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Negative curves of small genus on surfaces

Ted Chinburg[1] · **Matthew Stover**[2]

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Abstract
Let X be a smooth geometrically irreducible projective surface over a field. In this paper we give an effective upper bound in terms of the Néron–Severi rank of X for the number of irreducible curves C on X with negative self-intersection and geometric genus less than $b_1(X)/4$, where $b_1(X)$ is the first étale Betti number of X. The proof involves a hyperbolic analog of the theory of spherical codes. More specifically, we relate these curves to the *hyperbolic kissing number*, and then prove upper and lower bounds for the hyperbolic kissing number in terms of the classical Euclidean kissing number.

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1 Introduction

Let X be a smooth geometrically irreducible projective surface over a field k. By a negative curve on X we will mean a complete, reduced, irreducible curve with negative self intersection. The bounded negativity conjecture states that if $k = \mathbb{C}$, there is a lower bound depending only on X for the self-intersection of all negative curves on X. The origins of this conjecture, as well as various results concerning it, are discussed in [1,9,11] and their references. In this paper we study a different but related question that was asked explicitly in the introduction of [1]:

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✉ Ted Chinburg
ted@math.upenn.edu

Matthew Stover
mstover@temple.edu

[1] Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA

[2] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Question 1.1 *Let X be a smooth geometrically irreducible projective surface over an arbitrary field k. For which integers $g \geq 0$ are there infinitely many negative curves on X of genus g?*

Bogomolov proved in [2] that if $k = \mathbb{C}$, X has general type and $c_1(X)^2 > c_2(X)$, the number of negative curves C on X of geometric genus g is finite for every g. He did this by showing that, under these hypotheses, the C on X that have a given geometric genus form a bounded family. An effective version of the latter result was shown by Lu and Miyaoka in [10, Thm. 1(1)]. We refer the reader to [1, §2] for a more detailed account of subsequent related work. The results of Bogomolov and Lu–Miyaoka need not hold for arbitrary X even over $k = \mathbb{C}$. For example, it was shown in [1, Thm. 4.3] that for each $m > 1$ and each $g \geq 0$ there is an X over $k = \mathbb{C}$ containing infinitely many smooth irreducible curves of self-intersection $-m$ and genus g.

In this paper we prove that for all k and X, there is an effective finite upper bound on the number of negative curves C on X of geometric genus less than $b_1(X)/4$, where $b_1(X)$ is the first étale Betti number of X. The proof uses a different method than Bogomolov's and involves studying a hyperbolic analog of the theory of spherical codes in Euclidean space.

The theory of spherical codes arose from the classical question, going back to Newton and Gregory, of determining how many unit spheres in Euclidean n-space can touch a unit sphere centered at the origin without overlapping interiors. The centers of these spheres then form a spherical code on a sphere of radius 2 about the origin. The angle of such a code is at least $\pi/3$ in the sense that rays from the origin to two such centers must form an angle of at least $\pi/3$. There is an extensive literature concerning spherical codes of varying angles; see, for example, [4] and [5].

We will show how a hyperbolic analog of this theory arises naturally from studying negative curves of genus less than $b_1(X)/4$ on surfaces. The relevant angle is then $\pi/2$ rather than $\pi/3$, as we will explain below. The study and classification of which hyperbolic codes can in fact arise from negative curves in this way is a natural one when trying to determine the constraints on such curves coming from intersection theory.

To state an explicit result, recall that $b_1(X) = \dim_{\mathbb{Q}_p} H^1_{\text{ét}}(X, \mathbb{Q}_p)$ for any prime p different from $\text{char}(k)$. The Picard number $\rho(X)$ is the rank over $\mathbb{Z}$ of the Néron–Severi group $\text{NS}(X)$ of X. We prove:

Theorem 1.2 *Let X be a smooth geometrically irreducible projective surface over a field k. There are effective constants c_1 and c_2 for which the number $\tau(X)$ of negative curves C on X with geometric genus $g(C) < b_1(X)/4$ is finite and bounded above by $c_1 e^{c_2 \rho(X)}$. One has $\tau(X) \leq 2^{0.902\,\rho(X)}$ for sufficiently large $\rho(X)$.*

We do not know if the condition $g(C) < b_1(X)/4$ in Theorem 1.2 is sharp in characteristic zero. However, it is sharp in positive characteristic in view of the following example.

Example 1.3 Let X be the direct product $Y \times Y$, where Y is a smooth projective irreducible curve of genus $g \geq 2$ defined over $\mathbb{F}_p$. Set $k = \overline{\mathbb{F}}_p$, and let C_n be the graph of σ^n in $X(\overline{\mathbb{F}}_p)$, where $\sigma : Y \longrightarrow Y$ is the Frobenius automorphism. Then C_n is a reduced irreducible curve on X of arithmetic genus g, and [7, Ex. V.1.10] implies that $C_n^2 \longrightarrow -\infty$ as $n \longrightarrow \infty$. One has $b_1(X) = 4g$. In particular, there are infinitely many distinct reduced irreducible curves on X of genus $g = b_1(X)/4$ with negative self-intersection.

The proof of Theorem 1.2 is effective and involves analyzing how negative curves of small genus on X are constrained by the intersection pairing on the Neron–Severi group via the

Hodge index theorem. We will precisely define the set of constraints to be considered. In the course of proving Theorem 1.2, we will also show that one cannot use these constraints alone to improve the upper bound on $\tau(X)$ to one that is subexponential in $\rho(X)$.

We discuss below how the curves of Theorem 1.2 give rise to a hyperbolic code of angle at least $\pi/2$. Moreover, we will show that in fact, there do exist hyperbolic codes of angle at least $\pi/2$ that grow exponentially in size with $\rho(X)$. We state this in terms of the *hyperbolic kissing number*. Recall that the classical kissing number K_n arises from finding maximal spherical codes in the unit sphere $\mathbb{S}^{n-1}$ in Euclidean n-space which have a given angle. See Sect. 2 for more on the classical case. In Sect. 3, we define the hyperbolic kissing number in an analogous manner, and one of our key technical results is Theorem 4.6, which gives upper and lower bounds for the hyperbolic kissing number in terms of classical kissing numbers.

However, we do not know whether such large codes can arise from the intersection theory of negative curves of small genus on surfaces. This leads to the following question:

Question 1.4 *Is there a constant $c > 1$ and a sequence of surfaces X for which $\rho(X)$ tends to infinity such that $\tau(X) > c^{\rho(X)}$ for all X in this sequence? Can one find such a sequence in which $b_1(X)$ remains bounded?*

To describe the connection between negative curves and hyperbolic codes more precisely, recall that $\mathrm{NS}(X)$ is a finitely generated abelian group, and the group $\mathrm{Num}(X)$ of divisors modulo numerical equivalence on X is the quotient of $\mathrm{NS}(X)$ by its torsion subgroup. Thus $\mathrm{Num}(X)$ is a free $\mathbb{Z}$-module of rank $\rho(X) \geq 1$. The curves C in Theorem 1.2 map bijectively to their classes in $\mathrm{Num}(X)_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Num}(X)$. We show that these classes form a strict hyperbolic code of angle at least $\pi/2$ in $\mathrm{Num}(X)_{\mathbb{R}}$ in the sense of Definition 3.4. The maximal number of elements in such a code is the strict hyperbolic kissing number $R_{\rho(X)-1}(\pi/2)$. We will prove:

Theorem 1.5 *Let $R_n(\pi/2)$ be the strict hyperbolic kissing number of hyperbolic n-space. Then $R_0(\pi/2) = 0$ and $R_1(\pi/2) = 1$. For $n \geq 2$, $R_n(\pi/2)$ is bounded from above by $K_{n-1}(\phi_0) + 2$, where $K_{n-1}(\phi_0)$ is the classical kissing number for the Euclidean unit sphere $\mathbb{S}^{n-2}$ in $\mathbb{R}^{n-1}$ associated with the angle $\phi_0 = \arccos(3/4)$. Furthermore, if $n \geq 2$ then $R_n(\pi/2)$ is bounded below by $\lfloor K_{n-1}(2\phi_0)/2 \rfloor$, the greatest integer less than or equal to $K_{n-1}(2\phi_0)/2$.*

The bound in Theorem 1.2 then follows from an upper bound of Kabatiansky and Levenshtein on $K_{n-1}(\phi_0)$. This lower bound grows exponentially with n by work of Chabauty, Shannon and Wyner. However, as mentioned above, we do not know if such large hyperbolic codes can be realized by negative curves of small genus on surfaces. In the course of proving these results, we will show the following:

Theorem 1.6 *Let $\mathcal{F}$ be a set of at least two distinct irreducible curves C on X for which $C^2 < 0$. If $\mathcal{F}$ contains more than $R_{\rho(X)-1}(\pi/2)$ elements, there are two elements $C_1, C_2 \in \mathcal{F}$ together with positive integers a, b such that $aC_1 + bC_2$ is an effective connected nef divisor of positive self-intersection on X.*

It was shown in [12, Lem. 4.1] that if the set $\mathcal{F}$ in Theorem 1.6 has more than $\rho(X)^2 + \rho(X) + 1$ elements, then there is an effective nef divisor supported on the union of the elements of $\mathcal{F}$. However, using this replaces the genus bound $b_1(X)/4$ in Theorem 1.2 by the weaker bound $b_1(X)/(2\rho(X)^2 + 2\rho(X) + 2)$. In particular, it is crucial for the proof of Theorem 1.2 that we reduce down to two the number of curves in $\mathcal{F}$ involved in an effective connected nef divisor with positive self-intersection, which is clearly optimal.

We now outline the contents of the paper and the proofs of Theorems 1.6 and 1.2.

In Sect. 2 we recall the definition of spherical codes and some classical results concerning them. We define hyperbolic codes in Sect. 3. In Sect. 4 we state our results concerning the relation between negative curves of small genus and hyperbolic codes of angle at least $\pi/2$.

The proof of Theorem 1.6 involves the following steps. In Sect. 5 we study subsets $\mathcal{D} = \{D_i\}_i$ of $\mathrm{Num}(X)$ for which there is a class $h \in \mathrm{Num}(X)$ with $h^2 > 0$ such that $D_i^2 < 0 \leq D_i \cdot D_j$ and $h \cdot D_i > 0 \geq (aD_i + bD_j)^2$ for all $i \neq j$ and all integers $a, b \geq 0$. We show $\mathcal{D}$ has these properties if and only if it forms a strict hyperbolic code with angle at least $\pi/2$ in the hyperbolic space of dimension $n = \rho(X) - 1$ associated with the intersection pairing on $\mathrm{Num}(X)$.

