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Abelian Arithmetic Chern–Simons Theory and Arithmetic Linking Numbers

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Following the method of Seifert surfaces in knot theory, we define arithmetic linking numbers and height pairings of ideals using arithmetic duality theorems, and compute them in terms of n-th power residue symbols. This formalism leads to a precise arithmetic analogue of a "path-integral formula" for linking numbers.

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1 Introduction

Let M be an oriented three-manifold without boundary and α_1 and α_2 two knots that are homologically equivalent to zero in it. One way of computing the linking number of α_1 and α_2 uses the formula

$$\ell k(\alpha_1, \alpha_2) = \langle \Sigma_{\alpha_1}, \alpha_2 \rangle,$$

where Σ_{α_1} is a Seifert surface for α_1 transversal to α_2 and $\langle \Sigma_{\alpha_1}, \alpha_2 \rangle$ is the oriented intersection number. It is also suggestive to write this equality as

$$\ell k(\alpha_1, \alpha_2) = \langle d^{-1}\alpha_1, \alpha_2 \rangle,$$

d denoting the exterior derivative of currents. The pairing on the right is independent of the choice of (smooth, transversal) inverse image: because de Rham cohomology computed by forms and currents is the same, the ambiguity can be represented by closed 1-forms, which then integrate to zero on α_2, since the latter is assumed to be homologically equivalent to zero.

We can also define a pairing between two 1-forms A_1 and A_2 by

$$(A_1, A_2) := \langle A_1, dA_2 \rangle := \int_M A_1 \wedge dA_2.$$

Since

$$d(A_1 \wedge A_2) = dA_1 \wedge A_2 - A_1 \wedge dA_2,$$

we see right away that the pairing is symmetric by Stokes' theorem.

According to [1], the *Chern–Simons action*

$$(A, A) = \int_M A \wedge dA$$

for a 1-form A is related to the *helicity* of a magnetic field. Indeed, if M is a space-like slice of the spacetime $M \times \mathbb{R}$ and A the electromagnetic potential, we have the equality

$$\int_M A \wedge dA = \int_M \Phi \cdot B \, d\mathrm{vol},$$

where B is the magnetic field and Φ the magnetic vector potential.

Here is an aside about the meaning of the integral $\int_M A \wedge dA$ as "helicity." The choice of a volume form $d$vol on M determines an isomorphism $V \mapsto i_V d\mathrm{vol}$ from vector fields to 2-forms. The vector field V corresponding to dA will generate a flow so that we can consider the trajectory ℓ_p that starts from any given point p. Arnold and Khesin [1] define an asymptotic linking number $\ell k(\ell_p, \ell_q)$ and prove a formula of the form

$$\int_M A \wedge dA = \int_M d^{-1}(i_V d\mathrm{vol}) \wedge i_V d\mathrm{vol} = c \int_{M\times M} \ell k(\ell_p, \ell_q)\, d\mathrm{vol}_p\, d\mathrm{vol}_q.$$

That is, the helicity is an average asymptotic linking number between pairs of magnetic flows starting from two points in M.

Following Polyakov [13] and Schwarz [14], they also discuss the formal "Gaussian" path integral

$$\int \exp\left(-\pi \langle A, dA\rangle\right) DA = \det(*d)^{-\frac{1}{2}},$$

where $* : \Omega^2_M \to \Omega^1_M$ is the Hodge star operator with respect to a metric and the determinant is regularised (in this and the next formula, we will be somewhat vague with the precise definitions and computations, since we will not be using them in this article except as inspiration. In particular, [14] gives a careful discussion of the metric dependence and the possibility that d has non-trivial kernel. Also, we have normalised the constants slightly differently.) [1, p. 186]. Adding a linear term pairing the forms with homologically trivial currents ξ_i, we get (again formally)

$$\int \exp\left(-\pi \langle A, dA\rangle + 2\pi i \sum_i \langle A, \xi_i\rangle\right) DA = \det(*d)^{-\frac{1}{2}} \cdot \exp\left(-\pi \sum_{i,j} \langle d^{-1}\xi_i, \xi_j\rangle\right).$$

This can be viewed as an infinite dimensional analogue of a standard Gaussian integral formula in finite dimensional Euclidean space [12] (our main result uses a finite field analog of this formula). The pairings between currents on the right side are likely to be problematic in general. However, the case of interest is when the ξ_i are (oriented) knots and the pairing with A denotes an integral. The operator d acts on currents in a way compatible with boundary maps of singular chains. That is, if L, N are chains with $\partial N = L$ and $[L]$ and $[N]$ are the corresponding currents, then $d[N] = [\partial N] = [L]$. Hence, if ξ_i is a current corresponding to a homologically trivial knot, then $d^{-1}\xi_i$ will include a two-chain with boundary equal to ξ_i. Thus, each term $\langle d^{-1}\xi_i, \xi_j\rangle = \ell k(\xi_i, \xi_j)$ will be a linking number. The integral is thereby viewed as a correlation between the "Wilson loop functionals"

$$A \mapsto \exp(2\pi i \langle A, \xi_i \rangle),$$

associated to knots ξ_i with respect to a Chern–Simons measure

$$\exp(-\pi(A, A))\, DA.$$

In any case, the Gaussian integral with linear term provides one elementary explanation of how linking numbers come up in Chern–Simons theory.

The goal of this article is to present some preliminary investigations on arithmetic analogues of the preceding discussion. That is, when $X = \mathrm{Spec}(\mathcal{O}_F)$ for a totally imaginary number field F that contains the group μ_{n^2} of n^2-th roots of unity, we use arithmetic duality theorems to define a two term complex

$$d : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$$

as a mod n arithmetic analogue of the map $d : \Omega^1_M \to \Omega^2_M$. The Ext group is isomorphic to $Cl(F)/n$, the ideal class group of F mod n. Thus, every ideal I has a mod n class

$$[I]_n \in \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m),$$

and we define I to be *n-homologically trivial* if this class is in the image of d. On the other hand, there is a duality pairing

$$\langle \cdot, \cdot \rangle : H^1(X, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

and we define *the arithmetic linking number* of two prime ideals $\mathcal{P}$ and $\mathcal{Q}$ that are n-homologically trivial by

$$\ell k_n(\mathcal{P}, \mathcal{Q}) := \langle d^{-1}\mathcal{P}, \mathcal{Q} \rangle.$$

Of course one needs to check that this is well-defined and symmetric. We verify this in Section 2. In Section 3, we generalise the definition to arithmetic linking numbers on $X_S := \mathrm{Spec}(\mathcal{O}_F[1/S])$ for a finite set of primes S. We will see (Corollary 3.11) that this linking number can be computed in terms of n-th power residue symbols in a manner similar to Morishita's treatment in [10] (However, we do not carry out a direct comparison). This pairing can be defined also for non-prime ideals, in which case we call it the *arithmetic mod n height pairing*, denoted by $ht_n(I, J)$.

Parallel to the pairing on 1-forms, we also define a pairing

$$(\cdot,\cdot) : H^1(X,\mathbb{Z}/n\mathbb{Z}) \times H^1(X,\mathbb{Z}/n\mathbb{Z}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

as

$$(A,B) = \langle A, dB\rangle$$

and in such a way that (A,A) is the abelian arithmetic Chern–Simons function defined in [4, 6].

It is then pleasant to note a precise analogue of the Gaussian path integral in this arithmetic setting.

Theorem 1.1. Let p be an odd prime, $a = \dim H^1(X,\mathbb{Z}/p\mathbb{Z})$, $b = \dim \mathrm{Ker}(d)$, and $\{\xi_j\}$ a finite set of p-homologically trivial ideals. Denote by $\bar{d}$ the induced isomorphism

$$\bar{d} : H^1(X,\mathbb{Z}/p\mathbb{Z})/\,\mathrm{Ker}(d) \xrightarrow{\sim} \mathrm{Im}(d).$$

Then

$$\sum_{\rho\in H^1(X,\mathbb{Z}/p\mathbb{Z})} \exp\left[2\pi i((\rho,\rho) + \sum_j \langle\rho,\, [\xi_j]_p\rangle)\right]$$
$$= p^{(a+b)/2}\left(\frac{\det(\bar{d})}{p}\right) i^{[\frac{(a-b)(p-1)^2}{4}]} \exp\left[-2\pi i\left(\frac{1}{4}\sum_{i,j} ht_p(\xi_i,\xi_j)\right)\right]. \qquad \square$$

The determinant requires some commentary. The map $\bar{d}$ goes from $H^1(X,\mathbb{Z}/p\mathbb{Z})/\,\mathrm{Ker}(d)$ to its dual, since $\mathrm{Ker}(d)$ is the exact annihilator of $\mathrm{Im}(d)$. It is an easy exercise to check that the determinant is then well-defined modulo squares in $\mathbb{Z}/p\mathbb{Z}$ (it is just the discriminant of the corresponding quadratic form). Hence, its Legendre symbol is well-defined. This formula is essentially a formal consequence of the definitions. However, it does give indication that some notion of "quantisation" for arithmetic Chern–Simons theory might not be entirely empty, and furthermore, provide new interpretations of basic arithmetic invariants.

