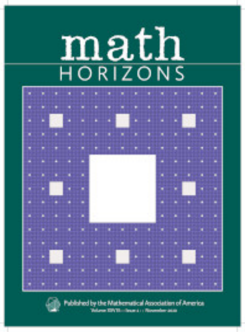

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KNOTRIS: A Game Inspired by Knot Theory

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A Game Inspired by Knot Theory

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When we began our CURM-sponsored research for the 2019–2020 academic year, we thought that we were going to work on pure math research (you know, making conjectures, proving theorems, and such) in the area of knot theory.

For those who are surprised to hear that knots are an object of study for mathematicians, here's a bit of background. *Knots* are typically considered to be knotted circles that live in three-dimensional space. *Links* are collections of these circles. Two knots or links are *equivalent* to each other if you can take one and bend, stretch, or generally rearrange it to produce the other. Cutting and regluing the knot or magically passing the knot through itself are not allowed. The simplest nontrivial knot is called the *trefoil*, and the simplest nontrivial link is called the *Hopf link*. Diagrams of these appear in figure 1.

We intended to study *knot mosaics*. Lomonaco and Kauffman first introduced knot mosaics in order to build a quantum knot system (Quantum knots and mosaics, *Quantum Information Processing*, 7 no. 2-3 [2008] 85–115). These objects attracted a lot of interest in the math community, not necessarily because of their applications in quantum knot theory, but because they're interesting combinatorial objects to play with and ask questions about.

So, what is a knot mosaic? It's a rectangular (or more often, square) arrangement of tiles, created from the 11 *basic mosaic tiles* in figure 2.

When we place tiles to form a knot mosaic, we require them to be *suitably connected*, meaning that each connection point on a tile matches up with a connection point of a neighboring tile. Figure 3 shows two suitably connected mosaics: the image on the left depicts the trefoil knot, and the one the right shows the Hopf link. Figure 4 shows a mosaic that fails to be suitably connected.

Recent research on knot mosaics has concerned board sizes: given a specific knot, what is the smallest $n \times n$ mosaic board on which that knot can be represented (Lee, Ludwig, Paat, and Peiffer, Knot mosaic tabulation, *Involve* **11** no. 1 [2017] 13–26)? For instance, figure 3 proves that the trefoil knot can be represented on a 4×4 mosaic, but the trefoil cannot be represented on a 3×3 mosaic because there is only room for one crossing tile on a 3×3 mosaic if the exterior edges of the mosaic aren't allowed to have connection points.

While trying to formulate some new questions about knot mosaics and consider strategies for

Figure 1. Depictions of the trefoil knot (left) and Hopf link (right).

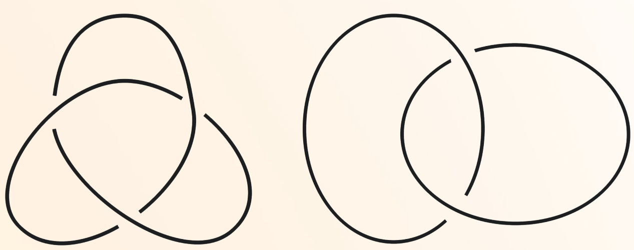


Figure 2. The 11 basic mosaic tiles.

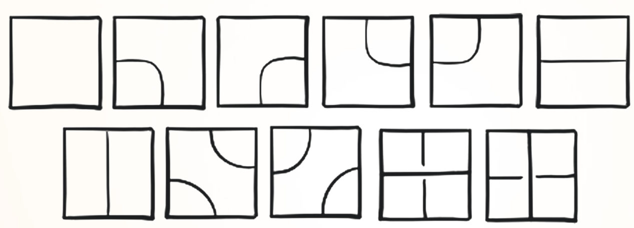


Figure 3. The trefoil knot and the Hopf link on a 4 × 4 mosaic board.

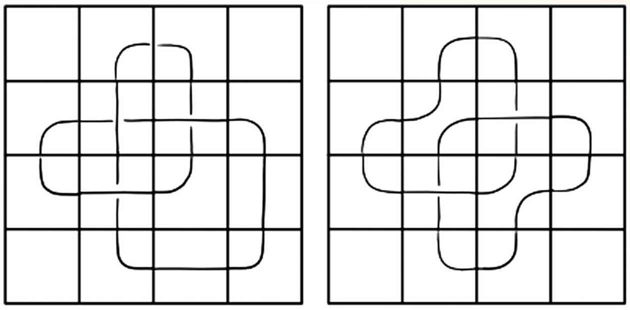
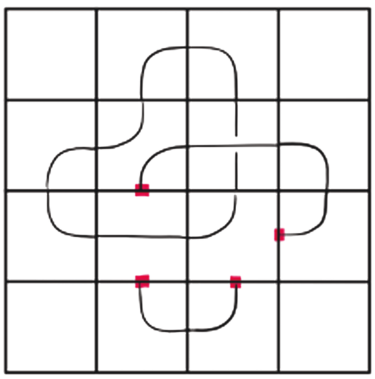


