Fisher Information and Logarithmic Sobolev Inequality for Matrix-Valued Functions

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Abstract. We prove a version of Talagrand's concentration inequality for subordinated sub-Laplacians on a compact Riemannian manifold using tools from noncommutative geometry. As an application, motivated by quantum information theory, we show that on a finite-dimensional matrix algebra the set of self-adjoint generators satisfying a tensor stable modified logarithmic Sobolev inequality is dense.

1. Introduction

Isoperimetric inequalities play an important role in geometry and analysis. In the last decades, the deep and beautiful connection between isoperimetric inequalities and functional inequalities has been discovered. This discovery started with the work of Meyer, Bakry and Émery on the famous 'carré du champs' or gradient form, and was brought to perfection by Varopoulos, Saloff-Coste [89,90], Coulhon [100], Diaconis [31], Bobkov and Götze [9,10], Barthe and his coauthors [5,6,12,16], and Ledoux [59–64]. It appears that the right framework of this analysis is given by abstract semigroup theory, i.e., starting with a semigroup of measure preserving maps on a measure space.

A crucial application of isoperimetric inequalities on compact manifolds is the famous *concentration of measure* phenomenon, used fundamentally in [35], and analyzed systematically by Milman and Schechtmann [73]. Thanks to the work of Gross [38–41], it is now well known that concentration of measure can occur in noncommutative spaces and infinite dimensions in the form of a logarithmic Sobolev inequality. Indeed, let $T_t = e^{-tA}$ be a measure-preserving

semigroup, acting on $L_\infty(\Omega, \mu)$ with energy form $\mathcal{E}(f) = (f, Af)$. Then, T_t (or its generator A) satisfies a *logarithmic Sobolev inequality*, in short λ-LSI, if

$$\lambda \int |f|^2 \log |f|^2 \mathrm{d}\mu \; \leq \; \mathcal{E}(f) \tag{1.1}$$

holds for all f with $\int |f|^2 \mathrm{d}\mu = 1$ in the domain of $A^{1/2}$. We will use the notation $\mathrm{Ent}(f) = \int f \log f \mathrm{d}\mu$ for the entropy of a probability density f. To simplify the exposition, we will assume throughout this paper that $\mathcal{A} \subset \mathrm{dom}(A) \cap L_\infty$ is a dense $*$-algebra in the domain and invariant under the semigroup. Semigroup techniques have been very successfully combined with the notion of hypercontractivity that

$$\|T_t : L_2 \to L_{q(t)}\| \; \leq \; 1 \quad \text{for} \quad q(t) \leq 1 + e^{ct} \; .$$

Indeed, the standard procedure to show that the Laplace–Beltrami operator on a compact Riemannian manifold satisfies λ-LSI is to derive hypercontractivity from heat kernel estimates and then use the Rothaus lemma to derive LSI from hypercontractivity. In this argument, ergodicity of the underlying semigroup appears to be crucial.

A major breakthrough in this development is Talagrand's inequality which connects entropic quantities with a given distance. A triple (Ω, μ, d) given by a measure and a metric satisfies Talagrand's inequality if

$$W_1(f\mu, \mu) \; \leq \; \sqrt{\frac{2\,\mathrm{Ent}(f)}{\lambda}} \; .$$

Here

$$W_1(\nu, \mu) \; = \; \inf_\pi \int \mathrm{d}(x, y) \mathrm{d}\pi(x, y) \; = \; \sup_{\|g\|_{Lip} \leq 1} \Big| \int g(x)\mathrm{d}\nu(x) - \int g(x)\mathrm{d}\mu(x)\Big| \tag{1.2}$$

is the Wasserstein 1-distance, and the second equality is the famous Kantorovich–Rubinstein duality (see [56,99]). The infimum is taken over all joint probability measures π on $\Omega \times \Omega$ with marginals as ν and μ. Using the triangle inequality for the Wasserstein distance, it is easy to derive the *geometric Talagrand's inequality*

$$\mathrm{d}(A, B) \; \geq \; h \quad \Longrightarrow \quad \mu(A)\mu(B) \; \leq \; e^{-\frac{h^2}{C}} \; .$$

If in addition $\mu(A) \geq 1/2$ and $B_h = \{x | \mathrm{d}(x, A) \geq h\}$, this inequality implies exponential decay of $\mu(B_h)$ in h, i.e., concentration of measure usually proved via isoperimetric inequalities. We refer to Tao's blog for applications of Talagrand's inequality [93] in particular to eigenvalues of random matrices [96,97].

As pointed out by Otto and Villani [76], Talagrand proved a much stronger inequality for 2-Wasserstein (in short λ-TA$_2$), namely

$$W_2(f\mu, \mu) \; \leq \; \sqrt{\frac{2\,\mathrm{Ent}(f)}{\lambda}} \tag{1.3}$$

for $\Omega = [0, 1]^n$ with respect to the Euclidean distance and for $\{0, 1\}^n$ with respect to the Hamming distance, with a constant λ not depending on n. Here

the 2-Wasserstein distance is obtained by replacing the L_1-norm by the L_2-norm of $L_2(\Omega \times \Omega)$ in the middle term of (1.2). Indeed, in the insightful paper by Otto and Villani [76], they point out that the correct way to understand Talagrand's inequality consists in pushing the semigroup into the state space of the underlying commutative C^*-algebra. Then, Talagrand's concentration inequality can be reformulated as a convexity condition for the Riemannian metric associated with the 2-Wasserstein distance. In that sense, Otto and Villani reconnects to the geometric aspect of concentration inequalities. The key idea in the Otto–Villani approach is to define the Riemannian metric such that the *relative entropy* function

$$D(\nu||\mu) \; = \; \int \log \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\nu$$

admits the semigroup $T_t(\frac{\mathrm{d}\nu}{\mathrm{d}\mu})$ as a path of steepest descent. Here, $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is the Radon–Nikodym derivative. A key tool in their analysis was to consider the modified version of the logarithmic Sobolev inequality, (in short λ-MLSI)

$$\lambda \operatorname{Ent}(f) \; \leq \; \int A(f) \log f \mathrm{d}\mu \; =: \; \mathcal{I}_A(f) \tag{1.4}$$

and show that it implies λ-TA$_2$. The right-hand side is known as Fisher information and turns out to be the energy functional for the relative entropy with respect to the Riemannian metric.

In this paper, we extend the theory of logarithmic Sobolev inequalities in *two directions*, by including matrix-valued functions and non-ergodic semigroups. The main road block, discovered in the quantum information theory literature, is that the Rothaus lemma ([86])

$$\left(\exists_{\lambda > 0} \; \forall_{E(f)=0} : \; \lambda D(f^2||E(f^2)) \; \leq \; \mathcal{E}(f) \right)$$
$$\overset{?}{\Longrightarrow} \quad \left(\exists_{\tilde{\lambda} > 0} \; \forall_f : \; \tilde{\lambda} D(f^2||E(f^2)) \; \leq \; \mathcal{E}(f) \right) \tag{1.5}$$

may fail for matrix-valued functions f. Here, $E(f^2)$ is the mean of the matrix-valued f^2. The failure of (1.5) forces us to introduce new tools. Recently, and in part parallel to the refereeing process of this paper, Talagrand's inequalities in the noncommutative setting have made very significant progress, in particular through the work of Rouzé and Datta [84], and the continuation of the seminal work [25] by Carlen and Maas in [26]. On the other hand, we are not aware of any investigation of Talagrand's inequality in the non-ergodic setting even in the commutative cases. For self-adjoint semigroups, the fixpoint algebra $N = \{x : \forall_t T_t(x) = x\}$ admits a normal conditional expectation E_{fix} onto N. This remains true in the noncommutative setting, i.e., for a semigroup (T_t) of (sub-)unital completely positive maps on a finite von Neumann algebra M provided each T_t is self-adjoint with respect to the inner product $\langle x, y \rangle = \tau(x^*y)$ of a normal faithful tracial state τ. The reader less familiar with von Neumann algebras is welcome to think of $M = L_\infty(\Omega, \mu; \mathbb{M}_m)$, the space of bounded random matrices equipped with $\tau(f) = \int_\Omega \frac{1}{m} tr(f(\omega))\mathrm{d}\mu(\omega)$ and $\tau(T_t(x^*)y) = \tau(x^* T_t(y))$.

Let us consider examples of the form $T_t = S_t \otimes id_{\mathbb{M}_m}$, where S_t is a nice ergodic semigroup. These examples are natural in the context of operator spaces (see [81] and [33] for more background), despite being obviously not ergodic. We say that a self-adjoint semigroup T_t or its generator A satisfies λ-MLSI (λ-modified logarithmic Sobolev inequality) if

$$\lambda D(\rho || E_{fix}(\rho)) \leq \mathcal{I}_A(\rho) = \tau(A(\rho) \ln \rho) .$$

The right-hand side is the noncommutative Fisher information introduced under the name entropy production by Spohn [92], which is well known in the quantum information theory. At the time of this writing, it is not known whether λ-MLSI is stable under tensorization. However, tensorization is an important feature and allows us to deduce the Gaussian log-Sobolev inequality from an elementary 2-point inequality, see, e.g., [7]. Therefore, we introduce the *complete logarithmic Sobolov inequality* (in short λ-CLSI) by requiring that $T_t \otimes id_{\mathbb{M}_m}$ satisfies λ-MLSI for all $m \in \mathbb{N}$. Using the data processing inequality, it is easy to show that the CLSI is stable under tensorization (c.f. Proposition 2.9). Before this paper, the list of examples which satisfy good tensorization properties could all be deduced from the following key example, due to Bardet [1] (see also [8,57]):

Lemma 1.1 (Examples 3.1 of [1]). *Let $E : M \to N$ be a conditional expectation. Then, $T_t = e^{-t(I-E)}$ satisfies 1-CLSI.*

Indeed, for conditional expectation, we have

$$\mathcal{I}_{I-E}(\rho) = D(\rho || E(\rho)) + D(E(\rho) || \rho) \geq D(\rho || E(\rho)) .$$

In this case, CLSI follows from non-negativity of relative entropy. The middle term is the original symmetrized divergence introduced by Kullback and Leibler [54], which is interesting from a historical point of view. Using the tensorization, one can now deduce that Gaussian systems (and certain depolarizing channels) also satisfy CLSI (see [8,23]).

Our new tool to prove CLSI is based on the gradient form:

$$2\Gamma_A(f,g) := A(f^*)g + f^*A(g) - A(f^*g) .$$

We say that the generator A satisfies λ-$\Gamma\mathcal{E}$ if

$$\lambda[\Gamma_{I-E_{fix}}(f_j, f_k)]_{i,j} \leq [\Gamma_A(f_j, f_k)]_{i,j}$$

holds for all finite families (f_k). The next lemma states the two new basic facts used in this paper. The second assertion (ii) is the tensorization property of CLSI.

Lemma 1.2. *(i) λ-$\Gamma\mathcal{E}$ implies λ-CLSI. (ii) If the generators A and B satisfy λ-CLSI, then $A \otimes id + id \otimes B$ satisfies λ-CLSI.*

Inspired by the work of Saloff-Coste [89], we find that $\Gamma\mathcal{E}$ is a strong condition that implies the following L_p-return time estimate, to our knowledge new even in the commutative setting.

Theorem 1.3. *If T_t satisfies λ-$\Gamma\mathcal{E}$, then for all x,*

$$\|T_t(x) - E(x)\|_1 \leq e^{-\lambda t}\|x - E(x)\|_1 \ .$$

Note that λ-MLSI implies exponential decay of relative entropy, $D(T_t(\rho)\|E(\rho))$ $\leq e^{-\lambda t}D(\rho\|E(\rho))$ and hence of L_1-norm via Pinsker inequality $D(\rho\|\sigma) \geq \frac{1}{2}\|$ $\rho - \sigma\|_1^2$. However, the initial term $D(\rho\|E(\rho))$ has linear growth with respect to the number of tensor products. Theorem 1.3 is strong in the sense that $\|\rho - E(\rho)\|_1 \leq 2$ for any density ρ.

Our main contribution is to identify large classes of examples from representation theory satisfying $\Gamma\mathcal{E}$. Recall that the definition of a Hörmander system on a Riemannian manifold $(\mathcal{M}, g)$ is given by a family of vector fields $X = \{X_1, \ldots, X_r\}$ such that the iterated commutators $[X_{i_1}, [X_{i_2}, \ldots]]$ generate the tangent space $T_x\mathcal{M}$ at every point $x \in \mathcal{M}$. Building on the famous heat kernel estimates from [87], see also [67], and the work of Saloff-Coste on return time, we find entropic concentration inequalities for subordinated sub-Laplacians.

Theorem 1.4. *Let X be a Hörmander system on a compact Riemannian manifold, and the self-adjoint generator $\Delta_X = \sum_j X_j{}^* X_j$ be the corresponding sub-Laplacian. Then, for any $0 < \theta < 1$, $(\Delta_X)^\theta$ satisfies λ-$\Gamma\mathcal{E}$, and hence λ-CLSI for some constant $\lambda = \lambda(X, \theta)$.*

It is widely open whether the Δ_X itself satisfies CLSI, even when Δ_X is the Laplace–Beltrami operator on a compact Riemannian manifold. For $\Omega = S^1$ the Torus the standard semigroup given by $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ satisfies 1-CLSI however fails λ-$\Gamma\mathcal{E}$. For more information on Bakry–Emery theory for sub-Laplacians, we refer to the deep work of Baudoin and his coauthors [2–4,11,13,15]. Subordinated semigroups (in a slightly different meaning) have also been investigated in the Gaussian setting, see [66,68]. From a rough kernel perspective (see [29,34,42]), it may appear less surprising that subordinated semigroups outperform their smooth counterparts.

In the context of group actions, we can transfer logarithmic Sobolev inequalities. Indeed, let $\alpha : G \to \mathrm{Aut}(M)$ be a trace-preserving action on a finite von Neumann algebra (M, τ), i.e., α is strongly continuous group homomorphism with values in the set of trace-preserving automorphisms on M. A semigroup $S_t : L_\infty(G) \to L_\infty(G)$ which is invariant under right translations is given by an integral operator of the form

$$S_t(f)(g) \ = \ \int k_t(gh^{-1})f(h)\mathrm{d}\mu(g)$$

where μ is the Haar measure. We will assume that G is compact and μ is a probability measure. Then, we may define the *transferred semigroup* on the von Neumann algebra M,

$$T_t(x) \ = \ \int k_t(g^{-1})\alpha_g(x)\mathrm{d}\mu(g) \ .$$

For ergodic S_t, the fixpoint algebra of the transferred semigroup T_t is then given by the fixpoint algebra of the action $N^G = \{x|\forall_g\alpha_g(x) = x\}$, which is in general not a trivial subalgebra.

Theorem 1.5. *Let G be a compact group acting on a finite von Neumann algebra (M, τ). If $S_t = e^{-tA}$ satisfies λ-CLSI (resp. λ-$\Gamma\mathcal{E}$), then T_t satisfies λ-CLSI (resp. λ-$\Gamma\mathcal{E}$).*

For a compact Lie group G, a generating set $X = \{X_1, \ldots, X_r\}$ of the Lie-algebra $\mathfrak{g}$ defines a Hörmander system $X = \{X_1, \ldots, X_r\}$ given by the corresponding right translation invariant vector fields. Then, we conclude that for *any* group representation the semigroup T_t^θ transferred from $S_t = e^{-t\Delta_X^\theta}$ satisfies λ-$\Gamma\mathcal{E}$. Our motivation is from quantum information theory and previous results of hypercontractivity. Starting with the seminal papers [30,65], Temme and his coauthors [22,57,58,94,95] made hypercontractivity on matrix algebras available in the ergodic setting (see [45–47,88] for results in group von Neumann algebras). Using transferred semigroups and the so-called Lindblad generators, we can prove the following density result of CLSI on matrix algebras:

Theorem 1.6. *The set of self-adjoint generators of semigroups on $\mathbb{M}_m$ satisfying $\Gamma\mathcal{E}$ and CLSI is dense.*

Indeed, combining all the results from above, we can show that for such self-adjoint generators A the subordinated A^θ satisfies $\lambda(\theta)$-$\Gamma\mathcal{E}$ for all $0 < \theta < 1$. Let us mention the deep work of Carlen-Maas [24,25]. They translate the work of Otto-Villani [76] to the state space of matrices and identify a truly noncommutative Wasserstein 2-distance $d_{A,2}(\rho, \sigma)$. They also showed (in the ergodic setting) that λ-MLSI implies

$$d_{A,2}(\rho, E(\rho)) \leq 2\sqrt{\frac{D(\rho||E(\rho))}{\lambda}} \ .$$

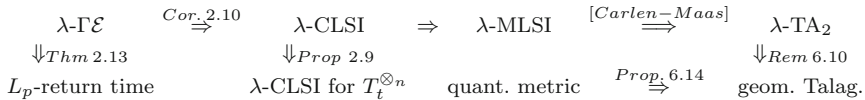An analogue of an intrinsic Wasserstein distance has already been introduced in [49,52]:

$$d_\Gamma(\rho, \sigma) = \sup_{f=f^*, \Gamma_A(f,f)\leq 1} |\tau(\rho f) - \tau(\sigma f)| \ .$$

Based on [48] we show that $d_\Gamma \leq 2\sqrt{2}d_{A,2}$, and hence, we see that λ-MLSI does indeed imply a noncommutative geometric Talagrand's inequality: Let e_1 and e_1 be projections in M such that for some test function f with $\Gamma_A(f,f) \leq 1$ we have

$$\left|\frac{\tau(e_1 f)}{\tau(e_1)} - \frac{\tau(e_2 f)}{\tau(e_2)}\right| \geq h \quad \Longrightarrow \quad \tau(e_1)\tau(e_2) \leq e^{-h^2/C} \ ,$$

where C only depends on the λ-MLSI constant of the generator A. Thus, we have identified large classes of new examples which satisfy the Talagrand's concentration inequality, not only for $T_t = e^{-tA}$, but also for the n-fold tensor product $(T_t^{\otimes n})$.

The paper is organized as follows: We discuss gradient forms, derivations, and Fisher information in Sect. 2. In Sect. 3, we first consider kernel and decay time estimates in the ergodic and then in the non-ergodic case. The latter analysis relies on the theory of mixed $L_p(L_q)$ spaces from [44], which has been recently used in [20]. We discuss group representations in Sect. 4 and the density result in Sect. 5. Section 6 is devoted to geometric applications and concentration inequalities. In Sect. 7, we discuss examples and counterexamples. A chart of the different properties considered in this paper is given in the following diagram:

$$\lambda\text{-}\Gamma\mathcal{E} \overset{Cor.\,2.10}{\Rightarrow} \lambda\text{-CLSI} \quad \Rightarrow \quad \lambda\text{-MLSI} \overset{[Carlen-Maas]}{\Longrightarrow} \lambda\text{-TA}_2$$

$$\Downarrow_{Thm\,2.13} \qquad\qquad \Downarrow_{Prop\,2.9} \qquad\qquad\qquad\qquad\qquad \Downarrow_{Rem\,6.10}$$

$$L_p\text{-return time} \qquad \lambda\text{-CLSI for } T_t^{\otimes n} \qquad \text{quant. metric} \overset{Prop.\,6.14}{\Rightarrow} \text{geom. Talag.}$$

Open problems will be mentioned at the end of Sect. 7. In fact, we expect CLSI to hold for the smooth generators of semigroups as well. Due to space restrictions, we ignore the deep and interesting connection to free Fisher information.

2. Gradient Forms and Fisher Information

2.1. Modules and Gradient Forms

Let (M, τ) be a finite von Neumann algebra M equipped with a normal faithful tracial state τ. We denote the noncommutative L_p-spaces by $L_p(M, \tau)$ or $L_p(M)$ if the trace is clear from the context. Throughout the paper, we consider that $T_t = e^{-tA} : M \to M$ is a strongly continuous semigroup of completely positive, unital and self-adjoint maps. Then, $\tau(x^* T_t(y)) = \tau(T_t(x)^* y)$ for all $x, y \in M$ and hence T_t is also trace preserving. The generator A is the (possibly unbounded) positive operator on $L_2(M, \tau)$ given by $Ax = \lim_{t \to 0} \frac{1}{t}(x - T_t(x))$ (see [28] for more background.) We will assume that there exists a weakly dense *-subalgebra $\mathcal{A} \subset M$ such that

(i) $\mathcal{A} \subset dom(A) \cap \{x | A(x) \in M\}$;
(ii) $T_t(\mathcal{A}) \subset \mathcal{A}$ for all $t > 0$.

In most cases, it is enough to assume that $\mathcal{A} \subset \mathrm{dom}(A^{1/2})$ and the Γ-regularity from [48]. The gradient form of A is defined as

$$\Gamma_A(x, y)(z) := \frac{1}{2}\Big(\tau(A(x)^* y z) + \tau(x^* A(y) z) - \tau(x^* y A(z))\Big)$$

We say the generator A satisfies Γ-regularity if $\Gamma_A(x, y) \in L_1(M, \tau)$ for all $x, y \in \mathrm{dom}(A^{1/2})$.

Theorem 2.1 [48]. *Suppose $\Gamma(x, x) \in L_1(M, \tau)$ for all $x \in \mathrm{dom}(A^{1/2})$. Then, there exists a finite von Neumann algebra $(\hat{M}, \hat{\tau})$ containing M with $\hat{\tau}|_{\mathcal{M}} = \tau$, and a self-adjoint derivation $\delta : \mathrm{dom}(A^{1/2}) \to L_2(\hat{M})$ such that for all $z \in M$,*

$$\tau(\Gamma_A(x, y) z) = \hat{\tau}(\delta(x)^* \delta(y) z) .$$

Equivalently, $E_M(\delta(x)^* \delta(y)) = \Gamma_A(x, y)$ where $E_M : \hat{M} \to M$ is the conditional expectation.

The Γ-regularity allows us to define the right Hilbert W^*-module Ω_Γ as the completion of $\mathrm{dom}(A^{1/2}) \otimes M$ with Γ-inner product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_\Gamma \;=\; x_2^* \Gamma(x_1, y_1) y_2 \,.$$

Here, we use the canonical right action $(x \otimes y) \cdot b = x \otimes yb$ and the left action is given by

$$\begin{aligned}
\Gamma(x, a \cdot y) &= E(\delta(x)^* a \delta(y)) = E(\delta(x)^* \delta(ay)) - E(\delta(x)^* \delta(a))y \\
&= \Gamma\,(x, ay) - \Gamma(x, a)y \,.
\end{aligned}$$

Namely, $a \cdot y = ay \otimes x - a \otimes xy$. Note that in Theorem 2.1 the completion of $\hat{M}$ with respect to M-valued inner product

$$\langle x, y \rangle_{E_M} := E_M(x^* y) \,,\; x, y \in \hat{M}$$

also gives rise to a W^*-module, which we denote as $L_\infty^c(M \subset \hat{M}) = \overline{\hat{M} \otimes_E M}^{STOP}$, see also [50,51]. Recall that for a W^*-module $\mathcal{H}$ of M, the norm is given by its M-valued inner product $\langle \cdot, \cdot \rangle_\mathcal{H}$ as follows:

$$\|\xi\|_\mathcal{H} \;=\; \|\langle \xi, \xi \rangle_\mathcal{H}\|_M^{1/2} \,.$$

Then, it is readily verified that the map

$$\pi_\delta : \Omega_\Gamma \to L_\infty^c(M \subset \hat{M}) \,,\; \pi_\delta(x \otimes y) \;=\; \delta(x)y$$

is an isometric right M-module map. Moreover, thanks to [77] (see also [50]), the range $\pi_\delta(\Omega_\Gamma)$ is 1-complemented in $L_\infty^c(M \subset \hat{M})$.

Remark 2.2. Our notation Ω_Γ is motivated by the universal bimodule of 1-forms

$$\Omega^1 \mathcal{A} \;=\; \{x \otimes y - 1 \otimes xy \mid x, y \in \mathcal{A}\} \subset \mathcal{A} \otimes \mathcal{A}$$

from noncommutative geometry (see [27]). Indeed, the map Π_δ induces a representation of the universal derivation $\delta(x) = x \otimes 1 - 1 \otimes x$.

We refer to [48] for the proof of Theorem 2.1. Here, we discuss the following special case. Let $T : M \to M$ be a unital completely positive self-adjoint map. Recall that the W^*-module $M \otimes_T M$ is given by GNS construction

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_T \;=\; x_2^* T(x_1^* y_1) y_2 \,.$$

with left module action $a \cdot (x \otimes y) = ax \otimes y$ and $(x \otimes y) \cdot b = x \otimes yb$ on $\mathcal{A} \otimes \mathcal{A}$. Note that this implies the 'differential left action' $a\delta(b) = \delta(ab) - \delta(a)b$ on $\Omega^1 \mathcal{A}$ which is shared by all representations. It is shown in [48](see also [91]) that there exists a finite von Neumann algebra $\hat{M}$ containing M and a self-adjoint element $\xi \in L_2(\hat{M})$ of norm 1 such that

$$T(x) \;=\; E_M(\xi x \xi) \,. \tag{2.1}$$

This gives an isometric M-module map $\phi : M \otimes_T M \to L_\infty^c(M \subset \hat{M})$

$$\phi(x \otimes y) \;=\; x \xi y$$

with respect to the right module action $\phi((x \otimes y) \cdot z) = x \xi y z = \phi(x \otimes y)z$. On the other hand, let $I : M \to M$ be the identity map. The map $A =$

$I - T$ is a generator of a semigroup of completely positive maps (see [79] for a characterization of generators of trace-preserving self-adjoint semigroups). Its gradient form is

$$\Gamma_{I-T}(x, y) = \frac{1}{2}\left(x^*y - T(x)^*y - x^*T(y) + T(x^*y)\right) .$$

The gradient form of this generator can be realized as a right submodule of $M \otimes_T M$ via the map

$$\psi : \Omega_{\Gamma_{I-T}} \to M \otimes_T M , \quad \psi(x \otimes y) = \frac{1}{\sqrt{2}}\left(x \otimes y - 1 \otimes T(x)y\right) . \qquad (2.2)$$

Indeed, this follows from

$$\langle \psi(x \otimes y), \psi(x' \otimes y')\rangle_T = \frac{1}{2}y^*(x^*x' - x^*T(x') - T(x^*)x'$$
$$+ T(x^*x'))y' = y^*\Gamma_{I-T}(x, x')y' .$$

This means for the semigroup generators of the form $A = I - T$, the deviation $\delta : M \to \hat{M}$ and the module isometry $\pi_\delta : \Omega_{\Gamma_{I-T}} \to \hat{M}$ can be obtained as a composition $\pi_\delta = \phi\psi$:

$$\delta(x) = \frac{1}{\sqrt{2}}(x\xi - \xi T(x)) , \quad \pi_\delta(x \otimes y) = \frac{1}{\sqrt{2}}(x\xi - \xi T(x))y .$$

where the vector ξ is as in (2.1).

In the following, we will use the completely positive order in two ways. For two completely positive maps T and S, we write $T \leq_{cp} S$ if $S - T$ is completely positive. For two gradient forms Γ, Γ', we write $\Gamma \leq_{cp} \Gamma'$ if

$$[\Gamma(x_i, x_j)]_{i,j} \leq [\Gamma'(x_i, x_j)]_{i,j}$$

holds for all finite sequence (x_j) in the domain of Γ.

Lemma 2.3. *(i) Let $T : M \to M$ be a completely positive unital map. Then, for any state ρ,*

$$\rho(\langle\xi, \xi\rangle_{\Gamma_{I-T}}) = \inf_{c \in M} \rho(\langle\xi - 1 \otimes c, \xi - 1 \otimes c\rangle_T) .$$

(ii) Let $T_1, T_2 : M \to M$ be two completely positive unital maps. Given $\lambda > 0$, $\lambda T_1 \leq_{cp} T_2$ implies $\lambda\Gamma_{I-T_1} \leq_{cp} \Gamma_{I-T_2}$.

