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BOUNDARY LAYER OF THE BOLTZMANN EQUATION IN 2-DIMENSIONAL CONVEX DOMAINS

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Consider the stationary Boltzmann equation in 2-dimensional convex domains with diffusive boundary condition. We establish the hydrodynamic limits while the boundary layers are present, and derive the steady Navier–Stokes–Fourier system with nonslip boundary conditions. Our contribution focuses on novel weighted $W^{1,\infty}$ estimates for the ϵ-Milne problem with geometric correction. Also, we develop stronger remainder estimates based on an L^{2m}-L^∞ framework.

1. Introduction

1A. ***Problem presentation.*** Consider the stationary Boltzmann equation in a 2-dimensional smooth convex domain $\Omega \ni \vec{x} = (x_1, x_2)$ with velocity $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$. The density function $\mathfrak{F}^\epsilon(\vec{x}, \vec{v})$ in the phase space satisfies

$$\begin{cases} \epsilon \vec{v} \cdot \nabla_x \mathfrak{F}^\epsilon = Q[\mathfrak{F}^\epsilon, \mathfrak{F}^\epsilon] & \text{in } \Omega \times \mathbb{R}^2, \\ \mathfrak{F}^\epsilon(\vec{x}_0, \vec{v}) = P^\epsilon[\mathfrak{F}^\epsilon](\vec{x}_0, \vec{v}) & \text{for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{v} \cdot \vec{n}(\vec{x}_0) < 0, \end{cases} \tag{1-1}$$

where $\vec{n}(\vec{x}_0)$ is the unit outward normal vector at $\vec{x}_0$, the Knudsen number ϵ satisfies $0 < \epsilon \ll 1$, and the diffusive boundary satisfies

$$P^\epsilon[\mathfrak{F}^\epsilon](\vec{x}_0, \vec{v}) = \mu_b^\epsilon(\vec{x}_0, \vec{v}) \int_{\vec{\mathfrak{u}} \cdot \vec{n}(\vec{x}_0) > 0} \mathfrak{F}^\epsilon(\vec{x}_0, \vec{\mathfrak{u}}) |\vec{\mathfrak{u}} \cdot \vec{n}(\vec{x}_0)| \, d\vec{\mathfrak{u}}. \tag{1-2}$$

The boundary Maxwellian

$$\mu_b^\epsilon(\vec{x}_0, \vec{v}) = \frac{\rho_b^\epsilon(\vec{x}_0)}{\theta_b^\epsilon(\vec{x}_0)\sqrt{2\pi}} \exp\left(-\frac{|\vec{v} - \vec{u}_b^\epsilon(\vec{x}_0)|^2}{2\theta_b^\epsilon(\vec{x}_0)}\right) \tag{1-3}$$

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is a perturbation of the standard Maxwellian

$$\mu_b(\vec{v})=\frac{1}{\sqrt{2\pi}}\exp\biggl(-\frac{|\vec{v}|^2}{2}\biggr). \tag{1-4}$$

It is normalized to satisfy

$$\int_{\vec{v}\cdot\vec{n}(\vec{x}_0)>0}\mu_b^\epsilon(\vec{x}_0,\vec{v})|\vec{v}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{v}=\int_{\vec{v}\cdot\vec{n}(\vec{x}_0)>0}\mu_b(\vec{v})|\vec{v}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{v}=1. \tag{1-5}$$

For simplicity, we just write $\mu=\mu_b$. We further assume that $(\rho_b^\epsilon,\vec{u}_b^\epsilon,\theta_b^\epsilon)$ are perturbations of the standard density, velocity and temperature $(1,\vec{0},1)$, and can be expanded into power series with respect to ϵ,

$$\rho_b^\epsilon(\vec{x}_0)=1+\sum_{k=1}^\infty\epsilon^k\rho_{b,k}(\vec{x}_0),\quad \vec{u}_b^\epsilon(\vec{x}_0)=0+\sum_{k=1}^\infty\epsilon^k\vec{u}_{b,k}(\vec{x}_0),\quad \theta_b^\epsilon(\vec{x}_0)=1+\sum_{k=1}^\infty\epsilon^k\theta_{b,k}(\vec{x}_0). \tag{1-6}$$

Consequently, we may also expand the boundary Maxwellian μ_b^ϵ into a power series with respect to ϵ,

$$\mu_b^\epsilon(\vec{x}_0,\vec{v})=\mu(\vec{v})+\mu^{\frac{1}{2}}(\vec{v})\biggl(\sum_{k=1}^\infty\epsilon^k\mu_k(\vec{x}_0,\vec{v})\biggr). \tag{1-7}$$

In particular, we have

$$\mu_1(\vec{x}_0,\vec{v})=\mu^{\frac{1}{2}}(\vec{v})\biggl(\rho_{b,1}(\vec{x}_0)+\vec{u}_{b,1}(\vec{x}_0)\cdot\vec{v}+\theta_{b,1}(\vec{x}_0)\frac{|\vec{v}|^2-2}{2}\biggr). \tag{1-8}$$

It is easy to check that

$$\biggl|\langle\vec{v}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{v}|^2}\frac{\mu_b^\epsilon-\mu}{\mu^{\frac{1}{2}}}\biggr|\leq C_0(\varrho,\vartheta)\epsilon \tag{1-9}$$

for any $0\leq\varrho<\frac{1}{4}$ and integer $\vartheta\geq 3$. We assume that the perturbations in ρ_b^ϵ, $\vec{u}_b^\epsilon$ and θ_b^ϵ are sufficiently small such that $C_0>0$ is also very small. Based on the expansion (1-7) and (1-5), we naturally have

$$\int_{\vec{v}\cdot\vec{n}(\vec{x}_0)>0}\mu_k(\vec{x}_0,\vec{v})\mu^{\frac{1}{2}}(\vec{v})|\vec{v}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{v}=0\quad\text{for } k\geq 1. \tag{1-10}$$

Following the notation in [Glassey 1996], here we have the nonlinear collision term

$$Q[F,G]=\int_{\mathbb{R}^2}\int_{\mathbb{S}^1}q(\vec{\omega},|\vec{\mathfrak{u}}-\vec{v}|)\bigl(F(\vec{\mathfrak{u}}_*)G(\vec{v}_*)-F(\vec{\mathfrak{u}})G(\vec{v})\bigr)\,\mathrm{d}\vec{\omega}\,\mathrm{d}\vec{\mathfrak{u}}, \tag{1-11}$$

with

$$\vec{\mathfrak{u}}_*=\vec{\mathfrak{u}}+\vec{\omega}((\vec{v}-\vec{\mathfrak{u}})\cdot\vec{\omega}),\qquad \vec{v}_*=\vec{v}-\vec{\omega}((\vec{v}-\vec{\mathfrak{u}})\cdot\vec{\omega}), \tag{1-12}$$

and the hard-sphere collision kernel

$$q(\vec{\omega},|\vec{\mathfrak{u}}-\vec{v}|)=q_0|\vec{\omega}\cdot(\vec{v}-\vec{\mathfrak{u}})| \tag{1-13}$$

for a positive constant q_0. We intend to study the behavior of $\mathfrak{F}^\epsilon$ as $\epsilon\to 0$.

Roughly speaking, the Boltzmann equation (1-1) and its evolutionary counterpart describe a rarefied gas confined in a bounded domain Ω, in contact with a surrounding gas reservoir modeled by (1-3) with density ρ_b^ϵ, velocity $\vec{u}_b^\epsilon$ and temperature θ_b^ϵ. It is well known that if the surrounding Maxwellian μ_b^ϵ is uniform on $\partial\Omega$, then the system has an equilibrium solution μ_b^ϵ. Also, such an equilibrium is reached exponentially fast, provided the initial state is sufficiently close to the above Maxwellian.

However, when the surrounding Maxwellian is not uniform, the procedure is much more complicated. There might be heat transfer, velocity exchange, or even the in-flow or out-flow of particles within Ω (e.g., a hotter place on $\partial\Omega$ might transfer heat to a colder place on $\partial\Omega$, so there is steady heat flow in Ω). The analysis itself may involve convection, oscillation, etc. Here there are mainly two issues to consider:

- The well-posedness of stationary solutions: This problem is not obvious now and we must analyze the effect of the nonuniform boundary Maxwellian on the interior solution.

- The asymptotic behavior of the solution when the Knudsen number ϵ shrinks to zero: $\epsilon \to 0$ roughly indicates that the gas undergoes more and more collisions (possibly denser and denser), so the overall behavior of the system may be modeled by classical fluid equations. Using expansion methods, it can be formally shown that the leading-order term is a local Maxwellian and the next-order evolution follows macroscopic equations, like the Navier–Stokes equations. Our main goal is to justify such an asymptotic convergence.

1B. *Linearization.* The solution $\mathfrak{F}^\epsilon$ can be expressed as a perturbation of the standard Maxwellian,

$$\mathfrak{F}^\epsilon(\vec{x},\vec{v}) = M_0\mu(\vec{v}) + \mu^{\frac{1}{2}}(\vec{v}) f^\epsilon(\vec{x},\vec{v}), \tag{1-14}$$

where M_0 is a constant. Then due to the conservation of mass, without loss of generality, we require that f^ϵ satisfies the normalization condition

$$\int_\Omega \int_{\mathbb{R}^2} f^\epsilon(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\vec{v}\,\mathrm{d}\vec{x} = 0. \tag{1-15}$$

Following the notation in [Glassey 1996], f^ϵ satisfies the equation

$$\begin{cases} \epsilon\vec{v}\cdot\nabla_x f^\epsilon + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(\vec{x}_0,\vec{v}) = \mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0,\vec{v}) \quad \text{for } \vec{x}_0\in\partial\Omega \text{ and } \vec{v}\cdot\vec{n}(\vec{x}_0)<0, \end{cases} \tag{1-16}$$

where

$$\mathcal{L}[f^\epsilon] = -2\mu^{-\frac{1}{2}} Q[\mu, \mu^{\frac{1}{2}} f^\epsilon], \qquad \Gamma[f^\epsilon, f^\epsilon] = \mu^{-\frac{1}{2}} Q[\mu^{\frac{1}{2}} f^\epsilon, \mu^{\frac{1}{2}} f^\epsilon] \tag{1-17}$$

and

$$\begin{aligned} &\mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0,\vec{v}) \\ &\quad = \mu_b^\epsilon(\vec{x}_0,\vec{v})\mu^{-\frac{1}{2}}(\vec{v}) \int_{\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)>0} \mu^{\frac{1}{2}}(\vec{\mathfrak{u}}) f^\epsilon(\vec{x}_0,\vec{\mathfrak{u}}) |\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathfrak{u}} + \mu^{-\frac{1}{2}}(\vec{v})\big(\mu_b^\epsilon(\vec{x}_0,\vec{v}) - \mu(\vec{v})\big). \end{aligned} \tag{1-18}$$

Hence, in order to study $\mathfrak{F}^\epsilon$, it suffices to consider f^ϵ.

1C. *Previous results.*

1C1. *Asymptotic analysis.* Hydrodynamic limits are central to connecting the kinetic theory and fluid mechanics. Since early 20th century, these types of problems have been extensively studied in many different settings: stationary or evolutionary, linear or nonlinear, strong solution or weak solution, etc.

The early result dates back to 1912 by Hilbert himself, using the so-called Hilbert's expansion, i.e., an expansion of the distribution function $\mathfrak{F}^\epsilon$ as a power series of the Knudsen number ϵ. Since then, a lot of works on the Boltzmann equation in $\mathbb{R}^n$ or $\mathbb{T}^n$ have been presented, including [Golse and Saint-Raymond 2004; de Masi et al. 1989; Bardos et al. 1991; 1993; 1998; 2000], for either smooth solutions or renormalized solutions.

The general theory of initial-boundary-value problems was first developed in [Grad 1963], and then extended in [Darrozes 1969; Sone 1969; 1971; 1991; Sone and Aoki 1987], for both the evolutionary and stationary equations. The classical books [Sone 2002; 2007] provide a comprehensive summary of previous results and give a complete analysis of such approaches.

For the stationary Boltzmann equation where the state of gas is close to a uniform state at rest, the expansion of the perturbation f^ϵ consists of two parts: the interior solution F, which is based on a hierarchy of linearized Boltzmann equations and satisfies a steady Navier–Stokes–Fourier system, and the boundary layer $\mathscr{F}$, which is based on a half-space kinetic equation and decays rapidly when it is away from the boundary.

The justification of hydrodynamic limits usually involves two steps:

(1) Expanding $F = \sum_{k=1}^{\infty} \epsilon^k F_k$ and $\mathscr{F} = \sum_{k=1}^{\infty} \epsilon^k \mathscr{F}_k$ as power series of ϵ and proving the coefficients F_k and $\mathscr{F}_k$ are well-defined. This is doable by inserting the above expansion ansatz into the Boltzmann equation to compare the order of ϵ and get a hierarchy of equations for F_k and $\mathscr{F}_k$. Traditionally, the estimates of interior solutions F_k are relatively straightforward. On the other hand, boundary layers $\mathscr{F}_k$ satisfy 1-dimensional half-space problems which lose some key structures of the original equations. The well-posedness of boundary layer equations is sometimes extremely difficult, and it is possible that they are actually ill-posed (e.g., certain types of Prandtl layers).

(2) Proving that $R = f^\epsilon - \epsilon F_1 - \epsilon \mathscr{F}_1 = o(\epsilon)$ as $\epsilon \to 0$. Ideally, this should be done just by expanding to the leading-order level F_1 and $\mathscr{F}_1$. However, in singular perturbation problems, the estimates of the remainder R usually involve negative powers of ϵ, which require expansion to higher-order terms F_N and $\mathscr{F}_N$ for $N \geq 2$ such that we have a sufficient power of ϵ. In other words, we define $R = f^\epsilon - \sum_{k=1}^{N} \epsilon^k F_k - \sum_{k=1}^{N} \epsilon^k \mathscr{F}_k$ for $N \geq 2$ instead of $R = f^\epsilon - \epsilon F_1 - \epsilon \mathscr{F}_1$ to get better estimate of R.

The above formulation is for the convergence in the L^∞ sense. If instead we consider L^p convergence for $1 \leq p < \infty$, then the boundary layer $\mathscr{F}_1$ is of order $\epsilon^{1/p}$ due to rescaling, which is negligible compared with F_1. In [Esposito et al. 2018] the authors justify the L^p convergence under the same formulation as ours without taking boundary layer expansion into consideration. On the other hand, the effect of boundary layers constitutes the major upshot of our paper.

1C2. *Classical approach.* The classical construction of boundary layers requires the analysis of the flat Milne problem. In the following, when we refer to the normal (tangential) variable (direction) without other specification, it usually means the variable (direction) in space. In detail, let η denote the rescaled normal variable with respect to the boundary, i.e., $\eta = \mathfrak{N}/\epsilon$ for $\mathfrak{N}$ the distance to the boundary $\partial\Omega$, let θ denote the tangential variable, and let $\vec{\mathfrak{v}} = (v_\eta, v_\phi)$ denote the normal and tangential velocities. The boundary layer $\mathscr{F}_1$ satisfies

$$v_\eta \frac{\partial \mathscr{F}_1}{\partial \eta} + \mathcal{L}[\mathscr{F}_1] = 0, \tag{1-19}$$

where $\mathcal{L}$ is the linearized Boltzmann operator in (1-17).

Although a rigorous proof of such expansions has not been presented, it is widely believed that the motivation of this approach is natural and the difficulties are purely technical. Besides the fact that this idea is an intuitive application of Hilbert's expansion, it is strongly supported by [Bardos et al. 1986], which justifies the well-posedness and decay of the above flat Milne problem.

This idea is easily adapted to other kinetic models. As a linear prototype of the Boltzmann equation, the case of the neutron transport equation was carefully investigated. In particular, the hydrodynamic limit was proved in the remarkable paper [Bensoussan et al. 1979] by Bensoussan, Lions and Papanicolaou. This is widely regarded as the foundation of rigorous analysis of boundary layers in kinetic equations.

Unfortunately, in [Wu and Guo 2015], we demonstrated that both the proof and results of this formulation in [Bensoussan et al. 1979] are invalid due to a lack of regularity in estimating $\partial \mathscr{F}_1/\partial \theta$. Similarly, counterexamples were proposed in [Wu 2016] that this idea is also invalid in the Boltzmann equation. The solution to (1-19) cannot correctly characterize the boundary layer, even when Ω is a unit disk. Basically, this pulls the whole study back to the starting point and any later results based on this type of boundary layer should be reexamined.

In detail, in order to show the hydrodynamic limits, we need $\partial \mathscr{F}_1/\partial \theta \in L^\infty$ since it is part of the remainder (3-43). However, though $\mathscr{F}_1 \in L^\infty$ as is shown in [Bardos et al. 1986], we do not necessarily have $\partial \mathscr{F}_1/\partial \eta \in L^\infty$. The bad behavior occurs near the grazing set $v_\eta = 0$ and $\vec{x} \in \partial\Omega$. If $\mathcal{L}[\mathscr{F}_1] \neq 0$ as $v_\eta \to 0$, then solving from (1-19), we have $\partial \mathscr{F}_1/\partial \eta \to \infty$. Furthermore, the singularity $\partial \mathscr{F}_1/\partial \eta \notin L^\infty$ will be transferred to $\partial \mathscr{F}_1/\partial \theta \notin L^\infty$. This singularity was rigorously shown in [Wu and Guo 2015] through a careful construction of the boundary data; i.e., the chain of estimates

$$R = o(\epsilon) \quad \Longleftarrow \quad \mathscr{F}_2 \in L^\infty \quad \Longleftarrow \quad \frac{\partial \mathscr{F}_1}{\partial \theta} \in L^\infty \quad \Longleftarrow \quad \frac{\partial \mathscr{F}_1}{\partial \eta} \in L^\infty \tag{1-20}$$

is broken since the rightmost estimate is wrong.

1C3. *Geometric correction.* While the classical method breaks down, a new approach with geometric correction to the boundary layer construction has been developed to ensure regularity in the cases of the disk and annulus in [Wu and Guo 2015; Wu et al. 2016; Wu 2016; 2017]. The new boundary layer $\mathscr{F}_1$ satisfies the ϵ-Milne problem with geometric correction,

$$v_\eta \frac{\partial \mathscr{F}_1}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_1}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_1}{\partial v_\phi}\right) + \mathcal{L}[\mathscr{F}_1] = 0, \tag{1-21}$$

where R_κ is the curvature on the boundary curve. We proved that the solution recovers the well-posedness and exponential decay as in the flat Milne problem, and the regularity in θ is indeed improved, i.e., $\partial \mathscr{F}_1/\partial \theta \in L^\infty$.

However, this new method fails to treat more general domains. Roughly speaking, we have two contradictory goals to achieve:

(1) To prove hydrodynamic limits, the remainder estimates for R require higher-order regularity estimates of the boundary layer.

(2) The geometric correction

$$\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\frac{\partial\mathscr{F}_1}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{F}_1}{\partial v_\phi}\right)$$

in the boundary layer equation is related to the curvature of the boundary curve, which prevents higher-order regularity estimates.

In other words, the above improvement of regularity is still not enough to close the proof. To be more specific, the discussion of domains is as follows:

- In the domain of the disk or annulus, when R_κ is constant, as in [Wu 2016], $\partial\mathscr{F}_1/\partial\theta$ is bounded, since the tangential derivative $\partial/\partial\theta$ commutes with the equation, and thus we do not need the estimate of the singular term $\partial\mathscr{F}_1/\partial\eta$.
- For general 2-dimensional smooth convex domains, when R_κ is a function of θ, $\partial/\partial\theta$ does not commute with the equation since it might hit R_κ. Then $\partial\mathscr{F}_1/\partial\theta$ relates to the normal and even velocity derivatives, which have been shown to be possibly unbounded in [Wu 2016]. Therefore, we get stuck again at the regularity estimates.

1D. ***Novel contribution of this paper.*** In this paper, we will push the above argument from both sides (remainder estimates and regularity estimates) and prove the hydrodynamic limits for the nonlinear Boltzmann equation in 2-dimensional smooth convex domains.

1D1. *Remainder estimates.* We first prove an almost optimal remainder estimate and reduce the regularity requirement of the expansion. Denote the remainder by $R(\vec{x},\vec{v})=f^\epsilon-F-\mathscr{F}$ for the interior solution $F(\vec{x},\vec{v})$ and the boundary layer

$$\mathscr{F}(\eta,\theta,\vec{v})\sim\epsilon\mathscr{F}_1(\eta,\theta,\vec{v})+\epsilon^2\mathscr{F}_2(\eta,\theta,\vec{v}). \tag{1-22}$$

Then the remainder equation is actually a nonhomogeneous linearized Boltzmann equation

$$\epsilon\vec{v}\cdot\nabla_x R+\mathcal{L}[R]=S. \tag{1-23}$$

The estimate in [Wu 2016] is

$$\|R\|_{L^\infty}\lesssim\frac{1}{\epsilon^3}\|S\|_{L^2}+\text{higher-order terms}. \tag{1-24}$$

Here the power 3 depends on the physical dimension of Ω and a standard energy argument. We intend to show that $\|R\|_{L^\infty}=o(\epsilon)$ as $\epsilon\to 0$. Since S in (3-43) contains the term related to $\partial\mathscr{F}_2/\partial\theta$ with order ϵ^2, the coefficient ϵ^{-3} is also a singularity.

Consider the general troublemaker $(1/\epsilon^q)\|S\|_{L^p(\Omega\times\mathbb{R}^2)}$ for some $p,q\in[1,\infty]$. It is easy to see that the smaller q is, the better estimate we get. On the other hand, a less obvious but key observation here is that due to the rescaled normal variable $\eta = \mathfrak{N}/\epsilon$,

$$\|S(\eta)\|_{L^p} \lesssim \left(\int_0^\infty S^p(\eta)\,\mathrm{d}\mathfrak{N}\right)^{\frac{1}{p}} \sim \epsilon^{\frac{1}{p}} \left(\int_0^\infty S^p(\eta)\,\mathrm{d}\eta\right)^{\frac{1}{p}}. \tag{1-25}$$

Therefore, the smaller p is, the better the estimate $\|S\|_{L^p}$ will be. We might improve the estimates in both directions.

Compared with [Wu 2016] which uses the so-called L^2-L^∞ framework, we employ a more general method, the L^{2m}-L^∞ framework for any $m \in \mathbb{N}$. This approach consists of two steps:

- Define

$$\mathbb{P}[R] = \mu^{\frac{1}{2}}\left(a + \vec{v}\cdot\vec{b} + \frac{|\vec{v}|^2 - 2}{2}c\right)$$

to be the orthogonal projection of R onto the null space of $\mathcal{L}$, and let $(\mathbb{I}-\mathbb{P})[R] = R - \mathbb{P}[R]$. Multiplying both sides of (1-23) by R, the direct energy estimates yield

$$\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2}^2 \leq \int_{\Omega\times\mathbb{R}^2} RS + \text{good terms}. \tag{1-26}$$

However, we do not have control of $\mathbb{P}[R]$. Then in order to bound $\mathbb{P}[R]$, we use nonstandard energy estimates in (1-23). We choose particular test functions like

$$\psi = \mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2 - \beta)(\vec{v}\cdot\nabla_x\phi(\vec{x})), \tag{1-27}$$

where $-\Delta_x\phi = \Upsilon^{2m-1}(\vec{x})$ satisfies Dirichlet's boundary condition $\phi = 0$ for $\Upsilon = a, \vec{b}, c$. Here, in the weak formulation of (1-23), we can write

$$-\int_{\Omega\times\mathbb{R}^2} R(\vec{v}\cdot\nabla_x\psi) = -\int_{\Omega\times\mathbb{R}^2} \mathbb{P}[R](\vec{v}\cdot\nabla_x\psi) - \int_{\Omega\times\mathbb{R}^2} (\mathbb{I}-\mathbb{P})[R](\vec{v}\cdot\nabla_x\psi) = I + II. \tag{1-28}$$

The first term I is roughly $\|\Upsilon\|_{L^{2m}}^{2m}$ and the second term II can be bounded as other terms in the weak formulation using Hölder's inequality. In total, this implies

$$\|\mathbb{P}[R]\|_{L^{2m}} \lesssim \|(\mathbb{I}-\mathbb{P})[R]\|_{L^{2m}} + \text{good terms}. \tag{1-29}$$

Note that we cannot directly insert (1-26) into (1-29) since $(\mathbb{I}-\mathbb{P})[R]$ are estimated in different norms. Then we hire an interpolation argument for the L^{2m} norm between the L^2 and L^∞ norms to bound

$$\|(\mathbb{I}-\mathbb{P})[R]\|_{L^{2m}} \lesssim \|(\mathbb{I}-\mathbb{P})[R]\|_{L^2} + o(1)\epsilon^{\frac{1}{m}}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^\infty} \lesssim \|(\mathbb{I}-\mathbb{P})[R]\|_{L^2} + o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty}, \tag{1-30}$$

where $o(1)$ denotes a sufficiently small constant and $\epsilon^{1/m}$ is for the next step. In total, at this step, inserting (1-30) into (1-29) and further (1-26), using Cauchy's inequality, we obtain

$$\|\mathbb{P}[R]\|_{L^{2m}} \lesssim o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty} + \frac{1}{\epsilon^2}\|\mathbb{P}[S]\|_{L^{2m/(2m-1)}} + \text{good terms}. \tag{1-31}$$

- Next, we bootstrap the L^{2m} estimates to L^∞ estimates. This is done through Duhamel's principle. Actually, we rewrite the solution along the characteristics, and rewrite the integrand in the nonlocal operator $\mathcal{L}[R]$ again tracking the characteristics. Then since the points on the characteristics never leave the domain Ω, a highly nontrivial substitution from velocity variable to spacial variable and Hölder's inequality lead to

$$\|R\|_{L^\infty} \lesssim \left(\int_\Omega \frac{1}{\epsilon^2} |\mathbb{P}[R]|^{2m}\right)^{\frac{1}{2m}} \lesssim \frac{1}{\epsilon^{\frac{1}{m}}} \|\mathbb{P}[R]\|_{L^{2m}}, \tag{1-32}$$

where $1/\epsilon^2$ is the Jacobian of this substitution. Here, actually the nonlocal operator $\mathcal{L}$ saves us since it contains a velocity integral. Then inserting (1-31) into (1-32), we obtain the desired estimates. In total, we obtain

$$\|R\|_{L^\infty} \lesssim \frac{1}{\epsilon^{2+\frac{1}{m}}} \|S\|_{L^{2m/(2m-1)}} + \text{higher-order terms.} \tag{1-33}$$

Definitely, this is much better than the estimate (1-24). In particular, the larger m is, the better estimate we will get.

1D2. *Well-posedness and regularity of boundary layers.* Recall the boundary layer expansion

$$\mathscr{F}(\eta,\theta,\vec{v}) \sim \epsilon\mathscr{F}_1(\eta,\theta,\vec{v}) + \epsilon^2\mathscr{F}_2(\eta,\theta,\vec{v}). \tag{1-34}$$

The diffusive boundary condition leads to an important simplification that $\mathscr{F}_1 = 0$, since μ_1 is in the null space of $\mathcal{L}$ and the interior solution F_1 can satisfy the boundary condition alone even without boundary layer correction. Thus the next-order boundary layer $\mathscr{F}_2$ must formally satisfy

$$v_\eta \frac{\partial \mathscr{F}_2}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_2}{\partial v_\phi}\right) + \mathcal{L}[\mathscr{F}_2] = 0. \tag{1-35}$$

The remainder estimate requires the estimate of $\partial\mathscr{F}_2/\partial\theta$, whose boundedness had remained open.

The key observation here is that the estimate of $\partial\mathscr{F}_2/\partial\theta$ relies on $v_\eta\partial\mathscr{F}_2/\partial\eta$, not $\partial\mathscr{F}_2/\partial\eta$ itself. This extra v_η saves us and avoids singularity. Intuitively, just as in the failure of the classical approach, the possible singularity occurs near the grazing set $\eta = 0$ and $v_\eta = 0$. Then directly solving (1-35) implies

$$v_\eta \frac{\partial \mathscr{F}_2}{\partial \eta} = \frac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_2}{\partial v_\phi}\right) - \mathcal{L}[\mathscr{F}_2]. \tag{1-36}$$

Here, it is possible that the right-hand side is nonzero when approaching the grazing set, but with the help of v_η, the left-hand side is always finite even if $\partial\mathscr{F}_2/\partial\eta \notin L^\infty$.

Similar to the analysis in [Wu 2016], we can show that $\mathscr{F}_2$ decays exponentially fast in L^∞. However, the regularity of $\mathscr{F}_2$ is much more delicate.

We cannot naively take η derivatives on both sides of (1-35) since the derivative might hit $\epsilon/(R_\kappa - \epsilon\eta)$ and we do not have bounds on the velocity derivatives. Instead, we design a delicate weight function

$$\zeta(\eta,\theta,v_\eta,v_\phi) = \left((v_\eta^2 + v_\phi^2) - \left(\frac{R_\kappa(\theta) - \epsilon\eta}{R_\kappa(\theta)}\right)^2 v_\phi^2\right)^{\frac{1}{2}}, \tag{1-37}$$

which satisfies

$$v_\eta \frac{\partial \zeta}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \frac{\partial \zeta}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \zeta}{\partial v_\phi}\right) = 0. \tag{1-38}$$

Hence, we have the weighted equation for $\mathscr{A}=\zeta(\partial\mathscr{F}_2/\partial\eta)$

$$v_\eta\frac{\partial\mathscr{A}}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\frac{\partial\mathscr{A}}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{A}}{\partial v_\phi}\right)+\zeta\mathcal{L}\left[\frac{\mathscr{A}}{\zeta}\right]=0. \tag{1-39}$$

At this stage, the nonlocal operator $\zeta\mathcal{L}[\mathscr{A}/\zeta]$ is a huge trouble since ζ may be zero and create a singularity. We cannot directly apply the well-posedness theorem for ϵ-Milne problem. Instead, we use the mild formulation and wrap the velocity integral of $\mathcal{L}$ with another spacial integral along the characteristics. An intuitive explanation is that the integral of a function is usually less singular than itself. After a very delicate analysis, we obtain

$$\|\mathrm{e}^{K\eta}\mathscr{A}\|_{L^\infty}=\left\|\mathrm{e}^{K\eta}\zeta\frac{\partial\mathscr{F}_2}{\partial\eta}\right\|_{L^\infty}\leq C. \tag{1-40}$$

Note that the weight function also has the good property that $\zeta\geq|v_\eta|$ (this is why we designed it this way), so using (1-35), we actually show

$$\left\|\mathrm{e}^{K\eta}v_\eta\frac{\partial\mathscr{F}_2}{\partial\eta}\right\|_{L^\infty}\leq C,\qquad \left\|\mathrm{e}^{K\eta}\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\frac{\partial\mathscr{F}_2}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{F}_2}{\partial v_\phi}\right)\right\|_{L^\infty}\leq C. \tag{1-41}$$

Our ultimate goal is to bound $\partial\mathscr{F}_2/\partial\theta$. Taking the θ-derivative of both sides of (1-35), noting that the curvature R_κ might depend on θ, the $\partial\mathscr{F}_2/\partial\theta$ equation is a nonhomogeneous ϵ-Milne problem with geometric correction, and the nonhomogeneous term is

$$\left(-\frac{\partial_\theta R_\kappa}{R_\kappa-\epsilon\eta}\right)\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\frac{\partial\mathscr{F}_2}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{F}_2}{\partial v_\phi}\right), \tag{1-42}$$

which can be shown to exponentially decay using (1-41). Then again by the well-posedness theorem of ϵ-Milne problem, we know $\partial\mathscr{F}_2/\partial\theta$ decays exponentially fast.

1E. *Main theorem.*

Theorem 1.1. *Let Ω be C^3 and convex. For given $M_0>0$ and $\mu_b^\epsilon>0$ satisfying (1-7) and (1-9) with $0<\epsilon\ll 1$, there exists a unique positive solution $\mathfrak{F}^\epsilon=M_0\mu+\mu^{1/2}f^\epsilon$ to the stationary Boltzmann equation (1-1) and (1-2), and f^ϵ fulfills that for an integer $\vartheta\geq 3$ and $0\leq\varrho<\frac{1}{4}$,*

$$\|\langle\vec{v}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{v}|^2}(f^\epsilon-\epsilon F)\|_{L^\infty}\leq C(\delta)\epsilon^{2-\delta} \tag{1-43}$$

for any $0<\delta\ll 1$, where

$$F=\mu^{\frac{1}{2}}\left(\rho+\vec{u}\cdot\vec{v}+\theta\frac{|\vec{v}|^2-2}{2}\right) \tag{1-44}$$

satisfies the steady Navier–Stokes–Fourier system

$$\begin{cases}\nabla_x(\rho+\theta)=0,\quad \nabla_x\cdot\vec{u}=0,\\ \vec{u}\cdot\nabla_x\vec{u}-\gamma_1\Delta_x\vec{u}+\nabla_x P=0,\quad \vec{u}\cdot\nabla_x\theta-\gamma_2\Delta_x\theta=0,\\ \rho(\vec{x}_0)=\rho_{b,1}(\vec{x}_0)+M(\vec{x}_0),\quad \vec{u}(\vec{x}_0)=\vec{u}_{b,1}(\vec{x}_0),\quad \theta(\vec{x}_0)=\theta_{b,1}(\vec{x}_0).\end{cases} \tag{1-45}$$

Here $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants, the pressure P can be solved from the system uniquely up to a constant, $M(\vec{x}_0)$ is such that the Boussinesq relation

$$\rho + \theta = \textit{constant} \tag{1-46}$$

and the normalization condition

$$\int_{\Omega}\int_{\mathbb{R}^2} F(\vec{x}, \vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\vec{v}\,\mathrm{d}\vec{x} = 0 \tag{1-47}$$

hold.