In Section 4, following up on the ideas of [2], we will also show how to realize the arithmetic linking pairing in the compact case by a simple construction that only involves Artin reciprocity and the "class invariant homomorphism," which gives a measure of the Galois structure of unramified Galois extensions. More precisely, we show that under the class field theory isomorphism $(Cl(F)/n)^\vee \simeq H^1(X, \mathbb{Z}/n\mathbb{Z})$ the map

$$d : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \simeq H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee$$

giving $(\cdot, \cdot)$ is identified with the class invariant homomorphism

$$(Cl(F)/n)^\vee = \mathrm{Hom}(Cl(F), \mathbb{Z}/n\mathbb{Z}) \to Cl(F)/n.$$

By definition, this sends the Artin map of a $\mathbb{Z}/n\mathbb{Z}$-unramified extension K/F to the class of the (locally free) $\mathcal{O}_F$-submodule of K consisting of $v \in K$ such that $a(v) = \zeta^a v$. Regarding Chern–Simons functionals, the first computation in terms of the Artin map was in Ref. [2]. Martin Taylor observed a relation to the class invariant homomorphism when $n = 2$, while Romyar Sharifi pointed out a connection to Bockstein maps.

As mentioned already, many of the ideas of the current article were explored in various forms and in considerable depth by Ref. [10]. What we view as the main contribution here, as in Ref. [4, 6], is an attempt to move beyond analogies to a precise correspondence of constructions and techniques used in topology (especially the ideas inspired by topological quantum field theory), and in arithmetic geometry. What is achieved is obviously modest. But we hope it is suggestive.

2 Arithmetic Linking Numbers in the Compact Case: Proof of Theorem 1.1

Let F be a totally imaginary algebraic number field with ring of integers $\mathcal{O}_F$ such that $\mu_{n^2} \subset F$, and let $X = \mathrm{Spec}(\mathcal{O}_F)$. We fix a trivialisation of the n-th roots of unity

$$\zeta : \mathbb{Z}/n\mathbb{Z} \simeq \mu_n.$$

We have various isomorphisms

$$\zeta_* : H^i(X, \mathbb{Z}/n\mathbb{Z}) \simeq H^i(X, \mu_n);$$
$$\zeta^* : \mathrm{Ext}^i_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \simeq \mathrm{Ext}^i_X(\mu_n, \mathbb{G}_m).$$

Let $\pi := \pi_1(X, b)$, where $b : \mathrm{Spec}(\bar{F}) \to \mathrm{Spec}(\mathcal{O}_F)$ is the geometric point coming from an algebraic closure $\bar{F}$ of F. For any natural number n, we have the isomorphism

$$\mathrm{Inv} : H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

and a perfect pairing [7]

$$\langle \cdot, \cdot \rangle : H^i(X, \mathcal{F}) \times \mathrm{Ext}^{3-i}_X(\mathcal{F}, \mathbb{G}_m) \to H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

for any n-torsion sheaf $\mathcal{F}$ in the étale topology (the pairing usually goes to $H^3(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$. But the statement that it is perfect means it induces an isomorphism

$$\mathrm{Ext}_X^{3-i}(\mathcal{F}, \mathbb{G}_m) \simeq \mathrm{Hom}(H^i(X, \mathcal{F}), H^3(X, \mathbb{G}_m)).$$

But $H^i(X, \mathcal{F})$ is n-torsion, which means that the image of any homomorphism lies in the n-torsion subgroup $H^3(X, \mathbb{G}_m)[n] \simeq H^3(X, \mu_n)$).

The cup product

$$\cup : H^1(X, \mathbb{Z}/n\mathbb{Z}) \times H^2(X, \mu_n) \to H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

induces a map

$$r : H^2(X, \mu_n) \to \mathrm{Ext}_X^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m),$$

such that

$$\mathrm{Inv}(a \cup b) = \langle a, r(b) \rangle.$$

The Bockstein operator

$$\delta : H^1(X, \mu_n) \to H^2(X, \mu_n),$$

comes from the exact sequences of sheaves

$$0 \to \mu_n \to \mu_{n^2} \to \mu_n \to 0.$$

Define the coboundary map d as the composition

$$d : H^1(X, \mathbb{Z}/n\mathbb{Z}) \overset{\zeta_*}{\simeq} H^1(X, \mu_n) \xrightarrow{\delta} H^2(X, \mu_n) \xrightarrow{r} \mathrm{Ext}_X^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m).$$

We view the two-term complex

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{d} \mathrm{Ext}_X^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m),$$

as a mod n arithmetic analogue of the complex

$$\Omega_M^1 \to \Omega_M^2$$

for three-manifolds. The idea that cohomology equipped with the Bockstein operation can have the nature of differential forms occurs in the theory of the de Rham-Witt complex for a variety in characteristic p: there, the de Rham-Witt differentials are sheaves of crystalline cohomology [5]. Also, recall that the curvature of a connection is the obstruction to deforming a bundle along a deformation of the space on which it lives. The Bockstein operator is a small piece of the obstruction to deforming it along a deformation of the coefficients.

There is also a Bockstein operator

$$\delta' : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z}),$$

associated with the exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

and a Bockstein in degree 2,

$$\delta_2 : H^2(X, \mu_n) \to H^3(X, \mu_n).$$

By choosing an isomorphism $\mathbb{Z}/n^2 \simeq \mu_{n^2}$ compatible with ζ, we see an equality of maps

$$\zeta_* \circ \delta' = \delta \circ \zeta_* : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mu_n).$$

The following fact is of course well-known, but it seems to be hard to find a reference for étale cohomology.

Lemma 2.1. The Bockstein operator δ_2 satisfies

$$\delta_2(\alpha \cup \beta) = \delta'\alpha \cup \beta - \alpha \cup \delta\beta$$

for all $\alpha \in H^1(X, \mathbb{Z}/n\mathbb{Z})$ and $\beta \in H^1(X, \mu_n)$. □

Proof. Since X is affine, the étale cohomology groups are isomorphic to the Čech cohomology groups (cf. [9, Theorem 10.2]). Thus, we can check the above formula using Čech cocycles (cf. [9, Section 22]).

Choose a sufficiently fine étale covering $(U_i)_{i\in I}$ of X. Define $U_{ij} = U_i \times_X U_j$, $U_{ijk} = U_i \times_X U_j \times_X U_k$ and so on. Typical elements of the index set I are denoted by i, j, k, and l. Represent α and β as Čech cocycles (α_{ij}) and (β_{ij}). For any pair (i, j) of distinct elements in I, choose a lifting $\tilde{\alpha}_{ij}$ of α_{ij} to $\mathbb{Z}/n^2\mathbb{Z}$, and similarly a lifting $\tilde{\beta}_{ij}$ of β_{ij} to μ_{n^2}. The class of $\delta'\alpha$ can be represented by the 2-cocycle whose section over U_{ijk} is

$(\delta'\alpha)_{ijk} := \tilde{\alpha}_{ij}|_{U_{ijk}} + \tilde{\alpha}_{jk}|_{U_{ijk}} - \tilde{\alpha}_{ik}|_{U_{ijk}}$ which takes values in $\mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}/n^2\mathbb{Z}$. We represent $\delta\beta$ in a similar way.