Proof. We choose a module basis $\{\xi_i\}_{i \in I}$ of $M \otimes_T M$ (see [50,77]) and let $\xi_0 = 1 \otimes 1$. Then,

$$\langle\xi_i, \xi_j\rangle_T = \delta_{ij}e_i ,$$

where $(e_i)_{i \in I} \subset M$ is a family of projections and $e_0 = 1$. Let $P : M \otimes_T M \to 1 \otimes M$ be the orthogonal projection given by

$$P(\xi) = P\left(\sum_i \xi_i(1 \otimes \alpha_i)\right) = \xi_0(1 \otimes \alpha_0) , \quad \alpha_i \in M$$

Note that

$$\langle 1 \otimes z, x \otimes y\rangle_T = z^*T(x)y = \langle 1 \otimes z, 1 \otimes T(x)y\rangle_T .$$

This implies that $P(x \otimes y) = 1 \otimes T(x)y$ and hence for $\xi = \sum_i \xi_i(1 \otimes \alpha_i)$, we find that

$$\langle \xi, \xi \rangle_{\Gamma_{I-T}} = \langle \psi(\xi), \psi(\xi) \rangle_T = \langle \xi - P(\xi), \xi - P(\xi) \rangle_T = \sum_{i \neq 0} \alpha_i^* e_i \alpha_i ,$$

where ψ is the isometric module map in (2.2). On the other hand, for any $c \in M$,

$$\langle \xi - 1 \otimes c, \xi - 1 \otimes c \rangle_T = (\alpha_0 - c)^*(\alpha_0 - c) + \sum_{i \neq 0} \alpha_i^* e_i \alpha_i .$$

This exactly implies that for any $c \in M$,

$$\langle \xi, \xi \rangle_{\Gamma_{I-T}} \leq \langle \xi - 1 \otimes c, \xi - 1 \otimes c \rangle_T$$

and clearly for any state ρ,

$$\rho(\langle \xi, \xi \rangle_{\Gamma_{I-T}}) = \inf_{c \in N} \rho(\langle \xi - 1 \otimes c, \xi - 1 \otimes c \rangle_T) .$$

For (ii), for any state ρ we can find a $c \in M$ such that

$$\rho(\langle \xi, \xi \rangle_{\Gamma_{I-T_2}}) = \rho(\langle \xi - 1 \otimes c, \xi - 1 \otimes c \rangle_{T_2}) \geq \lambda \rho(\langle \xi - 1 \otimes c, \xi - 1 \otimes c \rangle_{T_1})$$
$$\geq \lambda \rho(\langle \xi, \xi \rangle_{\Gamma_{I-T_1}}) .$$

Here, we used i) twice. The argument for arbitrary matrix levels is the same. □

Our next observation is based on operator integral calculus (see [82] and references therein for more information). Let $F : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and δ be a derivation as in Theorem 2.1. Then, for a positive $\rho \in \mathcal{A}$, the functional calculus for δ is given by the following operator integral:

$$\delta(F(\rho)) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{F(s) - F(t)}{s - t} dE_s^\rho \delta(\rho) dE_t^\rho . \tag{2.3}$$

where $E^\rho((s,t]) = 1_{(s,t]}(\rho)$ is the spectral projection of ρ. Indeed, this is obvious for monomials

$$\delta(\rho^n) = \sum_{j=0}^{n-1} \rho^j \delta(\rho) \rho^{n-j-1} = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{s^n - t^n}{s - t} dE_s^\rho \delta(\rho) dE_t^\rho .$$

The convergence of (2.3) in $L_2(\hat{M})$ follows from the boundedness of the derivative F' (and in $L_p(\hat{M})$ from the theory of singular integrals, see again [82]). Let us introduce the *double operator integral*

$$J_F^\rho(y) := \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{F(s) - F(t)}{s - t} dE_s^\rho y dE_t^\rho .$$

For $\rho = \sum_k \rho_k e_k$ with discrete spectrum, this simplifies to a Schur multiplier

$$J_F^\rho(y) = \sum_{k,l} \frac{F(\rho_k) - F(\rho_l)}{\rho_k - \rho_l} e_k y e_l .$$

At $k = l$, $\frac{F(\rho_k) - F(\rho_l)}{\rho_k - \rho_l}$ is understood as $F'(\rho_l)$.

Lemma 2.4. *Let $F : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable monotone increasing function and $\rho \in M$ be positive. Let A and B be two generators of semigroups on M with corresponding derivation δ_A and δ_B. Suppose their gradient forms satisfy that for some $\lambda > 0$,*

$$\lambda \Gamma_A \leq_{cp} \Gamma_B .$$

Then, $\mathrm{dom}(B^{\frac{1}{2}}) \subset \mathrm{dom}(A^{\frac{1}{2}})$ and for any $x \in \mathrm{dom}(B^{\frac{1}{2}})$,

$$\lambda \tau (E_M(\delta_A(x)^* J_F^\rho(\delta_A(x)))) \leq \tau(E_M(\delta_B(x)^* J_F^\rho(\delta_B(x)))) .$$

Proof. Let us first assume that $x = \sum_k \lambda_k e_k$ has discrete spectrum. Using (2.3) we find

$$\tau\left(E_M\big(\delta_A(x)^* J_F^\rho(\delta_A(x))\big)\right) = \sum_{k,l} \frac{F(\lambda_k) - F(\lambda_l)}{\lambda_k - \lambda_l} \hat\tau\left(\delta_A(x)^* e_k \delta_A(x) e_l\right)$$

$$= \sum_{k,l} \frac{F(\lambda_k) - F(\lambda_l)}{\lambda_k - \lambda_l} \|e_k \delta_A(x) e_l\|^2_{L_2(\hat M_A)} .$$

Recall that Ω_{Γ_A} (resp. Ω_{Γ_B}) is a submodule of $L_\infty^c(M \subset \hat M_A)$ (resp. $L_\infty^c(M \subset \hat M_B)$), and hence, there is an M-module projection P_B onto Ω_{Γ_B}. Our assumption implies that the right module map

$$\Phi : \Omega_{\Gamma_B} \to L_\infty^c(M \subset \hat M_A) , \quad \Phi(x \otimes y) = \delta_A(x) y$$

is well defined and of norm less than $\lambda^{-1/2}$. Now consider the composition

$$\tilde\Phi = \Phi \circ P_B : L_\infty^c(M \subset \hat M_B) \to L_\infty^c(M \subset \hat M_A) , \quad \tilde\Phi(\delta_B(x) y) = \delta_A(x) y .$$

It follows from the Leibniz rule that $\tilde\Phi$ is also a left $\mathcal{A}$-module map,

$$\tilde\Phi(a \delta_B(x) y) = \tilde\Phi(\delta_B(ax) y - \delta_B(a) x y) = \delta_A(ax) y - \delta_A(a) x y = a \delta_A(x) y .$$

Using strongly converging bounded nets from the weak*-dense algebra $\mathcal{A} \subset \mathrm{dom}(A^{1/2})$, we deduce that $\tilde\Phi$ extends to a M-bimodule map with $\| \tilde\Phi : L_2(\hat M_B) \to L_2(\hat M_A) \| \leq \lambda^{-1/2}$. Hence for all k, l,

$$\sqrt{\lambda} \|e_k \delta_A(x) e_l\|_{L_2(\hat M_A)} \leq \|e_k \delta_B(x) e_l\|_{L_2(\hat M_B)} .$$

Since F is increasing, $\frac{F(\lambda_k) - F(\lambda_l)}{\lambda_k - \lambda_l}$ and $F'(\lambda_k)$ are positive. Therefore, we obtain

$$\lambda \tau\left(E_M\big(\delta_A(x)^* J_F^\rho(\delta_A(x))\big)\right) \leq \tau\left(E_M\big(\delta_B(x)^* J_F^\rho(\delta_B(x))\big)\right) .$$

This implies the assertion for ρ with discrete spectrum.

Let $\rho \in \mathcal{M}$ be a general positive element. Then, we can approximate $F(s,t) = \frac{F(s) - F(t)}{s - t}$ by the sequence

$$F_n(s,t) = \frac{F(n\lfloor \frac{s}{n} \rfloor) - F(n\lfloor \frac{t}{n} \rfloor)}{n(\lfloor \frac{s}{n} \rfloor) - n(\lfloor \frac{t}{n} \rfloor)} ,$$

and find $\lim_{n \to \infty} J_{F_n}^\rho(x) = J_F^\rho(x)$ (with respect to convergence in L_2). $\qquad\square$

2.2. Fisher Information

Recall that the Fisher information of a generator A is defined as

$$\mathcal{I}_A(x) \ = \ \tau(A(x)\ln(x)) \ , \ x \in \mathcal{A} \cap M_+ \ ,$$

provided that $A(x) \in L_1(M)$ and $\ln x$ is bounded. Equivalently, one can define $\mathcal{I}_A(x) = \lim_{\varepsilon \to 0} \tau(A(x)\ln(x+\varepsilon 1))$. In the quantum information theory literature, $\mathcal{I}_A$ is also called *entropy production* (see [92]).

Corollary 2.5. *Let Γ_A, Γ_B be the gradient forms of two semigroups on M. Suppose $\lambda \Gamma_A \leq_{cp} \Gamma_B$. Then, for $x \in \mathcal{A} \cap M_+$,*

$$\lambda \mathcal{I}_A(x) \ \leq \ \mathcal{I}_B(x) \ .$$

Proof. Let $x \in \mathrm{dom}(B^{1/2}) \cap M$. Then, by the Leibniz rule,

$$\begin{aligned}
\|B^{1/2}(x^*x)\|_{L_2(M)} = \|\delta_B(x^*x)\|_{L_2(\hat{M}_B)} \ &= \ \|\delta_B(x^*)x + x^*\delta_B(x)\|_{L_2(\hat{M}_B)} \\
&\leq \|\delta_B(x)\|_2\|x\|_\infty + \|x^*\|_\infty\|\delta_B(x)\|_2 \\
&\leq 2\|x\|\|B^{1/2}(x)\| \ .
\end{aligned}$$

Hence, x^*x is also in the domain of $B^{1/2}$. Thus, we have enough positive elements in $\mathrm{dom}(B^{1/2}) \cap M$. Take the function $F(t) = \ln t$. Then, using Lemma 2.4,

$$\begin{aligned}
\tau\big(B(x)\ln(x+\varepsilon 1)\big) = \tau\Big(\delta_B(x)\delta_B\big(\ln(x+\varepsilon 1)\big)\Big) \ &= \ \tau\Big(\delta_B(x)J_F^{x+\varepsilon 1}\big(\delta_B(x)\big)\Big) \\
&\geq \ \lambda\tau\Big(\delta_A(x)J_F^{x+\varepsilon 1}\big(\delta_A(x)\big)\Big) \ = \ \lambda\tau\big(A(x)\ln(x+\varepsilon 1)\big) \ .
\end{aligned}$$

The assertion follows from sending $\varepsilon \to 0$. $\qquad\square$

Let $N \subset M$ be a von Neumann subalgebra and E_N be the conditional expectation (or shortly E for E_N if no ambiguity). We define the Fisher information for the subalgebra N with the help of the generator $I - E$:

$$\mathcal{I}_N(\rho) \ = \ \mathcal{I}_{I-E}(\rho) = \tau\Big((\rho - E(\rho))\ln\rho\Big) \ .$$

Recall that for two positive elements $\rho, \sigma \in M$, the relative entropy is

$$D(\rho||\sigma) := \begin{cases} \tau(\rho\ln\rho) - \tau(\rho\ln\sigma), & \text{if } \rho \ll \sigma \\ +\infty, & \text{otherwise.} \end{cases} \ ,$$

Equivalently, one can define $D(\rho||\sigma) = \lim_{\delta \to 0} D(\rho||\sigma + \delta 1)$. When $\tau(\rho) = \tau(\sigma)$, $D(\rho||\sigma)$ is always positive. The relative entropy with respect to N is defined as

$$D_N(\rho) \ = \ D(\rho||E(\rho)) \ = \ \inf_{\sigma \in N, \sigma \geq 0, \tau(\sigma) = \tau(\rho)} D(\rho||\sigma) \ .$$

See [37,74] for more information on D_N as an asymmetry measure. The following result is due to [1] (see [57] for the primitive case), but the simple proof is crucial for this paper.

Lemma 2.6. *The Fisher information satisfies*

$$\mathcal{I}_N(\rho) \;=\; D(\rho||E(\rho)) + D(E(\rho)||\rho)$$

and hence $D_N \leq \mathcal{I}_N$.

Proof. We first note that

$$\begin{aligned}
\mathcal{I}_N(\rho) &= \tau(\rho \ln \rho) - \tau(E(\rho) \ln \rho) \;=\; \tau(\rho \ln \rho) - \tau(\rho \ln E(\rho)) \\
&\quad + \tau(E(\rho) \ln E(\rho)) - \tau(E(\rho) \ln \rho) \\
&= D(\rho||E(\rho)) + D(E(\rho)||\rho) \;.
\end{aligned}$$

The non-negativity of the relative entropy implies the assertion. □

Now let $T_t = e^{-At} : M \to M$ be a self-adjoint semigroup of completely positive unital maps and $N \subset M$ be the fixpoint algebra of T_t. It is easy to see that

$$E \circ T_t = T_t \circ E = E \;.$$

It is well known that the Fisher information I_A appears as the negative derivative of relative entropy D_N under the semigroup T_t (see also [92]).

Proposition 2.7. *Suppose that*

$$\lambda D_N(\rho) \;\leq\; \mathcal{I}_A(\rho) \;, \; \forall \, \rho \geq 0 \;.$$

Then,

$$D_N(T_t(\rho)) \;\leq\; e^{-\lambda t} D_N(\rho) \;, \; \forall \, \rho \geq 0.$$

Proof. Take $f(t) = D_N(T_t(\rho))$. The idea is to differentiate

$$\begin{aligned}
f(t) = D_N\big(T_t(\rho)\big) &= \tau\big(T_t(\rho) \ln T_t(\rho)\big) - \tau\big(E(T_t(\rho)) \ln E(T_t(\rho))\big) \\
&= \tau\big(T_t(\rho) \ln T_t(\rho)\big) - \tau\big(E(\rho) \ln E(\rho)\big) \;.
\end{aligned}$$

For a function $F : \mathbb{R}_+ \to \mathbb{R}$ with bounded continuous derivatives F', we have (see, e.g., [101, Corollary 5.10])

$$\lim_{s \to 0} \frac{\tau(F(\rho + s\sigma)) - \tau(F(\rho))}{s} \;=\; \tau(F'(\rho)\sigma) \;.$$

Note that $\lim_{s \to 0} \frac{1}{s}(T_{t+s}(\rho) - T_t(\rho)) = -A\big(T_t(\rho)\big)$. Now we use the chain rule for $F(s) = s \ln s$ and $F'(s) = 1 + \ln s$ and deduce that

$$f'(t) = \tau\Big(-A\big(T_t(\rho)\big)\Big) + \tau\Big(-A\big(T_t(\rho)\big) \ln \big(T_t(\rho)\big)\Big) \;=\; -\mathcal{I}_A\big(T_t(\rho)\big) \;.$$

Here, the first term vanishes because A is self-adjoint and $A(1) = 0$. Thus, the assumption implies that

$$f'(t) = -\mathcal{I}_A(T_t(\rho)) \;\leq\; -\lambda D_N(T_t(\rho)) = -\lambda f(t).$$

Then, the assertion follows from Grönwall's lemma. □

Definition 2.8. The semigroup T_t or its generator A with fixpoint algebra N is said to satisfy:

(a) the gradient condition λ-$\Gamma\mathcal{E}$ for some $\lambda > 0$ if

$$\lambda\Gamma_{I-E_N} \leq_{cp} \Gamma_A .$$

(b) the modified logarithmic Sobolev inequality λ-MLSI if for all positive ρ,

$$\lambda D(\rho||E_N(\rho)) \leq \mathcal{I}_A(\rho) .$$

(c) the *complete* logarithmic Sobolev inequality λ-CLSI if $A \otimes id_{\mathbb{M}_m}$ satisfies λ-MLSI for all $m \in \mathbb{N}$.

We also say that T_t or A has $\Gamma\mathcal{E}$ (resp. MLSI, CLSI) if it satisfies λ-$\Gamma\mathcal{E}$ (resp. λ-MLSI, λ-CLSI) for some $\lambda > 0$.

It is an immediate consequence of the data processing inequality that CLSI has the tensorization property.

Proposition 2.9. *Let $T_t : M_1 \to M_1$ and $S_t : M_2 \to M_2$ be two self-adjoint semigroup that both satisfy λ-CLSI. Then, the tensor product semigroup $T_t \otimes S_t : M_1 \otimes M_2 \to M_1 \otimes M_2$ also satisfies λ-CLSI.*

Proof. Let A_1 (resp. A_2) be the generator of T_t (resp. S_t.). Then, $A = A_1 \otimes id + id \otimes A_2$ gives the generator of $T_t \otimes S_t$. Let $\rho \in L_1(M_1 \otimes M_2)$ and E_1, E_2 be the conditional expectation onto the fixpoint algebras. Then, we deduce from the data processing inequality that

$$\begin{aligned}
D(\rho||E_1 \otimes E_2(\rho)) &= \tau(\rho \ln \rho) - \tau(\rho \ln E_1 \otimes E_2(\rho)) \\
&= D(\rho||E_1 \otimes id(\rho)) + D(E_1 \otimes id(\rho)||E_1 \otimes E_2(\rho)) \\
&\leq D(\rho||E_1 \otimes id(\rho)) + D(\rho||id \otimes E_2(\rho)) \\
&\leq \lambda^{-1}\mathcal{I}_{A_1 \otimes id}(\rho) + \lambda^{-1}\mathcal{I}_{id \otimes A_2}(\rho) = \lambda^{-1}\mathcal{I}_A(\rho) . \qquad \square
\end{aligned}$$

Another immediate corollary is that λ-$\Gamma\mathcal{E}$ implies λ-CLSI.

Corollary 2.10. *If the generator A satisfies λ-$\Gamma\mathcal{E}$, then $A \otimes id_M$ has λ-$\Gamma\mathcal{E}$ for any finite von Neumann algebra M. In particular, λ-$\Gamma\mathcal{E}$ implies λ-CLSI.*

Proof. Let $\Gamma_M(a,b) = a^*b$ be trivial gradient form on M. Then, the generator $A \otimes I_M$ has gradient form $\Gamma_A \otimes \Gamma_M(x \otimes a, y \otimes b) = \Gamma_A(x,y) \otimes a^*b$. The first assertion follows from [52, Lemma 6.1]. The second follows from Corollary 2.5 and Lemma 2.6. $\qquad \square$

In the rest of this section, we discuss some interesting consequences of the $\Gamma\mathcal{E}$ condition. The first result is related to the symmetrized version of relative entropy.

Proposition 2.11. *Let $(T_t) : M \to M$ be a semigroup of completely positive unital self-adjoint maps with fixpoint algebra $N \subset M$. Then,*

$$\lambda\mathcal{I}_N(\rho) \leq \mathcal{I}_A(\rho)$$

implies

$$\mathcal{I}_N(T_t(\rho)) \leq e^{-\lambda t}\mathcal{I}_N(\rho) .$$

Proof. Let us consider the function

$$
\begin{aligned}
f(t) \; = \; \mathcal{I}_N\big(T_t(\rho)\big) \; &= \; D\big(T_t(\rho)||E(T_t(\rho))\big) \\
&+ D\big(E(T_t(\rho))||T_t(\rho)\big) \\
&= \; D\big(T_t(\rho)||E(\rho)\big) + D\big(E(\rho)||T_t(\rho)\big) \; .
\end{aligned}
$$

We have seen in Proposition 2.7 that the derivative of the first term is $-\mathcal{I}_A$ $(T_t(\rho))$. Write $h(t) = D\big(E(\rho)||T_t(\rho)\big)$ as the second term. By data processing inequality, we have

$$
h(s + t) = D\big(E(\rho)||T_{t+s}(\rho)\big) = D\Big(T_s\big(E(\rho)\big)||T_s\big(T_t(\rho)\big)\Big) \leq D\Big(E(\rho)||T_t(\rho)\Big) \; ,
$$

for any $s \geq 0$. Thus, $h'(t) \leq 0$, and hence,

$$
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}_N\big(T_t(\rho)\big) \; \leq \; -\mathcal{I}_A(\rho_t) \; \leq \; -\lambda \mathcal{I}_N(T_t(\rho)) \; .
$$

We conclude that f satisfies $f'(t) \leq -\lambda f(t)$ and hence, $f(t) \leq e^{-\lambda t} f(0)$. $\qquad \square$

Another application of noncommutative derivation calculus can be used to show $\Gamma\mathcal{E}$ gives exponential decay of L_p-distance for all $1 \leq p \leq \infty$.

Lemma 2.12. *Let $\lambda\Gamma_A \leq_{cp} \Gamma_B$ and $N \subset M$ be the fixpoint subalgebra of both semigroups e^{-tA} and e^{-tB}. Let $1 < p < \infty$. Then, for $x \in M$ self-adjoint, the functions*

$$
f_A(t) \; = \; \|e^{-tA}(x) - E(x)\|_{L_p(M)}^p, \qquad f_B(t) \; = \; \|e^{-tB}(x) - E(x)\|_{L_p(M)}^p
$$

satisfy $-\lambda f'_A(t) \leq -f'_B(t)$ for all $t \geq 0$.

Proof. Let $x \in M$ be self-adjoint. Then, $a = x - E(x)$ is again self-adjoint. We use the notation a_+ and a_- for the positive and negative part of a. Recall that the spectral projections of a_+ and a_- are mutually disjoint and commute with a. Thus, $|a|^p = a_+^p + a_-^p$. Note that

$$
f_A(t) = \|e^{-tA}(x) - E(x)\|_{L_p(M)}^p = \|e^{-tA}\big(x - E(x)\big)\|_{L_p(M)}^p = \|e^{-tA}(a)\|_{L_p(M)}^p
$$

Differentiating f_A at $t = 0$, we obtain that

$$
\begin{aligned}
f'_A(0) &= -p\tau\big(a|a|^{p-2}A(|a|)\big) \\
&= -p\Big(\tau\big(a_+^{p-1}A(a^+)\big) + \tau\big(a_+^{p-1}A(a_-)\big) + \tau\big(a_-^{p-1}A(a_-)\big) \\
&\quad + \tau\big(a_-^{p-1}A(a_+)\big)\Big) \; .
\end{aligned}
$$

Let δ_A be the derivation of A. Write $a_- = (\sqrt{a_-})^2 = b^2$. Then, (2.3) implies that

$$
\begin{aligned}
\tau(\delta(a_+^{p-1})\delta(b^2)) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \int_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{s^{p-1} - t^{p-1}}{s - t} \frac{r^2 - v^2}{r - v} \\
\tau(\mathrm{d}E_s\delta(a_+)\mathrm{d}E_t\mathrm{d}F_r\delta(b)\mathrm{d}F_v)
\end{aligned}
$$

where E_s (resp. F_r) are spectral projections of a_+ (resp. $\sqrt{a_-}$). Because E_s and F_r are orthogonal, we obtain $\tau(a_+^{p-1}A(a^-)) = 0$. The same argument applies to

$\tau(a_-^{p-1}A(a_+))$. By Lemma 2.4, $\lambda E\big(\delta_A(a_+^{p-1})\delta_A(a_+)\big) \le E\big(\delta_B(a_+^{p-1})\delta_B(a_+)\big)$, and

$$\lambda E\big(\delta_A(a_-^{p-1})\delta_A(a_-)\big) \le E\big(\delta_B(a_-^{p-1})\delta_B(a_-)\big)$$

This implies

$$-\lambda f_A'(0) = \lambda p\Big(\tau\big(\delta_A(a_+^{p-1})\delta_A(a_+)\big) + \tau\big(\delta_A(a_-^{p-1})\delta_A(a_-)\big)\Big)$$
$$\le p\Big(\tau\big(\delta_B(a_+^{p-1})\delta_B(a_+)\big) + \tau\big(\delta_B(a_-^{p-1})\delta_B(a_-)\big)\Big) = -f_B'(0) .$$

Replacing x by $e^{-tA}(x)$, we obtain that $-\lambda f_A'(t) \le -f_B'(t)$ for all $t \ge 0$. $\qquad\square$

Theorem 2.13. *Let $T_t = e^{-tA}$ be a self-adjoint semigroup satisfying λ-$\Gamma\mathcal{E}$. Then, for all $1 \le p \le \infty$,*

$$\|T_t(x) - E(x)\|_{L_p(M)} \le e^{-\lambda t}\|x - E(x)\|_{L_p(M)} .$$

Proof. Let us first assume that $x = T_s(y)$ for some y so that x belongs to the domain of A and $A^{\frac{1}{2}}$. Then, we note that

$$T_t((I - E)(x)) = T_t(x) - E(x) .$$

Write $a = x - E(x)$ and $S_t = e^{-t(I-E)}$. According to Lemma 2.12, we have that

$$-\lambda \frac{d}{dt}\|S_t(a)\|_{L_p(M)}^p \le -\frac{d}{dt}\|T_t(a)\|_{L_p(M)}^p .$$

Note that $E_N(a) = 0$, and hence, $S_t(a) = e^{-t}a$. Then,

$$-\frac{d}{dt}\|S_t(a)\|_{L_p(M)}^p = pe^{-tp}\|a\|_{L_p(M)}^p .$$

We apply Lemma 2.12 and deduce that

$$\lambda p f_A(0) = \lambda p\|a\|_p^p = -\lambda f_{I-E}'(0) \le -f_A'(0) .$$

Repeat the argument for $a_t = T_t(a)$ and deduce

$$\lambda p f_A(t) \le -f_A'(t) .$$

This implies $f_A(t) \le e^{-\lambda p t} f_A(0)$. Taking the p-root, we obtain that

$$\|T_t(a)\|_{L_p(M)} \le e^{-\lambda t}\|a\|_{L_p(M)} .$$

For general self-adjoint x, the assertion follows from the approximation $x = \lim_{s\to 0} T_s(x)$. By considering the 2×2 matrix $\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$, we deduce the assertion for all x. The cases $p = 1$ and $p = \infty$ are obtained by passing to the limit for $p \to 1$ or $p \to \infty$. $\qquad\square$

The next corollary studies the L_p-distance decay under tensor product.

