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On the odd order composition factors of finite linear groups

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Abstract. In this paper, we study the product of orders of composition factors of odd order in a composition series of a finite linear group. First we generalize a result by Manz and Wolf about the order of solvable linear groups of odd order. Then we use this result to find bounds for the product of orders of composition factors of odd order in a composition series of a finite linear group.

1 Introduction

The order of a finite group is perhaps the most fundamental quantity in group theory one can study. Accordingly, the concept of bounding the order of a finite group is a very natural one and has long been a subject of vigorous research. For example, Manz and Wolf obtained the following result [9, Theorem 3.5] in bounding the order of a solvable linear group by the size of the vector space on which it acts. For the rest of this paper, we let $\lambda = \sqrt[3]{24}$ and let

$$\alpha = (3 \cdot \log(48) + \log(24))/(3 \cdot \log(9)) \approx 2.25.$$

Theorem 1.1. *Let G be a finite solvable group, and let $V \neq 0$ be a finite, faithful, completely reducible G-module with* $\mathrm{char}(V) = p > 0$. *Then*

(a) $|G| \leq |V|^{\alpha}/\lambda$.

(b) *If* $2 \nmid |G|$ *or if* $3 \nmid |G|$, *then* $|G| \leq |V|^2/\lambda$.

(c) *If* $2 \nmid |G|$ *and* $p \neq 2$, *then* $|G| \leq |V|^{3/2}/\lambda$.

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In light of this result, it is natural to ask whether one can extend (b) and (c) to a similar result for the order of a subgroup H of a completely reducible linear group G (note that H need not be completely reducible on V).

It should be pointed out that several recent advances have improved the previous theorem. For instance, Guralnick, Maróti and Pyber [5] found a bound for the product of abelian composition factors of a primitive permutation group, and Halasi and Maróti [6] generalized part (a) of the above theorem to p-solvable groups.

Inspired by the above results and a sequence of papers written by the fifth author [8, 11], we consider the product of the orders of certain abelian composition factors. By combining the techniques used in [8, 9], we obtain an upper bound for the product of the orders of the odd order (abelian) composition factors of an arbitrary linear group, which generalizes part of Theorem 1.1 to an arbitrary finite linear group.

We define $a(G)$ to be the product of orders of composition factors of odd order in a composition series of a finite group G. By the Jordan–Hölder theorem, we see that this quantity is independent of the choice of composition series.

Our main result is the following.

Theorem 1.2. *Let G be a finite group acting on V faithfully and completely reducibly, where V is of characteristic p. Then the following hold.*

(1) $a(G) \leq |V|^2/\lambda$.

(2) *If $p \neq 2$, then* $a(G) \leq |V|^{3/2}/\lambda$.

The paper is organized as follows. In Section 2, we prove a slight generalization of [9, Theorem 3.5 (b) and (c)] which includes the solvable case of Theorem 1.2. In Section 3, we prove some properties of simple groups that are needed to reduce the general case to solvable groups. In Section 4, we prove a related result about permutation groups and then prove the main theorem of the paper.

We will use the following notation for the remainder of the paper. All groups in this paper are assumed to be finite. Given a group G, we use $F(G)$ to denote the Fitting subgroup of G, and $F^*(G)$ to denote the generalized Fitting subgroup of G. The layer of G is denoted by $E(G)$, and $\mathrm{Out}(G)$ is the outer automorphism group of G. In addition, for a prime p, we denote the order of Hall p'-subgroups of G by $|G|_{p'}$.

2 The solvable case

In this section, we generalize [9, Theorem 3.5 (b) and (c)] to a subgroup H of G that satisfies the respective conditions. We note that the action of H on V need not be completely reducible, and thus the generalization is not trivial.

