Journal of Functional Analysis 280 (2021) 108782

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

A noncommutative generalisation of a problem of Steinhaus

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A R T I C L E   I N F O

Article history:
Received 15 January 2020
Accepted 7 September 2020
Available online 28 September 2020
Communicated by Stefaan Vaes

MSC:
46L51
46L52
40A05
60F15

Keywords:
Noncommutative integration
Subsequence principle
Komlós theorem
Strong law of large numbers

A B S T R A C T

We extend the Révész and Komlós theorems to arbitrary finite von Neumann algebras, and in doing so solve an open problem of Randrianantoanina, removing the need for hyperfiniteness. The main result is the noncommutative Komlós theorem, which states that every norm-bounded sequence of operators in $L^1(\mathcal{M})$, for any finite von Neumann algebra $\mathcal{M}$, admits a subsequence, such that for any further subsequence, the Cesàro averages converge bilaterally almost uniformly. This is a natural extension of Komlós' original result to the noncommutative setting.

The necessary techniques which facilitate the proof also allow us to extend the Révész theorem to the noncommutative setting, which gives a similar subsequential law for series over bounded sequences in $L^2(\mathcal{M})$.

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[1] The second named author was supported by an Australian Government Research Training Program (RTP) Scholarship.
[2] The third named author was supported by ARC Laureate Fellowship grant FL170100052 — Breakthrough methods for noncommutative calculus.

https://doi.org/10.1016/j.jfa.2020.108782
0022-1236/

1. Introduction

In the new Scottish book, Steinhaus posed the following problem.

Problem 1.1 *(Problem 126, The New Scottish Book).* Does there exist a family F of measurable functions, such that for each $f \in F$, $|f(t)| = 1$, and for each sequence of functions $(f_n)_{n=1}^{\infty}$ in F, the sequence

$$\frac{1}{n}\sum_{k=1}^{n} f_k(t)$$

is divergent for almost all t?

Révész produced the following solution, phrased in the language of probability theory.

Theorem 1.2 *([75, Theorem 1]). Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of random variables, such that for some positive constant K, the expectations of the squares are bounded, that is $\mathbb{E}(\xi_n^2) \leq K$, for all $n \geq 1$.*

There then exists an increasing sequence $n_1 < n_2 < \cdots$ of integers, and a random variable η, such that the (Cesàro) averages converge to η, with probability equal to 1,

$$\mathbb{P}\left(\frac{\xi_{n_1} + \xi_{n_2} + \cdots + \xi_{n_k}}{k} \to \eta\right) = 1.$$

This result follows easily from the following more general result of Révész.

Theorem 1.3 *(Révész' Theorem [75, Theorem 2]). Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of random variables, such that for some positive constant K, the expectations of the squares are bounded, $\mathbb{E}(\xi_n^2) \leq K$, for all $n \geq 1$.*

There then exists an increasing sequence $n_1 < n_2 < \cdots$ of integers, and a random variable η, such that for any sequence $(c_n)_{n=1}^{\infty}$ of real numbers, satisfying $\sum_{n=1}^{\infty} c_n^2 < \infty$, the series

$$\sum_{k=1}^{\infty} c_k(\xi_{n_k} - \eta)$$

is convergent with probability equal to 1.

Révész' study concluded with the following problem: Can the condition $\mathbb{E}(\xi_n^2) < K$ be loosened to $\mathbb{E}(\xi_n^{1+\varepsilon}) < K$, for $0 < \varepsilon < 1$? Komlós not only provided an affirmative solution, but showed that the family of random variables need only be bounded in L^1.

Theorem 1.4 *(Komlós' Theorem [54, Theorem 1a]). Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of random variables, such that $\liminf_{n\to\infty} \mathbb{E}(\xi_n) < \infty$, then there exists a subsequence $(\eta_n)_{n=1}^{\infty}$ of*

$(\xi_n)_{n=1}^{\infty}$, *and an integrable random variable* η, *such that for any subsequence* $(\tilde{\eta}_n)_{n=1}^{\infty}$ *of* $(\eta_n)_{n=1}^{\infty}$, *we have that*

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{\tilde{\eta}_1+\tilde{\eta}_2+\cdots+\tilde{\eta}_n}{n}=\eta\right)=1.$$

Loosely, Komlós' result may be thought of as the claim that every sequence of integrable random variables contains a subsequence which behaves as a collection of independent random variables, in that it satisfies a kind of strong law of large numbers. However this elides the strength of the choice of subsequence. Consider that as we may choose arbitrary subsequences of $(\eta_n)_{n=1}^{\infty}$, we may choose two disjoint subsequences, and yet their Cesàro averages will still converge almost everywhere to the same limit.

Trautner [86] remarks that although the results of Révész and Komlós are stated in the language of probability theory, they are actually analytic claims. In particular, these results all state that, in essence, various L^p-spaces are compact with regard to Cesàro means, and almost everywhere convergence. Rephrased as such, what was an interesting statement about probability theory becomes a useful feature of the geometry of function spaces.

Even without this general motivation, the probabilistic interpretation is a strong impetus for the study of these results. The view of the Komlós theorem as an analogue of the strong law of large numbers has been sufficient to motivate a substantial body of research in the commutative setting, such as the generalisations of Aldous [1], Balder [5–7], Balder and Hess [8], Berkes [10,11], Cassese [17], and Chatterji [18], and the simplified proofs of Guessous [35], Schwartz [78], and Trautner [86]. See also the discussion of subsequence principles in [76, Chapter 5].

Our interest lies not only in the study of classical probability theory and analysis, but in the more general study of noncommutative integration. Driven by a desire to find a suitable framework for the description of quantum mechanical systems, it was already apparent in the original works of Murray and von Neumann that von Neumann algebras provided what could be understood as a noncommutative analogue of integration theory. However, it was with the pioneering work of Irving Segal [79], driven by the study of representations of noncommutative groups and harmonic analysis, that it was discovered that von Neumann algebras recovered and generalised the theory of integration. In the time since, von Neumann algebras have found substantial application not only in physics, through quantum mechanics [14,15], quantum statistical mechanics [59], and quantum information theory [36], but also in harmonic analysis [57,81], ergodic theory [52], geometry [21], and random matrix theory [62]. There have also been substantial developments in quantum and noncommutative probability, including deep results on martingales, probabilistic inequalities, and stochastic processes (see, for example, [12,23,47,45,65])

It is then natural for us to ask if more subtle results in probability such as the Komlós theorem can be extended to noncommutative probability, and indeed, Randrianantoanina was able to develop a partial noncommutative extension of the Komlós theorem. In order to make sense of the result in the noncommutative setting, almost everywhere convergence is replaced by bilateral almost uniform convergence. Noncommutative notions of almost everywhere convergence are defined and discussed in Section 2.

Theorem 1.5 *([73, Theorems 3.1, 3.8]). Let $\mathcal{M}$ be a semifinite von Neumann algebra, with a distinguished faithful normal semifinite trace τ. Fix $1 \leq p < \infty$. If $p = 1$, we additionally require that $\mathcal{M}$ is hyperfinite.*

Let $(f_n)_{n=1}^{\infty} \subseteq L^p(\mathcal{M}, \tau)$ satisfy $\sup_{n \geq 1} \|f\|_p < \infty$. There exists a subsequence $(g_n)_{n=1}^{\infty} \subseteq (f_n)_{n=1}^{\infty}$, and an operator $f \in L^p(\mathcal{M}, \tau)$, such that for every subsequence $(h_n)_{n=1}^{\infty} \subseteq (g_n)_{n=1}^{\infty}$, the sequence of Cesáro means,

$$\left(\frac{1}{n} \sum_{k=1}^{n} h_k \right)_{n=1}^{\infty},$$

converges to f bilaterally almost uniformly.

Randrianantoanina concludes his result by asking if the hyperfiniteness condition can be removed for L^1-bounded sequences [73, Problem 3.15]. This is an important question not only because it is often natural and necessary to work with von Neumann algebras which are not hyperfinite, but because the techniques Randrianantoanina used closely parallel those in classical analysis. A resolution of this problem then requires substantial new methods for the study of almost everywhere convergence. Here we give an affirmative answer, proving the Komlós theorem for arbitrary finite von Neumann algebras. We also extend the Révész theorem to the noncommutative setting, which had not been studied before.

The key to our resolution of Randrianantoanina's problem is an appeal to ultrapower techniques. In particular, the Révész and Komlós theorems are traditionally proved by passing to simple functions, each contained in a finite dimensional subspace. This is most clear in Randrianantoanina's proof [73], wherein the hyperfinite structure of the algebra induces a natural martingale filtration. At each stage of the filtration, the finite dimensional spaces allow one to pass from weak to strong convergence. Then, having strong convergence, one may apply the Doob martingale convergence theorem to pass to bilateral almost uniform convergence.

Rather than appealing to an internal structure, such as given by hyperfiniteness, to determine bilateral almost uniform convergence, we appeal to an external structure, given to us by iterated ultrapowers. Namely, we consider the countably iterated ultrapower of the noncommutative L^2-space, and then show that this forms a filtration of some large external von Neumann algebra. We then show in Section 4 that the maximal inequalities, given here by the Doob martingale convergence theorem, induces bilateral almost uniform convergence in the original algebra. This completes the missing part of the puzzle, allowing the result to be extended to arbitrary finite von Neumann algebras.

While ultrapower techniques stem from model theory, their use in Banach space theory is extensive, yielding many difficult results. The interested reader should consult the papers of Heinrich [39], and Sims [80], as well as the book of Diestel, Jarchow, and Tonge [26]. Moreover, while (finitely) iterated ultrapowers have been studied in the Banach space setting, they have been very poorly researched for von Neumann algebras and noncommutative integration.

After recalling the basic details of noncommutative integration, noncommutative martingales, and bilateral almost uniform convergence in Section 2, we develop the theory of iterated ultrapowers of von Neumann algebras, and their martingale structure, in Section 3. The key difficulty of the paper lies in the proof of Theorem 4.2, which requires a careful diagonalisation argument, and is proved in Section 4. Coarsely, we may understand Theorem 4.2 as a very specialised diagonalisation result. If a family of bounded sequences can be made to correspond to a martingale difference sequence, associated to the iterated ultrapower of a finite von Neumann algebra, then we may find a sequence of terms derived from the family of sequences, such that weighted series over these terms converge bilaterally almost uniformly, and so that the series also converge as such for series over any further subsequence.

Having shown this result, the proofs of the noncommutative Révész and Komlós theorems are not too far removed from the classical proofs, with the key changes being those necessary modifications for noncommutative integration, and some setup to work with the iterated ultrapower construction. In Section 5 we prove the noncommutative Révész theorem, and in Section 6 we prove the noncommutative Komlós theorem.

