

Sum of Squares based Convex Approach for Optimal Control Synthesis

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Abstract— We consider an optimal control synthesis problem for a class of control-affine nonlinear systems. We propose Sum-of-Square based computational framework for optimal control synthesis. The proposed computation framework relies on the convex formulation of the optimal control problem in the dual space of densities. The convex formulation to the optimal control problem is based on the duality results in dynamical systems' stability theory. We used the Sum-of-Square based computational framework for the finite-dimensional approximation of the convex optimization problem. The efficacy of the developed framework is demonstrated using simulation results.

I. INTRODUCTION

The control synthesis problem for nonlinear systems has been a longstanding challenge in the control community. The optimal control synthesis for general nonlinear dynamics aiming at designing a control law that minimizes some cost function is typically addressed using Pontryagin's maximum principle [1], or dynamic programming [2]. With dynamic programming, the optimal control is characterized by the celebrated Hamilton-Jacobi-Bellman equation [2]. In this control scheme, the problem described by partial differential equations is equal to the dimension of the corresponding state space. Thus, the complexity of the dynamic programming problem grows rapidly with the state dimension; this phenomenon is known as the curse of dimensionality. The Pontryagin's maximum principle, on the other hand, enjoys much better scalability. However, it only leads to local optimal control. Another limitation of the maximum principle is that the solution doesn't provide a feedback law; the solution must be recalculated each time for a different initial state.

The main results presented in this paper on the sum of square based computation framework for the computation of optimal control relies on the density-based formulation

of the optimal control problem. [3] originally explored the idea of carrying out control synthesis over the density space for the stabilization problems. The celebrated Lyapunov theory [4] is a powerful framework to certify the stability of a given nonlinear dynamics. Searching for a suitable Lyapunov function for a nonlinear system is a convex problem. However, when it comes to controller synthesis, this framework becomes less effective. The joint search of the Lyapunov function and the control law is, in general, a non-convex problem. To overcome this difficulty, [3] proposed to use the dual Lyapunov function known as Lyapunov density [5]. For control-affine dynamics, the control design can be formulated as a convex problem. The method we proposed in this paper can be viewed as a generalization of that in [3] to deal with optimal control problems. In [6], a novel framework was proposed towards the density-based formulation of the optimal controllers. The key idea of the results from [6] is to convert the nonlinear dynamics to linear dynamics in the lifted density space. By leveraging the linear operator theory for nonlinear systems and with proper reparametrization, the optimal control problem becomes a convex optimization over the lifted density space. The formulation of optimal control problem in the density space is made possible by viewing the duality in stability and stabilization results through the lens of linear operator theory involving Koopman and Perron-Frobenius operator [7]–[10].

The paper's main contribution is to exploit the convex formulation of the optimal control problem presented in [6] and provide a Sum of Square (SoS) based computational framework for optimal control synthesis. The results in this paper can hence be viewed as a natural extension of results presented in [3]. While [3] provides SoS based computational framework for stabilizing control synthesis, in this paper, we provide SoS based computational framework for optimal control synthesis. We show that (in Section III), the method in [3] is a special case of our framework.

The rest of the paper is structured as follows. In Section II, we provide a brief introduction to our framework's necessary ingredient. The problem formulation and the main theoretical results are given in Section III. In Section IV, we develop the algorithm details based on the SoS framework. This is followed by several numerical examples in Section V and a brief conclusion in Section VI.

II. BACKGROUND

A. Koopman and Perron-Frobenius Operators

For a given dynamical system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X} \subseteq \mathbb{R}^n, \quad (1)$$

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there are two different ways of linearly lifting the finite dimensional nonlinear dynamics from state space to infinite dimension space of functions, $\mathcal{F}$, namely Koopman and Perron-Frobenius operators. Denote the solution of system (1) at time t starting from initial condition $\mathbf{x}$ by $\mathbf{s}_t(\mathbf{x})$. The definitions of these operators along with the infinitesimal generators of these operators are defined as follows.

Definition 1 (Koopman Operator): $\mathbb{K}_t : \mathcal{L}_\infty(\mathbf{X}) \rightarrow \mathcal{L}_\infty(\mathbf{X})$ for dynamical system (1) is defined as

$$[\mathbb{K}_t \varphi](\mathbf{x}) = \varphi(\mathbf{s}_t(\mathbf{x})), \quad \varphi \in \mathcal{L}_\infty, \quad t \geq 0.$$

where $\mathcal{L}_\infty(\mathbf{X})$ is the space of all bounded functions [11]. The infinitesimal generator for the Koopman operator denoted by $\mathcal{K}_f$ is defined as

$$\lim_{t \rightarrow 0} \frac{(\mathbb{K}_t - I)\varphi}{t} = \mathbf{f}(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) =: \mathcal{K}_f \varphi \quad (2)$$