Corollary 2.14. *Let $T_t^j : M_j \to M_j$ be a family of semigroups with fixpoint subalgebras $N_j \subset M_j$. Then, the tensor product semigroup $T_t = T_t^1 \otimes T_t^2 \otimes$*

$\cdots \otimes T_t^n$ on $M = \otimes_{j=1}^n M_j$ has the fixpoint algebra $N = \otimes_{j=1}^n N_j$. Suppose for each j, T_j^t satisfies λ_j-ΓE. Then,

$$\|T_t - E_N(x)\|_{L_p(M)} \leq 2 \left(\sum_{j=1}^n e^{-t\lambda_j} \right) \|x\|_{L_p(M)} .$$

Proof. Let us consider a twofold tensor product $S_t = T_t^1 \otimes T_t^2$. Then,

$$S_t - E_{N_1} \otimes E_{N_2} = S_t(id - E_{N_1} \otimes E_{N_2}) = S_t \Big((id - E_{N_1}) \otimes id_{M_2}$$

$$+ E_{N_1} \otimes (id_{M_2} - E_{N_2}) \Big)$$

$$= (T_t^1 - E_{N_1}) \otimes T_t^2 + E_{N_1} \otimes (T_t^2 - E_{N_2}) .$$

Since T_t^2 and E_{N_1} are completely contractive on L_p spaces, we deduce from Theorem 2.13 that

$$\|S_t - E_{N_1} \otimes E_{N_2}(x)\|_{L_p(M)}$$

$$\leq \|(T_t^1 - E_{N_1}) \otimes T_t^2(x)\|_{L_p(M)} + \|id \otimes (T_t^2 - E_{N_2})(x)\|_{L_p(M)}$$

$$\leq 2e^{-\lambda_1 t}\|x\|_{L_p(M)} + \|id \otimes (T_t^2 - E_{N_2})(x)\|_{L_p(M)}$$

$$\leq 2(e^{-\lambda_1 t} + e^{-\lambda_2 t})\|x\|_{L_p(M)} .$$

For the n-fold tensor product, we may use induction. $\qquad\square$

3. Kernel Estimates and Module Maps

3.1. Kernels on Noncommutative Spaces

In this part, we derive kernel estimates for ergodic and non-ergodic semigroups and their matrix-valued extension. Let N, M be two von Neumann algebras and N_* be the predual of N. The kernel of maps between noncommutative measure spaces is given by the following theorem due to Effros and Ruan [33]:

$$CB(N_*, M) = N \bar{\otimes} M \tag{3.1}$$

which states that the space of completely bounded maps $CB(N_*, M)$ is completely isometrically isomorphic to the von Neumann algebra tensor product $N \bar{\otimes} M$. Let us now assume that N is semifinite with trace tr. The linear duality bracket $\langle x, y \rangle = tr(xy)$ gives a completely isometric pairing between $L_1(N, \tau)$ and N^{op}, the opposite algebra of N. More precisely, for every kernel $K \in N^{op} \bar{\otimes} M$ the linear map

$$T_K(x) = (tr \otimes id)\big(K(x \otimes 1)\big)$$

satisfies $\|K\|_{\min} = \|T_K : L_1(N, \tau) \to M\|_{cb}$. Let us pause for a moment and consider $M = \mathbb{M}_d$ and $N = \mathbb{M}_k$ and denote S_1^k for the trace class. For a linear map $T : S_1^k \to \mathbb{M}_d$, the Choi matrix is given by

$$\chi_T = \sum_{rs} |r\rangle\langle s| \otimes T(|r\rangle\langle s|) \in \mathbb{M}_k \otimes \mathbb{M}_d .$$

The map $\phi(a) = a^t$ given by the transpose is a *-homomorphism between $\mathbb{M}_k$ and $\mathbb{M}_k^{op}$. Therefore, we should consider

$$K_T = \sum_{rs} |s\rangle\langle r| \otimes T(|r\rangle\langle s|) \in \mathbb{M}_k^{op} \otimes \mathbb{M}_m$$

and find

$$T_{K_T}(|r\rangle\langle s|) = tr(K_T(|r\rangle\langle s| \otimes 1)) = \sum_{tv} tr(|t\rangle\langle v||r\rangle\langle s|)\, T(|v\rangle\langle t|) = T(|r\rangle\langle s|).$$

Equivalently, this shows that our description via kernels in $N^{op}\bar{\otimes}M$ is compatible with the standard choice of a Choi matrix, see also [78].

3.2. Saloff-Coste's Estimates

After these preliminary observation, we now assume that T_t is a semigroup of (sub-)unital completely positive, self-adjoint maps on a finite von Neumann algebra M so that $T_t : L_1(M) \to M$ are completely bounded. According to the previous section, the kernels of T_t are given by positive element $K_t \in M^{op}\bar{\otimes}M$. Let $N \subset M$ be the fixpoint subalgebra of the semigroup T_t and $E : M \to N$ be the conditional expectation. Recall that $T_t \circ E = E \circ T_t = E$ by the self-adjointness on $L_2(M)$.

Lemma 3.1. *Let* $(T_t) : M \to M$ *be a semigroup of self-adjoint *-preserving maps. Then, for any* $t \geq 0$,
  (i) $\|T_{2t} : L_1(M) \to L_\infty(M)\| = \|T_t : L_2(M) \to L_\infty(M)\|^2$;
  (ii) $\|T_{2t} - E : L_1(M) \to L_\infty(M)\| = \|(T_t - E) : L_2(M) \to L_\infty(M)\|^2$.
The same estimates also hold for cb-norm, instead of the operator norm.

Proof. We start by recalling a general fact. Let $v : X \to H$ be a linear map from a Banach space X to a Hilbert space H. By $H^* \cong \bar{H}$, we have for any $x, y \in X$,

$$\langle v(y), v(x)\rangle_H = \langle y, \bar{v}^*v(x)\rangle_{(X,\bar{X}^*)}.$$

Here, $\bar{v}^* : \bar{H} \to \bar{X}^*$ is the conjugate adjoint of v and the right-hand side is the sesquilinear bracket between X and its conjugate dual $\bar{X}^*$. Then, we have

$$\|v : X \to H\|^2 = \sup_{\|x\|\leq 1} \langle v(x), v(x)\rangle = \sup_{\|x\|,\|y\|\leq 1} |\langle y, \bar{v}^*v(x)\rangle|$$
$$= \|\bar{v}^*v : X \to \bar{X}^*\|. \qquad (3.2)$$

In our situation, we use $X = L_1(M)$ and $v = T_t : X \to L_2(M)$. Note that

$$\langle T_t(x), T_t(y)\rangle = \tau(T_t(x)^*T_t(y)) = (x^*, T_{2t}(y)).$$

and the anti-linear bracket

$$(x, y) = \tau(x^*y)$$

gives a complete isometry between M and $\overline{L_1(M)}^*$. Therefore,

$$\|T_{2t} : L_1(M) \to L_\infty(M)\| = \|T_t : L_1(M) \to L_2(M)\|^2.$$

Similarly, we deduce (ii) from

$$(T_t - E)(T_t - E) = T_{2t} - ET_t - T_tE + E = T_{2t} - E.$$

For the cb-norm estimate, we take $X = S_2(L_1(N))$ from [81] (see also Sect. 3.4). For the Schatten 2-space S_2, the trace bracket $\langle x, y \rangle = tr(x^*y)$ identifies S_2^* with $\bar{S}_2$. Thus,

$$\|id \otimes T_t : S_2(L_1(M)) \to S_2(L_2(M))\|^2$$
$$= \|id \otimes T_{2t} : S_2(L_1(M)) \to S_2(L_\infty(M))\|$$

follows again from the general principle. The same argument applies to $(T_t - E)^2 = T_{2t} - E$.   □

The following observation is essentially due to Saloff-Coste ([89]).

Proposition 3.2. *Let (T_t) be a semigroup of self-adjoint and *-preserving maps such that*

(i) $\|T_t : L_1(M) \to L_\infty(M)\|_{cb} \leq ct^{-d/2}$ *for some $c, d > 0$ and all $0 \leq t \leq 1$;*
(ii) *the self-adjoint positive generator A satisfies the λ-spectral gap condition:*

$$\|A^{-1}(I - E) : L_2(M) \to L_2(M)\| \leq \lambda^{-1} .$$

Then,

$$\|T_t - E : L_1(M) \to L_\infty(M)\|_{cb} \leq \begin{cases} 2ct^{-d/2} & 0 \leq t \leq 1 \\ C(d, \lambda)e^{-\lambda t} & 1 \leq t < \infty . \end{cases}$$

where $C(d, \lambda)$ is a constant depending only on d and λ.

Proof. First, note that $T_t(I - E) = T_t - E$ and $\|I - E : L_\infty(M) \to L_\infty(M)\|_{cb} \leq 2$. Then, the estimate for $t \leq 1$ follows from the assumption i). For $t \geq 1/2$, we use functional calculus so that

$$\|T_{t-1/4}(I - E) : L_2(M) \to L_2(M)\| \leq e^{-\lambda(t-1/4)} .$$

Note that $L_2(M)$ is an operator Hilbert space, and hence, the above operator norm coincides with the completely bounded norm ([80, Proposition 7.2]). Thus, we obtain

$$\|(T_t - E) : L_1 \to L_2\|_{cb}$$
$$\leq \|T_{t-1/4}(I - E) : L_2(M) \to L_2(M)\|_{cb}\|T_{1/4} : L_1(M) \to L_2(M)\|_{cb}$$
$$\leq e^{-\lambda(t-1/4)}\|T_{1/2} : L_1(M) \to L_\infty(M)\|_{cb}^{1/2} \leq \sqrt{c}2^{d/4}e^{\lambda/4}e^{-\lambda t} .$$

Applying Lemma 3.1 yields the assertion for $t' = 2t \geq 1$.   □

3.3. From Ultracontractivity to Gradient Estimates

In this part, we use the kernel estimate discussed above to prove $\Gamma\mathcal{E}$ for (generalized) fractional powers, including generators of so-called subordinated semigroups. This is a classical construction in harmonic analysis. Recall that $(I - T_t)$ is a semigroup generator. For a positive function F on $[0, \infty)$, we can define a new generator

$$\Phi_F(A) = \int_0^\infty (I - T_t)F(t)\frac{\mathrm{d}t}{t} ,$$

provided the integral is well defined. Then, the gradient form of $\Phi_F(A)$ is given by

$$\Gamma_{\phi_F(A)}(x,y) = \int_0^\infty \Gamma_{I-T_t}(x,y)F(t)\frac{\mathrm{d}t}{t} . \tag{3.3}$$

We also define the modified Laplace transform

$$\phi_F(\lambda) = \int_0^\infty (1-e^{-t\lambda})F(t)\frac{\mathrm{d}t}{t} .$$

For positive F, we may use the integrability (I), quasi-monotonicity (QM), or the well-known (Δ_2) conditions:

(I) $C_F := \int_0^\infty \min(1,t)F(t)\frac{\mathrm{d}t}{t} < \infty;$

(QM) For some $0 < \mu < 1$, there exists $C_\mu > 0$ such that $F(\mu t) \leq C_\mu F(t)$ for all $t > 0$.

(Δ_2) For some $0 < \mu < 1$, there exists $0 < \alpha < 1$, $t_\alpha > 0$, $c_\alpha > 0$ such that $F(\mu t) \leq c_\alpha \mu^{-\alpha} F(t)$ for $t_\alpha \leq \mu t \leq t$.

Since $1 - e^{-\lambda t} \leq \min(1, \lambda t) \leq (1+\lambda)\min(1,t)$, we deduce that

$$\phi_F(\lambda) \leq C_F(1+\lambda)$$

and hence

$$\Phi_F(A) \leq C_F(I+A) .$$

Then, $\Phi_F(A)$ is a closable operator well defined on the domain of A, and hence according to our assumptions also defined on the dense subalgebra $\mathcal{A}$.

Remark 3.3. Our calculus is closely related to the theory of symmetric positive definite functions on $\mathbb{R}$, which can be represented as:

$$\psi_G(\lambda) = \int_\mathbb{R} (1-\cos(s\lambda))G(s)\frac{\mathrm{d}s}{s} ,$$

where G is a positive function such that

$$\int_\mathbb{R} \min(1, s^2)G(s)\frac{\mathrm{d}s}{s} = \int_0^\infty \min(1,t)G(\sqrt{t})\frac{\mathrm{d}t}{t} < \infty .$$

Let g be a Gaussian distribution. Then, we obtain a randomized, new, positive definite function

$$\tilde{\psi}_G(\lambda) = \mathbb{E}\psi(g\lambda) = \int_\mathbb{R}(1-e^{-s^2\lambda^2})G(s)\frac{\mathrm{d}s}{s} = \int_0^\infty(1-e^{-t\lambda^2})G(\sqrt{t})\frac{\mathrm{d}t}{t} .$$

Thus, for any positive definite function ψ_G, the function $\phi(\lambda) = \tilde{\psi}(\sqrt{\lambda})$ can be used for the Laplace transform in (3.3) and hence defines a generalized subordinated semigroup.

In particular, all fractional power of generators are examples of our calculus.

Example 3.4. Let $0 < \alpha < 1$ and $F_\alpha(t) = c(\alpha)t^{-\alpha}$. Then,

$$\phi_\alpha(\lambda) \;=\; c(\alpha)\int_0^\infty (1 - e^{-\lambda t})t^{-\alpha}\frac{\mathrm{d}t}{t} \;=\; \lambda^\alpha c(\alpha)\int_0^\infty (1 - e^{-s})s^{-\alpha}\frac{\mathrm{d}s}{s} \;=\; \lambda^\alpha$$

holds for a suitable choice of the normalization $c(\alpha)$. It is clear that F_α satisfies the condition (I) and (QM). We refer to [36] for monotonicity results overlapping with our approach.

Let us now fix an F satisfying the condition (I) and (QM). A key technical tool for our estimates is the following family of unital completely positive maps:

$$\Psi_F(r) \;=\; g(r)^{-1}\int_0^\infty e^{-r/t}T_t F(t)\frac{\mathrm{d}t}{t} \ , \quad r > 0 \ ,$$

where $g(r)$ is the normalization constant given by

$$g(r) \;=\; \int_0^\infty e^{-r/t}F(t)\frac{\mathrm{d}t}{t} \ .$$

Lemma 3.5. *Let T_t be a semigroup of unital completely positive, self-adjoint maps. Suppose the generator A satisfies Γ-regularity and F satisfies (I). Then,*

(i) *For $r \geq s$, $g(r) \leq g(s)$ and $g(r)\Psi_F(r) \leq_{cp} g(s)\Psi_F(s)$;*

(ii) $\lim_{r\to 0} g(r)(I - \Psi_F(r)) = \Phi_F(A)$, $\lim_{r\to 0} g(r)\Gamma_{I-\Psi_F(r)} = \Gamma_{\Phi_F(A)}$;

(iii) $\Phi_F(A)$ *satisfies Γ-regularity.*

Proof. Let $r \geq s > 0$. Then, obviously $e^{-r/t} \leq e^{-s/t}$ and hence

$$g(r) \leq g(s) \ , \ g(r)\Psi_F(r) \leq_{cp} g(s)\Psi_F(s)$$

For (ii), since $\Psi_F(r)$ is completely positive and self-adjoint, then $A_r := g(r)\big(I - \Psi_F(r)\big)$ is a generator of a semigroup of unital completely positive and self-adjoint maps. Let us consider the function

$$\psi_r(\lambda) \;=\; g(r)^{-1}\int_0^\infty e^{-r/t}(1 - e^{-\lambda t})F(t)\frac{\mathrm{d}t}{t} \ .$$

Using

$$(1 - e^{-\lambda t}) \;\leq\; \min(1, \lambda t) \;\leq\; (1 + \lambda)\min(1, t)$$

we deduce from the dominated convergence theorem that

$$\lim_{r\to 0} g(r)\psi_r(\lambda) \;=\; \phi_F(\lambda) \ .$$

Note that $g(r)\psi_r \leq g(s)\psi_s$ if $r \geq s$. Applying the monotone convergence theorem, we have that for any $x \in \mathcal{A}$,

$$\lim_{r\to 0}\langle x, g(r)\psi_r(A)x\rangle \;=\; \lim_{r\to 0}\int_0^\infty g(r)\psi_r(\lambda)\mathrm{d}\mu_x(\lambda) \;=\; \langle x, \Phi_F(A)x\rangle \ .$$

Here, $\mathrm{d}\mu_x(\lambda)$ is the spectral measure $d\langle x, 1_{\{A\leq\lambda\}}x\rangle$. This implies that for any $x \in \mathcal{A}$,

$$\lim_{r\to 0} g(r)(I - \Psi_F(r))x = \Psi_F(A)x$$

in the weak topology on $L_2(M)$. Thus, for all $x, y \in \mathcal{A}$,

$$\lim_{r \to 0} g(r)\Gamma_{I-\Psi_F(r)}(x,y) = \lim_{r \to 0} \Gamma_{\Phi_F(A)}(x,y)$$

in the weak topology on $L_2(M)$ and also in the weak* topology in M. For (iii), we split the function ϕ_F into two pieces

$$\phi_F(\lambda) = \phi_F'(\lambda) + \phi_F''(\lambda) = \int_0^1 \frac{1-e^{-\lambda t}}{t} F(t)\mathrm{d}t + \int_1^\infty (1 - e^{-\lambda t})F(t)\frac{\mathrm{d}t}{t} .$$

and correspondingly $\Phi_F(A) = \Phi_F'(A) + \Phi_F''(A)$. According to our assumption, both $\int_0^1 F(t)\mathrm{d}t$ and $\int_1^\infty F(t)\frac{\mathrm{d}t}{t}$ are finite. For any $x, y \in \mathcal{A}$, $\frac{1}{t}\Gamma_{1-T_t}(x,y)$ is uniformly bounded in $L_1(M)$. Therefore, the gradient form of $\Phi_F'(A)$ is well defined on $\mathcal{A}$ and has range in $L_1(M)$. On the other hand, the map $T_t : M \to M$ is completely positive and bounded from $L_\infty(M)$ to itself. Then, $\Phi_F''(A)$ converges, and the gradient forms $\Gamma_{\Phi_F''(A)}$ take range in $L_\infty(M)$ and in particular also in $L_1(M)$. $\qquad \square$

A direct consequence of the above lemma is as follows:

Corollary 3.6. *For every $r > 0$,*

$$g(r)\Gamma_{I-\Psi_F(r)} \leq_{cp} \Gamma_{\Phi_F(A)} , \; g(r)\mathcal{I}_{I-\Psi_F(r)} \leq \mathcal{I}_{\Phi_F(A)} .$$

Proof. The previous lemma shows that $g(r)\Psi_F(r)$ is decreasing in cp-order as $r \to 0$. By Lemma 2.3, the gradient form $g(r)\Gamma_{I-\Psi_F(r)}$ is also decreasing in cp-order. Then, the assertion follows from the limits in Lemma 3.5 (iii) and Corollary 2.5. $\qquad \square$

We say a self-adjoint semigroup T_t is *ergodic* if its fixpoint subalgebra $\mathcal{N} = \mathbb{C}1$. In this situation, the conditional expectation is the trace $E_\tau(x) = \tau(x)1$ and the kernel is $K_E = 1 \otimes 1$ in $M^{op}\overline{\otimes}M$.

Proposition 3.7. *Assume that a semigroup (T_t) of unital completely positive and self-adjoint maps satisfy the conclusion of* Proposition 3.2 *with respect to $E_\tau(x) = \tau(x)1$. Suppose F satisfies (I) and (QM). Then, there exists a $r_0 > 0$ such that for all $r \geq r_0$,*

(i) $\|\Psi_F(r) - E_\tau : L_1(M) \to L_\infty(M)\|_{cb} \leq 1/2$;
(ii) $E_\tau \leq_{cp} 2\Psi_F(A)$;
(iii) $g(r)\Gamma_{I-E_\tau} \leq_{cp} 2\Gamma_{\Phi_F(A)}$.

Proof. Write $E_\tau = E$. For i) we use the assumption and Proposition 3.2,

$$\|\Psi_F(r) - E : L_1(M) \to L_\infty(M)\|_{cb}$$
$$\leq g(r)^{-1} \int_0^\infty e^{-r/t}\|T_t - E : L_1(M) \to L_\infty(M)\|_{cb}F(t)\frac{\mathrm{d}t}{t}$$
$$\leq g(r)^{-1}\left(2c\int_0^1 e^{-r/t}t^{-d/2}F(t)\frac{\mathrm{d}t}{t} + C(d,\lambda)\int_1^\infty e^{-r/t}e^{-\lambda t}F(t)\frac{\mathrm{d}t}{t}\right)$$
$$=: g(r)^{-1}(2c\mathrm{I} + C(d,\lambda)\mathrm{II}) .$$

Then, the condition (QM) of F implies that for some $0 < \mu < 1$,

$$
\begin{aligned}
I &= r^{-d/2} \int_r^\infty F(r/u) u^{d/2} e^{-u} \frac{\mathrm{d}u}{u} \\
&\leq r^{-d/2} c(d,\mu) \int_r^\infty F(r/\mu u) e^{-\mu u} \frac{\mathrm{d}u}{u} \\
&= r^{-d/2} c(d,\mu) \int_{\mu r}^\infty F(r/w) e^{-w} \frac{\mathrm{d}w}{w} \;\leq\; r^{-d/2} c(d,\mu) g(r) \;.
\end{aligned}
$$

Here, we have used the change of variables $u = r/t$ and $w = \mu u$. Indeed, the same change of variable shows that

$$
g(r) \;=\; \int_0^\infty F(r/u) e^{-u} \frac{\mathrm{d}u}{u} \;.
$$

For the second part, we see that

$$
II = \int_0^r e^{-\frac{\lambda}{u}} F(r/u) e^{-u} \frac{\mathrm{d}u}{u} \;\leq\; e^{-\lambda} \int_0^r F(r/u) e^{-u} \frac{\mathrm{d}u}{u} \;\leq\; e^{-\lambda} g(r) \;.
$$

Hence, there exists a r_0 which only depends to $c, d, C(d,\lambda)$ and C_μ so that for all $r \geq r_0$

$$
\|\Psi_F(r) - E\|_{cb} \;\leq\; \frac{1}{2} \;.
$$

This proves i). Let K_r be the kernel of $\Psi_F(r)$ in $M^{op} \overline{\otimes} M$ and recall that $K_E = 1 \otimes 1$ is the kernel of E. Then, by (3.1),

$$
\|K_r - 1 \otimes 1\|_{M^{op}\overline{\otimes}M} \;\leq\; \frac{1}{2}
$$

which implies $K_r \geq \frac{1}{2} 1 \otimes 1 = \frac{1}{2} K_E$, where (ii) follows. The assertion (iii) follow from Lemma 2.3 and Corollary 3.6. □

Remark 3.8. The polynomial decay $\|T_t : L_1 \to L_\infty\|_{cb} \leq t^{-d/2}$ for $0 < t < 1$ is not really needed. Indeed, if

$$
\|T_t : L_1 \to L_\infty\|_{cb} \;\leq\; C_\alpha e^{c_\alpha t^{-\alpha}}
$$

holds for some $\alpha < 1$ for all $t > 0$, then we can choose $\beta > 0$ such that $C_\alpha C_F(\mu) e^{-\frac{1-\mu}{2}\beta} \leq \frac{1}{4}$ and choose r large enough so that $c_\alpha r^{-\alpha} \leq \frac{1-\mu}{2}\beta^{1-\alpha}$. Thus, we find that

$$
\begin{aligned}
I_\beta &:= \int_0^{\frac{r}{\beta}} e^{-r/t} \|T_t - E : L_1(M) \to L_\infty(M)\|_{cb} F(t) \frac{\mathrm{d}t}{t} \\
&\leq \int_0^{\frac{r}{\beta}} C_\alpha e^{c_\alpha t^{-\alpha}} F(t) \frac{\mathrm{d}t}{t} \\
&\leq C_F(\mu) C_\alpha \int_\beta^\infty e^{u^\alpha(c_\alpha r^{-\alpha} - (1-\mu)u^{1-\alpha})} F(r/\mu u) e^{-\mu u} \frac{\mathrm{d}u}{u} \\
&\leq C_F(\mu) C_\alpha e^{-\frac{1-\mu}{2}\beta} \int_\mu^\infty F(r/w) e^{-w} \frac{\mathrm{d}w}{w} \;.
\end{aligned}
$$

Let II_β be the corresponding integral on the integral $[r/\beta, +\infty)$. Using a small modification in Proposition 3.2, the spectral gap allows us to estimate $g(r)^{-1} II_\beta \leq C(\beta, \lambda) e^{-\frac{r\lambda}{\beta}}$.

Motivated by the discussion above, let us introduce the return time

$$
\begin{aligned}
t_0 : &= \inf\{t \mid \|K_{T_t} - 1 \otimes 1\|_{M^{op}\overline{\otimes}M} \leq 1/2\} \\
&= \inf\{t \mid \|T_t - E_\tau : L_1(M) \to L_\infty(M)\|_{cb} \leq 1/2\} .
\end{aligned}
\tag{3.4}
$$

Under the assumption of Proposition 3.2, t_0 is always finite.

Proposition 3.9. *Assume that F satisfies (I) and (Δ_2). Let $r = \max\{t_0, t_\alpha\}$ where t_α is the parameter in (Δ_2). Then,*

$$
\Gamma_{\Phi_F(A)} \geq_{cp} \frac{F(r)}{2\alpha c_\alpha} \Gamma_{I-E} .
$$

Proof. Recall that Γ_B is linear with respect to B. Therefore, we deduce from $\|K_{T_t} - 1 \otimes 1\| \leq 1/2$ and Lemma 2.3 that $\Gamma_{I-T_t} \geq \frac{1}{2}\Gamma_{I-E}$ holds for $t \geq t_0$. Then, we note that the (Δ_2) condition implies

$$
F(r) = F\left(\frac{r}{t}t\right) \leq_{cp} c_\alpha (t/r)^\alpha F(t)
$$

for all $r \leq t$. Therefore,

$$
\begin{aligned}
\Gamma_{\Phi_F(A)} = \int_0^\infty \Gamma_{I-T_t} F(t)\frac{dt}{t} &\geq_{cp} \frac{\Gamma_{I-E}}{2} \int_r^\infty F(t)\frac{dt}{t} \\
&\geq_{cp} \frac{\Gamma_{I-E}}{2c_\alpha} F(r)r^\alpha \int_r^\infty t^{-(1+\alpha)}dt = \frac{\Gamma_{I-E}}{2\alpha c_\alpha} F(r)
\end{aligned}
$$

which completes the proof. $\square$

Theorem 3.10. *Let T_t be an ergodic semigroup of completely positive self-adjoint maps such that*

(i) *$\|T_t : L_1(M) \to L_\infty(M)\|_{cb} \leq ct^{-d/2}$ for $0 \leq t \leq 1$ and $c, d > 0$.*

(ii) *the generator A has a spectral gap $\sigma_{\min} > 0$.*

Let F be a function satisfying the condition $(I)+(QM)$ or $(I)+(\Delta_2)$. Then, $\Phi_F(A)$ satisfies λ-$\Gamma\mathcal{E}$ and hence λ-CLSI for some λ depending on c, d, F and $\sigma_{\min}$.

Proof. Let $E = E_\tau$ be the conditional expectation $E_\tau(x) = \tau(x)1$. For F satisfying $(I)+(QM)$, we deduce from Proposition 3.7 that

$$
g(r_0)\Gamma_{I-E_\tau} \leq_{cp} 2\Gamma_{\Phi_F(A)}
$$

for some r_0 depending on $\sigma_{\min}$, c and d. Then, one can choose $\lambda = \frac{g(r_0)}{2}$. Similarly, for F satisfying $(I)+(\Delta_2)$, we apply Proposition 3.9. $\square$

3.4. Non-ergodic Semigroups

In this part, we want to adapt the kernel techniques for ergodic maps to the non-ergodic situation. This requires more operator space theory from the work in [44] on vector-valued noncommutative L_p-spaces associated with an inclusion of von Neumann algebras. As usual, we assume that $(T_t) : M \to M$ is a semigroup of unital completely positive self-adjoint maps and $N \subset M$ is the fixpoint subalgebra. Whenever N is infinite dimensional, we can no longer hope for an ultracontractivity of $T_t : L_1(M) \to L_\infty(M)$ for operator norm or cb-norm, because the identity map

$$id : L_1(N) \to L_\infty(N)$$

is already unbounded. This leads us to use vector-valued L_p norms. Let $1 \leq p, q, r \leq \infty$ and fix the relation $\frac{1}{r} = |\frac{1}{p} - \frac{1}{q}|$. Recall that the $L_p(L_q)$ norms for the inclusion $N \subset \mathcal{M}$ is defined as:

$$\|x\|_{L_p^q(N \subset M)} = \begin{cases} \inf\limits_{x=ayb} \|a\|_{L_{2r}(N)} \|y\|_{L_q(M)} \|b\|_{L_{2r}(N)} & p \leq q , \\ \sup\limits_{\|a\|_{L_{2r}(N)}=\|b\|_{L_{2r}(N)} \leq 1} \|axb\|_{L_q(M)} & q \leq p . \end{cases} \tag{3.5}$$

Here, for $p \leq q$, the infimum takes over all factorization $x = ayb$ with $a, b \in L_{2r}(N), y \in L_q(M)$ and for $p \geq q$, the supremum runs over all $a, b \in L_{2r}(N)$ with $\|a\|_{L_{2r}(N)} = \|b\|_{L_{2r}(N)} \leq 1$. The Banach space $L_p^q(N \subset M)$ is then the completion of M with respect to the corresponding norm. It follows from the Hölder inequality that for $p = q$, $L_p^p(N \subset M) \cong L_p(M)$. These norms have been extensively studied in the quantum information theory and operator space community [20, 37, 72]. For the special cases $M = R \overline{\otimes} N$ and $M = \mathbb{M}_k(N)$,

$$L_p^q(N \subset R \overline{\otimes} N) = L_p(R, L_q(N)) , L_p^q(N \subset \mathbb{M}_k(N)) = S_p^k(L_q(N)) .$$

which are the vector-valued L_p spaces introduced in [80]. In the following, we mention the properties of $L_p^q(N \subset M)$ needed in our discussion and refer to [44] for a detail account of these L_p-spaces. First, we will use a duality relation that the anti-linear trace bracket $(x, y) = \tau(x^*y)$ provides an isometric embedding

$$L_p^q(N \subset M) \subset L_{p'}^{q'}(N \subset M)^* , \tag{3.6}$$

for $1 \leq p, q \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = \dfrac{1}{q} + \dfrac{1}{q'} = 1$, and it is indeed an equality when for $1 < p, q < \infty$. We will also need the following factorization property that

$$L_p(M) = L_{2p}(N) L_\infty^p(N \subset M) L_{2p}(N) \tag{3.7}$$

which reads as

$$\|x\|_{L_p(M)} = \inf_{x=ayb} \|a\|_{L_{2p}(N)} \|y\|_{L_\infty^p(N \subset M)} \|b\|_{L_{2p}(N)} .$$

This can be verified by interpolation. Indeed, it is obvious for $p = \infty$. For $p = 1$ let us assume that x is positive and $\tau(x) = 1$. Then, $\tau(E(x)) = 1$, and we may write

$$x = E(x)^{1/2}(E(x)^{-1/2}xE(x)^{-1/2})E(x)^{1/2} = E(x)^{1/2}yE(x)^{1/2} .$$

Note that for every $a \in L_2(N)$,

$$\tau(a^*ya) = \tau(E(x)^{-1/2}E(x)E(x)^{-1/2}aa^*) \;=\; \tau(aa^*) \;=\; \|a\|_2^2 \,.$$

Using the positivity of y and a Cauchy–Schwarz type argument, it follows that $\|y\|_{L_\infty^1(N\subset M)} \leq 1$. This factorization property is closely related to the following fact.