2. Background

In this section we review the necessary aspects of the theory of noncommutative integration. In particular, we discuss noncommutative L^p-spaces, and the various modes of convergence which we will study throughout the paper, and we will prove some supplemental lemmas. Moreover, we will review the necessary aspects of maximal functions in the noncommutative setting, as viewed through vector-valued noncommutative L^p-spaces, and their application through the noncommutative Doob maximal inequality.

For any necessary background on von Neumann algebras, see the books of Takesaki [83], and Strătilă and Zsidó [82], and for further information about noncommutative integration, see the survey of Pisier and Xu [72].

2.1. Noncommutative measure and integration

Let $\mathcal{M}$ be a semifinite von Neumann algebra, acting upon some fixed Hilbert space $\mathcal{H}$, and let τ be a faithful normal semifinite trace over $\mathcal{M}$.

A closed, densely defined operator $x : \mathrm{Dom}(x) \to \mathcal{H}$, with $\mathrm{Dom}(x) \subseteq \mathcal{H}$, is said to be *affiliated* to $\mathcal{M}$, if x commutes with every unitary operator in the commutant $\mathcal{M}'$ of $\mathcal{M}$.

An operator x, affiliated to $\mathcal{M}$, is said to be *τ-measurable* if for every $\varepsilon > 0$, there exists a projection $p \in \mathcal{M}$, such that $\tau(p) \leq \varepsilon$, and $(1-p)\mathcal{H} \subseteq \mathrm{Dom}(x)$. Let $\mathcal{S}(\mathcal{M},\tau)$ denote the space of all τ-measurable operators.

For any $0 < p < \infty$, the *noncommutative L^p-space* over $\mathcal{M}$ is defined by the set

$$L^p(\mathcal{M},\tau) = \{x \in \mathcal{S}(\mathcal{M},\tau) : \tau(|x|^p) < \infty\},$$

and is endowed with the norm

$$\|x\|_p = \tau(|x|^p)^{1/p},$$

for each $x \in L^p(\mathcal{M},\tau)$, where $|x| = (x^*x)^{1/2}$ is the modulus of x.

For $p = \infty$, let $L^\infty(\mathcal{M},\tau) = \mathcal{M}$, and is endowed with the norm $\|\cdot\|_\infty = \|\cdot\|_{\mathcal{M}}$.

For any fixed $1 \leq p \leq \infty$, we will call a sequence $(x_n)_{n=1}^\infty \subseteq L^p(\mathcal{M},\tau)$ *L^p-bounded* if $\sup_{n\geq 1} \|x_n\|_p < \infty$.

Much as for classical function space theory, we may define a weak and a strong topology upon $L^p(\mathcal{M},\tau)$, with $1 \leq p < \infty$. The *strong topology* is simply the topology induced by the L^p-norm, and the *weak topology* is the $\sigma(L^p, L^q)$-topology, where q is the conjugate exponent of p. As in the classical setting, $L^2(\mathcal{M},\tau)$ is weakly sequentially compact.

For any τ-measurable operator $x \in \mathcal{S}(\mathcal{M},\tau)$, the *decreasing rearrangement* of x is the function $\mu(t;x)$, defined by

$$\mu(t;x) = \inf\left\{s \geq 0 : \tau\left(\chi_{(s,\infty)}(|x|) \leq t\right)\right\},$$

for all real numbers $t > 0$, where $\chi_{(s,\infty)}(|x|)$ is the characteristic function of $|x|$, over the interval (s,∞), as given by the functional calculus.

2.2. The measure topology

In order to bridge convergence in the L^p-norm, and convergence "almost everywhere", we may appeal to a natural topology on the space of measurable operators. The *measure topology* on $\mathcal{S}(\mathcal{M},\tau)$ is the linear Hausdorff topology, generated by the fundamental system $\{V(\varepsilon,\delta) : \varepsilon,\delta > 0\}$ of neighbourhoods around zero, where for any $\varepsilon,\delta > 0$,

$$V(\varepsilon,\delta) = \{x \in \mathcal{S}(\mathcal{M},\tau) : \text{there exists a projection } p \in \mathcal{M}, \\ \text{such that } \|xp\|_\infty \leq \varepsilon \text{ and } \tau(1-p) \leq \delta\}.$$

The measure topology is used throughout in order to pass from convergence in the L^p-norm to bilateral almost uniform convergence.

Lemma 2.1 *([29, Theorem 3.7]). For any $0 < p < \infty$, let $(x_n)_{n=1}^\infty$ and x lie in $L^p(\mathcal{M},\tau)$. The following are equivalent.*

i. The sequence $(x_n)_{n=1}^{\infty}$ converges strongly to x, that is

$$\lim_{n\to\infty} \|x - x_n\|_p = 0.$$

ii. The sequence $(x_n)_{n=1}^{\infty}$ converges to x in the measure topology, and

$$\lim_{n\to\infty} \|x_n\|_p = \|x_n\|_p .$$

The measure topology also allows us to recover noncommutative analogues of important results in measure theory.

Lemma 2.2 *([29, Theorem 3.5]). Let $(x_n)_{n=1}^{\infty}$ be a sequence of positive τ-measurable operators, which are convergent in the measure topology to some $x \in \mathcal{S}(\mathcal{M}, \tau)$.*

i. (Fatou's Lemma)

$$\tau(x) \leq \liminf_{n\to\infty} \tau(x_n).$$

ii. (Monotone Convergence Theorem) If $x_n \leq x$ for each $n \geq 1$, or if $\mu(t; x_n) \leq \mu(t; x)$ for each $n \geq 1$ and $t > 0$, then

$$\tau(x) = \lim_{n\to\infty} \tau(x_n).$$

Lemma 2.3 *(Noncommutative Borel–Cantelli Lemma [77]). Let $(p_n)_{n=1}^{\infty}$ be any sequence of projections in $\mathcal{M}$. If $\sum_{n=1}^{\infty} \tau(p) < \infty$, then $\tau(\bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} p_k) = 0$, and* $\lim_{n\to\infty} \tau(\bigvee_{k=n}^{\infty} p_k) = 0$.

2.3. Almost uniform convergence

It is not immediate how one should define almost everywhere convergence in the noncommutative setting, as we cannot speak of points on which a function acts. While many noncommutative analogues of almost everywhere convergence have been studied, originating with Segal's investigation of "nearly everywhere" convergence [79, Section 2.6], we will study almost uniform convergence, which arises naturally through Egorov's theorem.

Theorem 2.4 *([38, §21, Theorem A]). For any measure space (X, μ), let E be a measurable set of finite measure. If $(f_n)_{n=1}^{\infty}$ is a set of measurable functions over E, which converge almost everywhere to a measurable function f, then for every $\varepsilon > 0$, there exists a measurable subset $F \subseteq E$, such that $\mu(F) < \varepsilon$, and $(f_n)_{n=1}^{\infty}$ converges uniformly to f over $E \setminus F$.*

This leads to a description of almost everywhere convergence without reference to individual points.

Definition 2.5. A sequence $(x_n)_{n=1}^\infty$ of τ-measurable operators is said to converge *almost uniformly* to $x \in \mathcal{S}(\mathcal{M},\tau)$ if for every $\varepsilon > 0$, there exists a projection $p_\varepsilon \in \mathcal{M}$, such that $\tau(1-p_\varepsilon) < \varepsilon$, and $\lim_{n\to\infty} \|(x-x_n)p_\varepsilon\|_\infty = 0$.

The sequence $(x_n)_{n=1}^\infty$ is said to converge *bilaterally almost uniformly* to x if for every $\varepsilon > 0$, there exists a projection $p_\varepsilon \in \mathcal{M}$, such that $\tau(1-p_\varepsilon) < \varepsilon$, and $\lim_{n\to\infty} \|p_\varepsilon(x-x_n)p_\varepsilon\|_\infty = 0$.

It is clear that every almost uniformly convergent sequence is necessarily bilaterally almost uniformly convergent, however there are bilaterally almost uniformly convergent sequences which are not almost uniformly convergent (see, for example, [25, Corollary 6.4, Example 6.5]).

In the commutative setting, almost uniform convergence recovers almost everywhere convergence, even for spaces with infinite measure.

Theorem 2.6 *([38, §21, Theorem B]). If $(f_n)_{n=1}^\infty$ is a sequence of measurable functions which converge to f almost uniformly, then $(f_n)_{n=1}^\infty$ converges to f almost everywhere.*

We note that over a measure space of infinite measure, almost uniform convergence is stronger than almost everywhere convergence. For example, consider the counting measure over the positive integers, and the sequence of characteristic functions $(\chi_{[1,n]})_{n=1}^\infty$. The sequence converges everywhere to the identity function $1 = \chi_{[1,\infty)}$, however it does not converge almost uniformly. That is to say that on spaces of infinite measure, almost everywhere convergence is strictly weaker than almost uniform convergence.

Throughout, the following results concerning almost uniform convergence will be useful.

As for function spaces, every sequence convergent in the measure topology contains a subsequence which converges almost uniformly.

Proposition 2.7 *([31, Proposition 1]). If $(x_n)_{n=1}^\infty \subseteq \mathcal{S}(\mathcal{M},\tau)$ converges to zero with respect to the measure topology, then there exists a subsequence $(y_n)_{n=1}^\infty \subseteq (x_n)_{n=1}^\infty$, which converges almost uniformly to zero.*

Almost uniform convergence is also well behaved with regard to addition of sequences, and Cesàro averages.

Lemma 2.8 *([77]). For any two sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq \mathcal{S}(\mathcal{M},\tau)$, such that $(x_n)_{n=1}^\infty$ converges (bilaterally) almost uniformly to $x \in \mathcal{S}(\mathcal{M},\tau)$, and $(y_n)_{n=1}^\infty$ converges (bilaterally) almost uniformly to $y \in \mathcal{S}(\mathcal{M},\tau)$, we have that $(x_n+y_n)_{n=1}^\infty$ converges (bilaterally) almost uniformly to $x+y$.*

Lemma 2.9 *([77]). Let $(x_n)_{n=1}^\infty \subseteq \mathcal{S}(\mathcal{M},\tau)$ be any sequence of measurable operators which converges (bilaterally) almost uniformly to zero. For any $1 \le p < \infty$, the sequence*

$$\left(\frac{1}{n^p}\sum_{k=1}^{n} x_k\right)_{k=1}^{\infty}$$

also converges (bilaterally) almost uniformly to zero.

It is interesting to note that even in the classical setting, convergence in measure is not well behaved with respect to the Cesàro averages. A sequence which is convergent in measure may have Cesàro averages which diverge in measure, and a sequence which vanishes in measure may have Cesàro averages which converge in measure to the identity. For details of these constructions, see Bikchentaev and Sabirova [13, Section 3.4].

The following lemma is a mild extension of [73, Proposition 2.3].

Lemma 2.10 *([77]). Let $1 \leq p < \infty$, and let $(x_n)_{n=1}^{\infty} \subseteq L^p(\mathcal{M}, \tau)$.*

i. If $\sum_{n=1}^{\infty} \|x_n\|_p < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ converges bilaterally almost uniformly.
ii. If $p = 2$, and $\sum_{n=1}^{\infty} \|x_n\|_2^2 < \infty$, then the sequence $(x_n)_{n=1}^{\infty}$ converges almost uniformly to zero.

