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Universal deformation rings, endo-trivial modules, and semidihedral and generalized quaternion 2-groups

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A B S T R A C T

Let k be a field of characteristic $p > 0$, and let W be a complete discrete valuation ring of characteristic 0 that has k as its residue field. Suppose G is a finite group and $G^{\mathrm{ab},p}$ is its maximal abelian p-quotient group. We prove that every endo-trivial kG-module V has a universal deformation ring that is isomorphic to the group ring $WG^{\mathrm{ab},p}$. In particular, this gives a positive answer to a question raised by Bleher and Chinburg for all endo-trivial modules. Moreover, we show that the universal deformation of V over $WG^{\mathrm{ab},p}$ is uniquely determined by any lift of V over W. In the case when $p = 2$ and G is a 2-group that is either semidihedral or (generalized) quaternion, we give an explicit description of the universal deformation of every indecomposable endo-trivial kG-module V.

1. Introduction

Let k be a field of positive characteristic p, let G be a finite group, and let V be a finitely generated kG-module. It is a classical problem to analyze when V can be lifted to a module for G over a complete discrete valuation ring W of characteristic 0 that has k as its residue field. For example, Green showed in [17] that if all 2-extensions of V by itself are trivial, then V can always be lifted over W. When G is a p-group, Alperin showed in [2] that every endo-trivial kG-module can be lifted to an endo-trivial WG-module. In [19, Thm. 1.3], Alperin's argument was generalized to endo-trivial modules for arbitrary finite groups. The proof given in [19, Sect. 2] uses that k contains enough roots of unity (see Example 3.2). We show how to combine the work in [19, Sect. 2] with obstruction theory to prove this result without any additional assumptions

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on k (see Lemma 3.1). The question of whether V can be lifted over W can be reformulated in terms of the versal deformation ring $R(G,V)$ of V by asking whether there is a W-algebra homomorphism $R(G,V) \to W$. It is then a natural problem to determine the full versal deformation ring $R(G,V)$ and the corresponding versal deformation. In this paper, we do this for all endo-trivial kG-modules V. In the case when k has characteristic 2 and G is either a semidihedral or a (generalized) quaternion 2-group, we moreover give an explicit description of the versal deformation of every indecomposable endo-trivial kG-module.

Recall that for arbitrary p and G, a finitely generated kG-module V is called endo-trivial if the kG-module $\mathrm{Hom}_k(V,V) \cong V^* \otimes_k V$ is isomorphic to a direct sum of the trivial simple kG-module k and a projective kG-module. Endo-trivial modules play an important role in the modular representation theory of finite groups. They arise in the context of derived equivalences and stable equivalences of block algebras, and also as building blocks for the more general endo-permutation modules (see for example [14,29]). In [12,11], Carlson and Thévenaz classified all endo-trivial kG-modules when G is a p-group. The classification of the endo-trivial kG-modules for arbitrary finite groups is still ongoing (see for example [9] and its references). Since the stable endomorphism ring of every endo-trivial kG-module is one-dimensional over k, it follows from [6, Prop. 2.1] that V has a well-defined universal deformation ring $R(G,V)$.

The topological ring $R(G,V)$ is universal with respect to deformations of V over all complete local commutative Noetherian W-algebras R with residue field k. A deformation of V over such a ring R is given by the isomorphism class of a finitely generated RG-module M which is free over R, together with a kG-module isomorphism $k \otimes_R M \to V$. There is also a universal deformation of V over $R(G,V)$ from which every other deformation of V over R arises via a unique specialization homomorphism $R(G,V) \to R$. (see Section 2 for more details).

In number theory, the main motivation for studying universal deformation rings of representations of finite groups is to provide evidence for and counter-examples to various possible conjectures concerning ring theoretic properties of universal deformation rings for profinite Galois groups. The idea is that universal deformation rings for finite groups can be more easily described using deep results from modular representation theory due to Brauer, Erdmann [16], Linckelmann [20,21], and others. Moreover, the results in [15] show that if Γ is a profinite group and V is a finite dimensional k-vector space with a continuous Γ-action which has a universal deformation ring, then $R(\Gamma,V)$ is the inverse limit of the universal deformation rings $R(G,V)$ when G ranges over all finite discrete quotient groups of Γ through which the Γ-action on V factors. Thus to answer questions about the ring structure of $R(\Gamma,V)$, it is natural to first consider the case when $\Gamma = G$ is finite.

The following is our first main result.

Theorem 1.1. *Let V be a finitely generated endo-trivial kG-module. Let V_1 be a non-projective indecomposable direct summand of V, which is unique up to isomorphism.*

- (i) *The universal deformation ring $R(G,V)$ is isomorphic to the group algebra $WG^{\mathrm{ab},p}$ where $G^{\mathrm{ab},p}$ is the maximal abelian p-quotient group of G.*
- (ii) *Let V_W be a lift of V over W. Then the universal deformation of V over $WG^{\mathrm{ab},p}$ is given by the isomorphism class of $V_W \otimes_W WG^{\mathrm{ab},p}$, as a module for G over $WG^{\mathrm{ab},p}$, on which $g \in G$ acts diagonally as multiplication by g on V_W and as multiplication by its image $\overline{g}$ on $WG^{\mathrm{ab},p}$.*
- (iii) *Suppose D is a defect group of the block of kG to which V_1 belongs. Then $R(G,V)$ is a quotient ring of the group ring WD, giving a positive answer to [6, Question 1.1] for endo-trivial modules.*

In [7,25], Broué and Puig introduced and studied so-called nilpotent blocks. Using [25], we obtain the following result as a consequence of Theorem 1.1, where we assume as in [25] that k is algebraically closed.

Corollary 1.2. *Suppose k is algebraically closed, and $\hat{B}$ is a nilpotent block of WG with a defect group D. Suppose V is a finitely generated kG-module belonging to $\hat{B}$ such that the stable endomorphism ring of V as a kG-module is isomorphic to k. Then the universal deformation ring $R(G,V)$ is isomorphic to $WD^{\mathrm{ab},p}$.*

The main tools in the proof of Theorem 1.1 are the results in [2,19] about lifting endo-trivial kG-modules to endo-trivial WG-modules, together with the result [4, Lemma 2.2.2] that shows that stable equivalences of Morita type over W preserve universal deformation rings.

Alperin's proof in [2], and its extended version in [19], of the existence of a lift of a given endo-trivial kG-module V over W is not constructive. Namely, after reducing to the case when the image of a representation $\rho : G \to \mathrm{GL}_n(k)$ giving rise to V lies in $\mathrm{SL}_n(k)$, endo-triviality of V is used to show that an infinite sequence of lifting obstructions vanishes, leading to a lift of this image to a subgroup of $\mathrm{SL}_n(W)$.

Since the description of the universal deformation in part (ii) of Theorem 1.1 depends on the description of a lift of V over W, the question arises how one can construct a lift V_W of V over W without using an infinite sequence of vanishing lifting obstructions. In other words, one would like to give an explicit description of the G-action on a W-basis of V_W. This problem can be compared to constructing an explicit solution by radicals to a polynomial equation rather than just proving that such a solution exists.

In this paper, we focus on the case when G is a p-group and the group $T(G)$ of equivalence classes of endo-trivial kG-modules has a non-trivial torsion subgroup. By [12, Thm. 1.1], this means that either G is cyclic of order at least 3, or $p = 2$ and G is a 2-group that is either semidihedral or (generalized) quaternion.

Our second main result is an explicit description of a lift over W of every indecomposable endo-trivial kG-module for these G. The following remark gives a summary of this result. For the precise versions, see Propositions 4.5–4.7 and 4.9 and Remark 4.10.

Remark 1.3. Let G be a p-group such that the group of equivalence classes of endo-trivial modules $T(G)$ has a non-trivial torsion subgroup. Let V be an indecomposable endo-trivial kG-module of k-dimension n.