The cup product $\delta'\alpha \cup \beta$ is represented by a family of sections $\gamma'_{ijkl} = (\delta'\alpha)_{ijk}|_{U_{ijkl}} \otimes \beta_{kl}|_{U_{ijkl}}$ and similarly $\alpha \cup \delta\beta$ is represented by $\gamma_{ijkl} = \alpha_{ij}|_{U_{ijkl}} \otimes (\delta\beta)_{jkl}|_{U_{ijkl}}$. On the other hand, we have

$$(\alpha \cup \beta)_{ijk} = \alpha_{ij}|_{U_{ijk}} \otimes \beta_{jk}|_{U_{ijk}}$$

which lifts to $\tilde{\alpha}_{ij}|_{U_{ijk}} \otimes \tilde{\beta}_{jk}|_{U_{ijk}}$ with values in $\mathbb{Z}/n^2\mathbb{Z} \otimes \mu_{n^2} \simeq \mu_{n^2}$. A μ_n-valued cocycle representing $\delta_2(\alpha \cup \beta)$ takes the form

$$(\delta_2(\alpha \cup \beta))_{ijkl} := \left(\tilde{\alpha}_{jk} \cdot \tilde{\beta}_{kl}|_{U_{jkl}}\right)|_{U_{ijkl}} - \left(\tilde{\alpha}_{ik} \cdot \tilde{\beta}_{kl}|_{U_{ikl}}\right)|_{U_{ijkl}} + \left(\tilde{\alpha}_{ij} \cdot \tilde{\beta}_{jl}|_{U_{ijl}}\right)|_{U_{ijkl}} - \left(\tilde{\alpha}_{ij} \cdot \tilde{\beta}_{jk}|_{U_{ijk}}\right)|_{U_{ijkl}}$$

where the isomorphism $\mathbb{Z}/n^2\mathbb{Z} \otimes \mu_{n^2} \simeq \mu_{n^2}$ sends $a \otimes b \mapsto a \cdot b$ by viewing μ_{n^2} an additive group. Since α_{ij} and β_{ij} are cocycles, $\tilde{\alpha}_{jk}|_{U_ijk} - \tilde{\alpha}_{ik}|_{U_ijk} = -\tilde{\alpha}_{ij}|_{U_ijk} + n\phi_{ijk}$ for some ϕ_{ijk} and similarly $\tilde{\beta}_{jl}|_{U_jkl} - \tilde{\beta}_{jk}|_{U_jkl} = \tilde{\beta}_{kl}|_{U_jkl} - n\psi_{jkl}$ for some ψ_{jkl}. Using these, the above simplifies to

$$\begin{aligned}(\delta_2(\alpha \cup \beta))_{ijkl} &= \left((-\tilde{\alpha}_{ij} + n\phi_{ijk}) \cdot \tilde{\beta}_{kl}\right)|_{U_{ijkl}} + \left(\tilde{\alpha}_{ij} \cdot (\tilde{\beta}_{kl} - n\psi_{jkl})\right)|_{U_{ijkl}} \\ &= \left(n\phi_{ijk} \cdot \tilde{\beta}_{kl}\right)|_{U_{ijkl}} + \left(\tilde{\alpha}_{ij} \cdot (-n\psi_{jkl})\right)|_{U_{ijkl}}\end{aligned}$$

which is equal to $\gamma'_{ijkl} - \gamma_{ijkl}$ via the isomorphisms $\mathbb{Z}/n\mathbb{Z} \simeq n\mathbb{Z}/n^2\mathbb{Z}$ sending $a \mapsto na$, and $\mu_n \simeq \mu_{n^2}/\mu_n$ sending $\xi \mapsto \xi^{1/n}$. Hence we have shown the desired property of δ_2. ■

Define the pairings

$$(\cdot,\cdot) : H^1(X, \mathbb{Z}/n\mathbb{Z}) \times H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z};$$

$$(\alpha, \beta) := \langle \alpha,\, d\beta \rangle \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Lemma 2.2. The pairing is symmetric:

$$(\alpha, \beta) = (\beta, \alpha)$$

for all $\alpha, \beta \in H^1(X, \mathbb{Z}/n\mathbb{Z})$. □

Proof. This follows from examining the second Bockstein operator above.

$$\delta_2 : H^2(X, \mu_n) \to H^3(X, \mu_n).$$

For the pro-sheaf $\mathbb{Z}_n(1) := \varprojlim_i \mu_{n^i}$, we have an exact sequence

$$0 \to \mathbb{Z}_n(1) \xrightarrow{n} \mathbb{Z}_n(1) \to \mu_n \to 0.$$

Because $H^3(X, \mathbb{Z}_n(1)) \simeq \mathbb{Z}_n$ is torsion-free, the boundary map $H^2(X, \mu_n) \to H^3(X, \mathbb{Z}_n(1))$ is zero, and the map $H^2(X, \mathbb{Z}_n(1)) \to H^2(X, \mu_n)$ is surjective. Hence, $H^2(X, \mu_{n^2}) \to H^2(X, \mu_n)$ is surjective, so that the map δ_2 is zero.

As a consequence, we have

$$0 = \delta_2(\alpha \cup \beta) = \delta'\alpha \cup \beta - \alpha \cup \delta\beta.$$

Therefore,

$$\begin{aligned}(a, b) &= \langle a,\, db\rangle = \mathrm{Inv}(a \cup \delta\zeta_* b) = \mathrm{Inv}(\delta' a \cup \zeta_* b) = \mathrm{Inv}(\zeta_*(\delta' a \cup b)) \\ &= \mathrm{Inv}(\zeta_*(b \cup \delta' a)) = \mathrm{Inv}(b \cup \zeta_*\delta' a)) = \mathrm{Inv}(b \cup \delta\zeta_* a)) = \langle b,\, da\rangle = (b, a).\end{aligned}$$ ■

Define $K = \mathrm{Ker}(d)$.

Corollary 2.3. If $a \in K$, then $(a, b) = 0$ for all b. □

Proof. If $a \in K$, then $(a, b) = (b, a) = \langle b,\, da\rangle = 0$. ■

According to duality, we have $\mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \simeq H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee \simeq Cl(X)/n$, where $Cl(X)$ is the ideal class group of $X = \mathrm{Spec}(\mathcal{O}_F)$. We will say an ideal $I \subset \mathcal{O}_F$ is *n-homologically trivial* if its class in $\mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$ is in the image of d. Even though there is some danger of confusion, when n is fixed for the discussion, we will also allow ourselves merely to say that I is "homologically trivial." If I and J are homologically trivial ideals, we define *the mod n height pairing* between I and J by

$$ht_n(I, J) = \langle d^{-1}[I]_n, [J]_n\rangle,$$

where $[I]_n$ denotes the class of I in $Cl(X)/n$. Writing $[J]_n = d(b)$ for some $b \in H^1(X, \mathbb{Z}/n\mathbb{Z})$, for any a such that $da = 0$, we have $\langle a,\, db\rangle = (a, b) = 0$ by Corollary 2.3. This implies that the mod n height pairing is well-defined. Using the pairing on $H^1(X, \mathbb{Z}/n\mathbb{Z})$, note that we can also write the height pairing as

$$(d^{-1}[I]_n, d^{-1}[J]_n),$$

rendering the symmetry evident. For two prime ideals $\mathcal{P}$ and $\mathcal{Q}$ (which are homologically trivial), we will also call their height pairing *their linking number*, and denote it

$$\ell k_n(\mathcal{P}, \mathcal{Q}) := ht_n(\mathcal{P}, \mathcal{Q}) = \langle d^{-1}[\mathcal{P}]_n, [\mathcal{Q}]_n \rangle.$$

In the articles [4, 6], we fixed a class $c \in H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and defined the *arithmetic Chern–Simons action* for homomorphisms

$$\rho : \pi = \pi_1(X, b) \to \mathbb{Z}/n\mathbb{Z}$$

as

$$CS_c(\rho) := \mathrm{Inv}(\zeta_*(j^3(\rho^*(c)))) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

where $j^i : H^i(\pi, \mathbb{Z}/n\mathbb{Z}) \to H^i(X, \mathbb{Z}/n\mathbb{Z})$ is the natural map from group cohomology to étale cohomology (cf. [8, Theorem 5.3 of Chapter I]). We can also define the *arithmetic Chern–Simons partition function* as

$$Z_c(X) := \sum_{\rho \in \mathrm{Hom}(\pi, \mathbb{Z}/n\mathbb{Z})} \exp(2\pi i \cdot CS_c(\rho)).$$

The class $c := Id \cup \tilde{\delta}(Id)$ is a generator of $H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, where Id is the identity from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ regarded as an element of $H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\tilde{\delta} : H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is a Bockstein operator induced from the exact sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n^2\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

There is a natural bijection between $\mathrm{Hom}(\pi, \mathbb{Z}/n\mathbb{Z})$ and $H^1(X, \mathbb{Z}/n\mathbb{Z})$ (defined by j^1) and we will simply identify the two. One then checks immediately that for the cocycle $c = Id \cup \tilde{\delta}(Id)$ we have

$$CS_c(\rho) = (\rho, \rho).$$

Thus for the partition function, we have

$$Z_c(X) = \sum_{\rho \in \mathrm{Hom}(\pi, \mathbb{Z}/n\mathbb{Z})} \exp(2\pi i \cdot CS_c(\rho)) = \sum_{\rho \in H^1(X, \mathbb{Z}/n\mathbb{Z})} \exp(2\pi i \cdot (\rho, \rho)).$$