Definition 2 (Perron-Frobenius Operator): $\mathbb{P}_t : \mathcal{L}_1(\mathbf{X}) \rightarrow \mathcal{L}_1(\mathbf{X})$ for dynamical system (1) is defined as

$$[\mathbb{P}_t \psi](\mathbf{x}) = \psi(\mathbf{s}_{-t}(\mathbf{x})) \left| \frac{\partial \mathbf{s}_{-t}(\mathbf{x})}{\partial \mathbf{x}} \right|, \quad \psi \in \mathcal{L}_1, \quad t \geq 0$$

where $|\cdot|$ stands for the determinant and $\mathcal{L}_1(\mathbf{X})$ is the space of integrable functions. The infinitesimal generator for the P-F operator denoted by $\mathcal{P}_f$ is given by

$$\lim_{t \rightarrow 0} \frac{(\mathbb{P}_t - I)\psi}{t} = -\nabla \cdot (\mathbf{f}(\mathbf{x})\psi(\mathbf{x})) =: \mathcal{P}_f \psi \quad (3)$$

These two operators are dual to each other where the duality is expressed as follows.

$$\int_{\mathbb{R}^n} [\mathbb{K}_t \varphi](\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} [\mathbb{P}_t \psi](\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}$$

B. Almost everywhere stability and Stabilization

The formulation for the optimal control problem we present in the dual space is intimately connected to density function and Lyapunov measure introduced for verifying the almost everywhere notion of stability defined below.

Definition 3 (Almost everywhere stable): The equilibrium point at $\mathbf{x} = 0$ is said to be almost everywhere stable w.r.t. measure, μ , if

$$\mu\{\mathbf{x} \in \mathbf{X} : \lim_{t \rightarrow \infty} \mathbf{s}_t(\mathbf{x}) \neq 0\} = 0$$

Following theorem from [5] provides the condition for almost everywhere stability with respect to (w.r.t.) Lebesgue measure.

Theorem 1: Given the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is continuous differentiable and $\mathbf{f}(0) = 0$, suppose there exists a nonnegative ρ is continuous differentiable for $\mathbf{x} \neq 0$ such that $\rho(\mathbf{x})\mathbf{f}(\mathbf{x})/|\mathbf{x}|$ is integrable on $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \geq 1\}$ and

$$[\nabla \cdot (\rho \mathbf{f})](\mathbf{x}) > 0 \text{ for almost all } \mathbf{x}.$$

Then, for almost all initial states $\mathbf{x}(0)$, the trajectory $\mathbf{x}(t)$ converge to zero as $t \rightarrow \infty$.

The density ρ serves as a stability certificate and can be viewed as a dual to the Lyapunov function [5]. Applying Theorem 1 to a control system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$, we arrive at

$$\nabla \cdot (\rho(\mathbf{f} + \mathbf{g}\mathbf{u})) > 0 \text{ for almost all } \mathbf{x}. \quad (4)$$

The control synthesis problem becomes searching for a pair $(\rho, \mathbf{u})$ such that (4) holds. Even though (4) is again bilinear, it becomes linear in terms of $(\rho, \rho\mathbf{u})$. Thus, the density function based method for control synthesis is a convex problem.

C. Sum of squares

Sum of squares optimization [12]–[15] is a relaxation of positive polynomial constraints appearing in polynomial optimization problems which are generally difficult to solve. SoS polynomials are in a set of polynomials which can be described as a finite linear combinations of monomials, i.e., $p = \sum_{i=1}^{\ell} d_i p_i^2$ where p is a SoS polynomial; p_i are monomials; and d_i are coefficients. Hence, SoS is a sufficient condition for nonnegativity of a polynomial and thus SoS relaxation provides a lower bound on the minimization problems of polynomial optimizations. Using the SoS relaxation, any polynomial optimization problems with positive constraints can be formulated as SoS optimization as follows:

$$\min_{\mathbf{d}} \mathbf{w}^\top \mathbf{d} \text{ s.t. } p_s(\mathbf{x}, \mathbf{d}) \in \Sigma[\mathbf{x}], p_e(\mathbf{x}; \mathbf{d}) = 0, \quad (5)$$

where $\Sigma[\mathbf{x}]$ denotes SoS set; $\mathbf{w}$ is weighting coefficients; p_s and p_e are polynomials with coefficients $\mathbf{d}$. The problem in (5) is translated into Semidefinite Programming (SDP) [13], [16]. There are readily available SoS optimization packages such as SOSTOOLS [17] and SOSOPT [18] to solve (5).