Now, suppose that $\mathcal{F}$ is a set of curves as in Theorem 1.6. The map sending $C \in \mathcal{F}$ to its class $[C]$ in $\mathrm{Num}(X)$ is injective, since $C_1 \cdot C_2 \geq 0 > C_1^2$ if C_1 and C_2 are distinct elements of $\mathcal{F}$. Since $\mathcal{F}$ has at least two distinct elements, we must have $n + 1 = \rho(X) \geq 2$. Suppose now that $\#\mathcal{F} \geq R_n(\pi/2)$ elements, so that $\mathcal{D} = \{[C] : C \in \mathcal{F}\}$ has more than $R_n(\pi/2)$ elements. Taking h to be the class of an ample effective divisor, we conclude that there are two curves $C_1, C_2 \in \mathcal{F}$ and integers $a, b \geq 0$ such that $E = aC_1 + bC_2$ has $E^2 > 0$. Since the C_i are irreducible, we can adjust a and b so that E becomes an effective connected nef divisor of positive self-intersection. This will prove Theorem 1.6.

To prove Theorem 1.2, we now let $\mathcal{F}$ be the set of irreducible curves C on X with $C^2 < 0$ and $g(C) < b_1(X)/4$. Suppose $\mathcal{F}$ has more than $R_n(\pi/2)$ elements. We show in Sect. 8 that this leads to a contradiction in the following way.

Theorem 1.6 implies there are $C_1, C_2 \in \mathcal{F}$ and $a, b \geq 0$ such that $E = aC_1 + bC_2$ is an effective connected nef divisor of positive self-intersection. An étale Lefschetz theorem (see [6], [3, §2]) implies that the induced homomorphism of étale fundamental groups

$$\pi_1^{\text{ét}}(|E|, x) \longrightarrow \pi_1^{\text{ét}}(X, x)$$

at some geometric point $x \in E$ has image of finite index in $\pi_1^{\text{ét}}(X, x)$, where $|E|$ is the reduction of E. In Theorem 8.3 of Sect. 8 we use a motivic weight argument to show that the natural morphism

$$\mathrm{Jac}(C_1^\sharp) \oplus \mathrm{Jac}(C_2^\sharp) \longrightarrow \mathrm{Alb}(X)$$

is surjective, where $\mathrm{Jac}(C_i^\sharp)$ is the Jacobian of the normalization $C_i^\sharp$ of C_i and $\mathrm{Alb}(X)$ is the Albanese variety of X. Since $g(C_i) = g(C_i^\sharp) = \dim(\mathrm{Jac}(C_i^\sharp))$ this implies that

$$\max(g(C_1), g(C_2)) \geq \dim(\mathrm{Alb}(X))/2 = b_1(X)/4.$$

This is impossible since the curves C in Theorem 1.2 are assumed to have geometric genus strictly less than $b_1(X)/4$.

This reduces the proof of Theorem 1.2 to bounding $R_n(\pi/2)$ from above. We do this in Sects. 5 and 6 using the upper half-space model of hyperbolic n-space to connect hyperbolic codes to spherical codes. The connection comes about from the fact that geodesic half-spaces in the upper half-space model are one side of either a vertical plane or a Euclidean sphere. The latter spheres have centers on the Euclidean space $\mathbb{R}^{n-1}$ of points at infinity different from ∞. In Sect. 6 we prove an upper bound for the size of a strict hyperbolic code of angle $\pi/2$ by showing that we can assume all the half-spaces associated with elements of the code have boundaries that are Euclidean spheres, and we can place the center of the smallest such sphere at the origin in $\mathbb{R}^{n-1}$. Then the rays outward from the origin to the centers associated to other spheres must intersect a unit sphere $\mathbb{S}^{n-2}$ in $\mathbb{R}^{n-1}$ in a spherical code with angle at least $\arccos(3/4) = \phi_0$. In Sect. 7 we prove a lower bound on the maximum size of a

hyperbolic code with angle at least $\pi/2$ by placing the above centers at a well-chosen subset of a spherical code in $\mathbb{R}^{n-1}$ with angle at at least $2\phi_0$ and by taking the radii of all the spheres around these centers to be $\sqrt{7/8}$.

2 Spherical codes

In this section we recall some definitions and results concerning spherical codes. See [5] for further details.

Definition 2.1 A *spherical code* is a subset $\mathcal{S}$ of the unit sphere $\mathbb{S}^{n-1}$ in n-dimensional Euclidean space $\mathbb{R}^n$. If $x, y \in \mathcal{S}$, the *angle* $\phi(x, y)$ between x and y is the unique number in the range $0 \le \phi(x, y) \le \pi$ such that $\cos(\phi(x, y)) = x \cdot y$. Define the *angle* $\phi(\mathcal{S})$ *of* $\mathcal{S}$ to be the infimum of $\phi(x, y)$ over all distinct $x, y \in \mathcal{S}$. For $0 < \phi \le \tau \le \pi$ define

$$K_n(\phi, \tau) = \max\{\#\mathcal{S} \ : \ \phi \le \phi(x, y) \le \tau \text{ for all } x, y \in \mathcal{S} \text{ with } x \ne y\}. \tag{2.1}$$

We set $K_n(\phi) = K_n(\phi, \pi)$.

Example 2.2 The kissing number $K_n = K_n(\pi/3)$ is the maximum number of spheres of a given positive radius that can touch a sphere of the same radius without having overlapping interiors.

The following result is due to Kabatiansky and Levenshtein [8]:

Theorem 2.3 *Suppose* $0 < \phi \le \pi/3$. *If* n *is sufficiently large, then* $K_n(\phi) \le c(\phi)^n$, *where*

$$c(\phi) = \frac{1}{2^{0.099}\sqrt{1 - \cos(\phi)}}. \tag{2.2}$$

In fact, [8] shows that the same conclusion holds for ϕ in the range from 0 to a number slightly larger than $\pi/3$. The following lower bound is due Chabauty, Shannon and Wyner [5, §1.6].

Theorem 2.4 *Suppose* $0 < \phi < \pi/2$ *and* $1 < c < \frac{1}{\sin(\phi)}$. *If* n *is sufficiently large, then*

$$K_n(\phi) \ge c^n. \tag{2.3}$$

3 Hyperbolic codes

The hyperbolic variant of spherical codes developed in this section is motivated by the following observation. A point w in a spherical code $W \subset \mathbb{S}^{n-1}$ determines and is determined by the geodesic half-space

$$Z(w) = \{w' \in \mathbb{S}^{n-1} \ : \ \langle w, w' \rangle \le 0\},$$

where $\langle , \rangle$ is the usual Euclidean inner product. One can thus reformulate spherical codes as collections of geodesic half-spaces of $\mathbb{S}^{n-1}$ whose outward normals form at least a certain angle at their intersections.

This interpretation carries over directly to hyperbolic space. One complication is that in hyperbolic space, half-spaces may not intersect and one half-space can properly contain

another. To formulate a precise definition, we first recall the definition of the hyperboloid model L^n and the ball model B^n of hyperbolic n-space. See [13, §3.2] for details.

Let $\langle\ ,\ \rangle : \mathbb{R}^n \otimes \mathbb{R}^n \longrightarrow \mathbb{R}$ be the usual Euclidean inner product with norm $\|\ \|^2 : \mathbb{R}^n \longrightarrow \mathbb{R}$. Define an inner product $I : (\mathbb{R}^n \perp \mathbb{R}) \oplus (\mathbb{R}^n \perp \mathbb{R}) \longrightarrow \mathbb{R}$ by

$$I((v;u),(v';u')) = -\langle v, v'\rangle + u \cdot u'.$$

For $q = (v;u)$ write $q^2 = I(q,q)$. The (upper) hyperboloid model of hyperbolic space is then

$$L^n = \left\{q = (v;u) \in \mathbb{R}^n \perp \mathbb{R} \ :\ q^2 = -\|v\|^2 + u^2 = 1 \text{ and } u > 0\right\}.$$

The line element for L^n is $ds = \sqrt{(dv)^2 - (du)^2}$.

Let $\underline{0}$ be the origin in $\mathbb{R}^n$. Projection

$$\pi : (\mathbb{R}^n \perp \mathbb{R}) \smallsetminus \{(\underline{0}; -1)\} \longrightarrow \mathbb{R}^n$$

from the point $(\underline{0}; -1)$ identifies L^n with the ball model

$$B^n = \{v \in \mathbb{R}^n \ :\ \|v\|^2 < 1\}$$

of hyperbolic space. The ideal boundary of B^n is the closed unit sphere $\partial B^n = \mathbb{S}^{n-1}$. Define $\overline{B}^n = B^n \cup \partial B^n$, which is the closed unit ball in $\mathbb{R}^n$.

Suppose $w \in \mathbb{R}^n \perp \mathbb{R}$ is a negative vector, i.e., that $I(w,w) = w^2 < 0$. The ray determined by w is

$$r(w) = \{tw \ :\ 0 < t \in \mathbb{R}\},$$

and the set of points

$$Y(w) = \{\pi(q) \in B^n \ :\ q = (v;u) \in L^n \text{ and } I(w,q) \le 0\} \tag{3.1}$$

is a closed geodesic half-space in B^n. Let

$$W(w) = \{\pi(q) \in B^n \ :\ q = (v;u) \in L^n \text{ and } I(w,q) = 0\}$$

be the boundary of $Y(w)$ in B^n and define $\partial Y(w) \subset \mathbb{S}^{n-1}$ (resp. $\partial W(w) \subset \mathbb{S}^{n-1}$) to be the ideal boundary of $Y(w)$ (resp. $W(w)$). Then set $\overline{Y}(w) = Y(w) \cup \partial Y(w)$ and $\overline{W}(w) = W(w) \cup \partial W(w)$. The following is well-known (e.g., see [14, §2.3] and Fig. 1 for an illustration).