Lemma 3.11 (Lemma 4.9 of [44]). *Let $x \in M$. Then*

$$\|x\|_{L_\infty^1(N\subset M)} \;=\; \inf_{x=x_1x_2} \|E(x_1x_1^*)\|_\infty^{1/2}\|E(x_2^*x_2)\|_\infty^{1/2} \,.$$

The following fact is an extension of Lemma 1.7 of [80]. Recall that $T : M \to M$ is a N-bimodule map if for any $a, b \in N, x \in M, T(axb) = aT(x)b$.

Lemma 3.12. *Let $T : M \to M$ be a N-bimodule map and $1 \leq p, q \leq \infty$. Then, for any $1 \leq s \leq \infty$,*

$$\|T : L_\infty^p(N\subset M) \to L_\infty^q(N\subset M)\| \;=\; \|T : L_s^p(N\subset M) \to L_s^q(N\subset M)\|$$

Proof. Let us introduce the short notation $\|\|T\|\|_s = \|T : L_s^p(N\subset M) \to L_s^q(N\subset M)\|$. We first prove "$\geq$" for $s = 1$. For an element $x \in L_1^p(N\subset M)$ of norm less than 1, we have a decomposition $x = ayb$ with $a, b \in L_{2p'}(N)$, and $y \in L_p(M)$ all of norm less than 1. Using the factorization (3.7), we may further write $y = \alpha Y\beta$ with $\alpha, \beta \in L_{2p}(N)$ and $Y \in L_\infty^p(M)$ all of norm less than 1. Therefore, we have shown that $x = (a\alpha)Y(\beta b)$ with $Y \in L_\infty^p(N\subset M)$, and

$$\|a\alpha\|_{L_2(N)} \leq 1 \,, \;\; \|\beta b\|_{L_2(N)} \leq 1 \,.$$

Thus, we may write $a\alpha = a'\alpha'$ and $\beta b = \beta'b'$ such that

$$\max\{\|b'\|_{L_{2q'}(N)}, \|\beta'\|_{L_{2q}(N)}, \|a'\|_{L_{2q'}(N)}, \|\beta'\|_{L_{2q}(N)}\} \leq 1 \,.$$

Then, we deduce from the module property that

$$T(x) \;=\; a'\alpha'T(Y)\beta'b' \;=\; a'(\alpha'T(Y)\beta')b'$$

and $\|T(Y)\|_{L_\infty^q(N\subset M)} \leq \|\|T\|\|_\infty$. Using (3.7) again, $(\alpha'T(Y)\beta') \in L_q(M)$ of norm less than $\|\|T\|\|_\infty$ and hence we have shown that $\| T(x) \|_{L_1^q(N\subset M)} \leq \|\|T\|\|_\infty$. By interpolation, we deduce that

$$\|\|T\|\|_s \;\leq\; \|\|T\|\|_\infty$$

for all $1 \leq s \leq \infty$. We dualize this inequality by applying it to T^* and obtain

$$\|\|T\|\|_s \;=\; \|T^* : L_{s'}^{q'}(N\subset M) \to L_{s'}^{p'}(N\subset M)\| \;\leq\; \|T^* : L_\infty^{q'}(N\subset M)$$
$$\to L_\infty^{p'}(N\subset M)\|$$
$$=\; \|\|T\|\|_1 \,.$$

Then, we dualize again to get

$$\|\|T\|\|_\infty \;\leq\; \|\|T^*\|\|_{s'} \;\leq\; \|\|T\|\|_s \;\leq\; \|\|T\|\|_\infty \,.$$

Hence, all these norms coincide. $\qquad\square$

Thanks to the independence of s, we may now introduce the short notation

$$\|T\|_{p\to q} = \|T : L^p_\infty(N\subset M) \to L^q_\infty(N\subset M)\| .$$

and similarly, the *cb*-version

$$\|T\|_{p\to q,cb} = \sup_m \|id_{\mathbb{M}_m} \otimes T : L^p_\infty\big(\mathbb{M}_m(N)\subset\mathbb{M}_m(M)\big)$$
$$\to L^q_\infty\big(\mathbb{M}_m(N)\subset\mathbb{M}_m(M)\big)\| .$$

In particular, we understand $L^p_\infty(N\subset M)$ as an operator space with operator space structure

$$\mathbb{M}_m\big(L^p_\infty(N\subset M)\big) = L^p_\infty\big(\mathbb{M}_m(N)\subset\mathbb{M}_m(M)\big) .$$

The analogue of Lemma 3.1 reads as follows:

Lemma 3.13. *Let (T_t) be a semigroup of self-adjoint *-preserving N-bimodule maps. Then,*

  (i) $\|T_{2t}\|_{1\to\infty} = \|T_t\|^2_{1\to 2}.$
  (ii) $\|T_{2t} - E\|_{1\to\infty} = \|(T_t - E)\|^2_{2\to\infty}$

The same equality holds for cb-norms.

Proof. Because T_t are N-bimodule maps, we know by Lemma 3.12 that

$$\|T_{2t}\|_{1\to\infty} = \|T_{2t} : L^1_2(N \subset \mathcal{M}) \to L^\infty_2(N \subset M)\| ,$$
$$\|T_t\|_{1\to 2} = \|T_t : L^1_2(N \subset M) \to L_2(M)\|$$

Take $X = L^1_2(N \subset M)$ and $H = L_2(M) \cong L^2_2(N \subset M)$. The anti-linear bracket $\langle x, y\rangle = \tau(x^*y)$ gives a complete isometric embedding $L^\infty_2(N\subset M) \subset \bar{X}^*$. Then, using the general principle in the proof of Lemma 3.1 implies the assertion because T_t is *-preserving and self-adjoint. The *cb*-norm case follows similarly with $X = S_2(L^1_2(N\subset M))$ and $H = S_2 \otimes_2 L_2(M)$, where S_2 is the Schatten 2-class. $\qquad\square$

We have seen in the last subsection that a complete positive order inequality $E_\tau \leq_{cp} T$ can be deduced from kernel estimates. For non-ergodic cases, we have to modify the argument by introducing the appropriate Choi matrix for bimodule maps. Let us recall that the conditional expectation $E : M \to N$ generates a Hilbert W^*-module $\mathcal{H}_E = L^c_\infty(N\subset M)$ with N-valued inner product

$$\langle x, y\rangle_{\mathcal{H}_E} = E(x^*y) .$$

As observed in [50,51], it is easy to identify the completion of this module in $\mathbb{B}(L_2(M))$, namely the strong closure of $\bar{\mathcal{H}}_E = \overline{Mp_E}$, where $p_E = E : L_2(M) \to L_2(N)$ is the Hilbert space projection onto the subspace $L_2(N) \subset L_2(M)$. The advantage of a complete W^*-module is the existence of a module basis $(\xi_i)_{i\in I}$ such that

$$\langle \xi_i, \xi_j\rangle_{\mathcal{H}_E} = \delta_{ij}p_i$$

where $p_i \in N$ are the projections. Note that in our situation the inclusion $Mp_E \subset L_2(M)$ is faithful and hence, the basis elements ξ_i (or more precisely $\hat{\xi}_i$ obtained from the GNS construction) are in $L_2(M)$. In particular, every element x in $L_2(M)$ has a unique decomposition

$$x = \sum_i \xi_i x_i$$

so that $x_i = p_i x_i \in N$. Indeed, we have $x_i = \langle \xi_i, x \rangle_{\mathcal{H}_E}$. For a N-bimodule map $T : L^1_\infty(N \subset M) \to M$, we may therefore introduce the Choi matrix

$$\chi_T = \sum_{i,j} |i\rangle\langle j| \otimes T(\xi_i^* \xi_j) \ .$$

Lemma 3.14. *Let $T : M \to M$ be a N-bimodule map. Then,*

$$\|T\|_{1\to\infty,cb} = \|\chi_T\|_{\mathbb{B}(\ell_2(I))\bar{\otimes}M} \ .$$

Proof. Let $q = \sum_{i,j} |i\rangle\langle j| \otimes \xi_i^* \xi_j \in \mathbb{B}(\ell_2(I))\bar{\otimes}M$. Viewing q as a kernel, the corresponding map $T_q : S_1(\ell_2(I)) \to M$ is given by

$$T_q(|i\rangle\langle j|) = \xi_i^* \xi_j$$

Let us show that $T_q : S_1(\ell_2(I)) \to L^1_\infty(N \subset M)$ is completely contractive. Indeed, using operator space version of (3.1)

$$\|T_q\|_{cb} = \|q\|_{\mathbb{B}(\ell_2(I))\otimes_{\min} L^1_\infty(N\subset M)} = \|q\|_{L^1_\infty(\mathbb{B}(\ell_2(I))\otimes_{\min}N\subset\mathbb{B}(\ell_2(I))\otimes_{\min}M)}$$
$$= \|id \otimes E(q)\|_{\mathbb{B}(\ell_2(I))\bar{\otimes}M} = \|\sum_i |i\rangle\langle i| \otimes p_i\|_{\mathbb{B}(\ell_2(I))\bar{\otimes}M} \le 1$$

Here, we have used the fact q is positive and p_i are projections. Note that the kernel of $T \circ T_q : S_1(\ell_2(I)) \to M$ is exactly the Choi matrix χ_T. Therefore, thanks to (3.1) again, we deduce that

$$\|\chi_T\| = \|T \circ T_q : S_1(\ell_2(I)) \to M\|_{cb} \le \|T\|_{1\to\infty,cb} \ .$$

Now let $x \in L^1_\infty(N \subset M)$ of norm less than 1. According Lemma 3.11, we have a factorization $x = y_1 y_2$ such that $E(y_1^* y_1) \le 1$ and $E(y_2^* y_2) \le 1$. This means we find coefficients a_i, b_j such that

$$x = \sum_{i,j} a_i^* \xi_i^* \xi_j b_j$$

and $\sum_i a_i a_i^* \le 1$ and $\sum_j b_j^* b_j \le 1$. Therefore, we deduce that

$$\|T(x)\|_M = \|\sum_{i,j} a_i^* T(\xi_i^* \xi_j) b_j\| = \|\left(\sum_i \langle i| \otimes a_i^*\right)\left(\sum_{i,j} |i\rangle\langle j| \otimes T(\xi_i^* \xi_j)\right)$$
$$\times \left(\sum_j |j\rangle \otimes b_j\right)\|$$
$$\le \|\sum_i a_i a_i^*\|^{1/2}\|\chi_T\|\|\sum_j b_j^* b_j\|^{1/2}.$$

This implies

$$\|T(x)\| \ \leq \ \|\chi_T\| \inf_{x=y_1 y_2} \|E(y_1 y_1^*)\|^{1/2} \|E(y_2^* y_2)\|^{1/2} \ ,$$

or equivalently

$$\|T\|_{1\to\infty} \ \leq \ \|\chi_T\| \ .$$

The same argument applies for $id_{\mathbb{M}_m} \otimes T$, and we have the equality $\|T\|_{1\to\infty,cb} \ = \ \|\chi_T\|$. $\qquad\square$

We are now in a position to prove a version of Proposition 3.7 ii) in the non-ergodic situation

Lemma 3.15. *Let $T : M \to M$ be a unital completely positive N-bimodule map such that*

$$\|T - E_N : L_\infty^1(N \subset M) \to M\|_{cb} \ \leq \ \frac{1}{2} \ .$$

Then, $E_N \leq_{cp} 2T$.

Proof. Let χ_T (resp. χ_E) be the Choi matrix of T (resp. E_N). We known by Lemma 3.14 that

$$\|\chi_T - \chi_E\|_{B(\ell_2(I))\bar{\otimes}N} \ \leq \ \frac{1}{2} \ .$$

Since T and E are completely positive, χ_T and χ_E are positive. Thus, we may write $\chi_E - \chi_T = \alpha - \beta$ with $0 \leq \alpha, \beta \leq \frac{1}{2}$. Write $\alpha = \sum_{i,j} |i\rangle\langle j| \otimes \alpha_{i,j}$ and $\beta = \sum_{i,j} |i\rangle\langle j| \otimes \beta_{i,j}$. It is clear that $\alpha_{i,j} - \beta_{i,j} = E(\xi_i^*\xi_j) - T(\xi_i^*\xi_j)$. Let $x = y^*y$ be a positive element in M and $y = \sum_j \xi_j y_j$ with coefficients $y_j \in N$ in the module basis. Then, we deduce that

$$\sum_j y_j^* p_j y_j = E(y^* y) \ = \ (E - T)(y^* y) + T(y^* y)$$

$$= \sum_{i,j} y_i^* p_i \alpha_{i,j} p_j y_j - \sum_{i,j} y_i^* p_i \beta_{i,j} p_j y_j + T(y^* y)$$

$$\leq \frac{1}{2} \left(\sum_j y_j^* p_j y_j \right) + T(y^* y) \ . \qquad (3.8)$$

Indeed, in the last step we use the fact that

$$\sum_{i,j} y_i^* p_i \alpha_{i,j} p_j y_j = \left(\sum_i \langle i| \otimes y_i^* p_i \right) \left(\sum_{i,j} |i\rangle\langle j| \otimes \alpha_{i,j} \right) \left(\sum_j |j\rangle \otimes p_j y_j \right)$$

$$\leq \frac{1}{2} \left(\sum_i \langle i| \otimes y_i^* p_i \right) \left(\sum_j |j\rangle \otimes p_j y_j \right)$$

$$= \frac{1}{2} \left(\sum_j y_j^* p_j y_j \right) \ .$$

Subtracting $\frac{1}{2}(\sum_j y_j^* p_j y_j)$ in (3.8), we obtain

$$E(y^*y) = \sum_j y_j^* p_j y_j \ \leq \ 2T(y^*y) \ .$$

The same argument holds for matrix coefficients. Hence, $E \leq_{cp} 2T$. □

Thus, in the non-ergodic situation, we can now state the analogue of Theorem 3.10.

Theorem 3.16. *Let T_t be a semigroup of completely positive self-adjoint maps and N be the fixpoint subalgebra. Suppose that*

(i) $\|T_t : L_\infty^1(N \subset M) \to L_\infty(M)\|_{cb} \ \leq \ ct^{-d/2}$ *for $0 \leq t \leq 1$ and $c > 0$, $d \geq 0$;*

(ii) $\|T_t(I - E_N) : L_2(M) \to L_2(M)\| \ \leq \ e^{-\sigma_{\min} t}$ *for some $\sigma_{\min} > 0$.*

Let F be a function satisfying (I)+(QM) or (I)+(Δ_2). Then, $\Phi_F(A)$ satisfies λ-$\Gamma\mathcal{E}$ and hence λ-CSLI for some λ depending on c, d, F and $\sigma_{\min}$.

Proof. The fixpoint algebra N is the common multiplicative domain of T_t for all t. Hence, T_t are N-bimodule maps. Then, the assertion follows from combining Lemma 3.15 with argument in Theorem 3.10. □

4. Riemannian Manifolds and Representation Theory

In this section, we find heat kernel estimates, which allow us to apply Theorem 3.16.

4.1. Riemannian Manifolds

Let $(\mathcal{M}, g)$ be a d-dimensional compact Riemannian manifold without boundary. A *Hörmander system* is a finite family of vector fields $X = \{X_1, \ldots, X_r\}$ such that for some global constant l_X, the set of iterated commutators (no commutator if $k = 1$)

$$\bigcup_{1 \leq k \leq l_X} \{[X_{j_1}, [X_{j_2}, \ldots, [X_{j_{k-1}}, X_{j_k}]]] \mid 1 \leq j_1, \ldots, j_k \leq r\}$$

spans the tangent space $T_x\mathcal{M}$ at every point $x \in \mathcal{M}$. We consider the sub-Laplacian

$$\Delta_X \ = \ \sum_{j=1}^r X_j^* X_j \ .$$

where X_j^* is the adjoint operator of X_j with respect to $L_2(\mathcal{M}, \mu)$. Here, $d\mu$ is the volume measure of the metric g, which in local coordinates is given by

$$d\mu(x) \ = \ \sqrt{|g|} dx_1 \wedge \cdots \wedge dx_n \ , |g|(x) = |\det(g_{ij}(x))| \ .$$

For compact $\mathcal{M}$, Δ_X extends to a self-adjoint operator on $L_2(\mathcal{M}, \mu)$. It follows from the famous Rothschild–Stein estimate [87] (see also [65]) that Δ_X is hypoelliptic. This leads to the estimate (see, e.g., [75])

$$\langle f, \Delta^{\frac{1}{l_X}} f \rangle \leq C(\langle f, \Delta_X f \rangle + \|f\|_2^2) \ , \tag{4.1}$$

where Δ is the Laplace–Beltrami operator on $\mathcal{M}$. Using the Hardy–Littlewood–Sobolev inequality in the Riemannian setting, we have the following Sobolev-type inequality (see, e.g., [67]):

Lemma 4.1. *Let $\mathcal{M}$ be a compact Riemannian manifold and X be a Hörmander system of $\mathcal{M}$. Let $q = \frac{2dl_X}{dl_X - 2}$. Then,*

$$\|f\|_q \;\leq\; C\big(\langle \Delta_X f, f\rangle + \|f\|_2^2\,\big)^{1/2}\,.$$

Now it is time to invoke the Varopoulos theorem about the dimension of semigroups.

Theorem 4.2 [100]. *Let $T_t : L_\infty(\Omega, \mu) \to L_\infty(\Omega, \mu)$ be a semigroup of measure preserving maps and A be its generator. The following conditions are equivalent: for $m \in \mathbb{N}$,*

(i) *$\|T_t : L_1(\Omega, \mu) \to L_\infty(\Omega, \mu)\| \;\leq\; C_1 t^{-m/2}$ for all $0 \leq t \leq 1$ and some C_1;*
(ii) *$\|f\|_{\frac{2m}{m-2}}^2 \;\leq\; C_2(\langle Af, f\rangle + \|f\|_2^2)$;*
(iii) *$\|f\|_2^{2+4/m} \;\leq\; C_3 \langle Af, f\rangle \|f\|_1^{4/m}$.*

Remark 4.3. Varopoulos theorem remains valid for semi-finite von Neumann algebras. For the proof, the only part which requires modification is i)$\Rightarrow$ ii) (see [52] and independently [102]). The completely bounded norm analog is significantly more involved [53], and it will be used later.

It is well known that the Laplace–Beltrami operator Δ_{LB} on a compact Riemannian manifold has a spectral gap. Similarly, Δ_X also has a spectral gap. Combining Lemma 4.1 and Theorem 4.2, we obtain the kernel estimates in Proposition 3.2 for $m = dl_X$. As a consequence of Theorem 3.10, we have:

Theorem 4.4. *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold and $X = \{X_1, \ldots, X_r\}$ be a Hörmander system. Then, there exists $m = dl_X \in \mathbb{N}$ and $c > 0$ such that $S_t = e^{-t\Delta_x}$ satisfies*

$$\|S_t : L_1(\mathcal{M}, \mu) \to L_\infty(\mathcal{M}, \mu)\|_{cb} \;\leq\; ct^{-m/2}\,.$$

Moreover, for every $0 < \theta < 1$, $S_t^\theta = e^{-t\Delta_x^\theta}$ satisfies λ-$\Gamma\mathcal{E}$ and λ-CLSI with $\lambda = c_0 t_0^{-\theta}(1-\theta)\theta^2$. Here, $t_0 = t_0(\Delta_X)$ is the return time of Δ_X in (3.4) and c_0 is an absolute constant.

As mentioned in Corollary 2.10, the $\Gamma\mathcal{E}$ condition automatically extends to the operator-valued setting for any finite von Neumann algebra M. Here, we note that the kernel estimates for Hörmander systems also extend to M-valued functions.

Corollary 4.5. *Let M be a finite von Neumann algebra with tracial state τ. Let $(\mathcal{M}, g)$ be a compact Riemannian manifold and X be a Hörmander system as above. Then,*

$$\|id \otimes S_t : L_\infty(M; L_1(\mathcal{M})) \to L_\infty(M; L_\infty(\mathcal{M}))\| \;\leq\; ct^{-m/2}$$

holds for $0 \leq t \leq 1$ and some $c > 0$.

Proof. Let $\mathbb{E}(f) = \dfrac{1}{Vol(\mathcal{M})} \displaystyle\int_{\mathcal{M}} f(x)\mathrm{d}\mu(x)$ be the conditional expectation onto M, i.e., $\mathbb{E}(f)1_{\mathcal{M}} = E_M(f)$. Then, a positive element $f \in L^1_\infty(M \subset L_\infty(\mathcal{M})\bar{\otimes}M)$ has norm ≤ 1 if $\|\mathbb{E}(f)\|_M \leq 1$. Let $h \in L_2(M)$ be a unit vector. Consider the scalar function $f_h(x) = \langle h, f(x)h\rangle_{L_2(M)}$. We deduce that

$$\mathbb{E}(f_h) = \int_{\mathcal{M}} f_h(x)\mathrm{d}\mu(x) = \langle h, \mathbb{E}(f)h\rangle_{L_2(M)} \leq 1$$

and therefore, $\|S_t(f)_h\|_{L_\infty(\mathcal{M})} \leq ct^{-m/2}$. This means

$$\sup_{\|h\|_2 \leq 1} \sup_{x \in \mathcal{M}} \langle h, S_t(f)(x)h\rangle_{L_2(M)} \leq ct^{-m/2} .$$

Interchanging the double supremums, with the help of the duality $L_1(\mathcal{M}, L_1(M))^* = L_\infty(\mathcal{M})\bar{\otimes}M$, implies the assertion $\qquad\square$

4.2. Group Representation

Let G be a compact group with Haar measure μ. We consider a semigroup of measure preserving maps $S_t : L_\infty(G) \to L_\infty(G)$ that is also right translation invariant. Suppose that S_t is given by the kernel

$$S_t(f)(g) = \int_G K_t(g, h)f(h)\mathrm{d}\mu(h) .$$

The right translation invariance means that for any $f \in L_\infty(G)$ and $g, s \in G$ we have

$$\int K_t(gs, h)f(h)\mathrm{d}\mu(h) = S_t(f)(gs) = \int K_t(g, h)f(hs)\mathrm{d}\mu(h)$$

$$= \int K_t(g, hs^{-1})f(h)\mathrm{d}\mu(h) .$$

Thus, $K_t(gs, h) = K_t(g, hs^{-1})$ and hence $K_t(g, h) = k_t(gh^{-1})$ for some single variable function k_t. Conversely, $K_t(g, h) = k_t(gh^{-1})$ implies right invariance.

Now let (M, τ) be a finite von Neumann algebra and $\alpha : G \to Aut(M)$ be an action G on M of trace-preserving automorphisms. Using the standard co-representation,

$$\pi : M \to L_\infty(G; M) , \quad \pi(x)(g) = \alpha_{g^{-1}}(x) .$$

we define the *transferred semigroup* on M

$$T_t(x) = \int_G k_t(g^{-1})\alpha_{g^{-1}}(x)\mathrm{d}\mu(g) .$$

Lemma 4.6. *The semigroups S_t and T_t satisfy the following factorization property:*

$$\pi \circ T_t = (S_t \otimes id_M) \circ \pi .$$

Proof. We include the proof for completeness. Indeed, for $x \in M$

$$\pi(T_t(x))(g) = \alpha_g^{-1} \int_G k_t(h^{-1})\alpha_{h^{-1}}(x)\mathrm{d}\mu(h)$$

$$= \int_G k_t(gg^{-1}h^{-1})\alpha_{(hg)^{-1}}(x)\mathrm{d}\mu(h)$$

$$= \int_G k_t(gh^{-1})\alpha_{h^{-1}}(x)\mathrm{d}\mu(h) \;=\; (S_t \otimes id)(\pi(x))(g)\,. \qquad \square$$

Let us denote by $N = \{x \,|\alpha_g(x) = x \,\forall\, g \in G\}$ the fixpoint subalgebra. Note that we have the following commuting diagram:

$$\begin{array}{ccc} M & \xrightarrow{\pi} & L_\infty(G,M) \\ \downarrow E_N & & \downarrow E_M \\ N & \xrightarrow{\pi} & M \end{array} \,. \qquad (4.2)$$

Here, $M \subset_\pi L_\infty(G,M)$ is considered as operator-valued constant functions and, as seen in Corollary 4.5, the conditional expectation is given by averaging. Then, for any $x \in M$,

$$\mathbb{E}(\pi(x)) \;=\; \int_G \alpha_{g^{-1}}(x)\mathrm{d}\mu(g) \;=\; E_N(x)$$

is exactly the conditional expectation form M onto the fixpoint algebra N. Since E_M is a unital completely positive N-bimodule map, we see that

$$E_M : L_p^q(M \subset L_\infty(G,M)) \to L_p^q(N \subset M)$$

is completely contractive for all $1 \le p,q \le \infty$. This implies that the inclusion $\pi : L_p^q(N \subset M) \subset L_p^q(M \subset L_\infty(G,M))$ is a completely isometric embedding (see [44] for details). The next proposition shows that λ-$\Gamma\mathcal{E}$ and λ-CLSI of the semigroup S_t on the group G pass to the transferred semigroup T_t on M.