Proposition 2.1. *Let G be a finite solvable group, and let $V \neq 0$ be a finite, faithful, completely reducible G-module with* $\mathrm{char}(V) = p > 0$. *Let H be a subgroup of G.*

(1) *If $2 \nmid |H|$ or if $3 \nmid |H|$, then $|H| \leq |V|^2/\lambda$.*

(2) *If $2 \nmid |H|$ and $p \neq 2$, then $|H| \leq |V|^{3/2}/\lambda$.*

Proof. Since G is solvable, we only need to consider the Hall $2'$-subgroup or the Hall $3'$-subgroup of G. The proof follows the arguments in [9, Theorem 3.5] with some slight adjustments in each of the steps. For consistency, we will adopt the notation used in [9, Theorem 3.5]. Step 1 shows that V is irreducible and the argument here is unchanged. Step 2 shows that V is quasi-primitive, and the argument there is unchanged as well. Step 3 shows that if we set $|V| = p^n$, then we may assume that $n \geq 2$ and $p^n \geq 16$. The calculation remains the same.

In step 4, we show that $G \not\leq \Gamma(p^n)$, $n > 3$, and if $p = 2$, then $n \geq 8$. All the arguments are the same with the exception of proving $n > 3$ for statement (2). Assume $n = 2$; we note that $e = 2$, $2 \mid p - 1$ and $p \geq 5$. We have

$$|G|_{2'} \leq 1/2 \cdot (p-1) \cdot 3 \leq p^3/3 \leq |V|^{3/2}/\lambda.$$

When $n = 3$, we have $e = 3$, $p \geq 7$, and thus $|V| \geq p^3$. Thus

$$|G| = |T||F/T||G/F| \mid (p-1) \cdot 9 \cdot 24.$$

We observe that $|G|_{2'} \leq \frac{p-1}{2} \cdot 27 \leq p^{4.5}/\lambda \leq |V|^{3/2}/\lambda$.

In step 5, by examining the proof of [9, Theorem 3.5] carefully, we only need to check a few cases when e is small for case (2).

(1) If $e = 2$, then $|G|$ is divisible by 8 and $A/F \leq \mathrm{GL}(2,2)$. Thus $|A/F|_{2'} \leq 3$. Since $|V| \geq 81$, we have

$$|G|_{2'} = (|G/A||A/F||F/T||T|)_{2'} \leq 3|U|^2 \leq 3|V| \leq \frac{|V|^{3/2}}{3}.$$

(2) If $e = 3$, then $|A/F| \leq \mathrm{GL}(2,3)$, $p \geq 4$. Thus $|A/F|_{2'} \leq 3$. Since $|V| \geq 256$, we have

$$|G|_{2'} = (|G/A||A/F||F/T||T|)_{2'} \leq 27 \cdot |U|^2 \leq 27 \cdot |V|^{2/3} \leq \frac{|V|^{3/2}}{3}.$$

(3) If $e = 4$, we note that $|A/F|_{2'} \leq 15$ and $|V| \geq 81$. Thus we have

$$|G|_{2'} \leq (|G/A||A/F||F/T||T|)_{2'} \leq 15 \cdot |U|^2 \leq 15 \cdot |V|^{1/2} \leq \frac{|V|^{3/2}}{3}.$$

This completes the proof. $\square$

3 Properties of simple groups

The following property about the odd order subgroups of simple groups is needed for the reduction of the main theorem to the solvable case, which also has some applications to the study of quantitative aspects of orbit structure of linear groups.

The general outline of the following proof is, for most finite simple groups of Lie type, we use results related to Zsigmondy primes to find two prime divisors L_1 and L_2 of $|G|$ such that there exist subgroups H_1, H_2 with $H_i = \mathrm{Syl}_{L_i}(G)$ satisfying the conditions required. There are some exceptional cases when either the rank or the size of the finite field is small. In these cases, one cannot find suitable Zsigmondy primes. We handle these exceptional cases by checking the bounds via direct calculation.

Lemma 3.1. *Let G be a finite non-abelian simple group, and let r be a fixed prime. Then there exists a solvable subgroup H of G such that*

$$(|H|, r) = 1 \quad \textit{and} \quad |H|_{2'} \geq 2|\mathrm{Out}(G)|_{2'}.$$

Proof. We now go through the Classification of Finite Simple Groups.