Our final lemma regarding almost uniform convergence is a minor extension Kronecker's lemma, showing that it holds much as expected, with regard to both convergence in the L^p-norm, and almost uniform convergence.

Lemma 2.11 *(Kronecker's Lemma for L^p-Operators [77]). Fix $1 \leq p \leq \infty$. For any sequence $(x_n)_{n=1}^{\infty} \subseteq L^p(\mathcal{M}, \tau)$, such that $\sum_{n=1}^{\infty} x_n \in L^p(\mathcal{M}, \tau)$, and any sequence $(b_n)_{n=1}^{\infty} \subseteq \mathbb{C}$, such that $(|b_n|)_{n=1}^{\infty}$ is an increasing sequence, with $\lim_{n\to\infty} b_n = \infty$, we have that*

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^{n} b_k x_k = 0,$$

where the limit is taken in the L^p-topology.

If $\sum_{n=1}^{\infty} x_n$ is converges (bilaterally) almost uniformly, then the sequence

$$\left(\frac{1}{b_n}\sum_{k=1}^{n} b_k x_k\right)_{n=1}^{\infty}$$

converges (bilaterally) almost uniformly to zero.

2.4. Integration for diffuse von Neumann algebras

A von Neumann algebra is said to be *diffuse*, or *(purely) non-atomic*, if it contains no minimal projections. While many results in noncommutative integration are stated for diffuse von Neumann algebras, they may be extended to arbitrary finite algebras through a careful choice of embedding.

In particular, we appeal to the following result, which is key to simplifying our proof of the Komlós theorem. The general theory of symmetric spaces is unnecessary for our work here, however the interested reader may consult Kreĭn, Petunīn, and Semënov [56, Section II.4] for the classical theory, and Dodds, Dodds, and de Pagter for the noncommutative theory, and the definition of $E(\mathcal{M},\tau)$ [27].

Proposition 2.12 *([28, Proposition 2.7]). Let E be a separable symmetric function space, which satisfies the Fatou property, and acts upon $(0,\infty)$. Let $\mathcal{M}$ be a diffuse von Neumann algebra, with a distinguished faithful normal semifinite trace τ.*

For any sequence $(x_n)_{n=1}^\infty \subseteq E(\mathcal{M},\tau)$, of self-adjoint operators, such that $\sup_{n\geq 1}\|x_n\|_{E(\mathcal{M},\tau)} < \infty$, there exists a subsequence $(x'_n)_{n=1}^\infty \subseteq (x_n)_{n=1}^\infty$, which satisfies the following properties.

i. The sequence $(x'_n)_{n=1}^\infty$ admits a splitting

$$x'_n = y_n + z_n + d_n,$$

for each $n \geq 1$, such that $(y_n)_{n=1}^\infty$, $(z_n)_{n=1}^\infty$, and $(d_n)_{n=1}^\infty$ are bounded sequences in $E(\mathcal{M},\tau)$.

ii. The sequence $(y_n)_{n=1}^\infty$ consists only of equimeasurable operators, which is to say that for every $n \geq 1$, $\mu(t;y_n) = \mu(t;y_1)$.

iii. The sequence $(z_n)_{n=1}^\infty$ converges to zero in the measure topology.

iv. The sequence $(d_n)_{n=1}^\infty$ converges to zero in the norm of $E(\mathcal{M},\tau)$.

If, additionally, $\mathcal{M}$ is a finite von Neumann algebra, and the sequence $(x_n)_{n=1}^\infty$ is weakly null, in the Banach space sense, then the sequences $(y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ may also be chosen as to be weakly null.

It is sufficient for us to find a subsequence which admits a splitting into equimeasurable operators, and a sequence which converges in measure to zero.

Consider, for an arbitrary von Neumann algebra $\mathcal{M}$, the tensor product $\mathcal{N} = \mathcal{M}\overline{\otimes}L^\infty([0,1];\mu)$, where μ is the Lebesgue measure. We may embed $\mathcal{M}$ into $\mathcal{N}$ with the map $x \mapsto x \otimes 1$, where 1 is the identity of $L^\infty([0,1];\mu)$.

Then let τ be a distinguished faithful normal tracial state on $\mathcal{M}$. If $\tilde{\tau} = \tau \otimes \int(\cdot)d\mu$, then $\mu(t;x) = \mu(t;x\otimes 1)$, for any τ-measurable operator x, where the rearrangement on the right is relative to $\tilde{\tau}$.

As the decreasing rearrangements are the same, any equimeasurable operators in $\mathcal{M}\otimes 1$ will also be equimeasurable in $\mathcal{M}$, and the measure topology on $\mathcal{M}\otimes 1$ is homeomorphic to that on $\mathcal{M}$.

The following lemma is then clear from Proposition 2.12.

Lemma 2.13. *Let $\mathcal{M}$ be an arbitrary finite von Neumann algebra, with a distinguished faithful normal tracial state τ. If $(x_n)_{n=1}^{\infty} \subseteq L^1(\mathcal{M}, \tau)$ is an L^1-bounded sequence of self-adjoint operators, then there exists a subsequence $(x'_n)_{n=1}^{\infty} \subseteq (x_n)_{n=1}^{\infty}$, satisfying the following properties.*

i. For each $n \geq 1$, there exists a splitting

$$x'_n = y_n + z_n,$$

such that the sequences $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ are L^1-bounded.

ii. The sequence $(y_n)_{n=1}^{\infty}$ consists only of equimeasurable operators, which is to say that for every $n \geq 1$, $\mu(t; y_n) = \mu(t; y_1)$.

iii. The sequence $(z_n)_{n=1}^{\infty}$ converges to zero in the measure

Moreover, if $(x_n)_{n=1}^{\infty}$ is weakly null, then the sequences $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ may also be chosen as to be weakly null.

For further details on the embedding of von Neumann algebras into a diffuse algebra, see [29,19,28].

2.5. Noncommutative vector-valued L^p-spaces and maximal inequalities

One of the substantial difficulties in the study of noncommutative analogues of almost everywhere convergence is that there is no notion of a maximal operator. Pisier [70] reconciled this difficulty by studying vector-valued noncommutative L^p-spaces associated to hyperfinite von Neumann algebras. This construction was later extended to arbitrary von Neumann algebras [47], and serves as a powerful substitute for the lack of maximal operators. Indeed, in the commutative setting, a sequence of functions lies in the vector-valued L^p-space if and only if the maximal function lies in the corresponding L^p-space [24]. Here we will review the necessary aspects of the theory of noncommutative vector-valued L^p-spaces.

Let $\mathcal{M}$ be a semifinite von Neumann algebra, with distinguished faithful normal semifinite trace τ. For $1 \leq p \leq \infty$, the *vector-valued noncommutative L^p-space* $L^p(\mathcal{M}; \ell^\infty)$ is the space of all sequences $(x_n)_{n\in\mathbb{N}} \subseteq L^p(\mathcal{M})$, which admit a factorisation of the following form. For some $a, b \in L^{2p}(\mathcal{M})$, and a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, $x_n = a y_n b$, for each $n \in \mathbb{N}$. The norm of a sequence $(x_n)_{n\in\mathbb{N}} \in L^p(\mathcal{M}; \ell^\infty)$ is defined by

$$\|(x_n)_{n\in\mathbb{N}}\|_{L^p(\mathcal{M};\ell^\infty)} = \inf \left\{ \|a\|_{2p} \sup_{n\in\mathbb{N}} \|y_n\|_\infty \|b\|_{2p} \right\},$$

where the infimum runs over all suitable factorisations of $(x_n)_{n\in\mathbb{N}}$. The space $L^p(\mathcal{M}; \ell^\infty)$ is then a Banach space under this norm, for all $1 \leq p \leq \infty$.

We note that if $(x_n)_{n\in\mathbb{N}}$ is a sequence of positive operators, then $(x_n)_{n\in\mathbb{N}} \in L^p(\mathcal{M};\ell^\infty)$ if and only if there exists a positive operator $a \in L^p(\mathcal{M})$, and a sequence of positive contractions $(y_n)_{n\in\mathbb{N}} \in \mathcal{M}$, such that $x_n = a^{1/2}y_n a^{1/2}$, for each $n \in \mathbb{N}$ [52, p. 392]. In particular, given such a factorisation, $x_n \le a$ for all $n \in \mathbb{N}$, and

$$\|(x_n)_{n\in\mathbb{N}}\|_{L^p(\mathcal{M};\ell^\infty)} \le \inf\left\{\|a\|_p\right\},$$

where the infimum is taken over all positive operators $a \in L^p(\mathcal{M})$, such that $x_n \le a$ for all $n \in \mathbb{N}$.

Note that we may also consider finite sequence variants of such vector-valued L^p-spaces. For any fixed integer $m \ge 1$, and any $1 \le p \le \infty$, let $L^p(\mathcal{M};\ell^\infty_m)$ denote the space of all sequences $(x_n)_{n=1}^m \subseteq L^p(\mathcal{M})$, which admit a factorisation of the following form. For some $a, b \in L^{2p}(\mathcal{M})$, and a sequence $(y_n)_{n=1}^m$ in $\mathcal{M}$, $x_n = ay_nb$, for each $1 \le n \le m$. The norm of a sequence $(x_n)_{n\in\mathbb{N}} \in L^p(\mathcal{M};\ell^\infty_m)$ is defined by

$$\|(x_n)_{n=1}^m\|_{L^p(\mathcal{M};\ell^\infty_m)} = \inf\left\{\|a\|_{2p} \sup_{1\le n\le m}\|y_n\|_\infty \|b\|_{2p}\right\},$$

where the infimum runs over all suitable factorisations of $(x_n)_{n=1}^m$. The space $L^p(\mathcal{M};\ell^\infty_m)$ is similarly a Banach space under this norm.

In order to pass from a maximal bound, such as knowing that $(x_n)_{n\in\mathbb{N}} \in L^p(\mathcal{M};\ell^\infty)$ to bilateral almost everywhere convergence, we will relate the sequence of operators to those support projections for which every operator in the sequence attains a minimal height. To do so, we modify the “column tail probability”

$$\mathrm{Prob}_C\left(\sup_{n\in\mathbb{N}}\|x_n\| > t\right) = \inf\left\{s > 0 : e \in \mathrm{Proj}(\mathcal{M});\ \tau(1-e) < s,\ \sup_{n\in\mathbb{N}}\|x_n e\|_\infty \le t\right\},$$

introduced in the thesis of Konwerska [55, Definition 3.15], which was also further studied in by Zeng [89]. We may then find a Chebyshev type inequality, which relates the norm of the maximal L^p-space and projections on which the sequence of operators are uniformly bounded.