Remark 1.2. Here the smoothness of Ω is to guarantee that the substitution (3-30) is valid in the construction of boundary layers. The convexity is to avoid the singularity of the solution in the interior of Ω; see [Kim 2011].

Remark 1.3. Note that $\rho_{b,1}$ and $\theta_{b,1}$ do not necessarily satisfy the Boussinesq relation, which determines M up to a constant, so that is why we need to introduce M. Then the normalization condition of F eventually fixes the unique M.

Remark 1.4. The pressure P actually does not play a significant role at the leading order. It provides the constant in the quasi-Boussinesq relation in the next order.

Remark 1.5. The case $\rho_{b,1}(\vec{x}_0) = 0$, $\vec{u}_{b,1}(\vec{x}_0) = 0$ and $\theta_{b,1}(\vec{x}_0) \neq 0$ is called the nonisothermal model, which represents a system that only has heat transfer through the boundary but has no work between the environment and the system. Based on the above theorem, the hydrodynamic limit is a steady Navier–Stokes–Fourier system with nonslip boundary condition. This provides a rigorous derivation of this important fluid model.

1F. ***Notation and structure of this paper.*** Throughout this paper, $C > 0$ denotes a constant that only depends on the parameter Ω, but does not depend on the data. It is referred to as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a \lesssim b$ to denote $a \leq Cb$.

This paper is organized as follows: In Section 2, we list some preliminary results on the linearized Boltzmann operator and the weak formulation. In Section 3, we present the asymptotic analysis of (1-16). In Section 4, we establish the L^∞ well-posedness of the linearized Boltzmann equation. In Section 5, we prove the well-posedness, decay and the weighted regularity of the ϵ-Milne problem with geometric correction. Finally, in Section 6, we prove the main theorem.

2. Preparation and discussions

2A. ***Preliminaries on linearized Boltzmann operator.*** Chapter 3 of [Glassey 1996] describes the linearized Boltzmann operator $\mathcal{L}$ as

$$\mathcal{L}[f] = -2\mu^{-\frac{1}{2}} Q[\mu, \mu^{\frac{1}{2}} f] = \nu(\vec{v}) f - K[f], \tag{2-1}$$

where

$$\nu(\vec{v}) = \int_{\mathbb{R}^2}\int_{\mathbb{S}^1} q(\vec{\omega}, |\vec{u} - \vec{v}|)\mu(\vec{u})\,\mathrm{d}\vec{\omega}\,\mathrm{d}\vec{u}, \tag{2-2}$$

$$K[f](\vec{v}) = K_2[f](\vec{v}) - K_1[f](\vec{v}) = \int_{\mathbb{R}^2} k(\vec{\mathrm{u}}, \vec{v}) f(\vec{\mathrm{u}}) \, \mathrm{d}\vec{\mathrm{u}}, \tag{2-3}$$

$$K_1[f](\vec{v}) = \mu^{\frac{1}{2}}(\vec{v}) \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} q(\vec{\omega}, |\vec{\mathrm{u}} - \vec{v}|) \mu^{\frac{1}{2}}(\vec{\mathrm{u}}) f(\vec{\mathrm{u}}) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\vec{\mathrm{u}}, \tag{2-4}$$

$$K_2[f](\vec{v}) = \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} q(\vec{\omega}, |\vec{\mathrm{u}} - \vec{v}|) \mu^{\frac{1}{2}}(\vec{\mathrm{u}}) \big(\mu^{\frac{1}{2}}(\vec{v}_*) f(\vec{\mathrm{u}}_*) + \mu^{\frac{1}{2}}(\vec{\mathrm{u}}_*) f(\vec{v}_*) \big) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\vec{\mathrm{u}} \tag{2-5}$$

for some kernel $k(\vec{\mathrm{u}}, \vec{v})$.

Let $\langle \cdot, \cdot \rangle$ be the standard L^2 inner product in $\Omega \times \mathbb{R}^2$. We define the L^p and L^∞ norms in $\Omega \times \mathbb{R}^2$ as usual:

$$\|f\|_{L^p} = \left(\int_\Omega \int_{\mathbb{R}^2} |f(\vec{x}, \vec{v})|^p \, \mathrm{d}\vec{v} \, \mathrm{d}\vec{x} \right)^{\frac{1}{p}}, \quad \|f\|_{L^\infty} = \sup_{(\vec{x}, \vec{v}) \in \Omega \times \mathbb{R}^2} |f(\vec{x}, \vec{v})|. \tag{2-6}$$

Define the weighted L^2 norm as

$$\|f\|_{L^2_\nu} = \|\nu^{\frac{1}{2}} f\|_{L^2}. \tag{2-7}$$

Define the weighted L^∞ norm as

$$\|f\|_{L^\infty_{\vartheta,\varrho}} = \sup_{(\vec{x}, \vec{v}) \in \Omega \times \mathbb{R}^2} \left(\langle \vec{v} \rangle^\vartheta \mathrm{e}^{\varrho |\vec{v}|^2} |f(\vec{x}, \vec{v})| \right). \tag{2-8}$$

Denote the Japanese bracket by

$$\langle \vec{v} \rangle = (1 + |\vec{v}|^2)^{\frac{1}{2}}. \tag{2-9}$$

Define the kernel operator $\mathbb{P}$ as

$$\mathbb{P}[f] = \mu^{\frac{1}{2}}(\vec{v}) \left(a_f(\vec{x}) + \vec{v} \cdot \vec{b}_f(\vec{x}) + \frac{|\vec{v}|^2 - 2}{2} c_f(\vec{x}) \right), \tag{2-10}$$

where $\mathbb{P}$ is the orthogonal projection onto the null space of $\mathcal{L}$, and the nonkernel operator $\mathbb{I} - \mathbb{P}$ as

$$(\mathbb{I} - \mathbb{P})[f] = f - \mathbb{P}[f], \tag{2-11}$$

which satisfies

$$\int_{\mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f] \begin{pmatrix} 1 \\ \vec{v} \\ |\vec{v}|^2 \end{pmatrix} \mathrm{d}\vec{v} = 0. \tag{2-12}$$

Lemma 2.1. *For the operator $\mathcal{L} = \nu I - K$, we have the estimates*

$$\left\| \frac{\partial \nu}{\partial |\vec{v}|} \right\|_{L^\infty} \le C, \tag{2-13}$$

$$\nu_0 (1 + |\vec{v}|) \le \nu(\vec{v}) \le \nu_1 (1 + |\vec{v}|), \tag{2-14}$$

$$\langle f, \mathcal{L}[f] \rangle = \langle (\mathbb{I} - \mathbb{P})[f], \mathcal{L}[(\mathbb{I} - \mathbb{P})[f]] \rangle \ge C \|\nu^{\frac{1}{2}} (\mathbb{I} - \mathbb{P})[f]\|_{L^2}^2, \tag{2-15}$$

$$\|\mathcal{L}[(\mathbb{I} - \mathbb{P})[f]]\|_{L^2}^2 \ge C \|\nu^{\frac{1}{2}} (\mathbb{I} - \mathbb{P})[f]\|_{L^2}^2, \tag{2-16}$$

$$\|\mathbb{P}[f]\|_{L^2} \le \|\nu \mathbb{P}[f]\|_{L^2} \le C \|\mathbb{P}[f]\|_{L^2} \tag{2-17}$$

for ν_0, ν_1 and C positive constants.

Proof. See [Glassey 1996, Chapter 3]. ∎

Lemma 2.2. *There exists* $0 \le C_0 < 1$ *such that for* $0 \le \delta \le C_0$

$$|k(\vec{\mathrm{u}},\vec{v})| \le C\mathrm{e}^{-\delta|\vec{\mathrm{u}}-\vec{v}|^2-\delta\frac{||\vec{\mathrm{u}}|^2-|\vec{v}|^2|^2}{|\vec{\mathrm{u}}-\vec{v}|^2}}. \tag{2-18}$$

Proof. See [Glassey 1996, Chapter 3] and [Guo 2010, Lemma 3]. □

Lemma 2.3. *Let* $w_\xi(\vec{v}) = w_{\xi,\beta,\varrho}(\vec{v}) = (1+\xi^2|\vec{v}|^2)^{\beta/2}\mathrm{e}^{\varrho|\vec{v}|^2}$ *for* $\xi,\beta>0$ *and* $0\le\varrho\le\frac14$. *Then there exists* $0 \le C_1(\varrho) < 1$ *and* $C_2(\varrho) > 0$ *such that for* $0 \le \delta \le C_1(\varrho)$

$$\int_{\mathbb{R}^2}\mathrm{e}^{\delta|\vec{\mathrm{u}}-\vec{v}|^2}k(\vec{\mathrm{u}},\vec{v})\frac{w_\xi(\vec{v})}{w_\xi(\vec{\mathrm{u}})}\,\mathrm{d}\vec{\mathrm{u}} \le \frac{C_2(\varrho)}{1+|\vec{v}|}, \tag{2-19}$$

$$\int_{\mathbb{R}^2}\mathrm{e}^{\delta|\vec{\mathrm{u}}-\vec{v}|^2}\frac{1}{|\vec{\mathrm{u}}|}k(\vec{\mathrm{u}},\vec{v})\frac{w_\xi(\vec{v})}{w_\xi(\vec{\mathrm{u}})}\,\mathrm{d}\vec{\mathrm{u}} \le C_2(\varrho), \tag{2-20}$$

$$\int_{\mathbb{R}^2}\mathrm{e}^{\delta|\vec{\mathrm{u}}-\vec{v}|^2}\nabla_v k(\vec{\mathrm{u}},\vec{v})\frac{w_\xi(\vec{v})}{w_\xi(\vec{\mathrm{u}})}\,\mathrm{d}\vec{\mathrm{u}} \le C_2(\varrho). \tag{2-21}$$

For $m\in\mathbb{N}$, *we have*

$$\|K[f]\|_{L^{2m}} \le C\|f\|_{L^{2m}}. \tag{2-22}$$

Proof. See [Guo 2010, Lemma 3]. □

Lemma 2.4. *We have*

$$\|K[f]\|_{L^\infty_{\vartheta,\varrho}} \le C\left\|\frac{f}{\nu}\right\|_{L^\infty_{\vartheta,\varrho}}, \tag{2-23}$$

$$\|\nabla_v K[f]\|_{L^\infty_{\vartheta,\varrho}} \le C\|f\|_{L^\infty_{\vartheta,\varrho}}. \tag{2-24}$$

Proof. Consider the fact that for $\vartheta=\beta$, we have

$$C_1\langle\vec{v}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{v}|^2} \le w_\xi \le C_2\langle\vec{v}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{v}|^2} \tag{2-25}$$

for some constant $C_1, C_2 > 0$. Then this is a natural corollary of Lemma 2.3. See [Guo 2010; Guo et al. 2017]. □

Lemma 2.5. *The nonlinear term* Γ *satisfies, for* $\beta>0$ *and* $0\le\varrho\le\frac14$,

$$\left\|\frac{\Gamma[f,f]}{\nu}\right\|_{L^\infty_{\vartheta,\varrho}} \le C\|f\|^2_{L^\infty_{\vartheta,\varrho}}, \tag{2-26}$$

$$\int_{\Omega\times\mathbb{R}^2}\Gamma[f,g]h \le C\|h\|_{L^2_\nu}\big(\|f\|_{L^2}\|g\|_{L^\infty_{\vartheta,\varrho}}+\|g\|_{L^2}\|f\|_{L^\infty_{\vartheta,\varrho}}\big), \tag{2-27}$$

$$\int_{\Omega\times\mathbb{R}^2}\Gamma[f,g]h \le C\|h\|_{L^2_\nu}\big(\|g\|_{L^2}\|f\|_{L^\infty_{\vartheta,\varrho}}+\|f\|_{L^\infty}\|g\|_{L^2_\nu}\big). \tag{2-28}$$

Proof. See [Guo 2002, Lemma 2.3] and [Glassey 1996, Chapter 3]. □

Based on the flow direction, we can divide the boundary $\gamma = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega,\ \vec{v} \in \mathbb{R}^2\}$ into the in-flow boundary γ_-, the out-flow boundary γ_+, and the grazing set γ_0, where

$$\gamma_- = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega,\ \vec{v} \cdot \vec{n}(\vec{x}_0) < 0\}, \tag{2-29}$$

$$\gamma_+ = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega,\ \vec{v} \cdot \vec{n}(\vec{x}_0) > 0\}, \tag{2-30}$$

$$\gamma_0 = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega,\ \vec{v} \cdot \vec{n}(\vec{x}_0) = 0\}. \tag{2-31}$$

It is easy to see $\gamma = \gamma_+ \cup \gamma_- \cup \gamma_0$. Also, the boundary condition is only given on γ_-.

Define $\mathrm{d}\gamma = |\vec{v} \cdot \vec{n}|\, \mathrm{d}\varpi\, \mathrm{d}\vec{v}$ on the boundary $\partial\Omega \times \mathbb{R}^2$ for ϖ as the curve measure. Define the L^p and L^∞ norms on the boundary as

$$|f|_{L^p} = \left(\iint_\gamma |f(\vec{x}, \vec{v})|^p \,\mathrm{d}\gamma \right)^{\frac{1}{p}}, \qquad |f|_{L^\infty} = \sup_{(\vec{x},\vec{v}) \in \gamma} |f(\vec{x}, \vec{v})|, \tag{2-32}$$

$$|f|_{L^p_\pm} = \left(\iint_{\gamma_\pm} |f(\vec{x}, \vec{v})|^p \,\mathrm{d}\gamma \right)^{\frac{1}{p}}, \qquad |f|_{L^\infty_\pm} = \sup_{(\vec{x},\vec{v}) \in \gamma_\pm} |f(\vec{x}, \vec{v})|. \tag{2-33}$$

2B. ***Discussion on the 3-dimensional problem.*** Physically, the more relevant case is when Ω is a 3-dimensional domain. We note that the 3-dimensional case actually has many differences from the 2-dimensional one. Based on our formulation, there are three key difficulties:

- Remainder estimates: The embedding theorem is much worse in three dimensions than in two. For example, the result (4-14) is only true when $1 \le m \le 3$. This restricts our choice of m and makes it hard to further improve the remainder estimates. The estimates in 3-dimensional neutron transport equation are provided in [Guo and Wu 2017b], and we can clearly see the loss of powers in ϵ.

- Boundary layer formulation: If $\Omega \subset \mathbb{R}^3$ is a smooth convex domain, then $\partial\Omega$ is a 2-dimensional manifold. For each point on $\partial\Omega$, there are two orthogonal principal directions corresponding to two principal curvatures. The ϵ-Milne problem with geometric correction actually depends on the detailed form of curvature. As [Guo and Wu 2017b] reveals, even the well-posedness of solution to the ϵ-Milne problem is highly nontrivial, let alone the regularity estimates.

- Singular kernel: A more serious issue is that 3-dimensional collision kernel $k(v, v')$ contains the singularity $1/|v - v'|$ which is absent in two dimensions (which can be derived much as in [Glassey 1996, Chapter 3]). Then the preliminary results in the previous section may not hold any more. In particular, for Lemma 2.3 in three dimensions, only the first inequality still holds (see [Guo 2010, Lemma 3]), and the other two are invalid now. Hence, the arguments in (5-78), (5-79) and (5-92), which highly depend on this lemma and cannot be easily adapted to a more singular operator k, must be replaced by new ones.

The 3-dimensional problem requires many more techniques, and it is not a natural generalization of the 2-dimensional proof.

3. Asymptotic analysis

In this section, we will construct the asymptotic expansion of the equation

$$\begin{cases}\epsilon\vec{v}\cdot\nabla_x f^\epsilon+\mathcal{L}[f^\epsilon]=\Gamma[f^\epsilon,f^\epsilon],\\ f^\epsilon(\vec{x}_0,\vec{v})=\mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0,\vec{v})\quad\text{for }\vec{x}_0\in\partial\Omega\text{ and }\vec{v}\cdot\vec{n}(\vec{x}_0)<0,\end{cases}\tag{3-1}$$

with the normalization condition

$$\int_\Omega\int_{\mathbb{R}^2}f^\epsilon(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\vec{v}\,\mathrm{d}\vec{x}=0.\tag{3-2}$$

3A. ***Interior expansion.*** We define the interior expansion

$$F(\vec{x},\vec{v})\sim\sum_{k=1}^{3}\epsilon^k F_k(\vec{x},\vec{v}).\tag{3-3}$$

Plugging it into (3-1) and comparing the order of ϵ, we obtain

$$\mathcal{L}[F_1]=0,\tag{3-4}$$

$$\mathcal{L}[F_2]=-\vec{v}\cdot\nabla_x F_1+\Gamma[F_1,F_1],\tag{3-5}$$

$$\mathcal{L}[F_3]=-\vec{v}\cdot\nabla_x F_2+4\Gamma[F_1,F_2].\tag{3-6}$$

The following analysis is standard and well known. We mainly refer to the method in [Sone 2002; 2007]. The solvability of

$$\mathcal{L}[F_k]=S\tag{3-7}$$

requires that

$$\int_{\mathbb{R}^2}S(\vec{v})\psi(\vec{v})\,\mathrm{d}\vec{v}=0\tag{3-8}$$

for any ψ satisfying $\mathcal{L}[\psi]=0$. Then each F_k consists of three parts:

$$F_k(\vec{x},\vec{v})=A_k(\vec{x},\vec{v})+B_k(\vec{x},\vec{v})+C_k(\vec{x},\vec{v}).\tag{3-9}$$

Here

$$A_k(\vec{x},\vec{v})=\mu^{\frac{1}{2}}(\vec{v})\bigg(A_{k,0}(\vec{x})+A_{k,1}(\vec{x})v_1+A_{k,2}(\vec{x})v_2+A_{k,3}(\vec{x})\Big(\frac{|\vec{v}|^2-2}{2}\Big)\bigg)\tag{3-10}$$

is the macroscopic part which must be determined at each order separately, and

$$B_k(\vec{x},\vec{v})=\mu^{\frac{1}{2}}(\vec{v})\bigg(B_{k,0}(\vec{x})+B_{k,1}(\vec{x})v_1+B_{k,2}(\vec{x})v_2+B_{k,3}(\vec{x})\Big(\frac{|\vec{v}|^2-2}{2}\Big)\bigg)\tag{3-11}$$

is the connection part, with B_k depending on A_s for $1\leq s\leq k-1$ as

$$\begin{gathered}B_{k,0}=0,\qquad B_{k,1}=\sum_{i=1}^{k-1}A_{i,0}A_{k-i,1},\qquad B_{k,2}=\sum_{i=1}^{k-1}A_{i,0}A_{k-i,2},\\ B_{k,3}=\sum_{i=1}^{k-1}\bigg(A_{i,0}A_{k-i,3}+A_{i,1}A_{k-i,1}+A_{i,2}A_{k-i,2}+\sum_{j=1}^{k-1-i}A_{i,0}(A_{j,1}A_{k-i-j,1}+A_{j,2}A_{k-i-j,2})\bigg).\end{gathered}\tag{3-12}$$

In other words, B_k only depends on the value of A_s from previous orders, and thus is not independent. This term is present due to the nonlinearity in Γ. $C_k(\vec{x},\vec{v})$ is the orthogonal part satisfying

$$\int_{\mathbb{R}^2} \mu^{\frac{1}{2}}(\vec{v})C_k(\vec{x},\vec{v}) \begin{pmatrix} 1 \\ \vec{v} \\ |\vec{v}|^2 \end{pmatrix} \mathrm{d}\vec{v} = 0, \tag{3-13}$$

with

$$\mathcal{L}[C_k] = -\vec{v}\cdot\nabla_x F_{k-1} + \sum_{i=1}^{k-1} \Gamma[F_i, F_{k-i}], \tag{3-14}$$

which can be uniquely determined. Hence, we only need to determine A_k. Traditionally, we write

$$A_k = \mu^{\frac{1}{2}}\left(\rho_k + \vec{u}_k\cdot\vec{v} + \theta_k\left(\frac{|\vec{v}|^2-2}{2}\right)\right), \tag{3-15}$$

where the coefficients ρ_k, u_k and θ_k represent density, velocity and temperature in the macroscopic scale. Then the analysis in [Sone 2002; 2007] shows that A_k satisfies the equations as follows:

First-order expansion:

$$P_1 - (\rho_1 + \theta_1) = 0, \tag{3-16}$$

$$\nabla_x P_1 = 0, \tag{3-17}$$

$$\nabla_x\cdot\vec{u}_1 = 0. \tag{3-18}$$

Second-order expansion:

$$P_2 - (\rho_2 + \theta_2 + \rho_1\theta_1) = 0, \tag{3-19}$$

$$\vec{u}_1\cdot\nabla_x\vec{u}_1 - \gamma_1\Delta_x\vec{u}_1 + \nabla_x P_2 = 0, \tag{3-20}$$

$$\vec{u}_1\cdot\nabla_x\theta_1 - \gamma_2\Delta_x\theta_1 = 0, \tag{3-21}$$

$$\nabla_x\cdot\vec{u}_2 + \vec{u}_1\cdot\nabla_x\rho_1 = 0. \tag{3-22}$$

Here P_1 and P_2 represent the pressure, and γ_1 and γ_2 are constants.

3B. ***Boundary layer expansion with geometric correction.*** We will use the Cartesian coordinate system for the interior solution, and a local coordinate system in a neighborhood of the boundary for the boundary layer.

Assume the Cartesian coordinates are $\vec{x}=(x_1,x_2)$. Using the polar coordinates $(r,\theta)\in[0,\infty)\times[-\pi,\pi)$ and choosing a pole in Ω, we assume $\vec{x}_0\in\partial\Omega$ is described by

$$\begin{cases} x_{1,0} = r(\theta)\cos\theta, \\ x_{2,0} = r(\theta)\sin\theta, \end{cases} \tag{3-23}$$

where $r(\theta)>0$ is a given function describing the boundary curve. Our local coordinate system is a modification of the polar coordinate system.

In the domain near the boundary, for each θ, we have the outward unit normal vector

$$\vec{n} = \left(\frac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}, \frac{r(\theta)\sin\theta - r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}\right), \tag{3-24}$$

where $r'(\theta)=\mathrm{d}r/\mathrm{d}\theta$. We can determine each point $\vec{x}\in\bar{\Omega}$ as $\vec{x}=\vec{x}_0-\mathfrak{N}\vec{n}$, where $\mathfrak{N}$ is the normal distance to the boundary point $\vec{x}_0$. In detail, this means

$$\begin{aligned} x_1&=r(\theta)\cos\theta-\mathfrak{N}\frac{r(\theta)\cos\theta+r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}},\\ x_2&=r(\theta)\sin\theta-\mathfrak{N}\frac{r(\theta)\sin\theta-r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}. \end{aligned} \tag{3-25}$$

It is easy to see that $\mathfrak{N}=0$ denotes the boundary $\partial\Omega$ and $\mathfrak{N}>0$ denotes the interior of Ω. Thus $(\mathfrak{N},\theta)$ is the desired local coordinate system.

Direct computations in [Guo and Wu 2017a] reveal that

$$\frac{\partial\theta}{\partial x_1}=\frac{MP}{P^3+Q\mathfrak{N}},\qquad \frac{\partial\mathfrak{N}}{\partial x_1}=-\frac{N}{P},\qquad \frac{\partial\theta}{\partial x_2}=\frac{NP}{P^3+Q\mathfrak{N}},\qquad \frac{\partial\mathfrak{N}}{\partial x_2}=\frac{M}{P}, \tag{3-26}$$

where

$$P=(r^2+r'^2)^{\frac12},\quad Q=rr''-r^2-2r'^2,\quad M=-r\sin\theta+r'\cos\theta,\quad N=r\cos\theta+r'\sin\theta. \tag{3-27}$$

Therefore, noting the fact that for C^2 convex domains the curvature satisfies

$$\kappa(\theta)=\frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{\frac32}}>0 \tag{3-28}$$

and the radius of curvature satisfies

$$R_\kappa(\theta)=\frac{1}{\kappa(\theta)}=\frac{(r^2+r'^2)^{\frac32}}{r^2+2r'^2-rr''}>0, \tag{3-29}$$

we define substitutions as follows:

Substitution 1: coordinate substitution. Let $(x_1,x_2)\to(\mathfrak{N},\theta)$ with $0\le\mathfrak{N}<R_{\min}$ for $R_{\min}=\min_\theta R_\kappa$ as

$$\begin{aligned} x_1&=r(\theta)\cos\theta-\mathfrak{N}\frac{r(\theta)\cos\theta+r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}},\\ x_2&=r(\theta)\sin\theta-\mathfrak{N}\frac{r(\theta)\sin\theta-r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}, \end{aligned} \tag{3-30}$$

and then (3-1) is transformed into

$$\begin{cases} \epsilon\left(v_1\dfrac{-r\cos\theta-r'\sin\theta}{(r^2+r'^2)^{\frac12}}+v_2\dfrac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)^{\frac12}}\right)\dfrac{\partial f^\epsilon}{\partial\mathfrak{N}}\\ \qquad+\epsilon\left(v_1\dfrac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)}+v_2\dfrac{r\cos\theta+r'\sin\theta}{(r^2+r'^2)}\right)\dfrac{1}{(1-\kappa\mathfrak{N})}\dfrac{\partial f^\epsilon}{\partial\theta}+\mathcal{L}[f^\epsilon]=\Gamma[f^\epsilon,f^\epsilon],\\ f^\epsilon(0,\theta,\vec{v})=\mathcal{P}^\epsilon[f^\epsilon](0,\theta,\vec{v})\quad\text{for }\vec{v}\cdot\vec{n}<0, \end{cases} \tag{3-31}$$

where

$$\vec{v}\cdot\vec{n}=v_1\frac{r\cos\theta+r'\sin\theta}{(r^2+r'^2)^{\frac12}}+v_2\frac{r\sin\theta-r'\cos\theta}{(r^2+r'^2)^{\frac12}}, \tag{3-32}$$

and

$$\mathcal{P}^{\epsilon}[f^{\epsilon}](0,\theta,\vec{v}) = \mu_b^{\epsilon}(\theta,\vec{v})\mu^{-\frac{1}{2}}(\vec{v})\int_{\vec{\mathfrak{u}}\cdot\vec{n}(\theta)>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})f^{\epsilon}(0,\theta,\vec{\mathfrak{u}})|\vec{\mathfrak{u}}\cdot\vec{n}(\theta)|\,d\vec{\mathfrak{u}}+\mu^{-\frac{1}{2}}(\vec{v})\big(\mu_b^{\epsilon}(\theta,\vec{v})-\mu(\vec{v})\big). \tag{3-33}$$

Substitution 2: velocity substitution. Define the orthogonal velocity substitution $\vec{v}=(v_1,v_2)\to\vec{\mathfrak{v}}=(v_\eta,v_\phi)$ as

$$\begin{aligned} v_1\frac{-r\cos\theta-r'\sin\theta}{(r^2+r'^2)^{\frac{1}{2}}}+v_2\frac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)^{\frac{1}{2}}}&=v_\eta,\\ v_1\frac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)^{\frac{1}{2}}}+v_2\frac{r\cos\theta+r'\sin\theta}{(r^2+r'^2)^{\frac{1}{2}}}&=v_\phi. \end{aligned} \tag{3-34}$$

Then using chain rule, we have

$$\frac{\partial}{\partial\theta}\to\frac{\partial}{\partial\theta}+\frac{\partial v_\eta}{\partial\theta}\frac{\partial}{\partial v_\eta}+\frac{\partial v_\phi}{\partial\theta}\frac{\partial}{\partial v_\phi}=\frac{\partial}{\partial\theta}-\kappa(r^2+r'^2)^{\frac{1}{2}}v_\phi\frac{\partial}{\partial v_\eta}+\kappa(r^2+r'^2)^{\frac{1}{2}}v_\eta\frac{\partial}{\partial v_\phi}. \tag{3-35}$$

The transport operator is

$$\vec{v}\cdot\nabla_x=v_\eta\frac{\partial}{\partial\mathfrak{N}}-\frac{v_\phi}{R_\kappa-\mathfrak{N}}\frac{R_\kappa}{(r^2+r'^2)^{\frac{1}{2}}}\frac{\partial}{\partial\theta}-\frac{v_\phi^2}{R_\kappa-\mathfrak{N}}\frac{\partial}{\partial v_\eta}+\frac{v_\eta v_\phi}{R_\kappa-\mathfrak{N}}\frac{\partial}{\partial v_\phi}. \tag{3-36}$$

Hence, (3-1) is transformed into

$$\begin{cases}\epsilon v_\eta\dfrac{\partial f^{\epsilon}}{\partial\mathfrak{N}}-\epsilon\dfrac{v_\phi}{R_\kappa-\mathfrak{N}}\dfrac{R_\kappa}{(r^2+r'^2)^{\frac{1}{2}}}\dfrac{\partial f^{\epsilon}}{\partial\theta}-\epsilon\dfrac{v_\phi^2}{R_\kappa-\mathfrak{N}}\dfrac{\partial f^{\epsilon}}{\partial v_\eta}+\epsilon\dfrac{v_\eta v_\phi}{R_\kappa-\mathfrak{N}}\dfrac{\partial f^{\epsilon}}{\partial v_\phi}+\mathcal{L}[f^{\epsilon}]=\Gamma[f^{\epsilon},f^{\epsilon}],\\ f^{\epsilon}(0,\theta,\vec{\mathfrak{v}})=\mathcal{P}^{\epsilon}[f^{\epsilon}](0,\theta,\vec{\mathfrak{v}})\quad\text{for } v_\eta>0,\end{cases} \tag{3-37}$$

where

$$\mathcal{P}^{\epsilon}[f^{\epsilon}](0,\theta,\vec{\mathfrak{v}})=\mu_b^{\epsilon}(\theta,\vec{\mathfrak{v}})\mu^{-\frac{1}{2}}(\vec{\mathfrak{v}})\int_{\vec{\mathfrak{u}}_\eta<0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}}_\eta)f^{\epsilon}(0,\theta,\vec{\mathfrak{u}}_\eta)|\vec{\mathfrak{u}}_\eta|\,d\vec{\mathfrak{u}}+\mu^{-\frac{1}{2}}(\vec{\mathfrak{v}})\big(\mu_b^{\epsilon}(\theta,\vec{\mathfrak{v}})-\mu(\vec{\mathfrak{v}})\big). \tag{3-38}$$

Substitution 3: scaling substitution. We define the rescaled variable $\eta=\mathfrak{N}/\epsilon$, which implies $\partial/\partial\mathfrak{N}=(1/\epsilon)(\partial/\partial\eta)$. Then, under the substitution $\mathfrak{N}\to\eta$, (3-1) is transformed into

$$\begin{cases}v_\eta\dfrac{\partial f^{\epsilon}}{\partial\eta}-\epsilon\dfrac{v_\phi}{R_\kappa-\epsilon\eta}\dfrac{R_\kappa}{(r^2+r'^2)^{\frac{1}{2}}}\dfrac{\partial f^{\epsilon}}{\partial\theta}-\epsilon\dfrac{v_\phi^2}{R_\kappa-\epsilon\eta}\dfrac{\partial f^{\epsilon}}{\partial v_\eta}+\epsilon\dfrac{v_\eta v_\phi}{R_\kappa-\epsilon\eta}\dfrac{\partial f^{\epsilon}}{\partial v_\phi}+\mathcal{L}[f^{\epsilon}]=\Gamma[f^{\epsilon},f^{\epsilon}],\\ f^{\epsilon}(0,\theta,\vec{\mathfrak{v}})=\mathcal{P}^{\epsilon}[f^{\epsilon}](0,\theta,\vec{\mathfrak{v}})\quad\text{for } v_\eta>0,\end{cases} \tag{3-39}$$

where

$$\mathcal{P}^{\epsilon}[f^{\epsilon}](0,\theta,\vec{\mathfrak{v}})=\mu_b^{\epsilon}(\theta,\vec{\mathfrak{v}})\mu^{-\frac{1}{2}}(\vec{\mathfrak{v}})\int_{\vec{\mathfrak{u}}_\eta<0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}}_\eta)f^{\epsilon}(0,\theta,\vec{\mathfrak{u}}_\eta)|\vec{\mathfrak{u}}_\eta|\,d\vec{\mathfrak{u}}+\mu^{-\frac{1}{2}}(\vec{\mathfrak{v}})\big(\mu_b^{\epsilon}(\theta,\vec{\mathfrak{v}})-\mu(\vec{\mathfrak{v}})\big). \tag{3-40}$$

We define the boundary layer expansion as

$$\mathscr{F}(\eta,\theta,\vec{\mathfrak{v}})\sim\sum_{k=1}^{2}\epsilon^k\mathscr{F}_k(\eta,\theta,\vec{\mathfrak{v}}), \tag{3-41}$$

where $\mathscr{F}_k$ can be defined by comparing the order of ϵ via plugging (3-41) into (3-39). Thus, in a neighborhood of the boundary, we have

$$v_\eta \frac{\partial \mathscr{F}_1}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_1}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_1}{\partial v_\phi}\right) + \mathcal{L}[\mathscr{F}_1] = 0, \tag{3-42}$$

$$v_\eta \frac{\partial \mathscr{F}_2}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_2}{\partial v_\phi}\right) + \mathcal{L}[\mathscr{F}_2] = 4\Gamma[F_1, \mathscr{F}_1] + \Gamma[\mathscr{F}_1, \mathscr{F}_1] + \frac{v_\phi}{R_\kappa - \epsilon\eta} \frac{R_\kappa}{(r^2 + r'^2)^{\frac{1}{2}}} \frac{\partial \mathscr{F}_1}{\partial \theta}. \tag{3-43}$$

3C. ***Expansion of boundary conditions.*** The bridge between the interior solution and boundary layer is the boundary condition

$$f^\epsilon(\vec{x}_0, \vec{v}) = \mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0, \vec{v}), \tag{3-44}$$

where

$$\mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0, \vec{v}) = \mu_b^\epsilon(\vec{x}_0, \vec{v})\mu^{-\frac{1}{2}}(\vec{v}) \int_{\vec{u}\cdot\vec{n}(\vec{x}_0)>0} \mu^{\frac{1}{2}}(\vec{u}) f^\epsilon(\vec{x}_0, \vec{u}) |\vec{u}\cdot\vec{n}(\vec{x}_0)| \,d\vec{u} + \mu^{-\frac{1}{2}}(\vec{v})\big(\mu_b^\epsilon(\vec{x}_0, \vec{v}) - \mu(\vec{v})\big). \tag{3-45}$$

Define

$$\mathcal{P}[f](\vec{x}_0, \vec{v}) = \mu^{\frac{1}{2}}(\vec{v}) \int_{\vec{u}\cdot\vec{n}(\vec{x}_0)>0} \mu^{\frac{1}{2}}(\vec{u}) f(\vec{x}_0, \vec{u}) |\vec{u}\cdot\vec{n}(\vec{x}_0)| \,d\vec{u}. \tag{3-46}$$

Plugging the combined expansion

$$f^\epsilon \sim \sum_{k=1}^{3} \epsilon^k F_k + \sum_{k=1}^{2} \epsilon^k \mathscr{F}_k \tag{3-47}$$

into the boundary condition and comparing the order of ϵ, we obtain

$$F_1 + \mathscr{F}_1 = \mathcal{P}[F_1 + \mathscr{F}_1] + \mu_1(\vec{x}_0, \vec{v}), \tag{3-48}$$

$$F_2 + \mathscr{F}_2 = \mathcal{P}[F_2 + \mathscr{F}_2] + \mu_1(\vec{x}_0, \vec{v}) \int_{\vec{u}\cdot\vec{n}(\vec{x}_0)>0} \mu^{\frac{1}{2}}(\vec{u})(F_1 + \mathscr{F}_1)|\vec{u}\cdot\vec{n}(\vec{x}_0)| \,d\vec{u} + \mu_2(\vec{x}_0, \vec{v}). \tag{3-49}$$

In particular, we do not further expand the boundary layer, so we directly require

$$F_3 = \mathcal{P}[F_3] + \mu_2(\vec{x}_0, \vec{v}) \int_{\vec{u}\cdot\vec{n}(\vec{x}_0)>0} \mu^{\frac{1}{2}}(\vec{u})(F_1 + \mathscr{F}_1)|\vec{u}\cdot\vec{n}(\vec{x}_0)| \,d\vec{u} + \mu_1(\vec{x}_0, \vec{v}) \int_{\vec{u}\cdot\vec{n}(\vec{x}_0)>0} \mu^{\frac{1}{2}}(\vec{u})(F_2 + \mathscr{F}_2)|\vec{u}\cdot\vec{n}(\vec{x}_0)| \,d\vec{u} + \mu_3(\vec{x}_0, \vec{v}). \tag{3-50}$$

These are the boundary conditions F_k and $\mathscr{F}_k$ need to satisfy.