Proof of Theorem 1.1. By Corollary 2.3 and the definition $(\rho, \rho) = \langle \rho, d\rho \rangle$, both (ρ, ρ) and $\langle \rho, [\xi_j]_p \rangle$ depend only on the class of ρ in $H^1(X, \mathbb{Z}/p\mathbb{Z})/K$, which we denote by $\bar{\rho}$. So we can write the sum as

$$p^b \sum_{\bar{\rho} \in H^1(X, \mathbb{Z}/p\mathbb{Z})/K} \exp[2\pi i((\bar{\rho}, \bar{\rho}) + \sum_j \langle \bar{\rho}, [\xi_j]_p \rangle)].$$

After a choice of basis for $H^1(X, \mathbb{Z}/p\mathbb{Z})/K$ and $\mathrm{Im}(d)$, this becomes a Gaussian integral over a finite field. Now the formula follows from [12, Proposition 3.2 of Chapter 9]. ∎

3 Boundaries

In this section, we fix a natural number n and a finite set S of places of F containing all the places that divide n and the Archimedean places. As before, we assume $\mu_{n^2} \subset F$. Put $U = \mathrm{Spec}(\mathcal{O}_{F,S})$, the spectrum of the ring of S-integers in F. Let $\pi_U := \pi_1(U)$ and $\pi_v := \pi_1(\mathrm{Spec}(F_v))$ for each place v of F. Denote by $C^*(U, \mathcal{G})$ the complex of continuous cochains of π_U with coefficients in a locally constant torsion $\mathbb{Z}_n = \varprojlim_i \mathbb{Z}/n^i\mathbb{Z}$-sheaf $\mathcal{G}$ on U and by $C^*(F_v, \mathcal{F})$, the complex of continuous cochains of π_v with coefficients in a sheaf $\mathcal{F}$ on $\mathrm{Spec}(F_v)$. As in [4, Section 2], we will use the "inclusion of the boundary" map

$$i_S = \prod_{v \in S} i_v : \partial U = \coprod_{v \in S} \mathrm{Spec}(F_v) \to U.$$

Let $\mathcal{G}$ be a sheaf on U, $\mathcal{F}$ a sheaf on ∂U, and $f : \mathcal{F} \to i_S^* \mathcal{G}$ a map of sheaves. In view of the applications in mind, we will refer to such a map as a *boundary pair*. Denote by $C^*(U, \mathcal{G} \times_S \mathcal{F})$, the two product of complexes defined by the following diagram:

$$\begin{array}{ccc}
C^*(U, \mathcal{G} \times_S \mathcal{F}) & \longrightarrow & \prod_{v \in S} C^*(F_v, \mathcal{F}) \\
\downarrow & \boxed{2} & \downarrow f_* \\
C^*(U, \mathcal{G}) & \xrightarrow[loc_S]{} & \prod_{v \in S} C^*(F_v, i_v^* \mathcal{G}),
\end{array}$$

where loc_S refers to the localisation map on cochains. Thus,

$$C^i(U, \mathcal{G} \times_S \mathcal{F}) = C^i(U, \mathcal{G}) \times \prod_{v \in S} C^i(F_v, \mathcal{F}) \times \prod_{v \in S} C^{i-1}(F_v, i_v^* \mathcal{G}),$$

and its elements will be denoted by (c, b_S, a_S), where $c \in C^i(U, \mathcal{G})$, $b_S = (b_v)_{v\in S} \in \prod_{v\in S} C^i(F_v, \mathcal{F})$, and $a_S = (a_v)_{v\in S} \in \prod_{v\in S} C^{i-1}(F_v, i_v^*\mathcal{G})$. The differential is defined by

$$d(c, b_S, a_S) = (dc, db_S, da_S + (-1)^i(f_*(b_S) - loc_S(c)).$$

Hence, a cocycle in $Z^i(U, \mathcal{G} \times_S \mathcal{F})$ consists of (c, b_S, a_S) such that $dc = 0, db_S = 0$, and

$$da_S = (-1)^i(loc_S(c) - f_*(b_S)).$$

Define

$$H^i(U, \mathcal{G} \times_S \mathcal{F}) := H^i(C^*(U, \mathcal{G} \times_S \mathcal{F})).$$

Here are some general properties that follow immediately from the definitions.

(1) When $\mathcal{F} = 0$, then $H^i(U, \mathcal{G} \times_S 0) = H^i_c(U, \mathcal{G})$, the compact support cohomology of $\mathcal{G}$.

(2) Given maps $\mathcal{F} \to \mathcal{F}'$, $\mathcal{G} \to \mathcal{G}'$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \downarrow & & \downarrow \\ i_S^*\mathcal{G} & \longrightarrow & i_S^*\mathcal{G}' \end{array}$$

we have an induced map of complexes

$$C^*(U, \mathcal{G} \times_S \mathcal{F}) \to C^*(U, \mathcal{G}' \times_S \mathcal{F}'),$$

and hence, a map of cohomologies

$$H^i(U, \mathcal{G} \times_S \mathcal{F}) \to H^i(U, \mathcal{G}' \times_S \mathcal{F}').$$

More precisely, the formation of the complex and the cohomology is functorial in the diagrams in an obvious sense.

(3) Suppose you have two exact sequences

$$0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$$

$$0 \to \mathcal{G}'' \to \mathcal{G} \to \mathcal{G}' \to 0$$

and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_S^*\mathcal{G}'' & \longrightarrow & i_S^*\mathcal{G} & \longrightarrow & i_S^*\mathcal{G}' & \longrightarrow & 0. \end{array}$$

Then you get an exact sequence of complexes

$$0 \to C^*(U, \mathcal{G}'' \times_S \mathcal{F}'') \to C^*(U, \mathcal{G} \times_S \mathcal{F}) \to C^*(U, \mathcal{G}' \times_S \mathcal{F}') \to 0,$$

and hence, a long exact sequence at the level of cohomology.

(4) Cup product is given by

$$C^i(U, \mathcal{G} \times_S \mathcal{F}) \times C^j(U, \mathcal{G}' \times_S \mathcal{F}') \to C^{i+j}(U, (\mathcal{G} \otimes \mathcal{G}') \times_S (\mathcal{F} \otimes \mathcal{F}'))$$

$$(c, b_S, a_S) \cup (c', b'_S, a'_S) = (c \cup c', b_S \cup b'_S, (-1)^j a_S \cup f_*(b'_S) + loc_S(c) \cup a'_S).$$

Another possibility for the cup product, temporarily denoted by $\cup'$, is

$$(c, b_S, a_S) \cup' (c', b'_S, a'_S) = (c \cup c', b_S \cup b'_S, (-1)^j a_S \cup loc_S(c') + f_*(b_S) \cup a'_S).$$

The difference is

$$\Delta = (0, 0, (-1)^j a_S \cup (f_*(b'_S) - loc_S(c')) + (loc_S(c) - f_*(b_S)) \cup a'_S).$$

It will be useful to note that

Lemma 3.1. When the two cochains are cocycles, the difference above is exact. □

Proof. The cocycle condition says

$$da_S = (-1)^i (loc_S(c) - f_*(b_S)); \qquad da'_S = (-1)^j (loc_S(c') - f_*(b'_S)).$$

Hence,

$$\begin{aligned} d(a_S \cup a'_S) &= (-1)^i (loc_S(c) - f_*(b_S)) \cup a'_S + (-1)^{i+j-1} a_S \cup (loc_S(c') - f_*(b'_S)) \\ &= (-1)^i (loc_S(c) - f_*(b_S)) \cup a'_S + (-1)^{i+j} a_S \cup (f_*(b'_S) - loc_S(c')) \\ &= (-1)^i \left((loc_S(c) - f_*(b_S)) \cup a'_S + (-1)^j a_S \cup (f_*(b'_S) - loc_S(c')) \right). \end{aligned}$$

Hence,

$$d(0, 0, (-1)^i a_S \cup a'_S) = \Delta. \qquad \blacksquare$$

The differential is compatible with the cup product:

Lemma 3.2. If $(c, b_S, a_S) \in C^i$ and $(c', b'_S, a'_S) \in C^j$, then

$$d[(c, b_S, a_S) \cup (c', b'_S, a'_S)] = [d(c, b_S, a_S)] \cup (c', b'_S, a'_S) + (-1)^i (c, b_S, a_S) \cup [d(c', b'_S, a'_S)]. \qquad \square$$