Definition 2.14. Given a sequence of self-adjoint operators $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathcal{M},\tau)$, the *maximal rearrangement function*

$$\mu\left(t;(x_n)_{n\in\mathbb{N}}\right) = \inf\left\{s > 0 : e \in \mathrm{Proj}(\mathcal{M}),\ \tau(1-e) < s,\ \sup_{n\in\mathbb{N}}\|e x_n e\|_\infty \le t\right\}.$$

We may now extend [55, Lemma 3.16] to a give bilateral tail variant of the Chebyshev inequality.

Lemma 2.15. *For any fixed $1 \le p < \infty$, if $(x_n)_{n\in\mathbb{N}} \in L^p(\mathcal{M};\ell^\infty)$ is a sequence of self-adjoint operators, then*

$$\mu\left(t;(x_n)_{n\in\mathbb{N}}\right)\leq t^{-2p}\left\|(x_n)_{n\in\mathbb{N}}\right\|^p_{L^p(\mathcal{M};\ell^\infty)},$$

for all $t>0$.

Proof. By characterisation of the space $L^p(\mathcal{M};\ell^\infty)$, there exists a sequence of positive contractions $(y_n)_{n\in\mathbb{N}}\subseteq\mathcal{M}$, and a positive operator $a\in L^p(\mathcal{M})$, such that $x_n=a^{1/2}y_na^{1/2}$ for each $n\in\mathbb{N}$ [52, p. 392]. As such, let $e=\chi_{[0,t]}(a^{1/2})$. We then have that

$$\begin{aligned}\|ex_ne\|_\infty&=\left\|\chi_{[0,t]}(a^{1/2})x_n\chi_{[0,t]}(a^{1/2})\right\|_\infty\\&=\left\|\chi_{[0,t]}(a^{1/2})a^{1/2}y_na^{1/2}\chi_{[0,t]}(a^{1/2})\right\|_\infty\\&\leq t,\end{aligned}$$

for each $n\in\mathbb{N}$. To see that e lies in the set of projections which the infimum of the maximal rearrangement runs over, let us find a suitable estimate,

$$\begin{aligned}\tau(1-e)&=\tau\left(\chi_{(t,\infty)}\left(a^{1/2}\right)\right)\\&=\tau\left(\chi_{(t,\infty)}\left(a^{1/2}\right)^{2p}\right)\\&\leq\tau\left(\left(\chi_{(t,\infty)}\left(a^{1/2}\right)\frac{a^{1/2}}{t}\right)^{2p}\right)\\&\leq t^{-2p}\|a\|_p^p.\end{aligned}$$

This inequality holds for all a for which the factorisation $a^{1/2}y_na^{1/2}=x_n$, and so if we take the infimum, the result follows from the characterisation of vector-valued noncommutative L^p-spaces. □

Remark 2.16. If Lemma 2.15 is an analogue of the Chebyschev inequality, then it is natural to ask about maximal analogues of weak L^p-spaces (see, for example, [9]). If one views the maximal rearrangement function as an extension of the decreasing rearrangement to sequences, then we see that the space $\Lambda^{p,\infty}(\mathcal{M};\ell^\infty)$ studied in [43,44,46] may be understood as the noncommutative weak L^p-space given when one substitutes the decreasing rearrangement by maximal rearrangement. Similarly, the "weak column quasi-norm", defined in [44, p. 1481], is given by substituting the decreasing rearrangement by Prob_C.

2.6. Noncommutative martingales

Finally, let us briefly recall the necessary details of noncommutative martingale theory, and present a suitable restatement of the noncommutative Doob maximal inequality. The interested reader should consult any of [22,71,47] for further details on noncommutative martingales.

Let $\mathcal{M}$ be a finite von Neumann algebra, with a distinguished faithful normal tracial state τ. Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$, such that $\bigcup_{n\in\mathbb{N}}\mathcal{M}_n$ is w*-dense in $\mathcal{M}$. We will call the sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ a *filtration* of $\mathcal{M}$.

Given a von Neumann subalgebra $\mathcal{N}\subseteq\mathcal{M}$, a *conditional expectation* is a map $\mathcal{E}:\mathcal{M}\to\mathcal{N}$, which is a positive contractive projection. Say that the conditional expectation $\mathcal{E}$ is *normal* if the adjoint map $\mathcal{E}^*$ satisfies $\mathcal{E}^*(\mathcal{M}_*)\subseteq\mathcal{N}_*$. For any normal conditional expectation $\mathcal{E}$, there exists a map $\mathcal{E}_*:\mathcal{M}_*\to\mathcal{N}_*$, whose adjoint is $\mathcal{E}$, and so we may assume without loss of generality that $\mathcal{E}$ extends to $L^1(\mathcal{M})$.

We will always assume that given a filtration $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of $\mathcal{N}$, there exists a normal conditional expectation $\mathcal{E}_n:\mathcal{M}\to\mathcal{M}_n$, for each $n\in\mathbb{N}$.

A *noncommutative martingale*, with respect to the filtration $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is a sequence $(x_n)_{n\in\mathbb{N}}$ in $L^1(\mathcal{N})$, such that for every $n\in\mathbb{N}$, $\mathcal{E}_n(x_{n+1})=x_n$. As for a classical martingale, we may equivalently consider a *noncommutative martingale difference sequence* $(dx_n)_{n=1}^\infty$, defined by $dx_{n+1}=x_{n+1}-x_n$, for all $n\geq 1$, and $dx_1=x_1$. It is immediate that a sequence $(dx_n)_{n=1}^\infty$ in $L^1(\mathcal{M})$ forms a martingale difference sequence if and only if $\mathcal{E}_n(dx_{n+1})=0$, for every $n\geq 1$.

A noncommutative martingale is said to be L^1-bounded if $\sup_{n\in\mathbb{N}}\|x_n\|_1<\infty$, in which case there exists an operator $x\in L^1(\mathcal{M})$, such that $x_n=\mathcal{E}_n(x)$, for each $n\in\mathbb{N}$. Similarly, a noncommutative martingale is said to be L^2-bounded if $\sup_{n\in\mathbb{N}}\|x_n\|_2<\infty$. It is expedient for us to appeal to Pisier and Xu's characterisation of L^2-bounded martingales.

Theorem 2.17 *([71, Theorem 2.1]). A noncommutative martingale sequence $(x_n)_{n\in\mathbb{N}}$ in $L^2(\mathcal{M})$ is L^2-bounded if and only if $(x_n)_{n\in\mathbb{N}}\in\mathcal{H}^2(\mathcal{M})$.*

Here, $\mathcal{H}^2(\mathcal{M})$ is the noncommutative Hardy space, for $p=2$, with the norm given by

$$\|(x)_{n\in\mathbb{N}}\|_2=\max\left\{\|(dx_n)_{n\in\mathbb{N}}\|_{L^2(\mathcal{M};\ell^2_C)},\|(dx_n)_{n\in\mathbb{N}}\|_{L^2(\mathcal{M};\ell^2_R)}\right\}.$$

The spaces $L^2(\mathcal{M};\ell^2_C)$ and $L^2(\mathcal{M};\ell^2_R)$ are column and row vector-valued noncommutative L^2-spaces, with these further norms given by

$$\|(dx_n)_{n\in\mathbb{N}}\|_{L^2(\mathcal{M};\ell^2_C)}=\left\|\left(\sum_{n\in\mathbb{N}}|dx_n|^2\right)^{1/2}\right\|_2,$$

and

$$\|(dx_n)_{n\in\mathbb{N}}\|_{L^2(\mathcal{M};\ell^2_R)}=\left\|\left(\sum_{n\in\mathbb{N}}|dx_n^*|^2\right)^{1/2}\right\|_2.$$

Key to our extension of the Révész and Komlós theorems to finite von Neumann algebras is the noncommutative Doob maximal inequality, which we present here in terms of vector-valued L^p-spaces.

Theorem 2.18 *([47, Theorem 0.2]). Let $\mathcal{M}$ be a finite von Neumann algebra, with a distinguished faithful normal tracial state τ. Assume that $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is an increasing filtration of $\mathcal{M}$, with associated normal conditional expectations $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$, for each $n \in \mathbb{N}$.*

For any fixed $1 < p \leq \infty$, and any operator $x \in L^p(\mathcal{M})$, there exist operators $a, b \in L^{2p}(\mathcal{M})$, and a sequence of contractions $(y_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$, such that $\mathcal{E}_n(x) = ay_nb$, for each $n \in \mathbb{N}$, and $\|a\|_{2p} \|b\|_{2p} \leq K_p \|x\|_p$, for some constant $K_p > 0$.

In particular, the martingale $(\mathcal{E}_n(x))_{n\in\mathbb{N}}$ is contained in $L^p(\mathcal{M}; \ell^\infty)$, and

$$\|(\mathcal{E}_n(x))\|_{L^p(\mathcal{M};\ell^\infty)} \leq K_p \|x\|_p .$$

3. Iterated ultrapowers of von Neumann algebras

The key to our proof of the noncommutative Komlós and Révész theorems lies in finding a substitute for the natural filtration induced by hyperfiniteness. While martingale structures are a convenient tool to prove almost uniform convergence (see, for example, [30]) such convergence is intrinsic to the von Neumann algebra. However, without an internal structure (e.g. the hyperfinite filtration) to reveal this behaviour, we look for a suitable external structure. This structure is provided by the theory of iterated ultrapowers, and studying martingales in the algebra given by the limit of these iterations. In this section we will review the construction of ultrapowers of finite von Neumann algebras, and the structure of the associated L_p-spaces. In doing so, we will also discuss the embeddings of noncommutative L^p-spaces into their ultrapowers, and how one may return to an algebra from its ultrapower. We will also discuss the theory of iterated ultrapowers of von Neumann algebras, and the corresponding martingale structure.

3.1. Ultrapowers in noncommutative integration

Let us start with the basic theory of ultrapowers of Banach spaces, as detailed in the paper of Heinrich [39], the book [26], and then we will discuss ultrapowers for von Neumann algebras, and noncommutative L^p-spaces.

An *ultrafilter* $\mathscr{U}$ over a set X is a collection of subsets of X, such that the following conditions are satisfied.

i. $X \in \mathscr{U}$, and $\varnothing \notin \mathscr{U}$.
ii. If $E, F \in \mathscr{U}$, then $E \cap F \in \mathscr{U}$.
iii. If $E \in \mathscr{U}$, and $E \subseteq F$, then $F \in \mathscr{U}$.
iv. For any set $E \subseteq X$, either $E \in \mathscr{U}$, or $X \setminus E \in \mathscr{U}$.

An ultrafilter $\mathscr{U}$ is said to be *non-principal*, or *free*, if there does not exist a set $A \subseteq X$ such that

$$\mathscr{U} = \{E \subseteq X : A \subseteq E\}.$$

For our purposes, ultrafilters are useful because they allow us to extend the notion of limits to non-convergent sequences.

