_N |x|^{-N} \int_{|\xi|\geq\delta} \sum_{l=0}^{N} \left| e^{-\frac{|\xi|}{2}} \sin\left(\frac{\sqrt{3}}{2}|\xi|\right) \right| |\xi|^{-l-1} + \sum_{l=0}^{N-1} \left| e^{-\frac{|\xi|}{2}} \cos\left(\frac{\sqrt{3}}{2}|\xi|\right) \right| |\xi|^{-l-1} d\xi$$
$$\leq C_N |x|^{-N} \int_{|\xi|\geq\delta} \sum_{l=0}^{N} e^{-\frac{|\xi|}{2}} |\xi|^{-l} d\xi, \tag{75}$$

where the cosine part has the index l going only up to $N-1$, since at least one ξ derivative must be used to turn sine into cosine.

We estimate II_B as follows:

$$\begin{aligned} II_B &\leq C_N|x|^{-N}\int_{\delta\leq|\xi|\leq 2\delta}\sum_{k=0}^{N-1}\delta^{-k}\left(\sum_{l=0}^{N-k}\left|e^{-\frac{|\xi|}{2}}\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)\right||\xi|^{-l-1}\right. \\ &\quad\left.+\sum_{l=0}^{N-k-1}\left|e^{-\frac{|\xi|}{2}}\cos\left(\frac{\sqrt{3}}{2}|\xi|\right)\right||\xi|^{-l-1}\right)d\xi \\ &\leq C_N|x|^{-N}\int_{\delta\leq|\xi|\leq 2\delta}\sum_{k=0}^{N-1}|\xi|^{-k}\left(\sum_{l=0}^{N-k}\left|e^{-\frac{|\xi|}{2}}\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)\right||\xi|^{-l-1}\right. \\ &\quad\left.+\sum_{l=0}^{N-k-1}\left|e^{-\frac{|\xi|}{2}}\cos\left(\frac{\sqrt{3}}{2}|\xi|\right)\right||\xi|^{-l-1}\right)d\xi, \end{aligned}$$

where we used the fact that $\delta^{-k}\leq 2^{N-1}|\xi|^{-k}$ for $k=0,1,...,N-1$ and for $\delta\leq|\xi|\leq 2\delta$. Continuing to estimate the above quantity, we get

$$\begin{aligned} II_B &\leq C_N|x|^{-N}\int_{\delta\leq|\xi|\leq 2\delta}\sum_{k=0}^{N-1}|\xi|^{-k}\left(\sum_{l=0}^{N-k}e^{-\frac{|\xi|}{2}}|\xi|^{-l}\right)d\xi \\ &= C_N|x|^{-N}\int_{\delta\leq|\xi|\leq 2\delta}\sum_{k=0}^{N-1}\sum_{l=0}^{N-k}e^{-\frac{|\xi|}{2}}|\xi|^{-k-l}d\xi, \end{aligned} \tag{76}$$

where k is the number of derivatives that fall on $\rho\left(\frac{\xi}{\delta}\right)$. Note that for all $j=0,1,...,N$, we have that there exists a constant C_N such that $0\leq e^{-\frac{|\xi|}{2}}|\xi|^j\leq C_N$, for all $j=0,1,...,N$ and for all $\xi\in\mathbb{R}^n$. Using this fact in (75) and (76), we get that

$$II_A\leq C_N|x|^{-N}\int_{|\xi|\geq\delta}|\xi|^{-N}d\xi \quad\text{and}\quad II_B\leq C_N|x|^{-N}\int_{\delta\leq|\xi|\leq 2\delta}|\xi|^{-N}d\xi. \tag{77}$$

By employing estimates (77) in (74), and recalling that N is an arbitrary positive integer such that $N\geq n+1$ so that the integral converges, we get

$$|II|\leq C_N|x|^{-N}\int_{|\xi|\geq\delta}|\xi|^{-N}d\xi = C_N|x|^{-N}\delta^{-N+n}. \tag{78}$$

Estimate of $K(x)$. Using the low frequency estimate (73) and high frequency estimate (78) in (72), we obtain

$$|K(x)|\leq C_N(|x|^{-1}\delta^n+|x|^{-N}\delta^{-N+n}).$$

Now we set $\delta=|x|^{-\frac{N-1}{N}}$ in the above inequality so that both terms in the sum are the same, to obtain

$$|K(x)|\leq C_N|x|^{-1-n\left(\frac{N-1}{N}\right)}. \tag{79}$$

Since $N\geq n+1$ was arbitrary, choose a positive integer N sufficiently large such that $\alpha:=-1-n\left(\frac{N-1}{N}\right)\leq -n-\frac{1}{2}$. Note that $K(x)$ is bounded uniformly in x, since $e^{-\frac{|\xi|}{2}}$ is integrable in $\mathbb{R}^n$. Then, we conclude that there exists a constant $C>0$ such that

$$|K(x)|\leq C(1+|x|)^{-n-\alpha},\ x\in\mathbb{R}^n,$$

for some $\alpha\geq 1/2$. Therefore, $K(x)\in L^p(\mathbb{R}^n)$ for all $1\leq p\leq\infty$. □

In fact, there is an explicit formula for $K(x)$, which will allow us to explicitly show a sharp decay rate for $K(x)$ in dimensions $n = 1, 2, 3$. To state this formula, we must first recall the fundamental solution for the wave equation, given by the following inverse Fourier transform,

$$K_t^W(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin(t|\xi|)}{|\xi|} e^{ix\cdot\xi} d\xi, \tag{80}$$

where superscript "W" stands for the wave equation. The explicit form of the fundamental solution for the wave equation is well-known for all dimensions, but is particularly easy to state in dimensions $n = 1, 2, 3$. We give the well-known formulas below, which can be deduced from the discussion on pg. 4-5 of Sogge [37].

$$\begin{aligned} &\text{For } n = 1, \qquad K_t^W(x) = \frac{1}{2} 1_{|x|\leq t}. \\ &\text{For } n = 2, \qquad K_t^W(x) = \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{|x|<t}. \\ &\text{For } n = 3, \qquad K_t^W(x) = \frac{1}{4\pi t} \sigma_t. \end{aligned} \tag{81}$$

Here, σ_t denotes Lebesgue measure on the sphere of radius t. There are more complicated formulas for higher dimensions, and in particular, for $n \geq 3$, $K_t^W(x)$ is distribution-valued.

Proposition 3.1. For arbitrary dimension n,

$$K(x) = c_n \left(P * K^W_{t=\frac{\sqrt{3}}{2}} \right)(x), \tag{82}$$

where $*$ denotes a spatial convolution and

$$P(x) = \frac{1}{(1 + 4|x|^2)^{\frac{n+1}{2}}}.$$

Proof. Recall that $K(x)$ is defined by the inverse Fourier transform given in (71). We first consider the inverse Fourier transform of $e^{-\frac{|\xi|}{2}}$ and $\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|}$ separately. The Poisson kernel, corresponding to the operator $e^{-\sqrt{-\Delta}}$, is well-understood and in particular,

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}} e^{ix\cdot\xi} d\xi = \frac{c_n}{(1 + 4|x|^2)^{\frac{n+1}{2}}} := c_n P(x),$$

for some constant c_n depending only on the dimension n. By formula (81),

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin(\frac{\sqrt{3}}{2}|\xi|)}{\frac{\sqrt{3}}{2}|\xi|} e^{ix\cdot\xi} d\xi = \frac{2}{\sqrt{3}} K^W_{t=\frac{\sqrt{3}}{2}}(x).$$

The result follows from the fact that the Fourier transform interchanges multiplication and convolution, where we absorbed the extra factor of $\frac{2}{\sqrt{3}}$ into the dimension-dependent constant c_n. □

Because the fundamental solution for the wave equation is particularly simple in dimensions $n = 1, 2, 3$, as given in (81), we can use Proposition 3.1 to show a sharp decay rate for $K(x)$. This sharp decay rate estimate will verify the result proved for general dimension n in Lemma 3.3 for the explicit dimensions $n = 1, 2, 3$.

Proposition 3.2. For $n = 1, 2, 3$, the convolution kernel $K(x)$ has the following pointwise estimate:

$$|K(x)| \leq \frac{C}{(1+|x|)^{n+1}},\ x \in \mathbb{R}^n,$$

for some constant $C > 0$. In particular, $K(x) \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ in dimensions $n = 1, 2, 3$.

Proof. We use the explicit formula for $K(x)$ given in Proposition 3.1. Note that $P(x)$ is bounded near the origin, and decays as $O(|x|^{-n-1})$ away from the origin. We begin with $n = 3$, then describe the case of $n = 1, 2$.

For $n = 3$, because $K^W_{t=\frac{\sqrt{3}}{2}}$ is Lebesgue measure on the sphere of radius $\frac{\sqrt{3}}{2}$, $K(x)$ in dimension $n = 3$ is an average over the boundary of spheres of radius $\frac{\sqrt{3}}{2}$,

$$K(x) = C\left(\frac{1}{(1+4|x|^2)^{\frac{n+1}{2}}} * \sigma_{\frac{\sqrt{3}}{2}}\right)(x) = C\int_{|y|=\frac{\sqrt{3}}{2}} \frac{1}{(1+4|x-y|^2)^{\frac{n+1}{2}}} dy,$$

from which we deduce the desired estimate for $n = 3$. We remark that in dimension $n = 3$, $K(x)$ (up to a constant factor) can be interpreted as spherical means of the Poisson kernel.

For $n = 1, 2$, one can use a similar interpretation of $K(x)$ as appropriate spatial averages of $P(x)$, and we sketch the idea. Note that $K^W_{t=\frac{\sqrt{3}}{2}}(x)$ is integrable and supported on the closed ball of radius t for dimensions $n = 1, 2$. Therefore, from the formula in Proposition 3.1, up to the constant factor, $K(x)$ in dimensions $n = 1$ is an average of $K(x)$ over closed intervals of radius $\frac{\sqrt{3}}{2}$,

$$K(x) = C\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{(1+4|x-y|^2)^{\frac{n+1}{2}}} dy,$$

and $K(x)$ in dimension $n = 2$ is a weighted average of $K(x)$ over open balls of radius $\frac{\sqrt{3}}{2}$,

$$K(x) = C\int_{|y|<\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{\frac{3}{4}-|y|^2}} \frac{1}{(1+4|x-y|^2)^{\frac{n+1}{2}}} dy.$$

Because $K(x)$ in both $n = 1, 2$ is, up to a constant factor, appropriate averages of values of $P(x)$, $K(x)$ hence shares the same decay rate as $P(x)$, which is $O(|x|^{-n-1})$, and $K(x)$ is also bounded near the origin, since $P(x)$ is. This establishes the desired estimate for $n = 1, 2$.

In particular, we see explicitly that $K(x) \in L^p$ for all $1 \leq p \leq \infty$ for $n = 1, 2, 3$. □

Next, we use Lemma 3.3 to prove an estimate on the action of $S(t)$ for $t > 0$. This will then be used to obtain the first part of the Strichartz estimate (52), namely to show

$$||u||_{L^q_t([0,T];L^r_x(\mathbb{R}^n))} \leq C_{q,\tilde{q},r,\tilde{r}} ||F||_{L^{\tilde{q}'}_t([0,T];L^{\tilde{r}'}_x(\mathbb{R}^n))}. \tag{83}$$

The following result is the analogue of Lemma 3.1 in [27].

Lemma 3.4. Let $1 \leq r \leq p \leq \infty$. There exists a constant $C > 0$, depending only on p and r, such that for all $\phi \in L^r(\mathbb{R}^n)$ and for all $t > 0$,

$$||S(t)\phi(x)||_{L^p_x(\mathbb{R}^n)} \leq Ct^{1-n\left(\frac{1}{r}-\frac{1}{p}\right)} ||\phi||_{L^r_x(\mathbb{R}^n)}.$$

Proof. This follows from the previous lemma, by Young's inequality, and the scaling property of our convolution kernels K_t. In particular, recall that

$$S(t)\phi(x) := e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}\phi = (K_t * \phi)(x), \text{ where } K_t(x) = t^{1-n}K\left(\frac{x}{t}\right),$$

with the unit scale kernel K given by (71). By Young's inequality,

$$||S(t)\phi(x)||_{L^p_x(\mathbb{R}^n)} = ||K_t * \phi||_{L^p_x(\mathbb{R}^n)} \leq ||K_t||_{L^q_x(\mathbb{R}^n)}||\phi||_{L^r_x(\mathbb{R}^n)}, \tag{84}$$

where

$$\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}.$$

Using the scaling of the kernel,

$$||K_t||_{L^q_x(\mathbb{R}^n)} = t^{1-n}\left(\int_{\mathbb{R}^n}\left|K\left(\frac{x}{t}\right)\right|^q dx\right)^{1/q} = t^{1-n+\frac{n}{q}}||K||_{L^q_x} = C_{p,r}t^{1+n\left(\frac{1}{p}-\frac{1}{r}\right)}$$

Substituting this into (84), we obtain the final result that

$$||S(t)\phi(x)||_{L^p_x(\mathbb{R}^n)} \leq C_{p,r}t^{1-n\left(\frac{1}{r}-\frac{1}{p}\right)}||\phi||_{L^r_x(\mathbb{R}^n)}$$

for $t > 0$ and for all $\phi \in L^r(\mathbb{R}^n)$. □

Remark 3.4. The result of Lemma 3.4 can be interpreted as a parabolic smoothing effect, due to the dissipative effects of the fluid viscosity. For parabolic equations, parabolic smoothing can often be captured by using energy estimates. Energy estimates for the linear viscous wave equation would correspond to the case of $r = 2$ in Lemma 3.4, as a basic energy estimate would consider initial data $(f, g) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $S(t)\phi$ in the statement of Lemma 3.4 corresponds to the solution of the linear viscous wave equation with zero initial displacement and initial velocity ϕ. We chose to use Fourier methods to establish Lemma 3.4 because they easily allow for more general ϕ in $L^r(\mathbb{R}^n)$ for $1 \leq r \leq \infty$, which allows for a wider range of exponents than basic energy estimates.

Now, we have the necessary components to establish inequality (83).

Lemma 3.5. Let $n \geq 2$ and let (q, r), $(\tilde{q}, \tilde{r})$ satisfy $1 < \tilde{q}' < q < \infty, 1 \leq \tilde{r}' < r \leq \infty$, and the gap condition (53):

$$\frac{1}{q} + \frac{n}{r} = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2.$$

Then, there exists a constant $C > 0$ depending only on $q, \tilde{q}, r, \tilde{r}$, such that for all $0 < T \leq \infty$, and for all $F \in L^{\tilde{q}'}_t([0,T]; L^{\tilde{r}'}_x(\mathbb{R}^n))$,

$$\left|\left|\int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}F(\tau,\cdot)d\tau\right|\right|_{L^q_t([0,T];L^r_x(\mathbb{R}^n))} \leq C||F||_{L^{\tilde{q}'}_t([0,T];L^{\tilde{r}'}_x(\mathbb{R}^n))}.$$

For the inhomogeneous linear viscous wave equation with zero initial data, notice that since

$$u(t,\cdot) := \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau,$$

this result implies

$$||u||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} \leq C_{q,\tilde{q},r,\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}.$$

Remark 3.5. In contrast with the Strichartz estimates for the wave equation and the Schrödinger equation, the result in Lemma 3.5 does not require any admissibility condition on (q,r) and $(\tilde{q},\tilde{r})$.

Proof. Similar to the approach used in [40] for fractional heat equations, the estimate in Lemma 3.5 is a consequence of the Hardy-Littlewood-Sobolev inequality.

More precisely, we first show the result in the case $T = \infty$. We begin by estimating:

$$\left|\left|\int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau\right|\right|_{L_t^q([0,\infty);L_x^r(\mathbb{R}^n))}$$
$$\leq \left|\left|\int_0^t \left|\left| e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)\right|\right|_{L_x^r(\mathbb{R}^n)} d\tau\right|\right|_{L_t^q[0,\infty)}.$$

From Lemma 3.4, we have that for $0 \leq \tau < t$, the integrand can be bounded as:

$$\left|\left| e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)\right|\right|_{L_x^r(\mathbb{R}^n))} \leq C_{r,\tilde{r}} |t-\tau|^{1-n\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)} ||F(\tau,\cdot)||_{L_x^{\tilde{r}'}(\mathbb{R}^n)},$$

which implies:

$$\left|\left|\int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau\right|\right|_{L_t^q([0,\infty);L_x^r(\mathbb{R}^n))}$$
$$\leq C_{r,\tilde{r}} \left|\left|\int_0^t |t-\tau|^{1-n\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)} ||F(\tau,\cdot)||_{L_x^{\tilde{r}'}(\mathbb{R}^n)} d\tau\right|\right|_{L_t^q[0,\infty)}. \quad (85)$$

We would like to apply the Hardy-Littlewood-Sobolev inequality in one dimension to the convolution in time above, which states that for all $0 < \alpha < 1$, and for s and $\tilde{s}$, $1 < \tilde{s} < s < \infty$, such that

$$1 + \frac{1}{s} = \frac{1}{\tilde{s}} + \alpha,$$

the following holds:

$$\left|\left|(|x|^{-\alpha} * f)\right|\right|_{L^s(\mathbb{R})} \leq C_{s,\tilde{s}} ||f||_{L^{\tilde{s}}(\mathbb{R})}.$$

We plan to apply this estimate in (85) by letting

$$\alpha = -1 + n\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right) = 1 - \frac{1}{\tilde{q}'} + \frac{1}{q},$$

where indeed, $0 < \alpha < 1$ since we assumed that $1 < \tilde{q}' < q < \infty$, and the second equality holds by the gap condition (53). We can then let $\tilde{s} = \tilde{q}'$ and $s = q$. To apply the Hardy-Littlewood-Sobolev inequality, which holds on $\mathbb{R}$, we extend the function F from $t \in [0, \infty)$ to $t \in \mathbb{R}$:

$$\tilde{F}(t,x) = F(t,x) \text{ for } t \geq 0 \text{ and } \tilde{F}(t,x) = 0 \text{ for } t < 0,$$

and rewrite estimate (85) in terms of $\tilde{F}$:

$$\left|\left| \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot) d\tau \right|\right|_{L_t^q([0,\infty);L_x^r(\mathbb{R}^n))}$$
$$\leq C_{r,\tilde{r}} \left|\left| \int_{\mathbb{R}} |t-\tau|^{1-n\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)} ||\tilde{F}(\tau,\cdot)||_{L^{\tilde{r}'}(\mathbb{R}^n)} d\tau \right|\right|_{L_t^q(\mathbb{R})}.$$

By applying the Hardy-Littlewood-Sobolev inequality we get:

$$\left|\left| \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot) d\tau \right|\right|_{L_t^q([0,\infty);L_x^r(\mathbb{R}^n))}$$
$$\leq C_{r,\tilde{r}} C_{q,\tilde{q}} ||\tilde{F}||_{L_t^{\tilde{q}'}(\mathbb{R};L_x^{\tilde{r}'}(\mathbb{R}^n))} = C_{q,\tilde{q},r,\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,\infty);L_x^{\tilde{r}'}(\mathbb{R}^n))},$$

which is what we wanted to prove.