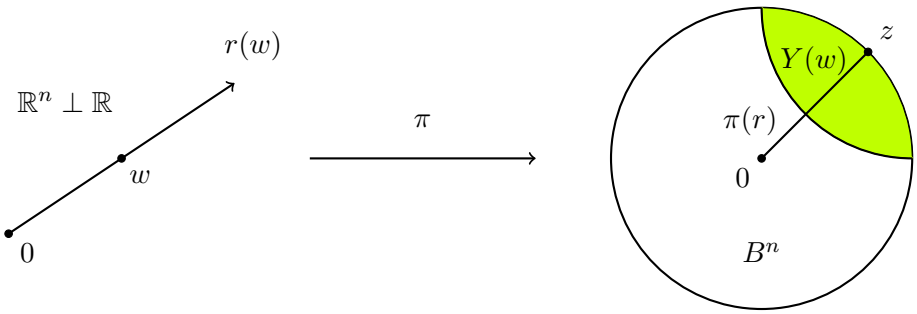


Fig. 1 Illustration of Lemma 3.1

Lemma 3.1 *The map identifying the ray $r(w)$ with the closed half-space $Y(w)$ defines a bijection between the set of negative rays in $\mathbb{R}^n \perp \mathbb{R}$ and the set of closed geodesic half-spaces in B^n. The set $\overline{W}(w)$ is the intersection of the closed unit ball $\overline{B}^n$ with a Euclidean sphere or with a hyperplane of dimension $n-1$. If $n \geq 2$ then $\overline{W}(w)$ intersects $\mathbb{S}^{n-1}$ at right angles and $\overline{W}(w) \cap \mathbb{S}^{n-1} = \partial W(w)$ is a Euclidean sphere of dimension $n-2$ and positive radius.*

Definition 3.2 Suppose that $q \in \overline{W}(w)$. If $q \in W(w) \subset B^n$, let $n_w(q)$ be the outward unit normal to $Y(w)$ at q in the tangent space $T_q B^n$ of q in B^n. If $q \in \partial W(w) \subset \mathbb{S}^{n-1}$ and $n \geq 2$, let $n_w(q)$ be the outward unit normal to $\overline{Y}(w) \cap \mathbb{S}^{n-1} = \partial Y(w)$ at q in the tangent space $T_q \mathbb{S}^{n-1}$.

We now need to understand more about the properties of the subspaces associated with a pair of negative vectors. Suppose that $w_1, w_2 \in \mathbb{R}^n \perp \mathbb{R}$ are negative vectors, and in what follows set $W_i = W(w_i)$ and $Y_i = Y(w_i)$, $i = 1, 2$. Suppose that there exists a point $q \in \overline{W}_1 \cap \overline{W}_2$. It is well-known that the angle $\theta(w_1, w_2) \in [0, \pi]$ between $n_{w_1}(q)$ and $n_{w_2}(q)$ satisfies

$$\cos(\theta(w_1, w_2)) = \frac{-I(w_1, w_2)}{\sqrt{I(w_1, w_1) \cdot I(w_2, w_2)}}. \tag{3.2}$$

See [13, §3.2], and note that our I is the negative of the form used there. If $\overline{W}(w_1) \cap \overline{W}(w_2) = \emptyset$, define $\theta(w_1, w_2) = -\infty$. We will need the following observations.

Lemma 3.3 *Suppose w_1 and w_2 are two negative elements of $\mathbb{R}^n \perp \mathbb{R}$. The following conditions are equivalent:*

(i.) $\theta(w_1, w_2) \geq \pi/2$;
(ii.) $I(w_1, w_2) \geq 0$ *and there exists a point* $q \in \overline{W}_1 \cap \overline{W}_2$;
(iii.) $I(w_1, w_2) \geq 0$ *and for all* $0 \leq a, b \in \mathbb{R}$ *one has*

$$I(aw_1 + bw_2, aw_1 + bw_2) \leq 0. \tag{3.3}$$

Now suppose that any (and hence all) of these conditions hold and that there exists $h \in L^n \subset \mathbb{R}^n \perp \mathbb{R}$ with $I(w_i, h) > 0$ for $i = 1, 2$. Then $\pi(h) \notin Y_1 \cup Y_2$. Let $P \in \mathbb{S}^{n-1} = \partial B^n$ be the limit point of the geodesic ray in B^n starting at $\pi(h)$ and perpendicular to W_1. Then P is a point of $\overline{Y}_1 \smallsetminus \overline{W}_1$ that is not in $\overline{Y}_2$.

Proof We will see that the proof reduces to checking the case $n = 2$. Since conditions (i), (ii) and (iii) are invariant under scaling, we can assume that $w_1^2 = w_2^2 = -1$. The fact that (i) implies (ii) is clear from (3.2). If (ii) holds, then (iii) follows from expanding $I(aw_1 + bw_2, aw_1 + bw_2)$ and using (3.2). Similarly, (iii) implies $|I(w_1, w_2)| \leq 1$, which means that $\overline{W}_1$ and $\overline{W}_2$ meet with angle given as in (3.2) and hence (iii) implies (i).

We now suppose that (i), (ii) and (iii) hold and that there is an $h \in L^n$ as in the last part of the lemma. The fact that $\pi(h) \notin Y_1 \cup Y_2$ follows immediately from the definition of the Y_i. Intersecting with the appropriate totally geodesic B^2 inside B^n, it suffices to prove the claim for P in the hyperbolic plane. Then we have the geometric arrangement shown in Fig. 2. If the geodesic ray ℓ from $\pi(h)$ intersecting W_1 orthogonally were to have endpoint on $\partial \overline{Y}_2$, then it would need to also meet $\overline{W}_2$.

Let z_1 be the point at which ℓ meets W_1 and z_2 the point where ℓ meets $\overline{W}_2$. When q, z_1, and z_2 are distinct, they form a triangle in $\overline{B}^2$, possibly with an ideal vertex at z_2, with interior angle θ at q and $\pi/2$ at z_1. Therefore the triangle has angle sum greater than or equal

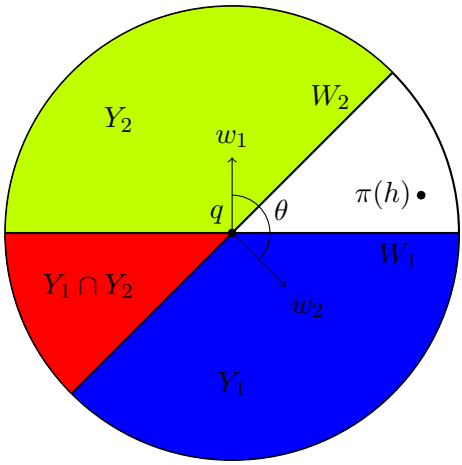


Fig. 2 Geometric picture for Lemma 3.3

to π, which is impossible for a triangle in $\overline{B}^2$ [13, §3.5]. In the degenerate case, $q = z_1 = z_2$, and the geodesic from $\pi(h)$ to q visibly makes an angle

$$\phi < \pi - \theta \leq \pi/2$$

with W_1 at q, and hence cannot be orthogonal to W_1. Since $\pi(h)$ is not in W_1 and W_1 is totally geodesic, it is also clear that the endpoint of ℓ cannot be in $\overline{W}_1$. This completes the proof of Lemma 3.3.

□

Definition 3.4 A *hyperbolic code* is a collection $\mathcal{S}$ of negative vectors $w \in \mathbb{R}^n \perp \mathbb{R}$. We say that $\mathcal{S}$ is *strict* if the union over all $w \in \mathcal{S}$ of the half-spaces $Y(w)$ is not all of B^n. Define $\theta(\mathcal{S}) \in \{-\infty\} \cup [0, \pi]$ to be the greatest lower bound over all pairs w_1, w_2 of distinct elements of $\mathcal{S}$ of the angle $\theta(w_1, w_2)$ defined above.

Definition 3.5 Let θ be an angle in the range $0 < \theta \leq \pi$. The *hyperbolic kissing number* (resp. *strict hyperbolic kissing number*) $\overline{R}_n(\theta)$ (resp. $R_n(\theta)$) in $\mathbb{Z} \cup \{\infty\}$ is the supremum of $\#\mathcal{S}$ over all hyperbolic codes $\mathcal{S}$ (resp. strict hyperbolic codes $\mathcal{S}$) for which $\theta(\mathcal{S}) \geq \theta$.

4 Negative curves, hyperbolic codes, and Theorem 1.2.

As in the introduction, let X be an irreducible smooth projective surface over a field k. The group $\mathrm{Num}(X)$ is torsion free and finitely generated. Let

$$\mathrm{Num}(X)_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Num}(X).$$

The Hodge index theorem implies that the intersection pairing on $\mathrm{Num}(X)$ extends to a pairing

$$I : \mathrm{Num}(X)_{\mathbb{R}} \times \mathrm{Num}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

with signature $(1, n)$, where $\dim(\mathrm{Num}(X)_{\mathbb{R}}) = n + 1$.

Definition 4.1 Let $L(X)$ be the hyperboloid model of hyperbolic n-space associated with the choice of isometry carrying I to the standard signature $(1, n)$ pairing on $\mathbb{R}^n \perp \mathbb{R}$.

Definition 4.2 Let $\mathcal{T}(X)$ be the set of all irreducible curves C on X for which $C^2 = I(C, C) < 0$ and $g(C) < b_1(X)/4$. For $C \in \mathcal{T}(X)$, let $[C]$ be the class of C in $\mathrm{Num}(X)_{\mathbb{R}}$ and define $\mathcal{S}(X) = \{[C] : C \in \mathcal{T}\}$.

The following is a well-known consequence of the fact that distinct curves have non-negative intersection.

Lemma 4.3 *The map $\mathcal{T}(X) \longrightarrow \mathcal{S}(X)$ sending C to $[C]$ is a bijection.*

We prove the following theorem, which is the key technical result connecting hyperbolic codes to negative curves, in Sect. 8.

Theorem 4.4 *The set $\mathcal{S}(X)$ is a strict hyperbolic code in $\mathrm{Num}(X)_{\mathbb{R}}$ of angle at least $\pi/2$.*

Recall that $\rho(X)$ is the rank of $\mathrm{Num}(X)$, i.e., the real dimension of $\mathrm{Num}(X)_{\mathbb{R}}$. We then have the following conclusion.

Corollary 4.5 *The number of elements of $\mathcal{T}(X)$ is bounded above by the strict hyperbolic kissing number $R_n(\pi/2)$, where $n = \rho(X) - 1$.*

Recall that $K_{n-1}(\theta)$ is the kissing number associated to the angle θ in the Euclidean space $\mathbb{R}^{n-1}$. The first statement of the following Theorem will be proved in Sect. 5. The second statement will be proved in Sects. 6 and 7.