To provide an explicit description of a G-action on W^n making W^n into a WG-module V_W that lifts V over W, one needs to solve at most one quadratic equation of the form

$$b^2 + f_1(t)\, b + f_0(t) = 0 \tag{1.1}$$

with $f_0(t), f_1(t) \in \mathbb{Z}[t]$ for an element b in $\mathbb{Z}_p[[t]]$. Moreover, if one needs to solve such an equation (1.1) then $p = 2$ and G is a generalized quaternion group whose order is at least 16. In this case, finding such a solution is equivalent to taking the square roots of explicitly given elements in certain cyclotomic extensions of $\mathbb{Q}_2$.

When p is odd, the isomorphism class of V_W, as a module for G over W, is the unique deformation of V over W. When $p = 2$, there are precisely four distinct deformations of V over W. In all cases, the isomorphism class of $V_W \otimes_W WG^{\mathrm{ab},p}$, as a module for G over $WG^{\mathrm{ab},p}$ on which G acts diagonally, is the universal deformation of V over $WG^{\mathrm{ab},p}$.

The significance of this result is that it bounds the computational complexity of producing universal deformations. We view this as analogous to how the quadratic formula gives a more explicit solution of a quadratic equation than saying the roots are the convergent limits of an infinite sequence of operations resulting from Newton's method. The latter is similar to Alperin's method in [2] of showing that an infinite sequence of lifting obstructions vanishes.

The main ingredient in the proof of Remark 1.3, and its more precise versions, is the explicit description of the indecomposable endo-trivial kG-modules V. If G is cyclic, this is given in [14, Cor. 8.8]. If $p = 2$ and G is either a semidihedral or a (generalized) quaternion 2-group, we use the description given in [10, Sects. 6 and 7].

The paper is organized as follows: In Section 2, we give a brief introduction to deformation rings and deformations. In particular, we show in Remark 2.3 how to use an explicit lift of a finitely generated kG-module V over W to construct an explicit lift of any (co-)syzygy $\Omega^i(V)$ over W. In Section 3, we show how to combine the work in [19, Sect. 2] with obstruction theory to prove that we can lift any endo-trivial kG-module V over W, without any additional assumption on k (see Example 3.2 and Lemma 3.1). Moreover, we prove Theorem 1.1 and Corollary 1.2. In Section 4, we prove the precise versions of Remark 1.3 and give an explicit description of the universal deformation of every indecomposable endo-trivial kG-module when G is a p-group and $T(G)$ has a non-trivial torsion subgroup. The case when $p = 2$ and G is a (generalized) quaternion group is proved in Propositions 4.5–4.7. The remaining cases are proved in Proposition 4.9 and Remark 4.10.

Part of this paper constitutes the Ph.D. thesis of the third author under the supervision of the first author [28]. We would like to thank the referee for their careful reading of our paper.

Unless stated otherwise, our modules are finitely generated left modules. On the other hand, when R and S are associative rings with 1 then an R–S-bimodule M is both a left R-module and a right S-module such that for all $r \in R$, $s \in S$ and $m \in M$ one has $r(ms) = (rm)s$. Our maps are written on the left such that the map composition $f \circ g$ means f after g.

2. Preliminaries

In this section, we give a brief introduction to versal and universal deformation rings and deformations. For more background material, we refer the reader to [24] and [15].

Let k be a field of positive characteristic p, and let W be a complete local commutative Noetherian ring of characteristic 0 that has k as its residue field. If k is a perfect field, one often chooses W to be the ring of infinite Witt vectors over k. Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian W-algebras R with unique maximal ideal m_R and fixed residue map $\pi_R : R \to k = R/m_R$. The morphisms in $\hat{\mathcal{C}}$ are continuous W-algebra homomorphisms which induce the identity map on k.

Suppose G is a finite group and V is a finitely generated kG-module. A lift of V over an object R in $\hat{\mathcal{C}}$ is a pair (M, ϕ) where M is a finitely generated RG-module that is free over R, and $\phi : k \otimes_R M \to V$ is an isomorphism of kG-modules. Two lifts (M, ϕ) and (M', ϕ') of V over R are said to be isomorphic if there is an RG-module isomorphism $f : M \to M'$ with $\phi = \phi' \circ (k \otimes f)$. The isomorphism class $[M, \phi]$ of a lift (M, ϕ) of V over R is called a deformation of V over R, and the set of all such deformations is denoted by $\mathrm{Def}_G(V, R)$. The deformation functor

$$\hat{F}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$$

is a covariant functor which sends an object R in $\hat{\mathcal{C}}$ to $\mathrm{Def}_G(V, R)$ and a morphism $\alpha : R \to R'$ in $\hat{\mathcal{C}}$ to the map $\mathrm{Def}_G(V, R) \to \mathrm{Def}_G(V, R')$ defined by $[M, \phi] \mapsto [R' \otimes_{R,\alpha} M, \phi_\alpha]$, where $\phi_\alpha = \phi$ after identifying $k \otimes_{R'} (R' \otimes_{R,\alpha} M)$ with $k \otimes_R M$.

Suppose there exists an object $R(G, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U(G, V), \phi_U]$ of V over $R(G, V)$ with the following property: For each R in $\hat{\mathcal{C}}$ and for each lift (M, ϕ) of V over R there exists a morphism $\alpha : R(G, V) \to R$ in $\hat{\mathcal{C}}$ such that $\hat{F}_V(\alpha)([U(G, V), \phi_U]) = [M, \phi]$, and moreover α is unique if R is the ring of dual numbers $k[\epsilon]$. Then $R(G, V)$ is called the versal deformation ring of V and $[U(G, V), \phi_U]$ is called the versal deformation of V. Note that $R(G, V)$ and $[U(G, V), \phi_U]$ are unique up to isomorphism.

If the morphism α is unique for all R and all lifts (M, ϕ) of V over R, then $R(G, V)$ is called the universal deformation ring of V and $[U(G, V), \phi_U]$ is called the universal deformation of V. In other words, $R(G, V)$ is universal if and only if $R(G, V)$ represents the functor $\hat{F}_V$ in the sense that $\hat{F}_V$ is naturally isomorphic to the Hom functor $\mathrm{Hom}_{\hat{\mathcal{C}}}(R(G, V), -)$. In this case, $R(G, V)$ and $[U(G, V), \phi_U]$ are unique up to a unique isomorphism.

By [24], every finitely generated kG-module V has a versal deformation ring. By a result of Faltings (see [15, Prop. 7.1]), V has a universal deformation ring if $\mathrm{End}_{kG}(V) = k$.

Proposition 2.1. ([6, Prop. 2.1], [5, Rem. 2.1]) *Suppose V is a finitely generated kG-module whose stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is isomorphic to k. Then V has a universal deformation ring $R(G,V)$. Moreover, if R is in $\hat{\mathcal{C}}$ and (M,ϕ) and (M',ϕ') are lifts of V over R such that M and M' are isomorphic as RG-modules then $[M,\phi] = [M',\phi']$.*

In particular, this means that if the stable endomorphism ring of V is isomorphic to k, then we do not need to distinguish between deformations $[M,\phi]$ of V over an object R in $\hat{\mathcal{C}}$, as defined above, and so-called *weak* deformations $[M]$ of V over R (i.e., isomorphism classes of RG-modules M with $k\otimes_R M \cong V$, without specifying a particular kG-module isomorphism $\phi : k\otimes_R M \to V$).

Suppose V is a finitely generated kG-module with $\underline{\mathrm{End}}_{kG}(V) \cong k$. Then there is a non-projective indecomposable kG-module V_1, which is unique up to isomorphism, such that V is isomorphic to $V_1 \oplus Q$ for some projective kG-module Q. In particular, $\underline{\mathrm{End}}_{kG}(V_1) \cong k$. Moreover, if $\Omega = \Omega_{kG}$ denotes the syzygy functor (also called Heller operator, see, for example, [1, §20]) then $\underline{\mathrm{End}}_{kG}(\Omega(V)) \cong k$. We have the following result.