The case for $0 < T < \infty$ follows from the case $T = \infty$ by considering

$$\tilde{F}_T(t,x) = F(t,x) \text{ for } 0 \leq t \leq T \text{ and } \tilde{F}_T(t,x) = 0 \text{ otherwise.}$$

This proves the lemma.

□

So we have just shown that $||u||_{L_t^q([0,T];L_x^r(\mathbb{R}^n))} \leq C_{q,\tilde{q},r,\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}$ for u given by (65). What remains to be shown to complete the inhomogeneous Strichartz estimate in Theorem 3.3 is given by the following lemma:

Lemma 3.6. Let $2 \leq \tilde{q} \leq \infty$, $2 \leq \tilde{r} < \infty$ and s satisfy

$$\frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2.$$

Then, there exists a constant C depending only on $\tilde{q}$ and $\tilde{r}$, such that for all $0 < T < \infty$ and for u as in (65):

$$||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \leq C_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}. \tag{86}$$

Proof. The proof uses similar arguments as in the proof of Theorem 3.2. First, recall from (65) that if u is the solution to the linear viscous wave equation with zero initial data and source term F, then

$$u(t,\cdot) := \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot) d\tau.$$

We define the Littlewood-Paley decomposition contributions as:

$$u_j(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\beta\left(\frac{|\xi|}{2^j}\right)\widehat{u}(t,\xi)d\xi, \quad F_j(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\beta\left(\frac{|\xi|}{2^j}\right)\widehat{F}(t,\xi)d\xi.$$

Step 1. We first claim that it is sufficient to show that estimate (86) holds for each integer j for all F with $\widehat{F}(t,\xi)$ for $0 \le t \le T$ supported in an annulus $2^{j-1} \le |\xi| \le 2^{j+1}$, where the constant in the estimate is *independent of* j *and* $0 < T < \infty$.

To see why this is so, we follow an argument similar to the one in the proof of Theorem 3.2. In particular, note that u_j is the solution to the linear viscous wave equation with zero initial data and source term F_j, where $\widehat{F_j}$ is supported in the annulus $2^{j-1} \le |\xi| \le 2^{j+1}$ for all $0 \le t \le T$. So if we had established that the estimate holds for each F with Fourier transform at all times supported in an annulus $2^{j-1} \le |\xi| \le 2^{j+1}$ for each j, for a constant independent of j, we would have for each j that

$$||u_j(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u_j(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \le C_{\tilde{q},\tilde{r}}||F_j||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))},$$

and thus

$$||u_j(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)} \le C_{\tilde{q},\tilde{r}}||F_j||^2_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} \text{ and } ||\partial_t u_j(T,\cdot)||^2_{\dot{H}^{s-1}(\mathbb{R}^n)} \le C_{\tilde{q},\tilde{r}}||F_j||^2_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))},$$

for all $0 < T < \infty$ and for any function F (since F_j now has Fourier transform at all times with support in $2^{j-1} \le |\xi| \le 2^{j+1}$). To get the estimate in Lemma 3.6, note that since $\sum_{j=-\infty}^{\infty}\beta^2\left(\frac{|\xi|}{2^j}\right) \ge c$ for some constant $c > 0$, and since $\widehat{\partial_t u_j}(t,\xi) = \beta(|\xi|/2^j)\widehat{\partial_t u}(t,\xi)$, we have

$$||u(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)} \le c^{-1}\sum_{j=-\infty}^{\infty}||u_j(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)} \text{ and } ||\partial_t u(T,\cdot)||^2_{\dot{H}^{s-1}(\mathbb{R}^n)} \le c^{-1}\sum_{j=-\infty}^{\infty}||\partial_t u_j(T,\cdot)||^2_{\dot{H}^{s-1}(\mathbb{R}^n)}.$$

In addition, since $\tilde{r} \ne \infty$, we have $\tilde{r}' \ne 1$, and from the assumptions on $\tilde{q}$ and $\tilde{r}$ from the lemma, we see that $1 \le \tilde{q}' \le 2$, $1 < \tilde{r}' \le 2$. So by estimate (55) in Lemma 3.1,

$$\sum_{j=-\infty}^{\infty}||F_j||^2_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} \le C||F||^2_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}.$$

Therefore, for general F and associated u,

$$\begin{aligned}\left(||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)}\right)^2 &\le 2c^{-1}\sum_{j=-\infty}^{\infty}\left(||u_j(T,\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u_j(T,\cdot)||^2_{\dot{H}^{s-1}(\mathbb{R}^n)}\right)\\ &\le 4c^{-1}C_{\tilde{q},\tilde{r}}\sum_{j=-\infty}^{\infty}||F_j||^2_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} \le C'_{\tilde{q},\tilde{r}}||F||^2_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}.\end{aligned}$$

Taking square roots gives the desired result.

Step 2. Because of the gap condition (53), we claim that it suffices to show estimate (86) for all F such that $\hat{F}$ is supported in $2^{j-1} \le |\xi| \le 2^{j+1}$, for only $j = 0$, i.e., for $\hat{F}$ supported in $1/2 \le |\xi| \le 2$ for all $0 \le t \le T$ with $0 < T < \infty$ arbitrary.

As in the proof of Theorem 3.2, we can use the scaling that preserves solutions of the linear viscous wave equation, that maps F and u into $\lambda^2 F(\lambda t, \lambda x), u(\lambda t, \lambda x)$. Under this scaling,

$$||u(\lambda T, \lambda x)||_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{-\frac{n}{2}+s}||u(\lambda T, x)||_{\dot{H}^s(\mathbb{R}^n)},$$

$$||\lambda \partial_t u(\lambda T, \lambda x)||_{\dot{H}^{s-1}(\mathbb{R}^n)} = \lambda^{-\frac{n}{2}+s}||\partial_t u(\lambda T, x)||_{\dot{H}^{s-1}(\mathbb{R}^n)},$$

$$||\lambda^2 F(\lambda t, \lambda x)||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} = \lambda^{2-\frac{n}{\tilde{r}'}-\frac{1}{\tilde{q}'}}||F(t,x)||_{L_t^{\tilde{q}'}([0,\lambda T];L_x^{\tilde{r}'}(\mathbb{R}^n))}.$$

Because of the gap condition (53), once we prove estimate (86) in the case where $\widehat{F}(t,\xi)$ is supported in $1/2 \leq |\xi| \leq 2$, we can use the scalings above to get that the inequality holds whenever $\widehat{F}(t,\xi)$ (for $0 \leq t \leq T$) is supported in any annulus $2^{j-1} \leq |\xi| \leq 2^{j+1}$ for any integer j, with the same constant $C_{\tilde{q},\tilde{r}}$.

Step 3. Thus, we must show that there exists a constant $C_{\tilde{q},\tilde{r}} > 0$, independent of $0 < T < \infty$, such that whenever $\widehat{F}(t,\xi)$ is supported in $1/2 \leq |\xi| \leq 2$ for all $0 \leq t \leq T$, the following holds:

$$||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} + ||\partial_t u(T,\cdot)||_{\dot{H}^{s-1}(\mathbb{R}^n)} \leq C_{\tilde{q},\tilde{r}}||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}.$$

We begin by estimating $||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)}$:

$$\begin{aligned}||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} &\leq \int_0^T \left|\left| e^{-\frac{\sqrt{-\Delta}}{2}(T-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(T-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)\right|\right|_{\dot{H}^s(\mathbb{R}^n)} d\tau \\ &= C\int_0^T \left(\int_{\mathbb{R}^n} |\xi|^{2s} \left| e^{-\frac{|\xi|}{2}(T-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(T-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|} \widehat{F}(\tau,\xi)\right|^2 d\xi\right)^{1/2} d\tau.\end{aligned}$$

To further estimate the right hand-side of this inequality, we first notice that on the support of $\widehat{F}(t,\xi)$, which is in $1/2 \leq |\xi| \leq 2$, we have $|\xi|^{2s} \leq C_s$, where the constant C_s can be expressed to depend only on $\tilde{q}, \tilde{r}$, i.e., $C_s = C_{\tilde{q},\tilde{r}}$ because of the gap condition. Furthermore, on the support of $\widehat{F}(t,\xi)$ and for $0 \leq \tau \leq T$ we have:

$$\left| e^{-\frac{|\xi|}{2}(T-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(T-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|}\right| \leq (T-\tau)e^{-\frac{|\xi|}{2}(T-\tau)} \leq (T-\tau)e^{-\frac{1}{4}(T-\tau)}.$$

Therefore, we now have:

$$||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} \leq C_{\tilde{q},\tilde{r}} \int_0^T \left(\int_{\mathbb{R}^n} |(T-\tau)e^{-\frac{1}{4}(T-\tau)}\chi_{1/2\leq|\xi|\leq 2}(\xi)|^2 |\widehat{F}(\tau,\xi)|^2 d\xi\right)^{1/2} d\tau.$$

By Holder's inequality with $\frac{\tilde{r}}{\tilde{r}-2} \geq 1$ and $\frac{\tilde{r}}{2} \geq 1$, we get:

$$\begin{aligned}||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} &\leq C_{\tilde{q},\tilde{r}} \int_0^T \left[\left(\int_{1/2\leq|\xi|\leq 2} \left|(T-\tau)e^{-\frac{1}{4}(T-\tau)}\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi\right)^{\frac{\tilde{r}-2}{2\tilde{r}}} ||\widehat{F}(\tau,\cdot)||_{L_\xi^{\tilde{r}}(\mathbb{R}^n)}\right] d\tau \\ &\leq C_{\tilde{q},\tilde{r}} \int_0^T (T-\tau)e^{-\frac{1}{4}(T-\tau)}||\widehat{F}(\tau,\cdot)||_{L_\xi^{\tilde{r}}(\mathbb{R}^n)} d\tau \leq C_{\tilde{q},\tilde{r}} \int_0^T (T-\tau)e^{-\frac{1}{4}(T-\tau)}||F(\tau,\cdot)||_{L_x^{\tilde{r}'}(\mathbb{R}^n)} d\tau,\end{aligned}$$

where we used that $1/2 \leq |\xi| \leq 2$ in $\mathbb{R}^n$ has finite volume and the fact that the Fourier transform is a bounded operator from $L^{\tilde{r}'}$ to $L^{\tilde{r}}$, since $\tilde{r} \geq 2$. Using Holder's inequality one more time with $\tilde{q}$ and $\tilde{q}'$, we obtain:

$$\begin{aligned}
||u(T,\cdot)||_{\dot{H}^s(\mathbb{R}^n)} &\leq C_{\tilde{q},\tilde{r}} \left(\int_0^T \left| (T-\tau) e^{-\frac{1}{4}(T-\tau)} \right|^{\tilde{q}} d\tau \right)^{1/\tilde{q}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} \\
&\leq C_{\tilde{q},\tilde{r}} \left(\int_0^T \left| \tau e^{-\frac{1}{4}\tau} \right|^{\tilde{q}} d\tau \right)^{1/\tilde{q}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} \\
&\leq C_{\tilde{q},\tilde{r}} \left(\int_0^\infty \left| \tau e^{-\frac{1}{4}\tau} \right|^{\tilde{q}} d\tau \right)^{1/\tilde{q}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} \leq C_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}.
\end{aligned}$$

Next, we estimate $||\partial_t u(T,\xi)||_{\dot{H}^{s-1}(\mathbb{R}^n)}$. First note that

$$\begin{aligned}
\widehat{\partial_t u}(t,\xi) &= \partial_t(\widehat{u}(t,\xi)) \\
&= \int_0^t e^{-\frac{|\xi|}{2}(t-\tau)} \left(\cos\left(\frac{\sqrt{3}}{2}|\xi|(t-\tau) \right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}|\xi|(t-\tau) \right) \right) \widehat{F}(\tau,\xi) d\tau.
\end{aligned}$$

By the support properties of $\widehat{F}(\tau,\xi)$ and boundedness of sines and cosines, we get:

$$\begin{aligned}
||\partial_t u(T,\xi)||_{\dot{H}^{s-1}(\mathbb{R}^n)} &\leq C \int_0^T \left(\int_{\mathbb{R}^n} |\xi|^{2s-2} \left| e^{-\frac{|\xi|}{2}(T-\tau)} \widehat{F}(\tau,\xi) \right|^2 d\xi \right)^{1/2} d\tau \\
&= C_{\tilde{q},\tilde{r}} \int_0^T \left(\int_{\mathbb{R}^n} \left| e^{-\frac{1}{4}(T-\tau)} \chi_{1/2 \leq |\xi| \leq 2}(\xi) \widehat{F}(\tau,\xi) \right|^2 d\xi \right)^{1/2} d\tau.
\end{aligned}$$

After applying Holder's inequality with $\frac{\tilde{r}}{\tilde{r}-2}$ and $\frac{\tilde{r}}{2}$, we get:

$$\begin{aligned}
||\partial_t u(T,\xi)||_{\dot{H}^{s-1}(\mathbb{R}^n)} &\leq C_{\tilde{q},\tilde{r}} \int_0^T \left(\int_{1/2 \leq |\xi| \leq 2} \left| e^{-\frac{1}{4}(T-\tau)} \right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi \right)^{\frac{\tilde{r}-2}{2\tilde{r}}} ||\widehat{F}(\tau,\cdot)||_{L_\xi^{\tilde{r}}(\mathbb{R}^n)} d\tau \\
&\leq C_{\tilde{q},\tilde{r}} \int_0^T e^{-\frac{1}{4}(T-\tau)} ||F(\tau,\cdot)||_{L_x^{\tilde{r}'}(\mathbb{R}^n)} d\tau \\
&= C_{\tilde{q},\tilde{r}} \left(\int_0^T \left| e^{-\frac{1}{4}\tau} \right|^{\tilde{q}} d\tau \right)^{1/\tilde{q}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} \leq C_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemmas 3.5 and 3.6 now imply the proof of Theorem 3.3, which completes the proof of the inhomogeneous Strichartz estimates for the linear viscous wave equation.

4 $C^0([0,T], H^s(\mathbb{R}^n))$ estimates for the linear viscous wave equation

In this section we present the H^s estimates on the solution of the linear viscous wave equation, which will be needed in probabilistic randomization, presented in Sec. 5. In contrast with the Strichartz estimates, where scale invariance plays an important role since the estimates hold globally in time, the $H^s(\mathbb{R}^n)$ norms are not invariant under the natural scaling map, and so the estimates will hold only on a finite time interval $[0,T]$. Since the focus of Sec. 5 will be on the case $n=2$, we present the estimates only for $n=2$.

Proposition 4.1. Let $n = 2$. Let $2 \leq \tilde{q} \leq \infty$, $2 \leq \tilde{r} < \infty$, and $s \geq 0$ satisfy the gap condition

$$\frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2.$$

Then, for all $0 < T \leq 1$, there exists a constant $C > 0$ independent of T, depending only on $\tilde{q}$ and $\tilde{r}$, such that

$$\left\| \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \right\|_{C^0([0,T];H^s(\mathbb{R}^n))} \leq C_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}. \quad (87)$$

Proof. We start by obtaining an $L^\infty([0,T], L^2(\mathbb{R}^n))$-estimate of the left hand-side in (87) and then use

$$||f||_{H^s} \leq C_s(||f||_{L^2} + ||f||_{\dot{H}^s})$$

along with the result of Theorem 3.3 to obtain an $L^\infty([0,T], H^s(\mathbb{R}^n))$ estimate. This estimate will then be improved by showing continuity in time, to get the desired $C^0([0,T], H^s(\mathbb{R}^n))$ estimate.

We first obtain the spatial H^s estimate. Since we already have the homogeneous $\dot{H}^s$-norm estimates from the inhomogeneous Strichartz estimates in Theorem 3.3, it suffices to obtain the corresponding L^2-estimate. For $0 < T \leq 1$ we have, by Hölder's inequality:

$$\left\| \int_0^T e^{-\frac{\sqrt{-\Delta}}{2}(T-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(T-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \right\|_{L^2(\mathbb{R}^n)}$$

$$\leq \int_0^T \left\| e^{-\frac{|\xi|}{2}(T-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(T-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|} \widehat{F}(\tau,\xi) \right\|_{L^2_\xi(\mathbb{R}^n)} d\tau \leq \int_0^T \|(T-\tau)e^{-\frac{|\xi|}{2}(T-\tau)}\|_{L_\xi^{\frac{2\tilde{r}}{\tilde{r}-2}}} \|\widehat{F}(\tau,\cdot)\|_{L_\xi^{\tilde{r}}} d\tau$$

$$\leq C_{\tilde{r}} \int_0^T \|(T-\tau)e^{-\frac{|\xi|}{2}(T-\tau)}\|_{L_\xi^{\frac{2\tilde{r}}{\tilde{r}-2}}(\mathbb{R}^n)} \|F(\tau,\cdot)||_{L_x^{\tilde{r}'}(\mathbb{R}^n)} d\tau = C_{\tilde{r}} \int_0^T (T-\tau)^{1-n\cdot\frac{\tilde{r}-2}{2\tilde{r}}} ||F(\tau,\cdot)||_{L_x^{\tilde{r}'}(\mathbb{R}^n)} d\tau.$$

Note that since $n = 2, 2 \leq \tilde{r} < \infty$, we have $1 - n \cdot \frac{\tilde{r}-2}{2\tilde{r}} > 0$. Then, using Hölder's inequality once more, we can continue estimating the right hand-side as

$$\leq C_{\tilde{r}} \left(\int_0^T (T-\tau)^{\tilde{q}\left(1-n\cdot\frac{\tilde{r}-2}{2\tilde{r}}\right)} d\tau \right)^{1/\tilde{q}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))} = C_{\tilde{q},\tilde{r}} T^{\alpha_{\tilde{q},\tilde{r}}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}, \quad (88)$$

for some $\alpha_{\tilde{q},\tilde{r}} > 0$ depending on $\tilde{q}$ and $\tilde{r}$. Therefore, we have shown that for all $0 < T \leq 1$,

$$\left\| \int_0^T e^{-\frac{\sqrt{-\Delta}}{2}(T-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(T-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \right\|_{L^2(\mathbb{R}^n)} \leq C_{\tilde{q},\tilde{r}} T^{\alpha_{\tilde{q},\tilde{r}}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))},$$

where $C_{\tilde{q},\tilde{r}}$ is independent of $0 < T \leq 1$. Now, from the inhomogeneous Strichartz estimates in Theorem 3.3, we have

$$\left\| \int_0^T e^{-\frac{\sqrt{-\Delta}}{2}(T-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(T-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \right\|_{\dot{H}^s(\mathbb{R}^n)} \leq C'_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))},$$

for all $0 < T < \infty$ where $C'_{\tilde{q},\tilde{r}}$ is independent of $0 < T < \infty$. Combining these two estimates we get that for all $0 < T \leq 1$,

$$\left\| \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \right\|_{L^\infty([0,T];H^s(\mathbb{R}^n))} \leq C_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'}(\mathbb{R}^n))}, \quad (89)$$

for some constant $C_{\tilde{q},\tilde{r}}$ independent of $0 < T \leq 1$.