III. CONVEX FORMULATION OF OPTIMAL CONTROL PROBLEM

We consider an optimal control problem for input in affine control system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \quad (6)$$

where, $\mathbf{x} \in \mathbf{X} \subseteq \mathbb{R}^n$ is the state, $\mathbf{u} \in \mathbb{R}^m$ is the control input and $\mathbf{g}(\mathbf{x}) = (\mathbf{g}_1(\mathbf{x}), \dots, \mathbf{g}_m(\mathbf{x}))$ with $\mathbf{g}_i \in \mathbb{R}^n$ is the input vector field. Both $\mathbf{f}$ and $\mathbf{g}_i$ are assumed to be $C^1(\mathbf{X}, \mathbb{R}^n)$ (set of all continuously differentiable functions from $\mathbf{X}$ to $\mathbb{R}^n$).

Assumption 2: We assume that the linearization of the nonlinear system at the origin i.e., the pair $(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0), \mathbf{g}(0))$ is stabilizable.

Following the above assumption with no loss of generality we can assume that the $\mathbf{x} = 0$ is locally stable equilibrium point of system (6) with $\mathbf{u} = 0$ with domain of attraction $\mathcal{N}$. Because if it is not the case then we can always design a locally stabilizing controller using Assumption 2. We denote, $\mathbf{X}_1 := \mathbf{X} \setminus \mathcal{N}$. With some abuse of notation we denote by $\mathbf{x}(t)$ solution of system (6) starting from initial condition $\mathbf{x}$.

$$J(\mu) = \int_{\mathbf{X}_1} \int_0^\infty [q(\mathbf{x}(t)) + \mathbf{u}^\top(t) \mathbf{R} \mathbf{u}(t)] dt d\mu(\mathbf{x}) \quad (7)$$

where μ is some given initial measure assumed to be equivalent to Lebesgue with density function h i.e., $\frac{d\mu}{d\mathbf{x}} = h(\mathbf{x})$. The $q : \mathbf{X} \rightarrow \mathbb{R}^+$ is a positive function such that $q(0) = 0$ and $\mathbf{R} > 0$ is positive definite. Some comments on the proposed cost function in (7) are in order. Note that the

cost function, $V(\mathbf{x})$, for the regular optimization problem is parameterized by initial state $\mathbf{x}$ as follows.

$$V(\mathbf{x}) = \int_0^\infty [q(\mathbf{x}(t)) + \mathbf{u}^\top(t) \mathbf{R} \mathbf{u}(t)] dt$$

Hence, we can write

$$J(\mu) = \int_{\mathbf{X}_1} V(\mathbf{x}) h(\mathbf{x}) d\mathbf{x}.$$

The measure μ or the density h corresponding to μ serves as a weighting function to the regular cost V . Hence, different initial conditions are weighted differently in our proposed formulation of the optimal control problem. The objective is to find the feedback controller $\mathbf{k}(\mathbf{x})$ to minimize the cost function (7) i.e.,

$$J^*(\mu) = \inf_{\mathbf{k}} \int_{\mathbf{X}_1} \int_0^\infty [q(\mathbf{x}(t)) + \mathbf{k}(\mathbf{x}(t))^\top \mathbf{R} \mathbf{k}(\mathbf{x}(t))] dt d\mu$$

Note that the cost function is optimized over set $\mathbf{X}_1$ and hence a small neighborhood, $\mathcal{N}$, around the origin is excluded from the optimization. The reason for this is explained in the form of Remark 1.

We now make the following assumption.

Assumption 3: (a) We assume that there exists a feedback controller for which the cost function in (7) is finite and that the optimal control input is feedback form i.e., $\mathbf{u} = \mathbf{k}(\mathbf{x})$ and the function $\mathbf{k}$ is assumed to be $\mathcal{C}^1(\mathbf{X}, \mathbb{R}^m)$. Furthermore, the feedback controller $\mathbf{k}$ is assumed to be almost everywhere stabilizing.

(b) The state cost function $q(\mathbf{x})$ is assumed to be uniformly bounded away from zero outside the neighborhood $\mathcal{N}$. With the assumed feedback form of optimal control input, the optimal control problem can be written as

$$\begin{aligned} \inf_{\mathbf{k}} \quad & \int_{\mathbf{X}_1} \left[\int_0^\infty q(\mathbf{x}) + \beta \|\mathbf{k}(\mathbf{x})\|_1 dt \right] d\mu(\mathbf{x}) \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{k}(\mathbf{x}). \end{aligned} \quad (8)$$

The solution to the above optimization problem with $\mathcal{L}_1$ norm on control input can be obtained by solving the following infinite-dimensional linear program. The proof of the theorem is omitted due to space constraints and we refer the readers to [6].