(1) Let G be one of the alternating groups A_n, $n \geq 5$. It is well known that $|\mathrm{Out}(\mathsf{A}_n)| = 2$ except when $n = 6$ and $|\mathrm{Out}(\mathsf{A}_6)| = 4$. Thus $|\mathrm{Out}(\mathsf{A}_n)|_{2'} = 1$. Since $5 \mid |\mathsf{A}_n|$ and $3 \mid |\mathsf{A}_n|$, the result follows.

(2) Let G be one of the sporadic or Tits groups. Then $|\mathrm{Out}(G)| \leq 2$, and the result can be confirmed by [2].

For simple groups of Lie type, we go through various families of Lie type. To illustrate the method, [8, Proposition 4.1] gives a detailed analysis for $\mathrm{A}_n(q)$ and shows how to handle most cases. For those finitely many exceptional cases, we will check that the required inequalities hold by direct calculation. Since these arguments are similar, for the remaining families of simple groups of Lie type, there is a table in [8, Proposition 4.1] that handles all the exceptional cases.

(3) Let $G = \mathrm{A}_1(q)$, where $q = p^f$. We have

$$|G| = q(q+1)(q-1)d^{-1},$$

where $d = (2, q-1)$ and $|\mathrm{Out}(G)| = df$.

Case (a). Suppose that q is even. Then $d = 1$ and $|\mathrm{Out}(G)| = f$.

Assume there exists a Zsigmondy prime L_1 for $p^{2f} - 1$. Then $L_1 \mid p^{2f} - 1$, and thus $L_1 \mid p^f + 1$ and $L_1 \geq 2f = 2|\mathrm{Out}(G)| \geq 2|\mathrm{Out}(G)|_{2'}$.

Assume there exists a Zsigmondy prime $L_2 \mid p^f - 1$, where $L_2 \geq f$. It is clear that $L_1 \neq L_2$. If $L_2 \geq 2f$, then we are done. Otherwise, if $L_2^2 \mid p^f - 1$, we consider the Sylow L_2-subgroup L. Then $|L| \geq 2f$. However, we have the following exceptions by [8, Lemma 3.1].

(i) $f = 4$, thus $|\mathrm{Out}(G)| = 4$, and $|\mathrm{Out}(G)|_{2'} = 1$. Since $2^4 + 1 = 17$ and $2^4 - 1 = 15 = 3 \cdot 5$, we may choose $L_1 = 17$ and $L_2 = 5$.

(ii) $f = 6$, thus $|\mathrm{Out}(G)| = 6$ and $|\mathrm{Out}(G)|_{2'} = 3$. Since $2^6 + 1 = 65 = 5 \cdot 17$ and $2^6 - 1 = 63 = 7 \cdot 3^2$, we may choose $L_1 = 13$ and $L_2 = 7$.

(iii) $f = 12$, thus $|\mathrm{Out}(G)| = 12$ and $|\mathrm{Out}(G)|_{2'} = 3$. Since

$$2^{12} + 1 = 4097 = 17 \cdot 241 \quad \text{and} \quad 2^{12} - 1 = 4095 = 3^2 \cdot 5 \cdot 7 \cdot 13,$$

we may choose $L_1 = 17$ and $L_2 = 7$.

Case (b). Suppose that q is odd. Then $d = 2$ and $|\mathrm{Out}(G)| = 2f$, implying that $|\mathrm{Out}(G)|_{2'} = f$. We apply the same idea as before,

$$L_1 \mid p^f + 1 \quad \text{and} \quad L_1 \geq 2f = 2|\mathrm{Out}(G)|_{2'}.$$

There exists an $L_2 \mid p^f - 1$, where $L_2 \geq f$ and $L_1 \neq L_2$. If $L_2 \geq 2f$, then we are done. Otherwise, if $L_2^2 \mid p^f - 1$, we consider the Sylow L_2-subgroup L. Then $|L| \geq 2f$. The following case is the exception by [8, Lemma 3.1]:

(i) When $p = 3$, $f = 4$, thus $|\mathrm{Out}(G)|_{2'} = 1$. Since $3^4 + 1 = 82 = 2 \cdot 41$ and $3^4 - 1 = 80 = 2^4 \cdot 5$, we may choose $L_1 = 41$ and $L_2 = 5$.