We need to show that the above estimate holds in terms of the $C^0([0,T];H^s(\mathbb{R}^n))$ norm on the left hand-side of (89). For this purpose, we first notice that estimate (89) establishes the continuity at $t = 0$, in the $H^s(\mathbb{R}^n)$ norm, of the map

$$u^t(\cdot) : t \to u(t,\cdot) = \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau.$$

We next show that this map is, in fact, a continuous map from $[0,1]$ to $H^s(\mathbb{R}^n)$. More precisely, fix an arbitrary $0 < t_1 \leq 1$. Given $\epsilon > 0$, we want to find $\delta > 0$ such that for all $\tilde{t} \in (t_1 - \delta, t_1 + \delta)$ (or $\tilde{t} \in (t_1 - \delta, t_1)$ if $t_1 = 1$), we have

$$\|u(t_1,\cdot) - u(\tilde{t},\cdot)\|_{H^s(\mathbb{R}^n)} < \epsilon.$$

Thus, we want to estimate:

$$\begin{aligned} \|u(t_1,\cdot) - u(\tilde{t},\cdot)\|_{H^s(\mathbb{R}^n)} = \Bigg\| & \int_0^{t_1} e^{-\frac{\sqrt{-\Delta}}{2}(t_1-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t_1-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \\ & - \int_0^{\tilde{t}} e^{-\frac{\sqrt{-\Delta}}{2}(\tilde{t}-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(\tilde{t}-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \Bigg\|_{H^s(\mathbb{R}^n)}. \end{aligned}$$

To do this, we introduce a $\delta' > 0$, which will be specified later, and then find $\delta < \delta'/2$ sufficiently small so that $\tilde{t} \in (t_1 - \delta, t_1 + \delta)$ will always be greater than $t_1 - \delta'$, and estimate the left hand side from above via the following three integrals:

$$\begin{aligned} & \|u(t_1,\cdot) - u(\tilde{t},\cdot)\|_{H^s(\mathbb{R}^n)} \\ & \leq \left\| \int_{t_1-\delta'}^{t_1} e^{-\frac{\sqrt{-\Delta}}{2}(t_1-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t_1-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \right\|_{H^s(\mathbb{R}^n)} \\ & + \left\| \int_0^{t_1-\delta'} \left(e^{-\frac{\sqrt{-\Delta}}{2}(t_1-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t_1-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} - e^{-\frac{\sqrt{-\Delta}}{2}(\tilde{t}-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(\tilde{t}-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} \right) F(\tau,\cdot)d\tau \right\|_{H^s(\mathbb{R}^n)} \\ & + \left\| \int_{t_1-\delta'}^{\tilde{t}} e^{-\frac{\sqrt{-\Delta}}{2}(\tilde{t}-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(\tilde{t}-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot)d\tau \right\|_{H^s(\mathbb{R}^n)} := I^{(1)}_{t_1,\tilde{t}} + I^{(2)}_{t_1,\tilde{t}} + I^{(3)}_{t_1,\tilde{t}}. \end{aligned} \quad (90)$$

We now choose $\delta' > 0$, $\delta' < t_1/2$ (and $\delta' < 1 - t_1$ if $0 < t_1 < 1$), sufficiently small so that on $[t_1 - \delta', t_1 + \delta']$,

$$C_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'}([t_1-\delta',t_1+\delta'],L_x^{\tilde{r}'}(\mathbb{R}^n))} < \frac{\epsilon}{3},$$

where $C_{\tilde{q},\tilde{r}}$ is the constant in (89). Then, using (89), the first and the third integral are immediately estimated as:

$$0 \leq I^{(1)}_{t_1,\tilde{t}} < \frac{\epsilon}{3}, \qquad 0 \leq I^{(3)}_{t_1,\tilde{t}} < \frac{\epsilon}{3}. \tag{91}$$

What remains is to estimate $I^{(2)}_{t_1,\tilde{t}}$. To simplify notation, we will use

$$\hat{K}_t(\xi) := e^{-\frac{|\xi|}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}$$

to denote the Fourier transform of the kernel K_t introduced in (68). By Hölder's inequality and Hausdorff-Young's inequality, we have that

$$\begin{aligned}
I^{(2)}_{t_1,\tilde{t}} &\leq \int_0^{t_1-\delta'} ||(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))\widehat{F}(\tau,\cdot)||_{L^2_\xi(\mathbb{R}^n)} d\tau \\
&\leq \int_0^{t_1-\delta'} ||(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))||_{L_\xi^{\frac{2\tilde{r}}{\tilde{r}-2}}(\mathbb{R}^n)} ||\widehat{F}(\tau,\cdot)||_{L^{\tilde{r}}_\xi(\mathbb{R}^n)} d\tau \\
&\leq C\int_0^{t_1-\delta'} ||(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))||_{L_\xi^{\frac{2\tilde{r}}{\tilde{r}-2}}(\mathbb{R}^n)} ||F(\tau,\cdot)||_{L^{\tilde{r}'}_x(\mathbb{R}^n)} d\tau \\
&\leq C||(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))||_{L^{\tilde{q}}_\tau([0,t_1-\delta'];L_\xi^{\frac{2\tilde{r}}{\tilde{r}-2}}(\mathbb{R}^n))} ||F||_{L^{\tilde{q}'}_t([0,1];L^{\tilde{r}'}_x(\mathbb{R}^n))}.
\end{aligned} \tag{92}$$

Since $||F||_{L^{\tilde{q}'}_t([0,1];L^{\tilde{r}'}_x(\mathbb{R}^n))}$ is a fixed constant for a given F, we just need to show that the factor multiplying this term is small, i.e., it can be bounded in terms of ϵ. The term we need to bound, in expanded form, reads:

$$\begin{aligned}
&C||(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))||_{L^{\tilde{q}}_\tau([0,t_1-\delta'];L_\xi^{\frac{2\tilde{r}}{\tilde{r}-2}}(\mathbb{R}^n))} \\
&\qquad = C\left(\int_0^{t_1-\delta'} \left(\int_{\mathbb{R}^n} \left|(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi\right)^{\tilde{q}\cdot\frac{\tilde{r}-2}{2\tilde{r}}} d\tau\right)^{1/\tilde{q}}.
\end{aligned} \tag{93}$$

First we bound the interior integral, namely the integral with respect to ξ. We divide the integral into two parts: the integral over $|\xi| \leq M$, and the integral over $|\xi| \geq M$, where M will be determined later:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi \\
&= \int_{|\xi|\leq M} \left|(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi \\
&+ \int_{|\xi|\geq M} \left|(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi.
\end{aligned}$$

We estimate each part separately, keeping in mind that $|\hat{K}_t(\xi)| \leq e^{-\frac{|\xi|}{2}t}|t|$, $|t_1 - \tilde{t}| < \delta$, and $\tau \in (0, t_1 - \delta')$. The smallness of the first integral will follow from the absolute continuity of the integrand as a function of time, and the smallness of the second integral will follow from the exponential decay of the kernel $\hat{K}_t(\xi)$, which can be made arbitrarily small for high frequencies ξ by the choice of $M > 1$.

Indeed, we start with the integral over $|\xi| \geq M$. We claim that we can choose $M > 1$ sufficiently large such that the contribution of that integral is smaller than a bound given in terms of ϵ. Namely, we will show that for every $\epsilon' > 0$, there is an $M > 1$, such that for all $0 \leq \tau \leq t_1 - \delta'$, $|\tilde{t} - t_1| < \delta < \delta'/2$, we have

$$\begin{aligned}
&\int_{|\xi|\geq M} \left|(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi \\
&\leq C_{\tilde{r}} \left(\int_{|\xi|\geq M} \left|(1+|\xi|^2)^{s/2}(t_1-\tau)e^{-(t_1-\tau)\frac{|\xi|}{2}}\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi + \int_{|\xi|\geq M} \left|(1+|\xi|^2)^{s/2}(\tilde{t}-\tau)e^{-(\tilde{t}-\tau)\frac{|\xi|}{2}}\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi \right) < \frac{\epsilon'}{2}.
\end{aligned} \tag{94}$$

Later, we will choose ϵ' to depend on ϵ and F so that the total integral $I^{(2)}_{t_1,\tilde{t}}$ is bounded by $\epsilon/3$. The contribution of each integral in the sum above can be estimated by noticing that both integrals have the form

$$\int_{|\xi|\geq M} \left|(1+|\xi|^2)^{s/2}\hat{t}e^{-\hat{t}\frac{|\xi|}{2}}\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi, \quad \text{for } \hat{t} \in [\frac{\delta'}{2}, 1],$$

where $\hat{t}$ plays the role of $t_1 - \tau$ and $\tilde{t} - \tau$. Indeed, because of the exponential decay in ξ, for every $\epsilon' > 0$, there exists a constant $M > 1$ such that

$$C_{\tilde{r}} \int_{|\xi|\geq M} \left|(1+|\xi|^2)^{s/2}\hat{t}e^{-\hat{t}\frac{|\xi|}{2}}\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi < \frac{\epsilon'}{4}, \quad \forall \hat{t} \in [\frac{\delta'}{2}, 1]. \tag{95}$$

Here, ϵ' will be chosen below in terms of ϵ and $||F||_{L^{\tilde{q}'}_t([0,1];L^{\tilde{r}'}_x(\mathbb{R}^n))}$.

To estimate the integral over $|\xi| \leq M$, we notice that the integrand $e^{-\frac{|\xi|}{2}\hat{t}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\hat{t}\right)}{\frac{\sqrt{3}}{2}|\xi|}$ is uniformly continuous on the compact set $|\xi| \leq M, \delta'/2 \leq \hat{t} \leq 1$. This implies that for every $\epsilon' > 0$, we can choose $\delta > 0$ such that $\delta < \delta'/2$ and such that for all $\tilde{t} \in (t_1 - \delta, t_1 + \delta)$ and $0 \leq \tau \leq t_1 - \delta'$, we have

$$\begin{aligned}
&\int_{|\xi|\leq M} \left|(1+|\xi|^2)^{s/2}(\hat{K}_{t_1-\tau}(\xi) - \hat{K}_{\tilde{t}-\tau}(\xi))\right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi \\
&\leq C_M \int_{|\xi|\leq M} \left| e^{-\frac{|\xi|}{2}(t_1-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t_1-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|} - e^{-\frac{|\xi|}{2}(\tilde{t}-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(\tilde{t}-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|} \right|^{\frac{2\tilde{r}}{\tilde{r}-2}} d\xi < \frac{\epsilon'}{2}.
\end{aligned} \tag{96}$$

By combining inequalities (94) and (96) with (92) and (93), we see that we can choose ϵ' small enough, depending on ϵ and $||F||_{L^{\tilde{q}'}_t([0,1];L^{\tilde{r}'}_x(\mathbb{R}^n))}$, such that for all $\tilde{t} \in (t_1 - \delta, t_1 + \delta)$,

$$0 \leq I^{(2)}_{t_1,\tilde{t}} < \frac{\epsilon}{3}. \tag{97}$$

Continuity in time now follows by combining the estimates (90), (91), and (97). This completes the proof. □

Remark 4.1. We remark that Proposition 4.1 also reflects the parabolic regularizing effects of the fluid viscosity. In fact, once one shows that the solution to the linear viscous wave equation with source term $F \in L^{\tilde{q}'}_t([0,T];L^{\tilde{r}'}_x(\mathbb{R}^n))$ and zero initial data gives rise to a solution that is in

$C^0([0,T];H^s(\mathbb{R}^n))$, one can use energy estimates to establish (87) for a couple of specific exponents in $n=2$. In particular, the case of $s=1$, $\tilde{q}=\infty$, and $\tilde{r}=2$ follows from usual basic energy estimates, while the case of $s=1$, $\tilde{q}=2$, and $\tilde{r}=4$ can be obtained from energy estimates along with the Sobolev embedding of $H^{1/2}(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$. However, the Fourier methods we have used to establish Proposition 4.1 provide a simpler way of covering a much larger range of potential parameters $(s,\tilde{q},\tilde{r})$ than basic energy estimates.

Remark 4.2. We have established the result in Proposition 4.1 for $n=2$. The main place we have used the fact that $n=2$ is in (88), where we need the time integral

$$\int_0^T (T-\tau)^{\tilde{q}\left(1-n\cdot\frac{\tilde{r}-2}{2\tilde{r}}\right)}d\tau$$

to be finite. For $n=2$, the exponent $\tilde{q}\left(1-n\cdot\frac{\tilde{r}-2}{2\tilde{r}}\right)$ is always nonnegative. If one were to attempt to extend the result to higher dimensions, this exponent could potentially be negative, and one would have to impose additional conditions on $\tilde{q}$ and $\tilde{r}$ to ensure that the resulting time integral above is finite.

5 Probabilistic well-posedness for the supercritical quintic nonlinear viscous wave equation

5.1 Description of the randomization and main result

In this section, we will specialize to the case of the quintic nonlinear viscous wave equation. In particular, we consider

$$\partial_{tt}u-\Delta u+\sqrt{-\Delta}\partial_t u+u^5=0 \qquad \text{on } \mathbb{R}^2, \tag{98}$$

with a randomization of the deterministic initial data in $\mathcal{H}^s(\mathbb{R}^2)=H^s\times H^{s-1}$:

$$u(0,x)=f(x)\in H^s, \qquad \partial_t u(0,x)=g(x)\in H^{s-1}.$$

The reason for considering the quintic nonlinearity is that 5 is the first odd exponent in $n=2$ for which we have ill-posedness as described by Theorem 2.1, as it is the first odd exponent for which $s_{cr}=\frac{n}{2}-\frac{2}{p-1}$ for $n=2$ is positive. Moreover, for this nonlinearity, we can use our results from the previous sections to generate Strichartz estimates for the solution space (106) that allow us to handle the quintic nonlinearity, as the Remarks 5.1 and 5.2 below explain.

For this specific case of the quintic viscous wave equation, the critical exponent is $s_{\text{cr}}=1/2$. It is likely that there is well posedness in the strong Hadamard sense for initial data in $\mathcal{H}^s$ for $s>1/2$, but in Sec. 2, we showed that there is a lack of continuity in the solution map for $0<s<1/2$. We will show that this lack of continuity in the solution map is in some sense a non-generic phenomenon by considering a randomization of the initial data, as described below. In particular, we will use what is called the *Wiener randomization*.

To define the randomization, denote by $\psi\in C_0^\infty(\overline{B_{r=2}(0)})$, $0\le\psi\le 1$, a function such that

$$\sum_{k\in\mathbb{Z}^2}\psi(\xi-k)\equiv 1.$$

This function defines a partition of unity created by translates of the same compactly supported function. Then, we note that for a given function f with Fourier transform $\widehat{f}(\xi)$,

$$\widehat{f}(\xi)=\sum_{k\in\mathbb{Z}^2}\psi(\xi-k)\widehat{f}(\xi).$$

For each $k \in \mathbb{Z}^2$ define

$$P_k f := \mathcal{F}^{-1}\left(\psi(\xi - k)\widehat{f}(\xi)\right)$$

to be the inverse Fourier transform of a localized portion of the function in frequency space. Note that for some positive constant $c > 0$,

$$0 < c \leq \sum_{k\in\mathbb{Z}^2} \psi^2(\xi - k) \leq 1,$$

and therefore, the two norms

$$||f||_{H^s(\mathbb{R}^2)} \sim (\sum_{k\in\mathbb{Z}^2} ||P_k f||^2_{H^s(\mathbb{R}^2)})^{1/2} \tag{99}$$

are equivalent.

Now, let $h_k(\omega)$ and $l_k(\omega)$ indexed by $k \in \mathbb{Z}^2$ be independent real random variables with mean zero on a probability space $(\Omega, \mathcal{F}, P)$. Assume that the independent random variables $h_k(\omega)$ and $l_k(\omega)$ have uniformly bounded sixth moments:

$$\int_\Omega (|h_k(\omega)|^6 + |l_k(\omega)|^6) dp(\omega) < C \qquad \text{for all } k \in \mathbb{Z}^2. \tag{100}$$

The uniform boundedness of sixth moments is associated with the quintic nonlinearity. It will provide sufficient regularity for the randomization of the initial data by the real independent random variables with mean zero and uniformly bounded sixth moments to improve the regularity of the *averaged* randomized free evolution solution in expectation, where the averaging is performed in terms of the averaged L^6-norms of the solution. See Lemma 5.3 below.

For each $\omega \in \Omega$, we define the randomized initial data, given by

$$f^\omega = \sum_{k\in\mathbb{Z}^2} h_k(\omega) P_k f, \qquad g^\omega = \sum_{k\in\mathbb{Z}^2} l_k(\omega) P_k g. \tag{101}$$

We introduce the map that associates to each possible outcome $\omega \in \Omega$, the corresponding initial data $\varphi^\omega = (f^\omega, g^\omega) \in \mathcal{H}^s(\mathbb{R}^2)$:

$$\omega \mapsto \varphi^\omega = (f^\omega, g^\omega), \quad \varphi^\omega : \Omega \to \mathcal{H}^s(\mathbb{R}^2). \tag{102}$$

One can show that the map φ^ω is a measurable map from $\Omega \to \mathcal{H}^s(\mathbb{R}^2)$, and $\varphi^\omega \in L^2(\Omega; \mathcal{H}^s(\mathbb{R}^2))$. Thus, φ^ω is a random variable taking values in $\mathcal{H}^s(\mathbb{R}^2)$. In particular, one can show that the following result holds:

Proposition 5.1. Let $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$, and let $\varphi^\omega = (f^\omega, g^\omega)$ be defined as in (101) and (102) via the independent random variables $h_k(\omega)$ and $l_k(\omega)$ with uniformly bounded sixth moments (100). Then

$$||\varphi^\omega||_{L^2(\Omega;\mathcal{H}^s(\mathbb{R}^2))} \leq C||\varphi||_{\mathcal{H}^s}. \tag{103}$$

Namely, φ^ω has almost surely finite $\mathcal{H}^s$ norm.