3D. ***Matching procedure.*** Define the length of boundary layer $L = \epsilon^{-1/2}$. Also, set $\mathscr{R}[v_\eta, v_\phi] = (-v_\eta, v_\phi)$. We divide the construction of the asymptotic expansion into several steps for each $k \geq 1$:

Step 1: construction of F_1 and $\mathscr{F}_1$. A direct computation reveals that $F_1 = A_1 + B_1 + C_1$, where $B_1 = C_1 = 0$. Based on our expansion,

$$\mu_1 = \mu^{\frac{1}{2}}\left(\rho_{b,1} + \vec{u}_{b,1}\cdot\vec{v} + \theta_{b,1}\frac{|\vec{v}|^2 - 2}{2}\right). \tag{3-51}$$

Define

$$F_1=\mu^{\frac{1}{2}}\left(\rho_1+\vec{u}_1\cdot\vec{v}+\theta_1\frac{|\vec{v}|^2-2}{2}\right), \tag{3-52}$$

satisfying the Navier–Stokes–Fourier system as (1-45) with the boundary condition

$$F_1(\vec{x}_0,\vec{v})=\mu_1(\theta,\vec{\mathfrak{v}})+M_1(\vec{x}_0)\mu^{\frac{1}{2}}(\vec{v}). \tag{3-53}$$

Here $M_1(\vec{x}_0)$ is such that the Boussinesq relation

$$\rho_1+\theta_1=\text{constant} \tag{3-54}$$

is satisfied. Note that this constant is determined by the normalization condition

$$\int_\Omega\int_{\mathbb{R}^2}F_1(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\mathrm{d}\vec{v}\,\mathrm{d}\vec{x}=0, \tag{3-55}$$

and we are able to add $M_1(\vec{x}_0)$ freely since $\mu^{1/2}=\mathcal{P}[\mu^{1/2}]$.

Then based on the compatibility condition of μ_1, which is

$$\int_{\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})\mu_1(\vec{x}_0,\vec{\mathfrak{u}})|\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathfrak{u}}=0, \tag{3-56}$$

we naturally obtain $\mathcal{P}[F_1]=M_1\mu^{1/2}$, which means

$$F_1=\mathcal{P}[F_1]+\mu_1\quad\text{on }\partial\Omega. \tag{3-57}$$

Therefore, compared with (3-48), it is not necessary to introduce the boundary layer at this order and we simply take $\mathscr{F}_1=0$.

<u>Step 2</u>: construction of F_2 and $\mathscr{F}_2$. Define $F_2=A_2+B_2+C_2$, where B_2 and C_2 can be uniquely determined following previous analysis, and

$$A_2=\mu^{\frac{1}{2}}\left(\rho_2+\vec{u}_2\cdot\vec{v}+\theta_2\frac{|\vec{v}|^2-2}{2}\right), \tag{3-58}$$

satisfying a more complicated fluid-type equation as in [Sone 2002, page 92]. On the other hand, $\mathscr{F}_2$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases}v_\eta\dfrac{\partial\mathscr{F}_2}{\partial\eta}-\dfrac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\dfrac{\partial\mathscr{F}_2}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathscr{F}_2}{\partial v_\phi}\right)+\mathcal{L}[\mathscr{F}_2]=0 & \text{for }(\eta,\theta,\vec{\mathfrak{v}})\in[0,L]\times[-\pi,\pi)\times\mathbb{R}^2,\\ \mathscr{F}_2(0,\theta,\vec{\mathfrak{v}})=h(\theta,\vec{\mathfrak{v}})-\tilde{h}(\theta,\vec{\mathfrak{v}}) & \text{for }v_\eta>0,\\ \mathscr{F}_2(L,\theta,\vec{\mathfrak{v}})=\mathscr{F}_2(L,\theta,\mathscr{R}[\vec{\mathfrak{v}}]),\end{cases} \tag{3-59}$$

with the in-flow boundary data

$$h(\theta,\vec{\mathfrak{v}})=\mu_1(\vec{x}_0,\vec{v})\int_{\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})(F_1+\mathscr{F}_1)|\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathfrak{u}}+\mu_2(\vec{x}_0,\vec{v})-((B_2+C_2)-\mathcal{P}[B_2+C_2]). \tag{3-60}$$

Based on Theorem 5.1, there exists a unique

$$\tilde{h}(\theta,\vec{\mathfrak{v}})=\mu^{\frac{1}{2}}\left(\widetilde{D}_0(\theta)+\widetilde{D}_1(\theta)v_\eta+\widetilde{D}_2(\theta)v_\phi+\widetilde{D}_3(\theta)\frac{|\vec{\mathfrak{v}}|^2-2}{2}\right) \tag{3-61}$$

such that (3-59) is well-posed and the solution decays exponentially fast. In particular, $\widetilde{D}_1=0$. Then we further require that A_2 satisfies the boundary condition

$$A_2(\vec{x}_0,\vec{v})=\tilde{h}(\theta,\vec{\mathfrak{v}})+M_2(\vec{x}_0)\mu^{\frac{1}{2}}(\vec{v}). \tag{3-62}$$

Here, the constant $M_2(\vec{x}_0)$ is chosen to enforce the Boussinesq relation

$$P_2-(\rho_2+\theta_2+\rho_1\theta_1)=0, \tag{3-63}$$

where P_2 is the pressure in (1-45). Similar to the construction of F_1, we can choose the constant to satisfy the normalization condition

$$\int_\Omega\int_{\mathbb{R}^2}(F_2+\mathscr{F}_2)(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\vec{v}\,\mathrm{d}\vec{x}=0. \tag{3-64}$$

Also, the construction implies that at the boundary, we have

$$\begin{aligned}A_2+\mathscr{F}_2&=M_2\mu^{\frac{1}{2}}+h\\&=M_2\mu^{\frac{1}{2}}+\mu_1\int_{\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})(F_1+\mathscr{F}_1)|\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathfrak{u}}+\mu_2-((B_2+C_2)-\mathcal{P}[B_2+C_2]).\end{aligned} \tag{3-65}$$

Comparing this with the desired boundary expansion (3-49), i.e.,

$$A_2+B_2+C_2+\mathscr{F}_2=\mathcal{P}[A_2+B_2+C_2+\mathscr{F}_2]+\mu_1\int_{\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})(F_1+\mathscr{F}_1)|\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathfrak{u}}+\mu_2, \tag{3-66}$$

we only need to verify that

$$\mathcal{P}[A_2+\mathscr{F}_2]=M_2\mu^{\frac{1}{2}}. \tag{3-67}$$

Equation (3-59) implies the zero mass-flux condition of $\mathscr{F}_2$ as

$$\int_{\mathbb{R}^2}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})\mathscr{F}_2(\vec{x},\vec{\mathfrak{u}})(\vec{\mathfrak{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathfrak{u}}=0, \tag{3-68}$$

and we know μ_1 and μ_2 satisfy the compatibility conditions

$$\int_{\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})\mu_1(\vec{x}_0,\vec{\mathfrak{u}})|\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathfrak{u}}=\int_{\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})\mu_2(\vec{x}_0,\vec{\mathfrak{u}})|\vec{\mathfrak{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathfrak{u}}=0. \tag{3-69}$$

Then based on (3-62), we have

$$\begin{aligned}\mathcal{P}[A_2+\mathscr{F}_2]&=\mu^{\frac{1}{2}}\int_{\vec{\mathfrak{u}}\cdot\vec{n}>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})A_2(\vec{x},\vec{\mathfrak{u}})(\vec{\mathfrak{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathfrak{u}}+\mu^{\frac{1}{2}}\int_{\vec{\mathfrak{u}}\cdot\vec{n}>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})\mathscr{F}_2(\vec{x},\vec{\mathfrak{u}})(\vec{\mathfrak{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathfrak{u}}\\&=M_2\mu^{\frac{1}{2}}+\mu^{\frac{1}{2}}\int_{\vec{\mathfrak{u}}\cdot\vec{n}>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})\tilde{h}(\vec{x},\vec{\mathfrak{u}})(\vec{\mathfrak{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathfrak{u}}+\mu^{\frac{1}{2}}\int_{\vec{\mathfrak{u}}\cdot\vec{n}>0}\mu^{\frac{1}{2}}(\vec{\mathfrak{u}})\mathscr{F}_2(\vec{x},\vec{\mathfrak{u}})(\vec{\mathfrak{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathfrak{u}}.\end{aligned} \tag{3-70}$$

Using (3-68) and (3-59), we know

$$\begin{aligned}\mathcal{P}[A_2+\mathscr{F}_2] &= M_2\mu^{\frac{1}{2}}+\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}>0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})\tilde{h}(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}-\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}<0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})\mathscr{F}_2(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}\\ &= M_2\mu^{\frac{1}{2}}+\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}>0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})\tilde{h}(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}-\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}<0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})(h-\tilde{h})(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}.\end{aligned}\tag{3-71}$$

Then direct computation reveals that

$$\begin{aligned}\mathcal{P}[A_2+\mathscr{F}_2] &= M_2\mu^{\frac{1}{2}}+\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}>0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})\tilde{h}(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}\\ &\qquad -\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}<0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})h(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}+\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}<0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})\tilde{h}(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}\\ &= M_2\mu^{\frac{1}{2}}+\mu^{\frac{1}{2}}\int_{\mathbb{R}^2}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})\tilde{h}(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}-\mu^{\frac{1}{2}}\int_{\vec{\mathrm{u}}\cdot\vec{n}<0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})h(\vec{x},\vec{\mathrm{u}})(\vec{\mathrm{u}}\cdot\vec{n})\,\mathrm{d}\vec{\mathrm{u}}.\end{aligned}\tag{3-72}$$

Finally, using (3-60), (3-69), and $\widetilde{D}_1=0$, we obtain

$$\mathcal{P}[A_2+\mathscr{F}_2]=M_2\mu^{\frac{1}{2}}+\widetilde{D}_1-0=M_2\mu^{\frac{1}{2}}.\tag{3-73}$$

F_3 can be defined in a similar fashion that satisfies an even more complicated fluid-type system; see [Sone 2002, page 92].

4. Remainder estimates

We consider the linearized stationary Boltzmann equation

$$\begin{cases}\epsilon\vec{v}\cdot\nabla_x f+\mathcal{L}[f]=S(\vec{x},\vec{v}) & \text{in } \Omega,\\ f(\vec{x}_0,\vec{v})=\mathcal{P}[f](\vec{x}_0,\vec{v})+h(\vec{x}_0,\vec{v}) & \text{for } \vec{x}_0\in\partial\Omega \text{ and } \vec{v}\cdot\vec{n}<0,\end{cases}\tag{4-1}$$

where

$$\mathcal{P}[f](\vec{x}_0,\vec{v})=\mu^{\frac{1}{2}}(\vec{v})\int_{\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac{1}{2}}(\vec{\mathrm{u}})f(\vec{x}_0,\vec{\mathrm{u}})|\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathrm{u}},\tag{4-2}$$

provided we have the compatibility conditions

$$\int_{\Omega\times\mathbb{R}^2}S(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\vec{v}\,\mathrm{d}\vec{x}=0,\qquad \int_{\gamma_-}h(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\gamma=0.\tag{4-3}$$

It is easy to see if f is a solution to (4-1), then $f+C\mu^{1/2}$ is also a solution for arbitrary $C\in\mathbb{R}$. Hence, we require that the solution satisfy the normalization condition

$$\int_{\Omega\times\mathbb{R}^2}f(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\vec{v}\,\mathrm{d}\vec{x}=0.\tag{4-4}$$

Our analysis is based on the ideas in [Esposito et al. 2013; 2018; Guo 2010]. Since the well-posedness of (4-1) is standard, we will focus on the a priori estimates here.

4A. L^{2m} estimates.

Lemma 4.1. *Define the near-grazing set of γ_+ or γ_- as*

$$\gamma_{\pm}^{\delta}=\left\{(\vec{x},\vec{v})\in\gamma_{\pm}:|\vec{n}(\vec{x})\cdot\vec{v}|\leq\delta \text{ or } |\vec{v}|\geq\frac{1}{\delta} \text{ or } |\vec{v}|\leq\delta\right\}. \tag{4-5}$$

Then

$$|f\mathbf{1}_{\gamma_{\pm}\backslash\gamma_{\pm}^{\delta}}|_{L^1}\leq C(\delta)\big(\|f\|_{L^1}+\|\vec{v}\cdot\nabla_x f\|_{L^1}\big). \tag{4-6}$$

Proof. See [Esposito et al. 2013, Lemma 2.1]. □

Lemma 4.2 (Green's identity). *Assume $f(\vec{x},\vec{v}),\ g(\vec{x},\vec{v})\in L^2(\Omega\times\mathbb{R}^2)$ and $\vec{v}\cdot\nabla_x f,\ \vec{v}\cdot\nabla_x g\in L^2(\Omega\times\mathbb{R}^2)$ with $f,\ g\in L^2(\gamma)$. Then*

$$\iint_{\Omega\times\mathbb{R}^2}\big((\vec{v}\cdot\nabla_x f)g+(\vec{v}\cdot\nabla_x g)f\big)\mathrm{d}\vec{x}\,\mathrm{d}\vec{v}=\int_{\gamma_+}fg\,\mathrm{d}\gamma-\int_{\gamma_-}fg\,\mathrm{d}\gamma. \tag{4-7}$$

Proof. See [Esposito et al. 2013, Lemma 2.2]. □

Here we consider $m\in\mathbb{N}$ and $m\geq 2$. Let $o(1)$ denote a sufficiently small constant.

Lemma 4.3. *The solution $f(\vec{x},\vec{v})$ to (4-1) satisfies the estimate*

$$\epsilon\|\mathbb{P}[f]\|_{L^{2m}}\leq C\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-8}$$

Proof. Apply Green's identity in Lemma 4.2 to the solution of (4-1). Then for any $\psi\in L^2(\Omega\times\mathbb{R}^2)$ satisfying $\vec{v}\cdot\nabla_x\psi\in L^2(\Omega\times\mathbb{R}^2)$ and $\psi\in L^2(\gamma)$, we have

$$\epsilon\int_{\gamma_+}f\psi\,\mathrm{d}\gamma-\epsilon\int_{\gamma_-}f\psi\,\mathrm{d}\gamma-\epsilon\int_{\Omega\times\mathbb{R}^2}(\vec{v}\cdot\nabla_x\psi)f=-\int_{\Omega\times\mathbb{R}^2}\psi\mathcal{L}[(\mathbb{I}-\mathbb{P})[f]]+\int_{\Omega\times\mathbb{R}^2}S\psi. \tag{4-9}$$

Since

$$\mathbb{P}[f]=\mu^{\frac{1}{2}}\left(a+\vec{v}\cdot\vec{b}+\frac{|\vec{v}|^2-2}{2}c\right), \tag{4-10}$$

our goal is to choose a particular test function ψ to estimate a, $\vec{b}$ and c. The main idea is to design the test function such that $-\epsilon\int_{\Omega\times\mathbb{R}^2}(\vec{v}\cdot\nabla_x\psi)f$ is roughly the L^{2m} norm of a, $\vec{b}$ and c. Also, we try to use the symmetry of $\vec{v}$ to eliminate other terms as much as possible.

Step 1: estimates of c. We choose the test function

$$\psi=\psi_c=\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_c)(\vec{v}\cdot\nabla_x\phi_c(\vec{x})), \tag{4-11}$$

where

$$\begin{cases}-\Delta_x\phi_c=c^{2m-1}(\vec{x}) & \text{in } \Omega,\\ \phi_c=0 & \text{on } \partial\Omega,\end{cases} \tag{4-12}$$

and β_c is a real number to be determined later. Based on the standard elliptic estimates (see [Krylov 2008]), we have

$$\|\phi_c\|_{W^{2,2m/(2m-1)}}\leq C\|c^{2m-1}\|_{L^{2m/(2m-1)}}\leq C\|c\|_{L^{2m}}^{2m-1}. \tag{4-13}$$

Also, we know

$$\|\psi_c\|_{L^2} \leq C\|\phi_c\|_{H^1} \leq C\|\phi_c\|_{W^{2,2m/(2m-1)}} \leq C\|c\|_{L^{2m}}^{2m-1}, \tag{4-14}$$

$$\|\psi_c\|_{L^{2m/(2m-1)}} \leq C\|\phi_c\|_{W^{1,2m/(2m-1)}} \leq C\|c\|_{L^{2m}}^{2m-1}. \tag{4-15}$$

With the choice of (4-11), the right-hand side (RHS) of (4-9) is bounded by

$$\text{RHS} \leq C\|c\|_{L^{2m}}^{2m-1}(\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2} + \|S\|_{L^2}). \tag{4-16}$$

We will choose β_c such that

$$\int_{\mathbb{R}^2} \mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2 - \beta_c)v_i^2 \,\mathrm{d}\vec{v} = 0 \quad \text{for } i = 1, 2. \tag{4-17}$$

The left-hand side (LHS) of (4-9) takes the form

$$\begin{aligned}\text{LHS} = \epsilon \int_{\partial\Omega\times\mathbb{R}^2} &\mathbf{1}_{\gamma_+}(1-\mathcal{P})[f]\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_c)\biggl(\sum_{i=1}^{2} v_i\partial_i\phi_c\biggr)(\vec{v}\cdot\vec{n}) \\ &+\epsilon \int_{\partial\Omega\times\mathbb{R}^2} \mathbf{1}_{\gamma_-}h\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_c)\biggl(\sum_{i=1}^{2} v_i\partial_i\phi_c\biggr)(\vec{v}\cdot\vec{n}) \\ &-\epsilon\sum_{i=1}^{2}\int_{\mathbb{R}^2}\mu(\vec{v})|v_i|^2(|\vec{v}|^2-\beta_c)\frac{|\vec{v}|^2-2}{2}\,\mathrm{d}\vec{v}\int_{\Omega}c(\vec{x})\partial_{ii}\phi_c(\vec{x})\,\mathrm{d}\vec{x} \\ &-\epsilon\int_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f]\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_c)\biggl(\sum_{i,j=1}^{2} v_iv_j\partial_{ij}\phi_c\biggr).\end{aligned} \tag{4-18}$$

Since

$$\int_{\mathbb{R}^2}\mu(\vec{v})|v_i|^2(|\vec{v}|^2-\beta_c)\frac{|\vec{v}|^2-2}{2}\,\mathrm{d}\vec{v} = C, \tag{4-19}$$

we have

$$\begin{aligned}\epsilon&\left|\int_{\Omega}\Delta_x\phi_c(\vec{x})c(\vec{x})\,\mathrm{d}\vec{x}\right| \\ &\leq C\|c\|_{L^{2m}}^{2m-1}\bigl(\epsilon|(1-\mathcal{P})[f]|_{L^m_+} + \|(\mathbb{I}-\mathbb{P})[f]\|_{L^2} + \epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}} + \|S\|_{L^2} + \epsilon|h|_{L^m_-}\bigr),\end{aligned} \tag{4-20}$$

where we have used the elliptic estimates, the Sobolev embedding theorem, Hölder's inequality $L^m/L^{m/(m-1)}$ and the trace estimate

$$|\nabla_x\phi_c|_{L^{m/(m-1)}} \leq C|\nabla_x\phi_c|_{W^{1/(2m),m/(m-1)}} \leq C\|\nabla_x\phi_c\|_{W^{1,m/(m-1)}} \leq C\|\phi_c\|_{W^{2,2m/(2m-1)}} \leq C\|c\|_{L^{2m}}^{2m-1}. \tag{4-21}$$

Since $-\Delta_x\phi_c = c^{2m-1}$, we know

$$\epsilon\|c\|_{L^{2m}}^{2m} \leq C\|c\|_{L^{2m}}^{2m-1}\bigl(\epsilon|(1-\mathcal{P})[f]|_{L^m_+} + \|(\mathbb{I}-\mathbb{P})[f]\|_{L^2} + \epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}} + \|S\|_{L^2} + \epsilon|h|_{L^m_-}\bigr), \tag{4-22}$$

which further implies

$$\epsilon\|c\|_{L^{2m}} \leq C\bigl(\epsilon|(1-\mathcal{P})[f]|_{L^m_+} + \|(\mathbb{I}-\mathbb{P})[f]\|_{L^2} + \epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}} + \|S\|_{L^2} + \epsilon|h|_{L^m_-}\bigr). \tag{4-23}$$

Step 2: estimates of $\vec{b}$. We further divide this step into several substeps:

Step 2.1: estimates of $(\partial_{ij}\Delta_x^{-1}b_j)b_i$ for $i,\ j=1,2$. We choose the test function

$$\psi=\psi_b^{i,j}=\mu^{\frac{1}{2}}(\vec{v})(v_i^2-\beta_b)\partial_j\phi_b^j, \tag{4-24}$$

where

$$\begin{cases}-\Delta_x\phi_b^j=b_j^{2m-1}(\vec{x}) & \text{in } \Omega,\\ \phi_b^j=0 & \text{on } \partial\Omega,\end{cases} \tag{4-25}$$

and β_b is a real number to be determined later. Based on the standard elliptic estimates (see [Krylov 2008]), we have

$$\|\phi_b^{i,j}\|_{W^{2,2m/(2m-1)}}\leq C\|b_j^{2m-1}\|_{L^{2m/(2m-1)}}\leq C\|b_j\|_{L^{2m}}^{2m-1}. \tag{4-26}$$

Also, we know

$$\|\psi_b^{i,j}\|_{L^2}\leq C\|\phi_b^{i,j}\|_{H^1}\leq C\|\phi_b^{i,j}\|_{W^{2,2m/(2m-1)}}\leq C\|b_j\|_{L^{2m}}^{2m-1}, \tag{4-27}$$

$$\|\psi_b^{i,j}\|_{L^{2m/(2m-1)}}\leq C\|\phi_b^{i,j}\|_{W^{1,2m/(2m-1)}}\leq C\|b_j\|_{L^{2m}}^{2m-1}. \tag{4-28}$$

With the choice of (4-24), the right-hand side (RHS) of (4-9) is bounded by

$$\text{RHS}\leq C\|\vec{b}\|_{L^{2m}}^{2m-1}(\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\|S\|_{L^2}). \tag{4-29}$$

We will choose β_b such that

$$\int_{\mathbb{R}^2}\mu(\vec{v})(|v_i|^2-\beta_b)\,\mathrm{d}\vec{v}=0\quad\text{for } i=1,2. \tag{4-30}$$

Hence, the left-hand side (LHS) of (4-9) takes the form

$$\begin{aligned}\text{LHS}=\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_+}(1-\mathcal{P})[f]\mu^{\frac{1}{2}}(\vec{v})(v_i^2-\beta_b)\partial_j\phi_b^j(\vec{v}\cdot\vec{n})+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_-}h\mu^{\frac{1}{2}}(\vec{v})(v_i^2-\beta_b)\partial_j\phi_b^j(\vec{v}\cdot\vec{n})\\ -\epsilon\sum_{l=1}^{2}\int_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_l^2(v_i^2-\beta_b)\partial_{lj}\phi_b^jb_l-\epsilon\sum_{l=1}^{2}\int_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f]\mu^{\frac{1}{2}}(\vec{v})(v_i^2-\beta_b)v_l\partial_{lj}\phi_b^j.\end{aligned}$$

For such β_b and any $i\neq l$, we can directly compute

$$\int_{\mathbb{R}^2}\mu(\vec{v})(|v_i|^2-\beta_b)v_l^2\,\mathrm{d}\vec{v}=0, \tag{4-31}$$

$$\int_{\mathbb{R}^2}\mu(\vec{v})(|v_i|^2-\beta_b)v_i^2\,\mathrm{d}\vec{v}=C\neq 0. \tag{4-32}$$

Then we deduce

$$\begin{aligned}&-\epsilon\sum_{l=1}^{2}\int_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_l^2(v_i^2-\beta_b)\partial_{lj}\phi_b^jb_i\\ &\qquad=-\epsilon\int_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_i^2(v_i^2-\beta_b)\partial_{ij}\phi_b^jb_l-\epsilon\sum_{l\neq i}\int_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_l^2(v_i^2-\beta_b)\partial_{lj}\phi_b^jb_l\\ &\qquad=C\int_{\Omega}(\partial_{ij}\Delta_x^{-1}b_j)b_i.\end{aligned} \tag{4-33}$$

Hence, by (4-23), we may estimate

$$\epsilon\left|\int_\Omega(\partial_{ij}\Delta_x^{-1}b_j)b_i\right| \le C\|\vec{b}\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-34}$$

Step 2.2: estimates of $(\partial_{jj}\Delta_x^{-1}b_i)b_i$ for $i\neq j$. We choose the test function

$$\psi=\mu^{\frac{1}{2}}(\vec{v})|\vec{v}|^2v_iv_j\partial_j\phi_b^i,\quad i\neq j. \tag{4-35}$$

The right-hand side (RHS) of (4-9) is still bounded by

$$\text{RHS}\le C\|\vec{b}\|_{L^{2m}}^{2m-1}(\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\|S\|_{L^2}). \tag{4-36}$$

Hence, the left-hand side (LHS) of (4-9) takes the form

$$\begin{aligned}\text{LHS}=\epsilon\int_{\partial\Omega\times\mathbb{R}^2}&\mathbf{1}_{\gamma_+}(1-\mathcal{P})[f]\mu^{\frac{1}{2}}(\vec{v})|\vec{v}|^2v_iv_j\partial_j\phi_b^i(\vec{v}\cdot\vec{n})+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_-}h\mu^{\frac{1}{2}}(\vec{v})|\vec{v}|^2v_iv_j\partial_j\phi_b^i(\vec{v}\cdot\vec{n})\\ &-\epsilon\int_{\Omega\times\mathbb{R}^2}\mu(\vec{v})|\vec{v}|^2v_i^2v_j^2(\partial_{ij}\phi_b^ib_j+\partial_{jj}\phi_b^ib_i)-\epsilon\sum_{l=1}^2\int_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f]\mu^{\frac{1}{2}}(\vec{v})|\vec{v}|^2v_iv_jv_l\partial_{lj}\phi_b^i.\end{aligned} \tag{4-37}$$

Then we deduce

$$-\epsilon\int_{\Omega\times\mathbb{R}^2}\mu(\vec{v})|\vec{v}|^2v_i^2v_j^2(\partial_{ij}\phi_b^ib_j+\partial_{jj}\phi_b^ib_i)=C\bigg(\int_\Omega(\partial_{ij}\Delta_x^{-1}b_i)b_j+\int_\Omega(\partial_{jj}\Delta_x^{-1}b_i)b_i\bigg). \tag{4-38}$$

Hence, we may estimate, for $i\neq j$,

$$\begin{aligned}\epsilon\left|\int_\Omega(\partial_{jj}\Delta_x^{-1}b_i)b_i\right|\le C\|\vec{b}\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big)\\ +C\epsilon\left|\int_\Omega(\partial_{ij}\Delta_x^{-1}b_i)b_j\right|,\end{aligned} \tag{4-39}$$

which implies

$$\epsilon\left|\int_\Omega(\partial_{jj}\Delta_x^{-1}b_i)b_i\right| \le C\|\vec{b}\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-40}$$

Moreover, by (4-34), for $i=j=1,2$,

$$\epsilon\left|\int_\Omega(\partial_{jj}\Delta_x^{-1}b_j)b_j\right| \le C\|\vec{b}\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-41}$$

Step 2.3: synthesis. Summarizing (4-40) and (4-41), we may sum up over $j=1,2$ to obtain, for any $i=1,2$,

$$\epsilon\|b_i\|_{L^{2m}}^{2m}\le C\|\vec{b}\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big), \tag{4-42}$$

which further implies

$$\epsilon\|\vec{b}\|_{L^{2m}}^{2m} \le C\|\vec{b}\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-43}$$

Then we have

$$\epsilon\|\vec{b}\|_{L^{2m}} \le C\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-44}$$

Step 3: estimates of a. We choose the test function

$$\psi=\psi_a=\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_a)(\vec{v}\cdot\nabla_x\phi_a(\vec{x})), \tag{4-45}$$

where

$$\begin{cases} -\Delta_x\phi_a = a^{2m-1}(\vec{x})-\dfrac{1}{|\Omega|}\displaystyle\int_\Omega a^{2m-1}(\vec{x})\,\mathrm{d}\vec{x} & \text{in } \Omega,\\ \dfrac{\partial\phi_a}{\partial\vec{n}}=0 & \text{on } \partial\Omega, \end{cases} \tag{4-46}$$

and β_a is a real number to be determined later. Based on the standard elliptic estimates (see [Krylov 2008]) with

$$\int_\Omega\left(a^{2m-1}(\vec{x})-\frac{1}{|\Omega|}\int_\Omega a^{2m-1}(\vec{x})\,\mathrm{d}\vec{x}\right)\mathrm{d}\vec{x}=0, \tag{4-47}$$

we have

$$\|\phi_a\|_{W^{2,2m/(2m-1)}} \le C\|a^{2m-1}\|_{L^{2m/(2m-1)}} \le C\|a\|_{L^{2m}}^{2m-1}. \tag{4-48}$$