Lemma 2.2. [6, Cors. 2.5 and 2.8] *Let V be a finitely generated kG-module with $\underline{\mathrm{End}}_{kG}(V) \cong k$.*

(i) *Then $R(G,\Omega(V))$ is universal and isomorphic to $R(G,V)$.*
(ii) *Let V_1 be a non-projective indecomposable kG-module such that V is isomorphic to $V_1 \oplus Q$ for some projective kG-module Q. Then $R(G,V_1)$ is universal and isomorphic to $R(G,V)$.*

One of our goals is to construct explicit deformations by constructing explicit lifts. The following remark shows how to construct an explicit lift of an arbitrary (co-)syzygy of a finitely generated kG-module V over W provided one knows an explicit lift of V over W.

Remark 2.3. Let V be a finitely generated kG-module, and suppose we know an explicit lift V_W of V over W with corresponding kG-module isomorphism $\phi_V : k\otimes_W V_W \to V$.

By [13, Props. 6.5 and 6.7], a projective kG-module can always be lifted over W. Let $\pi : P \to V$ (resp. $\iota : V \to E$) be a projective cover (resp. injective hull) of V as a kG-module. Since kG is self-injective, it follows that E is a projective kG-module. Let P_W (resp. E_W) be a lift of P (resp. E) over W, with corresponding kG-module isomorphism $\phi_P : k\otimes_W P_W \to P$ (resp. $\phi_E : k\otimes_W E_W \to E$). In particular, E_W is a projective WG-module.

Since P_W is a projective WG-module, there exists a WG-module homomorphism $\pi_W : P_W \to V_W$ such that $\phi_V \circ (k\otimes_W \pi_W)\circ \phi_P^{-1} = \pi$. By Nakayama's Lemma, it follows that π_W is surjective. Since $\mathrm{Ker}(\pi_W)$ is a free W-module, it follows that $\mathrm{Ker}(\pi_W)$ is a lift of $\Omega(V) = \mathrm{Ker}(\pi)$ over W. We use the notation $\Omega_{WG}(V_W) = \mathrm{Ker}(\pi_W)$. This is a variant for WG-lattices of the Heller operator defined in [23, §2.14].

We can use a similar argument as in the proof of [6, Prop. 2.4] to find an explicit lift of $\Omega^{-1}(V) = \mathrm{Coker}(\iota)$. Namely, we have a short exact sequence

$$0 \to m_W E_W \to E_W \to E \to 0 \tag{2.1}$$

and we have $\mathrm{Ext}^1_{WG}(V_W, m_W E_W) = 0$. This implies that there exists a WG-module homomorphism $\iota_W : V_W \to E_W$ such that $\phi_E \circ (k\otimes_W \iota_W)\circ \phi_V^{-1} = \iota$. Since ι is injective and since V_W and E_W are free over W, it follows by Nakayama's Lemma that ι_W is also injective. Thus we have a commutative diagram of WG-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_W & \xrightarrow{\iota_W} & E_W & \longrightarrow & \mathrm{Coker}(\iota_W) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \xrightarrow{\iota} & E & \longrightarrow & \Omega^{-1}(V) & \longrightarrow & 0 \end{array}$$

with exact rows, and $\mathrm{Coker}(\iota_W)$ is a free W-module. Therefore, $\mathrm{Coker}(\iota_W)$ is a lift of $\Omega^{-1}(V) = \mathrm{Coker}(\iota)$ over W. We use the notation $\Omega^{-1}_{WG}(V_W) = \mathrm{Coker}(\iota_W)$.

3. Endo-trivial modules

Assume the notation from the previous section. Moreover, assume that W is a complete discrete valuation ring of characteristic 0 that has k as its residue field. Suppose V is a finitely generated kG-module. We say V is endo-trivial if its k-endomorphism ring $\mathrm{End}_k(V)$ is, as a kG-module, stably isomorphic to the trivial kG-module k. In other words, $\mathrm{End}_k(V)$ is isomorphic to a direct sum of k and a projective kG-module.

Note that every endo-trivial kG-module satisfies $\underline{\mathrm{End}}_{kG}(V) \cong k$. If G is a p-group then it follows from [8] that the endo-trivial kG-modules coincide with the kG-modules V with $\underline{\mathrm{End}}_{kG}(V) \cong k$. However, for arbitrary finite groups, the set of isomorphism classes of kG-modules whose stable endomorphism rings are isomorphic to k usually properly contains the isomorphism classes of endo-trivial kG-modules.

In this section, we prove Theorem 1.1. One of the main ingredients in the proof is the following result. This was proved when G is a p-group by Alperin in [2] and in [19, Thm 1.3, pp. 144–145] for all G under the assumption that k is algebraically closed (see the assumptions of [19, Sect. 2]).

Lemma 3.1. *Let k be an arbitrary field of positive characteristic p, and let W be a complete discrete valuation ring of characteristic 0 that has k as its residue field. If V is an endo-trivial kG-module then V can be lifted to an endo-trivial WG-module V_W.*

This result was in fact stated for all k and G in [19, Thm. 1.3]. However, the proof of [19, Thm. 1.3] given in [19, Sect. 2] uses that k has enough roots of unity and does not apply for all k, as the following example shows.

Example 3.2. Suppose $p = 3$, $k = \mathbb{Z}/3$, and $G = \Sigma_3$ is the symmetric group on 3 letters. Let $V = \Omega(k)$ be the first syzygy of the trivial simple kG-module k, so that V is an endo-trivial kG-module. Let H be the subgroup of order 3 in G. The action of G on V is faithful, giving an injection $G \hookrightarrow \mathrm{GL}_2(k)$ after picking a basis of V. View G as a subgroup of $\mathrm{GL}_2(k)$ and let $n = 2$. On [19, p. 144] the groups $G_1 = GC$ and $G_0 = G_1 \cap \mathrm{SL}_2(k)$ are defined when one sets

$$C = \{aI_n : a^n = \det(g) \text{ for some } g \in G\}.$$

One checks that $G_0 = H \cdot \{\pm I_2\}$. However, G_1 is not the central product of G_0 and C, so the argument of [19, pp. 144–145] does not apply for arbitrary k of characteristic p.

Proof of Lemma 3.1. We indicate how to combine the work in [19, Sect. 2] with obstruction theory to deduce the general case from the case in which k is algebraically closed.

Both [19, Lemma 2.1] and [19, Prop. 2.4] are true for an arbitrary field k of characteristic p. Moreover, if N is a WG-module that is free over W such that $N/m_W N$ is a projective kG-module, then it follows, for example from [27, Sect. 14.2], that N is a projective WG-module without any additional assumptions on k. Write the order of G as $p^d m$, where p does not divide m. Let $n = \dim_k V$ and let $\rho : G \to \mathrm{GL}_n(k)$ be a representation of V. By [19, Lemma 2.1], n is relatively prime to p.

Suppose first that k contains all (nm)-th roots of unity. Then the arguments of [19, pp. 144–145] show that V can be lifted to an endo-trivial WG-module V_W.