(4) Let $G = \mathrm{A}_n(q)$, where $q = p^f$ and $n \geq 2$. Set $m = \prod_{i=1}^{n}(q^{i+1} - 1)$. Then $|G| = d^{-1}q^{n(n+1)/2}m$, $|\mathrm{Out}(G)| = 2fd$, where $d = (n+1, q-1)$.

With the exception of a finite number of cases, there exists a Zsigmondy prime L_1 for $p^{f(n+1)} - 1$ such that $L_1 \geq 2f(n+1)$ or $L_1^2 \mid p^{f(n+1)} - 1$. It follows that $L_1^2 \geq 2f(n+1)$. Let H_1 be a Sylow L_1-subgroup G. By [8, Lemma 3.2], with the exception of a finite number of cases, there exists a Zsigmondy prime L_2 for $p^{fn} - 1$ such that $L_2 \geq 3fn \geq 2f(n+1)$ or $L_2^2 \mid p^{fn} - 1$. This implies that $L_2^2 \geq 3fn \geq 2f(n+1)$. Let H_2 be a Sylow L_2-subgroup of G. Notice that $L_1 \neq L_2$.

Since $|\mathrm{Out}(G)| = 2fd$, $|\mathrm{Out}(G)|_{2'} \leq fd$. Also, $n + 1 \geq d = (n+1, q-1)$, and $|H_1|, |H_2| \geq 2|\mathrm{Out}(G)|_{2'}$. Therefore, the result follows. The exceptions are listed in Table 1 (by [8, Lemma 3.2]).

(5) Let $G = {}^2\mathrm{A}_n(q^2)$, where $n \geq 2$. Note that if $n = 2$, then $q > 2$. Set

$$m = \prod_{i=1}^{n}(q^{i+1} - (-1)^{i+1}), \quad q^2 = p^f \quad \text{and} \quad d = (n+1, q+1).$$

Then $|G| = d^{-1}mq^{n(n+1)/2}$, and $|\mathrm{Out}(G)| = df$. By [4, Theorem A] and [8, Lemma 3.2], there exists a Zsigmondy prime $L_1 \mid p^{f(n+1)/2} - (-1)^{n+1}$ such that $L_1 \geq 2(n+1)f \geq 2df$ or

$$L_1^2 \mid p^{f(n+1)/2} - (-1)^{n+1} \quad \text{and} \quad L_1^2 \geq 2(n+1)f \geq 2df.$$

p	n	f	d	$\lvert\mathrm{Out}(G)\rvert_{2'}$	$\lvert H_1\rvert$	$\lvert H_2\rvert$
2	3, 4, 6, 8, 12, 20	1	1	1	divides $2^{(n+1)}-1$	divides 2^n-1
2	2	2	3	3	9	7
2	2	3	1	3	73	7
2	3	2	1	1	7	17
2	2	4	3	3	17	13
2	4	2	1	1	31	17
2	2	6	3	9	73	19
2	3	4	1	1	257	17
2	4	3	1	3	151	31
2	6	2	1	1	127	43
2	2	10	3	15	331	151
2	4	5	1	5	31	11
2	5	4	1	1	31	11
2	10	2	1	1	31	11
3	2	2	1	1	7	5
3	2	3	1	3	13	7
3	3	2	4	1	41	13

Table 1. Exceptional cases for $\mathrm{A}_n(q)$.

Moreover, by [4, Theorem A] and [8, Lemma 3.2], with the exception of a finite number of cases, there exists a Zsigmondy prime $L_2 \mid p^{fn/2} - (-1)^{n+1}$ such that $L_2 \geq \frac{5}{2}(n+1)f \geq 2(n+1)f \geq 2df$ or

$$L_2^2 \mid p^{fn/2} - (-1)^{n+1} \quad \text{and} \quad L_2^2 \geq \frac{5}{2}(n+1)f \geq 2(n+1)f \geq 2df.$$

For all the exceptional cases, we can check the results via direct calculation and by considering the order of the group G and $\mathrm{Out}(G)$ (see Table 2).