Also, we know

$$\|\psi_a\|_{L^2} \le C\|\phi_a\|_{H^1} \le C\|\phi_a\|_{W^{2,2m/(2m-1)}} \le C\|a\|_{L^{2m}}^{2m-1}, \tag{4-49}$$

$$\|\psi_a\|_{L^{2m/(2m-1)}} \le C\|\phi_a\|_{W^{1,2m/(2m-1)}} \le C\|a\|_{L^{2m}}^{2m-1}. \tag{4-50}$$

With the choice of (4-45), the right-hand side (RHS) of (4-9) is bounded by

$$\text{RHS} \le C\|a\|_{L^{2m}}^{2m-1}(\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\|S\|_{L^2}). \tag{4-51}$$

We will choose β_a such that

$$\int_{\mathbb{R}^2}\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_a)\frac{|\vec{v}|^2-2}{2}v_i^2\,\mathrm{d}\vec{v}=0 \quad \text{for } i=1,2. \tag{4-52}$$

The left-hand side (LHS) of (4-9) takes the form

$$\begin{aligned} \text{LHS} = {} & \epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_+}(1-\mathcal{P})[f]\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_a)\left(\sum_{i=1}^2 v_i\partial_i\phi_a\right)(\vec{v}\cdot\vec{n})\\ & +\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_-}h\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_a)\left(\sum_{i=1}^2 v_i\partial_i\phi_a\right)(\vec{v}\cdot\vec{n})\\ & -\sum_{i=1}^2\epsilon\int_{\mathbb{R}^2}\mu(\vec{v})|v_i|^2(|\vec{v}|^2-\beta_a)\,\mathrm{d}\vec{v}\int_\Omega a(\vec{x})\partial_{ii}\phi_a(\vec{x})\,\mathrm{d}\vec{x}\\ & -\epsilon\int_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f]\mu^{\frac{1}{2}}(\vec{v})(|\vec{v}|^2-\beta_a)\left(\sum_{i,j=1}^2 v_iv_j\partial_{ij}\phi_a\right). \end{aligned} \tag{4-53}$$

Since

$$\int_{\mathbb{R}^2} \mu^{\frac{1}{2}}(\vec{v})|v_i|^2(|\vec{v}|^2-\beta_a)\,\mathrm{d}\vec{v} = C, \tag{4-54}$$

we have

$$-\epsilon\int_\Omega \Delta_x\phi_a(\vec{x})a(\vec{x})\,\mathrm{d}\vec{x} \\ \leq C\|a\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-55}$$

The normalization condition (4-4) implies

$$\int_\Omega a(\vec{x})\,\mathrm{d}\vec{x} = \int_{\Omega\times\mathbb{R}^2} f(\vec{x},\vec{v})\mu^{\frac{1}{2}}(\vec{v})\,\mathrm{d}\vec{v}\mathrm{d}\vec{x} = 0. \tag{4-56}$$

Since $-\Delta_x\phi_a = a^{2m-1}-(1/|\Omega|)\int_\Omega a^{2m-1}$, we know

$$\epsilon\|a\|_{L^{2m}}^{2m} \leq C\|a\|_{L^{2m}}^{2m-1}\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-57}$$

This implies

$$\epsilon\|a\|_{L^{2m}} \leq C\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-58}$$

<u>Step 4</u>: synthesis. Collecting (4-23), (4-44) and (4-58), we deduce

$$\epsilon\|\mathbb{P}[f]\|_{L^{2m}} \leq C\big(\epsilon|(1-\mathcal{P})[f]|_{L^m_+}+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+\epsilon\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}+\|S\|_{L^2}+\epsilon|h|_{L^m_-}\big). \tag{4-59}$$

This completes our proof. □

Theorem 4.4. *The solution $f(\vec{x},\vec{v})$ to (4-1) satisfies the estimate*

$$\frac{1}{\epsilon^{\frac{1}{2}}}|(1-\mathcal{P})[f]|_{L^2_+}+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2_\nu}+\|\mathbb{P}[f]\|_{L^{2m}} \\ \leq C\bigg(o(1)\epsilon^{\frac{1}{m}}(\|f\|_{L^\infty}+|f|_{L^\infty_+})+\frac{1}{\epsilon^2}\|\mathbb{P}[S]\|_{L^{\frac{2m}{2m-1}}}+\frac{1}{\epsilon}\|S\|_{L^2}+|h|_{L^m_-}+\frac{1}{\epsilon}|h|_{L^2_-}\bigg). \tag{4-60}$$

Proof. We divide it into several steps:

<u>Step 1</u>: energy estimate. Multiplying by f on both sides of (4-1) and applying Green's identity we get

$$\frac{\epsilon}{2}|f|_{L^2_+}^2+\langle\mathcal{L}[f],f\rangle = \frac{\epsilon}{2}|\mathcal{P}[f]+h|_{L^2_-}^2+\int_{\Omega\times\mathbb{R}^2} fS. \tag{4-61}$$

Considering the fact that

$$|f|_{L^2_+}^2-|\mathcal{P}[f]|_{L^2_+}^2 = |(1-\mathcal{P})[f]|_{L^2_+}^2, \tag{4-62}$$

we deduce from the spectral gap of $\mathcal{L}$ and Cauchy's inequality that

$$\frac{\epsilon}{2}|(1-\mathcal{P})[f]|_{L^2_+}^2+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2_\nu}^2 \leq \eta\epsilon^2|\mathcal{P}[f]|_{L^2_-}^2+\bigg(1+\frac{1}{\eta}\bigg)|h|_{L^2_-}^2+\int_{\Omega\times\mathbb{R}^2} fS. \tag{4-63}$$

<u>Step 2</u>: estimate of $|\mathcal{P}[f]|_{L^2}$. Multiplying by f on both sides of (4-1), we have

$$\epsilon\vec{v}\cdot\nabla_x(f^2) = -2f\mathcal{L}[f]+2fS. \tag{4-64}$$

Taking the absolute value and integrating (4-64) over $\Omega\times\mathbb{R}^2$, we deduce

$$\|\vec{v}\cdot\nabla_x(f^2)\|_1\le\frac{1}{\epsilon}\bigg(\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2+\int_{\Omega\times\mathbb{R}^2}fS\bigg).\tag{4-65}$$

On the other hand, by Lemma 4.1, for any $\gamma\backslash\gamma^\delta$ away from γ_0, we have

$$|\mathbf{1}_{\gamma\backslash\gamma^\delta}f|_2^2\le C(\delta)\big(\|f\|_{L^2}^2+\|\vec{v}\cdot\nabla_x(f^2)\|_1\big).\tag{4-66}$$

Based on the definition, we can rewrite $\mathcal{P}f=z_\gamma(\vec{x})\mu^{1/2}$ for a suitable function $z_\gamma(\vec{x})$ and for δ small, and we deduce

$$\begin{aligned}|\mathcal{P}[\mathbf{1}_{\gamma\backslash\gamma^\delta}f]|_2^2&=\int_{\partial\Omega}|z_\gamma(\vec{x})|^2\bigg(\int_{|\vec{v}\cdot\vec{n}(\vec{x})|\ge\delta,\delta\le|\vec{v}|\le\frac{1}{\delta}}\mu(\vec{v})|\vec{v}\cdot\vec{n}(\vec{x})|\,\mathrm{d}\vec{v}\bigg)\mathrm{d}\vec{x}\\&\ge\frac{1}{2}\bigg(\int_{\partial\Omega}|z_\gamma(\vec{x})|^2\,\mathrm{d}\vec{x}\bigg)\bigg(\int_{\mathbb{R}^2}\mu(\vec{v})|\vec{v}\cdot\vec{n}(\vec{x})|\,\mathrm{d}\vec{v}\bigg)=\frac{1}{2}|\mathcal{P}[f]|_2^2,\end{aligned}\tag{4-67}$$

where we utilized the fact that

$$\int_{|\vec{v}\cdot\vec{n}(\vec{x})|\le\delta}\mu(\vec{v})|\vec{v}\cdot\vec{n}(\vec{x})|\,\mathrm{d}\vec{v}\le C\delta,\tag{4-68}$$

$$\int_{|\vec{v}|\le\delta\text{ or }|\vec{v}|\ge\frac{1}{\delta}}\mu(\vec{v})|\vec{v}\cdot\vec{n}(\vec{x})|\,\mathrm{d}\vec{v}\le C\delta.\tag{4-69}$$

Therefore, from

$$|\mathcal{P}[\mathbf{1}_{\gamma\backslash\gamma^\delta}f]|_2\le C|\mathbf{1}_{\gamma\backslash\gamma^\delta}f|_2,\tag{4-70}$$

we conclude

$$\begin{aligned}\tfrac{1}{2}|\mathcal{P}[f]|_2^2\le|\mathcal{P}[\mathbf{1}_{\gamma\backslash\gamma^\delta}f]|_2^2\le C|\mathbf{1}_{\gamma\backslash\gamma^\delta}f|_2^2&\le C(\delta)\big(\|f\|_{L^2}^2+\|\vec{v}\cdot\nabla_x(f^2)\|_1\big)\\&\le C\bigg(\|f\|_{L^2}^2+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2+\frac{1}{\epsilon}\int_{\Omega\times\mathbb{R}^2}fS\bigg).\end{aligned}\tag{4-71}$$

Hence, we know

$$|\mathcal{P}[f]|_2^2\le C\bigg(\|f\|_{L^2}^2+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2+\frac{1}{\epsilon}\int_{\Omega\times\mathbb{R}^2}fS\bigg),\tag{4-72}$$

which can be further simplified as

$$|\mathcal{P}[f]|_2^2\le C\bigg(\|\mathbb{P}[f]\|_{L^2}^2+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2+\frac{1}{\epsilon}\int_{\Omega\times\mathbb{R}^2}fS\bigg).\tag{4-73}$$

Step 3: synthesis. Plugging (4-73) into (4-63), we obtain

$$\epsilon|(1-\mathcal{P})[f]|_{L^2_+}^2+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2_\nu}^2\le C\bigg(\eta\epsilon^2\|\mathbb{P}[f]\|_{L^2}^2+\bigg(1+\frac{1}{\eta}\bigg)|h|_{L^2_-}^2+\int_{\Omega\times\mathbb{R}^2}fS\bigg).\tag{4-74}$$

We square on both sides of (4-8) to obtain

$$\epsilon^2\|\mathbb{P}[f]\|_{L^{2m}}^2\le C\big(\epsilon^2|(1-\mathcal{P})[f]|_{L^m_+}^2+\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2+\epsilon^2\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}^2+\|S\|_{L^2}^2+\epsilon^2|h|_{L^m_-}^2\big).\tag{4-75}$$

Multiplying a small constant on both sides of (4-75) and adding to (4-74) with $\eta > 0$ sufficiently small to absorb $\epsilon^2\|\mathbb{P}[f]\|_{L^{2m}}^2$ and $\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}$ into the left-hand side, we obtain

$$\epsilon|(1-\mathcal{P})[f]|_{L^2_+}^2 + \|(\mathbb{I}-\mathbb{P})[f]\|_{L^2_\nu}^2 + \epsilon^2\|\mathbb{P}[f]\|_{L^{2m}}^2 \\ \leq C\bigg(\epsilon^2|(1-\mathcal{P})[f]|_{L^m_+}^2 + \epsilon^2\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}^2 + \|S\|_{L^2}^2 + \epsilon^2|h|_{L^m_-}^2 + |h|_{L^2_-}^2 + \int_{\Omega\times\mathbb{R}^2} fS\bigg). \tag{4-76}$$

Step 4: interpolation argument. By an interpolation estimate and Young's inequality, we have

$$\begin{aligned} |(1-\mathcal{P})[f]|_{L^m_+} &\leq |(1-\mathcal{P})[f]|_{L^2_+}^{\frac{2}{m}} |(1-\mathcal{P})[f]|_{L^\infty_+}^{\frac{m-2}{m}} \\ &= \bigg(\frac{1}{\epsilon^{\frac{m-2}{m^2}}}|(1-\mathcal{P})[f]|_{L^2_+}^{\frac{2}{m}}\bigg)\big(\epsilon^{\frac{m-2}{m^2}}|(1-\mathcal{P})[f]|_{L^\infty_+}^{\frac{m-2}{m}}\big) \\ &\leq C\bigg(\frac{1}{\epsilon^{\frac{m-2}{m^2}}}|(1-\mathcal{P})[f]|_{L^2_+}^{\frac{2}{m}}\bigg)^{\frac{m}{2}} + o(1)\big(\epsilon^{\frac{m-2}{m^2}}|(1-\mathcal{P})[f]|_{L^\infty_+}^{\frac{m-2}{m}}\big)^{\frac{m}{m-2}} \\ &\leq \frac{C}{\epsilon^{\frac{m-2}{2m}}}|(1-\mathcal{P})[f]|_{L^2_+} + o(1)\epsilon^{\frac{1}{m}}|(1-\mathcal{P})[f]|_{L^\infty_+} \\ &\leq \frac{C}{\epsilon^{\frac{m-2}{2m}}}|(1-\mathcal{P})[f]|_{L^2_+} + o(1)\epsilon^{\frac{1}{m}}|f|_{L^\infty_+}. \end{aligned} \tag{4-77}$$

Similarly, we have

$$\begin{aligned} \|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}} &\leq \|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^{\frac{1}{m}} \|(\mathbb{I}-\mathbb{P})[f]\|_{L^\infty}^{\frac{m-1}{m}} \\ &= \bigg(\frac{1}{\epsilon^{\frac{m-1}{m^2}}}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^{\frac{1}{m}}\bigg)\big(\epsilon^{\frac{m-1}{m^2}}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^\infty}^{\frac{m-1}{m}}\big) \\ &\leq C\bigg(\frac{1}{\epsilon^{\frac{m-1}{m^2}}}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^{\frac{1}{m}}\bigg)^{m} + o(1)\big(\epsilon^{\frac{m-1}{m^2}}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^\infty}^{\frac{m-1}{m}}\big)^{\frac{m}{m-1}} \\ &\leq \frac{C}{\epsilon^{\frac{m-1}{m}}}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2} + o(1)\epsilon^{\frac{1}{m}}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^\infty}. \end{aligned} \tag{4-78}$$

We need this extra $\epsilon^{1/m}$ for the convenience of the L^∞ estimate. Then we know for sufficiently small ϵ,

$$\begin{aligned} \epsilon^2|(1-\mathcal{P})[f]|_{L^m}^2 &\leq C\epsilon^{2-\frac{m-2}{m}}|(1-\mathcal{P})[f]|_{L^2}^2 + o(1)\epsilon^{2+\frac{2}{m}}|f|_{L^\infty_+}^2 \\ &\leq o(1)\epsilon|(1-\mathcal{P})[f]|_{L^2}^2 + o(1)\epsilon^{2+\frac{2}{m}}|f|_{L^\infty_+}^2. \end{aligned} \tag{4-79}$$

Similarly, we have

$$\begin{aligned} \epsilon^2\|(\mathbb{I}-\mathbb{P})[f]\|_{L^{2m}}^2 &\leq \epsilon^{2-\frac{2m-2}{m}}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2 + o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty}^2 \\ &\leq o(1)\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2 + o(1)\epsilon^{2+\frac{2}{m}}\|f\|_{L^\infty}^2. \end{aligned} \tag{4-80}$$

In (4-76), we can absorb $\epsilon|(1-\mathcal{P})[f]|_{L^2}^2$ and $\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}^2$ into the left-hand side to obtain

$$\epsilon|(1-\mathcal{P})[f]|_{L^2_+}^2 + \|(\mathbb{I}-\mathbb{P})[f]\|_{L^2_\nu}^2 + \epsilon^2\|\mathbb{P}[f]\|_{L^{2m}}^2 \\ \leq C\bigg(o(1)\epsilon^{2+\frac{2}{m}}(\|f\|_{L^\infty}^2 + |f|_{L^\infty_+}^2) + \|S\|_{L^2}^2 + \epsilon^2|h|_{L^m_-}^2 + |h|_{L^2_-}^2 + \int_{\Omega\times\mathbb{R}^2} fS\bigg). \tag{4-81}$$

We can take the decomposition

$$\int_{\Omega\times\mathbb{R}^2} fS = \iint_{\Omega\times\mathbb{R}^2} \mathbb{P}[S]\mathbb{P}[f] + \iint_{\Omega\times\mathbb{R}^2} (\mathbb{I}-\mathbb{P})S(\mathbb{I}-\mathbb{P})[f]. \tag{4-82}$$

Hölder's inequality and Cauchy's inequality imply

$$\int_{\Omega\times\mathbb{R}^2} \mathbb{P}[S]\mathbb{P}[f] \le \|\mathbb{P}[S]\|_{L^{2m/(2m-1)}}\|\mathbb{P}[f]\|_{L^{2m}} \le \frac{C}{\epsilon^2}\|\mathbb{P}[S]\|^2_{L^{2m/(2m-1)}} + o(1)\epsilon^2\|\mathbb{P}[f]\|^2_{L^{2m}} \tag{4-83}$$

and

$$\int_{\Omega\times\mathbb{R}^2} (\mathbb{I}-\mathbb{P})S(\mathbb{I}-\mathbb{P})[f] \le C\|\nu^{-\frac{1}{2}}(\mathbb{I}-\mathbb{P})S\|^2_{L^2} + o(1)\|(\mathbb{I}-\mathbb{P})[f]\|^2_{L^2_\nu}. \tag{4-84}$$

Hence, absorbing $\epsilon^2\|\mathbb{P}[f]\|^2_{L^{2m}(\Omega\times\mathbb{R}^2)}$ and $\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2_\nu}$ into the left-hand side of (4-81), we get

$$\begin{aligned}\epsilon|(1-\mathcal{P})[f]|^2_{L^2_+} &+ \|(\mathbb{I}-\mathbb{P})[f]\|^2_{L^2_\nu} + \epsilon^2\|\mathbb{P}[f]\|^2_{L^{2m}} \\ &\le C\biggl(o(1)\epsilon^{2+\frac{2}{m}}(\|f\|^2_{L^\infty} + |f|^2_{L^\infty_+}) + \frac{1}{\epsilon^2}\|\mathbb{P}[S]\|^2_{L^{2m/(2m-1)}} + \|S\|^2_{L^2} + \epsilon^2|h|^2_{L^m_-} + |h|^2_{L^2_-}\biggr).\end{aligned} \tag{4-85}$$

Therefore, we have

$$\begin{aligned}\frac{1}{\epsilon^{\frac{1}{2}}}|(1-\mathcal{P})[f]|_{L^2_+} &+ \frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2_\nu} + \|\mathbb{P}[f]\|_{L^{2m}} \\ &\le C\biggl(o(1)\epsilon^{\frac{1}{m}}(\|f\|_{L^\infty} + |f|_{L^\infty_+}) + \frac{1}{\epsilon^2}\|\mathbb{P}[S]\|_{L^{2m/(2m-1)}} + \frac{1}{\epsilon}\|S\|_{L^2} + |h|_{L^m_-} + \frac{1}{\epsilon}|h|_{L^2_-}\biggr),\end{aligned} \tag{4-86}$$

completing the proof. □

4B. *L^∞ estimates.* We first define the tracking back through the characteristics and diffusive reflection.

Definition 4.5 (stochastic cycle). For a fixed point $(\vec{x}, \vec{v})$ with $(\vec{x}, \vec{v}) \notin \gamma_0$, let $(t_0, \vec{x}_0, \vec{v}_0) = (0, \vec{x}, \vec{v})$. For $\vec{v}_{k+1}$ such that $\vec{v}_{k+1}\cdot\vec{n}(\vec{x}_{k+1}) > 0$, define the $(k+1)$-component of the back-time cycle as

$$(t_{k+1}, \vec{x}_{k+1}, \vec{v}_{k+1}) = (t_k + t_b(\vec{x}_k, \vec{v}_k), \vec{x}_b(\vec{x}_k, \vec{v}_k), \vec{v}_{k+1}), \tag{4-87}$$

where

$$t_b(\vec{x}, \vec{v}) = \inf\{t > 0 : \vec{x} - \epsilon t\vec{v} \notin \Omega\}, \tag{4-88}$$

$$x_b(\vec{x}, \vec{v}) = \vec{x} - \epsilon t_b(\vec{x}, \vec{v})\vec{v} \notin \Omega. \tag{4-89}$$

Set

$$X_{\mathrm{cl}}(s; \vec{x}, \vec{v}) = j\sum_k \mathbf{1}_{\{t_k\le s<t_{k+1}\}}(\vec{x}_k - \epsilon(t_k - s)\vec{v}_k), \tag{4-90}$$

$$V_{\mathrm{cl}}(s; \vec{x}, \vec{v}) = \sum_k \mathbf{1}_{\{t_k\le s<t_{k+1}\}}\vec{v}_k. \tag{4-91}$$

Define $\mathcal{V}_k = \{\vec{v}\in\mathbb{R}^2 : \vec{v}\cdot\vec{n}(\vec{x}_k) > 0\}$, and let the iterated integral for $k\ge 2$ be defined as

$$\int_{\prod_{k=1}^{k-1}\mathcal{V}_j}\prod_{j=1}^{k-1}\mathrm{d}\sigma_j = \int_{\mathcal{V}_1}\cdots\biggl(\int_{\mathcal{V}_{k-1}}\mathrm{d}\sigma_{k-1}\biggr)\cdots\mathrm{d}\sigma_1, \tag{4-92}$$

where $\mathrm{d}\sigma_j = \mu(\vec{v})|\vec{v}\cdot\vec{n}(\vec{x}_j)|\,\mathrm{d}\vec{v}$ is a probability measure. We define a weight function scaled with the parameter ξ

$$w_\xi(\vec{v}) = w_{\xi,\beta,\varrho}(\vec{v}) = (1+\xi^2|\vec{v}|^2)^{\frac{\beta}{2}}\mathrm{e}^{\varrho|\vec{v}|^2}, \tag{4-93}$$

and

$$\tilde{w}_\xi(\vec{v}) = \frac{1}{\mu^{\frac{1}{2}}(\vec{v})w_\xi(\vec{v})} = \sqrt{2\pi}\,\frac{\mathrm{e}^{(\frac{1}{4}-\varrho)|\vec{v}|^2}}{(1+\xi^2|\vec{v}|^2)^{\frac{\beta}{2}}}. \tag{4-94}$$

Lemma 4.6. *For $T>0$ sufficiently large, there exist constants $C_1, C_2>0$ independent of T_0 such that for $k=C_1T_0^{5/4}$ and $(\vec{x},\vec{v})\in\times\bar{\Omega}\times\mathbb{R}^2$*

$$\int_{\Pi_{k=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k(\vec{x},\vec{v},\vec{v}_1,\dots,\vec{v}_{k-1})<T_0/\epsilon\}}\prod_{j=1}^{k-1}\mathrm{d}\sigma_j \leq \left(\frac{1}{2}\right)^{C_2T_0^{5/4}}. \tag{4-95}$$

We also have, for $\beta>2$,

$$\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\langle\vec{v}_l\rangle\tilde{w}_\xi(\vec{v}_l)\prod_{j=1}^{k-1}\mathrm{d}\sigma_j \leq \frac{C(\beta,\varrho)}{\xi^3}, \tag{4-96}$$

$$\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k(\vec{x},\vec{v},\vec{v}_1,\dots,\vec{v}_{k-1})<T_0/\epsilon\}}\langle\vec{v}_l\rangle\tilde{w}_\xi(\vec{v}_l)\prod_{j=1}^{k-1}\mathrm{d}\sigma_j \leq \frac{C(\beta,\varrho)}{\xi^3}\left(\frac{1}{2}\right)^{C_2T_0^{5/4}}. \tag{4-97}$$

Proof. See [Esposito et al. 2013, Lemma 4.1]. □

Theorem 4.7. *The solution $f(\vec{x},\vec{v})$ to (4-1) satisfies the estimate for $\vartheta\geq 3$ and $0\leq\varrho<\frac{1}{4}$*

$$\begin{aligned}\|\langle\vec{v}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{v}|^2}f\|_{L^\infty} \leq C\bigg(&\frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathbb{P}[S]\|_{L^{\frac{2m}{2m-1}}}+\frac{1}{\epsilon^{1+\frac{1}{m}}}\|S\|_{L^2}+\left\|\frac{\langle\vec{v}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{v}|^2}S}{\nu}\right\|_{L^\infty}\\ &+\frac{1}{\epsilon^{1+\frac{1}{m}}}|h|_{L^2_-}+\frac{1}{\epsilon^{\frac{1}{m}}}|h|_{L^m_-}+|\langle\vec{v}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{v}|^2}h|_{L^\infty_-}\bigg).\end{aligned} \tag{4-98}$$

Proof. We divide the proof into several steps:

<u>Step 1</u>: mild formulation. Define

$$g(\vec{x},\vec{v}) = w_\xi(\vec{v})f(\vec{x},\vec{v}), \tag{4-99}$$

$$K_{w_\xi(\vec{v})}[g](\vec{x},\vec{v}) = w_\xi(\vec{v})K\left[\frac{g}{w_\xi}\right](\vec{x},\vec{v}) = \int_{\mathbb{R}^2}k_{w_\xi(\vec{v})}(\vec{v},\vec{\mathfrak{u}})g(\vec{x},\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}, \tag{4-100}$$

where

$$k_{w_\xi(\vec{v})}(\vec{v},\vec{\mathfrak{u}}) = k(\vec{v},\vec{\mathfrak{u}})\frac{w_\xi(\vec{v})}{w_\xi(\vec{\mathfrak{u}})}. \tag{4-101}$$

We can rewrite the solution of (4-1) along the characteristics by Duhamel's principle as

$$\begin{aligned}g(\vec{x},\vec{v}) = {}& w_\xi(\vec{v})h(\vec{x}-\epsilon t_1\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})t_1}+\int_0^{t_1}w_\xi(\vec{v})S(\vec{x}-\epsilon(t_1-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_1-s)}\,\mathrm{d}s\\ &+\int_0^{t_1}K_{w_\xi(\vec{v})}[g](\vec{x}-\epsilon(t_1-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_1-s)}\,\mathrm{d}s+\frac{\mathrm{e}^{-\nu(\vec{v})t_1}}{\tilde{w}_\xi(\vec{v})}\int_{\mathcal{V}_1}g(\vec{x}_1,\vec{v}_1)\tilde{w}_\xi(\vec{v}_1)\,\mathrm{d}\sigma_1,\end{aligned} \tag{4-102}$$

where the last term refers to $\mathcal{P}[f]$. We may further rewrite (4-1) along the stochastic cycle by applying Duhamel's principle k times as

$$\begin{aligned} g(\vec{x},\vec{v}) &= w_\xi(\vec{v})h(\vec{x}-\epsilon t_1\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})t_1}+\int_0^{t_1} w_\xi(\vec{v})S(\vec{x}-\epsilon(t_1-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_1-s)}\,\mathrm{d}s \\ &\quad+\int_0^{t_1} K_{w_\xi(\vec{v})}[g](\vec{x}-\epsilon(t_1-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_1-s)}\,\mathrm{d}s \\ &\quad+\frac{1}{\tilde{w}_\xi(\vec{v})}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{l}\mathcal{V}_j}(G[\vec{x},\vec{v}]+H[\vec{x},\vec{v}])\tilde{w}_\xi(\vec{v}_l)\biggl(\prod_{j=1}^{l}\mathrm{e}^{-\nu(\vec{v}_j)(t_{j+1}-t_j)}\,\mathrm{d}\sigma_j\biggr) \\ &\quad+\frac{1}{\tilde{w}_\xi(\vec{v})}\int_{\prod_{j=1}^{k}\mathcal{V}_j} g(\vec{x}_k,\vec{v}_k)\tilde{w}_\xi(\vec{v}_k)\biggl(\prod_{j=1}^{k}\mathrm{e}^{-\nu(\vec{v}_j)(t_{j+1}-t_j)}\,\mathrm{d}\sigma_j\biggr), \end{aligned} \tag{4-103}$$

where

$$G[\vec{x},\vec{v}]=h(\vec{x}_l-\epsilon t_{l+1}\vec{v}_l,\vec{v}_l)w_\xi(\vec{v}_l)+\int_{t_l}^{t_{l+1}}\bigl(S(\vec{x}_l-\epsilon(t_{l+1}-s)\vec{v}_l,\vec{v}_l)w_\xi(\vec{v}_l)\mathrm{e}^{\nu(\vec{v}_l)s}\bigr)\,\mathrm{d}s, \tag{4-104}$$

$$H[\vec{x},\vec{v}]=\int_{t_l}^{t_{l+1}}\bigl(K_{w_\xi(\vec{v}_l)}[g](\vec{x}_l-\epsilon(t_{l+1}-s)\vec{v}_l,\vec{v}_l)\mathrm{e}^{\nu(\vec{v}_l)s}\bigr)\,\mathrm{d}s. \tag{4-105}$$

Step 2: estimates of source terms and boundary terms. We set $k=CT_0^{5/4}$ and take the absolute value of both sides of (4-103). Then all the terms in (4-103) related to the source term S and boundary term h can be bounded as

$$\text{Part 1}\le C\biggl(|w_\xi h|_{L^\infty_-}+\biggl\|\frac{w_\xi S}{\nu}\biggr\|_{L^\infty}\biggr) \tag{4-106}$$

due to Lemma 4.6 and

$$\frac{1}{\tilde{w}_\xi}\le C(\beta,\varrho)\xi^\beta. \tag{4-107}$$

The last term in (4-103) can be decomposed as follows:

$$\begin{aligned} \text{Part 2} &= \frac{1}{\tilde{w}_\xi(\vec{v})}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}\le T_0/\epsilon\}}g(\vec{x}_k,\vec{v}_k)\tilde{w}_\xi(\vec{v}_k)\biggl(\prod_{j=1}^{k}\mathrm{e}^{-\nu(\vec{v}_j)(t_{j+1}-t_j)}\,\mathrm{d}\sigma_j\biggr) \\ &\quad+\frac{1}{\tilde{w}_\xi(\vec{v})}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}\ge T_0/\epsilon\}}g(\vec{x}_k,\vec{v}_k)\tilde{w}_\xi(\vec{v}_k)\biggl(\prod_{j=1}^{k}\mathrm{e}^{-\nu(\vec{v}_j)(t_{j+1}-t_j)}\,\mathrm{d}\sigma_j\biggr). \end{aligned} \tag{4-108}$$

Based on Lemma 4.6, we have

$$\frac{1}{\tilde{w}_\xi(\vec{v})}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}\le T_0/\epsilon\}}g(\vec{x}_k,\vec{v}_k)\tilde{w}_\xi(\vec{v}_k)\biggl(\prod_{j=1}^{k}\mathrm{e}^{-\nu(\vec{v}_j)(t_{j+1}-t_j)}\,\mathrm{d}\sigma_j\biggr)\le C\Bigl(\frac{1}{2}\Bigr)^{C_2T_0^{5/4}}\|g\|_{L^\infty}. \tag{4-109}$$

Based on Lemma 4.6 and $\nu_0(1+|\vec{v}|)\le\nu(\vec{v})\le\nu_1(1+|\vec{v}|)$, we obtain

$$\frac{1}{\tilde{w}_\xi(\vec{v})}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}\ge T_0/\epsilon\}}g(\vec{x}_k,\vec{v}_k)\tilde{w}_\xi(\vec{v}_k)\biggl(\prod_{j=1}^{k}\mathrm{e}^{-\nu(\vec{v}_j)(t_{j+1}-t_j)}\,\mathrm{d}\sigma_j\biggr)\le \mathrm{e}^{-\frac{\nu_0T_0}{\epsilon}}\|g\|_{L^\infty}. \tag{4-110}$$

For T_0 sufficiently large and ϵ sufficiently small, we get

$$\text{Part 2} \leq \delta\|g\|_{L^\infty} \tag{4-111}$$

for δ arbitrarily small.