Proposition 4.7. *Let $S_t : L_\infty(G) \to L_\infty(G)$ be an ergodic, right invariant semigroup and $T_t : M \to M$ be the transferred semigroup defined as above. Then,*

(i) $\|T_t - E_N : L_2(M) \to L_2(M)\| \le \|S_t - \mathbb{E} : L_2(G) \to L_2(G)\|$ *for all $t > 0$ and hence the spectral gap for T_t (with respect to E_N) is not less than the spectral gap of S_t.*

(ii) (T_t) *satisfies λ-$\Gamma\mathcal{E}$ (resp. λ-MLSI, λ-CLSI) if (S_t) does.*

Proof. By the diagram (4.2), the transferred semigroup T_t on M can be viewed as a restriction of semigroup $S_t \otimes id_M$ on $L_\infty(G,M)$. By definition, λ-$\Gamma\mathcal{E}$ and λ-CLSI of S_t on the group G naturally extends to $S_t \otimes id_M$, which implies corresponding property for T_t. $\qquad \square$

We obtain the following application of transference:

Theorem 4.8. *Let $S_t : L_\infty(G) \to L_\infty(G)$ be an ergodic right invariant semigroup with kernel function k_t. Let σ_{min} be the spectral gap of S_t and suppose $\sup_g |k_t(g)| \le ct^{-m/2}$ holds for some $c,m > 0$ and $0 \le t \le 1$. Then, the transferred semigroup $T_t : M \to M$ and its generator A satisfy:*

(i) $\|T_t : L_1(M) \to L_1^\infty(N \subset M)\|_{cb} \le ct^{-m/2}$ *for $0 \le t \le 1$, and*

$$\|T_t - E : L_1(M) \to L_1^\infty(N \subset M)\|_{cb} \le \begin{cases} 2ct^{-m/2} & 0 \le t \le 1\,, \\ c(m,\sigma_{\min})e^{-\sigma_{\min}t} & 1 \le t < \infty\,. \end{cases}$$

(ii) *For every function F satisfying condition (I)+(QM) or (I)+(Δ_2), the generator $\Phi_F(A)$ satisfies $\Gamma\mathcal{E}$ and hence CLSI.*

Proof. Let $T_K(f)(g) = \int K(g,h)f(h)dh$ be an integral operator. Then, we see that

$$|T_K(f)(g)| \ \leq \ \sup_h |K(g,h)| \int |f(h)|dh$$

implies that

$$\|T_K : L_1(G) \to L_\infty(G)\| \ = \ \text{ess sup}_{g,h} |K(g,h)|$$

is given by the essential supremum. For a right invariant kernel $K(g,h) = k_t(gh^{-1})$, we deduce $\|T_K\| = \|k_t\|_\infty$. Therefore, our assumption implies

$$\|S_t : L_1(G) \to L_\infty(G)\|_{cb} = \|S_t : L_1(G) \to L_\infty(G)\| \le ct^{-m/2}\,.$$

Combined with the assumption on spectral gap σ_{min}, we know by Proposition (3.2) that

$$\|S_t - \mathbb{E} : L_1(G) \to L_\infty(G)\|_{cb} \ \leq \ \begin{cases} 2ct^{-m/2} & 0 \le t \le 1 \\ C(d, \sigma_{min})e^{-\lambda t} & 1 \le t < \infty\,. \end{cases}$$

where $\mathbb{E}$ is the conditional expectation from $L_\infty(G)$ onto the constant functions. Note that $T_t = S_t \otimes id_M|_{\pi(M)}$, $E = \mathbb{E} \otimes id_M|_{\pi(M)}$ by restriction because the transference homomorphism π gives completely isometric inclusion (by the diagram (4.2))

$$L_1(M) \subset_\pi L_1(G, L_1(M)) \quad , \quad L_1^\infty(N\subset M) \subset_\pi L_1^\infty(M \subset L_\infty(G, M))\,,$$

Then, i) follows from

$$\|T_t - E : L_1(M) \to L_1^\infty(N \subset M)\|_{cb}$$
$$\le \|(S_t - \mathbb{E}) \otimes id_M : L_1(G, L_1(M)) \to L_1^\infty(M\subset L_\infty(G, M))\|_{cb}$$
$$= \|S_t - \mathbb{E} : L_1(G) \to L_\infty(G)\|_{cb}\,.$$

The assertion (ii) follows from (i) via Theorem 3.16. $\square$

Now we combine Theorem 4.8 with the kernel estimates for a Hörmander systems on a Lie group. Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra (of right invariant vector fields). A generating set $X = \{X_1, \ldots, X_r\}$ of $\mathfrak{g}$ is a right invariant Hörmander systems on G. Indeed,

$$X(f) \ = \ \frac{d}{dt}f(\exp(tX)g)|_{t=0}\,,$$

is right translation invariant because the left and right translations commute. Then, the sub-Laplacian $\Delta_X = \sum_{j=1}^{r} X_j{}^* X_j$ generates a right invariant semigroup $S_t = e^{-t\Delta_X}$

Corollary 4.9. *Let X be a generating set of $\mathfrak{g}$ and $S_t = e^{-t\Delta_X} : L_\infty(G) \to L_\infty(G)$ be the right invariant semigroup given by the sub-Laplacian Δ_X. Then, the transferred semigroup $T_t : M \to M$ and its generator A satisfy*

(i) *For every function F satisfying condition $(I) + (QM)$ or $(I) + (\Delta_2)$, the generator $\Phi_F(A)$ satisfies $\Gamma\mathcal{E}$ and hence CLSI.*

(ii) *In particular, for all $0 < \theta < 1$ the generator A^θ satisfies λ-$\Gamma\mathcal{E}$ with constant $\lambda(\theta, X) = c_0 t_0^{-\theta} \theta^2 (1 - \theta)$. Here, $t_0 = t_0(\Delta_X)$ is the return time of Δ_X and c_0 an absolute constant.*

Proof. This is an special case of Theorem 4.4 for a compact Lie group G.   □

4.3. Finite-Dimensional Representation of Lie Groups

Let $\mathbb{M}_m$ be the $m \times m$ matrix algebra and U_m be its unitary group. A unitary representation $u : G \to U_m$ induces a representation $\hat{u} : \mathfrak{g} \to \mathfrak{u}_m$ of the corresponding Lie algebra, where $\mathfrak{u}_m = i(\mathbb{M}_m)_{sa}$ is the Lie algebra of U_m and $(\mathbb{M}_m)_{sa}$ are the self-adjoint matrices in $\mathbb{M}_m$. Let $X = \{X_1, \ldots, X_r\}$ be a generating set of $\mathfrak{g}$ and $Y_1, \ldots, Y_r \in (\mathbb{M}_m)_{sa}$ be their images under $\hat{u}$. Indeed, for the exponential map, we have

$$u(\exp(tX_j)) = e^{itY_j}$$

and $iY_j \in i\mathbb{M}_m^{sa}$ is the corresponding generator for the one parameter unitary $u(\exp(tX_j)) \subset \mathbb{M}_m$. Let us consider the (self-adjoint) Lindblad generator given by

$$\mathcal{L}(\rho) = \sum_{j=1}^{r} Y_j^2 \rho + \rho Y_j^2 - 2Y_j \rho Y_j .$$

Then, we have a concrete realization of Lemma 4.6.

Lemma 4.10. *Let $\pi : \mathbb{M}_m \to L_\infty(G, \mathbb{M}_m)$ be given by*

$$\pi(x)(g) = u(g)^{-1} x u(g)$$

Then,

$$\Delta_X \otimes id_{\mathbb{M}_m}(\pi(\rho)) = \pi(\mathcal{L}(\rho)) , \ X_j \otimes id_{\mathbb{M}_m}(\pi(x)) = -i\pi([Y_j, x]) .$$

In particular, $e^{-t\mathcal{L}}$ is a transferred semigroup of $e^{-t\Delta_X}$ on G.

Proof. Let $x \in \mathbb{M}_m$ and $h, k \in l_2^m$ being two vectors. We consider the scalar function

$$f(g) = \langle h, u(g)^{-1} x u(g) \ k \rangle$$

Then, we have

$$X_j(f)(g) = \frac{d}{dt} f(\exp(tX_j)g)|_{t=0} = \frac{d}{dt} \langle h, u(g)^{-1} e^{-itY_j} x e^{itY_j} u(g)k \rangle|_{t=0}$$
$$= i\langle h, u(g)^{-1}(xY_j - Y_j x)u(g)k \rangle = -i\langle h, u(g)^{-1}[Y_j, x]u(g)k \rangle .$$

Since h, k are arbitrary, we deduce the second assertion. Note that $X_j = -X_j^*$. Then,

$$X_j^* X_j \pi(x) = \pi([Y_j, [Y_j, x]]) = \pi(Y_j^2 x + x Y_j^2 - 2Y_j x Y_j) .$$

and hence

$$\Delta_X(\pi(x)) \;=\; \pi\left(\sum_j Y_j^2 x + xY_j^2 - 2Y_j xY_j\right)\;.$$

This implies that the semigroup $S_t = e^{-t\Delta_X}$ on G satisfies

$$(S_t \otimes id) \circ \pi \;=\; \pi \circ e^{-t\mathcal{L}}\;. \qquad \square$$

Theorem 4.11. *Let $X = \{X_1, \ldots, X_r\}$ be a generating set of $\mathfrak{g}$ and $u : G \to U_m$ be a unitary representation such that $\hat{u}(X_j) = Y_j$. Let*

$$\mathcal{L}(x) \;=\; \sum_j Y_j^2 x + Y_j^2 - Y_j xY_j\;.$$

Then, for any $0 < \theta < 1$, $A = \mathcal{L}^\theta$ satisfies λ-$\Gamma\mathcal{E}$ and hence λ-CLSI with $\lambda(X, \theta) = c_0 t_0^{-\theta}\theta^2(1-\theta)$ depending on the return time $t_0 = t_0(e^{-t\Delta_X})$ defined in (3.4) and θ.

Proof. By Lemma 4.10, $e^{-t\mathcal{L}}$ is a transferred semigroup of S_t on G. The assertion follows from Proposition 4.7 and Corollary 4.9. $\qquad \square$

We obtain the following corollary from the *cb*-version of Varopoulos' theorem [53].

Corollary 4.12. *Let G be a d-dimensional Lie group and X be a generating set of $\mathfrak{g}$ using iterated Lie brackets up to order $l_X - 1$ (with l_X many elements from X). Let $u : G \to U_m$ be a unitary representation and $\mathcal{L}$ be as above. Suppose $S_t = e^{-tB}$ is a semigroup of completely positive self-adjoint trace-preserving maps on M_m such that*

  (i) *The fixpoint algebra $N_{\mathcal{L}}$ of $e^{-t\mathcal{L}}$ is contained in the fixpoint algebra N_B for e^{-tB};*
  (ii) *$\langle x, \mathcal{L}^\alpha x\rangle_{tr} \;\le\; c(\langle x, Bx\rangle_{tr} + \|x\|_2^2)$ for some $0 < \alpha < \frac{dl_X}{2}$.*

Then, for all $0 < \theta < 1$, B^θ satisfies $\Gamma\mathcal{E}$ and hence CLSI.

Proof. Denote $d_X = dl_X$. The cb-version of Varopoulos theorem implies that

$$\|(I + \mathcal{L})^{-\alpha/2} : L_2(M_m) \to L_2^q(N_{\mathcal{L}} \subset M_m)\|_{cb} \;\le\; c(q)$$

holds for $\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{d_X}$ provided $2\alpha < d_X$. By our assumption, we have

$$\|(I + \mathcal{L})^{\alpha/2}(x)\|_2 \sim (\|x\|_2 + \|\mathcal{L}^{\frac{\alpha}{2}}(x)\|_2) \;\le\; c(\|x\|_2 + \|B^{1/2}x\|_2)\;.$$

Using $N_{\mathcal{L}} \subset N_B$, we deduce that

$$\|(I + B)^{-1/2} : L_2(M_m) \to L_2^q(N_B \subset M_m)\|_{cb} \;\le\; c'(q)\;.$$

By $ii) \Rightarrow i)$ in Varopoulos Theorem (4.2), we deduce that

$$\|e^{-tB} : L_\infty^1(N_B \subset M)_m \to M_m\|_{cb} \;\le\; ct^{-d_X/2\alpha}\;.$$

Thanks to the spectral gap for B, we may again use Theorem 3.10 and deduce the assertion. $\qquad \square$

5. A Density Result

In this section, we show that on matrix algebras, the set of self-adjoint generators satisfying $\Gamma\mathcal{E}$ is dense. Let $T_t = e^{-tA} : \mathbb{M}_n \to \mathbb{M}_n$ be a semigroup of self-adjoint and unital completely positive maps. Using the Lindblad form, we may assume that

$$\mathcal{L}(x) = \sum_{k=1}^m a_k^2 x + x a_k^2 - 2a_k x a_k = \sum_{k=1}^m [a_k, [a_k, x]] .$$

with $a_1, \ldots, a_m$ self-adjoint. The corresponding derivation is given by

$$\delta : \mathbb{M}_n \to \oplus_{k=1}^m \mathbb{M}_n , \ \delta(x) = ([ia_k, x])_{k=1}^m ,$$

and the fixpoint algebra is

$$N = \{x \mid \delta(x) = 0\} = \{a_1, \ldots, a_m\}' .$$

It is easy to check that

$$\Gamma_{\mathcal{L}}(x, x) = \sum_k [ia_k, x^*][ia_k, x] = \sum_k [a_k, x]^*[a_k, x] .$$

Let $X = \{ia_1, \ldots, ia_m\}$, and $\mathfrak{g}$ be the matrix Lie algebra generated by X. Note that for two anti-selfadjoint operators $A^* = -A, B^* = -B$, the commutator is still anti-selfadjoint $[A, B]^* = (AB - BA)^* = B^* A^* - A^* B^* = [B, A]$. Then, $\mathfrak{g}$ is a Lie subalgebra of the anti-selfadjoint matrices $\mathfrak{u}_n$. The following lemma is probably well known. We include a proof for completeness.

Lemma 5.1. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{u}_n$. Then, $\mathfrak{g}$ is a Lie algebra of some connected compact Lie group.*

Proof. Recall that the Killing form on $\mathfrak{u}_n$ is given by

$$K(A, B) = tr(ad_A \circ ad_B) ,$$

where $ad_A(Y) = [A, Y]$ is the adjoint transformation on $\mathfrak{u}_n$. We first show that the killing form K is negative semi-definite. For matrices x and y, define the real inner product $(x, y) = \mathrm{Re}\,tr(x^* y)$. It is clear that $(\cdot, \cdot)$ is unitary invariant, i.e., for all unitary u, $(uxu^*, uyu^*) = (x, y)$. This yields that for anti-selfadjoint A

$$([A, x], y) + (x, [A, y]) = 0 ,$$

which means the adjoint transformation ad_A is skew-symmetric on some orthonormal basis with respect to the inner product. Then, the Killing form is negative semi-definite because for all skew-symmetric matrix T, $tr(T^2) = -tr(T^t T) \leq 0$. On the other hand, $\mathfrak{g} \subset \mathfrak{u}_n$ is matrix Lie algebra invariant under *-operation (conjugate transpose). According to [55, Prop 1.56], we see that $\mathfrak{g}$ is reductive. Hence, $\mathfrak{g} = \mathfrak{g}_o + [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{g}_0$ is abelian and $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple. Then, by [55, Theorem 1.42], we know that the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ is non-degenerate and hence negative-definite. According to [55, Prop 4.27], $[\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of some compact Lie group G_0. Therefore, the Lie group G corresponding to $\mathfrak{g}$ can be a product of a finite-dimensional tori and G_0, which is indeed compact. $\qquad\square$

Our aim now is to find a suitable approximation of the form $B_\epsilon = \phi_{\epsilon,\sigma}(\mathcal{L})$, which satisfies a $\Gamma\mathcal{E}$ estimate and is close to $\mathcal{L}$ in operator norm on $L_2(\mathbb{M}_n, tr)$. We apply the technique from Sect. 3 and define for a fixed $\sigma > 0$ the function

$$F_{\epsilon,\sigma}(t) = 1_{[\epsilon,1)}(t)t^{-2} + 1_{[1,\infty]}(t)t^{-(1+\sigma)} .$$

Lemma 5.2. *Let $\varepsilon > 0$. Define*

$$\phi_{\varepsilon,\sigma}(\lambda) = (-\ln\varepsilon)^{-1} \int_\varepsilon^\infty (1 - e^{-t\lambda})F_{\varepsilon,\sigma}(t)dt .$$

Then,

(1) $\|\mathcal{L} - \phi_{\varepsilon,\sigma}(\mathcal{L})\| \leq \frac{2\sigma^{-1} + \|\mathcal{L}\|^2}{|2\ln\varepsilon|}$;

(2) *If $c(\mathcal{L})\Gamma_{I-E} \leq_{cp} \Gamma_{I-T_t}$ holds for $t \geq t_0 \geq 1$, then*

$$\frac{c(\mathcal{L})}{\sigma|\ln\epsilon|t_0^\sigma}\Gamma_{I-E} \leq \Gamma_{\phi_{\varepsilon,\sigma}(\mathcal{L})} .$$

Proof. Using differentiation, we have that $x - \frac{x^2}{2} \leq 1 - e^{-x} \leq x$. Define $\psi(\lambda) = \int_\varepsilon^1 (1 - e^{-\lambda t})\frac{dt}{t^2}$. Then,

$$|\ln\varepsilon|\lambda - \frac{\lambda^2}{2} \leq \int_\varepsilon^1 \left(\lambda t - \frac{\lambda^2 t^2}{2}\right)\frac{dt}{t^2} \leq \psi(\lambda) \leq \int_\varepsilon^1 \lambda t \frac{dt}{t^2} = |\ln\varepsilon|\lambda .$$

Write $\tilde{\psi}(\lambda) = \int_1^\infty (1 - e^{-\lambda t})\frac{dt}{t^{1+\sigma}}$. Note that $0 \leq \tilde{\psi}(\lambda) \leq \sigma^{-1}$. Then, we find

$$\lambda - \frac{\lambda^2}{2|\ln\varepsilon|} \leq \phi_{\varepsilon,\sigma}(\lambda) \leq \lambda + \frac{1}{\sigma|\ln\varepsilon|} .$$

By functional calculus, we deduce that

$$\|\mathcal{L} - \phi_{\varepsilon,\sigma}(\mathcal{L})\| \leq \frac{1}{\sigma|\ln\varepsilon|} + \frac{\|\mathcal{L}\|^2}{2|\ln\varepsilon|} \leq \frac{2\sigma^{-1} + \|\mathcal{L}\|^2}{2|\ln\varepsilon|} .$$

For the second assertion, we observe that by linearity of Γ_A in A

$$\Gamma_{\phi_{\varepsilon,\sigma}(\mathcal{L})} \geq c(\mathcal{L})|\ln\varepsilon|^{-1}\left(\int_{t_0}^\infty \frac{dt}{t^{1+\sigma}}\right)\Gamma_{I-E} \geq \frac{c(\mathcal{L})}{\sigma|\ln\varepsilon|}t_0^{-\sigma}\Gamma_{I-E} .$$

This completes the proof of (ii).   □

Remark 5.3. (a) An interesting choice is $\sigma = \frac{1}{\ln t_0}$. Then, we find

$$\Gamma_{\phi_{\varepsilon,\sigma}(\mathcal{L})} \geq \frac{c(\mathcal{L})|\ln t_0|}{e|\ln\varepsilon|}\Gamma_{I-E} ,$$

and $\|\mathcal{L} - \phi_{\varepsilon,\sigma}(\mathcal{L})\| \leq \frac{2|\ln t_0| + \|\mathcal{L}\|^2}{2|\ln\varepsilon|}$. (b) We can also slightly improve the lower estimate. Let $\beta > 0$. The function $g(x) = 1 - e^{-x}$ is concave and hence $\frac{1-e^{-x}}{x} \geq (1 - \frac{x}{2})$ implies

$$\int_\varepsilon^{\varepsilon^\beta} (1 - e^{-\lambda t})\frac{dt}{t^2} \geq \frac{1 - e^{-\lambda\epsilon^\beta}}{\lambda\varepsilon^\beta} \int_\varepsilon^{\varepsilon^\beta} \lambda t \frac{dt}{t^2} \geq \left(1 - \frac{\lambda\varepsilon^\beta}{2}\right)(1 - \beta)\lambda|\ln\varepsilon| .$$

Thus, assuming $|\ln \varepsilon| \geq \frac{2}{\beta}|\ln \frac{\lambda}{2\beta}|$ implies

$$-2\beta\lambda \leq \phi_{\varepsilon,\sigma}(\lambda) - \lambda \leq \frac{1}{\sigma|\ln \epsilon|}$$

and hence, for $\sigma = \frac{1}{\ln t_0}$, $t_0 \geq 1$ we get

$$\|\phi_{\varepsilon,\sigma}(\mathcal{L}) - \mathcal{L}\| \leq 2\beta\|\mathcal{L}\| + \frac{\ln t_0}{|\ln \varepsilon|} .$$

(c) Estimating the return time t_0 through the Hörmander system may not be very concrete. Nevertheless, we only need to know an upper bound for $\|T_{1/2} : L_1(\mathbb{M}_n) \to L_1^\infty(N \subset \mathbb{M}_n)\|_{cb}$ and the spectral gap of $\mathcal{L}$ to control t_0.

Theorem 5.4. *Let $\mathcal{L}$ be the generator of a semigroup of unital completely positive and self-adjoint maps $T_t = e^{-t\mathcal{L}}$ on $\mathbb{M}_m$. Then, there exists a constant $\alpha(\mathcal{L})$ such that for every $\varepsilon > 0$ there exists a generator B_ε, obtained from functional calculus of $\mathcal{L}$, such that*

$$\|\mathcal{L} - B_\varepsilon : L_2(\mathbb{M}_m) \to L_2(\mathbb{M}_m)\| \leq \varepsilon \quad and \quad \varepsilon\alpha(\mathcal{L})\Gamma_{I-E_N} \leq \Gamma_{B_\varepsilon} .$$

Moreover, we have the estimate

$$\alpha(\mathcal{L}) \geq \frac{\ln t_0}{e(2\ln t_0 + \|\mathcal{L}\|^2)},$$

where t_0 is the return time of $\mathcal{L}$.

Proof. According to Lemma 5.1, we have a generating set $X = \{X_1, \ldots, X_r\}$ of a compact Lie algebra $\mathfrak{g}$ and a representation $\pi : \mathfrak{g} \to \mathfrak{u}_m$ satisfies $\pi(X_k) = ia_k$. Let G be the corresponding Lie group and Δ_X be the sub-Laplace of $X = \{X_1, \ldots, X_r\}$. Since $X = \{X_1, \ldots, X_r\}$ is generating and hence a Hörmander system, we have by Lemma 4.1 and Theorem 4.2 that S_t satisfies Proposition 3.2 and hence has a return time $t_0(\Delta_X)$ by Proposition 3.7. Note that $T_t = e^{-t\mathcal{L}}$ is a transferred semigroup of $S_t \otimes id_{\mathbb{M}_m}$ and the commutant $N = \{a_1, \ldots, a_r\}'$ is the fixpoint algebra of $e^{-t\mathcal{L}}$. Thus, we have $\Gamma_{I-T_t} \geq \frac{1}{2}\Gamma_{I-E_N}$ for $t \geq t_0$. Now, we choose $\sigma = \frac{1}{\ln t_0}$ and deduce that for $0 < \varepsilon_0 < 1$,

$$\|\phi_{\varepsilon_0,\sigma}(\mathcal{L}) - \mathcal{L}\| \leq \frac{2|\ln t_0| + \|\mathcal{L}\|^2}{2|\ln \varepsilon_0|}$$

and $\Gamma_{\phi_{\varepsilon_0,\sigma}(\mathcal{L})} \geq \frac{|\ln t_0|}{2e|\ln \varepsilon_0|}\Gamma_{I-E}$. Thus, we may choose $0 < \varepsilon_0 < 1$ such that $|\ln \varepsilon_0| = \frac{|\ln t_0| + \|\mathcal{L}\|^2/2}{\varepsilon}$ and obtain

$$\Gamma_{\phi_{\varepsilon_0,\sigma}(\mathcal{L})} \geq \frac{\varepsilon \ln t_0}{e(2\ln t_0 + \|\mathcal{L}\|^2)}\Gamma_{I-E} .$$

That completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 5.5. We can improve the dependence in $\|\mathcal{L}\|$ using Remark 5.3 (b). We choose $\beta = \frac{\varepsilon}{4\|\mathcal{L}\|}$ and ε_0 such that

$$|\ln \varepsilon_0| \geq \frac{2|\ln t_0|}{\varepsilon} , |\ln \varepsilon_0| \geq \frac{2}{\beta}|\ln \frac{\|\mathcal{L}\|}{2\beta}| = \frac{8\|\mathcal{L}\|}{\varepsilon}|\ln \frac{2\|\mathcal{L}\|^2}{\varepsilon}| .$$

Then, we obtain the estimate

$$\Gamma_{B_\varepsilon} \;\geq\; \frac{\varepsilon \ln t_0}{16e\|\mathcal{L}\|(2\ln\|\mathcal{L}\| + |\ln t_0|)}\Gamma_{I-E} \;.$$

Note that $t_0 = \frac{\ln c_0\|T_{1/2}\|_{cb}}{\sigma_{\min}(\mathcal{L})}$ only depends linearly on $1/\sigma_{\min}(\mathcal{L})$. Hence, our estimate just depends on the minimal and maximal eigenvalue of $\mathcal{L}$.

Corollary 5.6. *The set of generators of unital completely positive self-adjoint semigroups on $\mathbb{M}_m$ satisfying $\Gamma\mathcal{E}$ and hence CLSI is dense.*

Remark 5.7. In [43] it was shown that in primitive semigroup (with an unity full-rank invariant state) there exists an entanglement-breaking time t_{EB} such that T_t is entanglement-breaking for $t > t_{EB}$. A completely positive trace preserving map is called entanglement-breaking if its Choi matrix is a convex combination of tensor product positive matrices. Our kernel estimate can be used to estimate this entanglement breaking time t_{EB}.

6. Geometric Applications and Deviation Inequalities

The aim of this section is to derive several concentration inequalities for semi-groups satisfying MLSI in the non-ergodic and possibly infinite-dimensional situation. The starting point is a version of Rieffel's quantum metric space. Let $T_t : M \to M$ be a semigroup of unital completely positive and self-adjoint maps and A be the generator of T_t. As usual, we will assume that $\mathcal{A} \subset \mathrm{dom}(A^{1/2})$ is a dense *-algebra and invariant under T_t. On M we define the Lipschitz norm via the gradient form,

$$\|f\|_{Lip_\Gamma} \;=\; \max\{\|\Gamma_A(f,f)\|_M^{\frac{1}{2}}, \|\Gamma_A(f^*,f^*)\|_M^{\frac{1}{2}}\} \,, f \in \mathcal{A} \;.$$

This induces a quantum metric on the state space by duality

$$\|\rho\|_{\Gamma^*} \;=\; \sup\{|\tau(\rho f)| \mid E(f) = 0 \,, \; \|f\|_{Lip_\Gamma} \leq 1\} \;.$$

Usually, such a Lipschitz norm is considered in the ergodic setting, where the fixpoint subalgebra $N = \mathbb{C}1$ and hence the conditional expectation is given by $E(f) = \tau(f)1$. Since for states $\tau(\rho) = 1$, one can assume the additional condition $E(f) = 0$ when calculating the distance $d_\Gamma(\rho,\sigma) = \|\rho - \sigma\|_{\Gamma^*}$. This is crucial in the non-ergodic situation, see the last section of [49] for more detailed discussion. Let $\delta : \mathrm{dom}(A^{1/2}) \to L_2(\hat{M})$ be the derivation which implements the gradient form

$$\Gamma_A(x,y) \;=\; E_M(\delta(x)^*\delta(y)) \;.$$

In the construction of a derivation in [48], the following additional estimate was also proved.

$$\|\delta(x)\|_{\hat{M}} \;\leq\; 2\sqrt{2}\max\{\|\Gamma(x,x)\|_M^{1/2}, \|\Gamma(x^*,x^*)\|_M^{1/2}\} \;=\; 2\sqrt{2}\|x\|_{Lip_\Gamma} \;. \quad (6.1)$$

6.1. Wasserstein 2-Distance and Transport Inequalities

The Otto-Vilani's theory [76] of Wasserstein 2-distance transport inequality has been adapted to discrete commuting setting by Mass [32,69] and primitive finite-dimensional setting noncommutative by Carlen-Maas [25] (see also [24, 26]). In this part, we review and extend their approach to the non-ergodic self-adjoint setting. Let ρ be a positive density operator. Following [25, Lemma 5.8] we use the symbol

$$[\rho](x) \;=\; \int_0^1 \rho^s x \rho^{1-s} \mathrm{d}s$$

for the multiplier operator, and

$$[\rho]^{-1}(x) \;=\; \int_0^\infty (\rho + t)^{-1} x (\rho + t)^{-1} \mathrm{d}t \;.$$

for the inverse. The need of the symmetric two-sided multiplication is a major difference between the commutative and noncommutative setting. Let us recall a key formula which recovers the generator from the logarithm as follows:

$$A(\rho) = \delta^* \Big([\rho]\delta(\ln\rho - \ln\big(E(\rho)\big)) \Big) \tag{6.2}$$

Indeed, let us assume that ρ and $x \in \mathcal{A}$ and $\rho \geq c1$ for some $c > 0$. Write $\sigma = E(\rho)$. Using the operator integral J_F (as in Sect. 2.1) for $F(x) = \ln(x)$, we know $\delta(\ln\rho) = J_F(\delta(\rho))$ is well-defined in $L_2(\hat{M})$. Since $\ln(\sigma) \in N$, we deduce from $\delta(\ln\sigma) = 0$. Hence,

$$
\begin{aligned}
\Big\langle x, \delta^*[\rho]\delta\big(\ln\rho - \ln(E(\rho))\big) \Big\rangle &= \tau\Big(\delta(x^*)[\rho]\delta\big(\ln\rho\big)\Big) - \tau\Big(\delta(x^*)[\rho]\big(\delta(\ln\sigma)\big)\Big) \\
&= \tau\Big(\delta(x^*)[\rho]J_F^\rho\big(\delta(\rho)\big)\Big) \;=\; \tau\Big(\delta(x^*)[\rho][\rho]^{-1}\big(\delta(\rho)\big)\Big) \\
&= \tau(\Gamma(x,\rho)) \;=\; \frac{1}{2}\Big(\tau\big(A(x^*)\rho\big) + \tau\big(x^* A(\rho)\big) - \tau\big(A(x^*\rho)\big)\Big) \;=\; \tau(x^* A(\rho)) \;.
\end{aligned}
$$

which verifies (6.2) weakly in $L_2(M)$. Here, we used $A = A^*$ and $A(1) = 0$. The expression $\ln\rho - \ln E(\rho)$ itself occurs by differentiating the relative entropy $D_N(\rho) = D(\rho||E(\rho))$. Consider $g(t) = \rho + t\beta$ with a self-adjoint β with $\tau(\beta) = 0$. Using the derivation formula (2.3) for $F(x) = x\ln x$ with derivative $F'(x) = 1 + \ln x$, we deduce from the tracial property that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}D_N(\rho + t\beta)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\tau\big(F(\rho + t\beta)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\tau\big(F(E(\rho + t\beta))\big)|_{t=0} \\
&= \tau\big(F'(\rho)\beta\big) - \tau\big(F'(E(\rho))E(\beta)\big) \\
&= \tau(\beta) + \tau\big((\ln\rho)\beta\big) - \tau\big(E(\beta)\big) - \tau\big(\ln E(\rho)E(\beta)\big) \\
&= \tau\Big(\big(\ln\rho - \ln E(\rho)\big)\beta\Big) \;.
\end{aligned}
$$

This means the Radon–Nikodym derivative of D_N with respect to the trace satisfies

$$\frac{dD_N'(\rho)}{\mathrm{d}\tau} \;=\; \ln\rho - \ln E(\rho) \;. \tag{6.3}$$

In the following, we will identify a normal state $\phi_\rho(x) = \tau(x\rho)$ of M and its density operator ρ.