Proof. Indeed, using (99), we get:

$$\|f^\omega\|^2_{L^2(\Omega;H^s(\mathbb{R}^2))} = \int_\Omega \|f^\omega\|^2_{H^s} dP = \int_\Omega \| \sum_{k\in\mathbb{Z}^2} h_k(\omega) P_k f\|^2_{H^s} dP \leq C \int_\Omega \sum_{j\in\mathbb{Z}^2} \|P_j(\sum_{k\in\mathbb{Z}^2} h_k(\omega) P_k f)\|^2_{H^s} dP.$$

Now, by the support properties of $\psi(\xi)$, we have $P_jP_kf = 0$ when $|j-k| > c$. We use this property to continue the above estimate as follows:

$$= C\int_\Omega \sum_{j\in\mathbb{Z}^2} \| \sum_{|k-j|\le c} h_k(\omega)P_jP_kf\|_{H^s}^2 dP \le C\int_\Omega \sum_{j\in\mathbb{Z}^2} \left(\sum_{|k-j|\le c} ||h_k(\omega)P_jP_kf||_{H^s} \right)^2 dP.$$

Since for each j, the inner sum has finitely many terms with the same number of terms for each j, we can use the following inequality $\left(\sum_{k=1}^N a_k\right)^2 \le C_N \sum_{k=1}^N a_k^2$ for the sum of N terms, to continue the above estimate:

$$\le\ C\int_\Omega \sum_{j\in\mathbb{Z}^2}\sum_{|k-j|\le c} |h_k(\omega)|^2 ||P_jP_kf||_{H^s}^2 dP\ =\ C\sum_{j\in\mathbb{Z}^2}\sum_{|k-j|\le c}\left(\int_\Omega |h_k(\omega)|^2 dP\cdot ||P_jP_kf||_{H^s}^2\right).$$

Now, from the boundedness of the sixth moments (100), we have that the second moments are uniformly bounded, and we use this to continue to estimate the $L^2(\Omega;\mathcal{H}^s(\mathbb{R}^2))$-norm of f^ω as follows:

$$\le\ C'\sum_{j\in\mathbb{Z}^2}\sum_{|k-j|\le c} ||P_jP_kf||_{H^s}^2\ =\ C'\int_{\mathbb{R}^2}\sum_{j\in\mathbb{Z}^2}\sum_{|k-j|\le c}(1+|\xi|^2)^s|\psi(\xi-j)|^2|\psi(\xi-k)|^2|\widehat{f}(\xi)|^2 d\xi.$$

Denote by N be the number of terms in the sum with $|k-j|\le c$, which is the same for all j by translation. Then, since $|\psi(\xi-k)|^2 \le |\psi(\xi-k)| \le 1$, we finally obtain the following estimate for the $L^2(\Omega;\mathcal{H}^s(\mathbb{R}^2))$-norm of f^ω:

$$\le C'N\int_{\mathbb{R}^2}\sum_{j\in\mathbb{Z}^2}(1+|\xi|^2)^s|\psi(\xi-j)||\widehat{f}(\xi)|^2 d\xi = C''\int_{\mathbb{R}^2}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 d\xi = C||f||_{H^s}^2.$$

By repeating the above computation with g and g^ω, one obtains:

$$||\varphi^\omega||_{L^2(\Omega;\mathcal{H}^s(\mathbb{R}^2))} \le C||\varphi||_{\mathcal{H}^s}, \tag{104}$$

where $\varphi = (f,g) \in \mathcal{H}^s$ denotes the initial data before randomization, and $C > 0$ is *independent of* $\varphi = (f,g) \in \mathcal{H}^s$. □

The goal in this section is to prove **probabilistic well-posedness** for the quintic nonlinear viscous wave equation with initial data $\varphi^\omega \in L^2(\Omega;\mathcal{H}^s(\mathbb{R}^2))$ where $-1/6 < s \le 1/2$. We do this in two steps. First we will prove the existence of a unique solution stated in Theorem 5.1 below, and then show in Theorem 5.2 that the solution depends continuously on data in H^s, for $-1/6 < s \le 1/2$.

Theorem 5.1 (**Existence and uniqueness).** Let $-1/6 < s \le 1/2$, and let $\varphi^\omega \in L^2(\Omega;\mathcal{H}^s(\mathbb{R}^2))$ be the Wiener randomization of the initial data $\varphi = (f,g) \in \mathcal{H}^s(\mathbb{R}^2)$. Then, for almost all $\omega \in \Omega$, there exists $T_\omega > 0$ such that there is a unique solution u to the Cauchy problem of the nonlinear quintic viscous wave equation (98) with initial data φ^ω, where the solution belongs to the space

$$Z^\omega_{[0,T_\omega]} := e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)(f^\omega) + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g^\omega)$$
$$+ C^0([0,T_\omega];H^{1/2}(\mathbb{R}^2)) \cap L^6([0,T_\omega];L^6(\mathbb{R}^2)). \tag{105}$$

In particular, there exists $C > 0$ such that for every $0 < T \leq 1$, there is an event Ω_T with

$$P(\Omega_T) \geq 1 - CT^{s+1/6},$$

such that for all $\omega \in \Omega_T$, the Cauchy problem for (98) with initial data φ^ω has a unique solution in the space $Z^\omega_{[0,T]}$.

Theorem 5.2 (Continuous dependence). Let $-1/6 < s \leq 1/2$ and $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$. Then, for any choice of $\epsilon > 0$, $0 < T \leq 1$, and $0 < p < 1$, there exists an event $A_{\varphi,\epsilon}$ and $\delta > 0$ (with $\delta > 0$ depending only on ϵ, T, and p), such that for every $\omega \in A_{\varphi,\epsilon}$ the Cauchy problem for the nonlinear quintic viscous wave equation with initial data φ^ω has a solution $u \in C^0([0,T]; H^s(\mathbb{R}^2))$, which satisfies

$$||u||_{C^0([0,T];H^s(\mathbb{R}^2))} \leq \epsilon \text{ with probability} P(A_{\varphi,\epsilon}) > p, \text{ whenever } ||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta.$$

5.2 The solution space and preliminary estimates

We begin by justifying the choice for the function space (105) in the statement of the existence result in Theorem 5.1. For this purpose, we introduce the following notation:

Definition 5.1. For each $T > 0$, define the function space

$$X_T = C^0([0,T]; H^{1/2}(\mathbb{R}^2)) \cap L^6([0,T]; L^6(\mathbb{R}^2)). \tag{106}$$

Using this notation, the solution space (105) can be written as:

$$\begin{aligned} Z^\omega_{[0,T_\omega]} := e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)(f^\omega) \\ + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g^\omega) + X_{T_\omega}. \end{aligned}$$

To obtain the main existence result in Theorem 5.1, we will use Strichartz estimates to obtain estimates on the $L^q_t L^r_x$ norm of the solution to the *linear* viscous wave equation in terms of the $L^{\tilde{q}'}_t L^{\tilde{r}'}_x$ norm of the "right hand side," which will be the quintic nonlinearity $-u^5$. This will be the basis for a fixed point argument, which will provide existence of a unique solution to the nonlinear problem.

Remark 5.1. The choice of the function space X_T in (106) follows from the Strichartz estimates that we plan to use to obtain the well-posedness result, combined with a fixed point argument. In particular, we will be using the inhomogeneous Strichartz estimates for all $0 < T < \infty$ in Lemma 3.5, and the "local" $C_0([0,T]; H^s(\mathbb{R}^2))$ estimate for all $0 < T \leq 1$ from Proposition 4.1 to estimate the solution to the linear problem, with the nonlinearity $-u^5$ treated as a source term. For the inhomogeneous Strichartz estimates in Lemma 3.5 to hold, we need

$$1 < \tilde{q}' < q < \infty, \quad 1 \leq \tilde{r}' < r \leq \infty, \quad \frac{1}{q} + \frac{n}{r} = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2, \tag{107}$$

and for the "local" $C_0([0,T]; H^s(\mathbb{R}^2))$ estimate with $0 < T \leq 1$ in Proposition 4.1, we need $s \geq 0$, and

$$2 \leq \tilde{q} \leq \infty, \quad 2 \leq \tilde{r} < \infty, \quad \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2. \tag{108}$$

In particular, for $n = 2$, the following exponents satisfy all of the conditions above:

$$(q, r) = (\tilde{q}, \tilde{r}) = (6, 6), \text{ and } s = \frac{1}{2}.$$

More importantly, for this choice, we can indeed use the estimates in Lemma 3.5 to estimate $u \in L^r_x(\mathbb{R}^2) = L^6(\mathbb{R}^2)$ in terms of the $L^{\tilde{r}'}_x$-norm of the source term $-u^5$, since the conjugate exponent $\tilde{r}'$ to the exponent $r = 6$ is $\tilde{r}' = 6/5$, and we know that $u \in L^6(\mathbb{R}^2) \implies u^5 \in L^{6/5}(\mathbb{R}^2)$. Thus, the pair of exponents 6 and $6/5$ is well suited for the nonlinear quintic viscous wave equation.

Remark 5.2. Note that the fact that we are requiring our solutions to be in $L^6([0,T]; L^6(\mathbb{R}^2))$ allows us to interpret solutions of

$$\partial_{tt} u - \Delta u + \sqrt{-\Delta}\partial_t u + u^5 = 0,$$

$$u(0, x) = f(x) \in H^s, \qquad \partial_t u(0, x) = g(x) \in H^{s-1},$$

as weak solutions, defined in the usual way by integrating against a test function $\phi \in C_0^\infty([0,T) \times \mathbb{R}^2)$. If we require solutions u to be in $L^6([0,T]; L^6(\mathbb{R}^2)) = L^6([0,T]\times\mathbb{R}^2)$, then $u^5 \in L^{6/5}([0,T]\times\mathbb{R}^2)$ is indeed a distribution. In particular, it is also a tempered distribution in space, for almost every $t \in [0,T]$ (since it is in $L^{6/5}(\mathbb{R}^2)$ for a.e. $t \in [0,T]$). Therefore, all of the Fourier methods applied to the Fourier representation of the solution used above are applicable.

Strichartz estimate. We will need the following version of the Strichartz estimate:

Corollary 5.1 (Strichartz estimates for solution space X_T)**.** There exists a constant C (independent of T) such that for all $0 < T \leq 1$,

$$\left|\left| \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(\tau,\cdot) d\tau \right|\right|_{X_T} \leq C||F||_{L^{6/5}([0,T]\times\mathbb{R}^2)}.$$

Proof. The proof follows immediately from Lemma 3.5 and Proposition 4.1, where we let $q = r = \tilde{q} = \tilde{r} = 6$ and $s = 1/2$. □

Averaging effects. The following two lemmas will be useful in proving the averaging effects result in Lemma 5.3 below, which concerns the regularity in expectation of the averaged free evolution solution to the linear viscous wave equation for the random initial data φ^ω. The first is a probabilistic lemma, stated in Burq and Tzvetkov [5]:

Lemma 5.1. For $(h_n)_{n=1}^\infty$ independent, mean 0, complex-valued random variables that satisfy a uniform moment bound

$$\int_\Omega |h_n(\omega)|^{2k} dP \leq C \qquad \text{for all } n \geq 1$$

for some positive integer k, there exists a uniform constant C, such that for all $2 \leq p \leq 2k$ and for all $(c_n)_{n=1}^\infty \in \ell^2(\mathbb{C})$,

$$\|\sum_n h_n(\omega) c_n\|_{L^p(\Omega)} \leq C \left(\sum_n |c_n|^2\right)^{1/2} = C\|c_n\|_{\ell^2(\mathbb{C})}.$$

The next result is a unit-scale Bernstein's inequality for the frequency pieces $P_k f$. See Lemma 2.1 in Lührmann and Mendelson [26]:

Lemma 5.2 (Bernstein unit-scale inequality)**.** For all $2 \leq p_1 \leq p_2 \leq \infty$, there exists a constant $C = C(p_1, p_2) > 0$ such that for all $f \in L^2(\mathbb{R}^2)$, $k \in \mathbb{Z}^2$,

$$||P_k f||_{L^{p_2}(\mathbb{R}^2)} \leq C||P_k f||_{L^{p_1}(\mathbb{R}^2)}.$$

The proof is a simple consequence of Young's convolution inequality and the localization in the frequency domain of $P_k f$ [26].

We are now ready to state and prove the averaging lemma. It is well-known that the randomization does not improve the regularity of the initial data almost surely. In particular, if $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$ is not in $\mathcal{H}^{s+\epsilon}(\mathbb{R}^2)$, then φ^ω is a.s. not in $\mathcal{H}^{s+\epsilon}(\mathbb{R}^2)$ (see Lührmann and Mendelson [26], Burq and Tzvetkov [5]). Although the randomization almost surely does not improve the regularity of the initial data, it does improve the averaging of the L^6 norms of the randomized free evolution solution in expectation. More precisely, we have:

Lemma 5.3 (**Averaging Lemma**)**.** Let u_φ^ω be the free evolution associated with the randomization $\varphi^\omega = (f^\omega, g^\omega)$ of the initial data $\varphi = (f, g) \in \mathcal{H}^s(\mathbb{R}^2)$, with $-1/6 < s \leq 1/2$:

$$u_\varphi^\omega(t,x) = e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)(f^\omega) + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g^\omega).$$

Then, there exists a constant $C > 0$ depending only on s, independent of φ, such that for every $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$ and for all $0 < T \leq 1$,

$$||u_\varphi^\omega||_{L^6(\Omega\times[0,T]\times\mathbb{R}^2)} \leq CT^{\frac{1}{6}\left(s+\frac{1}{6}\right)}||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)}.$$

Proof. First, we show the result for the case with zero initial velocity $g = 0$:

$$u_f^\omega(t,x) = e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}f^\omega = \sum_{k\in\mathbb{Z}^2} h_k(\omega)e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}P_k f.$$

We recall that (h_k) are independent real random variables with mean zero on a probability space $(\Omega, \mathcal{F}, P)$, and calculate:

$$||u_f^\omega||_{L^6(\Omega\times[0,T]\times\mathbb{R}^2)} = ||u_f^\omega||_{L^6([0,T]\times\mathbb{R}^2;L^6(\Omega))}.$$

By the bounded sixth moment assumption on (h_k), we can apply Lemma 5.1 with $k = 3$ to obtain

$$\begin{aligned}||u_f^\omega||_{L^6(\Omega\times[0,T]\times\mathbb{R}^2)} &\leq C\left\|\sqrt{\sum_{k\in\mathbb{Z}^2}|e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}P_k f|^2}\right\|_{L^6([0,T]\times\mathbb{R}^2)}\\ &\leq C\left(\sum_{k\in\mathbb{Z}^2}\left\|e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}P_k f\right\|^2_{L^6((0,T];L^6(\mathbb{R}^2))}\right)^{1/2}. \qquad (109)\end{aligned}$$

We now want to apply the unit scale Bernstein inequality of Lemma 5.2, to estimate the L^6-norm in space with the L^2-norm. Since P_k commutes with the other Fourier multipliers that are acting on f, we will apply the unit scale Bernstein inequality to $P_k\left(e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}f\right)$. For this purpose,

as required by Lemma 5.2, we first need to show that $e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}f \in L^2(\mathbb{R}^2)$. This is, indeed, the case, since:

$$||e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}f||^2_{L^2(\mathbb{R}^2)} = ||e^{-\frac{|\xi|}{2}t}e^{i\frac{\sqrt{3}}{2}|\xi|t}\widehat{f}(\xi)||^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{-|\xi|t}|\widehat{f}(\xi)|^2 d\xi$$
$$= \int_{\mathbb{R}^2} e^{-|\xi|t}(1+|\xi|^2)^{-s}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 d\xi \leq C_{s,t}\int_{\mathbb{R}^2}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 d\xi = C_{s,t}||f||^2_{H^s(\mathbb{R}^2)},$$

where we used the fact that for all $s \in \mathbb{R}$, $e^{-|\xi|t}(1+|\xi|^2)^{-s}$ is bounded whenever $t > 0$.

Therefore, we can apply the unit scale Bernstein inequality in (109) to get:

$$C\left(\sum_{k\in\mathbb{Z}^2}\left|\left|e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}P_k f\right|\right|^2_{L^6((0,T];L^6(\mathbb{R}^2))}\right)^{1/2} \leq C\sqrt{\sum_{k\in\mathbb{Z}^2}\left|\left|e^{-\frac{\sqrt{-\Delta}}{2}t}e^{i\frac{\sqrt{3}}{2}\sqrt{-\Delta}t}P_k f\right|\right|^2_{L^6((0,T];L^2(\mathbb{R}^2))}}$$
$$= C\sqrt{\sum_{k\in\mathbb{Z}^2}||(1+|\xi|^2)^{-s/2}e^{-\frac{|\xi|}{2}t}(1+|\xi|^2)^{s/2}\widehat{P_k f}||^2_{L^6((0,T];L^2(\mathbb{R}^2))}}. \tag{110}$$

By Minkowski's integral inequality we can "change the order of integration" in (110) to obtain that (110) is bounded from above by:

$$\leq C\left(\sum_{k\in\mathbb{Z}^2}||(1+|\xi|^2)^{-s/2}e^{-\frac{|\xi|}{2}t}(1+|\xi|^2)^{s/2}\widehat{P_k f}||^2_{L^2(\mathbb{R}^2;L^6((0,T]))}\right)^{1/2}, \tag{111}$$

and we estimate the contribution in time of the exponential $e^{-\frac{|\xi|}{2}t}$ as follows:

$$||e^{-\frac{|\xi|}{2}t}||_{L^6((0,T])} = \left(\frac{1}{3|\xi|}(1-e^{-3|\xi|T})\right)^{1/6} = \frac{1}{(3|\xi|)^{-s}}\cdot\frac{(1-e^{-3|\xi|T})^{1/6}}{(3|\xi|)^{s+1/6}}.$$

Since we can bound the exponential $1-e^{-3x}$ in terms of monomial x^α, for any $\alpha \in [0,1]$, so that

$$1-e^{-3x} \leq C_\alpha x^\alpha,\ \forall x \geq 0,$$

we conclude that there exists a constant $C_s > 0$, such that

$$||e^{-\frac{|\xi|}{2}t}||_{L^6((0,T])} \leq C_s|\xi|^s|\xi|^{-s-\frac{1}{6}}(|\xi|T)^{\frac{1}{6}(s+\frac{1}{6})} = C_s|\xi|^s|\xi|^{-\delta(s)}T^{\frac{1}{6}\left(s+\frac{1}{6}\right)},$$

where we chose $0 < \alpha = s + 1/6 \leq 1$ and $\delta(s) = \frac{5}{6}\left(s+\frac{1}{6}\right) > 0$.