<u>Step 3</u>: estimates of K_{w_ξ} terms. So far, the only remaining terms are related to K_{w_ξ}. Define the back-time stochastic cycle from $(s, X_{\mathrm{cl}}(s;\vec{x},\vec{v}),\vec{v}')$ as $(t_i', \vec{x}_i', \vec{v}_i')$. Then we can rewrite K_{w_ξ} along the stochastic cycle as

$$\begin{aligned}
&K_{w_\xi(\vec{v})}[g](\vec{x}-\epsilon(t_1-s)\vec{v},\vec{v})\\
&\qquad= K_{w_\xi(\vec{v})}[g](X_{\mathrm{cl}},\vec{v}) = \int_{\mathbb{R}^2} k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')g(X_{\mathrm{cl}},\vec{v}')\,\mathrm{d}\vec{v}'\\
&\qquad\leq \left|\int_{\mathbb{R}^2}\int_0^{t_1'} k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')K_{w_\xi(\vec{v}')}[g](X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}',\vec{v}')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\right|\\
&\qquad\quad+\left|\int_{\mathbb{R}^2}\frac{1}{\tilde{w}_\xi(\vec{v}')}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{l}\mathcal{V}_j'} k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')H[X_{\mathrm{cl}},\vec{v}']\tilde{w}_\xi(\vec{v}_l')\left(\prod_{j=1}^{l}\mathrm{e}^{-\nu(\vec{v}_j')(t_{j+1}'-t_j')}\,\mathrm{d}\sigma_j'\right)\mathrm{d}\vec{v}'\right|\\
&\qquad\quad+\left|\int_{\mathbb{R}^2}k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')A\,\mathrm{d}\vec{v}'\right|\\
&\qquad= I+II+III.
\end{aligned}\tag{4-112}$$

Here, A in the last term is Part 1 + Part 2 in Step 2. Now III can be directly estimated as

$$III \leq C(|w_\xi h|_{L^\infty_-}+\|w_\xi S\|_{L^\infty}+\delta\|g\|_{L^\infty}). \tag{4-113}$$

We may further rewrite I as

$$I = \left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_0^{t_1'} k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')k_{w_\xi(\vec{v}')}(\vec{v}',\vec{v}'')g(X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}',\vec{v}'')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right|, \tag{4-114}$$

which will estimated in four cases:

$$I = I_1+I_2+I_3+I_4. \tag{4-115}$$

<u>Case I</u>: $|\vec{v}|\geq N$. Based on Lemma 2.3, we have

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')k_{w_\xi(\vec{v}')}(\vec{v}',\vec{v}'')\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right| \leq \frac{C}{1+|\vec{v}|}\leq\frac{C}{N}. \tag{4-116}$$

Hence, we get

$$I_1 \leq \frac{C}{N}\|g\|_{L^\infty}. \tag{4-117}$$

<u>Case II</u>: $|\vec{v}|\leq N$, $|\vec{v}'|\geq 2N$, or $|\vec{v}'|\leq 2N$, $|\vec{v}''|\geq 3N$. Notice this implies either $|\vec{v}'-\vec{v}|\geq N$ or $|\vec{v}'-\vec{v}''|\geq N$. Hence, either of the following is valid correspondingly:

$$|k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')| \leq C\mathrm{e}^{-\delta N^2}|k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')\mathrm{e}^{\delta|\vec{v}-\vec{v}'|^2}|, \tag{4-118}$$

$$|k_{w_\xi(\vec{v}')}(\vec{v}',\vec{v}'')| \leq C\mathrm{e}^{-\delta N^2}|k_{w_\xi(\vec{v}')}(\vec{v}',\vec{v}'')\mathrm{e}^{\delta|\vec{v}'-\vec{v}''|^2}|. \tag{4-119}$$

Then based on Lemma 2.3,

$$\int_{\mathbb{R}^2}|k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')\mathrm{e}^{\delta|\vec{v}-\vec{v}'|^2}|\,\mathrm{d}\vec{v}'<\infty, \tag{4-120}$$

$$\int_{\mathbb{R}^2}|k_{w_\xi(\vec{v}')}(\vec{v}',\vec{v}'')\mathrm{e}^{\delta|\vec{v}'-\vec{v}''|^2}|\,\mathrm{d}\vec{v}''<\infty. \tag{4-121}$$

Hence, we have

$$I_2\leq C\mathrm{e}^{-\delta N^2}\|g\|_{L^\infty}. \tag{4-122}$$

Case III: $t_1'-r\leq\delta$ and $|\vec{v}|\leq N$, $|\vec{v}'|\leq 2N$, $|\vec{v}''|\leq 3N$. In this case, since the integral with respect to r is restricted in a very short interval, there is a small contribution as

$$I_3\leq\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{t_1'-\delta}^{t_1'}k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')k_{w_\xi(\vec{v}')}(\vec{v}',\vec{v}'')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right|\|g\|_{L^\infty}\leq C\delta\|g\|_{L^\infty}. \tag{4-123}$$

Case IV: $t_1'-r\geq\delta$ and $|\vec{v}|\leq N$, $|\vec{v}'|\leq 2N$, $|\vec{v}''|\leq 3N$. Note we always have $X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}'\in\Omega$. Hence, we define the change of variable $\vec{v}'\to\vec{y}$ as $\vec{y}=(y_1,y_2)=X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}'$ such that

$$\left|\frac{\mathrm{d}\vec{y}}{\mathrm{d}\vec{v}'}\right|=\left\|\begin{matrix}\epsilon(t_1'-r) & 0\\ 0 & \epsilon(t_1'-r)\end{matrix}\right\|=\epsilon^2(t_1'-r)^2\geq\epsilon^2\delta^2. \tag{4-124}$$

In two dimensions, $k_{w_\xi(\vec{v})}(\vec{v},\vec{v}')$ does not have singularity of $1/|\vec{v}-\vec{v}'|$ and is bounded. Also, $|\vec{v}'|,|\vec{v}''|\leq 3N$. We estimate

$$\begin{aligned}
I_4&\leq C\left|\int_{|\vec{v}'|\leq 2N}\int_{|\vec{v}''|\leq 3N}\int_0^{t_1'}\mathbf{1}_{\{X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}'\in\Omega\}}f(X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}',\vec{v}'')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right|\\
&\leq C\left(\int_{|\vec{v}'|\leq 2N}\int_{|\vec{v}''|\leq 3N}\int_0^{t_1'}\mathbf{1}_{\{X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}'\in\Omega\}}\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right)^{\frac{2m-1}{2m}}\\
&\quad\times\left(\int_{|\vec{v}'|\leq 2N}\int_{|\vec{v}''|\leq 3N}\int_0^{t_1'}\mathbf{1}_{\{X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}'\in\Omega\}}(\mathbb{P}[f])^{2m}(X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}',\vec{v}'')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right)^{\frac{1}{2m}}\\
&\quad+C\left(\int_{|\vec{v}'|\leq 2N}\int_{|\vec{v}''|\leq 3N}\int_0^{t_1'}\mathbf{1}_{\{X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}'\in\Omega\}}\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_{|\vec{v}'|\leq 2N}\int_{|\vec{v}''|\leq 3N}\int_0^{t_1'}\mathbf{1}_{\{X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}'\in\Omega\}}((\mathbb{I}-\mathbb{P})[f])^2(X_{\mathrm{cl}}-\epsilon(t_1'-r)\vec{v}',\vec{v}'')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}r\,\mathrm{d}\vec{v}'\,\mathrm{d}\vec{v}''\right)^{\frac{1}{2}}\\
&\leq C\left|\int_0^{t_1'}\frac{1}{\epsilon^2\delta^2}\int_{|\vec{v}''|\leq 3N}\int_\Omega\mathbf{1}_{\{\vec{y}\in\Omega\}}(\mathbb{P}[f])^{2m}(\vec{y},\vec{v}'')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}\vec{y}\,\mathrm{d}\vec{v}''\,\mathrm{d}r\right|^{\frac{1}{2m}}\\
&\quad+C\left|\int_0^{t_1'}\frac{1}{\epsilon^2\delta^2}\int_{|\vec{v}''|\leq 3N}\int_\Omega\mathbf{1}_{\{\vec{y}\in\Omega\}}((\mathbb{I}-\mathbb{P})[f])^2(\vec{y},\vec{v}'')\mathrm{e}^{-\nu(\vec{v}')(t_1'-r)}\,\mathrm{d}\vec{y}\,\mathrm{d}\vec{v}''\,\mathrm{d}r\right|^{\frac{1}{2}}\\
&=\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{m}}}\|\mathbb{P}[f]\|_{L^{2m}}+\frac{C}{\epsilon\delta}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}.
\end{aligned} \tag{4-125}$$

Therefore, we've already proved

$$I \le (C\mathrm{e}^{-\delta N^2}+\delta)\|g\|_{L^\infty}+\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{m}}}\|\mathbb{P}[f]\|_{L^{2m}}+\frac{C}{\epsilon\delta}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}. \tag{4-126}$$

Choosing δ sufficiently small and then taking N sufficiently large, we have

$$I \le C\delta\|g\|_{L^\infty}+\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{m}}}\|\mathbb{P}[f]\|_{L^{2m}}+\frac{C}{\epsilon\delta}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}. \tag{4-127}$$

A similar technique can justify

$$II \le C\delta\|g\|_{L^\infty}+\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{m}}}\|\mathbb{P}[f]\|_{L^{2m}}+\frac{C}{\epsilon\delta}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}. \tag{4-128}$$

All the terms related to K_{w_ξ} can be estimated in a similar fashion.

Step 4: synthesis. Collecting all the above, based on the mild formulation (4-103), we have shown, for any $(\vec{x},\vec{v})\in\bar{\Omega}\times\mathbb{R}^2$,

$$|g(\vec{x},\vec{v})| \le C\delta\|g\|_{L^\infty}+\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{m}}}\|\mathbb{P}[f]\|_{L^{2m}}+\frac{C}{\epsilon\delta}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+C\left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty}+C|w_\xi h|_{L^\infty_-}. \tag{4-129}$$

Let δ be sufficiently small such that $C\delta\le\frac{1}{2}$. Taking the supremum over $(\vec{x},\vec{v})\in\gamma_+$ in (4-129), we have

$$|g|_{L^\infty_+} \le \tfrac{1}{2}\|g\|_{L^\infty}+\frac{C}{\epsilon^{\frac{1}{m}}}\|\mathbb{P}[f]\|_{L^{2m}}+\frac{C}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+C\left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty}+C|w_\xi h|_{L^\infty_-}. \tag{4-130}$$

Based on Theorem 4.4, we obtain

$$\begin{aligned}|g|_{L^\infty_+} \le \tfrac{1}{2}\|g\|_{L^\infty}+C\biggl(&o(1)(\|g\|_{L^\infty}+|g|_{L^\infty_+})+\frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathbb{P}[S]\|_{L^{2m/(2m-1)}}+\frac{1}{\epsilon^{1+\frac{1}{m}}}\|S\|_{L^2}\\&+\left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty}+\frac{1}{\epsilon^{1+\frac{1}{m}}}|h|_{L^2_-}+\frac{1}{\epsilon^{\frac{1}{m}}}|h|_{L^m_-}+|w_\xi h|_{L^\infty_-}\biggr).\end{aligned} \tag{4-131}$$

Absorbing $o(1)|g|_{L^\infty_+}$ into the left-hand side, we have

$$\begin{aligned}|g|_{L^\infty_+} \le \tfrac{1}{2}\|g\|_{L^\infty}+C\biggl(&o(1)\|g\|_{L^\infty}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathbb{P}[S]\|_{L^{2m/(2m-1)}}+\frac{1}{\epsilon^{1+\frac{1}{m}}}\|S\|_{L^2}\\&+\left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty}+\frac{1}{\epsilon^{1+\frac{1}{m}}}|h|_{L^2_-}+\frac{1}{\epsilon^{\frac{1}{m}}}|h|_{L^m_-}+|w_\xi h|_{L^\infty_-}\biggr).\end{aligned} \tag{4-132}$$

On the other hand, taking the supremum over $(\vec{x},\vec{v})\in\Omega\times\mathbb{R}^2$ in (4-129), we have

$$\|g\|_{L^\infty} \le \tfrac{1}{2}\|g\|_{L^\infty}+\frac{C}{\epsilon^{\frac{1}{m}}}\|\mathbb{P}[f]\|_{L^{2m}}+\frac{C}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f]\|_{L^2}+C\left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty}+C|w_\xi h|_{L^\infty_-}. \tag{4-133}$$

Based on Theorem 4.4, we obtain

$$\|g\|_{L^\infty} \le \tfrac{1}{2}\|g\|_{L^\infty} + C\Bigg(o(1)(\|g\|_{L^\infty} + |g|_{L^\infty_+}) + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathbb{P}[S]\|_{L^{\frac{2m}{2m-1}}} + \frac{1}{\epsilon^{1+\frac{1}{m}}}\|S\|_{L^2} + \left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty} + \frac{1}{\epsilon^{1+\frac{1}{m}}}|h|_{L^2_-} + \frac{1}{\epsilon^{\frac{1}{m}}}|h|_{L^m_-} + |w_\xi h|_{L^\infty_-}\Bigg). \tag{4-134}$$

Inserting (4-132) into (4-134), we get

$$\|g\|_{L^\infty} \le \tfrac{1}{2}\|g\|_{L^\infty} + C\Bigg(o(1)\|g\|_{L^\infty} + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathbb{P}[S]\|_{L^{2m/(2m-1)}} + \frac{1}{\epsilon^{1+\frac{1}{m}}}\|S\|_{L^2} + \left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty} + \frac{1}{\epsilon^{1+\frac{1}{m}}}|h|_{L^2_-} + \frac{1}{\epsilon^{\frac{1}{m}}}|h|_{L^m_-} + |w_\xi h|_{L^\infty_-}\Bigg). \tag{4-135}$$

Absorbing $\frac{1}{2}\|g\|_{L^\infty}$ and $o(1)\|g\|_{L^\infty}$ into the left-hand side, we have

$$\|g\|_{L^\infty} \le C\Bigg(\frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathbb{P}[S]\|_{L^{2m/(2m-1)}} + \frac{1}{\epsilon^{1+\frac{1}{m}}}\|S\|_{L^2} + \left\|\frac{w_\xi S}{\nu}\right\|_{L^\infty} + \frac{1}{\epsilon^{1+\frac{1}{m}}}|h|_{L^2_-} + \frac{1}{\epsilon^{\frac{1}{m}}}|h|_{L^m_-} + |w_\xi h|_{L^\infty_-}\Bigg). \tag{4-136}$$

It is easy to see for $\vartheta = \beta$ we have

$$C_1\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2} \le w_\xi \le C_2\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2} \tag{4-137}$$

for some constant $C_1, C_2 > 0$. Then we must have

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2} f\|_{L^\infty} \le C\Bigg(\frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathbb{P}[S]\|_{L^{2m/(2m-1)}} + \frac{1}{\epsilon^{1+\frac{1}{m}}}\|S\|_{L^2} + \left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2} S}{\nu}\right\|_{L^\infty} + \frac{1}{\epsilon^{1+\frac{1}{m}}}|h|_{L^2_-} + \frac{1}{\epsilon^{\frac{1}{m}}}|h|_{L^m_-} + |\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2} h|_{L^\infty_-}\Bigg), \tag{4-138}$$

completing the proof. □

5. ϵ-Milne problem with geometric correction

5A. *Well-posedness and decay.* We consider the ϵ-Milne problem with geometric correction for $g(\eta,\theta,\vec{\mathfrak{v}})$ in the domain $(\eta,\theta,\vec{\mathfrak{v}}) \in [0, L] \times [-\pi, \pi) \times \mathbb{R}^2$ as

$$\begin{cases} v_\eta \dfrac{\partial g}{\partial \eta} - \dfrac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \dfrac{\partial g}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g}{\partial v_\phi}\right) + \mathcal{L}[g] = 0, \\ g(0,\theta,\vec{\mathfrak{v}}) = h(\theta,\vec{\mathfrak{v}}) \quad \text{for } v_\eta > 0, \\ g(L,\theta,\vec{\mathfrak{v}}) = g(L,\theta,\mathscr{R}[\vec{\mathfrak{v}}]), \end{cases} \tag{5-1}$$

where R_κ is defined in (3-29), $\mathscr{R}[\vec{\mathfrak{v}}] = (-v_\eta, v_\phi)$ and $L = \epsilon^{-1/2}$. For simplicity, we temporarily ignore the dependence of θ; i.e., consider the ϵ-Milne problem with geometric correction for $g(\eta,\vec{\mathfrak{v}})$ in the domain

$(\eta, \vec{\mathfrak{v}}) \in [0, L] \times \mathbb{R}^2$ as

$$\begin{cases} v_\eta \dfrac{\partial g}{\partial \eta} - \dfrac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \dfrac{\partial g}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g}{\partial v_\phi}\right) + \mathcal{L}[g] = 0, \\ g(0, \vec{\mathfrak{v}}) = h(\vec{\mathfrak{v}}) \quad \text{for } v_\eta > 0, \\ g(L, \vec{\mathfrak{v}}) = g(L, \mathscr{R}[\vec{\mathfrak{v}}]). \end{cases} \tag{5-2}$$

The null space of the operator $\mathcal{L}$ is spanned by $\mathcal{N} = \mu^{1/2}\big\{1, v_\eta, v_\phi, \frac{1}{2}(|\vec{\mathfrak{v}}|^2 - 2)\big\} = \{\psi_0, \psi_1, \psi_2, \psi_3\}$. Our main goal is to find

$$\tilde{h}(\vec{\mathfrak{v}}) = \sum_{i=0}^{3} \widetilde{D}_i \psi_i \in \mathcal{N}, \tag{5-3}$$

with $\widetilde{D}_1 = 0$ such that the ϵ-Milne problem with geometric correction for $\mathcal{G}(\eta, \vec{\mathfrak{v}})$ in the domain $(\eta, \vec{\mathfrak{v}}) \in [0, L] \times \mathbb{R}^2$ as

$$\begin{cases} v_\eta \dfrac{\partial \mathcal{G}}{\partial \eta} - \dfrac{\epsilon}{R_\kappa - \epsilon\eta}\left(v_\phi^2 \dfrac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial \mathcal{G}}{\partial v_\phi}\right) + \mathcal{L}[\mathcal{G}] = 0, \\ \mathcal{G}(0, \vec{\mathfrak{v}}) = h(\vec{\mathfrak{v}}) - \tilde{h}(\vec{\mathfrak{v}}) \quad \text{for } v_\eta > 0, \\ \mathcal{G}(L, \vec{\mathfrak{v}}) = \mathcal{G}(L, \mathscr{R}[\vec{\mathfrak{v}}]) \end{cases} \tag{5-4}$$

is well-posed, and $\mathcal{G}$ decays exponentially fast as η becomes larger and larger. The estimates and decaying rate should be uniform in ϵ.

Let $G(\eta) = -\epsilon/(R_\kappa - \epsilon\eta)$. We define a potential function $W(\eta)$ as $G(\eta) = -\mathrm{d}W/\mathrm{d}\eta$ with $W(0) = 0$. It is easy to check that

$$W(\eta) = \ln\left(\frac{R_\kappa}{R_\kappa - \epsilon\eta}\right). \tag{5-5}$$

In this section, we introduce some special notation to describe the norms in the space $(\eta, \vec{\mathfrak{v}}) \in [0, L] \times \mathbb{R}^2$. Define the weighted L^∞ norm as

$$\|f(\eta)\|_{L^\infty_{\vartheta,\varrho}} = \sup_{\vec{\mathfrak{v}} \in \mathbb{R}^2} \big(\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2} |f(\eta, \vec{\mathfrak{v}})|\big), \tag{5-6}$$

$$|||f|||_{L^\infty L^\infty_{\vartheta,\varrho}} = \sup_{(\eta,\vec{\mathfrak{v}}) \in [0,L]\times\mathbb{R}^2} \big(\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2} |f(\eta, \vec{\mathfrak{v}})|\big). \tag{5-7}$$

Since the boundary data $h(\vec{\mathfrak{v}})$ is only defined on $v_\eta > 0$, we naturally extend the above definitions on this half-domain as

$$\|h\|_{L^\infty_{\vartheta,\varrho}} = \sup_{v_\eta > 0} \big(\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2} |h(\vec{\mathfrak{v}})|\big). \tag{5-8}$$

We assume

$$\|h\|_{L^\infty_{\vartheta,\varrho}} \le C \tag{5-9}$$

for some $C > 0$ uniform in ϵ.

Here, we mainly refer to the procedure in [Cercignani et al. 1998; Wu 2016], where $\eta \in [0, \infty)$. The proof is similar with obvious modifications, so we only present the main results.

Theorem 5.1. *There exists $\tilde h$ satisfying the condition (5-3) such that there exists a unique solution $\mathcal{G}(\eta,\vec{\mathfrak{v}})$ to the ϵ-Milne problem (5-4) satisfying for $\varrho\geq 0$ and an integer $\vartheta\geq 3$*

$$\||\mathcal{G}|\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C. \tag{5-10}$$

Theorem 5.2. *For sufficiently small K_0, there exists a unique solution $\mathcal{G}(\eta,\vec{\mathfrak{v}})$ to the ϵ-Milne problem (5-4) satisfying for $\varrho\geq 0$ and an integer $\vartheta\geq 3$*

$$\||\mathrm{e}^{K_0\eta}\mathcal{G}|\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C. \tag{5-11}$$

5B. *Weight function.* Now we begin to study the regularity of the solution $\mathcal{G}$ to (5-4). Let $p(\vec{\mathfrak{v}})=h(\vec{\mathfrak{v}})-\tilde h(\vec{\mathfrak{v}})$.

Define a weight function

$$\zeta(\eta,v_\eta,v_\phi)=\left((v_\eta^2+v_\phi^2)-\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa}\right)^2v_\phi^2\right)^{\frac12}. \tag{5-12}$$

It is easy to see that the closer a point (η,v_η,v_ϕ) is to the grazing set $(\eta,v_\eta,v_\phi)=(0,0,v_\phi)$, the smaller ζ is. In particular, at the grazing set, $\zeta(0,0,v_\phi)=0$.

Lemma 5.3. *We have*

$$v_\eta\frac{\partial\zeta}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\frac{\partial\zeta}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\zeta}{\partial v_\phi}\right)=0. \tag{5-13}$$

Proof. We may directly compute

$$\frac{\partial\zeta}{\partial\eta}=\frac1\zeta\frac{R_\kappa-\epsilon\eta}{R_\kappa}\epsilon v_\phi^2,\qquad \frac{\partial\zeta}{\partial v_\eta}=\frac1\zeta v_\eta,\qquad \frac{\partial\zeta}{\partial v_\phi}=\frac1\zeta\left(v_\phi-\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa}\right)^2v_\phi\right). \tag{5-14}$$

Then we know

$$\begin{aligned}v_\eta\frac{\partial\zeta}{\partial\eta}&-\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\frac{\partial\zeta}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\zeta}{\partial v_\phi}\right)\\&=\frac1\zeta\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa}\epsilon v_\eta v_\phi^2-\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\eta v_\phi^2-v_\eta v_\phi^2+v_\eta v_\phi^2\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa}\right)^2\right)\right)=0,\end{aligned} \tag{5-15}$$

completing the proof. □

5C. *Mild formulation.* Consider the ϵ-transport problem for $\mathscr{A}=\zeta(\partial\mathcal{G}/\partial\eta)$ as

$$\begin{cases}v_\eta\dfrac{\partial\mathscr{A}}{\partial\eta}+G(\eta)\left(v_\phi^2\dfrac{\partial\mathscr{A}}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathscr{A}}{\partial v_\phi}\right)+\nu\mathscr{A}=\widetilde{\mathscr{A}}+S_{\mathscr{A}},\\ \mathscr{A}(0,\vec{\mathfrak{v}})=p_{\mathscr{A}}(\vec{\mathfrak{v}})\quad\text{for } v_\eta>0,\\ \mathscr{A}(L,\vec{\mathfrak{v}})=\mathscr{A}(L,\mathscr{R}[\vec{\mathfrak{v}}]),\end{cases} \tag{5-16}$$

where $p_{\mathscr{A}}$ and $S_{\mathscr{A}}$ will be specified later with

$$\widetilde{\mathscr{A}}(\eta,\vec{\mathfrak{v}})=\int_{\mathbb{R}^2}\frac{\zeta(\eta,\vec{\mathfrak{v}})}{\zeta(\eta,\vec{\mathfrak{u}})}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\mathscr{A}(\eta,\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}. \tag{5-17}$$

Here we utilize Lemma 5.3. We need to derive the a priori estimate of $\mathscr{A}$. Define the energy

$$E_1 = v_\eta^2 + v_\phi^2, \tag{5-18}$$

$$E_2 = v_\phi \mathrm{e}^{-W(\eta)}. \tag{5-19}$$

We can easily check that the weight function satisfies $\zeta = \sqrt{E_1 - E_2^2}$. Along the characteristics, where E_1, E_2 and ζ are constants, (5-16) can be simplified as follows:

$$v_\eta \frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\eta} + \mathscr{A} = \widetilde{\mathscr{A}} + S_{\mathscr{A}}. \tag{5-20}$$

Let

$$v'_\phi(\eta, \vec{\mathfrak{v}}; \eta') = v_\phi e^{W(\eta') - W(\eta)}. \tag{5-21}$$

For $E_1 \geq v'^2_\phi$, define

$$v'_\eta(\eta, \vec{\mathfrak{v}}; \eta') = \sqrt{E_1 - v'^2_\phi(\eta, \vec{\mathfrak{v}}; \eta')}, \tag{5-22}$$

$$\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta') = (v'_\eta(\eta, \vec{\mathfrak{v}}; \eta'), v'_\phi(\eta, \vec{\mathfrak{v}}; \eta')), \tag{5-23}$$

$$\mathscr{R}[\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta')] = (-v'_\eta(\eta, \vec{\mathfrak{v}}; \eta'), v'_\phi(\eta, \vec{\mathfrak{v}}; \eta')). \tag{5-24}$$

Basically, this means (η, v_η, v_ϕ) and $(\eta', v'_\eta, v'_\phi)$, $(\eta', -v'_\eta, v'_\phi)$ are on the same characteristics. Also, this implies $v'_\eta \geq 0$. Moreover, define an implicit function $\eta^+(\eta, \vec{\mathfrak{v}})$ by the equation

$$E_1(\eta, \vec{\mathfrak{v}}) = v'^2_\phi(\eta, \vec{\mathfrak{v}}; \eta^+). \tag{5-25}$$

We know $(\eta^+, 0, v'_\phi)$ at the axis $v_\eta = 0$ is on the same characteristics as $(\eta, \vec{\mathfrak{v}})$. Finally put

$$H_{\eta,\eta'} = \int_{\eta'}^{\eta} \frac{\nu(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; y))}{v'_\eta(\eta, \vec{\mathfrak{v}}; y)} \,\mathrm{d}y, \tag{5-26}$$

$$\mathscr{R}[H_{\eta,\eta'}] = \int_{\eta'}^{\eta} \frac{\nu(\mathscr{R}[\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; y)])}{v'_\eta(\eta, \vec{\mathfrak{v}}; y)} \,\mathrm{d}y. \tag{5-27}$$

Actually, since ν only depends on $|\vec{\mathfrak{v}}|$, we must have $H_{\eta,\eta'} = \mathscr{R}[H_{\eta,\eta'}]$. This distinction is for the purpose of clarity and does not play a role in the estimates. We can define the solution along the characteristics as

$$\mathscr{A}(\eta, \vec{\mathfrak{v}}) = \mathcal{K}[p_{\mathscr{A}}] + \mathcal{T}[\widetilde{\mathscr{A}} + S_{\mathscr{A}}], \tag{5-28}$$

where:

Region I: For $v_\eta > 0$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0)) \exp(-H_{\eta,0}), \tag{5-29}$$

$$\mathcal{T}[\widetilde{\mathscr{A}} + S_{\mathscr{A}}] = \int_0^\eta \frac{(\widetilde{\mathscr{A}} + S_{\mathscr{A}})(\eta', \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))}{v'_\eta(\eta, \vec{\mathfrak{v}}; \eta')} \exp(-H_{\eta,\eta'}) \,\mathrm{d}\eta'. \tag{5-30}$$

Region II: For $v_\eta < 0$ and $v_\eta^2 + v_\phi^2 \geq v'^2_\phi(\eta, \vec{\mathfrak{v}}; L)$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0)) \exp(-H_{L,0} - \mathscr{R}[H_{L,\eta}]), \tag{5-31}$$

$$\mathcal{T}[\widetilde{\mathscr{A}}+S_{\mathscr{A}}]=\left(\int_0^L \frac{(\widetilde{\mathscr{A}}+S_{\mathscr{A}})(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\,\mathrm{d}\eta' \right.$$
$$\left. +\int_\eta^L \frac{(\widetilde{\mathscr{A}}+S_{\mathscr{A}})(\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta'\right). \tag{5-32}$$

Region III: For $v_\eta<0$ and $v_\eta^2+v_\phi^2\leq v_\phi'^2(\eta,\vec{\mathfrak{v}};L)$,

$$\mathcal{K}[p_{\mathscr{A}}]=p_{\mathscr{A}}(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};0))\exp(-H_{\eta^+,0}-\mathscr{R}[H_{\eta^+,\eta}]), \tag{5-33}$$

$$\mathcal{T}[\widetilde{\mathscr{A}}+S_{\mathscr{A}}]=\left(\int_0^{\eta^+} \frac{(\widetilde{\mathscr{A}}+S_{\mathscr{A}})(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta^+,\eta'}-\mathscr{R}[H_{\eta^+,\eta}])\,\mathrm{d}\eta' \right.$$
$$\left. +\int_\eta^{\eta^+} \frac{(\widetilde{\mathscr{A}}+S_{\mathscr{A}})(\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta'\right). \tag{5-34}$$

The different regions are based on whether the characteristics touch $\eta=L$ and $v_\eta=0$ before tracking back to the boundary. In Region I, the characteristics do not touch any of them; in Region II, it touches $\eta=L$ first; in Region III, it touches $v_\eta=0$ first. This distinction marks different estimating methods since the expression highly depends on the value of v_η and whether it is reflected at $\eta=L$. Then we need to estimate $\mathcal{K}[p_{\mathscr{A}}]$ and $\mathcal{T}[\widetilde{\mathscr{A}}+S_{\mathscr{A}}]$ in each region. We assume $0<\delta\ll 1$ and $0<\delta_0\ll 1$ are small quantities which will be determined later.

5D. ***Region I*: $v_\eta>0$.** Based on [Wu 2016, Lemmas 4.9 and 4.10], we can directly obtain

$$\|\mathcal{K}[p_{\mathscr{A}}]\|_{L^\infty_{\vartheta,\varrho}}\leq\|p_{\mathscr{A}}\|_{L^\infty_{\vartheta,\varrho}}, \tag{5-35}$$

$$\|\mathcal{T}[S_{\mathscr{A}}]\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq\left\|\frac{S_{\mathscr{A}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-36}$$

Hence, we only need to estimate

$$I=\mathcal{T}[\widetilde{\mathscr{A}}]=\int_0^\eta \frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta'. \tag{5-37}$$

Since we always assume that $(\eta,\vec{\mathfrak{v}})$ and $(\eta',\vec{\mathfrak{v}}')$ are on the same characteristics, in the following, we will simply write $\vec{\mathfrak{v}}'(\eta')$ or even $\vec{\mathfrak{v}}'$ instead of $\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')$ when there is no confusion. We can use this notation interchangeably when necessary.