Theorem 4.6 *For every $n \geq 2$, one has*

$$R_n(\pi/2) \leq \overline{R}_n(\pi/2) \leq 2R_n(\pi/2). \tag{4.1}$$

Let $0 < \phi < \tau \leq \pi$ be any choice of constants such that $\sqrt{2}\sin(\phi/2) = \sin(\tau/2)$. Then

$$\max\left\{\left\lfloor \frac{K_{n-1}(2\phi_0)}{2} \right\rfloor, K_{n-1}(\phi, \tau)\right\} \leq R_n(\pi/2) \leq K_{n-1}(\phi_0) + 2, \tag{4.2}$$

where $\phi_0 = \arccos(3/4)$.

From the results about spherical kissing numbers quoted in Sect. 2 we now have the following conclusion.

Corollary 4.7 *One has*

$$2^{0.011\,n}(1 + o(n)) \leq R_n(\pi/2) \leq 2^{0.901\,n}(1 + o(n)),$$

where $o(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof of Theorem 1.2 Combine Corollary 4.5, Theorem 4.6, and Corollary 4.7. □

5 Hyperbolic codes and the upper half-space model

We now use the upper half-space model

$$H^n = \{(z_1, \ldots, z_n) \ : \ z_i \in \mathbb{R}, z_n > 0\} \subset \mathbb{R}^n.$$

of hyperbolic space to give another description of hyperbolic codes.

Recall from [13, §4.4] that there is an isometry $f : L^n \longrightarrow H^n$ defined in the following way. For $x \in L^n \subset \mathbb{R}^n \perp \mathbb{R} = \mathbb{R}^{n+1}$, let y be the point $(y_1, \ldots, y_{n+1})$ on the unit $(n+1)$-sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ that is on the ray from the origin in $\mathbb{R}^{n+1}$ to x. Then $f(x)$ is the unique point $z = (z_1, \ldots, z_n) \in H^n$ such that $(1, z_1, \ldots, z_n) \in \mathbb{R}^{n+1}$ lies on the ray outward from $((-1, 0, 0, \ldots, 0); 0) \in \mathbb{R}^{n+1}$ through y.

Consider the one-point compactification

$$\partial H^n = \{\infty\} \cup \{(z_1, \ldots, z_{n-1}, 0) \ : \ z_i \in \mathbb{R}\}$$

of $\mathbb{R}^{n-1} = \{(z_1, \ldots, z_{n-1}, 0) \ : \ z_i \in \mathbb{R}\}$. Then ∂H^n is homeomorphic to $\mathbb{S}^{n-1}$, and the above construction identifies ∂H^n with the boundary of H^n. Geodesics in H^n are the intersection of H^n with either circles or vertical lines in $\mathbb{R}^n$ that intersect $\partial H^n \smallsetminus \{\infty\} = \mathbb{R}^{n-1}$ orthogonally. Geodesic hypersurfaces in H^n are the intersection of H^n with either

(i) vertical planes in $\mathbb{R}^n$ (i.e., planes intersecting $\partial H^n \smallsetminus \{\infty\} = \mathbb{R}^{n-1}$ orthogonally), or
(ii) Euclidean spheres with center on $\partial H^n \smallsetminus \{\infty\}$ (which then intersect $\partial H^n \smallsetminus \{\infty\}$ everywhere orthogonally).

Geodesic half-spaces are then formed by the set of all points of H^n that lie either on one chosen side of a geodesic hypersurface or on the hypersurface itself. Define $\overline{H}^n = H^n \cup \partial H^n$.

Definition 5.1 In (3.1) to each negative vector $w \in \mathbb{R}^n \perp \mathbb{R}$ we defined a geodesic half-space $Y(w)$ in the open ball model B^n of hyperbolic space with boundary $W(w)$, a geodesic hypersurface. Let $Y'(w)$ be the corresponding geodesic half-space in H^n with boundary $W'(w)$. Similarly, let $\partial Y'(w) \subset \partial H^n$ (resp. $\partial W'(w) \subset \partial H^n$) be the ideal boundary of $Y'(w)$ (resp. $W'(w)$). Finally, set $\overline{Y'}(w) = Y'(w) \cup \partial Y'(w)$ and $\overline{W'}(w) = W'(w) \cup \partial W'(w)$.

If $W'(w)$ lies in a vertical plane we will say that the center $z(w)$ of $Y'(w)$ is the point ∞ of ∂H^n and that the Euclidean radius of $\partial Y'(w)$ is ∞. Otherwise, $W'(w)$ is the intersection of H^n with a Euclidean sphere of some positive radius $d(w)$ centered at a point $z(w) \in \mathbb{R}^{n-1} = \partial H^n \smallsetminus \{\infty\}$. If $z(w) \neq \infty$ and $z(w) \in \overline{Y'}(w)$, then $Y'(w)$ is the intersection of H^n with the closed Euclidean ball of radius $d(w)$ about $z(w)$. Otherwise, $Y'(w)$ is the intersection of H^n with the complement of the interior of this ball.

We now reformulate the condition that $\theta(w_1, w_2) \geq \pi/2$ in Lemma 3.3 using the upper half-space model. To simplify notation in what follows, given a negative vector $w_i \in \mathbb{R}^n \perp \mathbb{R}$ we let $Y_i = Y(w_i)$ and similarly for the other notation from Definition 5.1.

Lemma 5.2 *Suppose w_1 and w_2 are two negative elements of $\mathbb{R}^n \perp \mathbb{R}$ such that neither W_1' nor W_2' lie in a vertical plane. For z_1, z_2, d_1, d_2 as in Definition 5.1, let $|z_1 - z_2|$ be the Euclidean distance between z_1 and z_2 in $\mathbb{R}^{n-1}$. Define $\delta_i = 1$ if $z_i \in \overline{Y}_i'$, and set $\delta_i = -1$ otherwise. Then $\theta(w_1, w_2) \geq \pi/2$ if and only if and only if*

$$\sqrt{d_1^2 + d_2^2} \leq |z_1 - z_2| \leq d_1 + d_2 \qquad \text{when} \quad \delta_1\delta_2 = 1 \tag{5.1}$$

$$|d_1 - d_2| \leq |z_1 - z_2| \leq \sqrt{d_1^2 + d_2^2} \qquad \text{when} \quad \delta_1\delta_2 = -1. \tag{5.2}$$

Finally, if $\theta(w_1, w_2) \geq \pi/2$ and $I(h, w_1), I(h, w_2) > 0$ for some $h \in L^n \subset \mathbb{R}^n \perp \mathbb{R}$, then $z_1 \neq z_2$.

Proof For $i = 1, 2$, the half-space $\overline{Y}_i'$ is either $\overline{H}^n \cap \overline{B}(z_i, d_i)$ (when $\delta_i = 1$) or the complement in $\overline{H}^n$ of the interior of $\overline{B}(z_i, d_i)$ (when $\delta(w_i) = -1$).

Suppose first that $\theta(w_1, w_2) \geq \pi/2$, so that there is a point $q \in \overline{W}_1 \cap \overline{W}_2$. If $\delta_1\delta_2 = 1$, the angle between the rays from q to z_1 and from q to z_2 is at least $\theta(w_1, w_2) \geq \pi/2$. Therefore $|z_1 - z_2| \geq \sqrt{d_1^2 + d_2^2}$. In this case, the existence of a point in $\overline{W}'_1 \cap \overline{W}'_2$ implies that $|z_1 - z_2| \leq d_1 + d_2$. This proves (5.1).

If $\delta_1\delta_2 = -1$, the angle between the rays from q to w_1 and from q to w_2 is at most $\pi/2$, rather than being at least $\pi/2$. This leads to $|z_1 - z_2| \leq \sqrt{d_1^2 + d_2^2}$. Since $\overline{W}'_1$ and $\overline{W'}_2$ must intersect, we see that $d_1 + d_2 \geq |z_1 - z_2| \geq |d_1 - d_2|$. Note that $|z_1 - z_2| \leq \sqrt{d_1^2 + d_2^2}$ already implies $d_1 + d_2 \geq |z_1 - z_2|$. This gives (5.2).

For the converse, one reverses the above reasoning to show that (5.1) and (5.2) imply that $\theta(w_1, w_2) \geq \pi/2$. If $z_1 = z_2$, then W'_1 and W'_2 are the the intersection of H^n with concentric spheres. Hence if $\theta(w_1, w_2) \geq \pi/2$, we would have $\overline{W}'_1 = \overline{W}'_2$ and $\theta(w_1, w_2) = \pi$. However, then $\overline{Y}_1 \cup \overline{Y}_2 = \overline{B}^n$, so there could be no $h \in L^n$ with $I(h, w_1), I(h, w_2) > 0$. □

We now have the following, which is one of the main technical results in this paper.

Theorem 5.3 *For all integers $m, n \geq 1$, the following are equivalent:*

(1) There are elements $w_0, \ldots, w_m \in \mathbb{R}^n \perp \mathbb{R}$ such that for some $h \in \mathbb{R}^n \perp \mathbb{R}$ one has

- *(a) $I(h, h) > 0 > I(w_i, w_i)$,*
- *(b) $I(h, w_i) > 0$,*
- *(c) $I(w_i, w_j) \geq 0$, and*
- *(d) $I(aw_i + bw_j, aw_i + bw_j) \leq 0$ for all distinct $0 \leq i, j \leq m$ and all positive $a, b \in \mathbb{R}$.*

(2) The subset $\{w_0, \ldots, w_m\} \subset \mathbb{R}^n \perp \mathbb{R}$ is a strict hyperbolic code having $m + 1$ elements and angle at least $\pi/2$.

(3) After replacing the $m + 1$ element subset $\{w_0, \ldots, w_m\} \subset \mathbb{R}^n \perp \mathbb{R}$ by their image under an isometry, the set $\{\overline{Y}'_0, \ldots, \overline{Y}'_m\}$ of half-spaces in H^n has the following description. The ideal boundary of each $\overline{Y}'_i$ is a sphere centered at a point $z_i \in \mathbb{R}^{n-1}$ of some radius $d_i > 0$. When $i = 0$, the point z_0 is the origin $\underline{0}$ of $\mathbb{R}^{n-1}$, and $\overline{Y}'_0$ is the exterior in $\overline{H}^n$ of the open ball of radius d_0.

If $1 \leq i \leq m$, $\overline{Y}'_i$ is the intersection of $\overline{H}^n$ with the closed ball of radius d_i around z_i. Finally, the following inequalities hold:

- *(a) $|z_j|^2 > \max(0, d_j^2 - d_0^2)$ if $1 \leq j \leq m$,*
- *(b) $|d_0 - d_i| \leq |z_i| \leq \sqrt{d_0^2 + d_i^2}$, and*
- *(c) $\sqrt{d_i^2 + d_j^2} \leq |z_i - z_j| \leq d_i + d_j$ if $1 \leq i < j \leq m$*

where $|z - z'|$ is the Euclidean distance between points $z, z' \in \mathbb{R}$.