We now continue to estimate (111) by separating the sum into the low frequency part $|k| \leq 4$, and the high frequency part $|k| > 4$. For the low frequency estimate, we note that

$$||e^{-\frac{|\xi|}{2}t}||_{L^6((0,T])} \leq T^{1/6}.$$

Moreover, in the $|k| \leq 4$ part, we can get rid of $(1+|\xi|^2)^{-s/2}$ using the support properties of $\widehat{P_k f}$ for $|k| \leq 4$ to obtain that (111) can be bounded from above by:

$$\leq \Bigg[C\sum_{|k|\leq 4}||T^{1/6}(1+|\xi|^2)^{s/2}\widehat{P_k f}||^2_{L^2(\mathbb{R}^2)}$$
$$+ C\left(\sum_{|k|>4}||(1+|\xi|^2)^{-s/2}|\xi|^s\cdot|\xi|^{-\delta(s)}\cdot T^{\frac{1}{6}\left(s+\frac{1}{6}\right)}\cdot(1+|\xi|^2)^{s/2}\widehat{P_k f}||^2_{L^2(\mathbb{R}^2)}\right)\Bigg]^{1/2}.$$

In the $|k|>4$ part, we have $|(1+|\xi|^2)^{-s/2}|\xi|^s| \leq C$ and $|\xi|^{-\delta(s)} \leq 1$ for all ξ in the support of $\widehat{P_k f}$. Therefore, we can further continue the estimate as follows:

$$\leq \left[CT^{1/3} \sum_{|k|\leq 4} ||(1+|\xi|^2)^{s/2} \widehat{P_k f}||^2_{L^2(\mathbb{R}^2)} + CT^{\frac{1}{3}\left(s+\frac{1}{6}\right)} \sum_{|k|>4} ||(1+|\xi|^2)^{s/2} \widehat{P_k f}||^2_{L^2(\mathbb{R}^2)} \right]^{1/2}.$$

Now, since $0 < 1/3(s+1/6) \leq 1/3$ for $-1/6 < s < 1/2$, and $0 < T \leq 1$ we get that the above expression is bounded from above by:

$$\leq \left(CT^{\frac{1}{3}\left(s+\frac{1}{6}\right)} \sum_{k\in\mathbb{Z}^2} ||(1+|\xi|^2)^{s/2} \widehat{P_k f}||^2_{L^2(\mathbb{R}^2)} \right)^{1/2} \leq CT^{\frac{1}{6}\left(s+\frac{1}{6}\right)} ||f||_{H^s(\mathbb{R}^2)},$$

where we used (99) in the last step. Thus, we have shown that

$$||u^\omega_f||_{L^6(\Omega\times[0,T]\times\mathbb{R}^2)} \leq CT^{\frac{1}{6}\left(s+\frac{1}{6}\right)} ||f||_{H^s(\mathbb{R}^2)},$$

where $C>0$ depends only on s.

For the term involving g, the same proof still holds, with slight modifications. In particular, instead of using the simple bound $\left| e^{i\frac{\sqrt{3}}{2}|\xi|t} \right| \leq 1$, we have to use the bound

$$\left| \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|} \right| \leq 1_{\{|\xi|\leq 1\}}(\xi) + \frac{1}{\frac{\sqrt{3}}{2}|\xi|} 1_{\{|\xi|>1\}}(\xi),$$

which holds for all $0 \leq t \leq T \leq 1$. This will give the low and high frequency estimate in the same way. Therefore, we have shown that there exists a constant $C>0$, depending only on s, such that

$$||u^\omega_\varphi||_{L^6(\Omega\times[0,T]\times\mathbb{R}^2)} \leq CT^{\frac{1}{6}\left(s+\frac{1}{6}\right)} ||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)}.$$

□

Crucial for the probabilistic well-posedness is the following corollary of the averaging lemma:

Corollary 5.2. Let u^ω_φ be the free evolution associated with the randomization $\varphi^\omega = (f^\omega, g^\omega)$ of the initial data $\varphi = (f,g) \in \mathcal{H}^s(\mathbb{R}^2)$, with $-1/6 < s \leq 1/2$ and $0 < T \leq 1$:

$$u^\omega_\varphi(t,x) = e^{-\frac{\sqrt{-\Delta}}{2}t} \left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) \right)(f^\omega) + e^{-\frac{\sqrt{-\Delta}}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g^\omega).$$

Define the set

$$E_{\lambda,T,\varphi} = \{\omega \in \Omega \mid ||u^\omega_\varphi||_{L^6([0,T]\times\mathbb{R}^2)} \geq \lambda\}. \tag{112}$$

Then,

$$P(E_{\lambda,T,\varphi}) \leq CT^{s+\frac{1}{6}} \lambda^{-6} ||\varphi||^6_{\mathcal{H}^s(\mathbb{R}^2)},$$

where the constant C depends only on s.

The proof is a direct consequence of Chebychev's inequality and Lemma 5.3.

Notice how, because we are using probabilistic methods, Chebyshev's inequality associates the size of the solution in the L^6-norm to the size of the initial data in the H^s-norm, with the probability of such a bound given in terms of the size of the initial data.

5.3 Proof of probabilistic well-posedness for $-1/6 < s \leq 1/2$

We are now ready to prove the probabilistic well-posedness results stated in Theorems 5.1 and 5.2.

5.3.1 Proof of Theorem 5.1 on existence of a unique solution

We want to prove that for $-1/6 < s \leq 1/2$, and for almost all $\omega \in \Omega$, there exists $T_\omega > 0$ such that there is a unique solution u to the Cauchy problem of the nonlinear quintic viscous wave equation (98) with initial data $\varphi^\omega \in L^2(\Omega; \mathcal{H}^s(\mathbb{R}^2))$, where the solution belongs to the space

$$Z^\omega_{[0,T_\omega]} := e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)(f^\omega) + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g^\omega)$$
$$+ C^0([0,T_\omega]; H^{1/2}(\mathbb{R}^2)) \cap L^6([0,T_\omega]; L^6(\mathbb{R}^2)).$$

In particular, we need to show that there exists $C > 0$ such that for every $0 < T \leq 1$, there is an event Ω_T with

$$P(\Omega_T) \geq 1 - CT^{s+1/6}, \tag{113}$$

such that for all $\omega \in \Omega_T$ and initial data φ^ω, the Cauchy problem for (98) has a unique solution in the space $Z^\omega_{[0,T]}$.

Proof. To prove this theorem we use a fixed point argument. We look for a solution to

$$(\partial_{tt} + \sqrt{-\Delta}\partial_t - \Delta)u + u^5 = 0, \qquad u(0) = f^\omega, \qquad \partial_t u(0) = g^\omega,$$

by expressing the solution u as the sum of the free evolution u^ω_φ, satisfying

$$(\partial_{tt} + \sqrt{-\Delta}\partial_t - \Delta)u^\omega_\varphi = 0, \qquad u^\omega_\varphi(0) = f^\omega, \qquad \partial_t u^\omega_\varphi(0) = g^\omega,$$

and the function $v := u - u^\omega_\varphi$, which satisfies:

$$(\partial_{tt} + \sqrt{-\Delta}\partial_t - \Delta)v = -(u^\omega_\varphi + v)^5, \qquad v(0) = 0, \qquad \partial_t v(0) = 0.$$

A solution to this equation on $[0,T]$ is a fixed point of the map

$$K^\omega_\varphi : v(t,\cdot) \Longrightarrow -\int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}((u^\omega_\varphi + v)^5)(\tau,\cdot)d\tau, \tag{114}$$

defined on X_T, where this map depends on ω. Therefore, if we can show that the map K^ω_φ has a fixed point $v^* \in X_T$, the solution u of the nonlinear quintic viscous wave equation will be given by $u = u^\omega_\varphi + v^*$, where u^ω_φ is the free evolution. More precisely, we need to show that for almost all $\omega \in \Omega$, there exists $T_\omega > 0$, such that K^ω_φ has a fixed point $v^* \in X_{T_\omega}$.

We will show this in three steps:

1. First we will show that there exists $\lambda = \lambda_0 > 0$ such that if ω is in the complement of $E_{\lambda_0,T,\varphi}$, i.e., $\omega \in E^c_{\lambda_0,T,\varphi}$, where $E_{\lambda,T,\varphi}$ is defined by (112) in Corollary 5.2, then the mapping K^ω_φ is a strict contraction on an appropriate closed subset of X_T, for arbitrary $0 < T \leq 1$.

2. Then, we will show that for almost all $\omega \in \Omega$, there exists a time $T_\omega > 0$, such that $\omega \in E^c_{\lambda_0,T_\omega,\varphi}$.

Steps 1 and 2 will imply that for almost all $\omega \in \Omega$, there exists $T_\omega > 0$, such that $u = u_\varphi^\omega + v^* \in Z^\omega_{[0,T_\omega]}$ is a solution to the quintic nonlinear viscous wave equation, where v^* is the fixed point of K_φ^ω. The third step is as follows:

3. Finally, we show that this solution $u = u_\varphi^\omega + v^* \in Z^\omega_{[0,T_\omega]}$ is unique in $Z^\omega_{[0,T_\omega]}$.

We start by taking ω in the complement of $E_{\lambda,T,\varphi}$, i.e., $\omega \in E^c_{\lambda,T,\varphi}$, where $E_{\lambda,T,\varphi}$ is defined by (112) in Corollary 5.2. Namely, ω is an outcome for which the initial data φ^ω has a free evolution solution with bounded $L^6([0,T]\times\mathbb{R}^2)$ norm:

$$||u_\varphi^\omega||_{L^6([0,T]\times\mathbb{R}^2)} \le \lambda, \quad \text{with} \quad P(E^c_{\lambda,T,\varphi}) \ge 1 - \tilde{C}T^{s+\frac{1}{6}}\lambda^{-6}||\varphi||^6_{\mathcal{H}^s(\mathbb{R}^2)}.$$

This holds for any $0 < T \le 1$.

For this fixed ω, we want to find $\lambda = \lambda_0 > 0$, such that the mapping K_φ^ω is a contraction. We start by estimating $K_\varphi^\omega(v)$. By using the Strichartz estimate in Corollary 5.1, for each $\omega \in E^c_{\lambda,T,\varphi}$, we have

$$\begin{aligned} ||K_\varphi^\omega(v)||_{X_T} &\le C||(u_\varphi^\omega + v)^5||_{L^{6/5}([0,T]\times\mathbb{R}^2)} \\ &\le C(||u_\varphi^\omega||^5_{L^6([0,T]\times\mathbb{R}^2)} + ||v||^5_{L^6([0,T]\times\mathbb{R}^2)}) \le C(\lambda^5 + ||v||^5_{X_T}). \end{aligned} \tag{115}$$

Similarly, one can show that for each $\omega \in E^c_{\lambda,T,\varphi}$, the following estimate holds:

$$\begin{aligned} ||K_\varphi^\omega(v) - K_\varphi^\omega(w)||_{X_T} &\le C||(u_\varphi^\omega + v)^5 - (u_\varphi^\omega + w)^5||_{L^{6/5}([0,T]\times\mathbb{R}^2)} \\ &\le C||v-w||_{X_T}(\lambda^4 + ||v||^4_{X_T} + ||w||^4_{X_T}). \end{aligned} \tag{116}$$

Notice that because the constant in Corollary 5.1 is independent of $0 < T \le 1$ and F, the constant C in the above two inequalities is also independent of $0 < T \le 1$ and $\omega \in E^c_{\lambda,T,\varphi}$.

Therefore, for $\omega \in E^c_{\lambda,T,\varphi}$, for any $0 < T \le 1$, the map K_φ^ω is a strict contraction on the ball of radius $2C\lambda^5$ in X_T provided that the following two conditions on λ hold:

1. The mapping K_φ^ω maps the ball of radius $2C\lambda^5$ in X_T into the same ball of radius $2C\lambda^5$ in X_T; from (115), this means that we need λ to satisfy:

$$C\lambda^5 + 2^5C^6\lambda^{25} < 2C\lambda^5; \tag{117}$$

2. The coefficient multiplying $||v-w||_{X_T}$ in (116) is strictly less than one:

$$C\lambda^4 + 2^5C^5\lambda^{20} < \frac{1}{2}. \tag{118}$$

One can see that this will be guaranteed if $\lambda^{20} << 1$.

Thus, there exists a $\lambda_0 > 0$ ($\lambda_0^{20} << 1$) such that whenever we choose an outcome ω so that the free evolution u_φ^ω associated with initial data φ^ω lies within the ball of radius λ_0 in $L^6([0,T]\times\mathbb{R}^2)$, the mapping K_φ^ω, defined on the closed ball of radius $2C\lambda_0^5$ in X_T, is a strict contraction (where C is the constant from (115) and (116), which is independent of $0 < T \le 1$). So there exists a fixed point v^* of K_φ^ω in the closed ball of radius $2C\lambda_0^5$ in X_T for every $\omega \in E^c_{\lambda_0,T,\varphi}$.

Notice that with this argument, we have shown the last part of the statement in the theorem, which is that there exists $C' > 0$ such that for every $0 < T \le 1$, there exists an event $\Omega_T := E^c_{\lambda_0,T,\varphi}$ with

$$P(E^c_{\lambda_0,T,\varphi}) \ge 1 - \tilde{C}\lambda_0^{-6}||\varphi||^6_{\mathcal{H}^s(\mathbb{R}^2)}T^{s+1/6} = 1 - C'T^{s+1/6}, \tag{119}$$

such that for all $\omega \in \Omega_T$ the existence of a solution in $Z^{\omega}_{[0,T]}$ holds, where $\tilde{C}$ is the constant from Corollary 5.2, and $C' = \tilde{C}\lambda_0^{-6}||\varphi||^6_{\mathcal{H}^s(\mathbb{R}^2)}$. By the fixed point argument above, for each $\omega \in \Omega_T := E^c_{\lambda_0,T,\varphi}$, the solution $u = u^{\omega}_{\varphi} + v^*$ that we constructed is unique within the class of solutions of the form $u = u^{\omega}_{\varphi} + v$ such that

$$||v||_{X_T} \leq 2C\lambda_0^5, \tag{120}$$

where C, the constant from (115) and (116), is independent of $0 < T \leq 1$.

We need to show more specifically that the solution $u = u^{\omega}_{\varphi} + v^*$ that we constructed is unique in the space of solutions $u^{\omega}_f + v$, where v more generally is any element in X_T and not just an element of X_T subject to condition (120). To see this, consider $\omega \in \Omega_T := E^c_{\lambda_0,T,\varphi}$ for some $0 < T \leq 1$. In particular,

$$||u^{\omega}_{\varphi}||_{L^6([0,T]\times\mathbb{R}^2)} \leq \lambda_0. \tag{121}$$

We know that there exists a solution $u_1 := u^{\omega}_{\varphi} + v_1^*$ to (98) with initial data φ^{ω}, where

$$||v_1^*||_{X_T} \leq 2C\lambda_0^5, \tag{122}$$

by our previous fixed point argument.

Assume for contradiction that there is another solution $u_2 = u^{\omega}_{\varphi} + v_2^*$ to (98) with initial data φ^{ω}, where $v_2^* \in X_T$ is different from v_1^*. Since both v_1^* and v_2^* are continuous in $H^{1/2}$ on $[0,T]$, we can define

$$T_* = \max\{t \in [0,T] : v_1^* = v_2^* \text{ on } [0,t] \text{ as functions in } H^{1/2}(\mathbb{R}^2)\}. \tag{123}$$

We use maximum instead of supremum in the definition (123) due to continuity in $H^{1/2}$. Because any two solutions to (98) with initial data φ^{ω} for our given ω are unique as long as the condition (120) holds, by continuity of the norms involved and the fact that v_1^* and v_2^* both have zero initial data, $T_* > 0$. Furthermore, $T_* < T$ by assumption.

Because v_1^* and v_2^* in X_T are different and the v component of any solution $u := u^{\omega}_{\varphi} + v$ to (98) with initial data φ^{ω} is unique up to condition (120), we conclude that

$$||v_2^*||_{X_t} > 2C\lambda_0^5 \qquad \text{for all } t > T_*, \tag{124}$$

since v_1^* satisfies (122). Furthermore, since v_1^* and v_2^* agree in $H^{1/2}(\mathbb{R}^2)$ up to time T_*,

$$||v_2^*||_{X_t} \leq 2C\lambda_0^5 \qquad \text{for all } t \leq T_* \tag{125}$$

by (122).

We will derive a contradiction by showing that v_1^* and v_2^* must agree past T_* to at least $T_* + \epsilon$, for a suitably small $\epsilon > 0$. To do this, we observe that the conditions (117) and (118) describing the choice of λ_0 are "open" conditions. More precisely, there exists $\tilde{\lambda}_0 > \lambda_0$ such that $\tilde{\lambda}_0$ also satisfies (117) and (118). As a result, the map K^{ω}_{φ} defined in (114) is a strict contraction on the ball of radius $2C\tilde{\lambda}_0^5$ in X_t for $\omega \in E^c_{\tilde{\lambda}_0,t,\varphi}$ for any $0 < t \leq 1$.

We now observe the following crucial fact: since $\tilde{\lambda}_0 > \lambda_0$, we have that for our $\omega \in \Omega_T$,

$$\omega \in \Omega_T := E^c_{\lambda_0,T,\varphi} \subset E^c_{\tilde{\lambda}_0,T,\varphi} \subset E^c_{\tilde{\lambda}_0,t,\varphi} \qquad \text{for all } 0 < t \leq T.$$

Hence, for our arbitrary $\omega \in \Omega_T$, K^{ω}_{φ} is a strict contraction on the ball of radius $2C\tilde{\lambda}_0^5$ in X_t for any $0 < t \leq T$. Since $2C\tilde{\lambda}_0^5 > 2C\lambda_0^5$, by (125) and continuity of the norm, there exists $\epsilon > 0$ such that $T_* + \epsilon < T$ and

$$||v_2^*||_{X_{T_*+\epsilon}} \leq 2C\tilde{\lambda}_0^5.$$

Since $||v_1^*||_{X_{T_*+\epsilon}} \leq 2C\tilde{\lambda}_0^5$ by (122), v_1^* and v_2^* are two different fixed points of the strict contraction K_φ^ω on the ball of radius $2C\tilde{\lambda}_0^5$ in $X_{T_*+\epsilon}$. Here, we used the crucial fact that $T_* + \epsilon < T$. This gives the desired contradiction, which proves that the solution to (98) with initial data φ^ω is unique within the space $Z^\omega_{[0,T]}$ for $\omega \in \Omega_T$.

We now show the second step of the proof, namely, that for *almost all* $\omega \in \Omega$, there exists $T_\omega > 0$, such that the existence of a unique solution $u \in Z^\omega_{[0,T_\omega]}$ holds. This follows from the fact that for $T = 1/n$, $n \geq 1$, the sets $E^c_{\lambda_0, T=1/n, \varphi}$ increase to a probability 1 subset of Ω as $n \to \infty$, since $P(E^c_{\lambda_0,T=1/n,\varphi}) \nearrow 1$ by (113). Therefore, for almost all $\omega \in \Omega$, we take $T_\omega = 1/n$ to be determined by the first n for which $E^c_{\lambda_0,T=1/n,\varphi}$ includes ω, and then we obtain existence of a unique solution $u \in Z^\omega_{[0,T_\omega]}$ from the first step of the proof.