We divide it into several steps:

Step 0: preliminaries. We have

$$E_2(\eta',v_\phi')=\frac{R_\kappa-\epsilon\eta'}{R_\kappa}v_\phi'. \tag{5-38}$$

Then we can directly obtain

$$\begin{aligned}\zeta(\eta',\vec{\mathfrak{v}}')&=\frac{1}{R_\kappa}\sqrt{R_\kappa^2(v_\eta'^2+v_\phi'^2)-((R_\kappa-\epsilon\eta')v_\phi')^2}=\frac{1}{R_\kappa}\sqrt{R_\kappa^2v_\eta'^2+(R_\kappa^2-(R_\kappa-\epsilon\eta')^2)v_\phi'^2},\\ &\leq\frac{1}{R_\kappa}\sqrt{R_\kappa^2v_\eta'^2}+\frac{1}{R_\kappa}\sqrt{(R_\kappa^2-(R_\kappa-\epsilon\eta')^2)v_\phi'^2}\leq C(v_\eta'+\sqrt{\epsilon\eta'}v_\phi')\leq C\nu(\vec{\mathfrak{v}}')\end{aligned} \tag{5-39}$$

and

$$\zeta(\eta',\vec{\mathfrak{v}}')\geq\frac{1}{2}\left(\frac{1}{R_\kappa}\sqrt{R_\kappa^2v_\eta'^2}+\frac{1}{R_\kappa}\sqrt{(R_\kappa^2-(R_\kappa-\epsilon\eta')^2)v_\phi'^2}\right)\geq C(v_\eta'+\sqrt{\epsilon\eta'}v_\phi')\geq C\sqrt{\epsilon\eta'}|\vec{\mathfrak{v}}|. \tag{5-40}$$

Also, we know for $0\leq\eta'\leq\eta$,

$$\begin{aligned} v_\eta'&=\sqrt{v_\eta^2+v_\phi^2-v_\phi'^2}=\sqrt{v_\eta^2+v_\phi^2-v_\phi^2\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa-\epsilon\eta'}\right)^2}\\ &=\frac{\sqrt{(R_\kappa-\epsilon\eta')^2v_\eta^2+(2R_\kappa-\epsilon\eta-\epsilon\eta')(\epsilon\eta-\epsilon\eta')v_\phi^2}}{R_\kappa-\epsilon\eta'}. \end{aligned} \tag{5-41}$$

Since

$$0\leq(2R_\kappa-\epsilon\eta-\epsilon\eta')(\epsilon\eta-\epsilon\eta')v_\phi^2\leq 2R_\kappa\epsilon(\eta-\eta')v_\phi^2, \tag{5-42}$$

we have

$$v_\eta\leq v_\eta'\leq 2\sqrt{v_\eta^2+\epsilon(\eta-\eta')v_\phi^2}, \tag{5-43}$$

which means

$$\frac{1}{2\sqrt{v_\eta^2+\epsilon(\eta-\eta')v_\phi^2}}\leq\frac{1}{v_\eta'}\leq\frac{1}{v_\eta}. \tag{5-44}$$

Therefore,

$$\begin{aligned} -\int_{\eta'}^{\eta}\frac{1}{v_\eta'(\eta,\vec{\mathfrak{v}};y)}\,\mathrm{d}y&\leq-\int_{\eta'}^{\eta}\frac{1}{2\sqrt{v_\eta^2+\epsilon(\eta-y)v_\phi^2}}\,\mathrm{d}y=\frac{1}{\epsilon v_\phi^2}\big(v_\eta-\sqrt{v_\eta^2+\epsilon(\eta-\eta')v_\phi^2}\big)\\ &=-\frac{\eta-\eta'}{v_\eta+\sqrt{v_\eta^2+\epsilon(\eta-\eta')v_\phi^2}}\leq-\frac{\eta-\eta'}{2\sqrt{v_\eta^2+\epsilon(\eta-\eta')v_\phi^2}}. \end{aligned} \tag{5-45}$$

Define a cut-off function $\chi\in C^\infty[0,\infty)$ satisfying

$$\chi(v_\eta)=\begin{cases}1 & \text{for } |v_\eta|\leq\delta,\\ 0 & \text{for } |v_\eta|\geq 2\delta.\end{cases} \tag{5-46}$$

In the following, we will divide the estimate of I into several cases based on the values of v_η, v_η', $\epsilon\eta'$ and $\epsilon(\eta-\eta')$. Let $\mathbf{1}$ denote the indicator function. Take the dummy variable $\vec{\mathfrak{u}}=(\mathfrak{u}_\eta,\mathfrak{u}_\phi)$. We write

$$\begin{aligned} I&=\int_0^\eta\mathbf{1}_{\{v_\eta\geq\delta_0\}}+\int_0^\eta\mathbf{1}_{\{0\leq v_\eta\leq\delta_0\}}\mathbf{1}_{\{\chi(\mathfrak{u}_\eta)<1\}}+\int_0^\eta\mathbf{1}_{\{0\leq v_\eta\leq\delta_0\}}\mathbf{1}_{\{\chi(\mathfrak{u}_\eta)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}|v_\phi'|\geq v_\eta'\}}\\ &\quad+\int_0^\eta\mathbf{1}_{\{0\leq v_\eta\leq\delta_0\}}\mathbf{1}_{\{\chi(\mathfrak{u}_\eta)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}|v_\phi'|\leq v_\eta'\}}\mathbf{1}_{\{v_\eta^2\leq\epsilon(\eta-\eta')v_\phi^2\}}\\ &\quad\quad+\int_0^\eta\mathbf{1}_{\{0\leq v_\eta\leq\delta_0\}}\mathbf{1}_{\{\chi(\mathfrak{u}_\eta)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}|v_\phi'|\leq v_\eta'\}}\mathbf{1}_{\{v_\eta^2\geq\epsilon(\eta-\eta')v_\phi^2\}}\\ &=I_1+I_2+I_3+I_4+I_5. \end{aligned} \tag{5-47}$$

<u>Step 1</u>: estimate of I_1 for $v_\eta\geq\delta_0$. In this step, we will prove estimates based on the characteristics of $\mathcal{G}$ itself instead of $\mathscr{A}$. Here, we rewrite (5-4) along the characteristics as

$$v_\eta\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\eta}+\nu\mathcal{G}=K[\mathcal{G}]. \tag{5-48}$$

In the following, we will repeatedly use simple facts (SF):

- Based on the well-posedness and decay theorem for $\mathcal{G}$, we know $\|\mathcal{G}\|_{L^\infty_{\vartheta,\varrho}} \leq C$.
- Based on Lemma 2.4, we get $\|K[\mathcal{G}]\|_{L^\infty_{\vartheta,\varrho}} \leq \|\mathcal{G}\|_{L^\infty_{\vartheta,\varrho}} \leq C$ and $\|\nabla_v K[\mathcal{G}]\|_{L^\infty_{\vartheta,\varrho}} \leq \|\mathcal{G}\|_{L^\infty_{\vartheta,\varrho}} \leq C$.
- Since E_1 is conserved along the characteristics, we must have $|\vec{\mathfrak{v}}| = |\vec{\mathfrak{v}}'|$.
- For $\eta' \leq \eta$, we must have $v'_\eta \geq v_\eta \geq \delta_0$.
- Using the substitution $y = H_{\eta,\eta'}$, we know

$$\left|\int_0^\eta \frac{\nu(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))}{v'_\eta(\eta, \vec{\mathfrak{v}}; \eta')} \exp(-H_{\eta,\eta'})\, d\eta'\right| \leq \left|\int_0^\infty e^{-y}\, dy\right| = 1. \tag{5-49}$$

For $v_\eta \geq \delta_0$, we do not need the mild formulation for $\mathscr{A}$. Instead, we directly estimate

$$|\langle\vec{\mathfrak{v}}\rangle^\vartheta e^{\varrho|\vec{\mathfrak{v}}|^2} I_1| \leq \left|\langle\vec{\mathfrak{v}}\rangle^\vartheta e^{\varrho|\vec{\mathfrak{v}}|^2} \frac{\partial\mathcal{G}}{\partial\eta}\right|. \tag{5-50}$$

We rewrite (5-4) along the characteristics as

$$\mathcal{G}(\eta, \vec{\mathfrak{v}}) = \exp(-H_{\eta,0})\left(p(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0)) + \int_0^\eta \frac{K[\mathcal{G}](\eta', \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))}{v'_\eta(\eta, \vec{\mathfrak{v}}; \eta')} \exp(H_{\eta',0})\, d\eta'\right), \tag{5-51}$$

where $(\eta', \vec{\mathfrak{v}}')$ and $(\eta, \vec{\mathfrak{v}})$ are on the same characteristic with $v'_\eta \geq 0$, and

$$H_{t,s} = \int_s^t \frac{\nu(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; y))}{v'_\eta(\eta, \vec{\mathfrak{v}}; y)}\, dy \tag{5-52}$$

for any $s, t \geq 0$.

Taking the η-derivative of both sides of (5-51), we have

$$\frac{\partial\mathcal{G}}{\partial\eta} = X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6, \tag{5-53}$$

where

$$X_1 = -\exp(-H_{\eta,0})\frac{\partial H_{\eta,0}}{\partial\eta}\left(p(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0)) + \int_0^\eta \frac{K[\mathcal{G}](\eta', \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))}{v'_\eta(\eta')} \exp(H_{\eta',0})\, d\eta'\right), \tag{5-54}$$

$$X_2 = \exp(-H_{\eta,0})\frac{\partial p(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0))}{\partial\eta}, \tag{5-55}$$

$$X_3 = \frac{K[\mathcal{G}](\eta, \vec{\mathfrak{v}})}{v_\eta}, \tag{5-56}$$

$$X_4 = -\exp(-H_{\eta,0})\int_0^\eta \left(K[\mathcal{G}](\eta', \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))\exp(H_{\eta',0})\frac{1}{v'^2_\eta(\eta, \vec{\mathfrak{v}}; \eta')}\frac{\partial v'_\eta(\eta, \vec{\mathfrak{v}}; \eta')}{\partial\eta}\, d\eta'\right), \tag{5-57}$$

$$X_5 = \exp(-H_{\eta,0})\int_0^\eta \frac{K[\mathcal{G}](\eta', \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))}{v'_\eta(\eta, \vec{\mathfrak{v}}; \eta')}\exp(H_{\eta',0})\frac{\partial H_{\eta',0}}{\partial\eta}\, d\eta', \tag{5-58}$$

$$X_6 = \exp(-H_{\eta,0})\int_0^\eta \frac{1}{v'_\eta(\eta, \vec{\mathfrak{v}}; \eta')}\left(\nabla_{\mathfrak{v}'}K[\mathcal{G}](\eta', \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))\frac{\partial\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta')}{\partial\eta}\right)\exp(H_{\eta',0})\, d\eta'. \tag{5-59}$$

We need to estimate each term. Note that

$$
\begin{aligned}
\frac{\partial H_{t,s}}{\partial\eta} &= \int_s^t \frac{\partial}{\partial\eta}\biggl(\frac{\nu(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};y))}{v_\eta'(\eta,\vec{\mathfrak{v}};y)}\biggr)\,\mathrm{d}y \\
&= \int_s^t \frac{1}{v_\eta'(\eta,\vec{\mathfrak{v}};y)}\frac{\partial\nu(|\vec{\mathfrak{v}}'|)}{\partial|\vec{\mathfrak{v}}'|}\frac{1}{|\vec{\mathfrak{v}}'|}\biggl(v_\eta'(\eta,\vec{\mathfrak{v}};y)\frac{\partial v_\eta'(\eta,\vec{\mathfrak{v}};y)}{\partial\eta}+v_\phi'(\eta,\vec{\mathfrak{v}};y)\frac{\partial v_\phi'(\eta,\vec{\mathfrak{v}};y)}{\partial\eta}\biggr)\,\mathrm{d}y \\
&\qquad - \int_s^t \frac{\nu(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};y))}{v_\eta'^2(\eta,\vec{\mathfrak{v}};y)}\frac{\partial v_\eta'(\eta,\vec{\mathfrak{v}};y)}{\partial\eta}\,\mathrm{d}y. \qquad (5\text{-}60)
\end{aligned}
$$

Considering

$$
v_\phi'(\eta,\vec{\mathfrak{v}};y) = v_\phi \mathrm{e}^{W(\eta')-W(\eta)} = v_\phi\frac{R_\kappa-\epsilon\eta}{R_\kappa-\epsilon\eta'}, \qquad (5\text{-}61)
$$

$$
v_\eta'(\eta,\vec{\mathfrak{v}};y) = \sqrt{v_\eta^2+v_\phi^2-v_\phi'^2} = \sqrt{v_\eta^2+v_\phi^2-v_\phi^2\biggl(\frac{R_\kappa-\epsilon\eta}{R_\kappa-\epsilon\eta'}\biggr)^2}, \qquad (5\text{-}62)
$$

we know

$$
\frac{\partial v_\phi'(\eta,\vec{\mathfrak{v}};y)}{\partial\eta} = \frac{\epsilon v_\phi}{R_\kappa-\epsilon\eta'}, \qquad \frac{\partial v_\eta'(\eta,\vec{\mathfrak{v}};y)}{\partial\eta} = \frac{2\epsilon v_\phi^2}{v_\eta'(\eta,\vec{\mathfrak{v}};y)}\frac{R_\kappa-\epsilon\eta}{R_\kappa-\epsilon\eta'}. \qquad (5\text{-}63)
$$

This implies

$$
\biggl|\frac{\partial v_\eta'(\eta,\vec{\mathfrak{v}};y)}{\partial\eta}\biggr| \le \frac{C\epsilon|\vec{\mathfrak{v}}|}{v_\eta'(\eta,\vec{\mathfrak{v}};y)} \le \frac{C\epsilon|\vec{\mathfrak{v}}|}{\delta_0}, \qquad \biggl|\frac{\partial v_\phi'(\eta,\vec{\mathfrak{v}};y)}{\partial\eta}\biggr| \le C\epsilon|\vec{\mathfrak{v}}|. \qquad (5\text{-}64)
$$

The method to estimate X_i is standard and we simply use the facts (SF) and direct computation, so we omit the details and only list the result:

$$
|\langle\vec{v}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{v}|^2}X_1| \le \frac{C}{\delta_0}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}, \qquad (5\text{-}65)
$$

$$
|\langle\vec{v}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{v}|^2}X_2| \le \frac{C}{\delta_0}\biggl(\biggl\|\frac{\partial p}{\partial v_\eta}\biggr\|_{L^\infty_{\vartheta,\varrho}}+\biggl\|\frac{\partial p}{\partial v_\phi}\biggr\|_{L^\infty_{\vartheta,\varrho}}\biggr), \qquad (5\text{-}66)
$$

$$
|\langle\vec{v}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{v}|^2}X_i| \le \frac{C}{\delta_0}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \quad \text{for } i=3,\dots,6. \qquad (5\text{-}67)
$$

In summary, we have

$$
|\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_1| \le \frac{C}{\delta_0}\biggl(\biggl\|\frac{\partial p}{\partial v_\eta}\biggr\|_{L^\infty_{\vartheta,\varrho}}+\biggl\|\frac{\partial p}{\partial v_\phi}\biggr\|_{L^\infty_{\vartheta,\varrho}}+\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\biggr). \qquad (5\text{-}68)
$$

Step 2: estimate of I_2 for $0\le v_\eta\le\delta_0$ and $\chi(\mathfrak{u}_\eta)<1$. We have

$$
\begin{aligned}
I_2 &= \int_0^\eta\biggl(\int_{\mathbb{R}^2}\frac{\zeta(\eta',\vec{\mathfrak{v}}')}{\zeta(\eta',\vec{\mathfrak{u}})}(1-\chi(\mathfrak{u}_\eta))k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\biggr)\frac{1}{v_\eta'}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta' \\
&= \int_0^\eta\biggl(\int_{\mathbb{R}^2}(1-\chi(\mathfrak{u}_\eta))k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\frac{\mathcal{G}(\eta',\vec{\mathfrak{u}})}{\partial\eta'}\,\mathrm{d}\vec{\mathfrak{u}}\biggr)\frac{\zeta(\eta',\vec{\mathfrak{v}}')}{v_\eta'}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta'. \qquad (5\text{-}69)
\end{aligned}
$$

Based on the ϵ-Milne problem with the geometric correction of $\mathcal{G}$ as

$$\mathrm{u}_\eta\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\eta'}+G(\eta')\left(\mathrm{u}_\phi^2\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\mathrm{u}_\eta}-\mathrm{u}_\eta\mathrm{u}_\phi\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\mathrm{u}_\phi}\right)+\nu\mathcal{G}(\eta',\vec{\mathrm{u}})-K[\mathcal{G}](\eta',\vec{\mathrm{u}})=0, \tag{5-70}$$

we have

$$\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\eta'}=-\frac{1}{\mathrm{u}_\eta}\left(G(\eta')\left(\mathrm{u}_\phi^2\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\mathrm{u}_\eta}-\mathrm{u}_\eta\mathrm{u}_\phi\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\mathrm{u}_\phi}\right)+\nu\mathcal{G}(\eta',\vec{\mathrm{u}})-K[\mathcal{G}](\eta',\vec{\mathrm{u}})\right). \tag{5-71}$$

Hence, we have

$$\begin{aligned}\tilde{A}&:=\int_{\mathbb{R}^2}(1-\chi(\mathrm{u}_\eta))k(\vec{\mathrm{u}},\vec{\mathfrak{v}}')\frac{\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\eta'}\,\mathrm{d}\vec{\mathrm{u}}\\&=-\int_{\mathbb{R}^2}(1-\chi(\mathrm{u}_\eta))k(\vec{\mathrm{u}},\vec{\mathfrak{v}}')\frac{1}{\mathrm{u}_\eta}\big(\nu\mathcal{G}(\eta',\vec{\mathrm{u}})-K[\mathcal{G}](\eta',\vec{\mathrm{u}})\big)\,\mathrm{d}\vec{\mathrm{u}}\\&\qquad-\int_{\mathbb{R}^2}(1-\chi(\mathrm{u}_\eta))k(\vec{\mathrm{u}},\vec{\mathfrak{v}}')\frac{1}{\mathrm{u}_\eta}G(\eta')\left(\mathrm{u}_\phi^2\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\mathrm{u}_\eta}-\mathrm{u}_\eta\mathrm{u}_\phi\frac{\partial\mathcal{G}(\eta',\vec{\mathrm{u}})}{\partial\mathrm{u}_\phi}\right)\mathrm{d}\vec{\mathrm{u}}\\&=\tilde{A}_1+\tilde{A}_2.\end{aligned} \tag{5-72}$$

Using $\mathcal{G}$ estimates and $|\mathrm{u}_\eta|\geq\delta$, we may directly obtain

$$\begin{aligned}\big|\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\tilde{A}_1\big|&\leq\left|\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}(1-\chi(\mathrm{u}_\eta))k(\vec{\mathrm{u}},\vec{\mathfrak{v}}')\frac{1}{\mathrm{u}_\eta}\big(\nu\mathcal{G}(\eta',\vec{\mathrm{u}})-K[\mathcal{G}](\eta',\vec{\mathrm{u}})\big)\,\mathrm{d}\vec{\mathrm{u}}\right|\\&\leq\frac{C}{\delta}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.\end{aligned} \tag{5-73}$$

On the other hand, an integration by parts yields

$$\begin{aligned}\tilde{A}_2=&\int_{\mathbb{R}^2}\frac{\partial}{\partial\mathrm{u}_\eta}\left(\frac{\mathrm{u}_\phi^2}{\mathrm{u}_\eta}G(\eta')(1-\chi(\mathrm{u}_\eta))k(\vec{\mathrm{u}},\vec{\mathfrak{v}}')\right)\mathcal{G}(\eta',\vec{\mathrm{u}})\,\mathrm{d}\vec{\mathrm{u}}\\&-\int_{\mathbb{R}^2}\frac{\partial}{\partial\mathrm{u}_\phi}\big(\mathrm{u}_\phi G(\eta')(1-\chi(\mathrm{u}_\eta))k(\vec{\mathrm{u}},\vec{\mathfrak{v}}')\big)\mathcal{G}(\eta',\vec{\mathrm{u}})\,\mathrm{d}\vec{\mathrm{u}},\end{aligned} \tag{5-74}$$

which further implies

$$\big|\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\tilde{A}_2\big|\leq\frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-75}$$

As before we can use substitution to show that

$$\left|\int_0^\eta\frac{\zeta(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta'\right|\leq\left|\int_0^\eta\frac{\nu(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta'\right|\leq 1, \tag{5-76}$$

and $|\vec{\mathfrak{v}}|$ is a constant along the characteristics. Then we have

$$\big|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_2\big|\leq\frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.$$

<u>Step 3</u>: estimate of I_3 for $0\leq v_\eta\leq\delta_0$, $\chi(\mathrm{u}_\eta)=1$ and $\sqrt{\epsilon\eta'}v'_\phi\geq v'_\eta$. Based on (5-39), this implies

$$\zeta(\eta',\vec{\mathfrak{v}}')\leq C\sqrt{\epsilon\eta'}v'_\phi.$$

Also, we know that

$$\zeta(\eta',\vec{\mathfrak{u}})\geq C\sqrt{\epsilon\eta'}|\vec{\mathfrak{u}}|.$$

Then we may take the decomposition

$$\begin{aligned}\tilde{A}&:=\int_{\mathbb{R}^2}\frac{\zeta(\eta',\vec{\mathfrak{v}}')}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\\&\leq\int_{|\vec{\mathfrak{u}}|\geq\sqrt{\delta}}\frac{v'_\phi}{|\vec{\mathfrak{u}}|}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}+\int_{|\vec{\mathfrak{u}}|\leq\sqrt{\delta}}\frac{v'_\phi}{|\vec{\mathfrak{u}}|}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\\&=v'_\phi(\tilde{A}_1+\tilde{A}_2).\end{aligned}\tag{5-77}$$

Using Lemma 2.3, we directly estimate

$$\begin{aligned}\left|\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\tilde{A}_1\right|&\leq C\left|\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{|\vec{\mathfrak{u}}|\geq\sqrt{\delta}}\frac{1}{|\vec{\mathfrak{u}}|}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right|\\&\leq\frac{C}{\sqrt{\delta}}\left|\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{|\mathfrak{u}_\eta|\leq2\delta}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right|\leq C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.\end{aligned}\tag{5-78}$$

Also, based on Lemma 2.3, we obtain

$$\left|\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\tilde{A}_2\right|\leq C\left|\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{|\vec{\mathfrak{u}}|\leq\sqrt{\delta}}\frac{1}{|\vec{\mathfrak{u}}|}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right|\leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.\tag{5-79}$$

Hence, since $|\vec{\mathfrak{v}}|$ is a constant along the characteristics, we have

$$\begin{aligned}\left|\langle\vec{\mathfrak{v}}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_3\right|&\leq C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\left(\int_0^\eta\frac{v'_\phi}{v'_\eta}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta'\right)\\&\leq C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\left(\int_0^\eta\frac{\nu(\vec{\mathfrak{v}}')}{v'_\eta}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta'\right)\leq C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.\end{aligned}\tag{5-80}$$

<u>Step 4</u>: estimate of I_4 for $0\leq v_\eta\leq\delta_0$, $\chi(\mathfrak{u}_\eta)=1$, $\sqrt{\epsilon\eta'}v'_\phi\leq v'_\eta$ and $v_\eta^2\leq\epsilon(\eta-\eta')v_\phi^2$. Based on (5-39), this implies

$$\zeta(\eta',\vec{\mathfrak{v}}')\leq Cv'_\eta.\tag{5-81}$$

Based on (5-45), we have

$$-H_{\eta,\eta'}=-\int_{\eta'}^\eta\frac{\nu(\vec{\mathfrak{v}})}{v'_\eta(y)}\,\mathrm{d}y\leq-\frac{\nu(\vec{\mathfrak{v}})(\eta-\eta')}{2v_\phi\sqrt{\epsilon(\eta-\eta')}}\leq-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta-\eta'}{\epsilon}}.\tag{5-82}$$

Hence, since $|\vec{\mathfrak{v}}|$ is a constant along the characteristics, we know

$$\begin{aligned}\left|\langle\vec{\mathfrak{v}}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_4\right|&\leq C\int_0^\eta\left|\left(\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{\zeta(\eta',\vec{\mathfrak{v}}')}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right)\frac{1}{v'_\eta}\exp(-H_{\eta,\eta'})\right|\mathrm{d}\eta'\\&\leq C\int_0^\eta\left|\left(\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{\epsilon\eta'}|\vec{\mathfrak{u}}|}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right)\frac{v'_\eta}{v'_\eta}\exp(-H_{\eta,\eta'})\right|\mathrm{d}\eta'\\&=C\int_0^\eta\left|\left(\langle\vec{\mathfrak{v}}'\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{1}{|\vec{\mathfrak{u}}|}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right)\frac{1}{\sqrt{\epsilon\eta'}}\exp(-H_{\eta,\eta'})\right|\mathrm{d}\eta'.\end{aligned}\tag{5-83}$$

Using an argument similar to that in Step 3, we obtain

$$\left|\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{1}{|\vec{\mathfrak{u}}|}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right|\le C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-84}$$

Hence,

$$\begin{aligned}|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_4|&\le C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\left(\int_0^\eta\frac{1}{\sqrt{\epsilon\eta'}}\exp(-H_{\eta,\eta'})\,\mathrm{d}\eta'\right)\\&\le C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\int_0^\eta\frac{1}{\sqrt{\epsilon\eta'}}\exp\left(-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta-\eta'}{\epsilon}}\right)\mathrm{d}\eta'.\end{aligned}$$

Define $z=\eta'/\epsilon$, which implies $\mathrm{d}\eta'=\epsilon\,\mathrm{d}z$. Substituting this into the above integral, we have

$$\begin{aligned}&|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_4|\\&\le C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\int_0^{\frac{\eta}{\epsilon}}\frac{1}{\sqrt{z}}\exp\left(-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta}{\epsilon}-z}\right)\mathrm{d}z\\&=C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\left(\int_0^1\frac{1}{\sqrt{z}}\exp\left(-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta}{\epsilon}-z}\right)\mathrm{d}z+\int_1^{\frac{\eta}{\epsilon}}\frac{1}{\sqrt{z}}\exp\left(-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta}{\epsilon}-z}\right)\mathrm{d}z\right).\end{aligned}\tag{5-85}$$

We can estimate these two terms separately:

$$\int_0^1\frac{1}{\sqrt{z}}\exp\left(-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta}{\epsilon}-z}\right)\mathrm{d}z\le\int_0^1\frac{1}{\sqrt{z}}\,\mathrm{d}z=2, \tag{5-86}$$

$$\begin{aligned}\int_1^{\frac{\eta}{\epsilon}}\frac{1}{\sqrt{z}}\exp\left(-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta}{\epsilon}-z}\right)\mathrm{d}z&\le\int_1^{\frac{\eta}{\epsilon}}\exp\left(-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}\sqrt{\frac{\eta}{\epsilon}-z}\right)\mathrm{d}z\\&\overset{t^2=\frac{\eta}{\epsilon}-z}{\le}2\int_0^\infty t\mathrm{e}^{-\frac{C'\nu(\vec{\mathfrak{v}})}{v_\phi}t}\,\mathrm{d}t<C\left(\frac{v_\phi}{\nu(\vec{\mathfrak{v}})}\right)^2\le C.\end{aligned}\tag{5-87}$$

Therefore, we know

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_4|\le C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-88}$$

<u>Step 5</u>: estimate of I_5 for $0\le v_\eta\le\delta_0$, $\chi(\mathfrak{u}_\eta)=1$, $\sqrt{\epsilon\eta'}v'_\phi\le v'_\eta$ and $v_\eta^2\ge\epsilon(\eta-\eta')v_\phi^2$. Based on (5-39), this implies

$$\zeta(\eta',\vec{\mathfrak{v}}')\le Cv'_\eta.$$

Based on (5-45), we have

$$-H_{\eta,\eta'}=-\int_{\eta'}^{\eta}\frac{\nu(\vec{\mathfrak{v}}')}{v'_\eta(y)}\,\mathrm{d}y\le-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}. \tag{5-89}$$

Hence, we know

$$\begin{aligned}&|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_5|\\&\quad\le C\int_0^\eta\left|\left(\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{\zeta(\eta',\vec{\mathfrak{v}}')}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right)\frac{1}{v'_\eta}\exp(-H_{\eta,\eta'})\right|\mathrm{d}\eta'\\&\quad\le C\int_0^\eta\left|\left(\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right)\frac{v'_\eta}{v'_\eta}\exp(-H_{\eta,\eta'})\right|\mathrm{d}\eta'\\&\quad\le C\int_0^\eta\left|\left(\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right)\exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\right)\right|\mathrm{d}\eta'.\end{aligned}\tag{5-90}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right|\\
&\leq\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\left|\frac{1}{\zeta^{\frac{1}{1+s}}(\eta',\vec{\mathfrak{u}})}\frac{1}{\zeta^{\frac{s}{1+s}}(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\\
&\leq\frac{C}{(\epsilon\eta')^{\frac{s}{1+s}}}\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\left|\frac{1}{\zeta^{\frac{1}{1+s}}(\eta',\vec{\mathfrak{u}})}\frac{1}{|\mathfrak{u}_\phi|^{\frac{s}{1+s}}}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\\
&\leq\frac{C}{(\epsilon\eta')^{\frac{s}{1+s}}}\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\left|\frac{1}{\zeta^{\frac{1}{1+s}}(\eta',\vec{\mathfrak{u}})}\frac{1}{|\mathfrak{u}_\phi|^{\frac{s}{1+s}}}\frac{1}{|\mathfrak{u}_\phi|^{\frac{1}{1+2s}}}\chi(\mathfrak{u}_\eta)|\mathfrak{u}_\phi|^{\frac{1}{1+2s}}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\\
&\leq\frac{C\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{(\epsilon\eta')^{\frac{s}{1+s}}}\left(\int_{\mathbb{R}^2}\left|\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\frac{1}{|\mathfrak{u}_\phi|^{s+\frac{1+s}{1+2s}}}\chi^{1+s}(\mathfrak{u}_\eta)\right|\mathrm{d}\vec{\mathfrak{u}}\right)^{\frac{1}{1+s}}\\
&\qquad\times\left(\left(\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\right)^{\frac{1+s}{s}}\int_{\mathbb{R}^2}\left|\frac{\mathfrak{u}_\phi^{\frac{1}{1+2s}}}{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\right|^{\frac{1+s}{s}}\mathrm{d}\vec{\mathfrak{u}}\right)^{\frac{s}{1+s}},
\end{aligned}\tag{5-91}$$

where $0<s\ll 1$. Note the fact that in two dimensions, $k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')$ does not contain the singularity of $|\vec{\mathfrak{u}}-\vec{\mathfrak{v}}'|^{-1}$. Using an argument similar to that in Step 3, we may directly compute

$$\begin{aligned}
&\left(\left(\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\right)^{\frac{1+s}{s}}\int_{\mathbb{R}^2}\left|\frac{\mathfrak{u}_\phi^{\frac{1}{1+2s}}}{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\right|^{\frac{1+s}{s}}\mathrm{d}\vec{\mathfrak{u}}\right)^{\frac{s}{1+s}}\\
&\qquad\leq\left(\left(\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\right)^{\frac{1+s}{s}}\int_{\mathbb{R}^2}\left|\frac{\langle\vec{\mathfrak{u}}\rangle^{\frac{1}{1+2s}}}{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\right|^{\frac{1+s}{s}}\mathrm{d}\vec{\mathfrak{u}}\right)^{\frac{s}{1+s}}\\
&\qquad\leq C\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.
\end{aligned}\tag{5-92}$$

Then we estimate

$$\int_{\mathbb{R}^2}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\frac{1}{|\mathfrak{u}_\phi|^{s+\frac{1+s}{1+2s}}}\chi^{1+s}(\mathfrak{u}_\eta)\,\mathrm{d}\vec{\mathfrak{u}}\leq\int_{\mathbb{R}}\left(\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)\,\mathrm{d}\mathfrak{u}_\eta\right)\frac{1}{|\mathfrak{u}_\phi|^{s+\frac{1+s}{1+2s}}}\,\mathrm{d}\mathfrak{u}_\phi.\tag{5-93}$$

We may directly compute

$$\begin{aligned}
\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)\,\mathrm{d}\mathfrak{u}_\eta&\leq\int_{-\delta}^{\delta}\frac{1}{\sqrt{R_\kappa^2(\mathfrak{u}_\eta)^2+(R_\kappa^2-(R_\kappa-\epsilon\eta')^2)(\mathfrak{u}_\phi)^2}}\chi(\mathfrak{u}_\eta)\,\mathrm{d}\mathfrak{u}_\eta\\
&\leq C\int_{-\delta}^{\delta}\frac{1}{\sqrt{(\mathfrak{u}_\eta)^2+r^2(\mathfrak{u}_\phi)^2}}\,\mathrm{d}\mathfrak{u}_\eta,
\end{aligned}\tag{5-94}$$

where

$$r=\sqrt{\frac{R_\kappa^2-(R_\kappa-\epsilon\eta')^2}{R_\kappa^2}}\leq C\sqrt{\epsilon\eta'}.\tag{5-95}$$

We may further compute

$$\begin{aligned}\int_{-\delta}^{\delta}\frac{1}{\sqrt{(\mathfrak{u}_\eta)^2+r^2(\mathfrak{u}_\phi)^2}}\,\mathrm{d}\mathfrak{u}_\eta&=2\int_0^{\delta}\frac{1}{\sqrt{(\mathfrak{u}_\eta)^2+r^2(\mathfrak{u}_\phi)^2}}\,\mathrm{d}\mathfrak{u}_\eta\\&=2\Big(\ln\Big(\mathfrak{u}_\eta+\sqrt{r^2(\mathfrak{u}_\phi)^2+(\mathfrak{u}_\eta)^2}\Big)-\ln(r\mathfrak{u}_\phi)\Big)\Big|_0^\delta\\&=2\Big(\ln\Big(\delta+\sqrt{r^2(\mathfrak{u}_\phi)^2+\delta^2}\Big)-\ln(r\mathfrak{u}_\phi)\Big)\\&\le C(1+\ln(r)+\ln(\mathfrak{u}_\phi)).\end{aligned}\tag{5-96}$$

Then considering $s+(1+s)/(1+2s)>1$, we know

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\frac{1}{|\mathfrak{u}_\phi|^{s+\frac{1+s}{1+2s}}}\chi^{1+s}(\mathfrak{u}_\eta)\,\mathrm{d}\vec{\mathfrak{u}}&\le C\int_{\mathbb{R}}(1+\ln(r)+\ln(\mathfrak{u}_\phi))\frac{1}{|\mathfrak{u}_\phi|^{s+\frac{1+s}{1+2s}}}\,\mathrm{d}\mathfrak{u}_\phi\\&\le C(1+|\ln(\epsilon)|+|\ln(\eta')|).\end{aligned}\tag{5-97}$$

Hence, we can obtain

$$\left|\langle\vec{\mathfrak{v}}'\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}'|^2}\int_{\mathbb{R}^2}\frac{1}{\zeta(\eta',\vec{\mathfrak{u}})}\chi(\mathfrak{u}_\eta)k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}')\mathscr{A}(\eta',\vec{\mathfrak{u}})\,\mathrm{d}\vec{\mathfrak{u}}\right|\le\frac{C\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{(\epsilon\eta')^{\frac{s}{1+s}}}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}(1+|\ln(\epsilon)|+|\ln(\eta')|).\tag{5-98}$$

Thus, we know

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_5|\le\frac{C}{\epsilon^s}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\int_0^\eta\frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}(1+|\ln(\epsilon)|+|\ln(\eta')|)\exp\bigg(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\bigg)\,\mathrm{d}\eta'.\tag{5-99}$$

Then we first estimate

$$\left|\int_0^\eta\frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}|\ln(\eta')|\exp\bigg(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\bigg)\,\mathrm{d}\eta'\right|.\tag{5-100}$$

If $0\le\eta\le 2$, using Hölder's inequality, we have

$$\begin{aligned}&\left|\int_0^\eta\frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}|\ln(\eta')|\exp\bigg(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\bigg)\,\mathrm{d}\eta'\right|\\&\qquad\le\bigg(\int_0^\eta\frac{1}{\eta'^{\frac12}}|\ln(\eta')|^{\frac{1+2s}{2s}}\,\mathrm{d}\eta'\bigg)^{\frac{2s}{1+2s}}\bigg(\int_0^\eta\langle\vec{\mathfrak{v}}'\rangle\exp\bigg(-\frac{(1+2s)\nu(\vec{\mathfrak{v}}')C(\eta-\eta')}{v_\eta}\bigg)\,\mathrm{d}\eta'\bigg)^{\frac{1}{1+2s}}\\&\qquad\le C\bigg(\frac{v_\eta\langle\vec{\mathfrak{v}}'\rangle}{\nu(\vec{\mathfrak{v}}')}\bigg)^{\frac{1}{1+2s}}\le v_\eta^{\frac{1}{1+2s}}\le C\delta_0^{\frac{1}{1+2s}}\le\sqrt{\delta_0}.\end{aligned}\tag{5-101}$$

If $\eta\ge 2$, it suffices to estimate

$$\begin{aligned}\left|\int_2^\eta\frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}|\ln(\eta')|\exp\bigg(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\bigg)\,\mathrm{d}\eta'\right|&\le\ln(L)\left|\int_2^\eta\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}\exp\bigg(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\bigg)\,\mathrm{d}\eta'\right|\\&\le C|\ln(\epsilon)|v_\eta\le C|\ln(\epsilon)|\delta_0.\end{aligned}\tag{5-102}$$

With a similar argument, we may justify

$$\left|\int_0^{\eta}(1+|\ln(\epsilon)|)\frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}\exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\right)\right|\leq C(\sqrt{\delta_0}+|\ln(\epsilon)|\delta_0). \tag{5-103}$$

Hence, we have

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I_5|\leq\frac{C}{\epsilon^s}(\sqrt{\delta_0}+|\ln(\epsilon)|\delta_0)\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-104}$$

<u>Step 6</u>: synthesis. Collecting all the terms in previous steps, we have proved

$$\begin{aligned}|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}I|\leq\frac{C}{\epsilon^s}(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}&\\ +\frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\frac{C}{\delta_0}\left(\left\|\frac{\partial p}{\partial v_\eta}\right\|_{L^\infty_{\vartheta,\varrho}}+\left\|\frac{\partial p}{\partial v_\phi}\right\|_{L^\infty_{\vartheta,\varrho}}+\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right).&\end{aligned} \tag{5-105}$$

5E. ***Region II*: $v_\eta<0$ *and* $v_\eta^2+v_\phi^2\geq v_\phi'^2(\eta,\vec{\mathfrak{v}};L)$.** Based on [Wu 2016, Lemmas 4.9 and 4.10], we can directly obtain

$$|\mathcal{K}[p_{\mathscr{A}}]|\leq\|p_{\mathscr{A}}\|_{L^\infty_{\vartheta,\varrho}}, \tag{5-106}$$

$$|\mathcal{T}[S_{\mathscr{A}}]|\leq\left\|\frac{S_{\mathscr{A}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-107}$$

Hence, we only need to estimate

$$\begin{aligned}II=\mathcal{T}[\widetilde{\mathscr{A}}]=\int_0^L\frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\,\mathrm{d}\eta'&\\ +\int_\eta^L\frac{\widetilde{\mathscr{A}}(\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta'.&\end{aligned} \tag{5-108}$$

In particular, we can take the decomposition

$$\begin{aligned}\mathcal{T}[\widetilde{\mathscr{A}}]=\int_0^\eta\frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\,\mathrm{d}\eta'&\\ +\int_\eta^L\frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\,\mathrm{d}\eta'&\\ +\int_\eta^L\frac{\widetilde{\mathscr{A}}(\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta'.&\end{aligned} \tag{5-109}$$

The integral $\int_0^\eta$ part can be estimated as in Region I, so we only need to estimate the integral $\int_\eta^L$ part. Also, noting that fact that

$$\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\leq\exp(-\mathscr{R}[H_{\eta',\eta}]), \tag{5-110}$$

we only need to estimate

$$\int_\eta^L\frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta',\eta})\,\mathrm{d}\eta'. \tag{5-111}$$

Here the proof is almost identical to that in Region I, so we only point out the key differences.