Lastly, if $\{z_1, \ldots, z_m\}$ is any set of m distinct points in $\mathbb{R}^{n-1}$ for which there are positive constants $d_1, \ldots, d_m > 0$ such that condition (c) of part (3) holds, then there exist $h, w_1, \ldots, w_m \in \mathbb{R}^n \perp \mathbb{R}$ for which the statements in condition (1) hold for $1 \leq i, j \leq m$.

Proof Lemma 3.3 shows the equivalence of (1) and (2). Indeed, if $h \in \mathbb{R}^n \perp \mathbb{R}$ and $I(h, h) > 0$, we can replace h by $h/\sqrt{I(h, h)}$ to make h an element of L^n.

To show that (1) implies (3), let h and $w_0, \ldots, w_m$ be as in (1), where as above we can assume $h \in L^n$. Let $\tilde{P}$ be the limit point on ∂H^n of the geodesic ray in H^n that starts at $f(h)$

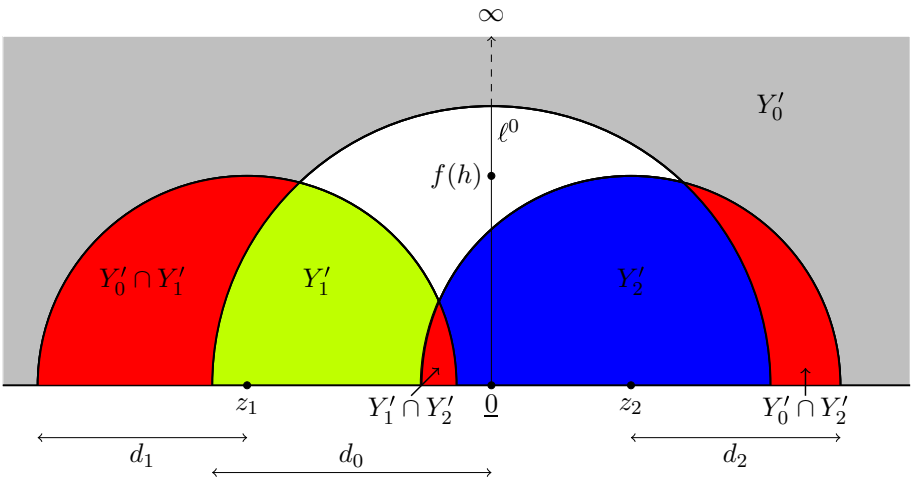


Fig. 3 Arrangement of half-spaces in Theorem 5.3

and is perpendicular to the geodesic hypersurface W_0', where $f : L^n \longrightarrow H^n$ is the above isometry. This geodesic ray is part of a geodesic line with another limit point $\tilde{P}'$ on ∂H^n. Applying an isometry, we can assume that $\tilde{P} = \infty$ and $\tilde{P}'$ is the origin $\overline{0}$ of $\mathbb{R}^{n-1} \subset \partial H^n$.

Translating the final statement of Lemma 3.3 to the upper half plane model, $\tilde{P}$ is a point of $\overline{Y}_0' \smallsetminus \overline{W}_0'$ that does not lie on $\overline{Y}_j'$ for any $j > 0$. Consider the points $z_0, \ldots, z_m$ associated with the w_i by Definition 5.1. It is clear from our assumptions that each z_j lies in $\mathbb{R}^{n-1} \subset \partial H^n$. Recall that W_j' is the intersection of H^n with a Euclidean sphere of radius $d_j > 0$ and center z_j.

When $j = 0$, we know $\infty = P' \in \overline{Y}_0'$, so $\overline{Y}_0'$ must be the complement in $\overline{H}^n$ of the interior $B(z_0, d_0)$ of the ball at z_0 of radius d_0. Thus $\delta_0 = -1$ in the terminology of Lemma 5.2. Furthermore, the sphere W_0 is perpendicular to the geodesic with limit points ∞ and $\overline{0}$, and this geodesic contains $f(h)$, and we conclude that $z_0 = \overline{0}$. Note also that now $f(h)$ must lie in the interior ℓ^0 of the vertical line segment of Euclidean length d_0 that has one endpoint at $\overline{0}$. See Fig. 3.

Suppose $1 \leq j \leq m$. Then $\infty \notin \overline{Y}_j'$ implies that $\overline{Y}_j' = \overline{H}^n \cap \overline{B}(z_j, d_j)$, so $\delta_j = 1$ for $1 \leq j \leq m$. Since $f(h)$ is not contained in $\overline{Y}_j'$, we find from the fact that $f(h)$ is on ℓ^0 that

$$d_0^2 + |z_j|^2 > d_j^2.$$

We now apply the criterion in Lemma 5.2 to every pair w_i, w_j with $0 \leq i \neq j \leq m$ to produce the inequalities in part (3) of Theorem 5.3.

Conversely, suppose all of the inequalities stated in part (3) of Theorem 5.3 are satisfied with $z_0 = \underline{0}$ and some of $\{z_i\}_{i=1}^m \subset \mathbb{R}^{n-1}$ and positive real numbers $\{d_i\}_{i=0}^m$. Then $z_0 \neq z_i$ for $1 \leq i \leq m$ because $|z_i| \neq 0$ was assumed in part (3c) of Theorem 5.3. We can choose negative vectors $w_0, \ldots, w_m$ in $\mathbb{R}^n \perp \mathbb{R}$ such that $\overline{Y}_0'$ is the complement in $\overline{H}^n$ of the open unit ball of radius d_0 about the origin z_0 and $\overline{Y}_i'$ is $\overline{H}^n \cap \overline{B}(z_i, d_i)$ for $1 \leq i \leq m$.

The assumption that $d_0^2 + |z_i|^2 > d_i^2$ for $1 \leq i \leq m$ in part (3) of Theorem 5.3 implies that if we choose $h \in L^n$ so that $f(h)$ lies in $B(\underline{0}, 1)$ and is close enough to the point that lies at distance 1 directly above the origin, then $f(h)$ will not be in $\overline{Y}_i'$ for $1 \leq i \leq m$ or

in $\overline{Y}_0'$. Now Lemma 5.2 shows that $h, w_0, w_1, \ldots, w_m$ satisfy the conditions in part (1) of Theorem 5.3.

The final statement we must prove is that if one has only points $z_1, \ldots, z_m$ in $\mathbb{R}^{n-1}$ and positive numbers $d_1, \ldots, d_m$ for which part (c) of condition (3) holds, then there are $h, w_1, \ldots, w_m \in N(X)_{\mathbb{R}}$ for which condition (1) holds for $1 \leq i, j \leq m$. In this case, we choose w_i so that $\overline{Y}_i'$ is $\overline{H}^n \cap \overline{B}(z_i, d_i)$ for $1 \leq i \leq m$. Then the vertical heights of points of each $\overline{Y}_i'$ are bounded, so we can find a point $f(h) \in H^n$ not in this union. Lemma 5.2 now shows that $h, w_1, \ldots, w_m$ satisfy the conditions in part (1) of the theorem for $1 \leq i, j \leq m$. □

We now give a number of corollaries to Theorem 5.3.

Corollary 5.4 *The strict hyperbolic kissing number $R_n(\pi/2)$ is the supremum of $m+1$ over all integers m for which there exist distinct points $z_1, \ldots, z_m \in \mathbb{R}^{n-1}$ and positive real constants $d_1, \ldots, d_m$ for which*

(a) $\max\{0, d_i^2 - 1\} < |z_i|$,
(b) $|1 - d_i| \leq |d_i| \leq \sqrt{1 + d_i^2}$, *and*
(c) $\sqrt{d_i^2 + d_j^2} \leq |z_i - z_j| \leq d_i + d_j$.

Proof The corollary follows from renormalizing the z_i and d_i as in part (3) of Theorem 5.3 by dividing each by d_0, $1 \leq i \leq m$. □

Corollary 5.5 *Suppose there is a possibly nonstrict hyperbolic code in L^n having m' elements and angle at least $\pi/2$. If $m = \lceil m'/2 \rceil$, then there is a strict hyperbolic code having at least m elements and angle at least $\pi/2$.*

Proof Applying an isometry, suppose we have a nonstrict code $\mathcal{S} = \{w_1, \ldots, w_{m'}\}$ with angle at least $\pi/2$ such that, in the upper half-space model each w_i gives a point $z_i \in \mathbb{R}^{n-1}$ together with a positive radius d_i. Removing at most half the w_i, we can replace m' by m and assume that all of the constants δ_i are the same. In other words, all of the half-spaces $\overline{Y}_i'$ come from either the interiors U_i of the open balls $B(z_i, d_i)$, or all of them come from the exterior of its closure of U_i. This means that we now satisfy the inequalities in (5.1) of Lemma 5.2 for all $1 \leq i, j \leq m$ with $i = j$. Replacing each $\overline{Y}_i'$ by U_i now produces, via the last statement of Theorem 5.3, a strict hyperbolic code with m elements, since the union of all the U_i cannot be all of H^n. □

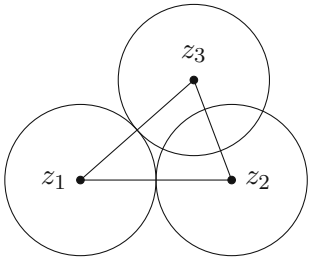
Corollary 5.6 *Suppose that there is a largest positive integer $m = m(n)$ for which the equivalent conditions (1), (2), and (3) in Theorem 5.3 can be satisfied by some choice of h, w_i, z_i and d_i as i ranges over $0 \leq i \leq m$. Then $m+1$ is the strict hyperbolic kissing number $R_n(\pi/2)$. The (nonstrict) hyperbolic kissing number $\overline{R}_n(\pi/2)$ satisfies*

$$R_n(\pi/2) \leq \overline{R}_n(\pi/2) \leq 2R_n(\pi/2). \tag{5.3}$$

6 The upper bound on $R_n(\pi/2)$

We begin with the following technical estimate.