This completes the proof. □

We are now in a position to prove the result on continuous dependence on H^s data, stated in Theorem 5.2. The two results, Theorem 5.1 and Theorem 5.2, imply probabilistic well-posedness for $-1/6 < s \leq 1/2$. This is an improvement over deterministic well-posedness, as the Cauchy problem for (98) is ill-posed for $0 < s < s_{cr} = 1/2$. The results in these two theorems show that the critical exponent for probabilistic well-posedness is pushed all the way down to $s_{cr}^{prob} = -1/6$, excluding $-1/6$.

5.3.2 Proof of Theorem 5.2 on continuous dependence on H^s data

We want to prove that for any fixed $-1/6 < s \leq 1/2$ and $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$, and for any choice of $\epsilon > 0$, $0 < T \leq 1$, and $0 < p < 1$, there exists an event $A_{\varphi,\epsilon}$ and $\delta > 0$, such that for any $\omega \in A_{\varphi,\epsilon}$ the Cauchy problem for the nonlinear quintic viscous wave equation with initial data φ^ω has a solution $u \in C^0([0,T]; H^s(\mathbb{R}^2))$ which satisfies

$$||u||_{C^0([0,T];H^s(\mathbb{R}^2))} \leq \epsilon,$$

with the probability of the event $A_{\varphi,\epsilon}$ being greater than p whenever

$$||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta.$$

Proof. Fix $-1/6 < s \leq 1/2$ and take $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$. We want to show that for every $\epsilon > 0$, $0 < T \leq 1$, and $0 < p < 1$, we can construct an event $A_{\varphi,\epsilon}$ and find $\delta > 0$, such that the nonlinear quintic viscous wave equation (98) with initial data φ^ω, where $\omega \in A_{\varphi,\epsilon}$, has a solution which satisfies $||u||_{C^0([0,T];H^s(\mathbb{R}^2))} \leq \epsilon$, with probability $P(A_{\varphi,\epsilon}) > p$, whenever $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta$.

For this purpose, we recall that the probabilistic solution u can be written as the sum of the free evolution u_φ^ω and the inhomogeneous part v^*. Therefore, the $C^0([0,T]; H^s(\mathbb{R}^2))$-norm of u is bounded from above by the $C^0([0,T]; H^s(\mathbb{R}^2))$-norm of u_φ^ω, plus the $C^0([0,T]; H^s(\mathbb{R}^2))$-norm of v^*. We would like to find $A_{\varphi,\epsilon}$ and δ so that the $C^0([0,T]; H^s(\mathbb{R}^2))$-norms of u_φ^ω and v^* are each bounded by $\epsilon/2$, with the probability of this happening being greater than p, whenever $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta$.

We start with the inhomogeneous part v^*. Recall that Theorem 5.1 implies that there exists λ_0 small enough such that for every $\omega \in E^c_{\lambda_0,T,\varphi}$, K_φ^ω is a strict contraction on the ball of radius $2C\lambda_0^5$ in X_T, where $C > 0$ is the constant from inequalities (115) and (116). Note that the constant $C > 0$ is independent of $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$. We can now choose λ_0 so small that it also satisfies:

$$2C\lambda_0^5 < \frac{\epsilon}{2}. \tag{126}$$

With this choice of λ_0, which depends on ϵ, we have that for $\omega \in E^c_{\lambda_0,T,\varphi}$, there exists a solution $u = u_\varphi^\omega + v^*$ for which the X_T-norm of v^* is bounded from above by $\epsilon/2$. Moreover, the probability

of this happening can be estimated from (119). Namely, (119) implies that there exists a $\tilde{C} > 0$ such that for the λ_0 above, and $0 < T \leq 1$,

$$P(E^c_{\lambda_0,T,\varphi}) \geq 1 - \tilde{C}\lambda_0^{-6}||\varphi||^6_{\mathcal{H}^s(\mathbb{R}^2)}T^{s+1/6} > 1 - \tilde{C}\lambda_0^{-6}\delta^6 T^{s+1/6}, \tag{127}$$

where we have used the assumption that $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta$. This will be used a bit later to determine $\delta > 0$ such that the probability of the event $A_{\varphi,\epsilon}$, which will be a subset of $E^c_{\lambda_0,T,\varphi}$, is greater than p, as required by the theorem.

Next, we find conditions under which the $C^0([0,T];H^s(\mathbb{R}^2))$ norm of the free evolution part of the solution can be made less than $\epsilon/2$. We claim that the free evolution part of the solution with arbitrary initial data $\varphi = (f,g)$ in $\mathcal{H}^s(\mathbb{R}^2)$, can be estimated as follows:

Proposition 5.2. Let $\varphi = (f,g) \in \mathcal{H}^s(\mathbb{R}^2)$. Then, the following estimate on the free evolution associated with φ holds:

$$\left|\left| e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)(f) \right.\right.$$
$$\left.\left. + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g)\right|\right|_{C^0([0,T];H^s(\mathbb{R}^2))} \leq C_{\text{free}}||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)},$$

where C_{free} is independent of $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$, and depends only on s.

We will prove this result after we finish the proof of the main result.

Proposition 5.2 implies that for each fixed ω, we have:

$$\|u^\omega_\varphi\|_{C^0([0,T];H^s(\mathbb{R}^2))} = \left|\left| e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)(f^\omega) \right.\right.$$
$$\left.\left. + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g^\omega)\right|\right|_{C^0([0,T];H^s(\mathbb{R}^2))} \leq C_{\text{free}}||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)}. \tag{128}$$

To find the conditions under which $\|u^\omega_\varphi\|_{C^0([0,T];H^s(\mathbb{R}^2))} \leq \epsilon/2$ whenever $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)}$ is small (less than δ), we need to associate the smallness of $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)}$ with the smallness of $||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)}$ in probability. Indeed, we recall from Proposition 5.1 and (104) that for any $\varphi \in \mathcal{H}^s(\mathbb{R}^2)$, the $L^2(\Omega;\mathcal{H}^s(\mathbb{R}^2))$ norm of φ^ω is bounded by a constant times the $\mathcal{H}^s$ norm of φ for our given randomization satisfying the conditions of Proposition 5.1. Moreover, one can estimate the probability that the $\mathcal{H}^s$ norm of φ^ω is smaller than a given value $\alpha > 0$ in terms of α and the size of the $\mathcal{H}^s$ norm of φ as follows:

$$P(||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} < \alpha) > 1 - C^2_\omega\alpha^{-2}\delta^2, \tag{129}$$

whenever $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta$. Estimate (129) is a direct consequence of Chebyshev's inequality:

$$P(||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} \geq \alpha) \leq C^2_\omega\alpha^{-2}||\varphi||^2_{\mathcal{H}^s(\mathbb{R}^2)} < C^2_\omega\alpha^{-2}\delta^2.$$

Therefore, given $\epsilon > 0$, we see from (128) that there exists $\alpha_0 > 0$ depending on ϵ, where α_0 measures the size of $||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)}$, such that

$$\|u^\omega_\varphi\|_{C^0([0,T];H^s(\mathbb{R}^2))} \leq \epsilon/2,$$

whenever $||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} < \alpha_0$, where the probability of the event that $||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} < \alpha_0$, is bounded from below by $1 - C_\omega^2 \alpha_0^{-2} \delta^2$ whenever $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta$, as specified in (129). This will be used to determine $\delta > 0$, depending on ϵ and p, such that the probability of the event $A_{\varphi,\epsilon}$, which will be a subset of the event $\{||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} < \alpha_0\}$, is greater than p.

We are now in a position where we can combine these two steps into one. We **define our event** $A_{\varphi,\epsilon}$ to be the intersection of the events specified by (127) and (129):

$$A_{\varphi,\epsilon} := \{||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} < \alpha_0(\epsilon)\} \cap E^c_{\lambda_0(\epsilon),T,\varphi} = \left[\{||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} \geq \alpha_0(\epsilon)\} \cup E_{\lambda_0(\epsilon),T,\varphi}\right]^c, \tag{130}$$

and **choose** $\delta > 0$ so that the probability of this event is greater than p:

$$P(A_{\varphi,\epsilon}) > p.$$

More precisely, given $\epsilon > 0$, $0 < T \leq 1$, and $0 < p < 1$, there exist $\lambda_0(\epsilon)$ and $\alpha_0(\epsilon)$ that define the event

$$A_{\varphi,\epsilon} := \{||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} < \alpha_0(\epsilon)\} \cap E^c_{\lambda_0(\epsilon),T,\varphi},$$

such that for every $\omega \in A_{\varphi,\epsilon}$ there exists a solution u associated with φ^ω, such that

$$\|u\|_{C^0([0,T];H^s(\mathbb{R}^2))} \leq \epsilon.$$

The probability of this event is associated with the size of the initial data $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)}$, and so we can choose $\delta > 0$, depending on ϵ and p, so that

$$P(A_{\varphi,\epsilon}) > p$$

whenever

$$||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta.$$

The choice of δ that guarantees $P(A_{\varphi,\epsilon}) > p$ whenever $||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)} < \delta$ is obtained from (129) and (127), and the calculation

$$\begin{aligned} P(A_{\varphi,\epsilon}) &> P\left(||\varphi^\omega||_{\mathcal{H}^s(\mathbb{R}^2)} < \alpha_0(\epsilon)\right) + P(E^c_{\lambda_0(\epsilon),T,\varphi}) - 1 \\ &> 1 - C_\omega^2 \alpha_0^{-2} \delta^2 + 1 - \tilde{C} \lambda_0^{-6} \delta^6 T^{s+1/6} - 1 > p. \end{aligned}$$

This concludes the proof of the main part of the theorem.

What remains is to prove Proposition 5.2. Namely, we want to obtain the following estimate:

$$\begin{aligned} \Bigg\| e^{-\frac{\sqrt{-\Delta}}{2}t} \left(\cos\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)(f) \\ + e^{-\frac{\sqrt{-\Delta}}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g) \Bigg\|_{C^0([0,T];H^s(\mathbb{R}^2))} \leq C_{\text{free}} ||\varphi||_{\mathcal{H}^s(\mathbb{R}^2)}. \end{aligned}$$

This inequality follows from the fact that for $0 \leq t \leq T \leq 1$, we have

$$\left| e^{-\frac{|\xi|}{2}t} \left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right)\right| \leq 2,$$

and from the following low and high frequency estimate:

$$\left\| e^{-\frac{\sqrt{-\Delta}}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}(g) \right\|^2_{H^s(\mathbb{R}^2)} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-|\xi|t}\left(\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\right)^2 |\widehat{g}(\xi)|^2(1+|\xi|^2)^s d\xi$$

$$\leq \frac{1}{(2\pi)^2}\left(\int_{|\xi|\leq 1} + \int_{|\xi|\geq 1} e^{-|\xi|t}\left(\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}\right)^2 |\widehat{g}(\xi)|^2(1+|\xi|^2)^s d\xi\right)$$

$$\leq \frac{1}{(2\pi)^2}\left(\int_{|\xi|\leq 1} 2t^2|\widehat{g}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi + \int_{|\xi|\geq 1}\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1+|\xi|^2}{|\xi|^2}\right)|\widehat{g}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi\right)$$

$$\leq C_s\left(\int_{|\xi|\leq 1}(1+|\xi|^2)^{s-1}|\widehat{g}(\xi)|^2 d\xi + \int_{|\xi|\geq 1}(1+|\xi|^2)^{s-1}|\widehat{g}(\xi)|^2 d\xi\right) = C_{\text{free}}||g||^2_{H^{s-1}(\mathbb{R}^2)},$$

where in the last step, we used that $0 < t \leq T \leq 1$ and for $|\xi| \geq 1$, we have $\frac{1+|\xi|^2}{|\xi|^2} \leq 2$.

Continuity in time with respect to the $H^s(\mathbb{R}^2)$ norm follows similarly from uniform continuity and a similar low and high frequency estimate. □

With this proof, we conclude the section in which we have shown probabilistic well-posedness for the supercritical quintic viscous wave equation, which holds for the initial data in H^s, where $-1/6 < s \leq s_{cr} = 1/2$.

Remark 5.3. For concreteness, we handled the case of $p = 5$ corresponding to the nonlinear quintic viscous wave equation, since this is the smallest positive odd integer p for which we get deterministic ill-posedness as described in Theorem 2.1. However, one can extend these results to encompass general power nonlinearities with p being a positive odd integer greater than or equal to five, for the equation

$$\partial_{tt}u - \Delta u + \sqrt{-\Delta}\partial_t u + u^p = 0 \qquad \text{on } \mathbb{R}^2.$$

In this case, the solution space X_T as defined in (106) would change to

$$X_T = C^0([0,T]; H^{1-\frac{2}{p-1}}) \cap L^{\frac{3}{2}(p-1)}([0,T]; L^{\frac{3}{2}(p-1)}(\mathbb{R}^2)).$$

This is because we must find (q,r), $(\tilde{q},\tilde{r})$, and $s \geq 0$ satisfying the various conditions in (107) and (108). As described in Remark 5.1, for the Strichartz estimate to work well with the power nonlinearity, we would want $\tilde{q}'$ and $\tilde{r}'$ to be p times q and p times r respectively. If we set $q = r$ for simplicity, this forces us to choose $q, r = \frac{3}{2}(p-1)$. Then, the condition on s in (108) forces us to choose $s = 1 - \frac{2}{p-1}$, which we note is exactly equal to s_{cr} for $n = 2$. One can carry out the remaining arguments in the section with very minor modifications to conclude a similar probabilistic well-posedness result like that of Theorem 5.1 for the exponents $-\frac{2}{3(p-1)} < s \leq 1 - \frac{2}{p-1}$.

6 Appendix

In the appendix, we provide the proof of the local existence result that was used in the proof of deterministic ill-posedness for $0 < s < s_{cr}$ in Sec. 2. Specifically, we recall that this proof relied crucially on the result of Proposition 2.1, which states that the solution $\phi(t,x)$ to the initial value

problem for (25) is close to the solution $\phi^{(0)}(t,x)$ to the initial value problem for the visco-dispersive limit (26) in $H^k(\mathbb{R}^n)$ norm for all visco-dispersive parameters ν sufficiently close to 0, and for a given range of times t. In the proof of Proposition 2.1, we did not explicitly justify why the initial value problem in (28) indeed has a solution that exists and is unique for the times for which we perform our analysis. This is what we establish in this appendix in the following lemma.

Note on notation. In what follows, we use multi-indices α in differentiation operators ∂_x^α to represent differentiations with respect to spatial variables and explicitly write out ∂_t for any differentiations with respect to time. In particular, *multi-indices α will never be used to represent any derivatives involving time.* In addition, we will use the shorthand notation w' to denote the *spacetime* gradient of w,

$$w' := (\partial_t w, \partial_{x_1} w, ..., \partial_{x_n} w), \tag{131}$$

which includes the derivative of w with respect to each spatial variable and also the derivative of w with respect to time.

Lemma 6.1. Let $k \geq n+1$ be an integer, let $p > 1$ be a positive odd integer, and let $0 < \nu \leq 1$. For initial data $(f,g) \in H^{k+1}(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$, consider the initial value problem on $\mathbb{R}^n$,

$$\begin{aligned} \partial_{tt} w - \nu^2 \Delta w + \nu\sqrt{-\Delta}\partial_t w = \nu^2 \Delta \phi^{(0)} - G(\phi^{(0)} + w) + G(\phi^{(0)}) - \nu\sqrt{-\Delta}\partial_t \phi^{(0)}, \\ w(0,x) = f(x), \qquad \partial_t w(0,x) = g(x), \end{aligned} \tag{132}$$

where $G(z) = z^p$, and $\phi^{(0)}(t,x)$, given by (27), is the solution to the dispersive limit problem given by (26) for fixed initial displacement $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ and zero initial velocity. This initial value problem (132) has a unique solution (w, w') in $C([0,T]; H^{k+1}(\mathbb{R}^n)) \times C([0,T]; H^k(\mathbb{R}^n))$ for some $T > 0$ sufficiently small. Furthermore, if we let T_* be the supremum of all such times for which this is true, then either $T_* = \infty$, or T_* is finite and

$$\sup_{0 \leq t < T_*} \left(\sum_{|\alpha| \leq k+1} ||\partial^\alpha w(t,\cdot)||_{L^2(\mathbb{R}^n)} + \sum_{|\alpha| \leq k} ||\partial^\alpha w_t(t,\cdot)||_{L^2(\mathbb{R}^n)} \right) = \infty$$

Intuitively, this lemma says that there is a solution to the initial value problem (132) which we can extend locally in time, as long as the $H^{k+1} \times H^k$ norm of the solution at a given time is bounded.

The proof below follows standard Picard iteration and energy methods, as one can find in Sogge [37] in the context of quasilinear wave equations. We divide the proof into the following steps.

Step 1: Energy Inequality

The first step is to obtain an energy inequality for the problem

$$w_{tt} - \nu^2 \Delta w + \nu\sqrt{-\Delta}\partial_t w = F(t,x),$$

where we emphasize that $\nu \in (0,1]$ is fixed. We have already done this in the proof of Proposition 2.1, so we simply tailor the result of that proof to fit our current problem.

We first start by defining the ν-wave energy of the solution w by

$$E_\nu(w(t)) := \int_{\mathbb{R}^2} \frac{1}{2}|w_t(t,x)|^2 + \frac{\nu^2}{2}|\nabla w(t,x)|^2 dx = \frac{1}{2}\|w_t(t,\cdot)\|_{L^2}^2 + \frac{\nu^2}{2}\|\nabla w(t,\cdot)\|_{L^2}^2. \tag{133}$$

The corresponding energy inequality in the proof of Proposition 2.1 shows that for the kth ν-wave energy defined as

$$E_{\nu,k}(w(t)) = \sum_{|\alpha|\leq k} E_\nu(\partial_x^\alpha w(t)), \tag{134}$$

which includes the energy of all derivatives of order less than or equal to k, we have that

$$\partial_t(E_{\nu,k}^{1/2}(w(t))) \leq C||F(t,\cdot)||_{H^k}.$$

Hence,

$$E_{\nu,k}^{1/2}(w(t)) \leq E_{\nu,k}^{1/2}(w(0)) + C\int_0^t ||F(\tau,\cdot)||_{H^k} d\tau \tag{135}$$

by integrating.