Definition 6.1. Given a faithful normal state $\rho \in M$, we define the weighted L_2-norm on $L_2(\hat{M})$ by the inner product

$$\langle \xi, \eta \rangle_\rho := \langle \xi, [\rho]\eta \rangle_{L_2(\hat{M})} = \int_0^1 \tau_{\hat{M}}(\xi^* \rho^{1-s} \eta \rho^s) \mathrm{d}s .$$

If ρ is invertible and $\mu 1 \le \rho \le \mu^{-1} 1$, we have

$$\mu \langle \xi, \xi \rangle \le \langle \xi, \xi \rangle_\rho \le \mu^{-1} \langle \xi, \xi \rangle .$$

Hence, for all invertible ρ, the weighted L_2-norm $\| \ \|_\rho$ is equivalent to the trace $L_2(\hat{M}, \tau)$-norm. However, this change of metric is crucial in introducing the following (pseudo-)Riemannian metric. Recall that $\Omega_\Gamma = \mathcal{H}$ is the W^*-submodule of $L_\infty^c(M \bar{\subset} \hat{M})$ generated by $\delta(\mathcal{A})\mathcal{A}$.

Lemma 6.2. *Let ρ be a faithful normal state of M. For $z \in Ran(\delta^*) \subset M$, define*

$$\|z\|_{\mathrm{Tan}_\rho} = \inf\{\|\xi\|_\rho \mid \delta^*([\rho]\xi) = z\} .$$

Here, the infimum is taken over all $\xi \in \mathcal{H}$ satisfying $\delta^*([\rho]\xi) = z$. Then, there exists a sequence $(a_n) \subset \mathcal{A}$ such that $\|\delta(a_n)\|_\rho \le \|z\|_{\mathrm{Tan}_\rho}$ and $z = \lim_n \delta^*([\rho]\delta(a_n))$ holds weakly.

Proof. We follow the argument of [25, Theorem 7.3] in the primitive case. Observe that for $x \in \mathcal{A}$, $[\rho](\delta(x))$ belongs to the closure of $\mathcal{H}$. We say that ξ in $\mathcal{H}$ is divergence-free if $\delta^*(\xi) = 0$. Let ξ_0 be in the closure of $\mathcal{H}$ such that

$$\|z\|_{\mathrm{Tan}_\rho}^2 = \|\xi_0\|_\rho^2 , \ \delta^*([\rho](\xi_0)) = z .$$

Write $\xi_\varepsilon = \xi_0 + \varepsilon[\rho]^{-1}(\xi)$. It satisfies

$$\|\xi_0\|_\rho^2 \le \|\xi_0 + \varepsilon[\rho]^{-1}(\xi)\|_\rho^2 = \|\xi_0\|_\rho^2 + 2\varepsilon \mathrm{Re}\langle \xi_0, [\rho]^{-1}(\xi) \rangle_\rho + \varepsilon^2 \|[\rho]^{-1}(\xi)\|_\rho^2$$

and hence $\langle \xi_0, [\rho]^{-1}(\xi) \rangle_\rho = 0$ for all divergence-free ξ. Equivalently, we find

$$\tau_{\hat{M}}(\xi_0^* \xi) = \tau(\xi_0^*[\rho][\rho]^{-1}(\xi)) = 0 .$$

Note that $\xi \in \mathrm{dom}(\delta^*)$ is divergence-free if and only if for all $x \in \mathcal{A}$,

$$\tau(\delta^*(\xi)x) = \hat{\tau}(\xi\delta(x)) = 0 .$$

Hence, ξ_0 is orthogonal to the divergence-free forms if and only if ξ_0 is in the closure of $\delta(\mathcal{A})$. In other words, there exists a sequence $(a_n) \subset \mathcal{A}$ such that $\xi_0 = \lim_n \delta(a_n)$ with respect to $\|\cdot\|_\rho$. This implies

$$\tau(b^* z) = \tau(\delta(b^*)[\rho]\xi_0) = \lim_n \tau\left(b^* \delta^*([\rho]\delta(a_n))\right)$$

for all $b \in \mathcal{A}$. Renormalizing a_n as $\tilde{a}_n = \frac{\|\xi_0\|_\rho}{\|\delta(a_n)\|_\rho} a_n$, we can assume $\|\delta(\tilde{a}_n)\|_\rho \le \|\xi_0\|_\rho = \|z\|_{\mathrm{Tan}_\rho}$ and deduce the assertion. $\square$

Remark 6.3. (a) If z is selfadjoint, we may use the fact that δ is *-preserving to show that $\xi_0 \in \hat{M}$ is also self-adjoint. Thus, we may replace a_n by their self-adjoint parts using the fact that $[\rho]$ also preserves self-adjointness.

(b) Since A is self-adjoint, we know that the range of A is dense in $(I - E)L_2(M)$, the orthogonal complement of $L_2(N)$, and hence contained in the closure of $\delta^*(\delta(\mathcal{A})\mathcal{A}) \subset L_2(N)^\perp$. In fact, the L_2-closure of $\delta^*(\delta(\mathcal{A})\mathcal{A}) = ran(\delta^*)$ is exactly $(I - E)L_2(M)$.

In the following, we denote by $\mathcal{H}_\rho$ the closure of $\delta(\mathcal{A})\mathcal{A}$ with respect to the $\|\cdot\|_\rho$ norm. $\mathcal{H}_\rho$ is viewed as the tangent space at the point ρ and $\|\cdot\|_{\mathrm{Tan}_\rho}$ gives a pseudo-Riemannian metric at ρ. (When N is not a trivial subalgebra, $ran(\delta^*)$ does not contain all traceless elements). The following inequalities in ergodic (primitive) setting were derived in [25,84].

Corollary 6.4. *Let ρ be a faithful normal state of M. Then,*

$$\|x\|_{\Gamma^*} \leq 2\sqrt{2}\|x\|_{\mathrm{Tan}_\rho} .$$

Proof. Let $a_n \in \mathcal{A}$ such that $\lim_{n\to\infty} \delta^*([\rho]\delta(a_n)) = x$. We may assume that

$$\|\delta(a_n)\|_\rho \leq (1+\varepsilon)\|x\|_{\mathrm{Tan}_\rho}$$

for a given $\varepsilon > 0$. Then, we deduce that for $f \in \mathcal{A}$ we have

$$|\tau(f^*x)| = \lim_n |\tau(f^*\delta^*([\rho]\delta(a_n)))| = \lim_n |\tau(\delta(f)^*[\rho]\delta(a_n))|$$

$$\leq \limsup_n \|\delta(f)\|_\rho \|\delta(a_n)\|_\rho \leq (1+\varepsilon)\|\delta(f)\|_\rho \|x\|_{\mathrm{Tan}_\rho} .$$

Furthermore, we deduce from the fact that the inclusion $M \subset \hat{M}$ is trace preserving that

$$\|\delta(f)\|_\rho^2 = \tau(\delta(f)^*[\rho]\delta(f)) = \int_0^1 \tau(\delta(f)^*\rho^{1-s}\delta(f)\rho^s)\mathrm{d}s$$

$$\leq \|\delta(f)\|_{L_\infty(\hat{M})}^2 \int_0^1 \|\rho^{1-s}\|_{L_{\frac{1}{1-s}}(\hat{M})} \|\rho^s\|_{\frac{1}{s}} \mathrm{d}s \leq \|\delta(f)\|_{L_\infty(\hat{M})}^2$$

$$\leq 8 \|f\|_{Lip_\Gamma}^2 .$$

Thus, the estimate (6.1) implies the assertion, after sending ε to 0. □

Denote $S(M)$ as the set of faithful normal states of M. Let $F : S(M) \to \mathbb{R}$ be a real function defined on $S(M)$. We say that F admits the *gradient* $\mathrm{grad}_\rho F$ with respect to the tangent metric, if for every ρ there is an vector $\xi \in \mathcal{H}_\rho$ such that for every differentiable path $\rho : (-\varepsilon, \varepsilon) \to S(M)$ with $\rho(0) = \rho$

$$\rho'(0) = \delta^*([\rho]\xi_0) \implies \frac{d}{dt}F(\rho(t))|_{t=0} = \langle \xi, \xi_0 \rangle_\rho .$$

and we write $\mathrm{grad}_\rho F = \xi$. Our control function is the relative entropy with respect to the fixpoint algebra

$$F(\rho) = D_N(\rho) = D(\rho\|E(\rho)) .$$

Let $\rho : (-\varepsilon, \varepsilon) \to S(M)$ be a smooth path. Using the directional derivative of D_N from (6.3), we find that at $\rho = \rho(0)$

$$\frac{dD_N(\rho(t))}{dt}\Big|_{t=0} = \tau\Big(\big(\ln\rho - \ln E(\rho)\big)\delta^*([\rho]\xi_0)\Big) \; = \; \tau\Big(\delta\big(\ln\rho - \ln E(\rho)\big)[\rho]\xi_0\Big)$$
$$= \; \big\langle\delta(\ln\rho), \xi_0\big\rangle_\rho \; .$$

By definition that means

$$\mathrm{grad}_\rho D_N \; = \; \delta(\ln\rho) \; . \tag{6.4}$$

Note that in [25] the inner product with the modified multiplication was exactly designed to satisfy this property. Moreover, we find that the corresponding tangent direction in the dual of the state space is given by

$$\delta^*([\rho]\,\mathrm{grad}_\rho D_N) \; = \; \delta^*([\rho]\delta(\ln\rho)) \; = \; \delta^*([\rho][\rho]^{-1}\delta(\rho)) \; = \; A(\rho) \; .$$

A curve γ in the state space is said to follow the path of *steepest descent* or gradient flow with respect to F if for any t,

$$\gamma'(t) = \delta^*([\gamma(t)]\,\mathrm{grad}_{\gamma(t)} F) \; .$$

This implies

$$\frac{\mathrm{d}F(\gamma(t))}{\mathrm{d}t} \; = \; -\|\,\mathrm{grad}_{\gamma(t)} F\|^2_{\gamma(t)} \; .$$

We denote $En(\rho) := \|\,\mathrm{grad}_\rho F\|^2_\rho$ as the energy function with respect to F. In our special case $F = D_N$, we find

$$En(\rho) = \|\,\mathrm{grad}_\rho D_N\|^2_\rho \; = \; \langle\delta(\ln\rho), \delta(\ln(\rho))\rangle_\rho \; = \; \big\langle[\rho]^{-1}\delta(\rho), [\rho][\rho]^{-1}\delta(\rho)\big\rangle_{L_2(\hat{M})}$$
$$= \tau\big(\delta(\rho)[\rho]^{-1}\delta(\rho)\big) \; = \; \tau\big(\rho\delta^*\delta(\ln\rho)\big) \; = \; \tau\big(\rho A(\ln(\rho))\big)$$
$$= \; \tau\big(A(\rho)\ln\rho\big) \; = \; \mathcal{I}_A(\rho) \; .$$

This means the pseudo-Riemannian metric is chosen so that the semigroup exactly follow the path of steepest descent with respect to $F = D_N$,

$$\frac{dD_N(T_t(\rho))}{dt}\Big|_{t=0} \; = \; -\mathcal{I}_A(\rho) \; = \; -\|\,\mathrm{grad}_\rho D_N\|^2_\rho \; .$$

With (6.2), we summarizes the above discussion as follows.

Proposition 6.5. *Suppose a differentiable curve* $\gamma : (a, b) \to S(M)$ *satisfies that*

$$\gamma'(t) \; = \; -A\big(\rho(t)\big)$$

Then, the curve γ follows the path of steepest descent with respect to D_N. In particular, the semigroup path $\gamma(t) = T_t(\rho)$ is a curve of steepest descent for D_N.

We include the following standard argument for completeness.

Lemma 6.6. *Let $F(\rho) = D_N(\rho)$ and $En(\rho) = \mathcal{I}_A(\rho)$. Let $\rho : [0, \infty) \to S(M)$ be a path of steepest descent with respect to F and $\lambda > 0$. Then,*

$$2\lambda F(\rho(t)) \; \leq \; En(\rho(t)) \quad \textit{implies} \quad F(\rho(t)) \; \leq \; e^{-2\lambda t}F(\rho(0))$$

Proof. According to the above discussion, we have

$$\rho'(t) = -A(\rho(t)) = -\delta^*\Big([\rho(t)]\,\mathrm{grad}_{\rho(t)}\,F\Big)\,,$$

$$\frac{\mathrm{d}F(\rho(t))}{\mathrm{d}t} = \langle\mathrm{grad}_{\rho(t)}\,F, -\mathrm{grad}_{\rho(t)}\,F\rangle_{\rho(t)} = -En(\rho(t))\,.$$

Then, our assumption implies that

$$\frac{\mathrm{d}F(\rho(t))}{\mathrm{d}t} = En(\rho(t)) \le -2\lambda F(\rho(t))$$

and hence $F(\rho(t)) \le e^{-2\lambda t}F(\rho(t))$ by Grönwall's Lemma. $\square$

Denote $S_+(M)$ be the space of all normal faithful states of M. The pseudo-Riemannian distance on $S_+(M)$ of our metric is given by

$$d_{A,2}(\rho,\sigma) = \inf\{L(\gamma) : \gamma(0) = \rho, \gamma(1) = \sigma\}$$

where the infimum runs over all piecewise smooth curve in $S(M)$ and the length function is defined by

$$L(\gamma) = \int_0^1 \|\gamma'(t)\|_{\mathrm{Tan}_\gamma(t)}\mathrm{d}t\,.$$

Thanks to Corollary 6.4 and the definition, we have the distance estimate

$$\|\rho - \sigma\|_{\Gamma^*} \le 2\sqrt{2}\,d_{A,2}(\rho,\sigma)\,. \tag{6.5}$$

The following result follows similarly from [25, Theorem 8.7] using the path of steepest descent and the relative entropy. Note that in [25] the modified log-Sobolev inequality is defined with constant 2λ.

Theorem 6.7. *The λ-MLSI inequality*

$$\lambda D\big(\rho\|E(\rho)\big) \le \mathcal{I}_A(\rho)$$

implies

$$d_{A,2}\big(\rho, E(\rho)\big) \le 2\sqrt{\frac{D(\rho\|E(\rho))}{\lambda}}\,. \tag{6.6}$$

We say the generator A satisfies λ-TA$_2$ if the above inequality (6.6) holds. Combining with the distance estimate (6.5), we have the following corollary of Γ-Lipschitz distance:

Corollary 6.8. *λ-MLSI implies*

$$\|\rho - E(\rho)\|_{\Gamma^*} \le 4\sqrt{\frac{2D(\rho\|E(\rho))}{\lambda}}$$

and

$$\|\rho_1 - \rho_2\|_{\Gamma^*} \le 4\sqrt{2}\left(\sqrt{\frac{D(\rho_1\|E(\rho_1))}{\lambda}} + \sqrt{\frac{D(\rho_2\|E(\rho_2))}{\lambda}}\right)\,.$$

The first inequality is just a combination of Corollary 6.4 and Theorem 6.7. For the second inequality, we observe that $\|E(\rho)\|_{\Gamma^*} = 0$, and hence the triangle inequality implies

$$\|\rho_1 - \rho_2\|_{\Gamma^*} \le \|\rho_1 - E(\rho_1)\|_{\Gamma^*} + \|\rho_2 - E(\rho_2)\|_{\Gamma^*} . \qquad \square$$

Remark 6.9. Let $e \in M$ be a projection. Then, $\rho_e = \frac{e}{\tau(e)}$ is the normalized state which satisfies

$$
\begin{aligned}
D(\rho_e\|E(\rho_e)) &= \tau(\rho_e \ln \rho_e) - \tau\big(E(\rho_e) \ln E(\rho_e)\big) \\
&\le -\ln \tau(e) - \tau(\rho_e) \ln \tau(\rho_e) = -\ln \tau(e) .
\end{aligned}
$$

Let $e_1, e_2 \in M$ be two projections. Assume that there exists a self-adjoint y such that

$$h \le \left| \frac{\tau(e_1 y)}{\tau(e_1)} - \frac{\tau(e_2 y)}{\tau(e_2)} \right| \quad \text{and} \quad \|\Gamma(y,y)\| \le 1 .$$

Then, we find the *geometric* version of Talagrand's inequality (see [93] and also [52])

$$\tau(e_1)\tau(e_2) \le e^{-\frac{\lambda h^2}{64}}$$

Indeed, this follows from Corollary 6.8

$$
\begin{aligned}
h &\le \|\rho_{e_1} - \rho_{e_2}\|_{\Gamma^*} \le 4\sqrt{2}\lambda^{-\frac{1}{2}}\big(\sqrt{-\ln \tau(e_1)} + \sqrt{-\ln \tau(e_2)}\big) \\
&\le 8\lambda^{-\frac{1}{2}}\sqrt{-\ln \tau(e_1) - \ln \tau(e_2)} .
\end{aligned}
$$

The constant 64 is probably not optimal in general.

6.2. Wasserstein 1-Distance and Concentration Inequalities

In [52], the commutative characterization of Wasserstein entropy estimates in terms of concentration inequalities was extended to the noncommutative setting. In the non-ergodic setting, we have the following result.

Theorem 6.10. *Let (M, τ) be a finite von Neumann algebra and (T_t) be a self-adjoint semigroup of completely positive trace reducing maps. Let N be the fixpoint subalgebra. Then, the following conditions are equivalent*

(i) *There exists a constant $C_1 > 0$ such that for all $p \ge 2$ and $f \in M$ with $E(f) = 0$,*

$$\|f\|_{L^p_\infty(N \subset M)} \le C_1 \sqrt{p}\|f\|_{Lip_\Gamma} ;$$

(ii) *There exists a constant $C_2 > 0$ such that for all normal states ρ*

$$\|\rho\|_{\Gamma^*} \le C_2 \sqrt{D(\rho\|E(\rho))} .$$

In the following, we say that (T_t) or its generator A *satisfies* λ-*WA$_1$* if

$$\|\rho\|_{\Gamma^*} \le 4\sqrt{2}\sqrt{\frac{D(\rho\|E(\rho))}{\lambda}} .$$

Note that the factor $4\sqrt{2}$ is chosen so that λ-MLSI implies λ-WA$_1$ (via λ-TA$_2$).

Proof. Fix $\frac{1}{p} + \frac{1}{p'} = 1$. Recall that the relative p-Rényi entropy

$$D_p(\rho||\sigma) \; = \; p' \ln \|\sigma^{-1/2p'} \rho \sigma^{-1/2p'}\|_p$$

is monotone over $p \in (1, \infty]$. Hence, $D_N^p(\rho) = \inf_{\sigma \in N, \tau(\sigma)=1, \sigma \geq 0} D_p(\rho||\sigma)$ satisfies

$$D(\rho||E(\rho)) \; \leq \; D_N^p(\rho) = p' \ln \|\rho\|_{L_1^p(N \subset M)} \; \leq \; \frac{p'}{\varepsilon} \|\rho\|_{L_1^p(N \subset M)}^\varepsilon .$$

for any $\varepsilon > 0$. Therefore, we deduce from (ii) that for positive ρ,

$$\|\rho\|_{\Gamma^*} \; \leq \; C \sqrt{\frac{p'}{\varepsilon}} \|\rho\|_1^{1-\frac{\varepsilon}{2}} \|\rho\|_{L_1^p}^{\frac{\varepsilon}{2}} .$$

Let us consider $\frac{1}{q} = \frac{1}{8} + \frac{7}{8p}$. We improve upon the well-known inequality

$$[L_\infty, L_\infty^{p'}]_{1/4,\infty} \subset [L_\infty, L_\infty^{p'}]_{7/8} \; = \; L_\infty^{q'}$$

by using the modified four term K_t functional from [JP05]. Indeed, $\|\alpha\|_{2q'} \leq 1$ implies

$$\|\sum_{j=1}^n \pi_j(\alpha^* x \alpha)\|_{L_{p'}(M, \ell_\infty^n)} \; \leq \; \|\alpha\|_{2p'} \|\sum_{j=1}^n \pi_j(x)\|_{L_\infty^{p'}(\ell_\infty^n)} .$$

By duality, for every $\rho = \rho^* \in L_1^p(N \subset M)$ of norm < 1, we can find a decomposition $\rho = \sum_j \lambda_j y_j$, $\sum_j \lambda_j \leq c$ and $y_j = a_j b_j$ such that $\|a_j a_j^*\|_1 + t_j \|a_j a_j^*\|_{L^p} \leq t_j^{1/4}$. The same estimate with the same t_j holds for $b_j^* b_j$. Using the positive 2×2 matrix $Y_j = \begin{pmatrix} a_j a_j^* & a_j b_j \\ b_j^* a_j & b_j^* b_j \end{pmatrix}$ we deduce

$$\|y_j\|_{\Gamma^*} \leq \|Y_j\|_{\Gamma^*} \leq 4C\sqrt{p'} \|Y_j\|_1^{3/4} \|Y_j\|_{L^p}^{1/4} \leq 4C\sqrt{p'} t_j^{3/16} t_j^{-3/16} \leq 4C\sqrt{p'}.$$

By convexity we deduce that

$$\|\rho\|_{\Gamma^*} \; = \; \|\sum_j \lambda_j y_j\|_{\Gamma^*} \; \leq \; 4c_0 \sqrt{p'} \|\rho\|_{L_1^q}.$$

By duality we deduce that for $E(f) = 0$ we have

$$\|f\|_{L_{q'}(N \subset M)} \; \leq \; 8c_0 \sqrt{2} \sqrt{p'} C \|f\|_\Gamma .$$

However, for a given $q' < \infty$ we may always choose $p' = \frac{8}{7}q'$, and hence replace $\sqrt{p'}$ by $\sqrt{\frac{8}{7}}\sqrt{q'}$. This concludes the proof of $ii) \Rightarrow i)$. Conversely, again we have by [44] that for $\frac{1}{q} = \frac{\varepsilon}{2s}$ the inclusion

$$L_\infty^q(N \subset M) \; = \; [L_\infty(M), L_\infty^s(N \subset M)]_{\varepsilon/2} \subset [L_\infty(M), L_\infty^s(N \subset M)]_{\varepsilon/2,\infty}$$

is bounded by a universal constant. This means i) implies that

$$\|f\|_{[L_\infty(M), L_\infty^s(N \subset M)]_{\varepsilon/2,\infty}} \; \leq \; C\sqrt{q} \|f\|_\Gamma \; = \; \sqrt{2}C \sqrt{\frac{s}{\varepsilon}} \|f\|_{Lip_\Gamma} .$$

By duality we deduce for $s = p'$ that

$$\|\rho\|_{\Gamma^*} \; \leq \; \sqrt{2e}C \sqrt{\frac{p'}{e\varepsilon}} \|\rho\|_1^{1-\frac{\varepsilon}{2}} \|\rho\|_{L_1^p(N \subset M)}^{\frac{\varepsilon}{2}} .$$

Thus, for a state ρ such that $D(\rho||E(\rho)) < \infty$, we may choose $\varepsilon = (\ln \|\rho\|_{L_1^p(N \subset M)})^{-1}$ and obtain that

$$\|\rho\|_{\Gamma^*} \leq \sqrt{2e}C\sqrt{p' \ln \|\rho\|_{L_1^p(N \subset M)}} \ .$$

By sending $p \to 1$, we deduce (ii). $\qquad\square$

Remark 6.11. Recall that in the ergodic case $N = \mathbb{C}1$, the relative entropy coincides with the entropy functional $\mathrm{Ent}(\rho) := \tau(\rho \ln \rho) = D(\rho||1)$. It was proved in [52] that $\|\rho\|_{\Gamma^*} \leq C\sqrt{\mathrm{Ent}(\rho)}$ for all density ρ is equivalent to that for $p \geq 2$,

$$\|f - E(f)\|_{L_p(M)} \leq C'\sqrt{p}\|f\|_{Lip_\Gamma} \ .$$

In that sense, the estimate in the non-ergodic case with respect to D_N is significantly stronger, because the inclusion $L_\infty^p(N \subset M) \subset L_p(M)$ is contractive for all finite M.

Lemma 6.12. *For a positive density ρ,*

$$D(\rho||E(\rho)) = \sup_\sigma \tau\Big(\rho\big(\ln \sigma - \ln E(\sigma)\big)\Big)$$

where the supremum is taken over all positive density σ with bounded inverse.