Step 0: preliminaries. We need to update one key result. For $0 \leq \eta \leq \eta'$,

$$v'_\eta = \sqrt{E_1 - v'^2_\phi} = \sqrt{E_1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa - \epsilon\eta'}\right)^2 v_\phi^2} \leq v_\eta. \tag{5-112}$$

Then we have

$$-\int_\eta^{\eta'} \frac{1}{v'_\eta(y)}\,\mathrm{d}y \leq -\frac{\eta' - \eta}{v_\eta}. \tag{5-113}$$

In the following, we will divide the estimate of II into several cases based on the values of v_η, v'_η and $\epsilon\eta'$. We write

$$\begin{aligned} II &= \int_\eta^L \mathbf{1}_{\{v_\eta \leq -\delta_0\}} + \int_\eta^L \mathbf{1}_{\{-\delta_0 \leq v_\eta \leq 0\}} \mathbf{1}_{\{\chi(\mathfrak{u}_\eta) < 1\}} \\ &\qquad + \int_\eta^L \mathbf{1}_{\{-\delta_0 \leq v_\eta \leq 0\}} \mathbf{1}_{\{\chi(\mathfrak{u}_\eta) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} v'_\phi \geq v'_\eta\}} + \int_\eta^L \mathbf{1}_{\{-\delta_0 \leq v_\eta \leq 0\}} \mathbf{1}_{\{\chi(\mathfrak{u}_\eta) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} v'_\phi \leq v'_\eta\}} \\ &= II_1 + II_2 + II_3 + II_4. \end{aligned} \tag{5-114}$$

Step 1: estimate of II_1 for $v_\eta \leq -\delta_0$. We first estimate v'_η. Along the characteristics, we know

$$\mathrm{e}^{-W(\eta')} v'_\phi = \mathrm{e}^{-W(\eta)} v_\phi, \tag{5-115}$$

which implies

$$|v'_\phi| = \mathrm{e}^{W(\eta') - W(\eta)} |v_\phi| \leq \mathrm{e}^{W(L) - W(0)} |v_\phi| \leq \mathrm{e}^{W(L) - W(0)} \sqrt{E_1 - \delta_0^2}. \tag{5-116}$$

Then we can further deduce that

$$|v'_\phi| \leq \left(1 - \frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^{-1} \sqrt{E_1 - \delta_0^2}. \tag{5-117}$$

Then we have

$$v'_\eta \geq \sqrt{E_1 - \left(1 - \frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^{-2} (E_1 - \delta_0^2)} \geq \delta_0 - C\epsilon^{\frac{1}{4}} > \frac{\delta_0}{2}, \tag{5-118}$$

when ϵ is sufficiently small. Then this implies that for $|E_1(\eta, \vec{\mathfrak{v}})| \geq |v'_\phi(\eta, \vec{\mathfrak{v}}; L)|$, for ϵ sufficiently small, we know $\min v'_\eta \geq \delta_0$, where $(\eta', \vec{\mathfrak{v}}')$ is on the same characteristics as $(\eta, \vec{\mathfrak{v}})$ with $v'_\eta \geq 0$.

Similar to the estimate of I_1, in this step, we will prove estimates based on the characteristics of $\mathcal{G}$ itself instead of $\mathscr{A}$. Here, we rewrite (5-4) along the characteristics as

$$v_\eta \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\eta} + \nu\mathcal{G} = K[\mathcal{G}]. \tag{5-119}$$

Also, we will still use simple facts (SF):

- Based on the well-posedness and decay theorem for $\mathcal{G}$, we know $\|\mathcal{G}\|_{L^\infty_{\vartheta,\varrho}} \leq C$.
- Based on Lemma 2.4, we get $\|K[\mathcal{G}]\|_{L^\infty_{\vartheta,\varrho}} \leq \|\mathcal{G}\|_{L^\infty_{\vartheta,\varrho}} \leq C$ and $\|\nabla_v K[\mathcal{G}]\|_{L^\infty_{\vartheta,\varrho}} \leq \|\mathcal{G}\|_{L^\infty_{\vartheta,\varrho}} \leq C$.
- Since E_1 is conserved along the characteristics, we must have $|\vec{\mathfrak{v}}| = |\vec{\mathfrak{v}}'|$.

- For $\eta' \le \eta$, we must have $v'_\eta \ge |v_\eta| \ge \delta_0$.
- Using substitution $y = H_{\eta,\eta'}$, we know

$$\left|\int_\eta^L \frac{\nu(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}\exp(H_{\eta,\eta'})\,\mathrm{d}\eta'\right| \le \left|\int_{-\infty}^0 \mathrm{e}^y\,\mathrm{d}y\right| = 1. \tag{5-120}$$

For $v_\eta \le -\delta_0$, we do not need the mild formulation for $\mathscr{A}$. Instead, we directly estimate

$$|\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2} II_1| \le \left|\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}\frac{\partial\mathcal{G}}{\partial\eta}\right|. \tag{5-121}$$

We rewrite the equation along the characteristics as

$$\begin{aligned}\mathcal{G}(\eta,\vec{\mathfrak{v}}) &= p(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};0))\exp(-H_{L,0}-\mathscr{R}[H_{L,\eta}])\\ &\quad+\int_0^L \frac{K[\mathcal{G}](\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\,\mathrm{d}\eta'\\ &\quad+\int_\eta^L \frac{K[\mathcal{G}](\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta',\end{aligned} \tag{5-122}$$

where $\vec{\mathfrak{v}}'(\eta') = \vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')$ satisfying $(\eta',\vec{\mathfrak{v}}')$ and $(\eta,\vec{\mathfrak{v}})$ are on the same characteristic with $v'_\eta \ge 0$, and

$$H_{t,s} = \int_s^t \frac{\nu(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};y))}{v'_\eta(\eta,\vec{\mathfrak{v}};y)}\,\mathrm{d}y$$

for any $s, t \ge 0$.

Then taking the η-derivative of both sides of (5-122) yields

$$\frac{\partial\mathcal{G}}{\partial\eta} = Y = Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7+Y_8+Y_9, \tag{5-123}$$

where

$$Y_1 = \frac{\partial p(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};0))}{\partial\eta}\exp(-H_{L,0}-\mathscr{R}[H_{L,\eta}]), \tag{5-124}$$

$$Y_2 = -p(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};0))\exp(-H_{L,0}-\mathscr{R}[H_{L,\eta}])\left(\frac{\partial H_{L,0}}{\partial\eta}+\frac{\partial\mathscr{R}[H_{L,\eta}]}{\partial\eta}\right), \tag{5-125}$$

$$Y_3 = \int_0^L \frac{K[\mathcal{G}](\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v'^2_\eta(\eta,\vec{\mathfrak{v}};\eta')}\frac{\partial v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}{\partial\eta}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\,\mathrm{d}\eta', \tag{5-126}$$

$$Y_4 = -\int_0^L \frac{K[\mathcal{G}](\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\left(\frac{\partial H_{L,\eta'}}{\partial\eta}+\frac{\partial\mathscr{R}[H_{L,\eta}]}{\partial\eta}\right)\mathrm{d}\eta', \tag{5-127}$$

$$Y_5 = \int_0^L \frac{1}{v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{L,\eta'}-\mathscr{R}[H_{L,\eta}])\left(\nabla_{\mathfrak{v}'}K[\mathcal{G}](\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))\frac{\partial\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')}{\partial\eta}\right)\mathrm{d}\eta', \tag{5-128}$$

$$Y_6 = \int_\eta^L \frac{K[\mathcal{G}](\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v'^2_\eta(\eta,\vec{\mathfrak{v}};\eta')}\frac{\partial v'_\eta(\eta,\vec{\mathfrak{v}};\eta')}{\partial\eta}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta', \tag{5-129}$$

$$Y_7 = \int_{\eta}^{L} \frac{K[\mathcal{G}](\eta', \mathscr{R}[\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta')])}{v_\eta'(\eta, \vec{\mathfrak{v}}; \eta')} \exp(\mathscr{R}[H_{\eta,\eta'}]) \frac{\mathscr{R}[H_{\eta,\eta'}]}{\partial \eta} \, d\eta', \tag{5-130}$$

$$Y_8 = \int_{\eta}^{L} \frac{1}{v_\eta'(\eta, \vec{\mathfrak{v}}; \eta')} \exp(\mathscr{R}[H_{\eta,\eta'}]) \left(\nabla_{\mathfrak{v}'} K[\mathcal{G}](\eta', \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta')) \frac{\partial \vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta')}{\partial \eta} \right) d\eta', \tag{5-131}$$

$$Y_9 = -\frac{K[\eta, \mathscr{R}[\vec{\mathfrak{v}}]]}{v_\eta}. \tag{5-132}$$

We need to estimate each term. Since the techniques are very similar to the estimate of I_1, without introducing new tricks we just list the results here:

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta} e^{\varrho|\vec{\mathfrak{v}}|^2} Y_1| \leq C\left(1 + \frac{1}{\delta_0}\right)\left(\left\|\frac{\partial p}{\partial v_\eta}\right\|_{L^\infty_{\vartheta,\varrho}} + \left\|\frac{\partial p}{\partial v_\phi}\right\|_{L^\infty_{\vartheta,\varrho}}\right), \tag{5-133}$$

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta} e^{\varrho|\vec{\mathfrak{v}}|^2} Y_2| \leq C\left(1 + \frac{1}{\delta_0}\right)\|p\|_{L^\infty_{\vartheta,\varrho}}, \tag{5-134}$$

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta} e^{\varrho|\vec{\mathfrak{v}}|^2} Y_i| \leq C\left(1 + \frac{1}{\delta_0}\right)\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \quad \text{for } i = 3, \dots 9. \tag{5-135}$$

In summary, we have

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta} e^{\varrho|\vec{\mathfrak{v}}|^2} II_1| \leq \frac{C}{\delta_0}\left(\left\|\frac{\partial p}{\partial v_\eta}\right\|_{L^\infty_{\vartheta,\varrho}} + \left\|\frac{\partial p}{\partial v_\phi}\right\|_{L^\infty_{\vartheta,\varrho}} + \|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right). \tag{5-136}$$

Step 2: estimate of II_2 for $-\delta_0 \leq v_\eta \leq 0$ and $\chi(\mathfrak{u}_\eta) < 1$. This is similar to the estimate of I_2 based on the integral

$$\int_{\eta}^{L} \frac{\nu(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; \eta'))}{v_\eta'(\eta, \vec{\mathfrak{v}}; \eta')} \exp(-H_{\eta',\eta}) \, d\eta' \leq 1. \tag{5-137}$$

Then we have

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta} e^{\varrho|\vec{\mathfrak{v}}|^2} II_2| \leq \frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.$$

Step 3: estimate of II_3 for $-\delta_0 \leq v_\eta \leq 0$, $\chi(\mathfrak{u}_\eta) = 1$ and $\sqrt{\epsilon\eta'}v_\phi' \geq v_\eta'$. This is identical to the estimate of I_3; we have

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta} e^{\varrho|\vec{\mathfrak{v}}|^2} II_3| \leq C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-138}$$

Step 4: estimate of II_4 for $-\delta_0 \leq v_\eta \leq 0$, $\chi(\mathfrak{u}_\eta) = 1$ and $\sqrt{\epsilon\eta'}v_\phi' \leq v_\eta'$. This step is different. We do not need to further decompose the cases. Based on (5-113), we have

$$-H_{\eta,\eta'} \leq -\frac{\nu(\vec{\mathfrak{v}})(\eta' - \eta)}{v_\eta}. \tag{5-139}$$

Then following the same argument in estimating I_5, we know

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta} e^{\varrho|\vec{\mathfrak{v}}|^2} II_4| \leq \frac{C}{\epsilon^s}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \int_{\eta}^{L} \frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}(1 + |\ln(\epsilon)| + |\ln(\eta')|) \exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta - \eta')}{v_\eta}\right) d\eta'. \tag{5-140}$$

Hence, we first estimate

$$\left|\int_\eta^L \frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}|\ln(\eta')|\exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\right)\mathrm{d}\eta'\right|. \tag{5-141}$$

If $\eta\geq 2$, we have

$$\begin{aligned}\left|\int_\eta^L \frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}|\ln(\eta')|\exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\right)\mathrm{d}\eta'\right| &\leq \ln(L)\left|\int_\eta^L \langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}\exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\right)\mathrm{d}\eta'\right|\\ &\leq C|\ln(\epsilon)|v_\eta\leq C|\ln(\epsilon)|\delta_0.\end{aligned} \tag{5-142}$$

If $0\leq\eta\leq 2$, using Hölder's inequality, it suffices to estimate

$$\begin{aligned}&\left|\int_0^2 \frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}|\ln(\eta')|\exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\right)\mathrm{d}\eta'\right|\\ &\quad\leq\left(\int_0^2\frac{1}{\eta'^{\frac{1}{2}}}|\ln(\eta')|^{\frac{1+2s}{2s}}\,\mathrm{d}\eta'\right)^{\frac{2s}{1+2s}}\left(\int_0^2\langle\vec{\mathfrak{v}}'\rangle\exp\left(-\frac{(1+2s)\nu(\vec{\mathfrak{v}}')C(\eta-\eta')}{v_\eta}\right)\mathrm{d}\eta'\right)^{\frac{1}{1+2s}}\\ &\quad\leq C\left(\frac{v_\eta\langle\vec{\mathfrak{v}}'\rangle}{\nu(\vec{\mathfrak{v}}')}\right)^{\frac{1}{1+2s}}\leq v_\eta^{\frac{1}{1+2s}}\leq C\delta_0^{\frac{1}{1+2s}}\leq\sqrt{\delta_0}.\end{aligned} \tag{5-143}$$

With a similar argument, we may justify

$$\left|\int_\eta^L(1+|\ln(\epsilon)|)\frac{\langle\vec{\mathfrak{v}}'\rangle^{\frac{1}{1+2s}}}{\eta'^{\frac{s}{1+s}}}\exp\left(-\frac{C\nu(\vec{\mathfrak{v}}')(\eta-\eta')}{v_\eta}\right)\right|\leq C(\sqrt{\delta_0}+|\ln(\epsilon)|\delta_0). \tag{5-144}$$

Hence, we have

$$|\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}II_4|\leq\frac{C}{\epsilon^s}(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \tag{5-145}$$

and

$$|\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}II_5|\leq\frac{C}{\epsilon^s}(\sqrt{\delta_0}+|\ln(\epsilon)|\delta_0)\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-146}$$

Step 5: synthesis. Collecting all the terms in previous steps, we have proved

$$\begin{aligned}|\langle\vec{\mathfrak{v}}\rangle^\vartheta \mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}II|&\leq\frac{C}{\epsilon^s}(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\\ &\quad+\frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\frac{C}{\delta_0}\left(\left\|\frac{\partial p}{\partial v_\eta}\right\|_{L^\infty_{\vartheta,\varrho}}+\left\|\frac{\partial p}{\partial v_\phi}\right\|_{L^\infty_{\vartheta,\varrho}}+\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right).\end{aligned} \tag{5-147}$$

5F. *Region III*: $v_\eta<0$ *and* $v_\eta^2+v_\phi^2\leq v_\phi'^2(\eta,\vec{\mathfrak{v}};L)$. Based on [Wu 2016, Lemmas 4.9 and 4.10], we still have

$$|\mathcal{K}[p_{\mathscr{A}}]|\leq\|p_{\mathscr{A}}\|_{L^\infty_{\vartheta,\varrho}}, \tag{5-148}$$

$$|\mathcal{T}[S_{\mathscr{A}}]|\leq\left\|\frac{S_{\mathscr{A}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-149}$$

Hence, we only need to estimate

$$III=\mathcal{T}[\widetilde{\mathscr{A}}]=\int_0^{\eta^+}\frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta^+,\eta'}-\mathscr{R}[H_{\eta^+,\eta}])\,\mathrm{d}\eta' \\ +\int_\eta^{\eta^+}\frac{\widetilde{\mathscr{A}}(\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta'. \tag{5-150}$$

Here η^+ is defined in (5-25). In particular, we can take the decomposition

$$\mathcal{T}[\widetilde{\mathscr{A}}]=\int_0^{\eta}\frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta^+,\eta'}-\mathscr{R}[H_{\eta^+,\eta}])\,\mathrm{d}\eta' \\ +\int_\eta^{\eta^+}\frac{\widetilde{\mathscr{A}}(\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(-H_{\eta^+,\eta'}-\mathscr{R}[H_{\eta^+,\eta}])\,\mathrm{d}\eta' \\ +\int_\eta^{\eta^+}\frac{\widetilde{\mathscr{A}}(\eta',\mathscr{R}[\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(\mathscr{R}[H_{\eta,\eta'}])\,\mathrm{d}\eta'. \tag{5-151}$$

Then the integral $\int_0^\eta(\cdots)$ is similar to the argument in Region I, and the integral $\int_\eta^{\eta^+}(\cdots)$ is similar to the argument in Region II. The only difference is in Step 1 when estimating the $\int_\eta^{\eta^+}(\cdots)$ part for $\eta\le-\delta_0$. Here, we introduce a special trick.

We first estimate v_η in terms of v_ϕ. Along the characteristics, we know

$$\mathrm{e}^{-W(L)}v_\phi'(L)=\mathrm{e}^{-W(\eta)}v_\phi, \tag{5-152}$$

which implies

$$|v_\phi'(L)|=\mathrm{e}^{W(L)-W(\eta)}|v_\phi|\le\mathrm{e}^{W(L)-W(0)}|v_\phi|=\left(1-\frac{\epsilon^{\frac12}}{R_\kappa}\right)^{-1}|v_\phi|. \tag{5-153}$$

Then we can further deduce that

$$v_\eta^2+v_\phi^2\le\left(1-\frac{\epsilon^{\frac12}}{R_\kappa}\right)^{-2}v_\phi^2\le\left(1-\frac{\epsilon^{\frac12}}{R_\kappa}\right)^{-2}v_\phi^2. \tag{5-154}$$

Then we have

$$|v_\eta|\le\sqrt{\left(1-\frac{\epsilon^{\frac12}}{R_\kappa}\right)^{-2}v_\phi^2-v_\phi^2}\le\epsilon^{\frac14}|v_\phi|\le\delta_0|v_\phi|, \tag{5-155}$$

when ϵ is sufficiently small.

- Therefore, if $|v_\phi|\le1$, then Step 1 is not necessary at all since we already have $|v_\eta|\le\delta_0$. We directly apply the argument in estimating II to obtain

$$|\langle\vec{\mathfrak{v}}\rangle^\vartheta\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}III|\le\frac{C}{\epsilon^s}(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-156}$$

- However, if $v_\phi\ge1$, let (η,v_η,v_ϕ) and $(\tilde\eta,-\delta_0,\tilde v_\phi)$ be on the same characteristics. Then we have the mild formulation

$$\mathcal{G}(\eta,\vec{\mathfrak{v}})=\mathcal{G}(\tilde\eta,-\delta_0,\tilde v_\phi)\exp(-H_{\tilde\eta,\eta})+\int_\eta^{\tilde\eta}\frac{K[\mathcal{G}](\eta',\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}};\eta')}\exp(H_{\eta',\eta})\,\mathrm{d}\eta'. \tag{5-157}$$

In other words, we try to use a mild formulation and avoid going through the η^+ point. Then similar to the estimate of II_1, taking the η-derivative in the mild formulation, we obtain

$$\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\frac{\partial\mathcal{G}}{\partial\eta}\right|\leq C\left(1+\frac{1}{\delta_0}\right)\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+C\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\zeta(\tilde{\eta},-\delta_0,\tilde{v}_\phi)\frac{\partial\mathcal{G}(\tilde{\eta},-\delta_0,\tilde{v}_\phi)}{\partial\eta}\right|.$$

Also, we may directly verify that

$$\begin{aligned}&\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\zeta(\tilde{\eta},-\delta_0,\tilde{v}_\phi)\frac{\partial\mathcal{G}(\tilde{\eta},-\delta_0,\tilde{v}_\phi)}{\partial\eta}\right|\\&\qquad\leq\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\zeta(\tilde{\eta},-\delta_0,\tilde{v}_\phi)\frac{\partial\mathcal{G}(\tilde{\eta},-\delta_0,\tilde{v}_\phi)}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial\eta}\right|+\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\zeta(\tilde{\eta},-\delta_0,\tilde{v}_\phi)\frac{\partial\mathcal{G}(\tilde{\eta},-\delta_0,\tilde{v}_\phi)}{\partial\tilde{v}_\phi}\frac{\partial\tilde{v}_\phi}{\partial\eta}\right|\\&\qquad\leq|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\widetilde{\mathscr{A}}(\tilde{\eta},-\delta_0,\tilde{v}_\phi)|,\end{aligned}\tag{5-158}$$

since $\partial\tilde{v}_\phi/\partial\eta=0$. The estimate of $|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\widetilde{\mathscr{A}}(\tilde{\eta},-\delta_0,\tilde{v}_\phi)|$ is achieved since now $|\tilde{v}_\eta|\leq\delta_0$.

Hence, we have

$$|\langle\vec{\mathfrak{v}}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{\mathfrak{v}}|^2}III|\leq\frac{C}{\epsilon^s}(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\frac{C}{\delta_0}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}.\tag{5-159}$$

5G. ***Estimates of normal derivative.*** Combining the analysis in these three regions, we have for $0<s\ll 1$

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq{}&\frac{C}{\epsilon^s}(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+C\sqrt{\delta}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\frac{C}{\delta^2}\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\\&+\frac{C}{\delta_0}\left(\left\|\frac{\partial p}{\partial v_\eta}\right\|_{L^\infty_{\vartheta,\varrho}}+\left\|\frac{\partial p}{\partial v_\phi}\right\|_{L^\infty_{\vartheta,\varrho}}+\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right)+\|p_{\mathscr{A}}\|_{L^\infty_{\vartheta,\varrho}}+\left\|\frac{S_{\mathscr{A}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}.\end{aligned}\tag{5-160}$$

Then we choose these constants to perform an absorbing argument. First we choose $0<\delta\ll 1$ sufficiently small such that

$$C\sqrt{\delta}\leq\tfrac{1}{4}.\tag{5-161}$$

Then we take $\delta_0=\sqrt{\delta}\epsilon^s|\ln(\epsilon)|^{-1}$ such that

$$\frac{C}{\epsilon^s}(1+|\ln(\epsilon)|)\sqrt{\delta_0}\leq 2C\delta\leq\tfrac{1}{2}\tag{5-162}$$

for ϵ sufficiently small. Note that this mild decay of δ_0 with respect to ϵ also justifies the assumption in Cases II and III that

$$\epsilon^{\frac{1}{4}}\leq\delta_0\tag{5-163}$$

for ϵ sufficiently small. Here since δ and C are independent of ϵ, there is no circular argument. Hence, we can absorb all the terms related to $\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}$ on the right-hand side of (5-160) to the left-hand side to obtain the desired result.

Lemma 5.4. *We have*

$$\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C\left(\|p_{\mathscr{A}}\|_{L^\infty_{\vartheta,\varrho}}+\left\|\frac{S_{\mathscr{A}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right)+C|\ln(\epsilon)|\epsilon^{-s}\left(\left\|\frac{\partial p}{\partial v_\eta}\right\|_{L^\infty_{\vartheta,\varrho}}+\left\|\frac{\partial p}{\partial v_\phi}\right\|_{L^\infty_{\vartheta,\varrho}}+\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right).\tag{5-164}$$

5H. ***Estimates velocity derivative.*** Consider the general ϵ-Milne problem with geometric correction for $\mathscr{B}=\zeta(\partial\mathcal{G}/\partial v_\eta)$ as

$$\begin{cases} v_\eta\dfrac{\partial\mathscr{B}}{\partial\eta}+G(\eta)\biggl(v_\phi^2\dfrac{\partial\mathscr{B}}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathscr{B}}{\partial v_\phi}\biggr)+\nu\mathscr{B}=\widetilde{\mathscr{B}}+S_{\mathscr{B}},\\ \mathscr{B}(0,\vec{\mathfrak{v}})=p_{\mathscr{B}}(\vec{\mathfrak{v}})\quad\text{for } v_\eta>0,\\ \mathscr{B}(L,\vec{\mathfrak{v}})=\mathscr{B}(L,\mathscr{R}[\vec{\mathfrak{v}}]),\end{cases}\tag{5-165}$$

where $p_{\mathscr{B}}$ and $S_{\mathscr{B}}$ will be specified later with

$$\widetilde{\mathscr{B}}(\eta,\vec{\mathfrak{v}})=\int_{\mathbb{R}^2}\zeta(\eta,\vec{\mathfrak{v}})\partial_{v_\eta}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\mathcal{G}(\eta,\vec{\mathfrak{u}})\,d\vec{\mathfrak{u}}.\tag{5-166}$$

This is much simpler than a normal derivative, since $\widetilde{\mathscr{B}}$ does not contain $\mathscr{B}$ directly. Then by an argument similar to one before, we obtain the desired result.

Lemma 5.5. *We have*

$$\|\mathscr{B}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C\biggl(\|p_{\mathscr{B}}\|_{L^\infty_{\vartheta,\varrho}}+\biggl\|\frac{S_{\mathscr{B}}}{\nu}\biggr\|_{L^\infty L^\infty_{\vartheta,\varrho}}\biggr)+C\biggl(\biggl\|\frac{\partial p}{\partial v_\eta}\biggr\|_{L^\infty_{\vartheta,\varrho}}+\biggl\|\frac{\partial p}{\partial v_\phi}\biggr\|_{L^\infty_{\vartheta,\varrho}}+\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\biggr).\tag{5-167}$$

In a similar fashion, consider the general ϵ-Milne problem with geometric correction for $\mathscr{C}=\zeta(\partial\mathcal{G}/\partial v_\phi)$ as

$$\begin{cases} v_\eta\dfrac{\partial\mathscr{C}}{\partial\eta}+G(\eta)\Bigl(v_\phi^2\dfrac{\partial\mathscr{C}}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathscr{C}}{\partial v_\phi}\Bigr)+\nu\mathscr{C}=\widetilde{\mathscr{C}}+S_{\mathscr{C}},\\ \mathscr{C}(0,\vec{\mathfrak{v}})=p_{\mathscr{C}}(\vec{\mathfrak{v}})\quad\text{for } v_\eta>0,\\ \mathscr{C}(L,\vec{\mathfrak{v}})=\mathscr{C}(L,\mathscr{R}[\vec{\mathfrak{v}}]),\end{cases}\tag{5-168}$$

where $p_{\mathscr{C}}$ and $S_{\mathscr{C}}$ will be specified later with

$$\widetilde{\mathscr{C}}(\eta,\vec{\mathfrak{v}})=\int_{\mathbb{R}^2}\zeta(\eta,\vec{\mathfrak{v}})\partial_{v_\phi}k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\mathcal{G}(\eta,\vec{\mathfrak{u}})\,d\vec{\mathfrak{u}}.\tag{5-169}$$

This is also much simpler than a normal derivative, since $\widetilde{\mathscr{C}}$ does not contain $\mathscr{C}$ directly. Then by an argument similar to one before, we obtain the desired result.

Lemma 5.6. *We have*

$$\|\mathscr{C}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C\biggl(\Bigl\|p_{\mathscr{C}}\Bigr\|_{L^\infty_{\vartheta,\varrho}}+\biggl\|\frac{S_{\mathscr{C}}}{\nu}\biggr\|_{L^\infty L^\infty_{\vartheta,\varrho}}\biggr)+C\biggl(\biggl\|\frac{\partial p}{\partial v_\eta}\biggr\|_{L^\infty_{\vartheta,\varrho}}+\biggl\|\frac{\partial p}{\partial v_\phi}\biggr\|_{L^\infty_{\vartheta,\varrho}}+\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\biggr).\tag{5-170}$$

5I. ***Estimates of tangential derivative.*** In this subsection, we combine above a priori estimates of normal and velocity derivatives.

Theorem 5.7. *We have*

$$\biggl\|\zeta\frac{\partial\mathcal{G}}{\partial\eta}\biggr\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\biggl\|\zeta\frac{\partial\mathcal{G}}{\partial v_\eta}\biggr\|_{L^\infty L^\infty_{\vartheta,\varrho}}+\biggl\|\zeta\frac{\partial\mathcal{G}}{\partial v_\phi}\biggr\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C|\ln(\epsilon)|\epsilon^{-s}\tag{5-171}$$

for some $0<s\ll 1$.