We want to estimate our solutions to (132) with $(w, w') \in H^{k+1} \times H^k$, so in particular, our kth ν-wave energy defined by (133) and (134) is missing a term that estimates $||w(t,\cdot)||_{L^2}$. However, we can estimate $||w(t,\cdot)||_{L^2}$, which is subsumed in $\|w(t,\cdot)\|_{H^k}$, by using the fundamental theorem of calculus:

$$\|w(t,\cdot)\|_{H^k} \leq \|w(0,\cdot)\|_{H^k} + \int_0^t \|\partial_t w(\tau,\cdot)\|_{H^k} d\tau. \tag{136}$$

Therefore, by (135) and (136),

$$\|w(t,\cdot)\|_{H^k} + E_{\nu,k}^{1/2}(w(t)) \leq \|w(0,\cdot)\|_{H^k} + E_{\nu,k}^{1/2}(w(0)) + \int_0^t \|\partial_t w(\tau,\cdot)\|_{H^k} d\tau + C\int_0^t \|F(\tau,\cdot)\|_{H^k} d\tau. \tag{137}$$

This expression (137) gives the desired energy inequality, as $\|w(t,\cdot)\|_{H^k} + E_{\nu,k}^{1/2}(w(t))$ is the appropriate energy for estimating (w, w') in $H^{k+1} \times H^k$. We can rewrite the energy $\|w(t,\cdot)\|_{H^k} + E_{\nu,k}^{1/2}(w(t))$ in the more concise form,

$$\mathcal{E}(w(t)) := ||\partial_t w(t,\cdot)||_{H^k} + \sum_{|\alpha|\leq 1} ||\partial_x^\alpha w(t,\cdot)||_{H^k}, \tag{138}$$

which is equivalent to $\|w(t,\cdot)\|_{H^k} + E_{\nu,k}^{1/2}(w(t))$ *since, as we emphasize,* $\nu \in (0,1]$ *is fixed.* Then, by (137), we conclude that for some constant C independent of w and t (which depends on the choice of k and ν),

$$\mathcal{E}(w(t)) \leq C\left(\mathcal{E}(w(0)) + \int_0^t \mathcal{E}(w(\tau))d\tau + \int_0^t ||F(\tau,\cdot)||_{H^k} d\tau\right).$$

Then,

$$\mathcal{E}(w(t)) \leq Ce^{Ct}\left(\mathcal{E}(w(0)) + \int_0^t ||F(\tau,\cdot)||_{H^k} d\tau\right)$$

by Gronwall's inequality. By taking

$$C_T := Ce^{CT} \tag{139}$$

where C_T depends on T, we have that for all $0 \leq t \leq T$,

$$\mathcal{E}(w(t)) \leq C_T\left(\mathcal{E}(w(0)) + \int_0^t ||F(\tau,\cdot)||_{H^k} d\tau\right), \tag{140}$$

where T is an arbitrary time in $0 < T < \infty$. Here, C_T depends on T and furthermore, C_T can be chosen to be strictly increasing in T, as seen by the definition (139).

Step 2: Picard iteration and uniform boundedness of iterates

Now that we have an energy inequality, the general strategy will be to show that our initial value problem has local existence for well-behaved initial data (f, g). Then, we will use the energy inequality (140) to extend this to more general data.

In particular, we first assume that $f, g \in \mathcal{S}(\mathbb{R}^n)$ and we wish to show that we have local existence. We use Picard iteration. Define the first iterate as $w_{-1} := 0$. Picard iteration then defines a sequence of solutions inductively, where we hope that the iterates converge to an actual solution. Explicitly, for $m \geq 0$, we define w_m to be the solution to the initial value problem

$$\begin{aligned} \partial_{tt} w - \nu^2 \Delta w + \nu\sqrt{-\Delta}\partial_t w = \nu^2 \Delta \phi^{(0)} - G(\phi^{(0)} + w_{m-1}) + G(\phi^{(0)}) - \nu\sqrt{-\Delta}\partial_t \phi^{(0)}, \qquad (141) \\ w(0,x) = f(x), \qquad \partial_t w(0,x) = g(x). \end{aligned}$$

This has a solution in all of time, as can be seen by Fourier methods, because the inhomogeneous term no longer depends on the solution w anymore. So we can indeed define w_m by this inductive procedure.

We will consider the following mth step energy:

$$\mathcal{E}(w_m(t)) := \|\partial_t w_m(t,\cdot)\|_{H^k} + \sum_{|\alpha| \leq 1} \|\partial_x^\alpha w_m(t,\cdot)\|_{H^k}. \qquad (142)$$

where $\mathcal{E}$ is the energy in (138). We want to show that there exists a time T and a constant A such that for all m, the following uniform energy estimate holds:

$$\mathcal{E}(w_m(t)) \leq A < \infty, \qquad \text{for all } 0 \leq t \leq T. \qquad (143)$$

This is the content of Step 2. We will show this by induction.

For the base case, estimate (143) is clearly true for $u_{-1} \equiv 0$ for any positive A, where we will choose A later in the inductive step. To establish the inductive step, we will use the energy inequality. Before using the energy inequality, we collect some estimates from the proof of Proposition 2.1 that we will need. By the previous estimates (33) and (36), there exists a constant C (depending on k and our fixed ν) such that

$$\begin{aligned} \|\nu^2 \Delta \phi^{(0)}\|_{H^k} \leq C(1+t)^C, \\ \|\nu\sqrt{-\Delta}\partial_t \phi^{(0)}\|_{H^k} \leq C(1+t)^C. \end{aligned}$$

In addition since $k > 1$, we can appeal to the previous estimate (34) to obtain

$$\|G(\phi^{(0)} + w_{m-1})(t) - G(\phi^{(0)})(t)\|_{H^k} \leq C(1+t)^C \|w_{m-1}(t,\cdot)\|_{H^k} \left(1 + \|w_{m-1}(t,\cdot)\|_{H^k}\right)^{p-1}.$$

Now that we have collected all of the necessary estimates, we use the energy inequality (140) to estimate $\mathcal{E}(w_m(t))$ as

$$\begin{aligned} \mathcal{E}(w_m(t)) &\leq C_T \left(\mathcal{E}(w_m(0)) + \int_0^t (C(1+s)^C + C(1+s)^C \|w_{m-1}(s,\cdot)\|_{H^k} \left(1 + \|w_{m-1}(s,\cdot)\|_{H^k}\right)^{p-1} ds \right) \\ &\leq C_T \left(\mathcal{E}(w_m(0)) + (C + CA(1+A)^{p-1}) \int_0^t (1+s)^C ds \right), \qquad (144) \end{aligned}$$

where we used the inductive assumption in the second inequality. Now, we will choose A to be any positive number such that

$$A > C_1 \mathcal{E}(w_m(0)), \qquad (145)$$

where C_1 is the constant C_T for $T = 1$ defined by (139). This is possible because the right hand side of (145) is the same for all m since w_m all have the same initial data (f, g). Upon choosing A, we can then choose $0 < T < 1$ sufficiently small so that

$$C_1 \left(\mathcal{E}(w_m(0)) + (C + CA(1+A)^{p-1}) \int_0^T (1+s)^C ds \right) < A. \tag{146}$$

Then because C_T is strictly increasing in T by (139), we get from (144) and (146) that $\mathcal{E}(w_m(t)) \leq A$ for all $0 \leq t \leq T$. So for this choice of A and $0 < T < 1$, we have the desired uniform bound (143) on the mth step energies of our iterates w_m.

Step 3: Convergence to a solution

Next, we show that the iterates w_m from Step 2 form a Cauchy sequence in $C([0, T_1]; H^{k+1}(\mathbb{R}^n)) \cap C([0, T_1]; H^k(\mathbb{R}^n))$, for some time $0 < T_1 < T$ where T is the time chosen in Step 2 in (146).

First note that since $f, g \in \mathcal{S}$, the w_m are all smooth since the inhomogeneous term is smooth and rapidly decreasing. Using the notation $\mathcal{E}$ from (138), it suffices to show that for all $t \in [0, T_1]$,

$$\begin{aligned} \mathcal{E}((w_m - w_{m-1})(t)) &:= \|\partial_t w_m(t,\cdot) - \partial_t w_{m-1}(t,\cdot)\|_{H^k} \\ &\quad + \sum_{|\alpha| \leq 1} \|\partial_x^\alpha w_m(t,\cdot) - \partial_x^\alpha w_{m-1}(t,\cdot)\|_{H^k} = O(2^{-m}), \end{aligned} \tag{147}$$

for some time $0 < T_1 < T$, where T is chosen in Step 2 in (146). Let A, chosen in the previous step in (145), be the constant in this $O(2^{-m})$. Thus, we claim that there exists a time $T_1 > 0$, smaller than the T from (146), such that

$$\mathcal{E}((w_m - w_{m-1})(t)) \leq A 2^{-m} \qquad \text{for all } t \in [0, T_1], m \geq 0. \tag{148}$$

As before, we will show this by induction. This inequality is indeed true for $m = 0$ because $w_{-1} = 0$, and thus the inequality for $m = 0$ follows directly from (143).

For the inductive step, note that $w_{m+1} - w_m$ is a solution to

$$\begin{aligned} \partial_{tt} v - \nu^2 \Delta v + \nu \sqrt{-\Delta} \partial_t v &= -G(\phi^{(0)} + w_m) + G(\phi^{(0)} + w_{m-1}), \\ v(0,x) = 0, \qquad \partial_t v(0,x) &= 0. \end{aligned} \tag{149}$$

By using the same inequality on pg. 11 of Christ, Colliander, and Tao [11] used to establish the inequality in (34), we get that

$$\begin{aligned} &\|G(\phi^{(0)} + w_m) - G(\phi^{(0)} + w_{m-1})\|_{H^k} \\ &\leq C((1+t)^C + A)^C \|w_m(t,\cdot) - w_{m-1}(t,\cdot)\|_{H^k} (1 + \|w_m(t,\cdot) - w_{m-1}(t,\cdot)\|_{H^k})^{p-1}. \end{aligned} \tag{150}$$

Then, by the energy inequality (140), for all $0 \leq t \leq T$ for T as in (146),

$$\begin{aligned} &\mathcal{E}((w_{m+1} - w_m)(t)) \\ &\leq C_T \left(C \int_0^t ((1+s)^C + A)^C \left(1 + \mathcal{E}((w_m - w_{m-1})(s))\right)^{p-1} \mathcal{E}((w_m - w_{m-1})(s)) ds \right), \end{aligned} \tag{151}$$

since there is zero initial data in (149). Choose $0 < T_1 < T$ such that

$$C_{T_1} C T_1 ((1+T_1)^C + A)^C (1+A)^{p-1} < 1/2, \tag{152}$$

where C_T is defined by (139). Then, for all $m \geq 0$,

$$C_{T_1} C T_1((1+T_1)^C + A)^C (1 + A2^{-m})^{p-1} A 2^{-m} < A 2^{-m-1}. \tag{153}$$

Then, by (151) and (153), we get that

$$\mathcal{E}((w_{m+1} - w_m)(t)) \leq A 2^{-(m+1)} \qquad \text{for all } 0 \leq t \leq T_1,$$

which establishes the inductive step. Hence, we have proved (147).

So (w_m, w_m'), where the prime denotes the spacetime gradient (131), forms a Cauchy sequence in $C([0,T_1]; H^{k+1}) \times C([0,T_1]; H^k)$. Therefore, $w_m \to w$ for some $w \in C([0,T_1]; H^{k+1})$ and $w_m' \to v$ for some $v \in C([0,T_1]; H^k)$. Hence $w_m \to w$ in $\mathcal{D}'([0,T_1) \times \mathbb{R}^n)$ and hence $w_m' \to w'$ in $\mathcal{D}'([0,T_1) \times \mathbb{R}^n)$ also. But by uniqueness, this means that $w' = v$. Therefore, $(w, w') \in C([0,T_1]; H^{k+1}) \times C([0,T_1]; H^k)$.

Note in particular that

$$\|\partial_t w(t,\cdot)\|_{H^k} + \sum_{|\alpha| \leq 1} \|\partial^\alpha w(t,\cdot)\|_{H^k} \leq A \qquad \text{for all } 0 < t \leq T_1, \tag{154}$$

since this is true for each of the w_m by the uniform bound (143) in Step 2. *In addition, we emphasize that the constants A and T_1 here depend only on the $H^{k+1} \times H^k$ norm of the initial data $(f,g) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$.* This fact will be important in the next step.

It remains to show that w, constructed as the limit of the iterates w_m, solves the original initial value problem (132). Because the inhomogeneous term in (132) depends on the solution w itself, we need to check that this inhomogeneous term converges appropriately as $m \to \infty$ to conclude that w is a weak solution. To see this, for all $0 \leq t \leq T_1$, by the argument on pg. 11 in Christ, Colliander, and Tao [11],

$$\begin{aligned} &\|G(\phi^{(0)} + w)(t) - G(\phi^{(0)} + w_m)(t)\|_{H^k} \\ &\qquad \leq C((1+t)^C + A)^C \|w(t,\cdot) - w_m(t,\cdot)\|_{H^k} (1 + \|w(t,\cdot) - w_m(t,\cdot)\|_{H^k})^{p-1} \to 0, \end{aligned}$$

where this convergence as $m \to \infty$ happens uniformly in $t \in [0,T_1]$ (since the convergence of $w_m \to w$ happens in $C([0,T_1]; H^{k+1})$). Therefore, $G(\phi^{(0)} + w_m) \to G(\phi^{(0)} + w)$ in $L^\infty([0,T_1]; H^k)$ and hence this convergence happens in the sense of weak convergence of distributions $\mathcal{D}'([0,T_1) \times \mathbb{R}^n)$. So we conclude that w is indeed a weak solution to the initial value problem (132).

Step 4: Approximation argument for general initial data

Now, we consider the initial value problem

$$\begin{gathered} \partial_{tt} w - \nu^2 \Delta w + \nu \sqrt{-\Delta} \partial_t w = \nu^2 \Delta \phi^{(0)} - G(\phi^{(0)} + w) + G(\phi^{(0)}) - \nu \sqrt{-\Delta} \partial_t \phi^{(0)}, \\ w(0,x) = f(x), \qquad \partial_t w(0,x) = g(x), \end{gathered} \tag{155}$$

for general $f, g \in H^{k+1}(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ and not just f, g that are in $\mathcal{S}(\mathbb{R}^n)$. For this more general class of initial data, we show that we still have local existence by approximating this general initial data by functions in $\mathcal{S}(\mathbb{R}^n)$.

Let $f_m \to f$ and $g_m \to g$ in H^{k+1} and H^k, where $f_m, g_m \in \mathcal{S}(\mathbb{R}^n)$. By refining this sequence as necessary, we can choose a subsequence (which we will continue to denote by f_m and g_m) so that

$$\|f_m - f_{m-1}\|_{H^{k+1}} \leq \frac{1}{n+2} (\max(2, C_1))^{-m}, \qquad \|g_m - g_{m-1}\|_{H^k} \leq \frac{1}{n+2} (\max(2, C_1))^{-m},$$

where C_1 is the constant C_T (139) for $T = 1$ from the energy inequality (140). Since the norms of f_m and g_m are uniformly bounded in H^{k+1} and H^k respectively, from our previous step, there exist uniform constants A and

$$0 < T_2 < 1, \tag{156}$$

(since the norms of $(f_m, g_m) \in H^{k+1} \times H^k$ are all uniformly bonded in m, see the remark immediately following (154)) such that:

- There exists a solution $(w_m, w_m') \in C([0, T_2]; H^{k+1}) \times C([0, T_2]; H^k)$ (with T_2 uniform) to the initial value problem

$$\begin{aligned} \partial_{tt} w - \nu^2 \Delta w + \nu\sqrt{-\Delta}\partial_t w = \nu^2 \Delta \phi^{(0)} - G(\phi^{(0)} + w) + G(\phi^{(0)}) - \nu\sqrt{-\Delta}\partial_t \phi^{(0)}, \\ w(0, x) = f_m(x), \qquad \partial_t w(0, x) = g_m(x). \end{aligned} \tag{157}$$

- The solutions w_m satisfy the uniform energy bound

$$\|\partial_t w_m(t, \cdot)\|_{H^k} + \sum_{|\alpha| \le 1} \|\partial_x^\alpha w_m(t, \cdot)\|_{H^k} \le A \qquad \text{for all } 0 \le t \le T_2, \tag{158}$$

  by (154).

We want to show that

$$(w_m, w_m') \text{ is Cauchy in } C([0, T_3]; H^{k+1}) \times C([0, T_3]; H^k) \text{ for some time } 0 < T_3 < T_2. \tag{159}$$

To do this, consider

$$\mathcal{E}((w_m - w_{m-1})(t)) := \|\partial_t w_m(t, \cdot) - \partial_t w_{m-1}(t, \cdot)\|_{H^k} + \sum_{|\alpha| \le 1} \|\partial_x^\alpha w_m(t, \cdot) - \partial_x^\alpha w_{m-1}(t, \cdot)\|_{H^k}. \tag{160}$$

Here, $\mathcal{E}$ is the energy from (138) and w_m is defined by (157). Then, $w_m - w_{m-1}$ satisfies the initial value problem

$$\begin{aligned} &v_{tt} - \nu^2 \Delta v + \nu\sqrt{-\Delta}\partial_t v = -G(\phi^{(0)} + w_m) + G(\phi^{(0)} + w_{m-1}), \\ &v(0, x) = f_m(x) - f_{m-1}(x), \quad \partial_t v(0, x) = g_m(x) - g_{m-1}(x), \end{aligned} \tag{161}$$

where we recall that by the choice of our subsequences f_m and g_m,

$$\|f_m - f_{m-1}\|_{H^{k+1}} \le \frac{1}{n+2}\rho^{-m}, \qquad \|g_m - g_{m-1}\|_{H^k} \le \frac{1}{n+2}\rho^{-m}, \quad \text{where } \rho := \max(2, C_1). \tag{162}$$

We claim that for some T_3 such that $0 < T_3 < T_2$, where T_2 satisfies the conditions in (157) and (158),

$$\mathcal{E}((w_m - w_{m-1})(t)) \le \rho^{-m+1} \qquad \text{for all } m \ge 2, \quad 0 \le t \le T_3. \tag{163}$$

We show this by using a bootstrap argument.