Fig. 4 Optimizing Lemma 6.1



Lemma 6.1 *Suppose* $n \geq 2$, $z_1, z_2, z_3 \in \mathbb{R}^{n-1}$, $0 < d_1 \leq d_2 \leq d_3$ *and that*

$$\sqrt{d_i^2 + d_j^2} \leq |z_i - z_j| \leq d_i + d_j \quad \text{for all} \quad i \neq j \tag{6.1}$$

as in condition (c) of Corollary 5.4 (cf. condition 2(c) of Theorem 5.3). Then $n \geq 3$ *and* z_1, z_2 *and* z_3 *are not collinear. Let* $0 < \theta_1 < \pi$ *be the angle of the triangle* (z_1, z_2, z_3) *at* z_1. *Then* $\theta_1 \geq \phi_0 = \arccos(3/4)$.

Proof Considering the subspace spanned by z_1, z_2, and z_3, we can reduce to the case where $n \leq 3$. If z_1, z_2, and z_3 are collinear, we can assume $n = 2$ and $z_1 < z_2 < z_3$ in $\mathbb{R}^{n-1} = \mathbb{R}$. Then (6.1) leads to a contradiction. Therefore after a translation and scaling, we can assume $n = 3$, $z_1 = (0, 0) = \underline{0}$ is the origin in $\mathbb{R}^2$ and $0 < d_1 \leq d_2 \leq d_3 = 1$. The input of z_2, z_3, d_1 and d_2 is now specified by 6 real variables, and we want to maximize the function $\cos(\theta_1)$ of these variables. It is a lengthy but elementary calculus exercise to show that the maximum is obtained when $\cos(\theta_1) = 3/4$. We list the steps involved here and include complete details in an appendix.

Regarding d_1, d_2 and $d_3 = 1$ as fixed for the moment, let $S(d_1, d_2)$ be the set of $(z_1, z_2, z_3) = (\underline{0}, z_2, z_3)$ that satisfy (6.1). One checks that $\cos(\theta_1)$ is a continuous function on the compact set $S(d_1, d_2)$, so that it attains its maximum at some point $(z_1, z_2, z_3) = (\underline{0}, z_2, z_3)$ in $S(d_1, d_2)$. To prove the lemma it suffices to show that $\theta_1 \geq \phi_0$.

The main fact we can now use is that since $(z_1, z_2, z_3) \in S(d_1, d_2)$ maximizes $\cos(\theta_1)$, we cannot move z_1, z_2 and z_3 in $\mathbb{R}^2$ and then translate z_1 back to $\underline{0}$ in such a way that the inequalities (6.1) still hold with the same d_1, d_2 and $d_3 = 1$ but with a smaller value for θ_1. By considering such moves, we show in the appendix that the maximum value of $\cos(\theta_1)$ over all possible choices of $0 < d_1 \leq d_2 \leq d_3 = 1$ is attained by the example in Remark 6.2 below. □

Remark 6.2 An angle of $\theta_1 = \phi_0$ can be achieved by setting $d_1 = d_2 = d_3 = 1$, $n = 3$, $z_1 = (0, 0) \in \mathbb{R}^{n-1} = \mathbb{R}^2$, $z_3 = (2, 0)$ and $z_2 = (2\cos(\theta_0), 2\sin(\phi_0)) = (3/2, \sqrt{7}/2)$. Then $|z_1 - z_3| = d_1 + d_3 = 2 = d_1 + d_2 = |z_1 - z_2|$ and $|z_2 - z_3| = \sqrt{d_2^2 + d_3^2} = \sqrt{2}$. See Fig. 4.

We now prove the following, which implies the upper bound in (4.2) of Theorem 4.6 as well as all the bounds in (4.1) via Corollary 5.6.

Corollary 6.3 *Suppose* $z_1, \ldots, z_m \in \mathbb{R}^{n-1}$ *and* $d_1, \ldots, d_m > 0$ *satisfy condition (c) of Corollary 5.4, so that* $\sqrt{d_i^2 + d_j^2} \leq |z_i - z_j| \leq d_i + d_j$ *for all* $i \neq j$. *Then* $m \leq K_{n-1}(\phi_0) + 1$. *In particular, the number* $m(n)$ *from Corollary 5.6 satisfies* $m(n) \leq K_{n-1}(\phi_0) + 2$.

Proof Without loss of generality, we can order $z_1, \ldots, z_m$ so that $d_1 \leq d_i$ for all $1 \leq i \leq m$. By condition (c), the points z_i are all distinct. Therefore, for $1 < i \leq m$ the points

$$\xi_i = (z_i - z_1)/|z_i - z_1|$$

lie on the unit sphere $\mathbb{S}^{n-2}$ in $\mathbb{R}^{n-1}$. Lemma 6.1 shows that for all $1 < i < j \le m$, the angle between the rays from the origin to ξ_i and to ξ_j must be at least ϕ_0. Therefore $\xi_2, \ldots, \xi_m$ must form a spherical code with angular separation at least ϕ_0, so $m - 1 \le K_{n-1}(\phi_0)$. Then the number $m(n)$ from Corollary 5.6 is the number of points $z_0, z_1, \ldots, z_m$ for which there are $d_0, \ldots, d_m$ as in Theorem 5.3, so we conclude $m(n) \le m + 1 \le K_{n-1}(\phi_0) + 2$. □

7 The lower bound on $R_n(\pi/2)$

Let $0 < \phi < \tau \le \pi$ be any choice of constants such that $\sqrt{2}\sin(\phi/2) = \sin(\tau/2)$ and define $m = K_{n-1}(\phi, \tau)$. We can therefore find a spherical code $S = \{z_1, \ldots, z_m\}$ on the unit sphere $\mathbb{S}^{n-2}$ in R^{n-1} such that the angular separation $\phi(z_i, z_j)$ between the rays from the origin to z_i and to z_j satisfies $\phi \le \phi(z_i, z_j) \le \tau$ for all $i \ne j$. Therefore,

$$4\sin^2(\phi/2) = 2 - 2\cos(\phi) \le |z_i - z_j|^2 = 2 - 2\cos(\phi(z_i, z_j)) \le 4\sin^2(\tau/2).$$

It follows that if we let $d_k = \sqrt{2}\sin(\phi/2) = \sin(\tau/2)$ for all $k = 1, \ldots, m$, then

$$\sqrt{d_i^2 + d_j^2} \le |z_i - z_j| \le d_i + d_j$$

for all $i \ne j$, as in condition (c) of Corollary 5.4. Theorem 5.3 now says that there are $h, w_1, \ldots, w_m \in N(X)_{\mathbb{R}}$ for which the statements in condition (1) of Theorem 5.3 hold for $1 \le i, j \le m$. Part (2) of Theorem 5.3 now says $\{w_1, \ldots, w_m\}$ is a strict hyperbolic code with angle at least $\pi/2$. Therefore Definition 3.5 gives that

$$m = K_{n-1}(\phi, \tau) \le R_n(\pi/2).$$

This is the first part of the lower bound (4.2) in Theorem 4.6.

To show the other lower bound in (4.2) of Theorem 4.6, it will suffice to show that when $\phi_0 = \arccos(3/4)$, we have

$$K_{n-1}(2\phi_0)/2 \le K_{n-1}(\phi, \tau)$$

for some ϕ and τ as above. Let $\phi = 2\phi_0 = 1.445\ldots$ and $\tau = \pi - \phi_0 = 2.418\ldots$, so $0 < \phi < \tau \le \pi$. We then have

$$2\sin^2(\phi/2) = 2\sin^2(\phi_0) = 2(1 - \cos^2(\phi_0)) = 2(1 - 9/16) = 7/8$$

$$\sin^2(\tau/2) = \sin^2(\pi/2 - \phi_0/2) = \cos^2(\phi_0/2) = \frac{\cos(\phi_0) + 1}{2} = \frac{3/4 + 1}{2} = 7/8.$$

Thus $\sqrt{2}\sin(\phi/2) = \sin(\tau/2)$ since both of these numbers are positive.

Recall that if z and w are points on the unit sphere $\mathbb{S}^{n-1}$, $\phi(z, w)$ is the angle between the rays $\tilde{z}$ and $\tilde{w}$ from the origin to z and to w, respectively. By the definition of $\ell = K_{n-1}(2\phi_0)$, we can find a spherical code $S' = \{r_1, \ldots, r_\ell\}$ on $\mathbb{S}^{n-2}$ such that

$$\phi(r_i, r_j) \ge 2\phi_0 \quad \text{if} \quad i \ne j. \tag{7.1}$$

For each i, consider the open cone $C(-r_i)$ of points $z \in \mathbb{S}^{n-2}$ such that $\phi(-r_i, z) < \phi_0$. If there were two distinct points r_j and r_q in $S' \cap C(-r_i)$, then

$$\phi(r_j, r_q) \le \phi(-r_i, r_j) + \phi(-r_i, r_q) < 2\phi_0,$$

which contradicts (7.1). Therefore there is at most point point of the form r_j in $S' \cap C(-r_i)$, and if such an r_j exists, r_i is the unique point in $S' \cap C(-r_j)$. Throwing away at most half

of the points in S' we then arrive at a spherical code $S = \{z_1, \ldots, z_{\ell'}\}$ with $\ell' \geq \ell/2 = K_{n-1}(2\phi_0)/2$ such that $S \cap C(-z_i) = \emptyset$ for all i. If $j \neq i$, then the angle $\phi(z_i, z_j)$ can be at most $\pi - \phi_0$, since z_j does not lie in $C(-z_i)$. We therefore have $\phi = 2\phi_0 \leq \phi(z_i, z_j) \leq \pi - \phi_0 = \tau$, which shows that $K_{n-1}(2\phi_0)/2 \leq K_{n-1}(\phi, \tau)$. This finishes the proof of the lower bound in (4.2) of Theorem 4.6.

8 The proofs of Theorems 1.6 and 4.4

Theorem 1.6 is equivalent to the following result.

Theorem 8.1 *Suppose $\mathcal{F}$ is a set of irreducible curves C on X such that $C^2 < 0$ and there is no connected effective nef divisor with positive self-intersection of the form $pC_1 + qC_2$ with $0 \leq p, q \in \mathbb{Z}$ and $C_1, C_2 \in \mathcal{F}$. Then the set $\{[C] : C \in \mathcal{F}\}$ is a strict hyperbolic code of angle at least $\pi/2$.*

Proof Recall from Lemma 4.3 that the elements $[C]$ are all distinct in $\mathrm{Num}(X)$. Let A be an ample effective divisor on X. Then $I([A], [C]) > 0$ for $C \in \mathcal{F}$, where I denotes the intersection pairing. Therefore $h = [A]/\sqrt{I(A, A)}$ is an element of the hyperbolic space $L(X)$ associated with the intersection pairing on $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Num}(X)$, and it does not lie in any of the geodesic half-spaces

$$H([C]) = \{q \in L(X) : I(q, [C]) \leq 0\},$$

hence

$$\mathcal{T} = \bigcup_{C \in \mathcal{F}} H([C])$$

is not all of $L(X)$.