Proof. We have

$$\begin{aligned} & d[(c, b_S, a_S) \cup (c', b'_S, a'_S)] = d(c \cup c', b_S \cup b'_S, (-1)^j a_S \cup f_*(b'_S) + loc_S(c) \cup a'_S) \\ &= (dc \cup c' + (-1)^i c \cup dc', db_S \cup b'_S + (-1)^i b_S \cup db'_S, (-1)^j da_S \cup f_*(b'_S) + (-1)^{i+j-1} a_S \cup f_*(db'_S) \\ &\quad + loc_S(dc) \cup a'_S + (-1)^i loc_S(c) \cup da'_S + (-1)^{i+j} (f_*(b_S \cup b'_S) - loc_S(c \cup c')), \end{aligned}$$

where the last component is the only thing we need to focus on. On the other hand, we have

$$\begin{aligned} & d(c, b_S, a_S) \cup (c', b'_S, a'_S) = (dc, db_S, da_S + (-1)^i (f_*(b_S) - loc_S(c)) \cup (c', b'_S, a'_S) \\ &= (dc \cup c', db_S \cup b'_S, (-1)^j (da_S + (-1)^i (f_*(b_S) - loc_S(c))) \cup f_*(b'_S) + loc_S(dc) \cup a'_S). \end{aligned}$$

Also,

$$\begin{aligned} & (c, b_S, a_S) \cup d(c', b'_S, a'_S) = (c, b_S, a_S) \cup (dc', db'_S, da'_S + (-1)^j (f_*(b'_S) - loc_S(c'))) \\ &= (c \cup dc', b_S \cup db'_S, (-1)^{j+1} a_S \cup f_*(db'_S) + loc_S(c) \cup [da'_S + (-1)^j (f_*(b'_S) - loc_S(c'))]). \end{aligned}$$

So the third component of

$$d(c, b_S, a_S) \cup (c', b'_S, a'_S) + (-1)^i (c, b_S, a_S) \cup d(c', b'_S, a'_S)$$

is

$$\begin{aligned} & (-1)^j (da_S + (-1)^i (f_*(b_S) - loc_S(c))) \cup f_*(b'_S) + loc_S(dc) \cup a'_S \\ &\quad + (-1)^i ((-1)^{j+1} a_S \cup f_*(db'_S) + loc_S(c) \cup [da'_S + (-1)^j (f_*(b'_S) - loc_S(c'))])) \end{aligned}$$

$$= (-1)^j da_S \cup f_*(b'_S) + (-1)^{i+j}(f_*(b_S \cup b'_S) + (-1)^{i+j-1} loc_S(c) \cup f_*(b'_S) + loc_S(dc) \cup a'_S$$
$$+ (-1)^{i+j-1} a_S \cup f_*(db'_S) + (-1)^i loc_S(c) \cup da'_S$$
$$+ (-1)^{i+j} loc_S(c) \cup f_*(b'_S) + (-1)^{i+j-1} loc_S(c \cup c').$$
$$= (-1)^j da_S \cup f_*(b'_S) + (-1)^{i+j}(f_*(b_S \cup b'_S) + loc_S(dc) \cup a'_S$$
$$+ (-1)^{i+j-1} a_S \cup f_*(db'_S) + (-1)^i loc_S(c) \cup da'_S + (-1)^{i+j-1} loc_S(c \cup c').$$

This is easily seen to be the third component of $d[(c, b_S, a_S) \cup (c', b'_S, a'_S)]$ above. ■

Corollary 3.3. The cup product of cocycles is a cocycle. □

Corollary 3.4. The cup product of cocycles induces a graded product map on cohomologies. □

Proof. Of course this is because if β is a cocycle, then $d(\alpha \cup \beta) = \pm(d\alpha) \cup \beta$ is a coboundary. ■

The main case of interest is when $\mathcal{F} = \mu_{n^2}$, $\mathcal{G} = \mathbb{Z}/n\mathbb{Z}$ and $f : \mu_{n^2} \to \mathbb{Z}/n\mathbb{Z}$ is the natural reduction followed by the trivialisation $\zeta^{-1} : \mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. From the exact sequence of pairs

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mu_{n^2} & \xrightarrow{Id} & \mu_{n^2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\zeta_*^{-1}} & \mu_{n^2} & \xrightarrow{f} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

and (1), (3) above, we get natural boundary maps

$$d : H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \to H^2_c(U, \mathbb{Z}/n\mathbb{Z})$$

and

$$d_2 : H^2(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \to H^3_c(U, \mathbb{Z}/n\mathbb{Z}).$$

Proposition 3.5. The map d_2 is zero. □

Proof. We will show that the previous map

$$H^2(U, \mu_{n^2} \times_S \mu_{n^2}) \to H^2(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2})$$

in the long exact sequence of cohomology is surjective. First we note that the map

$$H^2(U, \mu_{n^2}) \to H^2(U, \mathbb{Z}/n\mathbb{Z})$$

is surjective. To see this, use the map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(U, \mathbb{G}_m)[n^2] & \longrightarrow & \prod_{v\in S} \frac{1}{n^2}\mathbb{Z}/\mathbb{Z} & \xrightarrow{\Sigma} & \prod_{v\in S} \frac{1}{n^2}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & H^2(U, \mathbb{G}_m)[n^2] & \longrightarrow & \prod_{v\in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \xrightarrow{\Sigma} & \prod_{v\in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

from class field theory. The sum map is surjective from the kernel of the middle vertical map to the kernel of the right vertical map. The middle vertical map is trivially surjective. Hence, the left vertical map is surjective by the snake lemma. On the other hand, we also have the map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(U, \mathbb{G}_m)/n^2H^1(U, \mathbb{G}_m) & \longrightarrow & H^2(U, \mu_{n^2}) & \longrightarrow & H^2(U, \mathbb{G}_m)[n^2] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & H^1(U, \mathbb{G}_m)/nH^1(U, \mathbb{G}_m) & \longrightarrow & H^2(U, \mu_n) & \longrightarrow & H^2(U, \mathbb{G}_m)[n] & \longrightarrow & 0, \end{array}$$

where the left vertical map is the natural projection. Since the vertical maps on the left and right are surjective, so is the one in the middle.

Now let $(c, b_S, a_S) \in Z^2(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2})$. Choose $c' \in Z^2(U, \mu_{n^2})$ lifting c and $a'_S \in \prod_{v\in S} C^1(F_v, \mu_{n^2})$ lifting a_S under the map $f : \mu_{n^2} \to \mathbb{Z}/n\mathbb{Z}$. Then

$$da'_S = b_S - loc_S(c') + (b'_S)^n$$

for some $b'_S \in C^2(U, \mu_{n^2})$. However, $(b'_S)^n$ is a cocycle, since this is true of all other terms in the equality. Hence,

$$(c', b_S + (b'_S)^n, a'_S) \in Z^2(U, \mu_{n^2} \times_S \mu_{n^2})$$

is a lift of (c, b_S, a_S). ■

Lemma 3.6. For $\alpha \in H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2})$ and $\beta \in H^2_c(U, \mathbb{Z}/n\mathbb{Z})$, we have

$$\alpha \cup \beta = \beta \cup \alpha$$

in $H^3_c(U, \mathbb{Z}/n\mathbb{Z})$. □

Proof. Choose cocycle representatives (c, b_S, a_S) and $(c', 0, a'_S)$ for α and β. Then

$$\alpha \cup \beta = [(c, b_S, a_S) \cup (c', 0, a'_S)] = [(c \cup c', 0, loc_S(c) \cup a'_S)]$$

and

$$\beta \cup \alpha = [(c' \cup c, 0, -a'_S \cup b_S + loc_S(c') \cup a_S)] = [(c' \cup c, 0, -a'_S \cup loc_S(c))].$$

Here the last equality follows from Lemma 3.1. We have the map

$$\eta : H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \to H^1(U, \mathbb{Z}/n\mathbb{Z})$$

that sends (c, b_S, a_S) to c. So, using η, the desired commutativity $\alpha \cup \beta = \beta \cup \alpha$ reduces to the commutativity of the following two products:

$$\begin{array}{ccc} H^1(U, \mathbb{Z}/n\mathbb{Z}) \times H^2_c(U, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\cup} & H^3_c(U, \mathbb{Z}/n\mathbb{Z}) \\ ([(c,0,0)], [(c', 0, a'_S)]) & \longmapsto & [(c \cup c', 0, loc_S(c) \cup a'_S)] \\ H^2_c(U, \mathbb{Z}/n\mathbb{Z}) \times H^1(U, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\cup} & H^3_c(U, \mathbb{Z}/n\mathbb{Z}) \\ ([(c', 0, a'_S)], (c, 0, 0)]) & \longmapsto & [(c' \cup c, 0, -a'_S \cup loc_S(c))]. \end{array}$$