We consider $m \ge 2$. By applying the energy inequality (140), the uniform bound (158) on the w_m, and the previous estimate (150) to the initial value problem in (161),

$$\begin{aligned} \mathcal{E}((w_m - w_{m-1})(t)) \le C_{T_2} \Bigg(&\mathcal{E}((w_m - w_{m-1})(0) \\ &+ C \int_0^t ((1+s)^C + A)^C (1 + \mathcal{E}((w_m - w_{m-1})(s)))^{p-1} \mathcal{E}((w_m - w_{m-1})(s)) ds \Bigg), \end{aligned}$$

for all $0 \leq t \leq T_2$ for T_2 defined as in (157) and (158). Note that by the way we chose f_m and g_m in (162), we have $\mathcal{E}((w_m - w_{m-1})(0)) \leq \rho^{-m}$. Therefore, for all $0 \leq t \leq T_2$,

$$\begin{aligned}&\mathcal{E}((w_m - w_{m-1})(t))\\ &\leq C_{T_2}\left(\rho^{-m} + C\int_0^t ((1+s)^C + A)^C(1 + \mathcal{E}((w_m - w_{m-1})(s)))^{p-1}\mathcal{E}((w_m - w_{m-1})(s))ds\right).\end{aligned}$$

Let us make the *bootstrap assumption* that

$$\mathcal{E}((w_m - w_{m-1})(t)) \leq 1 \qquad \text{for all } 0 \leq t \leq T_2', \tag{164}$$

where $T_2' < T_2$ will be chosen later (*independently* of m). Then, for $t \in [0, T_2']$,

$$\mathcal{E}((w_m - w_{m-1})(t)) \leq C_{T_2'}\left(\rho^{-m} + C\int_0^t ((1+s)^C + A)^C 2^{p-1}\mathcal{E}((w_m - w_{m-1})(s))ds\right).$$

After consolidating constants and using the fact that $0 < T_2' < T_2 < 1$ (see (156)),

$$\mathcal{E}((w_m - w_{m-1})(t)) \leq C_{T_2'}\left(\rho^{-m} + C'\int_0^t \mathcal{E}((w_m - w_{m-1})(s))ds\right), \quad \text{for all } t \in [0, T_2'].$$

Using Gronwall's inequality, we get that for $t \in [0, T_2']$:

$$\mathcal{E}((w_m - w_{m-1})(t)) \leq C_{T_2'}\rho^{-m}\exp(C'C_{T_2'}t).$$

Then, we can choose the $T_2' < T_2$ appearing in the bootstrap assumption (164) so that

$$C_{T_2'}\exp(C'C_{T_2'}T_2') \leq \rho = \max(2, C_1).$$

This is possible because the constant C_T (139) in the energy inequality (140) is strictly increasing in T. Thus, we get that

$$\mathcal{E}((w_m - w_{m-1})(t)) \leq \rho^{-m+1} \qquad \text{for all } t \in [0, T_2']. \tag{165}$$

Since $m \geq 2$ and $\rho \geq 2$, this also closes the bootstrap assumption (164), since we have shown that if $\mathcal{E}((w_m - w_{m-1})(t)) \leq 1$ for all $t \in [0, T_2']$, then

$$\mathcal{E}((w_m - w_{m-1})(t)) \leq \rho^{-m+1} \leq \rho^{-1} \leq \frac{1}{2} \qquad \text{for all } t \in [0, T_2'].$$

In addition, since $\mathcal{E}((w_m - w_{m-1})(0)) \leq \rho^{-m}$, the bootstrap assumption (164) is true at $t = 0$. Note that indeed, $\mathcal{E}((w_m - w_{m-1})(t))$ is continuous on $[0, T_2']$, since we showed in the previous step that the solutions (w_m, w_m') are in $C([0, T_2]; H^{k+1}) \times C([0, T_2]; H^k)$.

So we have established the claim (163) for $T_3 := T_2'$. Therefore, (w_m, w_m') is Cauchy in $C([0, T_3]; H^{k+1}) \times C([0, T_3]; H^k)$. By the same argument in the previous step, there exists $w \in C([0, T_3]; H^{k+1})$ with $w' \in C([0, T_3]; H^k)$ such that $(w_m, w_m') \to (w, w')$ in the sense of weak convergence of distributions.

To show that the w we have constructed is indeed a weak solution to our desired initial value problem (155), we again must show appropriate convergence of the inhomogeneous terms which depend on the solution w. As before, an argument similar to that on pg. 11 in Christ, Colliander, and Tao [11] shows that for all $0 \leq t \leq T_3$,

$$\begin{aligned}&\|G(\phi^{(0)} + w)(t) - G(\phi^{(0)} + w_m)(t)\|_{H^k}\\ &\qquad \leq C((1+t)^C + A)^C ||w(t,\cdot) - w_m(t,\cdot)||_{H^k}(1 + \|w(t,\cdot) - w_m(t,\cdot)\|_{H^k})^{p-1} \to 0\end{aligned}$$

uniformly in $t \in [0, T_3]$ (since the convergence of $w_m \to w$ happens in $C([0,T_3]; H^{k+1})$) as $m \to \infty$, where we used the uniform bound on w_m in (158). So $G(\phi^{(0)}+w_m) \to G(\phi^{(0)}+w)$ in $L^\infty([0,T_3]; H^k)$. Finally, since $f_m \to f$ and $g_m \to g$ in H^{k+1} and H^k respectively, this convergence of the initial data happens in the sense of distributions on $\mathbb{R}^n$ also.

Therefore,

$$(w, w') \in C([0,T_3]; H^{k+1}) \times C([0,T_3]; H^k) \tag{166}$$

is indeed a weak solution to the given initial value problem with general initial data $(f, g) \in H^{k+1} \times H^k$.

Step 5: Uniqueness of solution

Next, we show that the solution (w, w') in $C([0,T]; H^{k+1}) \times C([0,T]; H^k)$ that we have constructed is unique for any given T for which the solution exists, where for simplicity of notation, we have replaced T_3 from Step 4 (166) with T. This uniqueness will play an important role in the next step.

Suppose w and $\tilde{w}$ are both solutions to the initial value problem given in (132). Then, their difference $v := w - \tilde{w}$ is a solution to

$$\begin{aligned} v_{tt} - \nu^2 \Delta v + \nu\sqrt{-\Delta}\partial_t v &= -G(\phi^{(0)} + w) + G(\phi^{(0)} + \tilde{w}), \\ v(0,x) = 0, \qquad \partial_t v(0,x) &= 0. \end{aligned} \tag{167}$$

Using an argument similar to that on pg. 11 of Christ, Colliander, and Tao [11], for all $0 \le t \le T$, the right hand side of equation (167) can be bounded as follows:

$$\begin{aligned} &\|G(\phi^{(0)} + w)(t) - G(\phi^{(0)} + \tilde{w})(t)\|_{H^k} \\ &\le C((1+t)^C + ||\tilde{w}(t,\cdot)||_{H^k})^C \|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{H^k} (1 + \|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{H^k})^{p-1}. \end{aligned}$$

Note that since $\tilde{w} \in C([0,T]; H^{k+1})$, we have that $\|\tilde{w}(t,\cdot)\|_{H^k} \le A$ for all $0 \le t \le T$ for some A. So then,

$$\begin{aligned} &\|G(\phi^{(0)} + w)(t) - G(\phi^{(0)} + \tilde{w})(t)\|_{H^k} \\ &\le C((1+t)^C + A)^C \|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{H^k} (1 + \|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{H^k})^{p-1} \\ &\le C' \|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{H^k} (1 + \|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{H^k})^{p-1} \qquad \text{for all } t \in [0,T], \end{aligned} \tag{168}$$

where $C' := C((1+T)^C + A)^C$.

For the difference $v := w - \tilde{w}$, consider

$$\mathcal{E}(v(t)) := \|\partial_t v(t,\cdot)\|_{H^k} + \sum_{|\alpha| \le 1} \|\partial^\alpha v(t,\cdot)\|_{H^k} = \|\partial_t (w - \tilde{w})(t,\cdot)\|_{H^k} + \sum_{|\alpha| \le 1} \|\partial^\alpha (w - \tilde{w})(t,\cdot)\|_{H^k},$$

where $\mathcal{E}$ is defined as in (138). Let us make the *bootstrap assumption* that

$$\|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{H^k} \le 1 \qquad \text{for all } t \in [0,T]. \tag{169}$$

Then, using the energy inequality (140), for all $t \in [0,T]$,

$$\begin{aligned} \mathcal{E}(v(t)) &\le C_T \left(\int_0^t C' ||w(s,\cdot) - \tilde{w}(s,\cdot)||_{H^k} (1 + ||w(s,\cdot) - \tilde{w}(s,\cdot)||_{H^k})^{p-1} ds \right) \\ &\le C_T C'' \int_0^t ||w(s,\cdot) - \tilde{w}(s,\cdot)||_{H^k} ds \qquad \text{(by the bootstrap assumption)} \\ &\le C_T C'' \int_0^t \mathcal{E}(v(s)) ds, \end{aligned}$$

since $\mathcal{E}(v(0)) = 0$. So by Gronwall's inequality, $\mathcal{E}(v(t)) \leq 0$ for $t \in [0,T]$. So therefore, $\mathcal{E}(v(t)) = 0$ for $t \in [0,T]$. This also closes the bootstrap assumption (169), since $\mathcal{E}(v(0)) = 0$ so the bootstrap assumption is satisfied for $t = 0$, and in addition, $\mathcal{E}(v(t))$ is a continuous function of t, since $(w - \tilde{w}, w' - \tilde{w}')$ is in $C([0,T]; H^{k+1}) \times C([0,T]; H^k)$. This shows that $(w, w') = (\tilde{w}, \tilde{w}')$ in $C([0,T]; H^{k+1}) \times C([0,T]; H^k)$.

Step 6: Existence as long as $H^{k+1} \times H^k$ norm is bounded

We have finished the proof of local existence and uniqueness. However, for the purposes of the proof of Proposition 2.1, we need something stronger: existence in $H^{k+1} \times H^k$ as long as this norm is bounded. This claim justifies our computations in the proof of Proposition 2.1. This will be the content of this final step.

We show that a solution to the initial value problem (132) exists as long as the $H^{k+1} \times H^k$ norm of the solution is bounded. So far, for given $(f, g) \in H^{k+1} \times H^k$, we have shown that there exists a $T > 0$ *depending only on the norm of the initial data* such that there is a unique solution w to the given initial value problem (132) with

$$\sum_{|\alpha| \leq k+1} \|\partial_x^\alpha w(t, \cdot)\|_{L^2} + \sum_{|\alpha| \leq k} \|\partial_x^\alpha w_t(t, \cdot)\|_{L^2} < \infty \qquad \text{for all } t \in [0,T], \tag{170}$$

where the left hand side is equivalent to the energy $\mathcal{E}(w(t))$ defined in (138).

Let T_* be the supremum of all such times $T > 0$ for which we have local existence and uniqueness of a solution for given initial data $(f, g) \in H^{k+1} \times H^k$ on $[0,T]$. We assert that either $T_* = \infty$ or

$$\sup_{0 \leq t < T_*} \left(\sum_{|\alpha| \leq k+1} \|\partial_x^\alpha w(t, \cdot)\|_{L^2} + \sum_{|\alpha| \leq k} \|\partial_x^\alpha w_t(t, \cdot)\|_{L^2} \right) = \infty.$$

To see this, suppose for contradiction that $T_* < \infty$ satisfies

$$\sup_{0 \leq t < T_*} \left(\sum_{|\alpha| \leq k+1} \|\partial_x^\alpha w(t, \cdot)\|_{L^2} + \sum_{|\alpha| \leq k} \|\partial^\alpha w_t(t, \cdot)\|_{L^2} \right) := M < \infty.$$

Recall that the time of existence that we found in Step 4 depends only on the norm of the initial data. Since $\|w(t, \cdot)\|_{H^{k+1}}$ and $\|\partial_t w(t, \cdot)\|_{H^k}$ are both bounded in their sum by M for $0 \leq t < T_*$, there exists a *uniform* time of existence $T_M > 0$ for the initial value problem (132) for any initial data with $H^{k+1} \times H^k$ norm less than or equal to M. Recalling that $(w, w') \in C([0,T]; H^{k+1}) \times C([0,T]; H^k)$ for $T \in [0, T_*)$, we can consider $0 < t_0 < T_*$ such that $t_0 > T_* - T_M$. We then consider the initial value problem (132) with initial data $w(t_0, x)$ and $\partial_t w(t_0, x)$. Gluing the resulting solution which exists for at least time T_M with the previous solution w from time 0 to t_0, we get a new solution that is extended past T_*. The uniqueness assertion from Step 5 shows that on the overlap, w and this newly constructed solution must be the same, and furthermore, the newly constructed solution is unique on the time interval on which it is defined. This contradicts the definition of T_*.

Acknowledgements

This work was partially supported by the National Science Foundation under grants DMS-1613757, DMS-1853340, and DMS-2011319, and by the UC Berkeley start-up funds.

References

[1] V. Barbu, Z. Grujić, I. Lasiecka, and A. Tuffaha. Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model. In *Fluids and waves*, volume 440 of *Contemp. Math.*, pages 55–82. Amer. Math. Soc., Providence, RI, 2007.

[2] V. Barbu, Z. Grujić, I. Lasiecka, and A. Tuffaha. Smoothness of weak solutions to a nonlinear fluid-structure interaction model. *Indiana Univ. Math. J.*, 57(3):1173–1207, 2008.

[3] H. Beirão da Veiga. On the existence of strong solutions to a coupled fluid-structure evolution problem. *J. Math. Fluid Mech.*, 6(1):21–52, 2004.

[4] A. Bényi, T. Oh, and O. Pocovnicu. Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS. In *Excursions in Harmonic Analysis*, volume 4 of *Applied and Numerical Harmonic Analysis*, pages 3–25. Springer International Puublishing, 2015.

[5] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations I: Local theory. *Invent. Math.*, 173(3):449–475, 2008.

[6] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(8):1245–1260, 2007.

[7] T. Cazenave and F. B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal.*, 14(10):807–836, 1990.

[8] A. Chambolle, B. Desjardins, M. J. Esteban, and C. Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *J. Math. Fluid Mech.*, 7(3):368–404, 2005.

[9] C. H. A. Cheng, D. Coutand, and S. Shkoller. Navier-Stokes equations interacting with a nonlinear elastic biofluid shell. *SIAM J. Math. Anal.*, 39(3):742–800, 2007.

[10] C. H. A. Cheng and S. Shkoller. The interaction of the 3D Navier-Stokes equations with a moving nonlinear Koiter elastic shell. *SIAM J. Math. Anal.*, 42(3):1094–1155, 2010.

[11] M. Christ, J. Colliander, and T. Tao. Ill-posedness for nonlinear Schrödinger and wave equations, 2003.

[12] D. Coutand and S. Shkoller. Motion of an elastic solid inside an incompressible viscous fluid. *Arch. Ration. Mech. Anal.*, 176(1):25–102, 2005.

[13] D. Coutand and S. Shkoller. The interaction between quasilinear elastodynamics and the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 179(3):303–352, 2006.

[14] Q. Du, M. D. Gunzburger, L. S. Hou, and J. Lee. Analysis of a linear fluid-structure interaction problem. *Discrete Contin. Dyn. Syst.*, 9(3):633–650, 2003.

[15] C. Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *SIAM J. Math. Anal.*, 40(2):716–737, 2008.

[16] C. Grandmont and M. Hillairet. Existence of global strong solutions to a beam-fluid interaction system. *Arch. Ration. Mech. Anal.*, 220(3):1283–1333, 2016.

[17] C. Grandmont, M. Lukáčová-Medvid'ová, and Š. Nečasová. Mathematical and numerical analysis of some FSI problems. In T. Bodnár, G. P. Galdi, and Š. Nečasová, editors, *Fluid-structure interaction and biomedical applications*, Advances in Mathematical Fluid Mechanics, pages 1–77. Birkhäuser, 2014.

[18] M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha. On well-posedness for a free boundary fluid-structure model. *J. Math. Phys.*, 53(11):115624, 13, 2012.

[19] M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha. On well-posedness and small data global existence for an interface damped free boundary fluid-structure model. *Nonlinearity*, 27(3):467–499, 2014.

[20] M. A. Keel and T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[21] I. Kukavica and A. Tuffaha. Solutions to a fluid-structure interaction free boundary problem. *DCDS-A*, 32(4):1355–1389, 2012.

[22] I. Kukavica, A. Tuffaha, and M. Ziane. Strong solutions for a fluid structure interaction system. *Adv. Differential Equations*, 15(3-4):231–254, 2010.

[23] D. Lengeler and M. Růžička. Weak solutions for an incompressible Newtonian fluid interacting with a Koiter type shell. *Arch. Ration. Mech. Anal.*, 211(1):205–255, 2014.

[24] J. Lequeurre. Existence of strong solutions to a fluid-structure system. *SIAM J. Math. Anal.*, 43(1):389–410, 2011.

[25] H. Lindblad and C. D. Sogge. On existence and scattering with minimal regularity for semi-linear wave equations. *J. Funct. Anal.*, 130(2):357–426, 1995.

[26] J. Lührmann and D. Mendelson. Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$. *Comm. Partial Differential Equations*, 39(12):2262–2283, 2014.

[27] C. Miao, B. Yuan, and B. Zhang. Well-posedness of the Cauchy problem for the fractional power dissipative equations. *Nonlinear Anal.*, 68(3):461–484, 2008.

[28] P. Moireau, N. Xiao, M. Astorino, C. A. Figueroa, D. Chapelle, C. A. Taylor, and J.-F. Gerbeau. External tissue support and fluid-structure simulation in blood flows. *Biomech Model Mechanobiol.*, 11(1-2):1–18, 2012.

[29] B. Muha and S. Čanić. Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls. *Arch. Ration. Mech. Anal.*, 207(3):919–968, 2013.

[30] B. Muha and S. Čanić. A nonlinear, 3D fluid-structure interaction problem driven by the time-dependent dynamic pressure data: a constructive existence proof. *Commun. Inf. Syst.*, 13(3):357–397, 2013.

[31] B. Muha and S. Čanić. Existence of a solution to a fluid-multi-layered-structure interaction problem. *J. Differential Equations*, 256(2):658–706, 2014.

[32] B. Muha and S. Čanić. Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy. *Interfaces Free Bound.*, 17(4):465–495, 2015.

[33] B. Muha and S. Čanić. Existence of a weak solution to a fluid-elastic structure interaction problem with the Navier slip boundary condition. *J. Differential Equations*, 260(12):8550–8589, 2016.

[34] J.-P. Raymond and M. Vanninathan. A fluid-structure model coupling the Navier-Stokes equations and the Lamé system. *J. Math. Pures Appl. (9)*, 102(3):546–596, 2014.

[35] C. G. Simader. Mean value formulas, Weyl's lemma and Liouville theorems for $\triangle^2$ and Stokes' system. *Results in Math.*, 22:761–780, 1992.

[36] C. D. Sogge. *Lectures on non-linear wave equations*. International Press, second edition, 2013.

[37] R. S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[38] J. Wang and K. Wang. Almost sure existence of global weak solutions to the 3D incompressible Navier-Stokes equation. *Discrete Contin. Dyn. Syst.*, 37(9):5003–5019, 2017.

[39] Y. Wang, S. Čanić, M. Bukač, and J. Tambača. Fluid-structure interaction between pulsatile blood flow and a curved stented coronary artery on a beating heart: a four stent computational study. *Comput. Methods Appl. Mech. Engrg.*, 350(15):679–700, 2019.

[40] Z. Zhai. Strichartz type estimates for fractional heat equations. *J. Math. Anal. Appl.*, 356(2):642–658, 2009.