Suppose next that k is an arbitrary field of characteristic p. Let ζ be a primitive (nm)-th root of unity in a fixed separable closure of k. Define $k' = k(\zeta)$ and $W' = W[\zeta]$. Note that since nm is relatively prime to p, W' is unramified over W. In particular, if ϖ is a uniformizer for W then ϖ is also a uniformizer for W'. Let $\rho' : G \to \mathrm{GL}_n(k')$ be the representation of G over k' obtained by composing ρ with the inclusion $\mathrm{GL}_n(k) \hookrightarrow \mathrm{GL}_n(k')$. Define G_ρ (resp. $G_{\rho'}$) to be the image of ρ (resp. ρ'). Our goal is to lift G_ρ to a subgroup of $\mathrm{GL}_n(W)$, by knowing that we can lift $G_{\rho'}$ to a subgroup of $\mathrm{GL}_n(W')$. Let $\ell \geq 1$. Suppose we have lifted G_ρ to a subgroup $G_{\rho,\ell}$ of $\mathrm{GL}_n(W/\varpi^\ell W)$ such that the injection $\mathrm{GL}_n(W/\varpi^\ell W) \to \mathrm{GL}_n(W'/\varpi^\ell W')$ provides a subgroup $G_{\rho',\ell}$ of $\mathrm{GL}_n(W'/\varpi^\ell W')$ lifting $G_{\rho'}$. The obstruction to lifting $G_{\rho,\ell}$ to $\mathrm{GL}_n(W/\varpi^{\ell+1} W)$ is a class c in $\mathrm{H}^2(G, \frac{\varpi^\ell W}{\varpi^{\ell+1} W} \otimes_k \mathrm{Ad}(\rho))$, where $\mathrm{Ad}(\rho)$ is the adjoint representation associated to ρ, i.e. $\mathrm{Ad}(\rho) = \mathrm{End}_k(V)$ with G acting by conjugation. On the other hand, the obstruction to lifting $G_{\rho',\ell}$ to $\mathrm{GL}_n(W'/\varpi^{\ell+1} W')$ is the class c' in

$$\mathrm{H}^2(G, \frac{\varpi^\ell W'}{\varpi^{\ell+1} W'} \otimes_{k'} \mathrm{Ad}(\rho')) = k' \otimes_k \mathrm{H}^2(G, \frac{\varpi^\ell W}{\varpi^{\ell+1} W} \otimes_k \mathrm{Ad}(\rho))$$

which is obtained from c by base changing from k to k'. (Note that we use here that W' is unramified over W.) Since by the arguments of [19, pp. 144–145], the class c' vanishes, this means that c must vanish as well, leading to a lift of G_ρ to a subgroup $G_{\rho,\ell+1}$ of $\mathrm{GL}_n(W/\varpi^{\ell+1} W)$ such that the injection $\mathrm{GL}_n(W/\varpi^{\ell+1} W) \to \mathrm{GL}_n(W'/\varpi^{\ell+1} W')$ provides a subgroup $G_{\rho',\ell+1}$ of $\mathrm{GL}_n(W'/\varpi^{\ell+1} W')$ lifting $G_{\rho'}$. Taking the limit as $\ell \to \infty$ of these lifts completes the proof of Lemma 3.1. □

Using Lemma 3.1 together with [4, Lemma 2.2.2], we can now prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.1, V can be lifted to an endo-trivial WG-module V_W. We may view the left WG-module V_W as a WG–WG-bimodule by letting G act trivially on the right. Define $M = V_W \otimes_W WG$. Then M is a WG–WG-bimodule, where the G-action on the left is diagonal and the G-action on the right is on the WG-factor. Hence M is projective (in fact, free) both as a left and as a right WG-module. Define $N = \mathrm{Hom}_W(M, W)$. Then

$$\begin{aligned} M \otimes_{WG} N &\cong WG \oplus P \quad \text{and} \\ N \otimes_{WG} M &\cong WG \oplus Q \end{aligned}$$

where P and Q are projective WG–WG-bimodules, and both of these isomorphisms are isomorphisms of WG–WG-bimodules. In particular, M and N define a stable autoequivalence of Morita type of the W-stable category WG-$\underline{\mathrm{mod}}$, which is the quotient category of the category WG-mod of finitely generated WG-modules by the subcategory of relatively W-projective modules (see [22] and [23, §2.13 and §4.13] for more details). If $V_0 = k$ is the trivial simple kG-module then

$$\begin{aligned} (k \otimes_W M) \otimes_{kG} V_0 &\cong (k \otimes_W V_W) \otimes_k (kG \otimes_{kG} V_0) \\ &\cong V \otimes_k V_0 \ \cong \ V \end{aligned}$$

where we use that G acts trivially on the right of V_W. Hence it follows by [4, Lemma 2.2.2] that $R(G, V) \cong R(G, V_0)$. By [24, Sect. 1.4], $R(G, V_0)$ is isomorphic to $WG^{\mathrm{ab},p}$ and the universal deformation of V_0 is given by the isomorphism class of $U(G, V_0) = WG^{\mathrm{ab},p}$ as an $R(G, V_0)G$-module on which $g \in G$ acts as multiplication by its image $\overline{g} \in G^{\mathrm{ab},p}$. The proof of [4, Lemma 2.2.2] shows that the isomorphism class of the universal deformation of V is given by the isomorphism class of

$$\begin{aligned}(M \otimes_W R(G,V_0)) \otimes_{R(G,V_0)G} U(G,V_0) &\cong (V_W \otimes_W R(G,V_0)G) \otimes_{R(G,V_0)G} U(G,V_0)\\ &\cong V_W \otimes_W U(G,V_0)\\ &\cong V_W \otimes_W WG^{\mathrm{ab},p}\end{aligned}$$

where $g \in G$ acts diagonally as multiplication by g on V_W and as multiplication by its image $\overline{g}$ on $WG^{\mathrm{ab},p}$. This proves parts (i) and (ii) of Theorem 1.1.

For part (iii) of Theorem 1.1, let V_1 be a non-projective indecomposable direct summand of V, as in the statement of the theorem. Then V_1 is unique up to isomorphism, and V is the direct sum of V_1 and a projective kG-module. In particular, we have $R(G,V) \cong R(G,V_1)$. Since by [19, Lemma 2.1] the k-dimension of V_1 is not divisible by p, it follows by [13, Thm. 19.26] that the vertices of V_1 are Sylow p-subgroups of G. Therefore, D must also be a Sylow p-subgroup. Since $G^{\mathrm{ab},p}$ is a quotient group of each Sylow p-subgroup of G, it follows that $R(G,V)$ is isomorphic to a quotient ring of WD. □

Remark 3.3. In the notation of Theorem 1.1(ii), the isomorphism class of V_W, as a module for G over W, is the unique deformation of V over W when $p > 2$. If $p = 2$ then the number of deformations of V over W equals the order of the maximal elementary abelian 2-quotient group of G. This follows since $R(G,V) \cong WG^{\mathrm{ab},p}$ by Theorem 1.1(i) and since the deformations of V over W are in one-to-one correspondence with the morphisms $R(G,V) \to W$ in the category $\hat{\mathcal{C}}$.

We now turn to nilpotent blocks, and the proof of Corollary 1.2. The following remark recalls the most important definitions and results for nilpotent blocks.

Remark 3.4. Assume that k is algebraically closed. Let D be a finite p-group, let G be a finite group and let $\hat{B}$ be a nilpotent block of WG that has D as a defect group. By [7, Def. 1.1], this means that whenever (D_1, e_1) is a $\hat{B}$-Brauer pair then the quotient group $N_G(D_1,e_1)/C_G(D_1)$ is a p-group. In other words, for all subgroups D_1 of D and for all block idempotents e_1 of $kC_G(D_1)$ associated with $\hat{B}$, the quotient of the stabilizer $N_G(D_1,e_1)$ of e_1 in $N_G(D_1)$ by the centralizer $C_G(D_1)$ is a p-group. In [25, Sect. 1.7], Puig rephrased this definition using the theory of local pointed groups (see also [29, Prop. 49.8]).

By [25, §1.4], $\hat{B}$ is Morita equivalent to WD. In [26, Thm. 8.2], Puig showed that the converse is also true in a very strong way. Namely, if $\hat{B}'$ is another block over W such that there is a stable equivalence of Morita type between $\hat{B}$ and $\hat{B}'$, then $\hat{B}'$ is also nilpotent. Hence Corollary 1.2 can be applied in particular if there is only known to be a stable equivalence of Morita type between $\hat{B}$ and WD.

Proof of Corollary 1.2. By Remark 3.4, the nilpotent block $\hat{B}$ of WG is Morita equivalent to WD. Suppose V is a finitely generated kG-module belonging to $\hat{B}$, and V' is the kD-module corresponding to V under this Morita equivalence. Then the stable endomorphism ring of V as a kG-module is isomorphic to k if and only if the stable endomorphism ring of V' as a kD-module is isomorphic to k. Suppose now that $\underline{\mathrm{End}}_{kG}(V) \cong k$. Then it follows for example from [3, Prop. 2.5] that $R(G,V) \cong R(D,V')$. By [8] and Theorem 1.1, this implies that $R(G,V) \cong WD^{\mathrm{ab},p}$. □

4. Explicit universal deformations of endo-trivial modules

Assume the notation from the previous section. Suppose G is a p-group such that the group $T(G)$ of equivalence classes of endo-trivial kG-modules has a non-trivial torsion subgroup. By [12, Thm. 1.1], this means that G is either cyclic of order at least 3, or $p = 2$ and G is a semidihedral or a (generalized) quaternion 2-group.