Figure 4. A mosaic board that is not suitably connected (due to the red dead-ends).



finding their answers, our team took some time to play. We obtained a big bag of woodcut mosaic tiles made by Lew Ludwig for the UnKnot Conference, and we made some of our own. For several weeks in the fall, we spent part of our research time physically playing with these tiles to build intuition and brainstorm questions.

One day, as we talked about potential research questions, playing idly with our tiles and trying to make collections of tiles be suitably connected, we realized that this simple act was actually pretty fun. “If we think putting together tiles in a suitably connected way is fun, maybe others would too!” “What if we created a game—kind of like Tetris (*tetris.com*)—that uses knot mosaic tiles instead of tetrominoes?” And that was how our project was born.

We dabbled in our pure math research and associated programming projects for several weeks after this idea came about, but quickly it became clear to us that working to develop *Knotris* should be our new goal.

How to Play Knotris

Our game begins on a 6-tile-wide by 13-tile-high board with a single, suitably connected row of mosaic tiles already in place at the bottom of the board. Just as in Tetris, new tiles fall from the top of the board one at a time. Each falling tile can be rotated any multiple of 90 degrees and shifted left or right as it falls; the player’s goal is to place this tile on top of another tile in such a way that the board remains suitably connected.

Once an internally suitably connected row (i.e., a row whose tiles are all suitably connected on their left and right boundaries) has been completed and this row is suitably connected to a completed row above it, the row gets cleared from the board and points are added to the player’s point total. (If a row is ever created so that it is not internally suitably connected, it can never be removed.)

Figure 5 illustrates two examples of game play early in a game. On the left, the falling tile cannot be placed anywhere to maintain a suitably connected board. On the right, the tile is poised to land in a spot that will allow the bottom row to be cleared from the board. Notice that the next three tiles the player can expect to see are previewed at the top right of the screen.

Figure 5. Two examples of mid-game play. The left image will result in a row that cannot be removed; the right image shows that the bottom row will clear when the piece falls in place.

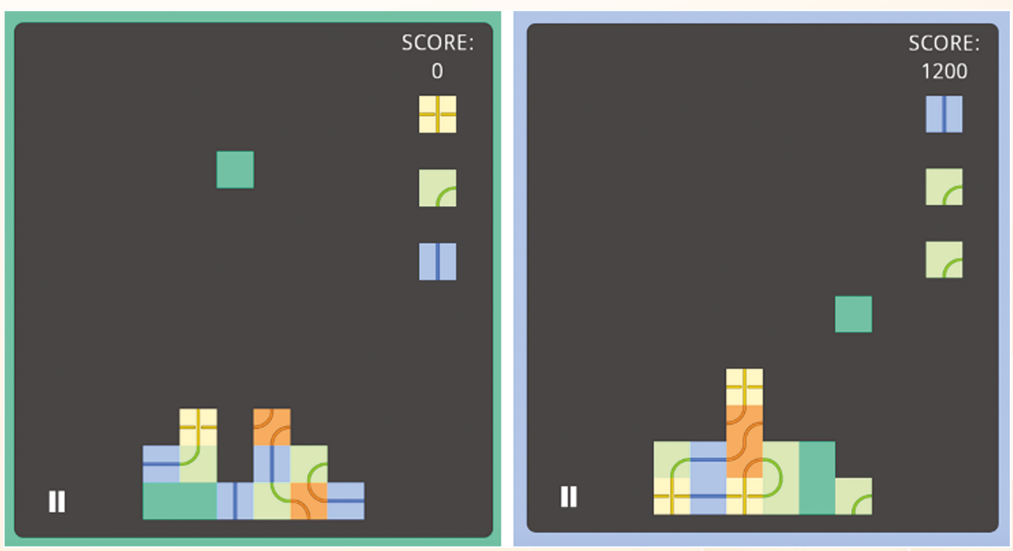
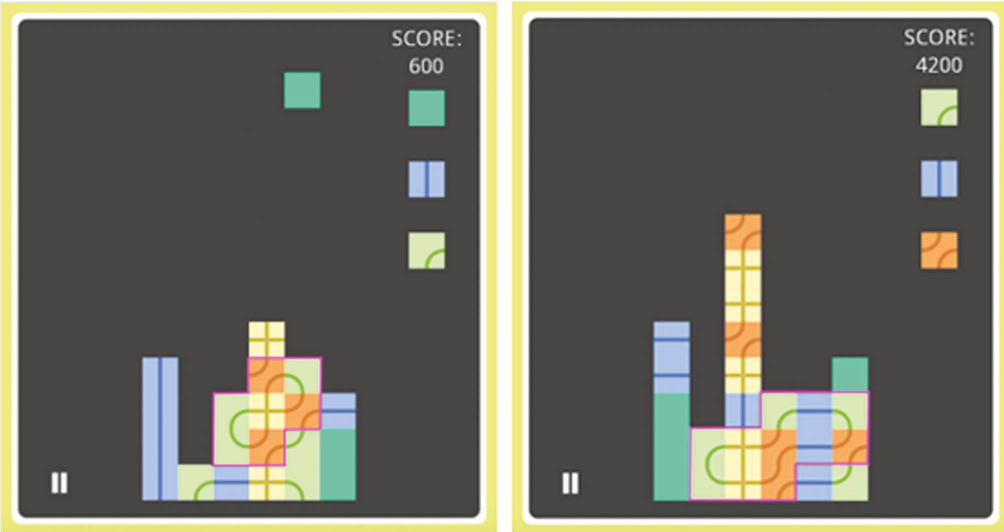