We now provide a table (Table 3) for the simple groups of Lie types other than $\mathrm{A}_n(q)$ and ${}^2\mathrm{A}(q^2)$. The second and the third columns in the table are two large prime divisors that correspond to Sylow subgroups. □

4 Composition factors of odd order

In this section, we prove the main result of the paper. Before doing so, we need the following proposition about permutation groups.

p	n	$f/2$	d	$\lvert \mathrm{Out}(G)\rvert_{2'}$	$\lvert H_1\rvert$	$\lvert H_2\rvert$
2	3	1	1	1	9	5
2	2	2	1	1	13	5
2	4	1	1	1	13	5
2	2	3	3	9	243	19
2	3	2	1	1	13	5
2	6	1	1	1	43	7
2	2	4	1	1	241	17
2	4	2	5	5	41	25
2	8	1	3	3	19	17
2	2	5	3	15	331	31
2	5	2	1	1	7	5
2	10	1	1	1	31	11
2	2	6	1	3	37	13
2	3	4	1	1	241	17
2	4	3	1	3	13	11
2	6	2	1	1	7	5
2	12	1	1	1	7	5
2	2	9	3	27	87211	73
2	3	6	1	3	37	13
2	6	3	1	3	19	7
2	9	2	5	5	41	31
2	18	1	1	1	19	7
2	2	10	1	5	61	41
2	4	5	1	5	31	11
2	5	4	1	1	13	9
2	10	2	1	1	31	11
2	20	1	3	3	41	31
2	2	14	1	7	127	43
2	4	7	1	7	71	43
2	7	4	1	1	257	17
2	14	2	5	5	41	17
2	28	1	1	1	59	17
3	3	1	4	1	5	13
3	2	2	1	1	73	5

Table 2. Exceptional cases for ${}^2\mathrm{A}_n(q^2)$.

p	n	$f/2$	d	$\lvert \mathrm{Out}(G)\rvert_{2'}$	$\lvert H_1\rvert$	$\lvert H_2\rvert$
3	4	1	1	1	61	5
3	2	3	1	3	13	7
3	3	2	2	1	73	41
3	6	1	1	1	13	7
5	2	2	1	1	601	13
5	4	1	1	1	3	313

Table 2 (continued).

Type	L_1	L_2	Exceptional cases
$B_n(q), q = p^f$	$p^{f(2n)} - 1$	$p^{f(2n-2)} - 1$	$(f = 1,\ n = 2)$; $(p = 2,\ f = 3, n = 2)$; $(p = 2,\ f = 1, n = 3)$
$C_n(q), q = p^f$	$p^{f(2n)} - 1$	$p^{f(2n-2)} - 1$	$(f = 1,\ n = 2)$; $(p = 2,\ f = 3, n = 2)$; $(p = 2,\ f = 1, n = 3)$
$D_n(q), q = p^f$	$p^{f(2n-2)} - 1$	$p^{f(2n-4)} - 1$	$(p = 2,\ f = 1, n = 5)$; $(p = 2,\ f = 2, n = 5)$; $(p = 2,\ f = 1, n = 3)$; $(p = 2,\ f = 1, n = 4)$; $(p = 2,\ f = 3, n = 4)$; $(p = 3,\ f = 1, n = 4)$; $(p = 5,\ f = 1, n = 4)$
${}^2D_n(q^2), q^2 = p^f$	$p^{f(n-1)} - 1$	$p^{f(n-2)} - 1$	$(p = 3,\ f = 2, n = 3, 4)$; $(p = 5,\ f = 2, n = 4)$
$E_6(q), q = p^f$	$p^{12f} - 1$	$p^{8f} - 1$	
$E_7(q), q = p^f$	$p^{18f} - 1$	$p^{14f} - 1$	
$E_8(q), q = p^f$	$p^{30f} - 1$	$p^{24f} - 1$	
$F_4(q), q = p^f$	$p^{12f} - 1$	$p^{8f} - 1$	
$G_2(q), q = p^f$	$p^{6f} - 1$	$p^{2f} - 1$	$(p = 2,\ f = 1, 2, 3)$
${}^2E_6(q), q^2 = p^f$	$p^{6f} - 1$	$p^{4f} - 1$	
${}^3D_4(q^3), q^3 = p^f$	$p^{4f} - 1$	$p^{2f} - 1$	$(p = 2,\ f = 3, 5, 7)$
${}^2B_2(2^{2n+1})$	$2^{4(2n+1)} - 1$	$2^{(2n+1)} - 1$	
${}^2F_4(2^{2n+1})$	$2^{4(2n+1)} - 1$	$2^{(2n+1)} - 1$	
${}^2G_2(3^{2n+1})$	$3^{3(2n+1)} + 1$	$3^{(2n+1)} - 1$	