Theorem 4: Under the Assumption 3, the optimal control problem (8) can be written as following infinite dimensional linear optimization problem

$$\begin{aligned} \inf_{\rho \geq 0, \bar{\rho}} \quad & \int_{\mathbf{X}_1} q(\mathbf{x})\rho(\mathbf{x}) + \beta \|\bar{\rho}(\mathbf{x})\|_1 d\mathbf{x} \\ \text{s.t.} \quad & \nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\rho}) = h, \end{aligned} \quad (9)$$

and the optimal feedback control input recovered from the solution of the above linear program as $\mathbf{k}(\mathbf{x}) = \frac{\bar{\rho}(\mathbf{x})}{\rho(\mathbf{x})}$.

We next consider an optimization problem with $\mathcal{L}_2$ norm on control input,

$$\begin{aligned} \inf_{\mathbf{k}} \quad & \int_{\mathbf{X}_1} \left[\int_0^\infty q(\mathbf{x}) + \mathbf{k}(\mathbf{x})^\top \mathbf{R} \mathbf{k}(\mathbf{x}) dt \right] d\mu(\mathbf{x}) \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{k}(\mathbf{x}) \end{aligned} \quad (10)$$

We now present the main theorem on the convex formulation of the optimal control problem.

Theorem 5: Under the Assumption 3, the optimal control problem (10) can be written as following infinite dimensional convex optimization problem

$$\begin{aligned} \inf_{\rho \geq 0, \bar{\rho}} \quad & \int_{\mathbf{X}_1} q(\mathbf{x})\rho(\mathbf{x}) + \frac{\bar{\rho}(\mathbf{x})^\top \mathbf{R} \bar{\rho}(\mathbf{x})}{\rho} d\mathbf{x} \\ \text{s.t.} \quad & \nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\rho}) = h \end{aligned} \quad (11)$$

and the optimal feedback control input recovered as

$$\mathbf{k}(\mathbf{x}) = \frac{\bar{\rho}(\mathbf{x})}{\rho(\mathbf{x})}.$$

Furthermore the controller $\mathbf{k}$ is a.e stabilizing the equilibrium point at the origin.

Remark 1: The cost function is optimized in the region excluding the small neighborhood around the origin. The reason for this can be explained as follows. The optimal density function serves a density function for occupation measure. In particular, $\bar{\mu}(A) := \int_A \rho(\mathbf{x}) d\mathbf{x}$ signifies the amount of time system trajectories for the close loop system spend in set A . Hence the optimal density function has singularity at the origin as all the trajectories are funnelled to the origin. This singularity at the origin create computation challenge for the approximation of the cost function near the origin. There are two potential approaches to address this challenge due to singularity. The first approach is to ignore the optimality at the origin as the neighborhood around the origin is small the performance of closed loop system is not compromised. The second approach is to design a locally optimal stabilizing controller based on the linearization of the nonlinear system at the origin. The local optimal controller can be designed as a linear quadratic regulator. In this paper, we adopt the later approach.

IV. SoS-BASED COMPUTATION FRAMEWORK FOR OPTIMAL CONTROL

In this section, we provide SoS-based computational framework for the finite dimensional approximation of optimal control formulation involving $\mathcal{L}_1$ norm (Eq. (8)) and $\mathcal{L}_2$ norm (Eq. (10)) for the control cost.

A. Optimal Control with $\mathcal{L}_1$ norm of feedback control

Consider the parameterization of $\rho(\mathbf{x})$ and $\bar{\rho}(\mathbf{x})$ as follows:

$$\rho(\mathbf{x}) = \frac{a(\mathbf{x})}{b(\mathbf{x})^\alpha}, \quad \bar{\rho}(\mathbf{x}) = \frac{\mathbf{c}(\mathbf{x})}{b(\mathbf{x})^\alpha}, \quad (12)$$

where the polynomials $a(\mathbf{x}) \geq 0$ and $\mathbf{c}(\mathbf{x}) = [c_1(\mathbf{x}), \dots, c_m(\mathbf{x})]^\top$, having some arbitrary degrees. By notation, we denote the minimum and maximum degrees $\underline{d}_{\mathbf{p}}$ and $\bar{d}_{\mathbf{p}}$ of any polynomials or polynomial vectors $\mathbf{p}(\mathbf{x})$ by $\deg(\mathbf{p}(\mathbf{x})) = \underline{d}_{\mathbf{p}} : \bar{d}_{\mathbf{p}}$. Typically, it requires $\deg(a(\mathbf{x})) \geq \deg(\mathbf{c}(\mathbf{x}))$ for high-order control $u = \mathbf{k}(\mathbf{x})$. Here, $b(\mathbf{x})$ is an arbitrary positive polynomial (positive at $\mathbf{x} \neq 0$), and α is a positive constant which is sufficiently large so that the integrability condition in Theorem 1 holds. Using (12),

we can restate the left-hand side of the constraint in (9) as follows [3]:

$$\begin{aligned}\nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\rho}) &= \nabla \cdot \left[\frac{1}{b^\alpha} (\mathbf{f}\mathbf{a} + \mathbf{g}\mathbf{c}) \right] \\ &= \frac{1}{b^{\alpha+1}} [b\nabla \cdot (\mathbf{f}\mathbf{a} + \mathbf{g}\mathbf{c}) - \alpha \nabla b \cdot (\mathbf{f}\mathbf{a} + \mathbf{g}\mathbf{c})].\end{aligned}$$