Figure 7. Examples creating multipliers for tiles.



A Knotris game ends when a tile touches the top of the game board.

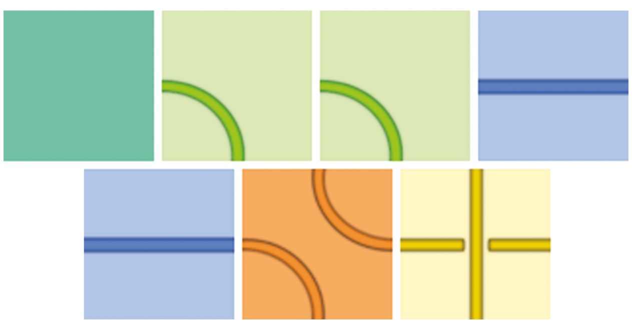
We only use 5 of the 11 basic mosaic tiles in Knotris because we allow rotation of tiles. These tiles do not come up randomly, with equal probability. We determined through our early play with physical wooden tiles that random tile generation would quickly lead to a very lopsided board, and games would end quickly.

Tiles that can be especially difficult to place are blanks (tiles that have no connection points) and 4-connected tiles (those that have connection points on all four sides). The tiles with two connection points, which we call *elbows* and *lines*, are more versatile. Owing to these characteristics, we determined that the collection of seven tiles shown in figure 6 should constitute our *basic game bag*.

This seven-tile game bag—from which the probability of choosing a blank is 1/7 and the probability of choosing each other tile type (i.e., elbows, lines, or 4-connected tiles) is equal to 2/7—forms the basic building block to construct Knotris.

We combined three basic game bags to create a larger, 21-tile game bag from which tiles are chosen. The larger game bag is replenished and reshuffled when it runs out during game play. Combining game bags allows interesting (and sometimes frustrating) things to happen, such as three blank tiles appearing in a row.

Figure 6. The seven tiles in a game bag for Knotris.



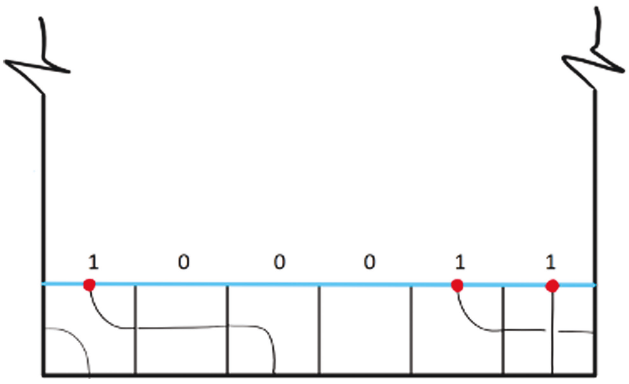
We intentionally chose our seven-tile game bag in relation to the six-tile width of the game board. If the basic game bag were the same size as the board width (or some multiple) and if it were possible to use the bag to create an internally suitably connected row, this row could just be repeated (right side up or upside down) to continue game play forever without much effort. Considerations like these went into decisions about both the basic game bag and the game board size.