These products are the same as the ones defined in [11, Section 5.3.3]. Moreover, Nekovar defined the involution

$$\mathcal{T} : C^*(U, \mathbb{Z}/n\mathbb{Z}) \to C^*(U, \mathbb{Z}/n\mathbb{Z}), \quad \mathcal{T} : C^*(U, \mathbb{Z}/n\mathbb{Z} \times_S 0) \to C^*(U, \mathbb{Z}/n\mathbb{Z} \times_S 0),$$

which are homotopic to the identity, and showed that the following diagram is commutative:

$$\begin{array}{ccc} C^*(U, \mathbb{Z}/n\mathbb{Z}) \times C^*(U, \mathbb{Z}/n\mathbb{Z} \times_S 0) & \xrightarrow{\cup} & C^*(U, \mathbb{Z}/n\mathbb{Z} \times_S 0) \\ \downarrow {\scriptstyle s_{12} \circ (\mathcal{T} \otimes \mathcal{T})} & & \downarrow {\scriptstyle \mathcal{T} \circ (s_{12})_*} \\ C^*(U, \mathbb{Z}/n\mathbb{Z} \times_S 0) \times C^*(U, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\cup} & C^*(U, \mathbb{Z}/n\mathbb{Z} \times_S 0), \end{array}$$

where s_{12} is the permutation between $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z} \times_S 0$ defined similarly as in [11, 3.4.5.4]. This finishes the proof. ■

The proof of the previous lemma makes use of the natural map

$$\eta : H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \to H^1(U, \mathbb{Z}/n\mathbb{Z})$$

that sends (c, b_S, a_S) to c. In fact, we have proved:

Lemma 3.7. The cup product

$$H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \times H^2_c(U, \mathbb{Z}/n\mathbb{Z}) \to H^3_c(U, \mathbb{Z}/n\mathbb{Z})$$

factors through the product

$$H^1(U, \mathbb{Z}/n\mathbb{Z}) \times H^2_c(U, \mathbb{Z}/n\mathbb{Z}) \to H^3_c(U, \mathbb{Z}/n\mathbb{Z})$$

via the map η. This is also true with the factors switched. □

We now use the tools developed above to define a pairing

$$H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \times H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

given by

$$(a, b) = \mathrm{Inv} \circ \zeta_*(a \cup db).$$

The pairing goes through

$$H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \times H^2(U, \mathbb{Z}/n\mathbb{Z} \times_S 0) \simeq H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2}) \times H^2_c(U, \mathbb{Z}/n\mathbb{Z}),$$

and hence, through $H^3(U, \mathbb{Z}/n\mathbb{Z} \times_S 0) = H^3_c(U, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\zeta_*} H^3_c(U, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Lemma 3.8. The pairing is symmetric. □

Proof. We have $0 = d_2(a \cup b) = da \cup b - a \cup db$. So

$$(a, b) = \mathrm{Inv} \circ \zeta_*(a \cup db) = \mathrm{Inv} \circ \zeta_*(da \cup b) = \mathrm{Inv} \circ \zeta_*(b \cup da) = (b, a).$$ ■

Lemma 3.9. If $a \in \mathrm{Ker}(d)$, then

$$(a, b) = 0$$

for all b. □

Proof.

$$(a, b) = (b, a) = \mathrm{Inv} \circ \zeta_*(b \cup da) = 0.$$ ■

Denote by $H^1(U,[\mathbb{Z}/n\mathbb{Z}]') \subset H^1(U,\mathbb{Z}/n\mathbb{Z})$ the classes that locally (at all $v \in S$) lift to μ_{n^2}. Equivalently, $H^1(U,[\mathbb{Z}/n\mathbb{Z}]')$ is the image of η. Because the pairing $(\cdot,\cdot)$ is symmetric and given by the form $(a,b) = \mathrm{Inv}\circ\zeta_*(\eta(a)\cup db)$ by Lemma 3.7, it follows that it factors to a pairing

$$H^1(U,[\mathbb{Z}/n\mathbb{Z}]') \times H^1(U,[\mathbb{Z}/n\mathbb{Z}]') \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Let $A_S = (\pi_U)^{ab}$ be the maximal Abelian quotient of π_U. By the Poitou–Tate duality, we have $H^2_c(U,\mathbb{Z}/n\mathbb{Z}) \simeq A_S/n$. Given an ideal I coprime to S, we can consider its class $[I]_{S,n} \in H^2_c(U,\mathbb{Z}/n\mathbb{Z})$ via class field theory and the previous isomorphism. We will say I is (S,n)-*homologically trivial* if $[I]_{S,n}$ is in the image of d. We can now define *the height pairing* of two (S,n)-homologically trivial ideals that are coprime to S via

$$ht_{S,n}(I,J) := (d^{-1}[I]_{S,n}, d^{-1}[J]_{S,n}) = \mathrm{Inv}\circ\zeta_*(d^{-1}[I]_{S,n}\cup[J]_{S,n}) =: \langle d^{-1}[I]_{S,n}, [J]_{S,n}\rangle,$$

which is well-defined by the discussion above.

Let I be an ideal such that I^n is principal in $\mathcal{O}_{F,S}$. Write $I^n = (f^{-1})$. Then the Kummer cocycles $k_n(f)$ will be in $Z^1(U,\mathbb{Z}/n\mathbb{Z})$. For any $a \in F$, denote by a_S its image in $\prod_{v\in S} F_v$. Thus, we get an element

$$[f]_{S,n} := [(k_n(f), k_{n^2}(f_S), 0)] \in Z^1(U,\mathbb{Z}/n\mathbb{Z}\times_S \mu_{n^2})$$

which is well-defined in cohomology independently of the choice of roots used to define the Kummer cocycles.

Proposition 3.10. We have $d[f]_{S,n} = [I]_{S,n}$ in $H^2_c(U,\mathbb{Z}/n\mathbb{Z})$. In particular, for any ideal I such that I^n is principal in $\mathcal{O}_{F,S}$, $[I]_{S,n}$ is (S,n)-homologically trivial. □

Proof. Let $T = S\cup S'$ be large enough that for $V = U\setminus S'$, $H^1(V,\mathbb{G}_m)[n] = 0$, and such that the support of I is still in V. Then I defines a class $[I]_{T,n}$ in $H^2_c(V,\mathbb{Z}/n\mathbb{Z})$. Similarly, f defines a class $[f]_{T,n} = [(k_n(f), k_{n^2}(f_T), 0)]$ in $H^1(V,\mathbb{Z}/n\mathbb{Z}\times_S \mu_{n^2})$. It is clear that the elements $[I]_{T,n}$ and $d[f]_{T,n}$ map to $[I]_{S,n}$ and $d[f]_{S,n}$ under the pushforward map $H^2_c(V,\mathbb{Z}/n\mathbb{Z}) \to H^2_c(U,\mathbb{Z}/n\mathbb{Z})$. Hence, it suffices to prove equality of the elements on V. We will prove that the two elements pair the same way with elements of $H^1(V,\mu_n)$. On V, by the exact sequence

$$0 \to \mathbb{G}_m(V)/\mathbb{G}_m(V)^n \to H^1(V,\mu_n) \to H^1(V,\mathbb{G}_m)[n] \to 0,$$

every element of $H^1(V, \mu_n)$ comes from $g \in \mathbb{G}_m(V)$ via the Kummer map. For this, we can compute the pairing between $\beta = (k_n(g), k_{n^2}(g_T), 0)$, which lifts note that $[k_n(g)]$ along η, and the cocycle representative α of $d[f]_{T,n}$

$$\alpha = (d\widetilde{k_n(f)}, 0, (\widetilde{k_n(f)})_T - k_{n^2}(f_T)),$$

where $\widetilde{k_n(f)}$ is a lift of $k_n(f)$ to μ_{n^2}. We find

$$\beta \cup \alpha = (k_n(g) \cup d\widetilde{k_n(f)}, 0, -(k_n(g))_T \cup [(\widetilde{k_n(f)})_T - k_{n^2}(f_T)]).$$

We note that the cup product $k_n(g) \cup \widetilde{k_n(f)}$ takes values in $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n \subset \mu_{n^2}$. So we have the cochain $(k_n(g) \cup \widetilde{k_n(f)}, 0, 0)$ whose differential is $(k_n(g) \cup d\widetilde{k_n(f)}, 0, -(k_n(g))_T \cup (\widetilde{k_n(f)})_T)$. Hence, it suffices to compute the invariant of

$$(0, 0, (k_n(g))_T \cup k_{n^2}(f_T))$$

which is homologous to $\beta \cup \alpha$.