Let $\mathscr{U}$ denote an ultrafilter over the index set I. A family $(x_i)_{i\in I} \subseteq X$ is said to *converge along* $\mathscr{U}$ to $x \in X$ if for any open neighbourhood O of X, the set

$$\{i \in I : x_i \in O\}$$

is an element of $\mathscr{U}$. Denote this point by

$$x = \lim_{i,\mathscr{U}} x_i.$$

The limit is unique, if the topology is Hausdorff.

The limit along an ultrafilter may be defined as such with respect to any topology on X.

If X is a Banach space, a family $(x_i)_{i\in I} \subseteq X$ is said to *converge along* $\mathscr{U}$ to $x \in X$, if for every $\varepsilon > 0$, the set

$$\{i \in I : \|x_i - x\|_X < \varepsilon\}$$

is an element of $\mathscr{U}$. Again, this is denoted by

$$x = \lim_{i,\mathscr{U}} x_i.$$

For a given Banach space X, and an index set I, let

$$\ell^\infty(I; X) = \left\{(x_i)_{i\in I} : x_i \in X \text{ for all } i \in I, \text{ and } \sup_{i\in I} \|x_i\|_X < \infty\right\}.$$

Under the norm

$$\|(x_i)_{i\in I}\| = \sup_{i\in I} \|x_i\|_X,$$

$\ell^\infty(I; X)$ forms the Banach space of all bounded X-valued sequences over I.

Given a non-principal ultrafilter $\mathscr{U}$ over $\mathbb{N}$, and a Banach space X, the *(Banach space) ultrapower* is defined to be the quotient space

$$X^\omega = \ell^\infty(\mathbb{N}; X)/\mathscr{N}_{\mathscr{U}},$$

where

$$\mathscr{N}_{\mathscr{U}} = \left\{ (x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}; X) : \lim_{n,\mathscr{U}} \|x_n\|_X = 0 \right\}.$$

We may also denote X^ω by $X^\omega_{\mathscr{U}}$ when we wish to make the choice of ultrafilter explicit.

Note that the quotient norm on X^ω may be evaluated simply as the limit over the ultrafilter [39],

$$\|(x_i)^\bullet_{i\in I}\|_{X^\omega} = \lim_{i,\mathscr{U}} \|x_i\|_X .$$

It is tempting to define the ultrapower of a von Neumann algebra in a similar manner, and indeed one may very successfully do so following the Groh–Raynaud construction [34,74]. However, there are substantial costs associated to this construction, in that the ultrapower of a semifinite and σ-finite von Neumann algebra need not again be semifinite, nor σ-finite [2]. Instead, we will study the Ocneanu construction [64], which extends the prior constructions of McDuff [61] and Vesterstrøm [87]. For complete proofs of the properties of the Ocneanu construction, see [64,2,37].

Let $\mathscr{U}$ be a non-principal ultrafilter over some index set I, and let $\mathcal{M}$ be a σ-finite von Neumann algebra, with a distinguished faithful normal semifinite trace τ.

Consider the two-sided closed ideal

$$\mathscr{I}_{\mathscr{U}} = \left\{ (x_i)_{i\in I} : \lim_{i,\mathscr{U}} \tau(x_i^* x_i) = 0 \right\},$$

contained in $\ell^\infty(I; \mathcal{M})$. The quotient space

$$\mathcal{M}^\omega = \mathcal{M}^\omega_{\mathscr{U}} = \ell^\infty(I; \mathcal{M})/\mathscr{I}_{\mathscr{U}}$$

is not only a C*-algebra, but also a von Neumann algebra. Moreover, if $\mathcal{M}$ is finite, then so too is $\mathcal{M}^\omega$. Let

$$\pi : \ell^\infty(I; \mathcal{M}) \to \mathcal{M}^\omega \tag{3.1}$$

denote the quotient map, defined for each sequence $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(I; \mathcal{M})$ by

$$\pi\left((x_i)_{i\in I}\right) = (x_i)^\bullet_{i\in I},$$

where $(x_i)^\bullet_{i\in I}$ denotes the equivalence class of $(x_i)_{i\in I}$ in $\mathcal{M}^\omega$.

The algebra $\mathcal{M}^\omega$ admits a faithful normal tracial state,

$$\tau_{\mathscr{U}}(\widetilde{x}) = \lim_{i,\mathscr{U}} \tau(x_i),$$

where $\widetilde{x} = (x_i)^\bullet_{i\in I}$.

Note that we never consider the Banach space ultraproduct of $\mathcal{M}$, and so the notation $\mathcal{M}^\omega$ is unambiguous for our purposes.

Before we may consider the iterated ultraproduct construction, we must find what Hilbert space $\mathcal{M}^\omega$ acts upon. Assume that a von Neumann algebra acts upon the Hilbert space $\mathcal{H}$. Following Raynaud [74, Section 1], let $\mathcal{H}^\omega$ denote the ultrapower of $\mathcal{H}$. There exists a unital isometry

$$j : \mathcal{B}(\mathcal{H})^\omega \hookrightarrow \mathcal{B}(\mathcal{H}^\omega),$$

defined by the map

$$(j(\widetilde{T}))(\widetilde{x}) = (T_i(x_i))^\bullet_{i\in I},$$

where $\widetilde{T} = (T_i)^\bullet_{i\in I} \in \mathcal{B}(\mathcal{H})^\omega$, and $\widetilde{x} = (x_i)^\bullet_{i\in I} \in \mathcal{H}^\omega$. Note that $\mathcal{H}^\omega$ is indeed a Hilbert space, as it may easily be verified that it satisfies the parallelogram identity. This inclusion will be necessary for us to define the iterated ultrapower of a von Neumann algebra.

Let us now return to noncommutative L^p-spaces. Given that $L^p(\mathcal{M}^\omega) \neq L^p(\mathcal{M})^\omega$, how should we understand the relationship between the McDuff ultrapower, and the ultrapowers of noncommutative L^p-spaces? To start, we do have the following inclusion,

$$L^p(\mathcal{M}^\omega, \tau_{\mathscr{U}}) \hookrightarrow L^p(\mathcal{M}, \tau)^\omega,$$

and more precisely, the noncommutative L^p-space associated to the McDuff ultrapower is a corner of the ultrapower of the L^p-space associated to original algebra. We are then able to find an explicit condition for when an element of $L^p(\mathcal{M}, \tau)^\omega$ lies in $L^p(\mathcal{M}^\omega, \tau_{\mathscr{U}})$, following the technique of [37].

For $1 \leq p < \infty$, let Y^p denote the closure of $\ell^\infty(I; \mathcal{M})$ in $\ell^\infty(I; L^p(\mathcal{M}))$. By [37, Lemma 2.13], the quotient map π, defined by (3.1), admits a unique bounded extension, $\widetilde{\pi} : Y^p \to L^p(\mathcal{M}^\omega, \tau_{\mathscr{U}})$, such that for any $(x_i)_{i\in I} \in Y^p$,

$$\|\widetilde{\pi}\left((x_i)_{i\in I}\right)\|_{L^p(\mathcal{M}^\omega, \tau_{\mathscr{U}})} = \lim_{i,\mathscr{U}} \|x_n\|_{L^p(\mathcal{M},\tau)}.$$

Let $\rho : \ell^\infty(L^p(\mathcal{M}, \tau)) \to L^p(\mathcal{M}, \tau)^\omega$ denote the quotient map given by the ultrapower construction. As noted following [37, Lemma 2.13], there exists a unique isometric embedding $\iota : L^p(\mathcal{M}^\omega) \to L^p(\mathcal{M})^\omega$, such that the following diagram commutes,

$$\begin{array}{ccc} & & L^p(\mathcal{M}^\omega) \\ & \overset{\widetilde{\pi}}{\nearrow} & \big\downarrow \iota \\ Y^p & \xrightarrow{\ \rho\ } & L^p(\mathcal{M})^\omega \end{array}.$$

As this embedding holds for all $1 \leq p < \infty$, we may unambiguously represent any element of $L^p(\mathcal{M}^\omega)$ as an equivalence class of sequences, by considering its representation $\iota(x) \in$

$L^p(\mathcal{M})^\omega$. We will not distinguish between an element of $L^p(\mathcal{M}^\omega)$ and its equivalence class representation in $L^p(\mathcal{M})^\omega$.

We may determine if some element of $L^p(\mathcal{M})^\omega$ is an element of $L^p(\mathcal{M}^\omega)$, using the following lemma.

Lemma 3.1 *([51, Lemma 18]). Let $1 \leq p < \infty$, and let $\widetilde{x} = (x_i)^\bullet_{i\in I} \in L^p(\mathcal{M},\tau)^\omega$. If $((|x_i|^p)_{i\in I})^\bullet \in L^1(\mathcal{M},\tau)^\omega$ is uniformly integrable, in the sense that for all $\varepsilon > 0$, there exists $K > 0$ such that*

$$\tau\left(|x_i|^p\,\chi_{(K,\infty)}(|x_i|)\right) < \varepsilon,$$

for all $i \in I$, then

$$\widetilde{x} \in L^p(\mathcal{M}^\omega, \tau_{\mathscr{U}}).$$

Remark 3.2. Note that if a sequence $(x_k)_{k\in\mathbb{N}}$ is equimeasurable, then $(|x_k|^p)_{k\in\mathbb{N}}$ is uniformly integrable. This is immediate from the definition of the decreasing rearrangement.

We wish to consider an increasing sequence of the form

$$\mathcal{M} \subseteq \mathcal{M}^\omega \subseteq (\mathcal{M}^\omega)^\omega \subseteq ((\mathcal{M}^\omega)^\omega)^\omega \subseteq \cdots,$$

contained in some enveloping algebra, such that we have a filtration of a finite von Neumann algebra. This sequence of embeddings provides the foundation of the external structure that we require in order to prove our results. We will first consider the inclusions, and then we will consider the enveloping structure.

It is easy to verify that the constant mapping $\kappa_0 : \mathcal{M} \to \mathcal{M}^\omega$, defined by $x \mapsto \widetilde{x} = (x)^\bullet_{i\in I}$ defines an isometric and injective $*$-homomorphism. In the reverse direction, we expect that as $\mathcal{M}^\omega$ is finite, there should be a normal conditional expectation $\mathbb{E}_0 : \mathcal{M}^\omega \to \mathcal{M}$.

Proposition 3.3. *The mapping $\mathbb{E}_0 : \mathcal{M}^\omega \to \mathcal{M}$, defined by*

$$\mathbb{E}_0(\widetilde{x}) = \lim_{i,\mathscr{U}} x_i,$$

where $\widetilde{x} = (x_i)^\bullet_{i\in I} \in \mathcal{M}^\omega$, and with the limit taken with respect to the weak operator topology on $\mathcal{M}$, forms a normal conditional expectation, and extends to $L^1(\mathcal{M}^\omega)$, by

$$\mathbb{E}_0(\widetilde{x}) = w\text{-}\lim_{i,\mathscr{U}} x_i,$$

for $\widetilde{x} = (x_i)^\bullet_{i\in I} \in L^1(\mathcal{M}^\omega)$, where the limit is taken L^1-weak topology, that is the $\sigma(L^1(\mathcal{M}),\mathcal{M})$-topology.