So, how does scoring work? Each tile is initially assigned a multiplier of 1, which can be increased if the player arranges for a special configuration to occur on the game board. When a row clears, the player's score increases by 100 times the sum of the multipliers of the six tiles in the cleared row. Typically, clearing a row simply adds 600 to a player's score, but sometimes a player can do better.

One way that a player can increase the multipliers of the tiles is by creating a knot or a link, L, on the game board. Once L has been completed, each tile contributing to L has its multiplier increased by one plus the number of crossings in L.

Consider, for instance, the two examples in figure 7. On the left, seven tiles in the middle create a knot (in the shape of a figure-8) with one crossing. The multipliers of these seven tiles each increase from 1 to 3—we add 1 for completing the knot and an additional 1 for the crossing in the knot. On the right, 11 tiles are used to create a knot (which is just a circle, or the unknot). Because the knot completes a closed loop and uses two crossing tiles, we assign a multiplier of 4 to each of our 11 tiles to both reward the completion of the knot and add a crossing tile bonus.

Figure 8. A row with upper boundary condition (1,0,0,0,1,1).



A multiplier of a tile is also increased if the tile is involved in a row that initially fails to be suitably connected to the row above it but ends up being cleared later in the game, in a cascade of clearing rows. The idea of increasing multipliers in this situation motivated us to wonder if this phenomenon is even possible, especially given the constraints of our tile bag. It turns out that we can answer this question by investigating the tiles and our game bag more closely.

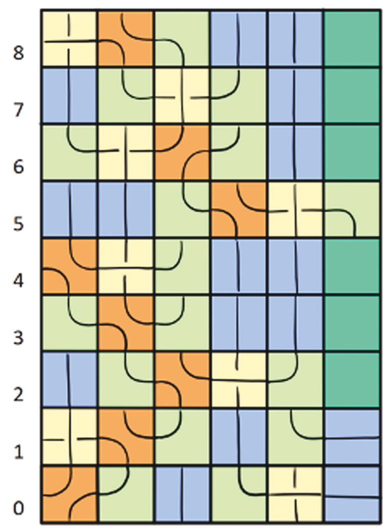
An Interesting Result

While developing Knotris, we've studied various probabilities that inform game design and strategy. One question we've asked is: given a game board configuration with a flat top, such as the starting configuration of the game, what is the probability that the next six tiles can form another complete, suitably connected row? This probability depends on the boundary condition dictated by the top row of the game board configuration.

In figure 8, the tiles that are placed in the first, fifth, and sixth spots should have connection points along their bottom edges. This *boundary condition* can be represented by the six-tuple (1,0,0,0,1,1). It turns out that one of the easiest boundary conditions to satisfy by any six of the seven tiles in our basic game bag is (1,1,1,1,1,0); many such rows are pictured in figure 9.

Our investigation of the boundary conditions helped us find an answer to our question about clearing multiple rows by placing a single tile. Indeed, we can clear multiple rows; moreover, we can clear arbitrarily many rows at once! Figure 9 demonstrates an attainable game board where rows 0–7 will not clear before row 8 is placed.

Figure 9. A proof by picture that we can, in theory, clear as many rows as we like at once.



While rows 0–7 are all internally suitably connected, they fail to be suitably connected along their upper and lower boundaries.

In figure 9, the upper boundary conditions for rows 0–7 are identical: they are all (1,1,1,1,1,0). Moreover, the 8th row has lower boundary condition (1,1,1,1,1,0), so once this row is placed, all eight rows below it will clear in a cascade! This configuration is also realizable (according to our bag constraints), and the strategy can be expanded to clear even more rows!

When multiple rows are cleared at once, a player can generate a much larger number of points than clearing rows one at a time—all the tiles in each row cleared during a cascade are assigned a multiplier of n, where n represents the total number of rows that have been cleared. In the example above, all 48 of the tiles in rows 0–7 get assigned a multiplier of 8. So, this is a highly desirable configuration.

Can *you* clear multiple rows at once? Try it by playing Knotris at *https://izook.github.io/knotris/*. Let us know what you think!

***Allison Henrich**, professor of mathematics at Seattle University, won a grant from the Center for Undergraduate Research in Mathematics (CURM) to work on this yearlong project with her dream team of Seattle University undergraduates: **Alex Ionescu, Brooke Mathews, Isaac Ortega, and Kelemua Tesfaye**. Each student brought their unique expertise (in game design, programming, probability, and more!) to make this project's vision a reality.*

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