Let T' be the support of I. Then $T \cup T'$ is the full set of places where the global cocycle $(k_n(g))_T \cup k_{n^2}(f_T)$ with coefficients in $\mu_n \subset \mu_{n^2}$ is possibly ramified. By global reciprocity, we have

$$\sum_{v \in T} \mathrm{Inv}_v((k_n(g))_T \cup k_{n^2}(f_T)) = -\sum_{v \in T'} \mathrm{Inv}_v((k_n(g))_{T'} \cup k_{n^2}(f_{T'})).$$

Let $\mathrm{ord}_v(I) = e_v$ and let ϖ_v be a uniformiser at v. Then $f_v = u_v \varpi_v^{ne_v}$ for a unit $u_v \in F_v$, so that $k_{n^2}(f_v) = k_n(u_v \varpi_v^{e_v})$. Also, $F(\sqrt[n]{g})$ is unramified at $v \in T'$. Hence, for $v \in T'$, we get

$$\mathrm{Inv}_v((k_n(g))_{T'} \cup k_{n^2}(f_{T'})) = (g_v, u_v \varpi_v^{e_v})_{v,n} = (g_v, \varpi_v^{e_v})_{v,n},$$

where the bracket $(\cdot, \cdot)_{v,n}$ now refers to the n-th Hilbert symbol in F_v.

Therefore, we conclude that

$$\sum_{v \in T} \mathrm{Inv}_v(k_n(g)_{T'} \cup k_{n^2}(f_{T'})) = -\sum_{v \in T'} k_n(g)_v(\mathrm{rec}_v(\varpi_v^{e_v})) = k_n(g)(\mathrm{rec}(I)) = \langle [\beta], [I]_{T,n} \rangle,$$

where rec_v is the local Artin map and rec is the global Artin map (cf. [3, p. 174–176]), finishing the proof. ∎

Corollary 3.11. Let I, J be ideals in $\mathcal{O}_F$ supported outside S that are n-torsion in the Picard group of U. Choose any $f \in F^*$ such that $I^n = (f^{-1})$ as ideals of $\mathcal{O}_{F,S}$. Let T be the support of J, ϖ_v be a uniformiser at v, and $e_v = \mathrm{ord}_v(J)$. Then

$$ht_{S,n}(I,J) = \sum_{v \in T} (f_v, \varpi_v^{e_v})_{v,n},$$

where the bracket denotes the n-th Hilbert symbol in F_v. □

Proof. By Proposition 3.10, we have $[f]_{S,n} \in H^1(U, \mathbb{Z}/n\mathbb{Z} \times_S \mu_{n^2})$ such that $\eta([f]_{S,n}) = k_n(f)$ and $d[f]_{S,n} = [I]_{S,n} \in \pi_U^{ab} \simeq H_c^2(U, \mathbb{Z}/n\mathbb{Z})$. The pairing $ht_{S,n}(I,J)$ is given by the Poitou-Tate pairing $\langle k_n(f), [J]_{S,n} \rangle$, which is equal to $k_n(f)([J]_{S,n}) = \sum_{v \in T} (f_v, \varpi_v^{e_v})_{v,n}$ by the local-global compatibility of Artin maps. ■

The referee points out that the definition of n-th power residue symbols for non-principal ideals is a long-standing problem in algebraic number theory. This corollary indicates that the linking pairing for homologically trivial ideals is a modest solution.

4 Arithmetic Linking, Class Invariants and the Artin Map

We continue with the assumption of a fixed trivialization $\zeta : \mathbb{Z}/n\mathbb{Z} \simeq \mu_n$ over the totally imaginary number field F.

Let us recall the construction of the class invariant homomorphism

$$\Psi : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to Cl(X) := Cl(F)$$

of Waterhouse [16] and Taylor [15]. Suppose $x \in H^1(X, \mathbb{Z}/n\mathbb{Z})$ is the class of the $\mathbb{Z}/n\mathbb{Z}$-torsor given as the spectrum of an étale $\mathcal{O}_F$-algebra $\mathcal{O}$ with $\mathbb{Z}/n\mathbb{Z}$-action. To avoid confusion we will write $\sigma_a(v)$ for the effect of the action of $a \in \mathbb{Z}/n\mathbb{Z} = \mathrm{Gal}(\mathcal{O}/\mathcal{O}_F)$ on $v \in \mathcal{O}$. We consider the $\mathcal{O}_F$-module $\mathcal{L}$ consisting of all elements $v \in \mathcal{O}$ such that

$$\sigma_a(v) = \zeta(a) \cdot v$$

for all $a \in \mathbb{Z}/n\mathbb{Z}$. Using étale descent along the extension $\mathcal{O}/\mathcal{O}_F$ we can easily see that $\mathcal{L}$ is $\mathcal{O}_F$-locally free of rank 1. Then we set $\Psi(x) = \Psi(\mathcal{O}/\mathcal{O}_F)$ to be the class of $\mathcal{L}$ in $\mathrm{Pic}(X) = Cl(X)$. This homomorphism Ψ can also be viewed as follows: The $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$-torsor over X that corresponds to x induces by $\mu_n \to \mathbb{G}_m$ a $\mathbb{G}_m$-torsor, that is, a line bundle whose class is $\Psi(x)$.

This construction plays a central role in the theory of Galois module structure; indeed, $\Psi(x)$ is an important invariant of the structure of $\mathcal{O}$ as an $\mathcal{O}_F[\mathbb{Z}/n\mathbb{Z}] = \mathcal{O}_F[x]/(x^n - 1)$-module. The general form of the class invariant homomorphism for the constant group scheme $\mathbb{Z}/n\mathbb{Z}$ with Cartier dual $\boldsymbol{\mu}_n$ is

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Pic}((\boldsymbol{\mu}_n)_X) = \mathrm{Pic}(\mathcal{O}_F[x]/(x^n - 1)).$$

(See e.g., [16]). The map Ψ above is obtained by composing the above with the restriction along the section $X \to (\boldsymbol{\mu}_n)_X$ given by $x \mapsto \zeta(1)$.

Combining this with class field theory allows us to define the class invariant pairing

$$(\cdot,\cdot)_c : (Cl(X)/n)^\vee \times (Cl(X)/n)^\vee \to \mathbb{Z}/n\mathbb{Z}$$

as follows: Take $f, f' \in (Cl(X)/n)^\vee = \mathrm{Hom}_{\mathbb{Z}}(Cl(X), \mathbb{Z}/n\mathbb{Z})$. By class field theory, f and f' correspond to unramified $\mathbb{Z}/n\mathbb{Z}$-extensions K_f and $K_{f'}$ of F. Let $\mathcal{O}_f$ and $\mathcal{O}_{f'}$ be the normalisations of $\mathcal{O}_F$ in K_f and $K_{f'}$ respectively; these are étale $\mathcal{O}_F$-algebras with $\mathbb{Z}/n\mathbb{Z}$-action. By definition, the class invariant pairing is

$$(f,f')_c := f'(\Psi(\mathcal{O}_f/\mathcal{O}_F)).$$

Theorem 4.1. Under the class field theory isomorphism

$$\mathrm{Ar} : H^1(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} (Cl(X)/n)^\vee,$$

the class invariant pairing

$$(\cdot,\cdot)_c : (Cl(X)/n)^\vee \times (Cl(X)/n)^\vee \to \mathbb{Z}/n\mathbb{Z}$$

is identified with the pairing

$$(\cdot,\cdot) : H^1(X, \mathbb{Z}/n\mathbb{Z}) \times H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}, \quad (\alpha,\beta) = \mathrm{Inv} \circ \zeta_*(\alpha \cup \delta'\beta),$$

defined as in Section 2. □

Remark 4.2.

(a) It follows that the arithmetic Chern–Simons invariant

$$CS_c : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}, \quad CS_c(x) = (x,x),$$

for $c = Id \cup \tilde{\delta}(Id)$ can be identified under $(Cl(X)/n)^\vee \simeq H^1(X, \mathbb{Z}/n\mathbb{Z})$ with the quadratic form $(Cl(X)/n)^\vee \to \mathbb{Z}/n\mathbb{Z}$, $f \mapsto (f, f)_c$, of the class invariant pairing $(\cdot,\cdot)_c$. This statement was first shown in [2] by a different argument. This result of [2] inspired us to obtain the above theorem.