Proof. Using the convexity of $F(\rho) = D(\rho||E(\rho))$, we know that

$$F(\rho) \geq F(\sigma) + F'(\sigma)(\rho - \sigma)$$

We observed in (6.3) that the total derivative is

$$F'(\sigma)(\beta) = \tau\Big(\beta(\ln \sigma - \ln E(\sigma))\Big)$$

Then, for $\beta = \rho - \sigma$,

$$\begin{aligned} F(\sigma) + F'(\sigma)(\rho - \sigma) &= \tau(\sigma \ln \sigma - \sigma \ln E(\sigma)) + \tau\Big((\rho - \sigma)(\ln \sigma - \ln E(\sigma))\Big) \\ &= \tau(\rho \ln \sigma - \rho \ln E(\sigma)) \ , \end{aligned}$$

which proves one direction. Conversely, we have equality for $\sigma = \rho$ as states, and hence, homogeneity implies the assertion for strictly positive ρ. Note also that we may replace σ by $\sigma + \epsilon 1$ to guarantee that the relative entropy is well-defined. The extra scaling factor $\tau(\sigma) + \varepsilon$ cancels thanks to the logarithm. $\qquad\square$

Proposition 6.13. *Let (T_t) be a semigroup as in* Theorem 6.10. *The condition*

(iii) *There exists a $c > 0$ such that*

$$E_N(e^{tf}) \leq e^{ct^2}$$

for all self-adjoint f with $\Gamma(f,f) \leq 1$, $E_N(f) = 0$ and $t > 0$.

implies λ-WA_1 for some λ. If in addition N is contained in the center of M, then (iii) *is equivalent to WA_1.*

Proof. Let us assume that (iii) holds and that $f = f^*$ satisfies $\Gamma(f, f) \leq 1$. We define $\rho = \frac{e^{tf}}{\tau(e^{tf})}$ and deduce that for every state ψ

$$D(\psi \| E(\psi)) \geq \tau\Big(\psi\big(\ln \rho - \ln E(\rho)\big)\Big) = \tau\Big(\psi\big(tf - \ln E(e^{tf})\big)\Big).$$

This implies

$$\tau(\psi f) \leq \frac{D(\psi \| E(\psi))}{t} + \frac{\tau(\psi \ln E(e^{tf}))}{t} \leq \frac{D(\psi \| E(\psi))}{t} + ct.$$

Now we may choose $t = \sqrt{\frac{D(\psi \| E(\psi))}{c}}$ to deduce the condition (ii) in Theorem 6.10 with constant $C = 2\sqrt{c}$. For the converse, we assume that N is in the center, $f = f^*$ and $\Gamma(f, f) \leq 1$ and $E(f) = 0$. Then, we deduce from condition (i) in Theorem 6.10 that

$$\tau(\sigma f) \leq \tau(\sigma |f|^p)^{1/p} = \|\sigma^{1/2p} f \sigma^{1/2p}\|_p \leq C\sqrt{p}$$

for all $\sigma \in N_+$, $\tau(\sigma) = 1$. $E(f) = 0$ implies that the first-order term in the exponential expansion vanishes and hence

$$\tau(\sigma E(e^{tf})) \leq 1 + \sum_{k \geq 2} \frac{(Ct)^k \sqrt{k}^k}{k!} \leq 1 + \sum_{k \geq 2} \frac{(Cet)^k}{k^{k/2}} \leq 1 + \sum_{j=1}^{\infty} \frac{(KCet)^{2j}}{j!}.$$

Here, we use that for $k = 2j$ we have $(2j)^{\frac{2j}{2}} \geq 2^j j^j \geq j!$. A slightly more involved estimate works for $k = 2j - 1$, $j \geq 2$ and leads to the constant K. $\qquad \square$

Let us recall the definition of the Orlicz space $L_\Phi(M, \tau)$ of a Young function Φ by the Luxembourg norm

$$\|x\|_{L_\Phi} = \inf\left\{\nu | \tau\left(\Phi\left(\frac{|x|}{\nu}\right)\right) \leq 1\right\}.$$

It is well known that for the convex function $\mathrm{Exp}_2(t) = e^{t^2} - 1$, the Orlicz norm $\|x\|_{L_{\mathrm{Exp}_2}}$ is equivalent to $\sup_{p \geq 2} \frac{\|x\|_p}{\sqrt{p}}$.

Corollary 6.14. *Assume that the generator A satisfies λ-WA_1. Then,*

$$\|f\|_{L_{Exp_2}} \leq K\lambda^{-2}\|f\|_{Lip_\Gamma}$$

holds for all f with $E(f) = 0$ and some universal constant K.

Proof. Indeed, we have two ways of proving this. By Hölder's inequality, we have a contraction

$$L_\infty^p(N \subset M) \subset L_p(M).$$

Then, the assertion follows from the equivalence in Theorem 6.10. On the other hand, we note that λ-WA_1 implies that

$$\|\rho\|_{\Gamma^*} \leq 2\sqrt{\frac{\mathrm{Ent}(\rho)}{\lambda}}.$$

Then, Remark 6.11 also implies the assertion. $\qquad \square$

Remark 6.15. Similar concentration inequalities for a fixed reference state σ can be found in [84]. They deduced an estimate for $\tau(\sigma e^f)$ using the gradient norm of $\sigma^{1/2} f \sigma^{-1/2}$. Here, we also need information for σ^{-1}, unless N is central.

In [49], the cb-version of having finite diameter was used for approximation by finite-dimensional systems. It is shown that the famous rotation algebras A_θ have finite *cb*-diameter. Let us recall that for the intrinsic metric $\|f\|_{Lip_\Gamma} \simeq \|\delta(f)\|$ one can define a natural operator space structure as intersection of a column and a row space in a Hilbert C^*-module, or cb-equivalently as a subspace $\delta(\text{dom}(A^{1/2})) \subset \hat{M}$. Thus, it makes sense to say that $(\mathcal{A}, \| \|_{Lip_\Gamma}, M)$ has *finite* cb-diameter D_{cb} if

$$\|I - E_N : (\mathcal{A}, \| \|_{Lip_\Gamma}) \to M\|_{cb} \leq D_{cb} .$$

Corollary 6.16. *Let A be a generator of a self-adjoint semigroup on a finite von Neumann algebra M. If $(\mathcal{A}, \| \|_{Lip_\Gamma})$ as a quantum metric space has finite cb-diameter, then A satisfies WA_1 for $A \otimes id_{\mathbb{M}_m}$ for all $m \in \mathbb{N}$ and $A \otimes id_{\tilde{M}}$ for any finite von Neumann algebra $\tilde{M}$.*

Proof. We just have to note that the inclusion

$$L_\infty(M \bar{\otimes} \tilde{M}) \subset L_\infty^p(N \bar{\otimes} \tilde{M} \subset M \bar{\otimes} \tilde{M})$$

is a contraction. Then, Theorem 6.10 implies the assertion. In particular, the norm from the Lipschitz functions to L_∞^p space is smaller than $2\sqrt{p} D_{cb}$. $\square$

We see that both conditions λ-CLSI and $D_{cb} < \infty$ imply λ-WA$_1$ on all matrix levels. We say a semigroup (T_t) or its generator A satisfies λ-CWA$_1$ if for all n, $id_{M_n} \otimes T_t$ satisfies λ-WA$_1$. Note that according to Remark 6.9, λ-CWA$_1$ implies the geometric Talagrand's inequality on all matrix levels, which we will call matricial Talagrand's inequality.

Let $(\mathcal{M}, g)$ be a d-dimensional compact Riemannian manifold with sub-Laplacian Δ_X and sub-Riemannian (or Carnot–Caratheodory) metric d_X induced by a Hörmander system X (see [85]). This gives a corresponding gradient form:

$$\Gamma_X(f, f) = \sum_{j=1}^k |X_j(f)|^2 .$$

For matrix-valued functions $f : \mathcal{M} \to \mathbb{M}_m$, the natural operator space structure is given by

$$\|f\|_{\mathbb{M}_m(Lip_\Gamma)} = \max\left\{ \|\sum_j |X_j(f)|^2\|^{1/2}, \|\sum_j |X_j(f)^*|^2\|^{1/2} \right\} .$$

Thanks to Voiculescu's inequality, this is equivalent to

$$\|f\|_{\mathbb{M}_m(Lip_\Gamma)} \sim \|\sum_j g_j \otimes X_j(f)\|$$

where g_j are freely independent semicircular (or circular) random variables ([81]). For matrix-valued functions, it is therefore better to use the free Dirac operator $D = \sum_j g_j \otimes X_j$ and the Laplace–Beltrami operator in contrast to the spin Dirac operator $D = \sum_j c_j \otimes X_j$ which is more common in noncommutative geometry [27]. Let us now consider a manifold $\mathcal{M}$ with finite diameter $\mathrm{diam}_X(\mathcal{M}) = \sup_{x,y} d_X(x,y)$ and a normalized volume form μ. Here, d_X is the Carnot–Caratheodory distance given by the Hörmander system. Let $f : \mathcal{M} \to M$ be an M-valued Lipschitz function. Let $h, k \in L_2(M)$. Then, $f_{h,k}(x) = (h, f(x)k)$ is a complex valued function and hence (following Connes' [27])

$$
|(h, (f(x) - \mathbb{E}_{\mathcal{M}} f)k)| = |f_{h,k}(x) - \int_{\mathcal{M}} f_{h,k}(y) \mathrm{d}\mu(y)|
$$

$$
\leq \int_{\mathcal{M}} |f_{h,k}(x) - f_{h,k}(y)| \mathrm{d}\mu(y) \leq \mathrm{diam}_X(\mathcal{M}) \sup_z \left| \left(\sum_j |X_j f_{h,k}(z)|^2 \right)^{1/2} \right|
$$

$$
= \mathrm{diam}_X(\mathcal{M}) \sup_z \left(\sum_j \langle h, X_j(f)(z)k \rangle \right)^{1/2}
$$

$$
\leq \mathrm{diam}_X(\mathcal{M}) \|h\| \|k\| \| \sum_j |X_j(f)|^2 \|^{1/2} .
$$

Actually, the inequality $|f(x) - f(y)| \leq \|f\|_{Lip} d_X(x, y)$ follows directly from the definition of the distance using connecting path. Therefore, we have shown the following easy fact:

Lemma 6.17. *Let Δ_X be the sub-Laplacian on $\mathcal{M}$ given by a Hörmander system X. Then,*

$$
D_{cb}(\Delta_X) \leq \mathrm{diam}_X(\mathcal{M}) .
$$

Theorem 6.18. *Let X be a Hörmander system on a connected compact Riemannian manifold. Then, Δ_X satisfies CWA_1.*

Proof. According to the Chow-Rashevskii theorem (see [85, Theorem 3.29] and [83]), the Carnot–Caratheodory distance $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is continuous with respect to the original topology of the Riemannian metric. Thus, by compactness $\mathrm{diam}_X(\mathcal{M})$ is finite. Then, Corollary 6.16 and Lemma 6.17 imply the assertion. $\square$

Corollary 6.19. *Let $\mathcal{L}$ be the generator of a self-adjoint semigroup on $\mathbb{M}_m$. Then, $\mathcal{L}$ satisfies CWA_1.*

Proof. According to Lemma 5.1, we find a connected compact Lie group G and a generating set X of $\mathfrak{g}$ such that the transference Theorem 4.8 applies. That is, via the co-representation $\pi(x)(g) = u(g)xu(g)^{-1}$, $e^{-t\mathcal{L}}$ is a subdynamical system of $e^{-t\Delta_X} \otimes id_{\mathbb{M}_m}$. According to Theorem 6.18, we know that Δ_X has $\lambda(X)$-CWA_1 for some constant $\lambda(X)$, and hence, $\mathcal{L}$ inherits this property (compare to Proposition 4.7). $\square$

Remark 6.20. We conjecture that on compact Riemannian manifolds the Laplace–Beltrami operator satisfies λ-CLSI. However, since $\Gamma\mathcal{E}$ fails in general, new techniques will be needed to approach this problem.

We end up this section with the "complete" analogues of the 'triangle' inequality for the Wasserstein 1-distance from [70,71].

Proposition 6.21. *Assume that (M_j, Γ_{A_j}) satisfies C_j^{-1}-CWA$_1$ for $j = 1, \ldots, n$. Let $E = E_1 \otimes \cdots \otimes E_n : \otimes_{j=1}^n M_j \to \otimes_{j=1}^n N_j$ be the conditional expectation onto the fixpoint algebra of tensor product. Then, for any density operator ψ and self-adjoint f in $\otimes_{j=1}^n M_j$ with $E(f) = 0$,*

$$|\tau(\psi f)| \leq 2\sqrt{2}\sqrt{D(\psi)\|E(\psi)\|} \left(\sum_j C_j \|\Gamma_{A_j}(f, f)\|^2\right)^{1/2}.$$

In particular, the tensor product generator $A(n) = A_1 \otimes I \otimes \cdots \otimes I + I \otimes A_2 \otimes \cdots \otimes I + I \otimes \cdots \otimes A_n$ satisfies C-CWA$_1$ for $C = (\sum_j C_j)^{-1/2}$.

Proof. We use a martingale argument by denoting

$$\tilde{E}_j : M_1 \otimes \cdots \otimes M_n \to M_1 \otimes \cdots \otimes M_j \otimes N_{j+1} \otimes \cdots \otimes N_n ,$$

the corresponding conditional expectation and $\tilde{E}_0 = E, \tilde{E}_n = id_{\otimes_{j=1}^n M_j}$. Let f be a mean 0 element and write $d_j(f) = \tilde{E}_j(f) - \tilde{E}_{j-1}(f)$. The gradient form Γ_{A_j} trivially extends to the tensor product, by identifying

$$A_j = I \otimes \cdots \otimes A_j \otimes \cdots \otimes I \tag{6.7}$$

with the generator applied in the jth-component. For a positive ψ, we deduce from the CWA property that

$$|\tau(\psi f)| \leq \sum_j |\tau(\psi d_j(f))| = \sum_j |\tau(\tilde{E}_j(\psi)d_j(f))|$$

$$\leq 2\sqrt{2}\sum_j \sqrt{C_j D(\tilde{E}_j(\psi)\|\tilde{E}_{j-1}(\psi))} \|\Gamma_{A_j}(d_j(f), d_j(f))\|^{1/2}$$

$$\leq 2\sqrt{2}\left(\sum_j D\left(\tilde{E}_j(\psi)\|\tilde{E}_{j-1}(\psi)\right)\right)^{1/2} \left(\sum_j C_j \|\Gamma_{A_j}(d_j(f), d_j(f))\|\right)^{1/2}.$$

Note, however, that $\tilde{E} = \tilde{E} \circ \tilde{E}_j$ implies

$$D(\psi\|E(\psi)) = \sum_{j=1}^n D(\tilde{E}_j(\psi)\|\tilde{E}_{j-1}(\psi)) .$$

For each A_j, we have $\Gamma_{A_j}(x, x) = E_j(\delta_j(x)^*\delta_j(x))$ for the self-adjoint derivation δ_j from Theorem 2.1 extended canonically to the tensor product. Recall that the derivation δ_j only depends on the j-the tensor component and hence commutes with $\tilde{E}_j$. By Kadison's inequality,

$$\Gamma_{A_j}\left(d_j(f), d_j(f)\right)$$
$$= \Gamma_{A_j}\left(\tilde{E}_j(f) - \tilde{E}_{j-1}(f), \tilde{E}_j(f) - \tilde{E}_{j-1}(f)\right) = \Gamma_{A_j}\left(\tilde{E}_j(f), \tilde{E}_j(f)\right)$$

$$= E_j\Big(\delta_j\big(\tilde{E}_j(f)\big)^*\delta_j\big(\tilde{E}_j(f)\big)\Big) \;=\; E_j\Big(\tilde{E}_j\big(\delta_j(f)\big)^*\tilde{E}_j\big(\delta_j(f)\big)\Big)$$
$$\leq E_j\big(\tilde{E}_j(\delta_j(f)^*\delta_j(f))\big) \;=\; \tilde{E}_j\big(\Gamma_{A_j}(f,f)\big)$$

Taking norms implies the first assertion. For the second we just observe that for each j

$$\|\Gamma_{A_j}(f,f)\| \;\leq\; \|\sum_j \Gamma_{A_j}(f,f)\|$$

holds by positivity. $\qquad\qquad\square$

This implies that if T_t satisfies λ-CWA$_1$, its tensor product $T_t^{\otimes n}$ satisfies $\frac{\lambda}{n}$-CWA$_1$. This is enough to imply Talagrand's inequality for matrix-valued functions on $\{-1,1\}^n$ and $[-1,1]^n$, see [62] for details in the scalar case.

7. Examples and Counterexamples

This section discusses examples of $\Gamma\mathcal{E}$ and CLSI and some related counterexamples. Section 7.1 proves the stability of $\Gamma\mathcal{E}$ with respect to free products. Section 7.2 considers $\Gamma\mathcal{E}$ for the graph Laplacian on weighted finite graphs. Section 7.3 discusses the Schur multiplier semigroup on group von Neumann algebras. We also provide counterexamples of additivity of Fisher information in Sect.7.4. Section 7.5 gives a counterexample of Rothaus lemma for matrices. We end up the discussion with a summary and some open questions in Sect. 7.6.

7.1. Free Products

Following the lead of [52], we discuss the stability of $\Gamma\mathcal{E}$ with respect to free products. We refer to [98] for the definition and general facts on amalgamated free products. Let $N \subset M_j$ be finite von Neumann algebras with trace preserving conditional expectation $E_j : M_j \to N$. According to [18], a family of unital completely positive $T^j : M_j \to M_j$ that leave N invariant can be extended to the free product with amalgamation $M = *_N^j M_j$ via

$$T(a_1 \cdots a_m) \;=\; T^{i_1}(a_1) \cdots T^{i_m}(a_m)$$

provided $a_j \in M_{i_j}$ and $i_1 \neq i_2 \neq \cdots \neq i_m$.

Lemma 7.1. *Let $A_j, B_j, 1 \leq j \leq n$ be generator of self-adjoint semigroup on M_j with same fixpoint algebra $N \subset M_j$. Let $A(n)$ (resp. $B(n)$) be the generator of the free product semigroup $e^{-tA(n)} := *_{j=1}^n e^{-tA_j}$ (resp. $e^{-tB(n)} := *_{j=1}^n e^{-tA_j}$) on amalgamated free products. If $\Gamma_{A_j} \leq \Gamma_{B_j}$ for all j, then $\Gamma_{A(n)} \leq \Gamma_{B(n)}$.*

Proof. Let us briefly sketch the argument for readers familiar with free probability. For simplicity of notations, we assume that all the algebras $M_j = M$ are the same, and all the generators $A = A_j$ and $B = B_j$ are the same. Our first task is to identify the derivation for free product generator $A(n)$. Let δ_A be the derivation for A. We observe that $\delta_{A_j}(xb) = x\delta_{A_j}(b)$ holds for $x \in N$.

Let us recall from [51, Proposition 2.8] that for the conditional expectation $E : M \to N$, there exists a right N-module map $u : L_\infty^c(N \subset M) \to L_\infty(N, l_2^c)$ such that

$$u(x) = (u_j(x))_j , \ E(x^*y) = u(x)^* u(y) := \sum_j u_j(x)^* u_j(y) .$$

For a word $\omega = a_1 \cdots a_m$ so that $a_j \in M_{i_j}$ are mean zero element and $i_1 \neq i_2 \neq \cdots \neq i_m$, we define the vectors $\xi_l = (e_{i_1}, \ldots, e_{i_l}) \in \ell_2(\mathbb{N}^l)$ for each $1 \leq l \leq m$. Then, the derivation of $A(n)$ can be defined as follows,

$$v(\omega) \ = \ \sum_l v_l(\omega) := \ \sum_{l=1}^m \xi_l \otimes \Big(u(a_1) \otimes \cdots \otimes u(a_{l-1}) \Big) \delta_A(a_l) a_{l+1} \cdots a_m .$$

Here we view

$$u(a_1) \otimes \cdots \otimes u(a_{l-1}) = \Big(u_{j_1}(a_1) \cdots u_{j_{l-1}}(a_{l-1}) \Big)_{(j_1, \ldots, j_{l-1})} \in L_\infty(N, l_2^c(\mathbb{N}^{l-1}))$$

and one can check that

$$\Big(u(b_{l-1}) \otimes \cdots \otimes u(b_1) \Big)^* \Big(u(a_1) \otimes \cdots \otimes u(a_{l-1}) \Big)$$
$$= E(b_{l-1} E(b_{l-2} \cdots E(b_1 a_1) \cdots a_{l-2}) a_{l-1})$$
$$= E(b_{l-1} \cdots b_1 a_1 \cdots a_{l-1})$$

To explain the cancellation in the gradient form, let us consider the example $\omega = b_1^* b_2^*$, $b_k \in A_{l_k}$, and $\omega' = a_1 a_2 a_3$, $a_k \in A_{r_k}$. We use the notation $a^\circ = a - E(a)$ for the mean 0 part. We have the following decomposition in terms of mean 0 words,

$$\omega^* \omega' = b_2 b_1 a_1 a_2 a_3$$
$$= b_2 (b_1 a_1)^\circ a_2 a_3 + b_2 E(b_1 a_2) a_2 a_3$$
$$= b_2 (b_1 a_1)^\circ a_2 a_3 + (b_2 E(b_1 a_1) a_2)^\circ a_3 + E_N(b_2 b_1 a_1 a_2) a_3 .$$

Here the second equality holds if $l_1 = r_1$ and the third equality holds if $l_1 = r_1$ and $l_2 = r_2$. If $l_1 \neq r_1$ and $l_2 \neq r_2$ we think $(b_1 a_1)^\circ = b_1 a_1$, $(b_2 a_2)^\circ = b_2 a_2$ the latter terms vanish. Then, for the free product generator $A(n)$

$$A(n)(\omega^* \omega')$$
$$= A(n)(b_2 (b_1 a_1)^\circ a_2 a_3) + A(n)((b_2 E(b_1 a_1) a_2)^\circ a_3) + A(n)(E(b_2 b_2 a_1 a_2) a_3)$$
$$= A(b_2)(b_1 a_1)^\circ a_2 a_3 + b_2 A(b_1 a_1) a_2 a_3 + b_2 (b_1 a_1)^\circ A(a_2) a_3 + b_2 (b_1 a_1)^\circ a_2 A(a_3)$$
$$\quad + A((b_2 E(b_1 a_1) a_2)^\circ) a_3 + (b_2 E(b_1 a_1) a_2)^\circ A(a_3) + E(b_2 b_2 a_1 a_2) A(a_3) .$$

Recall that $2\Gamma(\omega, \omega') = A(\omega^*) \omega' + \omega^* A(\omega') - A(\omega^* \omega')$. We calculate

$$A(n)(\omega^*) \omega' + \omega^* A(n)(\omega')$$
$$= A(b_2) b_1 a_1 a_2 a_3 + b_2 A(b_1) a_1 a_2 a_3 + b_2 b_1 A(a_1) a_2 a_3 + b_2 b_1 a_1 A(a_2) + b_2 b_1 a_1 a_2 A(a_3)$$
$$= A(b_2)(b_1 a_1)^\circ a_2 a_3 + A(b_2) E(b_1 a_1) a_2 a_3 + b_2 (b_1 a_1)^\circ A(a_2) a_3 + b_2 E(b_1 a_1) A(a_2) a_3$$
$$\quad + b_2 (b_1 a_1)^\circ a_2 A(a_3) + b_2 E(b_1 a_1) a_2 A(a_3) + b_2 A(b_1) a_1 a_2 a_3 + b_2 b_1 A(a_1) a_2 a_3$$

We first observe that the $A(a_3)$ terms cancel, because they can not interact with anything from b. If $l_1 = r_1$ and $r_2 = l_2$ we have the additional term

$$A(b_2)E(b_1a_1)a_2a_3 + b_2E(b_1a_1)A(a_2)a_3 - A((b_2E(b_1a_1)a_2)^\circ)a_3$$
$$= \Gamma_A(b_2^*, E(b_1a_1)a_2)a_3 .$$

So the gradient form of $A(n)$ is

$$\Gamma_{A(n)}(\omega, \omega') = \begin{cases} 0, & \text{if } l_1 \neq r_1 \\ b_2\Gamma_A(b_1^*, a_1)a_2a_3, & \text{if } l_1 = r_1, l_2 \neq r_2 \\ b_2\Gamma_A(b_1^*, a_1)a_2a_3 + \Gamma_A(b_2^*, E(b_1a_1)a_2)a_3, & \text{if } l_1 = r_1, l_2 = r_2. \end{cases}$$

and for each case $\Gamma_{A(n)}(\omega, \omega') = v(\omega)^*v(\omega')$. In full generality, we have to use an inductive procedure and obtain

$$\Gamma(b_1 \cdots b_m, a_1 \cdots a_n) = v(b_1 \cdots b_m)^*v(a_1 \cdots a_n) .$$

For a word $\omega \in A_{i_1}^\circ \cdots A_{i_m}^\circ$, we denote $\sigma_l(\omega) = (i_1, \ldots, i_l)$ for the first l indices. Let $x = \sum_{(i_1, \cdots, i_m)} \omega(i_1, \ldots, i_m)$ with $\omega(i_1, \ldots, i_m)$ in the linear span $A_{i_1}^\circ \cdots A_{i_m}^\circ$. Then

$$\Gamma_{A(n)}(x, x) = \sum_{\omega, \omega'} \sum_l \delta_{\sigma_l(\omega), \sigma_l(\omega')} v_l^A(\omega)^* v_l^A(\omega')$$

$$\leq \sum_{\omega, \omega'} \sum_l \delta_{\sigma_l(\omega), \sigma_l(\omega')} v_l^B(\omega)^* v_l^B(\omega')$$

$$= \Gamma_{B(n)}(x, x) .$$

Here for each l we have used that for any fixed $\alpha_k, \beta_k, \gamma_k \in M$,

$$\sum_{k,k'} \beta_k^* \Gamma_A(\alpha_k, E(\gamma_k^*\gamma_{k'})\alpha_{k'})\beta_{k'} = \sum_{k,k'} \sum_j \beta_k^* \Gamma_A(w_j(\gamma_k)\alpha_k, w_j(\gamma_{k'})\alpha_{k'})\beta_{k'}$$

$$\leq \sum_{k,k'} \sum_j \beta_k^* \Gamma_B(w_j(\gamma_k)\alpha_k, w_j(\gamma_{k'})\alpha_{k'})\beta_{k'}$$

$$= \sum_{k,k'} \beta_k^* \Gamma_B(\alpha_k, E(\gamma_k^*\gamma_{k'})\alpha_{k'})\beta_{k'} .$$

which follows from the assumption $\Gamma_A \leq_{cp} \Gamma_B$. $\square$

A particularly interesting case is given by $T_t^j = e^{-t(I-E_N)}$ for all j. The corresponding free product semigroup is the so-called block-length semigroup

$$T_t(a_1 \cdots a_n) = e^{-tn} a_1 \cdots a_n$$

for all (free) products of mean 0 terms $a_1, \ldots, a_n$. Note that here we use $E : M \to N$ for the conditional expectation on a single component and $E_N : \ast_{j=1}^n M \to N$ on the free product.

Lemma 7.2. *Let $A(n)$ be the generator of the block-length semigroup Then*

$$\frac{1}{n}\Gamma_{I-E} \leq \Gamma_{A(n)} .$$

Proof. For $I - E$ the gradient form

$$2\Gamma_{I-E}(x,y) = (x - E(x))^*(y - E(y)) + E((x - E(x))^*(y - E(y)))$$
$$= v_1(x)^*v_1(x) + v_2(x)v_2(x)$$

splits into two forms $v_1(x) = (x - E(x))$ and $v_2(x) = u(x - E(x))$. Therefore, of we may use our argument from above and find two orthogonal forms

$$v^1(a_1 \cdots a_m) = \sum_{l=1}^{m} \xi_l \otimes u(a_1 \otimes \cdots \otimes a_{l-1})a_l \cdots a_m$$

$$v^2(a_1 \cdots a_m) = \sum_{l=1}^{m} \xi_l \otimes u(a_1 \otimes \cdots \otimes a_l)a_{l+1} \cdots a_m ,$$

and

$$2\Gamma_{A(n)}(\omega, \omega') = v^1(\omega)^*v^1(\omega') + v^2(\omega)^*v^2(\omega') .$$

Let $x = \sum_{(i_1,\ldots,i_m)} \omega(i_1,\ldots,i_m)$ with $\omega(i_1,\ldots,i_m)$ in the linear span of $A_{i_1}^\circ \cdots A_{i_m}^\circ$. We deduce from the mean 0 property of the products that

$$E_N(x^*x) = \sum_{\omega} E(\omega^*\omega) \leq |v^2(x)|^2 . \tag{7.1}$$

Moreover, let P_j be the projection onto words starting with $i_1 = j$. Then, we see that for mean 0 words

$$x^*x = \sum_{j,k} P_j(x)^*P_k(x) \leq n\sum_j P_j(x)^*P_j(x)$$

$$\leq n\left(\sum_{l \geq 1, \omega, \omega'} \delta_{\sigma_l(\omega),\sigma_l(\omega')} v_l^1(\omega)^*v_l^1(\omega')\right) \leq n|v^1(x)|^2 ,$$

because the term $l = 1$ exactly corresponds to $i_1 = i_1'$. Therefore, we find that for mean 0 element x,

$$\Gamma_{I-E_N}(x,x) = \frac{x^*x + E_N(x^*x)}{2}$$

$$\leq \frac{n}{2}|v^1(x)|^2 + \frac{1}{2}|v^2(x)|^2 \leq \frac{n}{2}(|v^1(x)|^2 + |v^2(x)|^2)$$

$$= n\Gamma_{(I-E)^{*n}}(x,x) .$$

By taking words a_j of length 1 and $E(a_j^*a_j)$ very small, we see that n is indeed optimal.   □

Combining Lemma 7.1 and Lemma 7.2, we have the similar result for the general free products.