Proof. Collecting the estimates for $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ in Lemmas 5.4, 5.5, and 5.6, we have

$$\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \leq C\left(\|p_{\mathscr{A}}\|_{L^\infty_{\vartheta,\varrho}} + \left\|\frac{S_{\mathscr{A}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right) + C_0|\ln(\epsilon)|\epsilon^{-s}, \tag{5-172}$$

$$\|\mathscr{B}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \leq C\left(\|p_{\mathscr{B}}\|_{L^\infty_{\vartheta,\varrho}} + \left\|\frac{S_{\mathscr{B}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right) + C_0, \tag{5-173}$$

$$\|\mathscr{C}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \leq C\left(\|p_{\mathscr{C}}\|_{L^\infty_{\vartheta,\varrho}} + \left\|\frac{S_{\mathscr{C}}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right) + C_0, \tag{5-174}$$

where

$$C_0 = \|p\|_{L^\infty_{\vartheta,\varrho}} + \left\|\frac{\partial p}{\partial v_\eta}\right\|_{L^\infty_{\vartheta,\varrho}} + \left\|\frac{\partial p}{\partial v_\phi}\right\|_{L^\infty_{\vartheta,\varrho}} + \|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-175}$$

Taking derivatives on both sides of (5-4) and multiplying by ζ, we have

$$p_{\mathscr{A}} = -\frac{\epsilon}{R_\kappa}\left(v_\phi^2\frac{\partial p}{\partial v_\eta} - v_\eta v_\phi\frac{\partial p}{\partial v_\phi}\right) + \nu p - K[\mathcal{G}](0,\vec{\mathfrak{v}}), \tag{5-176}$$

$$p_{\mathscr{B}} = v_\eta\frac{\partial p}{\partial v_\eta}, \tag{5-177}$$

$$p_{\mathscr{C}} = v_\eta\frac{\partial p}{\partial v_\phi}, \tag{5-178}$$

$$S_{\mathscr{A}} = \frac{\partial G}{\partial \eta}(v_\phi^2\mathscr{B} - v_\eta v_\phi\mathscr{C}), \tag{5-179}$$

$$S_{\mathscr{B}} = \mathscr{A} - G v_\phi\mathscr{C}, \tag{5-180}$$

$$S_{\mathscr{C}} = G(2v_\phi\mathscr{B} - v_\eta\mathscr{C}). \tag{5-181}$$

We can directly verify that

$$\|p_{\mathscr{A}}\|_{L^\infty_{\vartheta,\varrho}} + \|p_{\mathscr{B}}\|_{L^\infty_{\vartheta,\varrho}} + \|p_{\mathscr{C}}\|_{L^\infty_{\vartheta,\varrho}} \leq C_0. \tag{5-182}$$

Since $|G(\eta)| + |\partial G/\partial\eta| \leq \epsilon$, from (5-181), we obtain

$$\begin{aligned}\|\mathscr{C}\|_{L^\infty L^\infty_{\vartheta,\varrho}} &\leq C_0 + C_0\epsilon\left(\left\|\frac{v_\phi\mathscr{B}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \left\|\frac{v_\eta\mathscr{C}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right)\\ &\leq C_0 + C_0\epsilon(\|\mathscr{B}\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \|\mathscr{C}\|_{L^\infty L^\infty_{\vartheta,\varrho}}),\end{aligned} \tag{5-183}$$

which further implies

$$\|\mathscr{C}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \leq C_0 + C_0\epsilon\|\mathscr{B}\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-184}$$

Plugging (5-184) into (5-180), we obtain

$$\begin{aligned}\|\mathscr{B}\|_{L^\infty L^\infty_{\vartheta,\varrho}} &\leq C_0 + C_0\left(\left\|\frac{\mathscr{A}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \epsilon\left\|\frac{v_\phi\mathscr{C}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right)\\ &\leq C_0 + C_0\left(\left\|\frac{\mathscr{A}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \epsilon\|\mathscr{C}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right)\\ &\leq C_0 + C_0\left(\left\|\frac{\mathscr{A}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \epsilon^2\|\mathscr{B}\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right),\end{aligned} \tag{5-185}$$

which further implies

$$\|\mathscr{B}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \le C_0 + C_0\left\|\frac{\mathscr{A}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}, \tag{5-186}$$

$$\|\mathscr{C}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \le C_0 + C_0\epsilon\left\|\frac{\mathscr{A}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}. \tag{5-187}$$

Plugging (5-186) and (5-187) into (5-179), we get

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}} &\le C_0|\ln(\epsilon)|\epsilon^{-s} + C_0\epsilon\left(\left\|\frac{v_\phi^2\mathscr{B}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \left\|\frac{v_\eta v_\phi\mathscr{C}}{\nu}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right)\\ &\le C_0|\ln(\epsilon)|\epsilon^{-s} + C_0\epsilon\left(\left\|\frac{v_\phi^2\mathscr{A}}{\nu^2}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \epsilon\left\|\frac{v_\eta v_\phi\mathscr{A}}{\nu^2}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\right)\\ &\le C_0|\ln(\epsilon)|\epsilon^{-s} + C_0\epsilon\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}},\end{aligned} \tag{5-188}$$

which implies

$$\|\mathscr{A}\|_{L^\infty L^\infty_{\vartheta,\varrho}} \le C_0|\ln(\epsilon)|\epsilon^{-s}. \tag{5-189}$$

Hence, we derive

$$\mathscr{A},\ \mathscr{B},\ \mathscr{C} \le C|\ln(\epsilon)|\epsilon^{-s}, \tag{5-190}$$

completing the proof. □

The above theorems only provide a priori estimates. The rigorous proof relies on a penalty method and an iteration argument. This step is standard as in [Guo and Wu 2017a], so we omit it here.

Theorem 5.8. *For $K_0 > 0$ sufficiently small, we have*

$$\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathcal{G}}{\partial\eta}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathcal{G}}{\partial v_\eta}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} + \left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathcal{G}}{\partial v_\phi}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} \le C|\ln(\epsilon)|\epsilon^{-s} \tag{5-191}$$

for some $0 < s \ll 1$.

Proof. This proof is almost identical to Theorem 5.7. The only difference is that $S_{\mathscr{A}}$ is added by $K_0v_\eta\mathscr{A}$, $S_{\mathscr{B}}$ is added by $K_0v_\eta\mathscr{B}$, and $S_{\mathscr{C}}$ is added by $K_0v_\eta\mathscr{C}$. When K_0 is sufficiently small, we can also absorb them into the left-hand side. Hence, this is obvious. □

Now we pull the θ-dependence back and study the tangential derivative.

Theorem 5.9. *We have*

$$\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathcal{G}}{\partial\theta}(\eta,\theta,\phi)\right\|_{L^\infty L^\infty_{\vartheta,\varrho}} \le C|\ln(\epsilon)|\epsilon^{-s} \tag{5-192}$$

for some $0 < s \ll 1$.

Proof. Let $\mathscr{W} = \partial\mathcal{G}/\partial\theta$. Taking the θ-derivative of both sides of (5-4), we have that $\mathscr{W}$ satisfies

$$\begin{cases} v_\eta\dfrac{\partial\mathscr{W}}{\partial\eta}+G(\eta)\left(v_\phi^2\dfrac{\partial\mathscr{W}}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathscr{W}}{\partial v_\phi}\right)+\nu\mathscr{W}-K[\mathscr{W}] = \dfrac{R_\kappa'}{R_\kappa-\epsilon\eta}G(\eta)\left(v_\phi^2\dfrac{\partial\mathcal{G}}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathcal{G}}{\partial v_\phi}\right),\\ \mathscr{W}(0,\theta,\vec{\mathfrak{v}}) = \dfrac{\partial p}{\partial\theta}(\theta,\vec{\mathfrak{v}}) \quad \text{for } \sin\phi>0,\\ \mathscr{W}(L,\theta,\vec{\mathfrak{v}}) = \mathscr{W}(L,\theta,\mathscr{R}[\vec{\mathfrak{v}}]),\end{cases} \tag{5-193}$$

where R'_κ is the θ-derivative of R_κ. For $\eta\in[0,L]$, we have

$$\frac{R'_\kappa}{R_\kappa-\epsilon\eta}\leq C\max_\theta R'_\kappa\leq C. \tag{5-194}$$

Since $\zeta(\eta,\vec{\mathfrak{v}})\geq v_\eta$, based on Theorem 5.8 and (5-4), we know

$$\left\|\mathrm{e}^{K_0\eta}v_\eta\frac{\partial\mathscr{W}}{\partial\eta}\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C|\ln(\epsilon)|\epsilon^{-s}, \tag{5-195}$$

which further implies

$$\left\|\mathrm{e}^{K_0\eta}G(\eta)\left(v_\phi^2\frac{\partial\mathscr{W}}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{W}}{\partial v_\phi}\right)\right\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C|\ln(\epsilon)|\epsilon^{-s} \tag{5-196}$$

for some $0<s\ll 1$. Therefore, the source term in (5-193) is in L^∞ and decays exponentially. By Theorem 5.2, we have

$$\|\mathrm{e}^{K_0\eta}\mathscr{W}(\eta,\theta,\phi)\|_{L^\infty L^\infty_{\vartheta,\varrho}}\leq C|\ln(\epsilon)|\epsilon^{-s} \tag{5-197}$$

for some $0<s\ll 1$. □

6. Proof of the main theorem

Now we turn to the proof of the main result, Theorem 1.1. The asymptotic analysis already reveals that the construction of the interior solution and boundary layer is valid. Here, we focus on the remainder estimates. We divide the proof into several steps:

Step 1: remainder definitions. Define the remainder as

$$R=\frac{1}{\epsilon^3}\big(f^\epsilon-(\epsilon F_1+\epsilon^2F_2+\epsilon^3F_3)-(\epsilon\mathscr{F}_1+\epsilon^2\mathscr{F}_2)\big)=\frac{1}{\epsilon^3}(f^\epsilon-Q-\mathscr{Q}), \tag{6-1}$$

where

$$Q=\epsilon F_1+\epsilon^2F_2+\epsilon^3F_3, \tag{6-2}$$

$$\mathscr{Q}=\epsilon\mathscr{F}_1+\epsilon^2\mathscr{F}_2. \tag{6-3}$$

In other words, we have

$$f^\epsilon=Q+\mathscr{Q}+\epsilon^3R. \tag{6-4}$$

We write $\mathscr{L}$ to denote the linearized Boltzmann operator,

$$\begin{aligned}\mathscr{L}[f]&=\epsilon\vec{v}\cdot\nabla_xu+\mathcal{L}[f]\\&=v_\eta\frac{\partial f}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\left(v_\phi^2\frac{\partial f}{\partial v_\eta}-v_\eta v_\phi\frac{\partial f}{\partial v_\phi}\right)-\frac{\epsilon}{R_\kappa-\epsilon\eta}\frac{R_\kappa}{(r^2+r'^2)^{\frac12}}v_\phi\frac{\partial f}{\partial\theta}+\mathcal{L}[f].\end{aligned} \tag{6-5}$$

Step 2: representation of $\mathscr{L}[R]$. Equation (1-16) is actually

$$\mathscr{L}[f^\epsilon]=\Gamma[f^\epsilon,f^\epsilon], \tag{6-6}$$

which means

$$\mathscr{L}[Q+\mathscr{Q}+\epsilon^3R]=\Gamma[Q+\mathscr{Q}+\epsilon^3R,\,Q+\mathscr{Q}+\epsilon^3R]. \tag{6-7}$$

Note that the nonlinear term can be decomposed as

$$\Gamma[Q+\mathscr{Q}+\epsilon^3 R,\, Q+\mathscr{Q}+\epsilon^3 R]=\epsilon^6\Gamma[R,\, R]+2\epsilon^3\Gamma[R,\, Q+\mathscr{Q}]+\Gamma[Q+\mathscr{Q},\, Q+\mathscr{Q}]. \tag{6-8}$$

The interior contribution can be represented as

$$\begin{aligned}\mathscr{L}[Q]&=\epsilon\vec{v}\cdot\nabla_x(\epsilon F_1+\epsilon^2F_2+\epsilon^3F_3)+\mathcal{L}[\epsilon F_1+\epsilon^2F_2+\epsilon^3F_3]\\&=\epsilon^4\vec{v}\cdot\nabla_x F_3+\epsilon^2\Gamma[F_1,\, F_1]+2\epsilon^3\Gamma[F_1,\, F_1].\end{aligned} \tag{6-9}$$

The nonlinear term will be handled by $\Gamma[Q+\mathscr{Q},\, Q+\mathscr{Q}]$. On the other hand, we consider the boundary layer contribution. Since $\mathscr{F}_1=0$, we may directly compute

$$\begin{aligned}\mathscr{L}[\mathscr{Q}]&=\epsilon^2\biggl(v_\eta\frac{\partial\mathscr{F}_2}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\biggl(v_\phi^2\frac{\partial\mathscr{F}_2}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{F}_2}{\partial v_\phi}\biggr)-\frac{\epsilon}{R_\kappa-\epsilon\eta}\frac{R_\kappa}{(r^2+r'^2)^{\frac12}}v_\phi\frac{\partial\mathscr{F}_2}{\partial\theta}+\mathcal{L}[\mathscr{F}_2]\biggr)\\&=-\frac{\epsilon^3}{R_\kappa-\epsilon\eta}\frac{R_\kappa}{(r^2+r'^2)^{\frac12}}v_\phi\frac{\partial\mathscr{F}_2}{\partial\theta}.\end{aligned} \tag{6-10}$$

Therefore, we have

$$\mathscr{L}[R]=\epsilon^3\Gamma[R,\, R]+2\Gamma[R,\, Q+\mathscr{Q}]+S_1+S_2, \tag{6-11}$$

where

$$S_1=-\epsilon\vec{v}\cdot\nabla_x F_3+\frac{1}{R_\kappa-\epsilon\eta}\frac{R_\kappa}{(r^2+r'^2)^{\frac12}}v_\phi\frac{\partial\mathscr{F}_2}{\partial\theta}, \tag{6-12}$$

$$S_2=2\Gamma[F_1,\mathscr{F}_2]+2\epsilon\Gamma[F_1,\, F_3]+\epsilon\Gamma[\mathscr{F}_2,\mathscr{F}_2]+2\epsilon\Gamma[F_2,\mathscr{F}_1]+2\epsilon^2\Gamma[F_2,\, F_3]+\epsilon^3\Gamma[F_3,\, F_3]. \tag{6-13}$$

Step 3: representation of $R-\mathcal{P}[R]$. Since

$$f^\epsilon(\vec{x}_0,\vec{v})=\mu_b^\epsilon(\vec{x}_0,\vec{v})\mu^{-\frac12}(\vec{v})\int_{\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac12}(\vec{\mathrm{u}})f^\epsilon(\vec{x}_0,\vec{\mathrm{u}})|\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathrm{u}}+\mu^{-\frac12}(\vec{v})(\mu_b^\epsilon(\vec{x}_0,\vec{v})-\mu(\vec{v})),$$

where both sides are linear, we may directly write

$$R(\vec{x}_0,\vec{v})-\mathcal{P}[R](\vec{x}_0)=H[R](\vec{x}_0,\vec{v})+h(\vec{x}_0,\vec{v}), \tag{6-14}$$

where

$$H[R](\vec{x}_0,\vec{v})=(\mu_b^\epsilon(\vec{x}_0,\vec{v})-\mu(\vec{v}))\mu^{-\frac12}(\vec{v})\int_{\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac12}(\vec{\mathrm{u}})R(\vec{x}_0,\vec{\mathrm{u}})|\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathrm{u}} \tag{6-15}$$

and

$$\begin{aligned}h(\vec{x}_0,\vec{v})&=(\mu_b^\epsilon(\vec{x}_0,\vec{v})-\mu(\vec{v}))\mu^{-\frac12}(\vec{v})\int_{\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac12}(\vec{\mathrm{u}})F_3(\vec{x}_0,\vec{\mathrm{u}})|\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathrm{u}}\\&\quad+\bigl(\mu_b^\epsilon(\vec{x}_0,\vec{v})-\mu(\vec{v})-\epsilon\mu_1(\vec{x}_0,\vec{v})\bigr)\mu^{-\frac12}(\vec{v})\int_{\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac12}(\vec{\mathrm{u}})(\epsilon^{-1}F_2+\epsilon^{-1}\mathscr{F}_2+F_3)(\vec{x}_0,\vec{\mathrm{u}})|\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathrm{u}}\\&\quad+\bigl(\mu_b^\epsilon(\vec{x}_0,\vec{v})-\mu(\vec{v})-\epsilon\mu_1(\vec{x}_0,\vec{v})-\epsilon^2\mu_2(\vec{x}_0,\vec{v})\bigr)\mu^{-\frac12}(\vec{v})\\&\qquad\qquad\times\int_{\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)>0}\mu^{\frac12}(\vec{\mathrm{u}})\epsilon^{-3}(Q+\mathscr{Q})(\vec{x}_0,\vec{\mathrm{u}})|\vec{\mathrm{u}}\cdot\vec{n}(\vec{x}_0)|\,\mathrm{d}\vec{\mathrm{u}}\\&\quad+\epsilon^{-3}\mu^{-\frac12}(\vec{v})\bigl(\mu_b^\epsilon(\vec{x}_0,\vec{v})-\mu(\vec{v})-\epsilon\mu_1(\vec{x}_0,\vec{v})-\epsilon^2\mu_2(\vec{x}_0,\vec{v})-\epsilon^3\mu_3(\vec{x}_0,\vec{v})\bigr).\end{aligned} \tag{6-16}$$

Step 4: L^{2m} estimates of R. Using Theorem 4.4, we have the L^{2m} estimate of R

$$
\begin{aligned}
&\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}+\frac{1}{\epsilon^{\frac{1}{2}}}|(1-\mathcal{P})[R]|_{L^2_+}+\|\mathbb{P}[R]\|_{L^{2m}}\\
&\quad\le C\bigg(o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty}+\frac{1}{\epsilon^2}\|\mathbb{P}[\mathscr{L}[R]]\|_{L^{2m/(2m-1)}}+\frac{1}{\epsilon}\|\mathscr{L}[R]\|_{L^2}+|R-\mathcal{P}[R]|_{L^m_-}+\frac{1}{\epsilon}|R-\mathcal{P}[R]|_{L^2_-}\bigg)\\
&\quad\le C\bigg(o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty}+\frac{1}{\epsilon^2}\|\mathbb{P}[S_1]\|_{L^{2m/(2m-1)}}\\
&\qquad\qquad+\frac{1}{\epsilon}\big(\|\epsilon^3\Gamma[R,R]\|_{L^2}+\|2\Gamma[R,Q+\mathscr{Q}]\|_{L^2}+\|S_1\|_{L^2}+\|S_2\|_{L^2}\big)\\
&\qquad\qquad\qquad+|H[R]|_{L^m_-}+|h|_{L^m_-}+\frac{1}{\epsilon}|H[R]|_{L^2_-}+\frac{1}{\epsilon}|h|_{L^2_-}\bigg).
\end{aligned}
\tag{6-17}
$$

Note that here we do not have other source terms in the $L^{2m/(2m-1)}$ norm because for any $f, g\in L^2$

$$\mathbb{P}[\Gamma(f,g)]=0. \tag{6-18}$$

We need to estimate each term. It is easy to check

$$\|\epsilon\vec{v}\cdot\nabla_x F_3\|_{L^2}\le C\epsilon, \tag{6-19}$$

$$\|\epsilon\vec{v}\cdot\nabla_x F_3\|_{L^{2m/(2m-1)}}\le C\epsilon, \tag{6-20}$$

and also by Theorem 5.9, using the rescaling and exponential decay, we have

$$
\begin{aligned}
\left\|\frac{1}{R_\kappa-\epsilon\eta}\frac{R_\kappa}{(r^2+r'^2)^{\frac{1}{2}}}v_\phi\frac{\partial\mathscr{F}_2}{\partial\theta}\right\|_{L^2}&\le C\bigg(\int_{-\pi}^{\pi}\int_0^{R_{\min}\epsilon^{1/2}}(R_\kappa-\mathfrak{N})\left\|\frac{\partial\mathscr{F}_2}{\partial\theta}(\mathfrak{N},\theta)\right\|^2_{L^\infty_{\vartheta,\varrho}}\mathrm{d}\mathfrak{N}\,\mathrm{d}\theta\bigg)^{\frac{1}{2}}\\
&\le C\epsilon^{\frac{1}{2}}\bigg(\int_{-\pi}^{\pi}\int_0^{R_{\min}\epsilon^{-1/2}}\left\|\frac{\partial\mathscr{F}_2}{\partial\theta}(\eta,\theta)\right\|^2_{L^\infty_{\vartheta,\varrho}}\mathrm{d}\eta\,\mathrm{d}\theta\bigg)^{\frac{1}{2}}\\
&\le C\epsilon^{\frac{1}{2}}\bigg(\int_{-\pi}^{\pi}\int_0^{R_{\min}\epsilon^{-1/2}}\mathrm{e}^{-2K_0\eta}|\ln(\epsilon)|^2\epsilon^{-2s}\,\mathrm{d}\eta\,\mathrm{d}\theta\bigg)^{\frac{1}{2}}\\
&\le C\epsilon^{\frac{1}{2}-s}|\ln(\epsilon)|
\end{aligned}
\tag{6-21}
$$

for some $0<s\ll 1$. Similarly, we can prove that

$$\left\|\frac{1}{R_\kappa-\epsilon\eta}\frac{R_\kappa}{(r^2+r'^2)^{\frac{1}{2}}}v_\phi\frac{\partial\mathscr{F}_2}{\partial\theta}\right\|_{L^{2m/(2m-1)}}\le C\epsilon^{1-\frac{1}{2m}-s}|\ln(\epsilon)|. \tag{6-22}$$

In total, we have

$$\|S_1\|_{L^2}\le C\epsilon^{\frac{1}{2}-s}|\ln(\epsilon)|, \tag{6-23}$$

$$\|\mathbb{P}[S_1]\|_{L^{2m/(2m-1)}}\le C\epsilon^{1-\frac{1}{2m}-s}|\ln(\epsilon)|. \tag{6-24}$$

On the other hand, by Lemma 2.5, we know

$$\|2\Gamma[R,Q+\mathscr{Q}]\|_{L^2}\le C\big(\|R\|_{L^2}\|Q+\mathscr{Q}\|_{L^\infty_{\vartheta,\varrho}}+\|R\|_{L^2_\nu}\|Q+\mathscr{Q}\|_{L^\infty}\big). \tag{6-25}$$

Based on the smallness assumption (1-9) on the boundary Maxwellian, it is easy to check that

$$\|R\|_{L^2}\|Q+\mathscr{Q}\|_{L^\infty_{\vartheta,\varrho}}\leq o(1)\epsilon\|R\|_{L^2}. \tag{6-26}$$

Also, we may take the decomposition

$$\begin{aligned}\|R\|_{L^2_\nu}\|Q+\mathscr{Q}\|_{L^\infty}&\leq\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}\|Q+\mathscr{Q}\|_{L^\infty}+\|\mathbb{P}[R]\|_{L^2_\nu}\|Q+\mathscr{Q}\|_{L^\infty}\\&\leq o(1)\epsilon\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}+o(1)\epsilon\|\mathbb{P}[R]\|_{L^2}.\end{aligned} \tag{6-27}$$

Then we may derive that

$$\|2\Gamma[R,\,Q+\mathscr{Q}]\|_{L^2}\leq o(1)\epsilon\big(\|\mathbb{P}[R]\|_{L^2}+\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}\big). \tag{6-28}$$

Also, using the smallness assumption (1-9) again, we can directly estimate

$$|H[R]|_{L^m_-}\leq o(1)\epsilon|R|_{L^m_-}\leq o(1)\epsilon\|R\|_{L^\infty}, \tag{6-29}$$

$$|H[R]|_{L^2_-}\leq o(1)\epsilon|\mathcal{P}[R]|_{L^2_+}. \tag{6-30}$$

Using Lemma 2.5, it is easy to check

$$\|S_2\|_{L^2}\leq C\epsilon^{\frac12-s}|\ln(\epsilon)|, \tag{6-31}$$

$$|h|_{L^m_-}\leq C\epsilon, \tag{6-32}$$

$$|h|_{L^2_-}\leq C\epsilon. \tag{6-33}$$

Summarizing, we have proved that

$$\begin{aligned}&\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}+\frac{1}{\epsilon^{\frac12}}|(1-\mathcal{P})[R]|_{L^2_+}+\|\mathbb{P}[R]\|_{L^{2m}}\\&\quad\leq C\big(o(1)\epsilon^{\frac1m}\|R\|_{L^\infty}+\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\Gamma[R,R]\|_{L^2}+o(1)\|\mathbb{P}[R]\|_{L^2}+o(1)\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}\\&\qquad+\epsilon^{-\frac12-s}|\ln(\epsilon)|+\epsilon^{-\frac12-s}|\ln(\epsilon)|+o(1)\epsilon\|R\|_{L^\infty}+\epsilon+o(1)|\mathcal{P}[R]|_{L^2_+}+1\big).\end{aligned} \tag{6-34}$$

Absorbing $\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}$ into the left-hand side, we obtain

$$\begin{aligned}&\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}+\frac{1}{\epsilon^{\frac12}}|(1-\mathcal{P})[R]|_{L^2_+}+\|\mathbb{P}[R]\|_{L^{2m}}\\&\quad\leq C\big(o(1)\epsilon^{\frac1m}\|R\|_{L^\infty}+\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\Gamma[R,R]\|_{L^2}+o(1)\|\mathbb{P}[R]\|_{L^2}+o(1)|\mathcal{P}[R]|_{L^2_+}\big).\end{aligned} \tag{6-35}$$

Here, we apply the estimate (4-72) to bound

$$\begin{aligned}|\mathcal{P}[R]|_{L^2_+}&\leq C\left(\|\mathbb{P}[R]\|_{L^2}+\frac{1}{\epsilon^{\frac12}}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2}+\frac{1}{\epsilon^{\frac12}}\left(\int_{\Omega\times\mathbb{R}^2}R\mathscr{L}[R]\right)^{\frac12}\right)\\&\leq C\left(\|\mathbb{P}[R]\|_{L^2}+\frac{1}{\epsilon^{\frac12}}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2}+\frac{1}{\epsilon^{\frac12}}\|(\mathbb{I}-\mathbb{P})[\mathscr{L}[R]]\|_{L^2}+\frac{1}{\epsilon}\|\mathbb{P}[\mathscr{L}[R]]\|_{L^2}\right).\end{aligned} \tag{6-36}$$

Then using L^2 version of Lemma 4.3 (see [Wu 2016; Guo and Wu 2017a]), we obtain

$$\|\mathbb{P}[R]\|_{L^2}\leq C\left(|(1-\mathcal{P})[R]|_{L^2_+}+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2}+\frac{1}{\epsilon}\|\mathscr{L}[R]\|_{L^2}+|R-\mathcal{P}[R]|_{L^2_-}\right). \tag{6-37}$$

Therefore, we have

$$\begin{aligned}|\mathcal{P}[R]|_{L^2_+}&\leq C\bigg(\|\mathbb{P}[R]\|_{L^2}+\frac{1}{\epsilon^{\frac{1}{2}}}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2}+\frac{1}{\epsilon^{\frac{1}{2}}}\bigg(\int_{\Omega\times\mathbb{R}^2}R\mathscr{L}[R]\bigg)^{\frac{1}{2}}\bigg)\\&\leq C\bigg(|(1-\mathcal{P})[R]|_{L^2_+}+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2}\\&\qquad+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[\mathscr{L}[R]]\|_{L^2}+\frac{1}{\epsilon}\|\mathbb{P}[\mathscr{L}[R]]\|_{L^2}+|R-\mathcal{P}[R]|_{L^2_-}\bigg).\end{aligned}\tag{6-38}$$

Note that the right-hand side of (6-38) has been estimated in (6-17) and (6-35), so we obtain

$$\begin{aligned}&|\mathcal{P}[R]|_{L^2_+}\\&\quad\leq C\big(o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty}+\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\Gamma[R,R]\|_{L^2}+o(1)\|\mathbb{P}[R]\|_{L^2}+o(1)|\mathcal{P}[R]|_{L^2_+}\big).\end{aligned}\tag{6-39}$$

Absorbing $|\mathcal{P}[R]|_{L^2_+}$ into the left-hand side, we obtain

$$|\mathcal{P}[R]|_{L^2_+}\leq C\big(o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty}+\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\Gamma[R,R]\|_{L^2}+o(1)\|\mathbb{P}[R]\|_{L^2}\big).\tag{6-40}$$

Plugging (6-40) into (6-35), we have

$$\begin{aligned}&\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}+\frac{1}{\epsilon^{\frac{1}{2}}}|(1-\mathcal{P})[R]|_{L^2_+}+\|\mathbb{P}[R]\|_{L^{2m}}\\&\qquad\leq C\big(o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty}+\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\Gamma[R,R]\|_{L^2}+o(1)\|\mathbb{P}[R]\|_{L^2}\big).\end{aligned}\tag{6-41}$$

Note that the estimate of $\|\mathbb{P}[R]\|_{L^2}$ has been incorporated in above analysis for $|\mathcal{P}[R]|_{L^2_+}$, so we may further simplify

$$\begin{aligned}&\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2_\nu}+\frac{1}{\epsilon^{\frac{1}{2}}}|(1-\mathcal{P})[R]|_{L^2_+}+\|\mathbb{P}[R]\|_{L^{2m}}\\&\qquad\leq C\big(o(1)\epsilon^{\frac{1}{m}}\|R\|_{L^\infty}+\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\Gamma[R,R]\|_{L^2}\big).\end{aligned}\tag{6-42}$$

<u>Step 5</u>: L^∞ estimates of R. Based on Theorem 4.7, we have

$$\begin{aligned}&\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}\\&\quad\leq C\bigg(\frac{1}{\epsilon^{\frac{1}{m}}}\|\mathbb{P}[R]\|_{L^{2m}}+\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[R]\|_{L^2}\\&\qquad+\bigg\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\epsilon^3\Gamma[R,R]}{\nu}\bigg\|_{L^\infty}+\bigg\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\Gamma[R,Q+\mathscr{Q}]}{\nu}\bigg\|_{L^\infty}+\bigg\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}S_1}{\nu}\bigg\|_{L^\infty}\\&\qquad+\bigg\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}S_2}{\nu}\bigg\|_{L^\infty}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}H[R]|_{L^\infty_-}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}h|_{L^\infty_-}\bigg).\end{aligned}\tag{6-43}$$

Hence, using (6-42), we know

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}$$
$$\leq C\Bigg(o(1)\|R\|_{L^\infty}+\epsilon^{-1-\frac{1}{2m}-4s}|\ln(\epsilon)|^8+\epsilon^2\|\Gamma[R,R]\|_{L^2}$$
$$+\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\epsilon^3\Gamma[R,R]}{\nu}\right\|_{L^\infty}+\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\Gamma[R,Q+\mathscr{Q}]}{\nu}\right\|_{L^\infty}+\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}S_1}{\nu}\right\|_{L^\infty}$$
$$+\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}S_2}{\nu}\right\|_{L^\infty}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}H[R]|_{L^\infty_-}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}h|_{L^\infty_-}\Bigg). \quad (6\text{-}44)$$

We can directly estimate

$$\|\epsilon^2\Gamma[R,R]\|_{L^2}\leq C\epsilon^2\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|^2_{L^\infty}, \quad (6\text{-}45)$$

$$\epsilon^3\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\Gamma[R,R]}{\nu}\right\|_{L^\infty}\leq C\epsilon^3\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|^2_{L^\infty}, \quad (6\text{-}46)$$

$$\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\Gamma[R,Q+\mathscr{Q}]}{\nu}\right\|_{L^\infty}\leq C\epsilon\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}. \quad (6\text{-}47)$$

Also, we know

$$\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}S_1}{\nu}\right\|_{L^\infty}\leq C, \quad (6\text{-}48)$$

$$\left\|\frac{\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}S_2}{\nu}\right\|_{L^\infty}\leq C, \quad (6\text{-}49)$$

$$|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}H[R]|_{L^\infty_-}\leq\epsilon|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}, \quad (6\text{-}50)$$

$$|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}h|_{L^\infty_-}\leq\epsilon. \quad (6\text{-}51)$$

Hence, in total, we have

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}$$
$$\leq C\big(o(1)\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}+o(1)|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}+\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|^2_{L^\infty}\big). \quad (6\text{-}52)$$

Absorbing $o(1)\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}$ and $o(1)|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}$ into the left-hand side, we obtain

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}\leq C(\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|+\epsilon^2\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|^2_{L^\infty}), \quad (6\text{-}53)$$

which further implies

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R\|_{L^\infty}+|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}R|_{L^\infty_+}\leq C\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)| \quad (6\text{-}54)$$

for ϵ sufficiently small. This means we have shown

$$\frac{1}{\epsilon^3}\big\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}\big(f^\epsilon-(\epsilon F_1+\epsilon^2F_2+\epsilon^3F_3)-(\epsilon\mathscr{F}_1+\epsilon^2\mathscr{F}_2)\big)\big\|_{L^\infty}\leq C\epsilon^{-1-\frac{1}{2m}-s}|\ln(\epsilon)|. \quad (6\text{-}55)$$

Therefore, we know

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}(f^{\epsilon}-\epsilon F_1-\epsilon\mathscr{F}_1)\|_{L^\infty}\leq C\epsilon^{2-\frac{1}{2m}-s}|\ln(\epsilon)|. \tag{6-56}$$

Since $\mathscr{F}_1=0$, we naturally have for $F=F_1$

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}(f^{\epsilon}-\epsilon F)\|_{L^\infty}\leq C\epsilon^{2-\frac{1}{2m}-s}|\ln(\epsilon)|. \tag{6-57}$$

Here $0<s\ll 1$, so we may further bound

$$\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\varrho|\vec{v}|^2}(f^{\epsilon}-\epsilon F)\|_{L^\infty}\leq C(\delta)\epsilon^{2-\delta} \tag{6-58}$$

for any $0<\delta\ll 1$. Also, this justifies that the solution f^{ϵ} to (1-16) exists and is well-posed. The uniqueness and positivity follow from a standard argument as in [Esposito et al. 2013].

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