The goal of this section is to give an explicit description of the universal deformations of all endo-trivial kG-modules V. In particular, we will prove the precise versions of Remark 1.3. In Section 4.1, we will consider the case when $p = 2$ and G is a (generalized) quaternion 2-group. In Section 4.2, we will consider the remaining cases.

4.1. The (generalized) quaternion 2-groups

Fix an integer $d \geq 3$, and let Q be a (generalized) quaternion group of order 2^d with the following presentation

$$\mathrm{Q} = \langle x, y \mid x^{2^{d-1}} = 1, x^{2^{d-2}} = y^2, yxy^{-1} = x^{-1}\rangle. \tag{4.1}$$

Then $\mathrm{Q} = \langle yx, y\rangle$ where yx and y both have order 4. Define

$$\overline{\mathrm{Q}} = \mathrm{Q}^{\mathrm{ab},2} = \langle \overline{yx}, \overline{y}\rangle = \{\overline{1}, \overline{x}, \overline{y}, \overline{yx}\} \tag{4.2}$$

where for $g \in \mathrm{Q}$, $\overline{g}$ denotes its image in $\overline{\mathrm{Q}}$.

Let k be a field of characteristic 2 and let W be a complete discrete valuation ring of characteristic 0 that has k as its residue field. Let $\mathbb{F}_2$ be the prime subfield of k with two elements, and let $\mathbb{Z}_2$ be the 2-adic integers such that $\mathbb{Z}_2 \subseteq W$.

Definition 4.1. Let $X = \langle x\rangle$ be a maximal cyclic subgroup of Q, which is unique when $d \geq 4$. Let $\mathrm{Tr}_X = \sum_{j=0}^{2^{d-1}-1} x^j$ be the trace element of X, and let $\overline{S}$ be the ring $kX/k \cdot \mathrm{Tr}_X$. Let $*$ denote the involution of $\overline{S}$ that is induced by inversion on X. Suppose $\overline{\beta} \in \overline{S}$ is an element satisfying

$$\overline{\beta}\,\overline{\beta}^* = x^{2^{d-2}} \tag{4.3}$$

and define

$$y \cdot s = \overline{\beta}\, s^* \quad \text{for all } s \in \overline{S}. \tag{4.4}$$

As in the proof of [10, Lemma 6.4], this makes $\overline{S} = kX/k \cdot \mathrm{Tr}_X$ into a kQ-module, which we denote by $L(\overline{\beta})$. Since $\mathrm{Res}^{\mathrm{Q}}_{\langle x\rangle} L(\overline{\beta}) \cong \Omega^1_{k\langle x\rangle}(k)$ and $\langle x\rangle$ contains the unique elementary abelian subgroup of Q, it follows from [10, Lemma 2.9] that $L(\overline{\beta})$ is an endo-trivial kQ-module of k-dimension $2^{d-1} - 1$.

Remark 4.2. Concerning the existence of an element $\overline{\beta} \in \overline{S}$ satisfying (4.3), we have the following by [10, Sect. 6].

(a) If $d \geq 4$ then it is shown in the proof of [10, Lemma 6.4] that the element

$$\overline{\beta} = 1 + (x + x^2)(1 + x^{2^{d-2}+2}) \left(\sum_{i=0}^{2^{d-4}-1} x^{4i}\right)$$

in $\overline{S}$ satisfies (4.3).

(b) Suppose now that $d = 3$ and that k contains a primitive cube root ω of unity. Then

$$\overline{\beta} = \omega^2 + \omega x$$

in $\overline{S}$ satisfies (4.3). Moreover, consider the k-basis of $\overline{S}$ given by $\{1, 1 + x, 1 + x^2\}$. Then the actions of x and y on $L(\overline{\beta})$ with respect to this basis are given by the matrices

$$\rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ \omega & 1 & 0 \\ 0 & \omega^2 & 1 \end{pmatrix}.$$

Remark 4.3. Let $T(\mathrm{Q})$ be the group of equivalence classes of endo-trivial $k\mathrm{Q}$-modules. By [10, Sect. 6], we have the following result.

(a) If $d = 3$ and k does not contain a primitive cube root of unity, then $T(\mathrm{Q})$ is a cyclic group of order 4 given by $\{[\Omega^i_{k\mathrm{Q}}(k)]\}_{i=0}^3$. Since the k-dimensions of $\Omega^i_{k\mathrm{Q}}(k)$ are congruent to ± 1 modulo $2^d = 8$, there cannot be an endo-trivial $k\mathrm{Q}$-module of dimension $2^{d-1} - 1 = 3$. By Definition 4.1, this means that there cannot exist an element $\overline{\beta} \in \overline{S}$ satisfying (4.3).

(b) Suppose now that either $d \geq 4$ or $d = 3$ and k contains a primitive cube root ω of unity. Then $T(\mathrm{Q})$ is isomorphic to $\mathbb{Z}/4 \oplus \mathbb{Z}/2$. More precisely, let $L = L(\overline{\beta})$ be as in Definition 4.1 for some $\overline{\beta} \in \overline{S}$ satisfying (4.3). Then $T(\mathrm{Q})$ is generated by $[\Omega^1_{k\mathrm{Q}}(k)]$ and $[\Omega^1_{k\mathrm{Q}}(L)]$, and $[\Omega^1_{k\mathrm{Q}}(L)]$ has order 2. In particular, each element of $T(\mathrm{Q})$ is of the form $[\Omega^i_{k\mathrm{Q}}(k)]$ or $[\Omega^i_{k\mathrm{Q}}(L)] = [\Omega^i_{k\mathrm{Q}}(k) \otimes L]$ for some $i \in \{0, 1, 2, 3\}$.

By Theorem 1.1, we know that the universal deformation ring of every endo-trivial $k\mathrm{Q}$-module is isomorphic to $W\overline{\mathrm{Q}}$ where $\overline{\mathrm{Q}}$ is as in (4.2). We now give an explicit description of the universal deformation of every endo-trivial $k\mathrm{Q}$-module V. Since projective $k\mathrm{Q}$-modules are free, and hence can easily be lifted over W, we can concentrate on the indecomposable endo-trivial $k\mathrm{Q}$-modules.

The crucial step in constructing an explicit lift over W of the endo-trivial $k\mathrm{Q}$-module $L = L(\overline{\beta})$ as in Remark 4.3(b) when $d \geq 4$ is to give a constructive version in Proposition 4.6 of the following result:

Proposition 4.4. *Suppose $d \geq 4$. Let $X = \langle x \rangle$, let $\mathrm{Tr}_X = \sum_{j=0}^{2^{d-1}-1} x^j$, and let $S_2 = \mathbb{Z}_2 X / \mathbb{Z}_2 \cdot \mathrm{Tr}_X$. Let $*$ denote the involution of S_2 that is induced by inversion on X. There exists an element $\beta \in S_2$ satisfying*

$$\beta\beta^* = x^{2^{d-2}}. \tag{4.5}$$

The existence of β is subtle because S_2 is not a regular ring. We use a surjection $\pi : \mathbb{Z}_2[[t]] \to S_2^+$ where S_2^+ is the subring of S_2 that is invariant under $*$. To construct β we need Proposition 4.5 below which shows that a particular quadratic equation, given by (4.7), with coefficients in $\mathbb{Z}[t]$ has a solution in $\mathbb{Z}_2[[t]]$. In Proposition 4.6, we then use this solution to find $\beta \in S_2$ satisfying (4.5).