Table 3. Other Lie type groups.

Proposition 4.1. *Let G be a group of permutations on a set Ω of order n. Then $a(G)$ is at most 2^{n-1}.*

Proof. We first check that the result is true for $n \leq 4$ ($|\mathsf{S}_2|_{2'} \leq 2$, $|\mathsf{S}_3|_{2'} \leq 4$, $|\mathsf{S}_4|_{2'} \leq 8$). We may assume that $n \geq 5$. We shall proceed by induction on $|G|$.

We first suppose that G is intransitive on Ω. Write $\Omega = \Gamma_1 \cup \Gamma_2$ for nonempty subsets Γ_i of Ω such that G permutes Γ_1 and also Γ_2. Write $n_i = |\Gamma_i|$ for $i = 1, 2$, so clearly $n_1 + n_2 = n$. Let L be the kernel of G on Γ_1 so that L acts faithfully on Γ_2. By induction, we then have $a(G/L) \leq 2^{n_1-1}$ and $a(L) \leq 2^{n_2-1}$, and so $a(G) \leq 2^{n_1-1} \cdot 2^{n_2-1} = 2^{n-2} < 2^{n-1}$, as desired. So now, we may assume that G is transitive on Ω.

Suppose that G is imprimitive on Ω. Then there is a nontrivial decomposition $\Omega = \bigcup_i \Omega_i$ with G permuting $X = \{\Omega_1, \ldots, \Omega_r\}$ and $N_G(\Omega_1)$ acting primitively on Ω_1, where $|\Omega_1| = m$ and $n = mr$. Let π be the permutation representation of G on X and $K = \ker \pi$. Set

$$K_0 = K \quad \text{and} \quad K_{i+1} = \{g \in K_i \mid g \text{ acts trivially on } \Omega_{i+1}\}.$$

We note that $a(G) = a_0 a_1 \ldots a_r$, where $a_0 = a(G/K)$ and $a_i = a(K_{i-1}/K_i)$ for $i \geq 1$. By induction, $a_0 \leq 2^{r-1}$ and $a_i \leq 2^{m-1}$ for $i \geq 1$. It is easy to see that $a(G) \leq 2^{n-1}$.

Thus we may assume G is primitive; then we know that G either contains A_n or is one of the groups in the exceptional list by [10, Corollary 1.4]. If G contains A_n, then the result is clear since $n \geq 5$. Otherwise, G is one of the groups in the exceptional list; we verify the result in Table 4. □

With this result and the work done in the previous sections, we now can prove the main result.

Proof of Theorem 1.2. Let S be the maximal normal solvable subgroup of G. Consider $\bar{G} = G/S$. It is easy to see that $F(\bar{G}) = 1$. Therefore,

$$F^*(\bar{G}) = F(\bar{G})E(\bar{G}) = E(\bar{G}).$$

Moreover, we know that $Z(E(\bar{G}))$ is trivial; otherwise, S is not maximal. Since $E(\bar{G})/Z(E(\bar{G}))$ is the unique largest semi-simple subgroup of $\bar{G}$, $E(\bar{G})$ is the product of simple non-abelian subgroups.