For $1 < p < \infty$, it follows by interpolation that the conditional expectation extends to $L^p(\mathcal{M}^\omega)$, and is defined by

$$\mathbb{E}_0(\widetilde{x}) = w^*\text{-}L^p\text{-}\lim_{i,\mathscr{U}} x_i,$$

for $\widetilde{x} = (x_i)^\bullet_{i\in I} \in L^p(\mathcal{M}^\omega)$, where the limit is taken in the weak*-topology on $L^p(\mathcal{M}^\omega)$.

Proof. These results are known, but are detailed here for the reader's convenience. We repeat here the details of the construction given in [49, Section 2.2], see also [3, Proposition 2.4]. Consider the constant mapping $\kappa_0 : L^1(\mathcal{M}) \to L^1(\mathcal{M}^\omega)$, which is trace preserving, and as such must be isometric. Let

$$\mathbb{E}_0 = (\kappa_0)^* : \mathcal{M}^\omega \to \mathcal{M}$$

be the adjoint map, explicitly given by

$$\mathbb{E}_0(\widetilde{x}) = wo\text{-}\lim_{i,\mathscr{U}} x_i,$$

for each $\widetilde{x} = (x_i)_{i\in I} \in \mathcal{M}^\omega$, where the limit is taken with respect to the weak operator topology. Note that such a limit exists on any representative element $(x_i)_{i\in I} \in \ell^\infty(I;\mathcal{M})$, as the sequence is bounded, and is well-defined, as the equivalence class $(x_i)^\bullet_{i\in I}$ is defined to be the set of all sequences with null differences in the L^2-topology.

To verify that this is indeed the conditional expectation, let us consider trace invariance. For any operator $y \in \mathcal{M}$, and any $\widetilde{x} = (x_n)_{n\in\mathbb{N}} \in \mathcal{M}^\omega$, we have that

$$\tau(\mathbb{E}_0(\widetilde{x}\cdot y)) = \tau(\mathbb{E}_0(\widetilde{x})y = \tau_\omega\left((x_n)_{n\in\mathbb{N}} \cdot \mathbb{E}_0^*(y)\right) = \lim_{n,\mathscr{U}} \tau(x_n y),$$

where the multiplication $\widetilde{x}\cdot y$ is given by the natural $\mathcal{M}$-bimodule structure of $\mathcal{M}^\omega$, such that $\widetilde{x}\cdot y = (x_n y)^\bullet_{n\in\mathbb{N}}$. It is then clear that this is the correct construction for the conditional expectation.

It is immediate that $\mathbb{E}_0$ is a normal conditional expectation on $\mathcal{M}^\omega$, as it is necessarily contractive, trace preserving, and invariant on the embedding $\kappa_0(\mathcal{M})$, in the sense that $\mathbb{E}_0 \circ \kappa_0$ is the identity map on $\mathcal{M}$.

We then wish to extend $\mathbb{E}_0$, using the density of $\mathcal{M}^\omega$ in $L^1(\mathcal{M}^\omega)$, and continuity given by normality. We will denote the natural extension by

$$\mathbb{E}_0 : L^1(\mathcal{M}^\omega) \to L^1(\mathcal{M}),$$

which is defined by

$$\mathbb{E}_0(\widetilde{x}) = w\text{-}L^1\text{-}\lim_{i,\mathscr{U}} x_i,$$

for all $\widetilde{x} = (x_i)^\bullet_{i\in I} \in L^1(\mathcal{M}^\omega)$, where the limit is taken over the ultrafilter, and with respect to the weak-L^1 topology. Note that as $L^1(\mathcal{M}^\omega)$ embeds into $L^1(\mathcal{M})^\omega$, we may see that this limit is again well defined, in much the same way as for the conditional expectation from $\mathcal{M}^\omega$ to $\mathcal{M}$.

As $L^p(\mathcal{M}^\omega)$ is constructed with respect to a tracial state, we have the inclusion $L^p(\mathcal{M}^\omega) \subseteq L^q(\mathcal{M}^\omega)$, for any $1 \le p \le q \le \infty$. Again, let us note that for any element $\widetilde{x} = (x_n)^\bullet_{n\in\mathbb{N}} \in L^p(\mathcal{M}^\omega)$, and any $1 < p < \infty$, every representative sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in the L^p-norm. Then, we may apply [88, Theorem V.1.3], which states that for a norm-bounded sequence, it is sufficient for weak convergence to only consider functionals from a strongly dense subset of the dual. In particular, the convergence holds for all functionals in $L^\infty(\mathcal{M}^\omega)$, the weak limit in $L^1(\mathcal{M}^\omega)$ is sufficient to determine weak-L^p convergence. That is to say that the conditional expectation $\mathbb{E}_0$ then may be restricted from $L^1(\mathcal{M}^\omega)$, and explicitly calculated by

$$\mathbb{E}_0(\widetilde{x}) = w^*\text{-}L^p\text{-}\lim_{i,\mathscr{U}} x_i,$$

for $\widetilde{x} = (x_i)^\bullet_{i\in I} \in L^p(\mathcal{M}^\omega)$, where we may now consider the weak*-topology, as this coincides with the weak topology. □

Let us now consider a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of non-principal ultrafilters, each over the natural numbers $\mathbb{N}$. Let $\mathcal{M}^{\omega,1} = \mathcal{M}^\omega_{\mathscr{U}_1}$, and let us inductively define $\mathcal{M}^{\omega,n+1}$ for each $n \ge 1$ by $\mathcal{M}^{\omega,n+1} = (\mathcal{M}^{\omega,n})^\omega_{\mathscr{U}_n}$. For each von Neumann algebra $\mathcal{M}^{\omega,n}$, let τ_n denote the trace given inductively by the ultrapower construction, starting with the trace τ on $\mathcal{M}$.

It is clear that the constant inclusion mapping $\mathcal{M}^{\omega,n} \hookrightarrow \mathcal{M}^{\omega,n+1}$ may be used to define the sequence of inclusions

$$\mathcal{M} \subseteq \mathcal{M}^{\omega,1} \subseteq \mathcal{M}^{\omega,2} \subseteq \cdots \mathcal{M}^{\omega,n} \subseteq \mathcal{M}^{\omega,n+1} \subseteq \cdots,$$

however it is not apparent how we may easily express the elements of $\mathcal{M}^{\omega,n}$, which are equivalence classes of equivalence classes of equivalence classes. To make these spaces easier to work with, let us consider the (tensor) products of ultrafilters.

Definition 3.4. Given ultrafilters $\mathscr{U}$ and $\mathscr{V}$ over the natural numbers, the *product ultrafilter* is defined to be the set

$$\mathscr{U} \otimes \mathscr{V} = \{S \subseteq \mathbb{N} \times \mathbb{N} : \{s \in \mathbb{N} : \{t \in \mathbb{N} : (s,t) \in S\} \in \mathscr{U}\} \in \mathscr{V}\}.$$

Although not immediate, we will see that this is a natural definition for the product of ultrafilters. It is also important to note that if $\mathscr{U}$ and $\mathscr{V}$ are non-principal, then $\mathscr{U} \otimes \mathscr{V}$ is also non-principal [80, Proposition 13.1]. The reader is also warned that the ordering of the product $\mathscr{U} \otimes \mathscr{V}$ is sometimes reversed in the literature.

We would like to prove the finite iteration theorem for ultrapowers of von Neumann algebras, which states that the ultrapower of an ultrapower is again an ultrapower. To do so, let us first consider the iteration of limits over ultrafilters. The following lemma is an immediate modification of [16, Theorem 1.3], which is stated for convergence of real numbers, although the proof remains identical for convergence in a Banach space.

Lemma 3.5. *For any two ultrafilters $\mathscr{U}$ and $\mathscr{V}$, we have that for any sequence $(x_{j,k})_{j,k\in\mathbb{N}}$ in a given Banach space X, the iterated limits over $\mathscr{U}$ and $\mathscr{V}$ may be amalgamated into a limit over $\mathscr{U}\otimes\mathscr{V}$, in the sense that*

$$\lim_{j,\mathscr{U}}\lim_{k,\mathscr{V}} x_{j,k} = \lim_{j,k,\mathscr{U}\otimes\mathscr{V}} x_{j,k},$$

where each limit exists if and only if the other does.

It is also clear that the amalgamation of limits also applies to convergence over ultrafilters in a topological space. While we may replace the two limits with one limit over a product ultrafilter, it is not true in general that we may commute the two limits. For further details, see [48], which discusses a Fubini theorem for tensors in noncommutative L^p-spaces, and [60], which shows the failure of the interchange for sums.

Proposition 3.6 *([80, Theorem 13.2]). Let $\mathscr{U}$ and $\mathscr{V}$ be non-principal ultrafilters over $\mathbb{N}$. For any Banach space X, there exists an isometric isomorphism*

$$(X^\omega_{\mathscr{U}})^\omega_{\mathscr{V}} \cong X^\omega_{\mathscr{V}\otimes\mathscr{U}}.$$

3.2. Martingales and infinitely iterated ultrapowers

Having established the basic machinery of ultrapowers of von Neumann algebras and noncommutative L^p-spaces, let us detail the constructions which we will work with, in order to find our necessary external structure for the proof of our key results.

We may consider the first n iterated ultrapowers of $\mathcal{M}$, $\mathcal{M}^{\omega,1},\dots,\mathcal{M}^{\omega,n}$, as von Neumann subalgebras of $\mathcal{B}(\mathcal{H}_{\omega,n})$, where $\mathcal{H}_{\omega,n}$ is the n-th iterated ultrapower of $\mathcal{H}$, the Hilbert space that $\mathcal{M}$ acts upon. Namely, given a sequence of non-principal ultrafilters, $(\mathscr{U}_n)_{n\in\mathbb{N}}$, we let the iterated ultrapower of $\mathcal{H}$ be the inductively defined Hilbert space $\mathcal{H}_{\omega,n+1} = (\mathcal{H}_{\omega,n})^\omega$, where the $n+1$-th ultrapower is taken with respect to the ultrafilter $\mathscr{U}_{n+1}$. Similarly, we may inductively define the iterated von Neumann algebras, such that $\mathcal{M}^{\omega,n+1} = (\mathcal{M}^{\omega,n})^\omega$, again with the $n+1$-th ultrapower taken with respect to the ultrafilter $\mathscr{U}_{n+1}$. This gives a family of inclusions, following from repeated application of Raynaud's isometric embedding of ultrapowers of Hilbert spaces [74], such that

$$\mathcal{B}(\mathcal{H}_{\omega,1}) \hookrightarrow \mathcal{B}(\mathcal{H}_{\omega,2}) \hookrightarrow \cdots \mathcal{B}(\mathcal{H}_{\omega,n}) \hookrightarrow \mathcal{B}(\mathcal{H}_{\omega,n+1}) \hookrightarrow \cdots,$$

which in turn gives the sequence of unital inclusions

$$\mathcal{M} \hookrightarrow \mathcal{M}^{\omega,1} \hookrightarrow \mathcal{M}^{\omega,2} \hookrightarrow \dots \mathcal{M}^{\omega,n} \hookrightarrow \mathcal{M}^{\omega,n+1} \hookrightarrow \cdots.$$

This sequence of inclusions suggests how we may find a sensible enveloping algebra for the family $(\mathcal{M}^{\omega,n})_{n\in\mathbb{N}}$.