Proposition 4.5. *Suppose $d \geq 4$. For $j \geq 0$, define $p_j \in \mathbb{Z}[t]$ inductively by $p_0 = 2$, $p_1 = t$ and the recurrence relation*

$$p_{j+1} = tp_j - p_{j-1} \qquad \text{for } j \geq 1. \tag{4.6}$$

Let $\tau = \sum_{j=0}^{2^{d-2}-1} p_j$, and let $a = 1 + 2^{d-4}t$. Then the equation

$$b^2 + tab + a^2 = 1 - \tau \tag{4.7}$$

has a solution b in $\mathbb{Z}_2[[t]]$. Moreover, the discriminant $\Delta(t)$ of (4.7) is a polynomial in $\mathbb{Z}[t]$ such that $\Delta(t) = t^2((1-t)^2 - 8)$ if $d = 4$, and if $d \geq 5$ then $\Delta(t) = t^2(1 + 4\,m(t))$ for some polynomial $m(t) \in \mathbb{Z}[t]$ whose constant coefficient is divisible by 2.

Proof. Consider the recurrence relation for p_j in (4.6). The associated quadratic equation

$$Y^2 - tY + 1 = 0 \tag{4.8}$$

in the variable Y has two solutions y_1, y_2 which belong to a quadratic extension of the field $\mathbb{Q}_2((t))$ and which are integral units over the ring $\mathbb{Z}_2[[t]]$. Note that $y_1^0 + y_2^0 = 2 = p_0$ and $y_1^1 + y_2^1 = t = p_1$. Since y_1 and y_2 are roots of (4.8), it follows that the recurrence relation (4.6) holds for $y_1^j + y_2^j$, which implies that

$$p_j = y_1^j + y_2^j \tag{4.9}$$

for all $j \geq 0$. Using that $(y_1 - 1)(y_2 - 1) = 2 - t$ and that y_1, y_2 satisfy (4.8), we rewrite τ as

$$\begin{aligned}\tau = \sum_{j=0}^{2^{d-2}-1} (y_1^j + y_2^j) &= \frac{y_1^{2^{d-2}} - 1}{y_1 - 1} + \frac{y_2^{2^{d-2}} - 1}{y_2 - 1} \\ &= \frac{(y_2 - 1)\left(\sum_{a=1}^{2^{d-3}} \binom{2^{d-3}}{a}(ty_1)^a(-1)^a\right) + (y_1 - 1)\left(\sum_{a=1}^{2^{d-3}} \binom{2^{d-3}}{a}(ty_2)^a(-1)^a\right)}{2 - t} \\ &= \frac{t\,g(t)}{2 - t}\end{aligned}$$

for some $g(t) \in \mathbb{Z}_2[[t]]$. Since $\tau \in \mathbb{Z}_2[[t]]$ and $\mathbb{Z}_2[[t]]$ is a unique factorization domain, there exists $h(t) \in \mathbb{Z}_2[[t]]$ such that $g(t) = (2 - t)\,h(t)$. Let $A = \mathbb{Z}_2[[t]][y_1, y_2]$. Using that y_1, y_2 satisfy (4.8), we have $t^2 y_1^2 \equiv -t^2 \mod t^3 A$ and $t^2 y_2^2 \equiv -t^2 \mod t^3 A$. Hence we obtain the following congruence relations:

$$\begin{aligned}t(2-t)h(t) &\equiv (y_2 - 1)\left(\sum_{a=1}^{2} \binom{2^{d-3}}{a}(ty_1)^a(-1)^a\right) + (y_1 - 1)\left(\sum_{a=1}^{2} \binom{2^{d-3}}{a}(ty_2)^a(-1)^a\right) \mod t^3 A \\ &\equiv (y_2 - 1)\left(2^{d-3}(-ty_1) - \binom{2^{d-3}}{2}t^2\right) + (y_1 - 1)\left(2^{d-3}(-ty_2) - \binom{2^{d-3}}{2}t^2\right) \mod t^3 A \\ &\equiv t\,2^{d-3}\left((y_2 - 1)(-y_1) + (y_1 - 1)(-y_2)\right) + t^2\binom{2^{d-3}}{2}\left(-(y_2 - 1) - (y_1 - 1)\right) \mod t^3 A \\ &\equiv t\,2^{d-3}(-2 + t) + t^2\binom{2^{d-3}}{2}(2 - t) \mod t^3 A \\ &\equiv t\,(2 - t)\left(-2^{d-3} + t\binom{2^{d-3}}{2}\right) \mod t^3 A.\end{aligned}$$

This implies

$$\tau \equiv t\left(-2^{d-3} + t\binom{2^{d-3}}{2}\right) \mod t^3\,\mathbb{Z}_2[[t]]. \tag{4.10}$$

Setting $a = 1 + 2^{d-4}t$ in the quadratic equation (4.7) gives the discriminant

$$\Delta(t) = t^2(1 + 2^{d-4}t)^2 - 4(2^{2d-8}t^2 + 2^{d-3}t + \tau) \tag{4.11}$$

which is a polynomial in $\mathbb{Z}[t]$, since p_j, $j \geq 0$, and hence also τ, are polynomials in $\mathbb{Z}[t]$.

If $d \geq 5$ then we obtain from (4.10) the congruence relation

$$\Delta(t) \equiv t^2\left(1 - 4\left(2^{2d-8} + 2^{d-4}\left(2^{d-3} - 1\right)\right)\right) \mod 4t^3\,\mathbb{Z}_2[[t]].$$

If $d = 4$ then $\tau = \sum_{j=0}^{3} p_j = t^3 + t^2 - 2t$ and

$$\Delta(t) = t^2(1 - t)^2\left(1 - \frac{8}{(1 - t)^2}\right) \quad \text{in } \mathbb{Z}_2[[t]].$$

In other words, for all $d \geq 4$, there exists a polynomial $f(t) \in \mathbb{Z}_2[t]$ and a power series $m(t)$ in the maximal ideal $(2,t)\,\mathbb{Z}_2[[t]]$ such that

$$\Delta(t) = f(t)^2(1 + 4\,m(t)). \tag{4.12}$$

Note that if $d \geq 5$ then $m(t)$ is actually a polynomial in $\mathbb{Z}[t]$ whose constant coefficient is divisible by 2. Using the binomial series for $(1+T)^{1/2}$ when $T = 4\,m(t)$ and noticing that $\binom{1/2}{j}4^j$ is an element in $2\,\mathbb{Z}_2$ for all $j \geq 1$, it follows that there exists a power series $\delta(t) \in \mathbb{Z}_2[[t]]$ satisfying

$$(1 + 2\,\delta(t))^2 = 1 + 4\,m(t).$$

Hence, if $d \geq 5$ then

$$b = -2^{d-5}\,t^2 + t\,\delta(t) \qquad (\text{resp. } b = -t - 2^{d-5}\,t^2 - t\,\delta(t)) \tag{4.13}$$

and if $d = 4$ then

$$b = -t^2 + t(1-t)\,\delta(t) \qquad (\text{resp. } b = -t - t(1-t)\,\delta(t)) \tag{4.14}$$

are power series in $\mathbb{Z}_2[[t]]$ satisfying (4.7).

Recall that a polynomial in $\mathbb{Z}_2[t]$ is said to be distinguished (or a Weierstrass polynomial) if its leading coefficient is a unit in $\mathbb{Z}_2$ and all its non-leading coefficients are divisible by 2 (see [18, §IV.9]).