(b) Under the additional hypothesis that $\mu_{n^2} \subset F$, the pairing $(\cdot,\cdot)$ is symmetric and agrees with the pairing defined in Section 2. This follows from Lemma 2.2 and its proof. □

Corollary 4.3. Assuming $\mu_{n^2} \subset F$, the class invariant pairing $(\cdot,\cdot)_c$ is symmetric. □

Proof. This follows from Lemma 2.2 and its proof and Theorem 4.1. ■

Proof of Theorem 4.1. Recall that Artin–Verdier duality [7] gives isomorphisms

$$H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee \simeq \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m), \quad H^2(X, \mathbb{Z}/n\mathbb{Z})^\vee \simeq \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m). \tag{4.1}$$

Applying $\mathrm{Ext}^i_X(-, \mathbb{G}_m)$ to $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ gives an exact sequence

$$0 \to \mathrm{Ext}^0_X(\mathbb{Z}, \mathbb{G}_m)/n \xrightarrow{\partial} \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \mathrm{Ext}^1_X(\mathbb{Z}, \mathbb{G}_m)[n] = Cl(X)[n] \to 0, \tag{4.2}$$

where the connecting ∂ is given via the Yoneda product

$$\mathrm{Ext}^0_X(\mathbb{Z}, \mathbb{G}_m) \times \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$$

with the class of $R(n) = (0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0)$. This combined with (4.1) induces a surjective homomorphism

$$h : H^2(X, \mathbb{Z}/n\mathbb{Z})^\vee \to Cl(X)[n].$$

Similarly, we have

$$0 \to \mathrm{Ext}^1_X(\mathbb{Z}, \mathbb{G}_m)/n \xrightarrow{\partial'} \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \mathrm{Ext}^2_X(\mathbb{Z}, \mathbb{G}_m)[n] = \mathrm{Br}(X)[n] = 0,$$

in other words, an isomorphism

$$\partial' : Cl(X)/n \xrightarrow{\simeq} \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m).$$

The composition of ∂' with the duality $\mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \simeq H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee$ is the dual $\mathrm{Ar}^\vee$ of the isomorphism

$$\mathrm{Ar} : H^1(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} (Cl(X)/n)^\vee$$

given by the Artin map of class field theory, that is, $\mathrm{Ar}(x)$ is the Artin reciprocity map $Cl(X) \to \mathbb{Z}/n\mathbb{Z}$ for the $\mathbb{Z}/n\mathbb{Z}$-torsor given by x (see [7, p. 539]).

Taking Yoneda product with the class

$$[E(n)] = (0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0)$$

in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ gives the Bockstein homomorphisms:

$$\delta'' : \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m),$$

$$\delta' : \mathrm{Ext}^1_X(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Ext}^2_X(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = H^2(X, \mathbb{Z}/n\mathbb{Z}).$$

Under the duality isomorphisms (4.1), the dual $\delta'^\vee$ is identified with the Bockstein δ''. This easily follows from the fact that the Artin–Verdier duality pairings are also given via Yoneda products.

Proposition 4.4. The dual $\delta'^\vee : H^2(X, \mathbb{Z}/n\mathbb{Z})^\vee \to H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee$ of the Bockstein homomorphism δ' is equal to the composition

$$H^2(X, \mathbb{Z}/n\mathbb{Z})^\vee \xrightarrow{h} Cl(X)[n] \to Cl(X)/n \xrightarrow{\mathrm{Ar}^\vee} H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee.$$

where the map $Cl(X)[n] \to Cl(X)/n$ is induced by the identity on $Cl(X)$. □

Proof. Consider the composition $\delta' \circ \partial$ where ∂ is as in (4.2). The connecting ∂ is given as Yoneda product

$$\mathrm{Ext}^0_X(\mathbb{Z}, \mathbb{G}_m) \times \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$$

with the class of $R(n) = (0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0)$. Hence the composition $\delta'' \circ \partial$ is given by

$$(\beta' \circ \partial)(a) = a \cup [R(n)] \cup [E(n)].$$

But $[R(n)] \cup [E(n)] = 0$, since $\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = (0)$. Therefore, δ'' factors through the quotient by the image of ∂:

$$\delta'' : \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \mathrm{Ext}^1_X(\mathbb{Z}, \mathbb{G}_m)[n] = Cl(X)[n] \to \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m).$$

Combining this with the isomorphism ∂' gives a factorization of δ'' as a composition

$$\mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to Cl(X)[n] \xrightarrow{\epsilon} Cl(X)/n \xrightarrow{\sim} \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m).$$

Since duality identifies $\delta'^{\vee}$ with δ'' it remains to see that, in the above, ϵ is induced by the identity map on $Cl(X)$:

Write an element $y \in Cl(X)[n]$ as the extension $1 \to \mathbb{G}_m \to J' \to \mathbb{Z} \to 0$ coming from pulling back $x = (1 \to \mathbb{G}_m \to J \to \mathbb{Z}/n\mathbb{Z} \to 0) \in \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$ via $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Then $\delta''(x)$ is the class of

$$1 \to \mathbb{G}_m \to J \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

concatenating x with $E(n)$. On the other hand, y corresponds under $Cl(X)/n \xrightarrow{\simeq} \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$ to the extension

$$1 \to \mathbb{G}_m \to J' \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

obtained by concatenating $1 \to \mathbb{G}_m \to J' \to \mathbb{Z} \to 0$ with $R(n) : 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. Pushing out $R(n) : 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ by $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ gives $E(n) : 0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ and so we have a commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & J & \longrightarrow & \mathbb{Z}/n^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & J' & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

which shows the statement. This concludes the proof of the Proposition. ■

By the definition of the arithmetic linking pairing

$$(\cdot,\cdot) : \ H^1(X, \mathbb{Z}/n\mathbb{Z}) \times H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}$$

the corresponding homomorphism $D : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^1(X, \mathbb{Z}/n\mathbb{Z})^{\vee}$ (i.e., with $D(x)(x') = (x, x')$) is given as the composition

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z})^{\vee} \xrightarrow{\delta'^{\vee}} H^1(X, \mathbb{Z}/n\mathbb{Z})^{\vee}$$

of the homomorphism given by cup product and Artin–Verdier duality with the dual of the Bockstein. By combining this with Proposition 4.4 we see that D is the composition

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z})^{\vee} \xrightarrow{h} Cl(X))[n] \to Cl(X)/n \xrightarrow{\mathrm{Ar}^{\vee}} H^1(X, \mathbb{Z}/n\mathbb{Z})^{\vee}.$$

Lemma 4.5. Suppose that the $\mathbb{Z}/n\mathbb{Z}$-torsor $x \in H^1(X, \mathbb{Z}/n\mathbb{Z})$ has generic fiber $F(\xi^{1/n})/F$ where $\xi \in F^*$ is a Kummer generator. Then the fractional ideal of F generated by ξ is the

n-th power $(\xi) = I^n$ of a well-defined fractional ideal I of F; the class $[I] = [I(x)] \in Cl(X)$ only depends on x, is n-torsion, and is equal to the image $\Psi(x)$ of the class invariant homomorphism. The image of x under the composition of the first two maps above

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z})^\vee \xrightarrow{h} Cl(X)[n]$$

is $\Psi(x) = [I(x)]$. □

Proof. The first part of the statement is standard. In fact, we have $i : \mathcal{L} \otimes_{\mathcal{O}_F} F \simeq F \cdot \xi^{1/n} \simeq F$ and, by definition, $I = i(\mathcal{L})$ and so $\Psi(x) = [I(x)]$.

The rest of the statement of the lemma follows from Artin–Verdier duality, the computation of the group $H^2(X, \mathbb{Z}/n\mathbb{Z})^\vee \simeq \mathrm{Ext}^1_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$, and of the local duality pairings via Hilbert symbols, in Ref. [7, p. 540–541, 550–551]. A more detailed statement appears in Ref. [2]. ■

It now follows that $D : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee$ is the map

$$x \mapsto (x' \mapsto \mathrm{Ar}(x')([I(x)])) \in \mathrm{Hom}(H^1(X, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z}),$$

where $\mathrm{Ar}(x') : Cl(X) \to \mathbb{Z}/n\mathbb{Z}$ is the Artin (reciprocity) homomorphism associated to the $\mathbb{Z}/n\mathbb{Z}$-torsor given by x'. The statement of the theorem follows. ■

Remark 4.6.

(a) It follows from the above description of the map

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{D} H^1(X, \mathbb{Z}/n\mathbb{Z})^\vee \simeq \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \simeq Cl(X)/n$$

that the group of n-homologically trivial ideal classes in $Cl(X)/n$ coincides with the image of the class invariant homomorphism Ψ in $Cl(X)/n$. In the theory of Galois module structure, ideal classes which are in the image of the class invariant homomorphism are often called "realisable."

(b) Assuming $\mu_{n^2} \subset F$, the class invariant pairing can be viewed as a canonical symmetric tensor

$$c(F, n) \in \mathrm{TS}^2_{\mathbb{Z}/n\mathbb{Z}}(Cl(F)/n) := (Cl(F)/n \otimes Cl(F)/n)^{S_2}.$$

It would be interesting to study this tensor and its variation in families of number fields. □

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