Let us recall the basic aspects of the theory of inductive limits of Banach spaces. For further details on these constructions, see [69, Section 10], and [85, Chapter XIV].

Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces, paired with a sequence $(j_n : E_n \to E_{n+1})_{n\in\mathbb{N}}$ of isometric embeddings. Let X denote the space of all sequences $(x_n)_{n\in\mathbb{N}}$ in the set product $\prod_{n\in\mathbb{N}} E_n$, such that for all sufficiently large n, $x_{n+1} = j_n(x_n)$. As $\|x_n\|_{E_n} = \|x_{n+1}\|_{E_{n+1}}$ for all sufficiently large n, for any sequence $(x_n)_{n\in\mathbb{N}} \in X$, the seminorm $\|(x_n)_{n\in\mathbb{N}}\|_X = \lim_{n\in\mathbb{N}} \|x_n\|_{E_n}$ is well-defined. Let $X_0 = \{x \in X : \|x\|_X = 0\}$ denote the kernel of the seminorm. The quotient X/X_0 is then a normed space, and we may define E_∞ to be the completion of X/X_0. We will call the space E_∞ the *(inductive) limit* of the sequence $\{E_n, j_n\}_{n\in\mathbb{N}}$. One may verify that the inductive limit of a sequence of Hilbert spaces satisfies the parallelogram identity, and so is again a Hilbert space.

Let $h_n : \mathcal{H}_{\omega,n} \to \mathcal{H}_{\omega,n+1}$ denote the constant embedding map for the ultrapower Hilbert spaces. The system $\{\mathcal{H}_{\omega,n}, h_n\}_{n\in\mathbb{N}}$ defines the inductive limit $\mathcal{H}_{\omega,\infty}$, which is again a Hilbert space. Let $h_{\infty,n} : \mathcal{H}_{\omega,n} \to \mathcal{H}_{\omega,\infty}$ denote the induced isometric embeddings of each $\mathcal{H}_{\omega,n}$. These embeddings extend naturally to unital embeddings $\mathcal{B}(\mathcal{H}_{\omega,n}) \hookrightarrow \mathcal{B}(\mathcal{H}_{\omega,\infty})$. As such, there exists an embedding of every space $\mathcal{M}^{\omega,n}$ into $\mathcal{B}(\mathcal{H}_{\omega,\infty})$. Moreover, this embedding is compatible with the embeddings of $\mathcal{M}^{\omega,n}$ into $\mathcal{M}^{\omega,n+1}$, in the sense that the following diagram commutes, for all $n, k \in \mathbb{N}$,

$$\begin{array}{ccc} \mathcal{M}^{\omega,n} & \hookrightarrow & \mathcal{B}(\mathcal{H}_{\omega,n}) \\ \downarrow & & \downarrow \\ \mathcal{M}^{\omega,n+k} & \hookrightarrow & \mathcal{B}(\mathcal{H}_{\omega,n+k}) \\ & \searrow & \downarrow \\ & & \mathcal{B}(\mathcal{H}_{\omega,\infty}) \end{array} .$$

This follows immediately from the construction of each of these embeddings.

Under the embedding into $\mathcal{B}(\mathcal{H}_{\omega,\infty})$, we may consider the union

$$\mathcal{M}^\omega_\cup = \bigcup_{n\in\mathbb{N}} \mathcal{M}^{\omega,n},$$

and define $\mathcal{M}^{\omega,\infty}$ to be the w*-closure of $\mathcal{M}^\omega_\cup$. We will call $\mathcal{M}^{\omega,\infty}$ the *iterated ultrapower* of a von Neumann algebra, although one could in principle continue to iterate the ultrapower construction so far as set theory allows.

Proposition 3.7. *The space $\mathcal{M}^{\omega,\infty}$ is a finite von Neumann algebra, with some faithful normal tracial state τ_∞ defined on the positive cone $\mathcal{M}^{\omega,\infty}_+$ by*

$$\tau_\infty(x) = \sup_{\lambda\in\Lambda} \tau_{n(\lambda)}(x_\lambda),$$

where $(x_\lambda)_{\lambda\in\Lambda}$ *is an increasing net of positive operators in* $\bigcup_{n\in\mathbb{N}} \mathcal{M}^{\omega,n}$*, which converge to* x *in the strong operator topology, and* $n(\lambda)$ *is the least integer such that* $x_{n(\lambda)} \in \mathcal{M}^{\omega,n}$*.*

Proof. It is immediate that $\mathcal{M}^{\omega,\infty}$ is a von Neumann algebra, and it will follow that it is finite if we may show that τ_∞ is a well-defined faithful normal tracial state, as claimed.

As $\mathcal{M}^\omega_\cup$ is dense in $\mathcal{M}^{\omega,\infty}$, there must exist, for any $x \in \mathcal{M}^{\omega,\infty}$, a bounded and increasing net in $\mathcal{M}^\omega_\cup$ which converges in the strong operator topology to x. As the trace is necessarily normal on bounded increasing nets, the definition is natural, and well-defined, as the conditional expectation between any two subalgebras $\mathcal{M}^{\omega,n}$ and $\mathcal{M}^{\omega,n}$ is trace preserving.

The construction of τ_∞ makes clear that it is tracial, normal, and faithful. Finally, it follows from the sequence of unital embeddings which defines $\mathcal{M}^\omega_\cup$ that the identity of $\mathcal{M}^{\omega,\infty}$ lies in every $\mathcal{M}^{\omega,n}$, and as each τ_n is a state, so too is τ_∞. □

As such, we now have a "noncommutative probability space" $(\mathcal{M}^{\omega,\infty}, \tau_\infty)$, and a filtration $(\mathcal{M}^{\omega,n})_{n\in\mathbb{N}}$. The restriction $\tau_\infty \restriction_{\mathcal{M}^{\omega,n}}$ recovers τ_n, by construction, and so for each $n \in \mathbb{N}$, there exists a conditional expectation $\mathcal{E}_n : \mathcal{M}^{\omega,\infty} \to \mathcal{M}^{\omega,n}$ [84, Theorem IX.4.2]. Moreover, if $x_{n+k} \in \mathcal{M}^{\omega,n+k}$, for any $n, k \in \mathbb{N}$, then by uniqueness of the conditional expectation,

$$\mathcal{E}_n(x_{n+k}) = (\mathbb{E}_n \circ \mathbb{E}_{n+1} \circ \cdots \circ \mathbb{E}_{n+k-2} \circ \mathbb{E}_{n+k-1})(x_{n+k}). \tag{3.2}$$

As we do not consider the space $L^1(\mathcal{M}^{\omega,\infty})$, we do not need to show that $\mathcal{E}_n$ is normal.

This construction forms the foundation of our proof, wherein we will show that suitable maximal inequalities hold, as a result of the noncommutative Doob martingale convergence theorem, at the level of $\mathcal{M}^{\omega,\infty}$.

While it will be more or less straightforward to show that the necessary bilateral almost uniform convergence holds in the iterated ultrapower, it is not at all clear how to show that this implies bilateral almost uniform convergence in the original algebra. This is a subtle problem, and requires a careful construction. The following result will be key to allowing us to drag almost everywhere convergence back to the original algebra. Before we prove Proposition 3.9, we will need a small technical lemma. This will allow us to decompose an element of $\mathcal{M}^\omega$ into a bounded sequence in $\mathcal{M}$, and a fixed element of $L^q(\mathcal{M})$ of arbitrarily small norm.

Lemma 3.8. *Let* $\widetilde{y} = (y(j))^\bullet_{j\in\mathbb{N}} \in \mathcal{M}^\omega$*. For any* $\varepsilon > 0$*, and any fixed* $1 < q < \infty$*, there exists a sequence* $(z(j))_{j\in\mathbb{N}} \subseteq \mathcal{M}$*, and an operator* $\sigma \in L^q(\mathcal{M})$*, such that*

$$y(j) = z(j) + \sigma,$$

for all $j \in \mathbb{N}$*,*

$$\sup_{j\in\mathbb{N}} \|z(j)\|_{\mathcal{M}} \leq (1+\varepsilon) \|y\|_{\mathcal{M}^\omega},$$

and $\|\sigma\|_q \leq \varepsilon$.

Proof. In order to find a suitable decomposition, we will use the Kaplansky density theorem. It is immediate that the quotient map π, given by (3.1), is a surjective contraction map,

$$\pi : \ell^\infty(\mathcal{M}) \twoheadrightarrow \mathcal{M}^\omega.$$

By duality, and the isomorphism $L^1(\mathcal{M}^\omega)^* \cong \mathcal{M}^\omega$, we have that

$$L^1(\mathcal{M}^\omega) \hookrightarrow L^1(\mathcal{M}^\omega)^{**} \cong (\mathcal{M}^\omega)^* \hookrightarrow \ell^\infty(\mathcal{M}^\omega)^*.$$

Applying duality again, now to the induced map

$$s : L^1(\mathcal{M}^\omega) \hookrightarrow \ell^\infty(\mathcal{M}^\omega)^*,$$

we find that there exists a contractive normal surjection

$$s^* : \ell^\infty(\mathcal{M})^{**} \twoheadrightarrow \mathcal{M}^\omega.$$

There then exists, for any $x \in \mathcal{M}^\omega$, with $\|x\| \leq 1$, some $x^{**} \in \ell^\infty(\mathcal{M})^{**}$, such that $\|x^{**}\| \leq 1$, and $s^*(x^{**}) = x$.

By the Kaplansky density theorem (see, for example, [53, Theorem 5.3.5]), there exists a net $(x_\lambda)_{\lambda\in\Lambda} \in \ell^\infty(\mathcal{M})$, such that $(x_\lambda)_{\lambda\in\Lambda}$ converges in the strong operator topology to x^{**}. In turn, $(s^*(x_\lambda))_{\lambda\in\Lambda}$ converges to x in the strong operator topology. Applying [4, Lemma 2.5], given that $\mathcal{M}$ is a finite von Neumann algebra, that convergence in the strong operator topology implies that the net $(s^*(x_\lambda) \cdot 1)_{\lambda\in\Lambda}$, where 1 is the identity operator, converges in the L^1-topology. Moreover, by [50, Lemma 2.3], the net converges in the $L^q(\mathcal{M}^\omega)$-topology. As this is a metrisable topology, it is sufficient to consider a sequence.