Proposition 4.6. *Suppose $d \geq 4$. Let $X = \langle x \rangle$, let $\mathrm{Tr}_X = \sum_{j=0}^{2^{d-1}-1} x^j$, and let $S_2 = \mathbb{Z}_2 X/\mathbb{Z}_2 \cdot \mathrm{Tr}_X$. Let $*$ denote the involution of S_2 that is induced by inversion on X, and let S_2^+ be the subring of S_2 that is invariant under $*$. The map*

$$\pi: \quad \mathbb{Z}_2[[t]] \to S_2^+ \qquad \textit{given by} \quad \pi(t) = x + x^{-1} \tag{4.15}$$

defines a surjective $\mathbb{Z}_2$-algebra homomorphism. The kernel of π is the ideal of $\mathbb{Z}_2[[t]]$ generated by the monic distinguished polynomial $\Phi(t)$ in $\mathbb{Z}_2[t]$ with coefficients in $\mathbb{Z}$ that is the product of the irreducible polynomials $q_j(t)$ over $\mathbb{Q}$ of the numbers $\zeta_{2^j} + \zeta_{2^j}^{-1}$ when ζ_{2^j} is a root of unity of order 2^j and $j = 1, \ldots, d-1$.

Let $\tau, a \in \mathbb{Z}[t]$ and $b \in \mathbb{Z}_2[[t]]$ be as in Proposition 4.5. Then

$$\pi(1-\tau) = x^{2^{d-2}} \tag{4.16}$$

and

$$\beta = \pi(a) + x\,\pi(b)$$

is an element of S_2 satisfying (4.5), i.e. $\beta\beta^* = x^{2^{d-2}}$.

Finding $\pi(b)$ explicitly as an element of S_2^+ is equivalent to finding an explicit square root of $\pi(\Delta(t))$ inside $S_2^+ \subset S_2$, where $\Delta(t)$ is the discriminant of (4.7). The latter is equivalent to taking particular square roots of explicitly given elements inside the image of S_2 under the injective $\mathbb{Z}_2$-algebra homomorphism

$$\iota_{S_2}: \quad S_2 \to \mathbb{Q}_2 \otimes_{\mathbb{Z}_2} S_2 = \prod_{j=1}^{d-1} \mathbb{Q}_2(\zeta_{2^j}) \tag{4.17}$$

which sends x in S_2 to the tuple $(\zeta_{2^j})_{j=1}^{d-1}$.

Proof. Let $J = \langle * \rangle$ be the group of order two generated by the involution $*$. Then J acts on $\mathbb{Z}_2 X$ and we have an exact sequence of J-modules

$$0 \to \mathbb{Z}_2 \cdot \mathrm{Tr}_X \to \mathbb{Z}_2 X \to S_2 \to 0$$

in which J acts trivially on $\mathbb{Z}_2 \cdot \mathrm{Tr}_X$. Taking J-cohomology and using that $\mathrm{H}^1(J, \mathbb{Z}_2 \cdot \mathrm{Tr}_X) = 0$ since J acts trivially on $\mathbb{Z}_2 \cdot \mathrm{Tr}_X \cong \mathbb{Z}_2$ and $\mathbb{Z}_2$ is torsion free, we obtain that $(\mathbb{Z}_2 X)^J \to S_2^J = S_2^+$ is surjective. Note that $(\mathbb{Z}_2 X)^J$ is a free $\mathbb{Z}_2$-module generated by

$$\{x^j + x^{-j}\}_{j=1}^{2^{d-2}-1} \cup \{1, x^{2^{d-2}}\}.$$

Let $\pi : \mathbb{Z}_2[t] \to S_2^+$ be the $\mathbb{Z}_2$-algebra homomorphism given by $\pi(t) = x + x^{-1}$. The recurrence relation (4.6) gives immediately that $\pi(p_j) = x^j + x^{-j}$ for all $j \geq 0$. Moreover, the definition of τ in Proposition 4.5 shows (4.16).

Define $\Phi(t) \in \mathbb{Z}[t]$ to be the product of the irreducible polynomials $q_j(t)$ over $\mathbb{Q}$ of the numbers $\zeta_{2^j} + \zeta_{2^j}^{-1}$ when ζ_{2^j} is a root of unity of order 2^j and $j = 1, \ldots, d-1$. Since S_2 can be identified with the image of the injective $\mathbb{Z}_2$-algebra homomorphism ι_{S_2} from (4.17), it follows that $\pi(\Phi(t)) = 0$. Note that $q_1(t) = t+2$ and $q_2(t) = t$. Moreover, using $(\zeta_{2^{j+1}} + \zeta_{2^{j+1}}^{-1})^2 - 2 = \zeta_{2^{j+1}}^2 + (\zeta_{2^{j+1}}^2)^{-1}$, it follows that $q_{j+1}(t) = q_j(t^2-2)$ for $j \geq 2$. Therefore, we see that $\Phi(t)$ is a monic distinguished polynomial in $\mathbb{Z}_2[t]$ with coefficients in $\mathbb{Z}$ of degree 2^{d-2}. This implies that π induces a well-defined surjective $\mathbb{Z}_2$-algebra homomorphism $\pi : \mathbb{Z}_2[[t]] \to S_2^+$ such that the kernel of π contains the ideal of $\mathbb{Z}_2[[t]]$ generated by $\Phi(t)$. Since $\mathbb{Z}_2[[t]]/(\Phi(t))$ is a free $\mathbb{Z}_2$-module of rank 2^{d-2} and the rank of $\mathbb{Q}_2 \otimes_{\mathbb{Z}_2} S_2^+$ is also 2^{d-2}, it follows that $\Phi(t)$ generates the kernel of π.

Letting $a = 1 + 2^{d-4}\, t$, it follows from Proposition 4.5 that the equation

$$b^2 + tab + a^2 = 1 - \tau$$

has a solution b in $\mathbb{Z}_2[[t]]$. Applying the homomorphism π from (4.15) to a and b we obtain an element

$$\beta = \pi(a) + x\,\pi(b)$$

in S_2. Since $\beta^* = \pi(a) + x^{-1}\pi(b)$, we have

$$\begin{aligned}\beta\beta^* &= \pi(a)^2 + \pi(a)\pi(b)(x + x^{-1}) + \pi(b)^2 \\ &= \pi(b^2 + tab + a^2).\end{aligned}$$

Using (4.16), it follows that β is an element of S_2 satisfying the relation (4.5).

For the last statement of Proposition 4.6, we use the injective $\mathbb{Z}_2$-algebra homomorphism ι_{S_2} from (4.17). By Proposition 4.5, $\Delta(t) = t^2((1-t)^2 - 8)$ if $d = 4$, and if $d \geq 5$ then $\Delta(t) = t^2(1 + 4\,m(t))$ for some polynomial $m(t) \in \mathbb{Z}[t]$ whose constant coefficient is divisible by 2.

If $d = 4$ then the inverse of $\pi(1-t) = 1 - x - x^{-1}$ in S_2 is given by

$$u_1 = \frac{1}{3}\,(-1 - 2x - x^2 + x^3 + 2x^4 + x^5 - x^6 - 2x^7).$$

This means that

$$\pi(\Delta(t)) = (x + x^{-1})^2\,(1 - x - x^{-1})^2\,\left(1 - 8\,u_1^2\right) \qquad \text{for } d = 4.$$

Therefore, for all $d \geq 4$,

$$\pi(\Delta(t)) = s^2\,(1+4r)$$

for explicitly given elements $r, s \in S_2$, where r is in the Jacobson radical of S_2. Let $(1+4r)_j$ be the image of $1+4r$ in $\mathbb{Q}_2(\zeta_{2^j})$ under the injection ι_{S_2} in (4.17). Then we can take $\sqrt{1+4r} \in S_2$ to be the unique element whose component in $\mathbb{Q}_2(\zeta_{2^j})$ is the square root of $(1+4r)_j$ which is congruent to 1 modulo $2\,\mathfrak{m}_j$ when $\mathfrak{m}_j = \mathbb{Z}_2[\zeta_{2^j}](1-\zeta_{2^j})$ is the maximal ideal of the ring of integers $\mathbb{Z}_2[\zeta_{2^j}]$.