Thus, it becomes finding a , and $\mathbf{c}$, such that

$$b\nabla \cdot (\mathbf{f}\mathbf{a} + \mathbf{g}\mathbf{c}) - \alpha \nabla b \cdot (\mathbf{f}\mathbf{a} + \mathbf{g}\mathbf{c}) > 0 \quad (13)$$

is a non-negative polynomial function. Furthermore, $b(\mathbf{x})$ can be chosen as a quadratic control Lyapunov function for the linearized dynamics at $\mathbf{x} = 0$ to guarantee a local stabilization near the origin. Combining (12)–(13), (9) can be rewritten as follows:

$$\begin{aligned}\inf_{\rho \geq 0, \bar{\rho}} \int_{\mathbf{X}_1} \frac{q(\mathbf{x})a(\mathbf{x})}{b(\mathbf{x})^\alpha} + \frac{\beta \|\mathbf{c}(\mathbf{x})\|_1}{b(\mathbf{x})^\alpha} d\mathbf{x} \\ \text{s.t. } (13) \geq 0, a(\mathbf{x}) \geq 0,\end{aligned} \quad (14)$$

where a small neighborhood of the origin

$$\mathcal{N} = \{\mathbf{x} \in \mathbf{X} : |\mathbf{x}| \leq \epsilon, \epsilon > 0\}$$

is chosen as a polytope and excluded from the integration of the cost function to remove singularity at the origin for the reason mentioned in Remark 1. Although the proposed feedback control stabilizes the system for the entire $\mathbf{X}$, we design an optimal linear quadratic regulator (LQR) controller for $\mathcal{N}$ from the linearized dynamics of the model to guarantee the optimality of control costs in $\mathcal{N}$.

By calculating the integrals in the cost function in terms of coefficient vectors of the polynomials and substituting non-negativity constraints with SoS constraints, (14) can be expressed as a SoS problem as below:

$$\begin{aligned}\min_{\substack{\mathbf{c}_a, \mathbf{c}_{c_j}, \mathbf{c}_{s_j} \\ j=1, \dots, m}} \mathbf{d}_1^\top \mathbf{c}_a + \beta \sum_{j=1}^m \mathbf{d}_2^\top \mathbf{c}_{s_j} \\ \text{s.t. } (13) \in \Sigma[\mathbf{x}], a(\mathbf{x}) \in \Sigma[\mathbf{x}], \\ (\mathbf{s}(\mathbf{x}) - \mathbf{c}(\mathbf{x})) \in \Sigma[\mathbf{x}], \\ (\mathbf{s}(\mathbf{x}) + \mathbf{c}(\mathbf{x})) \in \Sigma[\mathbf{x}],\end{aligned} \quad (15)$$

where $\mathbf{s}(\mathbf{x})$ are the polynomials equal to $|\mathbf{c}(\mathbf{x})|$; $\mathbf{c}_a$, $\mathbf{c}_{c_j}$ and $\mathbf{c}_{s_j}$ are the vectors containing coefficients of $a(\mathbf{x})$, $c_j(\mathbf{x})$, and $s_j(\mathbf{x})$ in terms of the monomial vector, $\Psi(\mathbf{x}) = [1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_n^2, \dots]^\top$ such that

$$a(\mathbf{x}) = \mathbf{c}_a^\top \Psi(\mathbf{x}), c_j(\mathbf{x}) = \mathbf{c}_{c_j}^\top \Psi(\mathbf{x}), s_j(\mathbf{x}) = \mathbf{c}_{s_j}^\top \Psi(\mathbf{x}),$$

for $j = 1, \dots, m$, with $\Psi(\mathbf{x})$ having the maximum degree of $a(\mathbf{x})$, $c_j(\mathbf{x})$, and $s_j(\mathbf{x})$; and the vectors of the coefficients $\mathbf{d}_1$ and $\mathbf{d}_2$ are calculated as

$$\mathbf{d}_1 = \int_{\mathbf{X}_1} \frac{q(\mathbf{x})\Psi(\mathbf{x})}{b(\mathbf{x})^\alpha} d\mathbf{x}, \quad \mathbf{d}_2 = \int_{\mathbf{X}_1} \frac{\Psi(\mathbf{x})}{b(\mathbf{x})^\alpha} d\mathbf{x}. \quad (16)$$