Suppose that $\mathcal{T}$ is not a strict hyperbolic code with angle at least $\pi/2$. Then we have that $\theta([C_1], [C_2]) < \pi/2$ for some distinct elements C_1, C_2 of $\mathcal{F}$, and Lemma 3.3 shows that there are $0 \leq a, b \in \mathbb{R}$ such that

$$I(a[C_1] + b[C_2], a[C_1] + b[C_2]) = \alpha a^2 + 2\beta ab + \gamma b^2 > 0,$$

where

$$\begin{aligned} \alpha &= I([C_1], [C_1]) < 0 \\ \gamma &= I([C_2], [C_2]) < 0 \\ \beta &= I([C_1], [C_2]) \geq 0. \end{aligned}$$

Therefore $\beta > 0$ and $\beta^2 > \alpha\gamma$. There will be positive integers p and q such that

$$0 < -\gamma/\beta < p/q < -\beta/\alpha.$$

Then $I([C_1], p[C_1]+q[C_1]) = p\alpha+\beta q > 0$ and $I([C_2], p[C_1]+q[C_2]) = p\beta+q\gamma > 0$. It is then clear that pC_1+qC_2 is an effective connected nef divisor of positive self-intersection, contradicting the hypothesis of Theorem 8.1. This proves the theorem. □

Remark 8.2 The referee noticed that one can also give the following simple argument for the last part of the proof of Theorem 8.1. Our setup implies that the quadratic form $\alpha a^2+2\beta ab+\gamma b^2$ does not take positive values on the first quadratic in $\mathbb{R}^2$. Then any positive value can be obtained with a, b of the same sign, since $\alpha, \gamma < 0$ and $\beta \geq 0$, hence the form takes no positive values at all. This implies that $\beta^2 < \alpha\gamma$, which gives $\cos\theta < 0$.

As in the statement of Theorem 4.4, let $\mathcal{T}(X)$ be the set of all irreducible curves C on X for which $C^2 = I(C, C) < 0$ and $g(C) < b_1(X)/4$, and set

$$\mathcal{S}(X) = \{[C] : C \in \mathcal{T}\} \subset \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Num}(X).$$

We must show that $\mathcal{S}(X)$ is a strict hyperbolic code in $L(X)$ with angle at least $\pi/2$. We suppose throughout this section that this is not the case, and we will derive a contradiction.

Theorem 8.1 implies there is an effective connected nef divisor of positive self-intersection on X of the form $pC_1 + qC_2$ in which C_1 and C_2 are elements of $\mathcal{T}(X)$ and $0 < p, q \in \mathbb{Z}$. We will prove the following result below:

Theorem 8.3 *Suppose that E is a connected effective nef divisor on a smooth projective geometrically integral surface X over a field k with positive self-intersection . Let $E^\sharp$ be the normalization of the reduction $|E|$ of E. Let $J(E^\sharp)$ be the direct sum of the Jacobians of the irreducible components of $E^\sharp$. Then the natural morphism from $J(E^\sharp)$ to the Albanese variety* $\mathrm{Alb}(X)$ *of X is surjective.*

Before giving the proof, we note how it implies Theorem 4.4. If $E = pC_1 + qC_2$ as above, we obtain a surjection

$$J(E^\sharp) = J(C_1^\sharp) \oplus J(C_2^\sharp) \longrightarrow \mathrm{Alb}(X).$$

Since $\mathrm{Alb}(X)$ has dimension $b_1(X)/4$ and $J(C_i^\sharp)$ has dimension the geometric genus $g(C_i)$, we see that $g(C_1) + g(C_2) \geq b_1(X)/2$. However, we supposed that every curve $C \in \mathcal{T}(X)$ has $g(C) < b_1(X)/4$, and this contradiction proves Theorem 4.4. This also completes the proof of Theorem 1.6.

Proof of Theorem 8.3 It suffices to prove the theorem for the base change of E and X to an algebraic closure of k. We assume for the rest of the proof that k is algebraically closed.

Let f be the pullback morphism from the Picard variety $\mathrm{Pic}^{0,red}(X)$ of X to the direct sum $\mathrm{Pic}^{0,red}(E^\sharp)$ of the Picard varieties of the irreducible components of $E^\sharp$. By duality, it will be enough to show that $\mathrm{Ker}(f)$ is a finite group scheme. We suppose in what follows that $\mathrm{Ker}(f)$ is not finite and we will derive a contradiction.

Since $\mathrm{Ker}(f)$ is a subgroup scheme of an abelian variety, it is an extension of an abelian variety B of positive dimension by a finite group scheme. Let ℓ be a prime different from the characteristic of k. Then the ℓ-adic Tate module $T_\ell(\mathrm{Ker}(f))$ is isomorphic to $T_\ell(B)$, and it is a positive rank submodule of $T_\ell(\mathrm{Pic}^{0,red}(X))$. The pullback morphism from $T_\ell(\mathrm{Ker}(f)) = T_\ell(B)$ to $T_\ell(\mathrm{Pic}^{0,red}(E^\sharp))$ is trivial.

We know from the étale Lefschetz theorem that the morphism

$$\pi_1^{\text{ét}}(|E|, x) \longrightarrow \pi_1^{\text{ét}}(X, x)$$

of étale fundamental groups at a geometric point x in the support of $|E|$ is surjective, since E is connected, nef, and effective. Results of this kind go back to Grothendieck in [6]; see Bost [3, §2] for an excellent discussion, particularly Prop. 2.3. This means that

$$\mathrm{Hom}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}/\ell^n) \longrightarrow \mathrm{Hom}(\pi_1^{\text{ét}}(|E|, x), \mathbb{Z}/\ell^n)$$

is injective for all n. Since k is algebraically closed, $\mathbb{Z}/\ell^n$ is isomorphic to the group scheme μ_{ℓ^n} of $(\ell^n)^{th}$ roots of unity. Hence the Kummer sequence shows that

$$\mathrm{Hom}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}/\ell^n) = \mathrm{Pic}(X)[\ell^n] \longrightarrow \mathrm{Hom}(\pi_1^{\text{ét}}(|E|, x), \mathbb{Z}/\ell^n) = \mathrm{Pic}(|E|)[\ell^n]$$

is injective for all n.

Taking inverse limits over n we see that the pullback homomorphism

$$T_\ell(\mathrm{Pic}(X)) \longrightarrow T_\ell(\mathrm{Pic}(|E|))$$

is injective. On the other hand, $T_\ell(B) \subseteq T_\ell(\mathrm{Pic}(X))$ maps to 0 in $T_\ell(\mathrm{Pic}^{0,red}(E^\sharp))$, so the pullback of line bundles must induce an injection

$$\xi : T_\ell(B) \longrightarrow U = \mathrm{Ker}\left(T_\ell(\mathrm{Pic}(|E|)) \longrightarrow T_\ell(\mathrm{Pic}^{0,red}(E^\sharp)\right). \tag{8.1}$$

We will derive our contradiction from this statement.

All of the above schemes are defined over finitely generated algebras over $\mathbb{Z}$. By increasing ℓ, if necessary, we can find a specialization of all of the above schemes over a finite field k' of characteristic p not equal to ℓ so that it will suffice to show the map ξ in (8.1) is not injective for k an algebraic closure of k'.

We now analyze U using the map $\pi : E^\sharp \longrightarrow |E|$ coming from the fact that $E^\sharp$ is the normalization of $|E|$. We have an exact sequence of sheaves of groups in the étale topology of $|E|$ given by

$$1 \longrightarrow \mathbb{G}_{m,|E|} \longrightarrow \pi_*\mathbb{G}_{m,E^\sharp} \longrightarrow V \longrightarrow 1$$

in which V has support of dimension 0. Since π is finite, when we take the étale cohomology of this sequence, we find that U is a quotient of

$$M = \varprojlim_n H^0(k \otimes_{k'} |E|, V)[\ell^n],$$

where $H^0(k \otimes_{k'} |E|, V)[\ell^n]$ is the ℓ^n torsion in the $H^0(k \otimes_{k'} |E|, V)$.

Recall that ℓ is prime to the residue characteristic of the finite field k' over which we are working. There is a filtration of $H^0(k \otimes_{k'} |E|, V)$ by $\mathrm{Gal}(k/k')$-stable submodules such that each graded quotient is isomorphic to either k^* or the additive group k^+. Therefore, if Φ is the arithmetic Frobenius of $\mathrm{Gal}(k/k')$, then the eigenvalues of Φ on M are all equal to the order $\#k'$ of k'. This implies that the eigenvalues of Φ on U equal $\#k'$.

On the other hand $T_\ell(B)$ is the Tate module of an abelian variety B over k', so the eigenvalues of Φ on $T_\ell(B)$ have absolute value the square root of $\#k'$ by the Weil conjectures. It follows from this that ξ cannot be injective, since $T_\ell(B)$ has positive rank. This contradiction completes the proof. □

Remark 8.4 We note that one can prove Theorem 8.3 using only the classical Lefschetz theorem when $k = \mathbb{C}$, as the maps

$$\pi_1(E^\sharp) \longrightarrow \pi_1(E) \longrightarrow \pi_1(X)$$

are surjective. Our proof adapts this idea to arbitrary characteristic.

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9 Appendix: A calculus exercise

In this appendix we complete the proof of Lemma 6.1, whose notation we now assume. As in that proof, we begin by fixing $0 < d_1 \leq d_2 \leq d_3 = 1$. Let $\underline{0} = (0, 0)$ be the origin in $\mathbb{R}^2 = \mathbb{R}^{n-1}$, and let $S(d_1, d_2)$ be the set of triples $(z_1, z_2, z_3) = (\underline{0}, z_2, z_3)$ with $z_2, z_3 \in \mathbb{R}^2$ that satisfy (6.1). Then (6.1) implies that $|z_1 - z_2| = |z_2|$, $|z_1 - z_3| = |z_3|$, and $|z_2 - z_3|$ are

bounded above and below by positive constants. It follows that $S(d_1, d_2)$ is compact. The law of cosines gives

$$\cos(\theta_1) = \frac{|z_1 - z_2|^2 + |z_1 - z_3|^2 - |z_2 - z_3|^2}{2 \cdot |z_1 - z_2| \cdot |z_1 - z_3|}, \tag{9.1}$$

where the denominator on the right is bounded away from 0. Thus $\cos(\theta_1)$ is a continuous function on $S(d_1, d_2)$, so it attains its maximum. We now assume this maximum occurs at $(z_1, z_2, z_3) = (\underline{0}, z_2, z_3)$. As noted in Sect. 6, to prove Lemma 6.1 it will suffice to show that $\theta_1 \geq \phi_0$.