Proposition 4.7. *Let $R = W\overline{\mathrm{Q}}$ where $\overline{\mathrm{Q}}$ is as in (4.2). If $V_0 = k$ is the trivial simple $k\mathrm{Q}$-module, let $V_{0,W} = W$ with trivial Q-action. Define $S = WX/W\cdot \mathrm{Tr}_X$ and $\overline{S} = kX/k\cdot\mathrm{Tr}_X$.*

(a) *If $d \geq 4$, let $\beta \in S_2$ be the element from Proposition 4.6, and let $\overline{\beta}$ be the reduction of β modulo 2. Using the natural injections $S_2 \hookrightarrow S$ and $\mathbb{F}_2X/\mathbb{F}_2\cdot\mathrm{Tr}_X \hookrightarrow \overline{S}$, we view $\beta \in S$ and $\overline{\beta} \in \overline{S}$.*
(b) *If $d = 3$ and k contains a primitive cube root ω of unity, let $\hat{\omega} \in W$ be a primitive cube root of unity lifting ω. Let $\beta = \hat{\omega}^2 - \hat{\omega}x \in WX/W\cdot\mathrm{Tr}_X$, and let $\overline{\beta} = \omega^2 + \omega x$ be its image in $\overline{S}$.*

In both cases (a) and (b), let $V_1 = L(\overline{\beta})$ be the corresponding $k\mathrm{Q}$-module as in Definition 4.1. Define a $W\mathrm{Q}$-module structure on $V_{1,W} = S = WX/W\cdot\mathrm{Tr}_X$ by letting y act as

$$y\cdot v_1 = \beta\, v_1^* \qquad \textit{for all } v_1 \in V_{1,W}. \tag{4.18}$$

Let $i \in \{0,1,2,3\}$. Let $j = 0$ if $d = 3$ and k does not contain a primitive cube root of unity, and let $j \in \{0,1\}$ in all other cases. Then $\Omega^i_{W\mathrm{Q}}(V_{j,W})$, as defined in Remark 2.3, is a lift of $\Omega^i_{k\mathrm{Q}}(V_j)$ over W. Moreover, there are precisely four distinct deformations of $\Omega^i_{k\mathrm{Q}}(V_j)$ over W. The universal deformation of $\Omega^i_{k\mathrm{Q}}(V_j)$ is given by the isomorphism class of the $R\mathrm{Q}$-module $U(\mathrm{Q},\Omega^i_{k\mathrm{Q}}(V_j)) = \Omega^i_{W\mathrm{Q}}(V_{j,W})\otimes_W R$ on which x (resp. y) acts diagonally as multiplication by $x\otimes\overline{x}$ (resp. $y\otimes\overline{y}$).

Proof. In both cases (a) and (b), β satisfies (4.5). To check that (4.18) defines a $W\mathrm{Q}$-module structure on $V_{1,W} = S$, we follow the corresponding argument in the proof of [10, Lemma 6.4]. The remaining statements of Proposition 4.7 now follow from Theorem 1.1 and Remarks 2.3 and 3.3.

4.2. The remaining cases

We now consider the remaining cases of p-groups G such that $T(G)$ has a non-trivial torsion subgroup.

Suppose first that k is a field of characteristic 2, and let W be a complete discrete valuation ring of characteristic 0 that has k as its residue field. Fix an integer $d \geq 4$, and let SD be a semidihedral group of order 2^d with the following presentation

$$\mathrm{SD} = \langle x, y \mid x^{2^{d-1}} = y^2 = 1, yxy^{-1} = x^{2^{d-2}-1}\rangle. \tag{4.19}$$

Then $\mathrm{SD} = \langle yx, y\rangle$ where yx has order 4 and y has order 2. Define

$$\overline{\mathrm{SD}} = \mathrm{SD}^{\mathrm{ab},2} = \langle \overline{yx}, \overline{y}\rangle = \{\overline{1}, \overline{x}, \overline{y}, \overline{yx}\}. \tag{4.20}$$

Remark 4.8. Let $T(\mathrm{SD})$ denote the group of equivalence classes of endo-trivial $k\mathrm{SD}$-modules. By [10, Sect. 7], $T(\mathrm{SD})$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/2$. More precisely, define

$$Y = k\,\mathrm{SD}/\langle y\rangle \qquad \text{and} \qquad L = \mathrm{rad}(Y), \tag{4.21}$$

where $\mathrm{SD}/\langle y\rangle$ denotes the set of distinct left cosets of $\langle y\rangle$ in SD, and Y is the corresponding permutation module for SD over k. Then $T(\mathrm{SD})$ is generated by $[\Omega^1_{k\mathrm{SD}}(k)]$ and $[\Omega^1_{k\mathrm{SD}}(L)]$, and $[\Omega^1_{k\mathrm{SD}}(L)]$ has order 2. In particular, each element of $T(\mathrm{SD})$ is of the form $[\Omega^i_{k\mathrm{SD}}(k)]$ or $[\Omega^i_{k\mathrm{SD}}(L)]$ for some $i \in \mathbb{Z}$.

By Theorem 1.1, we know that the universal deformation ring of every endo-trivial kSD-module is isomorphic to $W\overline{\mathrm{SD}}$ where $\overline{\mathrm{SD}}$ is as in (4.20). We now give an explicit description of the universal deformation of every endo-trivial kSD-module V. Since projective kSD-modules are free, and hence can easily be lifted over W, we can concentrate on the indecomposable endo-trivial kSD-modules. The following result is proved using Theorem 1.1 and Remarks 2.3 and 3.3.

Proposition 4.9. *Let $R = W\overline{\mathrm{SD}}$. If $V_0 = k$ is the trivial simple kSD-module, let $V_{0,W} = W$ with trivial SD-action. If $V_1 = L$ is as in (4.21), let $V_{1,W} = \mathrm{Ker}(\pi_W)$ where $\pi_W : W\,\mathrm{SD}/\langle y\rangle \to W$ is the WSD-module homomorphism defined by $\pi_W(x^a\langle y\rangle) = 1_W$ for all $0 \le a \le 2^{d-1} - 1$.*

Let $i \in \mathbb{Z}$ and $j \in \{0, 1\}$. Then $\Omega^i_{W\mathrm{SD}}(V_{j,W})$, as defined in Remark 2.3, is a lift of $\Omega^i_{k\mathrm{SD}}(V_j)$ over W. Moreover, there are precisely four distinct deformations of $\Omega^i_{k\mathrm{SD}}(V_j)$ over W. The universal deformation of $\Omega^i_{k\mathrm{SD}}(V_j)$ is given by the isomorphism class of the RSD-module $U(\mathrm{SD}, \Omega^i_{k\mathrm{SD}}(V_j)) = \Omega^i_{W\mathrm{SD}}(V_{j,W}) \otimes_W R$ on which x (resp. y) acts diagonally as multiplication by $x \otimes \overline{x}$ (resp. $y \otimes \overline{y}$).

The following remark deals with the final case of cyclic groups.

Remark 4.10. Let p be a prime number, and let G be a cyclic group of order $p^d \ge 3$. Let k be a field of characteristic p and let W be a complete discrete valuation ring of characteristic 0 that has k as its residue field. Let $T(G)$ denote the group of equivalence classes of endo-trivial kG-modules. By [14, Cor. 8.8], $T(G)$ is a cyclic group of order 2 given by $\{[k], [\Omega^1_{kG}(k)]\}$.

Let $V_0 = k$ be the trivial simple kG-module, and let $V_{0,W} = W$ with trivial G-action. If $i \in \{0, 1\}$, then $\Omega^i_{WG}(V_{0,W})$, as defined in Remark 2.3, is a lift of $\Omega^i_{kG}(V_0)$ over W. By Remark 3.3, the isomorphism class of $\Omega^i_{WG}(V_{0,W})$, as a module for G over W, is the unique deformation of $\Omega^i_{kG}(V_0)$ over W when $p > 2$. On the other hand, if $p = 2$ then there are precisely two distinct deformations of $\Omega^i_{kG}(V_0)$ over W. The universal deformation of $\Omega^i_{kG}(V_0)$ is given by the isomorphism class of the RG-module $U(G, \Omega^i_{kG}(V_0)) = \Omega^i_{WG}(V_{0,W}) \otimes_W R$ on which x (resp. y) acts diagonally as multiplication by $x \otimes \overline{x}$ (resp. $y \otimes \overline{y}$).

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