B. Optimal Control with $\mathcal{L}_2$ norm of feedback control

Following the same parameterization in (12), $\mathcal{L}_2$ optimal control problem in (11) is restated as:

$$\begin{aligned}\min \int_{\mathbf{X}_1} \frac{q(\mathbf{x})a(\mathbf{x})}{b(\mathbf{x})^\alpha} + \frac{\mathbf{c}(\mathbf{x})^\top \mathbf{R}\mathbf{c}(\mathbf{x})}{a(\mathbf{x})b(\mathbf{x})^\alpha} d\mathbf{x} \\ \text{s.t. } (13) \geq 0, a(\mathbf{x}) \geq 0,\end{aligned} \quad (17)$$

where we exclude a small neighborhood of the origin to avoid singularity at the origin (refer to Remark 1). To convert (17) into a SoS problem, we first reformulate (17) as follows:

$$\begin{aligned}\min \int_{\mathbf{X}_1} \frac{q(\mathbf{x})a(\mathbf{x})}{b(\mathbf{x})^\alpha} + \frac{w(\mathbf{x})}{b(\mathbf{x})^\alpha} d\mathbf{x} \\ \text{s.t. } (13) \geq 0, a(\mathbf{x}) \geq 0, \\ \mathbf{M}(\mathbf{x}) = \begin{bmatrix} w(\mathbf{x}) & \mathbf{c}(\mathbf{x})^\top \\ \mathbf{c}(\mathbf{x}) & a(\mathbf{x})\mathbf{R}^{-1} \end{bmatrix} \succcurlyeq 0,\end{aligned} \quad (18)$$

where the positive semidefinite constraint of the polynomial matrix $\mathbf{M}(\mathbf{x})$ is a result of applying the Schur complement lemma on the $\mathcal{L}_2$ control cost bounded by $w(\mathbf{x})$, i.e., $\frac{\mathbf{c}(\mathbf{x})^\top \mathbf{R}\mathbf{c}(\mathbf{x})}{a(\mathbf{x})} \leq w(\mathbf{x})$; Now, to algebraically express the positive semidefinite matrix $\mathbf{M}(\mathbf{x})$, we first introduce the following lemma:

Lemma 6 (Positive semidefinite polynomial matrix): A $p \times p$ matrix $\mathbf{H}(\mathbf{x})$ whose entries are polynomials is positive semidefinite with respect to the monomial vector $\mathbf{z}(\mathbf{x})$, if and only if, there exist $\mathbf{D} \succcurlyeq 0$ such that

$$\mathbf{H}(\mathbf{x}) = (\mathbf{z}(\mathbf{x}) \otimes \mathbf{I}_p)^\top \mathbf{D} (\mathbf{z}(\mathbf{x}) \otimes \mathbf{I}_p),$$

where $\otimes$ denotes a Kronecker product (tensor product) and $\mathbf{I}_p$ is an identity matrix with dimension p [19].

Following Lemma 6, let $\mathbf{z}(\mathbf{x})$ be a monomial vector with $\deg(\mathbf{z}(\mathbf{x})) = \text{floor}\left(\frac{\max(\deg(\Psi(\mathbf{x})))}{2}\right) + 1$ where $\max(\deg(\Psi(\mathbf{x})))$ denotes the maximum degree of the polynomial vector $\Psi(\mathbf{x})$, then $\mathbf{M}(\mathbf{x})$ in (18) is positive semidefinite when there exists $\mathbf{D} \succcurlyeq 0$ such that $\mathbf{M}(\mathbf{x}) = \mathbf{H}(\mathbf{x})$. Using this result and also the integrals of the cost functions in terms of $\Psi(\mathbf{x})$ shown in (16), a SoS problem equivalent to (18) can be formulated as follows:

$$\begin{aligned}\min_{\substack{\mathbf{c}_a, \mathbf{c}_w, \mathbf{c}_{c_j} \\ j=1, \dots, m}} \mathbf{d}_1^\top \mathbf{c}_a + \mathbf{d}_2^\top \mathbf{c}_w \\ \text{s.t. } (13) \in \Sigma[\mathbf{x}], a(\mathbf{x}) \in \Sigma[\mathbf{x}], \\ w(\mathbf{x}) - \mathbf{H}_{11}(\mathbf{x}) = 0, \\ \mathbf{c}(\mathbf{x}) - \mathbf{H}_{12}(\mathbf{x}) = 0, \\ a(\mathbf{x})\mathbf{R}^{-1} - \mathbf{H}_{22}(\mathbf{x}) = 0, \\ \mathbf{D} \succcurlyeq 0,\end{aligned} \quad (19)$$

where $\mathbf{H}_{ij}(\mathbf{x})$ denotes a polynomial in ij th entry of $\mathbf{H}(\mathbf{x})$; and $\mathbf{c}_w$ is a vector of coefficient of $w(\mathbf{x})$ in terms of $\Psi(\mathbf{x})$, i.e., $w(\mathbf{x}) = \mathbf{c}_w^\top \Psi(\mathbf{x})$.