Let θ_2 and θ_3 be the angles at z_2 and at z_3 between the sides of the triangle with vertices at z_1, z_2, z_3, respectively. Since we observed in Sect. 6 that (6.1) implies that z_1, z_2, and z_3 are not collinear, all of θ_1, θ_2 and θ_3 lie in the open interval $(0, \pi)$. Suppose $\theta_3 \geq \pi/2$. Then (6.1) gives

$$\begin{aligned}(d_1 + d_2)^2 &\geq |z_1 - z_2|^2 \\ &\geq |z_1 - z_3|^2 + |z_2 - z_3|^2 \qquad (\text{since } \theta_3 \geq \pi/2) \\ &\geq d_1^2 + d_3^2 + d_2^2 + d_3^2. \end{aligned} \tag{9.2}$$

This gives

$$2d_1 d_2 \geq 2d_3^2 = 2.$$

However, $d_1 \leq d_2 \leq d_3 = 1$, so this is only possible if $d_1 = d_2 = d_3 = 1$ and if all of the inequalities in (9.2) are equalities. Hence $(z_1, z_2, z_3) = (\underline{0}, z_2, z_3)$ is a right triangle with side lengths $|z_1 - z_2| = |z_2| = d_1 + d_2 = 2$, $|z_1 - z_3| = \sqrt{d_1^2 + d_3^2} = \sqrt{2}$, and $|z_2 - z_3| = \sqrt{d_2^2 + d_3^2} = \sqrt{2}$. This means that $\theta_1 = \pi/4 \geq \phi_0$ in this case, as claimed.

As noted in Sect. 6, the main fact we can now apply is that since $(z_1, z_2, z_3) \in S(d_1, d_2)$ minimizes θ_1, we cannot move z_1, z_2 and z_3 in $\mathbb{R}^2$ and then translate z_1 back to $\underline{0}$ in such a way that the inequalities (6.1) still hold with the same d_1, d_2 and $d_3 = 1$ but a smaller value for θ_1.

We may assume that z_3 is a point on the positive real axis by rotating both z_2 and z_3 around $z_1 = \underline{0}$. Since $0 < \theta_1 < \phi_0 < \pi/2$, the point z_2 now lies in the upper right quadrant. Let C_1 be the circle of radius $|z_1 - z_2| = |z_2|$ around z_2 in $\mathbb{R}^2$. Suppose that

$$|z_1 - z_3| < d_1 + d_3. \tag{9.3}$$

Let $z_2' = z_2$ and $z_3' = z_3$. We now move $z_1 = \underline{0}$ to a point z_1' on C_1 that lies in the upper left quadrant and is very close to z_1. The only side length which changes is then $|z_1 - z_3|$, which becomes $|z_1' - z_3'| > |z_1 - z_3|$ since $z_3' = z_3$ lies on the positive part of the real line. Since $|z_1 - z_3| < d_1 + d_3$, all of the inequalities in (6.1) will hold with the same d_1, d_2, d_3 when we replace (z_1, z_2, z_3) by $(z_1', z_2', z_3') = (z_1', z_2, z_3)$ if z_1' is a point on C_1 that lies in the upper left quadrant and is close enough to z_1. We now show that the angle θ_1' between the sides meeting at z_1' of the triangle (z_1', z_2', z_3') satisfies $\theta_1' < \theta_1$. This will contradict the minimality of θ_1 and show that (9.3) cannot hold.

Let θ_3' be the angle between the sides of (z_1', z_2', z_3') meeting at $z_3' = z_3$. Since z_1' lies in the upper left quadrant and on the same side of the line between $z_2' = z_2$ and $z_3' = z_3$ as the origin $z_1 = (0, 0)$, we have $0 < \theta_3' < \theta_3 < \pi/2$. Thus $0 < \sin(\theta_3') < \sin(\theta_3)$. The law of sines now gives

$$\frac{\sin(\theta_1')}{\sin(\theta_3')} = \frac{|z_2' - z_3'|}{|z_1' - z_2'|}$$

$$= \frac{|z_2 - z_3|}{|z_1 - z_2|}$$
$$= \frac{\sin(\theta_1)}{\sin(\theta_3)}. \tag{9.4}$$

Since $\sin(\theta_3') < \sin(\theta_3)$ we conclude that $\sin(\theta_1') < \sin(\theta_1)$. Since we took z_1' to be close to $z_1 = (0, 0)$ on C_1, we can ensure that that θ_1' is close to θ_1. Since $0 < \theta_1 < \phi_0 < \pi/2$, we conclude that $\theta_1' < \theta_1$, contradicting the minimality of θ_1. Thus (9.3) is false, so

$$z_1 = \underline{0} = (0, 0) \quad \text{and} \quad z_3 = (d_1 + d_3, 0) \tag{9.5}$$

after rotating z_3 as above so that it lies on the positive real line.

Now suppose that

$$d_1^2 + d_2^2 < |z_1 - z_2|^2 < (d_1 + d_2)^2. \tag{9.6}$$

Recall that z_2 is a point in the upper right quadrant, and that we have reduced to the case in which (9.5) holds. We let $z_2' = z_2$ and $z_3' = z_3$. Define C_2 to be the circle with center $z_3 = (d_1 + d_3, 0)$ and radius $d_1 + d_2$, so that C_2 contains $z_1 = \underline{0}$ by (9.5). Consider points z_1' very close to z_1 on C_2. The only edge distance that can change on replacing (z_1, z_2, z_3) by (z_1', z_2', z_3') is $|z_1 - z_2|$. Since $|z_1' - z_2'| = |z_1' - z_2|$ will be close to $|z_1 - z_2| = |z_2|$ if z_1' is close to z_1, we conclude from (9.6) that all the inequalities in (6.1) will hold if (z_1, z_2, z_3) is replaced by $(z_1', z_2', z_3') = (z_1', z_2, z_3)$ and z_1' is any point on C_2 sufficiently close to z_1.

Recall that θ_2 is the angle at z_2 between the sides of the triangle (z_1, z_2, z_3) adjoining z_2. Let θ_2' be the corresponding angle for the triangle (z_1', z_2', z_3'). If z_1' lies in the upper half plane and is sufficiently close to z_1, it is on the other side of the line between z_2 and $z_1 = \underline{0}$ from z_3. It follows that $\theta_2' > \theta_2$ in this case. We find similarly that $\theta_2' < \theta_2$ in case z_1' is a point of C_2 that lies in the lower half plane and is sufficiently close to z_1. Thus we can in either case choose a z_1' on C_2 arbitrarily close to z_1 for which

$$0 < \sin(\theta_2') < \sin(\theta_2). \tag{9.7}$$

Since $|z_1 - z_3| = d_1 + d_2 = |z_1' - z_3|$ and $|z_2 - z_3| = |z_2' - z_3'|$, the law of sines gives

$$\frac{\sin(\theta_1')}{\sin(\theta_2')} = \frac{|z_2' - z_3'|}{|z_1' - z_3'|}$$
$$= \frac{|z_2 - z_3|}{|z_1 - z_3|}$$
$$= \frac{\sin(\theta_1)}{\sin(\theta_2)}. \tag{9.8}$$

Now (9.7) shows $\sin(\theta_1') < \sin(\theta_1)$. Since θ_1' will be close to $\theta_1 < \phi_0 < \pi/2$ for z_1' close to z_1, we conclude that $\theta_1' < \theta_1$, which contradicts the minimality of θ_1. Thus the hypothesis (9.6) must be false, and so

$$d_1^2 + d_2^2 = |z_1 - z_2|^2 \quad \text{or} \quad |z_1 - z_2|^2 = (d_1 + d_2)^2. \tag{9.9}$$

We now apply the law of cosines, together with (9.5) and $d_2^3 + d_3^2 \leq |z_2 - z_3|^2$ from (6.1). This gives

$$\cos(\theta_1) = \frac{|z_1 - z_2|^2 + |z_1 - z_3|^2 - |z_2 - z_3|^2}{2 \cdot |z_1 - z_2| \cdot |z_1 - z_3|}$$

$$\leq \frac{|z_1 - z_2|^2 + (d_1 + d_3)^2 - d_2^2 - d_3^2}{2 \cdot |z_1 - z_2| \cdot (d_1 + d_3)} \tag{9.10}$$

where $d_1 \leq d_2 \leq d_3 = 1$.

Suppose first that $d_1^2 + d_2^2 = |z_1 - z_2|$ in (9.9). Then (9.10) becomes

$$\cos(\theta_1) \leq \frac{d_1^2 + d_2^2 + (d_1 + 1)^2 - d_2^2 - 1}{2 \cdot \sqrt{d_1^2 + d_2^2} \cdot (d_1 + 1)} = \frac{d_1}{\sqrt{d_1^2 + d_2^2}} = \frac{1}{\sqrt{1 + (d_2/d_1)^2}} \leq \frac{1}{\sqrt{2}} \tag{9.11}$$

since $0 < d_1 \leq d_2$. This forces $\theta_1 \geq \pi/4$, contradicting $\theta_1 < \phi_0 < \pi/4$.

The remaining possibility in (9.9) is that $|z_1 - z_2| = d_1 + d_2$. Then (9.10) gives

$$\begin{aligned} \cos(\theta_1) &\leq \frac{(d_1 + d_2)^2 + (d_1 + 1)^2 - d_2^2 - 1^2}{2 \cdot (d_1 + d_2) \cdot (d_1 + 1)} \\ &= \frac{(d_1 + d_2 + 1)d_1}{(d_1 + d_2) \cdot (d_1 + 1)} \\ &\leq \left(1 + \frac{1}{d_1 + d_2}\right) \cdot \left(\frac{1}{1 + 1/d_1}\right) \\ &\leq \frac{3}{4}. \end{aligned} \tag{9.12}$$

since $0 < d_1 \leq d_2 \leq d_3 = 1$. This gives $\theta_1 \geq \phi_0 = \arccos(3/4)$, which completes the proof of Lemma 6.1.

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