Theorem 7.3. *Let A_j be generators such that*

$$\lambda\Gamma_{I-E} \leq_{cp} \Gamma_{A_j}$$

holds for $j = 1, \ldots, n$. Then, the generator $A(n)$ of the free product semigroup $*_{j=1}^{n} T_t^j$ satisfies

$$\frac{\lambda}{n} \Gamma_{I-E_N} \leq_{cp} \Gamma_{A(n)} .$$

7.2. Graph Laplacians

Let (V, E) be a finite graph and $w : E \to \mathbb{R}_+$ be a positive symmetric weight function on edges. Let us consider the commutator derivation $\delta(a) = [\xi, a]$ with $\xi \in \mathbb{B}(l_2(V))$. The induced gradient form for $f \in l_\infty(V)$ is

$$\Gamma(f, f)(x) = \sum_y |\xi_{xy}|^2 |f(x) - f(y)|^2 . \tag{7.2}$$

is always given by a set of weights $w_{xy} = |\xi_{xy}|^2$. Using the uniform probability measure μ on $l_\infty(V)$, we have

$$\langle \delta(f_1), \delta(f_2) \rangle_\mu = \frac{1}{|V|} \left(2 \sum_{x \neq y} w_{xy} \bar{f}_1(x) f_2(x) - 2 \sum_{x \neq y} w_{xy} \bar{f}_1(x) f_2(y) \right) .$$

Thus we have a one to one correspondence between the weight w, the derivation $\delta(\cdot) = [\xi, \cdot]$ and the weighted graph Laplacian

$$A_{x,y} = \begin{cases} \sum_{y \neq x} w_{xy}, & \text{if } x = y \\ -w_{xy}, & \text{otherwise.} \end{cases}$$

When the graph is connected, the semigroup $T_t = e^{-At} : l_\infty(V) \to l_\infty(V)$ is ergodic because the only invariant elements are the constant functions on V. For this situation the conditional expectation $E : l_\infty(V) \to \mathbb{C}1$ is given by the trace $E(f) = \frac{1}{|V|} \sum_{i \in V} f(i)$ and the gradient form Γ_{I-E} corresponds to weights $w_{xy}^{I-E} = \frac{1}{2|V|}$. It follows from (7.2) that for two gradient forms $\Gamma_A \leq \Gamma_B$ if and only if the weights $w_{xy}^A \leq w_{xy}^B$ for any $x, y \in V$.

Corollary 7.4. *An ergodic graph Laplacian A satisfies λ-$\Gamma\mathcal{E}$ if and only if*

$$w_{xy} \geq \frac{\lambda}{2|V|} \quad \text{for all } x \neq y .$$

In particular, the weights $w_{xy}(\theta)$ for the subordinated generator A^θ are strictly positive.

Proof. For an ergodic graph Laplacian we have a spectral gap σ and $\|T_1 : l_1(V) \to l_\infty(V)\| \leq c_1 < \infty$. Thanks to Proposition 3.2 and Theorem 3.10, we obtain that A^θ satisfies $\lambda(\theta)$-$\Gamma\mathcal{E}$ for some $\lambda(\theta) > 0$ and hence $w_{xy}(\theta) \geq \frac{\lambda(\theta)}{2|V|}$ is strictly positive. Conversely, we know that by the above discussion that $w_{xy}(\theta) \geq \frac{\lambda}{2|V|}$ for all $x, y \in E$ implies $\lambda \Gamma_{I-E} \leq_{cp} \Gamma_A$. $\qquad\square$

Remark 7.5. Of course, we expect CLSI for every finite graph. The above corollary shows that a finite graph with positive weights has $\Gamma\mathcal{E}$ if and only if it is a complete graph. Also, it is not clear how expander graphs fit into this picture since they are in certain sense opposite to complete graphs (see [21] for more information).

7.3. Fourier Multiplier and Discrete Groups

In this part we discuss group von Neumann algebras and their Fourier multipliers. Let G be a discrete group and $L(G)$ be its group von Neumann algebra. Denote $\lambda(g)$ as the left shifting unitary of $g \in G$. We consider the multiplier semigroup

$$T_t : L(G) \to L(G) , \ T_t(\lambda(g)) = e^{-t\psi(g)}\lambda(g)$$

for a conditional negative function ψ (see [17,19] for more information). The gradient form is given by

$$\Gamma_\psi(\lambda(g), \lambda(h)) \ = \ K(g,h)\lambda(g^{-1}h) .$$

where K is the Gromov distance

$$2K_\psi(g,h) \ = \ \psi(g) + \psi(h) - \psi(g^{-1}h) ,$$

It is easy to see that for two multiplier generators ψ and $\tilde{\psi}$ the relation $\lambda\Gamma_{\tilde{\psi}} \leq_{cp} \Gamma_\psi$ is equivalent to $\lambda K_{\tilde{\psi}} \leq K_\psi$ in the usual order of matrices. The conditional expectation onto $\mathbb{C}$ is the canonical trace $E(\lambda(g)) = \tau(\lambda(g))1 = \delta_{g,1}1$. Then, $I - E$ is a Fourier multiplier and

$$K_{I-E}(g,h) \ = \ \frac{1}{2}(1 - \delta_{g,1})(1 - \delta_{h,1})(1 + \delta_{g,h}) .$$

It therefore suffices to consider the matrix K_{I-E} on $G\backslash\{1\}$. Let us now consider the specific example $G = \mathbb{Z}$ and $\psi(k) = |k|$ given by the Poisson semigroup. Then

$$K_\psi(k,j) \ = \ \frac{|k| + |j| - |k - j|}{2} \ = \ \begin{cases} 0 & \text{if } k < 0 < j \text{ or } j < 0 < k \\ \min(|j|, |k|) & \text{else.} \end{cases} .$$

Let $B(j,k) = \min(j,k)$ be the matrix on $l_2(\mathbb{N})$ and $\alpha = (\alpha_j) \in l_2(\mathbb{N})$ be a finite sequence. Then, we see that

$$(\alpha, B(\alpha)) \ = \ \sum_{j,k} \bar{\alpha}_j \alpha_k \min(j,k) \ = \ \sum_{j,k} \bar{\alpha}_j \alpha_k \sum_{1 \leq l \leq \min(j,k)} 1 \ = \ \sum_{l \geq 1} |\sum_{j \, \geq \, l} \alpha_j|^2 .$$

Using $|a - b|^2 \leq 2a^2 + 2b^2$, we deduce that

$$\sum_{l \geq 1} |\alpha_l|^2 \ = \ \sum_{l \geq 1} \left| \sum_{j \, \geq \, l} \alpha_j - \sum_{j > l} \alpha_j \right|^2$$

$$\leq 2 \sum_l \left(\left| \sum_{j \, \geq \, l} \alpha_j \right|^2 + \left| \sum_{j > l} \alpha_j \right|^2 \right) \leq 4 \, (\alpha, B\,(\alpha)) .$$

This means $4B \geq 1_\mathbb{N}$ where $1_\mathbb{N}$ is the identity matrix on $l_2(\mathbb{N})$. Note that K_ψ restricted on either the $j, k \geq 0$ part or the $j, k \leq 0$ part is a copy of B. Hence

$$2K_\psi \ = \ 2(B \oplus B) \ \geq \ \frac{1}{2}1_{\mathbb{Z}\backslash 0} .$$

where $1_{\mathbb{Z}\setminus 0}$ is the identity matrix on $l_2(\mathbb{Z}\setminus 0)$. On the other hand, let $\mathbb{1}$ denote the matrix with all entries 1 on $l_2(\mathbb{N})$. Then, we certainly have

$$B \geq \mathbb{1}_{\mathbb{N}}$$

and therefore

$$K_{\psi} = B \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq \frac{1}{2} B \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} 1_{\mathbb{Z}\setminus\{0\}} .$$

Because $2K_{I-E} = 1_{\mathbb{Z}\setminus 0} + 1_{\mathbb{Z}\setminus\{0\}}$, we deduce that

$$\frac{1}{3} K_{I-E} \leq K_{\psi} , \frac{1}{3} \Gamma_{I-E} \leq \Gamma_{\psi} .$$

Corollary 7.6. *The Poisson semigroup on* $L(\mathbb{Z}) = L_{\infty}(\mathbb{T})$ *satisfies* $\frac{1}{3}$-$\Gamma\mathcal{E}$, *and hence* $\frac{1}{3}$-$CLSI$. *In particular, the Fourier multiplier associated with* $\psi_n(k_1,\ldots,k_n) = \sum_{j=1}^{n} |k_j|$ *on* $\mathbb{Z}^n$ *satisfies* $\frac{1}{3}$-$CLSI$, *but not* $\Gamma\mathcal{E}$ *for* $n \geq 2$. *The free product* $L(\mathbb{F}_n)$ *with the word length function satisfies* $\frac{1}{3n}$-$\Gamma\mathcal{E}$.

Proof. Let $n = 2$ and define the sequence $\alpha = (\alpha_{j,k}) = (\varepsilon_j \varepsilon_k)$ in $l_2(\mathbb{Z}^2)$ so that

$$\sum_{j=1}^{m} \varepsilon_j = 0 , \sum_{j=1}^{m} \varepsilon_j^2 = m , \text{ and } \varepsilon_j = 0 \text{ if } j > m .$$

Then, for

$$(\alpha, K_{\psi_2}\alpha) = (\varepsilon, K_{\psi_1}\varepsilon)(\varepsilon, \mathbb{1}\varepsilon) + (\varepsilon, \mathbb{1}\varepsilon)(\varepsilon, K_{\psi_1}\varepsilon) = 0 .$$

On the other hand $(\alpha, 1_{\mathbb{N}^2}\alpha) = m^2$ and $K_{I-E} \geq \frac{1}{2} 1_{\mathbb{N}^2}$. Here $\Gamma\mathcal{E}$ cannot holds for any constant for ψ_2. For free group, $L(\mathbb{F}_n) = *_{j=1}^{n} L(\mathbb{Z})$ and moreover the free product of Poisson semigroup is the Poisson semigroup of $L(\mathbb{F}_n)$. Then, the last fact follows from Theorem 7.2. $\qquad\square$

Proposition 7.7. *Let* $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ *be the cyclic group of cardinality* $n \in \mathbb{N}$ *and the multiplier function be* $\psi(j) = (1 - \cos(\frac{2\pi j}{n}))$. *Then,* A_{ψ} *satisfies 1-$\Gamma\mathcal{E}$ for* $n = 2$, $1/6$-$\Gamma\mathcal{E}$ *for* $n = 3$. *For* $n > 3$, A_{ψ} *fails* $\Gamma\mathcal{E}$, *although* $A_{\psi}^{1-\theta}$ *satisfies* $\lambda_{\theta}(n)$-$\Gamma\mathcal{E}$ *for some* $\lambda_{\theta}(n)$.

Proof. Since we are working with a Fourier multiplier, we find

$$2K_n(j,l) = \left(1 - \cos\left(\frac{2\pi j}{n}\right)\right) + \left(1 - \cos\left(\frac{2\pi l}{n}\right)\right) - \left(1 - \cos\left(\frac{2\pi(j-l)}{n}\right)\right)$$

$$= 1 + \cos\left(\frac{2\pi(j-l)}{n}\right) - \cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{2\pi l}{n}\right)$$

$$= 1 + \cos\left(\frac{2\pi j}{n}\right)\cos\left(\frac{2\pi l}{n}\right) + \sin\left(\frac{2\pi j}{n}\right)\sin\left(\frac{2\pi l}{n}\right) - \cos\left(\frac{2\pi j}{n}\right)$$

$$- \cos\left(\frac{2\pi l}{n}\right)$$

$$= \left(1 - \cos\left(\frac{2\pi j}{n}\right)\right)\left(1 - \cos\left(\frac{2\pi l}{n}\right)\right) + \sin\left(\frac{2\pi j}{n}\right)\sin\left(\frac{2\pi l}{n}\right) .$$

The gradient matrix K_{I-E} corresponding to Γ_{I-E} is given by $2(1_{\mathbb{Z}_n \setminus 0} + 1_{\mathbb{Z}_n \setminus \{0\}})$ and we shall consider them as $(n-1) \times (n-1)$ matrices supported on $\{|0\rangle\}^{\perp}$. Since K_{ψ} has rank at most 2, we deduce that for no $n > 3$ and $\lambda > 0$ the property λ-$\Gamma\mathcal{E}$ is satisfied. For $n = 2$ we have the standard Walsh system and 1-$\Gamma\mathcal{E}$. For $n = 3$ we find the matrix

$$2K_\psi = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} .$$

We have to compare this to

$$2K_{I-E} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} .$$

It is easy to see that for $\lambda = 6$ we have

$$6(2K_\psi) \geq 2K_{I-E} .$$

Thus we have $\frac{1}{6}$-$\Gamma\mathcal{E}$ and $\frac{1}{6}$-CLSI. For $n > 3$ and $0 < \theta < 1$. Then, Theorem 3.16 applies for $d = 0$ because $\|T_t : L_1 \to L_\infty\| \leq n$. Moreover, and A_ψ has a spectral gap of order n^{-2}. We refer to [31] for the return time estimates $t_0 \sim n$ and hence Corollary 4.9 gives an estimates of the order $\lambda_\theta(n) \sim n^{-2\theta}(1-\theta)$ for $\Gamma\mathcal{E}$ constant. $\qquad\square$

7.4. Non-additivity of $\mathcal{I}_{I-E_N}$

In the proof of tensorization of CLSI, we have used the following subadditivity of relative entropy

$$D_{N_1 \otimes N_2}(\rho) \leq D_{N_1 \otimes M_2}(\rho) + D_{M_1 \otimes N_2}(\rho) ,$$

where $N_j \subset M_j, j = 1, 2$ are finite von Neumann subalgebras. This is not true for the symmetrized Kullback-Leibler divergence $\mathcal{I}_N$.

Proposition 7.8. *The inequality*

$$\mathcal{I}_{N_1 \otimes N_2} \leq \mathcal{I}_{N_1 \otimes M_2} + \mathcal{I}_{M_1 \otimes N_2} . \tag{7.3}$$

is not valid in general.

Proof. Let $E_j : M_j \to N_j, j = 1, 2$ be the conditional expectation. We note that (7.3) is equivalent to

$$\tau\Big((E_1 \otimes id)(x) \ln x\Big) + \tau\Big((id \otimes E_2)(x) \ln x\Big) \leq \tau(x \ln x) + \tau\Big(E_1 \otimes E_2(x) \ln x\Big) .$$

Let $N_1 = N_2 = \mathbb{C}$ and $M_1 = M_2 = \ell_\infty(\{1, 2, 3\})$. We can write $x \in M_1 \otimes M_2$ in a 3×3 matrix form

$$x = \sum_{i,j} x_{i,j} e_i \otimes e_j = [x_{i,j}]_{i,j=1}^3 .$$

Then, the conditional expectation is given by row- and column-average. Therefore it suffices to decide whether $\tau((x + 1 - E_1 \otimes id(x) - id \otimes E_2(x)) \ln(x))$ is always positive. Let $\delta > 0$ and

$$[x_{i,j}] = \begin{bmatrix} \delta & \alpha & \alpha \\ \alpha & \gamma & \gamma \\ \alpha & \gamma & \gamma \end{bmatrix}$$

where $\alpha = 3/8$ and $\gamma = 15/8 - \frac{\delta}{4}$. Then, the $(1,1)$-entry of $1 + x - E_1 \otimes id(x) - id \otimes E_2(x)$ is given by

$$1 + \delta - 2/3(\delta + \alpha + \alpha) = \frac{1}{2} + \frac{\delta}{3}$$

Note that $\lim_{\delta \to 0} \gamma = 15/8$ is away from 0 and $\frac{1}{2}(\ln \delta) \to -\infty$. Thus for $\delta \to 0$, $\tau((x + 1 - E_1 \otimes id(x) - id \otimes E_2(x)) \ln(x))$ converges to $-\infty$. (Although written in a matrix form, x and $x + 1 - E_1 \otimes id(x) - id \otimes E_2(x)$ are really scalar functions on $\{1,2,3\} \times \{1,2,3\}$. Hence it suffices to argue that there is one entry goes to $-\infty$). $\qquad \square$

7.5. Failure of Rothaus Lemma for Matrix-Valued Functions

Let $N = \mathbb{M}_n \otimes \mathbb{C}1 \subset \mathbb{M}_n \otimes \mathbb{M}_n$. We will always use the normalized trace on matrix algebras and the conditional expectation is the normalized partial trace $E = id \otimes \frac{1}{n} tr$.

Proposition 7.9. *For $n \geq 2$, there exists no constant C_1, C_2 such that*

$$D_N(|x|^2) \leq C_1 \tau(x^* A(x)) + C_2 \|x - E(x)\|^2 . \tag{7.4}$$

Moreover, there are no constants C_1, C_2 such that

$$D_N(|x|^2) \leq C_1 D_N(|x - E(x)|^2) + C_2 \|x - E(x)\|^2 , \tag{7.5}$$

holds for all self-adjoint x.

Let us start with the non-selfadjoint element in "bracket" notation

$$y = \frac{n}{\sqrt{n-1}} \sum_{j=2}^{n} |11\rangle\langle jj| .$$

The corresponding conditional expectation is given by

$$E(|y|^2) = \frac{n^2}{n-1} E\left(\sum_{j,k=2}^{n} |jj\rangle\langle kk| \right) = \frac{n}{n-1} \sum_{j=2}^{n} |j\rangle\langle j| = \frac{n}{n-1} 1_{n-1} ,$$

where $1_{n-1} = \sum_{j=2}^{n} |j\rangle\langle j|$ has rank $n-1$. Since y is rank one, we get

$$D_N(|y|^2) = \tau(|y|^2 \ln |y|^2) - \tau(E|y|^2 \ln E|y|^2) = \frac{1}{n^2} n^2 \ln n^2 - \ln \frac{n}{n-1}$$

$$= 2\ln n - \ln \frac{n}{n-1} .$$

Now we modify this element

$$x = \alpha(|1\rangle\langle 1| \otimes 1) + y$$

by adding an element in $\mathbb{M}_n \otimes 1$. Thus $x - E(x) = y$. We have to calculate $D_N(|x|^2)$. Let us denote by $f = |1\rangle\langle 1| \otimes 1$ the projection. First we observe that

$$x^* x = \alpha^2 f + \alpha y + y^* \alpha + y^* y$$

and hence

$$E(x^* x) = \alpha^2 |1\rangle\langle 1| + \frac{n}{n-1} 1_{n-1} .$$

This implies

$$\tau(E(|x|^2)\ln E(|x|^2)) = \frac{\alpha^2}{n}\ln\alpha^2 + \ln\frac{n}{n-1} .$$

In order to calculate the entropy for $|x|^2$, we decompose $f = |11\rangle\langle 11| + 1_{n-1}\otimes|1\rangle\langle 1|$. The second projection is orthogonal to the support of y, which we denote by g. Hence x^2 is unitarily equivalent to

$$\begin{pmatrix} \alpha^2 & n\alpha & 0 \\ n\alpha & n^2 & 0 \\ 0 & 0 & \alpha^2 g \end{pmatrix} .$$

The upper corner is of rank 1 with size $n^2\alpha^2$ and hence

$$\tau(|x|^2\ln|x|^2) = \frac{n^2+\alpha^2}{n^2}\ln(n^2+\alpha^2) + \frac{\alpha^2(n-1)}{n^2}\ln(\alpha^2) .$$

This yields

$$D_N(|x|^2) = \frac{n^2+\alpha^2}{n^2}\ln(n^2+\alpha^2) + \frac{\alpha^2}{n}\ln(\alpha^2) - \frac{\alpha^2}{n^2}\ln(\alpha^2)$$
$$- \frac{\alpha^2}{n}\ln(\alpha^2) - \ln\frac{n}{n-1}$$
$$= \ln(n^2+\alpha^2) + \frac{\alpha^2}{n^2}\ln\left(1+\frac{n^2}{\alpha^2}\right) - \ln\frac{n}{n-1}$$

In order to contradict (7.4) and (7.5), we observe that $\tau(xA(x)) = \tau(xA(y)) = \tau(yA(y))$. Hence, the right-hand side in (7.4) and (7.5) is bounded, but the left-hand side converges to $+\infty$ for $\alpha\to\infty$, as long as $n \geq 2$. For self-adjoint x see below. □

We will now address cb-hypercontractivity (in the sense of [14]) at $p = 2$ by considering the self-adjoint element

$$z = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} .$$

We have $\tau(|z|^2\ln|z|^2) = \tau(|x|^2\ln|x|^2)$. For the conditional expectation, we find

$$E(|z|^2) = \begin{pmatrix} E(xx^*) & 0 \\ 0 & E(x^*x) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha^2 f + E(n^2|11\rangle\langle 11|) & 0 \\ 0 & \alpha^2 f + \frac{n}{n-1}1_{n-1} \end{pmatrix} .$$

This gives

$$\tau(|z|^2\ln|z|^2) = \frac{\alpha^2}{2n}\ln\alpha^2 + \frac{1}{2}\ln\frac{n}{n-1} + \frac{\alpha^2+n}{2n}\ln(\alpha^2+n) .$$

The new part here is

$$D_N(xx^*) = \frac{n^2+\alpha^2}{n^2}\ln(n^2+\alpha^2) + \frac{\alpha^2(n-1)}{n^2}\ln(\alpha^2) - \frac{\alpha^2+n}{n}\ln(\alpha^2+n)$$

$$= \ln\left(\frac{n^2 + \alpha^2}{n + \alpha^2}\right) + \frac{\alpha^2}{n^2}\ln\frac{\alpha^2 + n^2}{\alpha^2} - \frac{\alpha^2}{n}\ln\left(\frac{\alpha^2 + n}{\alpha^2}\right).$$

Following our previous calculation, we find that

$$D_N(|z|^2) = \frac{1}{2}\left(\ln(n^2 + \alpha^2) + \frac{\alpha^2}{n^2}\ln\left(1 + \frac{n^2}{\alpha^2}\right) - \ln\frac{n}{n-1}\right)$$
$$+ \frac{1}{2}\left(\ln\left(\frac{n^2 + \alpha^2}{n + \alpha^2}\right) + \frac{\alpha^2}{n^2}\ln\frac{\alpha^2 + n^2}{\alpha^2} - \frac{\alpha^2}{n}\ln\left(\frac{\alpha^2 + n}{\alpha^2}\right)\right)$$
$$= \frac{1}{2}\ln(n^2 + \alpha^2) + \frac{1}{2}\ln\left(\frac{n^2 + \alpha^2}{n + \alpha^2}\right) + \frac{\alpha^2}{2n^2}\ln\left(\frac{n^2 + \alpha^2}{n + \alpha^2}\right)$$
$$+ \frac{\alpha^2}{n^2}\ln\left(1 + \frac{n^2}{\alpha^2}\right) - \frac{1}{2}\ln\frac{n}{n-1} - \frac{\alpha^2}{2n}\ln\left(\frac{\alpha^2 + n}{\alpha^2}\right).$$

In order to keep the last term, we choose $\alpha_n^2 = n$ and then find

$$D_N(|z|^2) = \frac{1}{2}\ln n + \frac{1}{2}\ln(n+1) + \frac{1}{2}\ln(n+1) - \frac{1}{2}\ln 2 + \frac{1}{2n}(\ln(n+1) - \ln 2)$$
$$+ \frac{1}{n}\ln(n+1) - \frac{1}{2}\ln\frac{n}{n-1} - \frac{1}{2}\ln 2$$
$$= \frac{1}{2}\ln n + \left(1 + \frac{3}{2n}\right)\ln(n+1) - \left(1 + \frac{1}{2n}\right)\ln 2 - \frac{1}{2}\ln\frac{n}{n-1}.$$

Note that the $\log n$ term is the optimal rate for entropy as $n \to \infty$, and hence, the example is rather extreme. Following the work of [14], we may formulate this observation as follows.

Proposition 7.10. *Let (A_n) be sequence of self-adjoint generators on $\mathbb{M}_n$ such that $\sup_n \|A_n : L_2(\mathbb{M}_n) \to L_2(\mathbb{M}_n)\| < \infty$. Then, the 2-cb-hypercontractivity constant of A_n always converges to ∞.*

For example, we may choose $A_n = I - \tau_n$ on $\mathbb{M}_n$ has norm 1. In fact, we only have to control the behavior of A_n on some version of the maximally entangled state.

Proof. We recall [14] that the cb-hypercontractivity constant $\lambda_2^{cb} = \inf \lambda_2^m$, where λ_2^m is the best constant such that for all $x \in \mathbb{M}_m \otimes \mathbb{M}_n$,

$$\lambda_2^m D_{\mathbb{M}_n}(|x|^2) \leq 4\tau(x^*(id_m \otimes A_n)x) = 4\mathcal{E}(x).$$

Note the trivial bound $\lambda_2^n \geq \frac{1}{2\ln n}$. Hence our choice of $|x|$ in above discussion shows that $\frac{\ln n}{2}\lambda_2^n \leq \|A_n\|$ and hence, up to constant $4\sup_n \|A_n\|$, the trivial bound cannot be improved as n tends to infinity. A limiting channel violates 2-cb-hypercontractivity. $\square$

Remark 7.11. A counterexample of similar nature was constructed in [20], which also shows that $S_2(S_p)$ is not uniformly convex. Since our example is the tracial setting, we think they are of independent interests.

7.6. Conclusion and Open Problems

We end our discussion with a summary of examples that satisfy CLSI and $\Gamma\mathcal{E}$ obtained in this paper and also questions on CLSI that remain open.

Conclusion 7.12.   (i) *The (infinite) tensor products of self-adjoint semigroup satisfying λ-CLSI still satisfy λ-CLSI.*
(ii) *The n-fold free product of generators satisfying λ-$\Gamma\mathcal{E}$ satisfies $\frac{\lambda}{n}$-$\Gamma\mathcal{E}$.*
(iii) *The weighted graph Laplacian of a complete graph satisfies $\Gamma\mathcal{E}$ and hence CLSI.*
(iv) *The subordinated semigroups of sub-Laplacians of Hörmander systems on compact Riemannian manifolds satisfy $\Gamma\mathcal{E}$ and CLSI.*
(iv) *The subordinated semigroups of self-adjoint semigroups on matrix algebras satisfy $\Gamma\mathcal{E}$ and CLSI.*

In particular, combining (i)+(iii), (i)+(iv) and (i)+(v) gives infinite-dimensional examples of semigroups satisfying CLSI and Talagrand's inequality TA_2, but not necessarily $\Gamma\mathcal{E}$. Based on iv), a natural open question is

Problem 7.13. *Does every Laplace–Beltrami operator on a compact Riemannian manifold satisfy CLSI?*

If it is true, how does the optimal constant of CLSI compares to the optimal constant of the non-complete version? Since $\Gamma\mathcal{E}$ fails for the heat semigroup on torus, new techniques are needed to approach this problem. The similar question for matrix algebra is

Problem 7.14. *Does every generator of a (self-adjoint) semigroup on a matrix algebra satisfy CLSI?*

Bardet proved in [1] via a compactness argument that every semigroup (not necessarily self-adjoint) on a matrix algebra satisfies λ-MLSI, and inquired about the CLSI version. Motivated by the stability of $\Gamma\mathcal{E}$ under free product, one can ask

Problem 7.15. *Is CLSI stable under free products?*

Acknowledgements

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