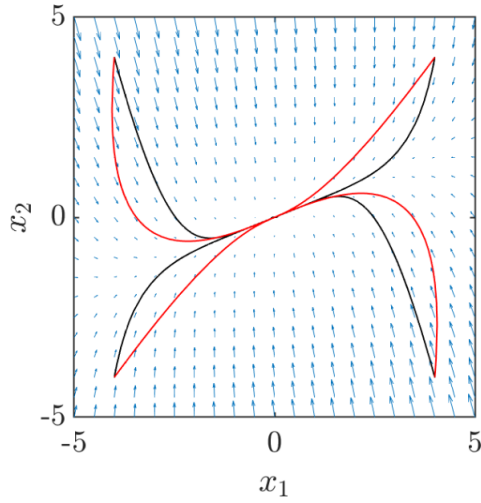


Fig. 1: Closed-loop trajectories of the system in Ex. 1 obtained using LQR control (red) SoS based computational method (black).

V. SIMULATION RESULTS

In this section, we present simulation results for the model-based optimal control involving examples of systems with polynomial vector field.

Example 1: Our first example is that of linear system. The example is used to compare the analytically derived optimal control obtained using linear quadratic regulator (LQR) with the one obtained using the proposed SoS based computational framework.

$$\dot{x}_1 = -2x_1 + x_2, \quad \dot{x}_2 = 3x_2 + u.$$

The finite-dimensional approximation of the infinite dimensional optimization problem is obtained as follows. The various polynomials along with the degrees are chosen as follows: $\deg(a(\mathbf{x})) = 0 : 1$, $\deg(c(\mathbf{x})) = 0 : 1$, $\deg(s(\mathbf{x})) = 0 : 2$, $\alpha = 4$, and $b(\mathbf{x}) = 0.1959x_1^2 - 0.0374x_1x_2 + 0.1304x_2^2$. Also, the parameters for $\mathcal{L}_2$ optimal control costs are chosen with reference to (14) as follows: $\beta = 0.1$, $\mathbf{R} = 1$, and $q(\mathbf{x}) = x_1^2 + x_2^2$. By solving the SoS optimization problems (18) for $\mathcal{L}_2$ control cost described in Section IV, an optimal control solution is obtained, $u(\mathbf{x}) = \frac{c(\mathbf{x})}{a(\mathbf{x})}$. In Fig. 1, we compare closed loop trajectories with control obtained using LQR formula (red) and proposed SoS based computational framework (black). In Fig. 2, we show the plot comparison of the control inputs.

Example 2: Consider the dynamics of controlled Van der Pol oscillator as follows:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = (1 - x_1^2)x_2 - x_1 + u.$$

In this example, we will consider $\mathcal{L}_2$ feedback control. For the procedure of formulating finite-dimensional optimal control problems described in Section IV for the feedback

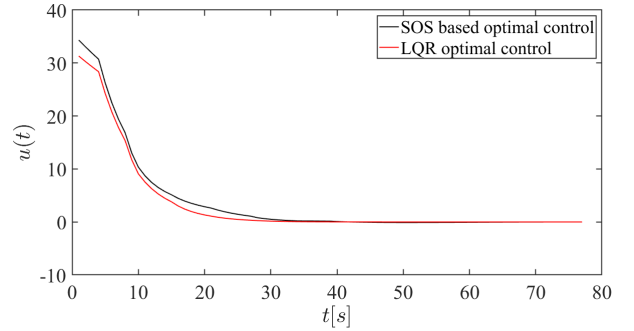


Fig. 2: Comparison of control inputs: LQR optimal control (red) and optimal control obtained using SoS based computational framework (black) for Ex. 1 with initial condition at $(-4, -4)$.

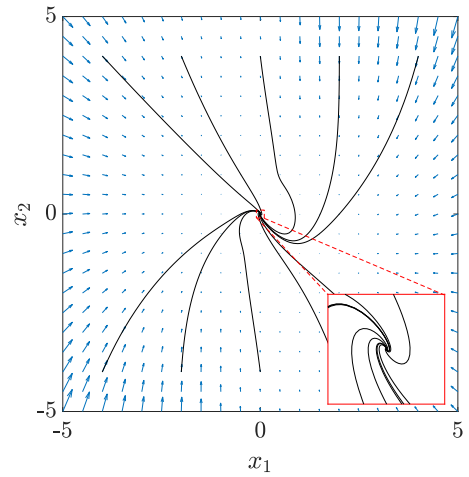


Fig. 3: Closed loop dynamics of Van der Pol oscillator using optimal control with $\mathcal{L}_2$ norm for control input.