Let $\bar{C} = C_{\bar{G}}(E(\bar{G}))$. Since $F^*(G)$ is self-centralizing, we have $\bar{C} < F^*(G)$. Let $K = \bar{G}/\bar{C}$. Then K acts faithfully on $E(\bar{G})$. We may assume that K acts transitively on $L_1 = E_{11} \times \cdots \times E_{1k_1}$, where $E_{11}, \ldots, E_{1k_1}$ are non-abelian simple components of $E(\bar{G})$. Let $K_1 = C_K(L_1)$. We may assume that K_1 acts transitively on $L_2 = E_{21} \times \cdots \times E_{2k_2}$, where $E_{21}, \ldots, E_{2k_2}$ are non-abelian sim-

G	$\lvert G\rvert_{2'}$	n	2^{n-1}
AGL(1, 5)	5	5	16
AGL(3, 2)	21	8	128
AGL(2, 3)	27	9	256
AGL(4, 2)	315	16	32768
AΓL(1, 8)	21	8	128
$2^4 : \mathsf{A}_7$	315	16	32768
PSL(2, 5)	15	6	32
PSL(3, 2)	21	7	64
PSL(2, 7)	21	8	128
PSL(3, 3)	351	13	4096
PSL(4, 2)	315	15	16384
PGL(2, 5)	15	6	32
PGL(2, 7)	21	8	128
PGL(2, 9)	45	10	512
PΓL(2, 8)	189	9	256
PΓL(2, 9)	45	10	512
M_{10}	45	10	512
M_{11}	495	11	1024
M_{11}	495	12	2048
M_{12}	1485	12	2048
M_{23}	79695	23	4194304
M_{24}	239085	24	8388608
S_6	45	10	512

Table 4. List of exceptional primitive groups not containing A_n.

ple components of $E(\bar{G})$. Let $K_2 = C_{K_1}(L_2)$, and inductively, we may define $L_3, K_3, \ldots, L_t, K_t$. Then $E(\bar{G}) = L_1 \times \cdots \times L_t$, and K_{i-1}/K_i acts transitively on L_i.

Since G acts on V completely reducibly and S is a normal subgroup of G, we know that S acts on V completely reducibly. Since $E_1, \ldots, E_m$ are non-abelian simple subgroups, there exists a solvable subgroup $H = H_1 \times \cdots \times H_m$ by Lemma 3.1, where $H_i < E_i$ such that $(H_i, p) = 1$ and $|H_i|_{2'} \geq 2|\mathrm{Out}(E_i)|_{2'}$. Therefore, $|H|_{2'} = \prod_i |H_i|_{2'} \geq 2^m |\mathrm{Out}(E(\bar{G}))|_{2'}$.

Moreover, K is a permutation group permuting $E_1, \ldots, E_m$. By Proposition 4.1, the product of the orders of all the odd order composition factors of K is less than

2^{m-1}. Thus $|H|_{2'} \geq 2^m |\mathrm{Out}(E(\bar{G}))|_{2'}$, which is greater than the product of the orders of all the odd order composition factors of K and the $2'$ part of the outer automorphism of $E(\bar{G})$.

Let $\phi: G \to G/S$ be the canonical homomorphism. Since $(|H|, p) = 1$, we know that $\phi^{-1}(H)$ acts on V completely reducibly by the generalized Maschke theorem (cf. [7, Problem 1.8]). Therefore, by Proposition 2.1, we have

$$|\phi^{-1}(H)|_{2'} \leq |V|^2/\lambda,$$

and if $p \neq 2$, then $|\phi^{-1}(H)|_{2'} \leq |V|^{3/2}/\lambda$.

We observe that the odd order composition factors of G are distributed in the maximal normal solvable subgroup S, the outer automorphism of the direct product of simple groups $E(\bar{G})$, and the odd order composition factors of the permutation group $\bar{G}/\bar{C}$. Therefore, $a(G) \leq |\phi^{-1}(H)|_{2'}$, and the result follows. □

From this result, we derive the following corollary.

Corollary 4.2. *Let G be a finite group acting on V faithfully and completely reducibly (V is possibly of mixed characteristic). Then $a(G) \leq |V|^2/\lambda$.*

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