control, we consider the parameters to (12)–(14) as follows: $\deg(a(\mathbf{x})) = 0 : 3$, $\deg(c(\mathbf{x})) = 0 : 6$, $\deg(s(\mathbf{x})) = 0 : 7$, $b(\mathbf{x}) = 5.8x_1^2 + 2.9x_1x_2 + 2.9x_2^2$, $\alpha = 4$, $q(\mathbf{x}) = 3.5x_1^2 + x_2^2$, and $\mathbf{R} = 1$. By solving the resulting SoS optimization problems in Section IV, we get the stabilized trajectories starting from arbitrary initial points in $\mathbf{X} = [-4, 4] \times [-4, 4]$, converging to the origin, as shown in Fig. 3. Similar to the previous example, the small red box in the figures represents the small neighborhood of the origin, $\mathcal{N} = [-0.1, 0.1] \times [-0.1, 0.1]$, where the control is switched to the local LQR controller once the trajectories are attracted to $\mathcal{N}$ towards the origin by the nonlinear optimal feedback controls synthesized from the proposed method. This neighbourhood is explicitly characterised a priori (see *Remark 2*).

Example 3: Consider the of controlled Lorentz attractor:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2 + u, \\ \dot{x}_3 &= x_1x_2 - \gamma x_3, \end{aligned}$$

where $\sigma = 10$, $\rho = 28$, and $\gamma = \frac{8}{3}$. To formulate finite-dimensional $\mathcal{L}_2$ optimal control problem, consider $\deg(a(\mathbf{x})) = 0 : 2$, $\deg(c(\mathbf{x})) = 0 : 9$, and $\deg(s(\mathbf{x})) = 0 : 10$, $b(\mathbf{x}) = 39.15x_1^2 + 60.82x_1x_2 + 23.7x_2^2 + 0.2x_3^2$, $\alpha = 4$, $q(\mathbf{x}) = 3.5x_1^2 + x_2^2 + x_3^2$, and $\mathbf{R} = 1$. The result of $\mathcal{L}_2$ control synthesized from our proposed method is shown in Fig. 5. The closed-loop trajectories are stabilized, starting from arbitrary initial conditions in $\mathbf{X} = [-4, 4] \times [-4, 4] \times [-4, 4]$. The LQR control for the linearized system is applied to the region near the origin, $\mathcal{N} = [-0.1, 0.1] \times [-0.1, 0.1] \times [-0.1, 0.1]$.

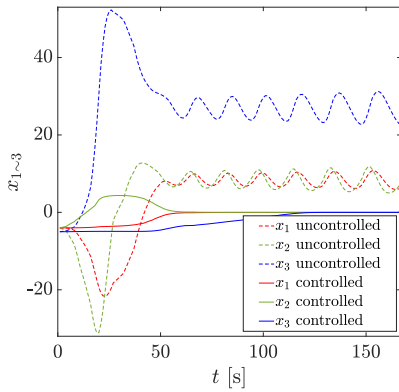


Fig. 4: Trajectories in states vs. time of Lorentz attractor simulated from open-loop as well as optimal control with $\mathcal{L}_2$ control norm, starting from some disturbed initial points converge to the origin while open-loop dynamics shows chaotic behavior.

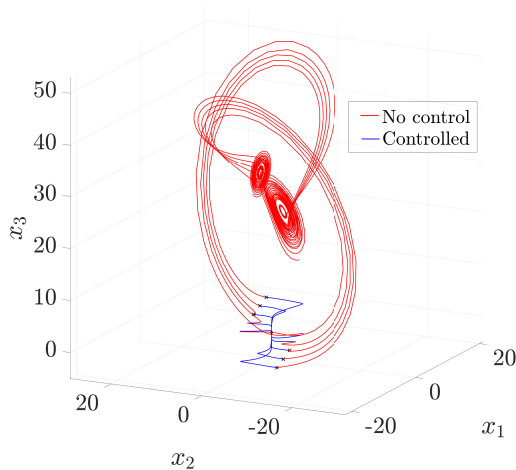


Fig. 5: Dynamics of Lorentz attractor, for both closed-loop system using optimal control with $\mathcal{L}_2$ norm for control input (blue) converging to the origin (denoted by black dot) as well as the open-loop dynamics (red), starting from initial points (denoted by $\times$).

VI. CONCLUSION

A systematic convex optimization-based framework is provided for optimal control of nonlinear system. The

optimal control problem is formulated in the dual space of density function leading to infinite dimensional convex optimization problem for optimal control. The proposed approach uses the SoS optimization framework for the finite-dimensional approximation of the optimization problem. Future research efforts will focus on extending the framework to data-driven setting, where the explicit knowledge of system dynamics is not assumed to be known.

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