In particular, if we apply the quotient mapping π, we see that every element $\widetilde{y} \in \mathcal{M}^\omega$ is approximated in the strong L^q-topology by some sequence $(\widetilde{y}_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}^\omega$, and so there exists some $\widetilde{y}_m \in \mathcal{M}^\omega$, such that $\|\widetilde{y} - \widetilde{y}_m\|_q < \varepsilon$. We may then set $\sigma = w^*\text{-}L^q\text{-}\lim_{n\in\mathbb{N}}(y(n) - y_m(n))$, where $\widetilde{y} = (y(n))^\bullet_{n\in\mathbb{N}}$, and $\widetilde{y}_m = (y_m(n))^\bullet_{n\in\mathbb{N}}$, for each $m \in \mathbb{N}$.

By construction of the sequence $(\widetilde{y}_m)_{m\in\mathbb{N}}$, using the Kaplansky density theorem, we have that $\|\widetilde{y}_m\|_{\mathcal{M}^\omega} \leq \|\widetilde{y}\|_{\mathcal{M}^\omega}$. As the norm on $\mathcal{M}^\omega$ is given by a quotient norm on $\ell^\infty(\mathcal{M})$, there must exist a representative sequence $(z_n)_{n\in\mathbb{N}}$ of the equivalence class $\widetilde{y}_m$, such that $\sup_{n\in\mathbb{N}} \|z_n\|_{\mathcal{M}} \leq (1+\epsilon)\,\|\widetilde{y}_m\|_{\mathcal{M}^\omega}$, and in particular such that $\sup_{n\in\mathbb{N}} \|z_n\|_{\mathcal{M}} \leq (1+\epsilon)\,\|\widetilde{y}\|_{\mathcal{M}^\omega}$.

This gives the necessary decomposition. □

Proposition 3.9. *For any finite von Neumann algebra $\mathcal{M}$, and any sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of non-principal ultrafilters over $\mathbb{N}$, let $(\mathcal{M}^{\omega,n})_{n\in\mathbb{N}}$ denote the sequence of iterated ultrapowers of $\mathcal{M}$.*

For any integer $m \geq 1$, and any choice of indices $1 < \widetilde{p} < p < \infty$, consider the sequence $(\widetilde{y}_k)_{k=1}^m \in L^p(\mathcal{M}^{\omega,1}; \ell_m^\infty)$, where $\widetilde{y}_k = (y_k(j))^\bullet_{j\in\mathbb{N}}$, for each k. We then have that

$$\lim_{j,\mathscr{U}_1} \|y_k(j)\|_{L^{\widetilde{p}}(\mathcal{M};\ell_m^\infty)} \leq \|\widetilde{y}_k\|_{L^p(\mathcal{M}^{\omega,1};\ell_m^\infty)}\,. \tag{3.3}$$

Moreover, for any $n \geq 1$, if there exists a sequence $(\widetilde{y}_k)_{k=1}^m \in L^p(\mathcal{M}^{\omega,n}; \ell_m^\infty)$, such that for each $1 \leq k \leq m$, $\widetilde{y}_k = (y(1_1, \ldots, i_n))^\bullet_{i_1,\ldots,i_n\in\mathbb{N}}$, then we have that

$$\lim_{j_n,\mathscr{U}_n} \lim_{j_{n-1},\mathscr{U}_{n-1}} \cdots \lim_{j_1,\mathscr{U}_1} \|y_k(j_1, j_2, \ldots, j_{n-1}, j_n)\|_{L^{\widetilde{p}}(\mathcal{M};\ell_m^\infty)} \leq \|\widetilde{y}_k\|_{L^p(\mathcal{M}^{\omega,n};\ell_m^\infty)}\,. \tag{3.4}$$

Proof. Given $(\widetilde{y}_k)_{k=1}^m \in L^p(\mathcal{M}^\omega; \ell_m^\infty)$, let $\alpha = \|(\widetilde{y}_k)_{k=1}^m\|_{L^p(\mathcal{M}^\omega;\ell_m^\infty)}$. By definition of the space $L^p(\mathcal{M}^\omega; \ell_m^\infty)$, see Subsection 2.5, there exists a decomposition for any $\varepsilon > 0$ and each $1 \leq k \leq m$, $\widetilde{y}_k = \widetilde{a}\widetilde{z}_k\widetilde{b}$, such that

$$\|\widetilde{a}\|_{L^{2p}(\mathcal{M}^\omega)} = \sqrt{(1+\varepsilon)\alpha}, \qquad \|\widetilde{b}\|_{L^{2p}(\mathcal{M}^\omega)} = \sqrt{(1+\varepsilon)\alpha},$$

and

$$\|\widetilde{z}_k\|_{\mathcal{M}^\omega} \leq 1.$$

Let us fix any $\widetilde{p}$, with $1 < \widetilde{p} < p$, and then let q be such that $1/\widetilde{p} = 1/p + 1/q$.

Let us apply Lemma 3.8 to each $\widetilde{z}_k$. Let $(z_k(n))_{n\in\mathbb{N}}$ be a representative sequence for $\widetilde{z}_k$, for each k. Then there exists a sequence $(z'_k(n))_{n\in\mathbb{N}}$, and an operator $\sigma_k \in L^q(\mathcal{M})$, for each k, such that $z_k(n) = z'_k(n) + \sigma_k$, and $\|\sigma_k\|_q \leq 2^{-k}\varepsilon$. We also have that

$$\|(z'_k(n))_{n\in\mathbb{N}}\|_{\ell^\infty(\mathcal{M})} \leq (1+\varepsilon)\|\widetilde{z}_k\|_{\mathcal{M}^\omega}\,.$$

Combining our decompositions,

$$y_k(n) = a(n)z'_k(n)b(n) + a(n)\sigma_k b(n), \tag{3.5}$$

for each $1 \leq k \leq m$, and $n \in \mathbb{N}$.

Setting $x_k(n) = a(n)z'_k(n)b(n)$, for each $1 \leq k \leq m$, and $n \in \mathbb{N}$, we have that

$$\lim_{n,\mathscr{U}_1} \|(x_k(n))_{k=1}^m\|_{L^{\widetilde{p}}(\mathcal{M};\ell_m^\infty)} \leq (1+\varepsilon)\alpha = (1+\varepsilon)\|(\widetilde{y}_k)_{k=1}^m\|_{L^p(\mathcal{M}^\omega;\ell_m^\infty)}\,. \tag{3.6}$$

Let us then find a suitable estimate on the sequence $((a(n)\sigma_k b(n))_{n\in\mathbb{N}})_{k=1}^m$,

$$\lim_{n,\mathscr{U}_1} \|(a(n)\sigma_k b(n))_{k=1}^m\|_{L^{\widetilde{p}}(\mathcal{M};\ell_m^\infty)} \leq \lim_{n,\mathscr{U}_1} \sup_{1\leq k\leq m} \|a(n)\|_{2p}\|\sigma_k\|_q\|b(n)\|_{2q}$$

$$
\begin{aligned}
&\leq \lim_{n,\mathscr{U}_1} \sum_{k=1}^{m} \|a(n)\|_{2p} \|\sigma_k\|_q \|b(n)\|_{2p} \\
&= \sum_{k=1}^{m} \lim_{n,\mathscr{U}_1} \|a(n)\|_{2p} \|\sigma_k\|_q \|b(n)\|_{2p} \\
&= \sum_{k=1}^{m} \|\widetilde{a}\|_{2p} \|\sigma_k\|_q \left\|\widetilde{b}\right\|_{2p} \\
&\leq \sum_{k=1}^{m} 2^{-k}\varepsilon(1+\varepsilon)\alpha \\
&\leq \varepsilon(1+\varepsilon)\alpha.
\end{aligned}
$$

Taking the limit over all decompositions as in (3.5), and taking the limit $\varepsilon \to 0$ in the choice of such decompositions, we have that

$$
\lim_{n,\mathscr{U}_1} \|(a(n)\sigma_k b(n))_{k=1}^m\|_{L^{\tilde{p}}(\mathcal{M};\ell_m^\infty)} = 0.
$$

By the triangle inequality, and equation (3.6)

$$
\begin{aligned}
&\lim_{n,\mathscr{U}_1} \|(y_k(n))_{k=1}^m\|_{L^{\tilde{p}}(\mathcal{M};\ell_m^\infty)} \\
&\quad \leq \lim_{\varepsilon\to 0} \lim_{n,\mathscr{U}_1} \left(\|(a(n)z_k(n)b(n))_{k=1}^m\|_{L^{\tilde{p}}(\mathcal{M};\ell_m^\infty)} + \|(a(n)\sigma_k b(n))_{k=1}^m\|_{L^{\tilde{p}}(\mathcal{M};\ell_m^\infty)} \right) \\
&\quad = \lim_{\varepsilon\to 0} (1+\varepsilon) \|(\widetilde{y}_k)_{k=1}^m\|_{L^p(\mathcal{M}^\omega;\ell_m^\infty)} + 0 \\
&\quad = \|(\widetilde{y}_k)_{k=1}^m\|_{L^p(\mathcal{M}^\omega;\ell_m^\infty)},
\end{aligned}
$$

which gives (3.3).

It is clear that the result may be iterated to give (3.4). □

4. An iterated ultrapower martingale convergence theorem

The following result is a substantial application of the Doob maximal inequality at the level of the iterated ultrapower. It allows us to take a weakly null sequence, such that some natural martingale in $L^2(\mathcal{M}^{\omega,\infty})$ is L^2-bounded, and find a subsequence such that the series over all further subsequences converge bilaterally almost uniformly. This is not difficult in classical proofs of the Komlós and Révész theorems, however the process of passing back down from the iterated ultrapower is substantial, and requires a careful diagonalisation argument. While the following result is hard, once it is shown, the proofs of the Komlós and Révész theorems are more or less straightforward.

With the goal of applying the Doob maximal inequality, let us consider a sequence of embeddings of a given element $\widetilde{w} \in L^1(\mathcal{M}^{\omega,1})$ into $L^1(\mathcal{M}^{\omega,k})$, such that we have a martingale difference sequence.

For each $k \geq 1$, define the mapping

$$\pi_k : \ell^\infty(\mathbb{N}; \mathcal{M}) \to \ell^\infty(\mathbb{N}^k; \mathcal{M})$$

by $\pi_k(x)(a_1, \ldots, a_k) = x(a_k)$, for any indices $a_1, \ldots, a_k \in \mathbb{N}$, and each sequence $(x(k))_{k\in\mathbb{N}} \in \ell^\infty(\mathcal{M})$. It is easy to see that π_k induces a well-defined mapping on $\mathcal{M}^\omega$, which we will again denote by $\pi_k : \mathcal{M}^{\omega,1} \to \mathcal{M}^{\omega,k}$.

Moreover, π_k naturally extends to a mapping $\pi_k : L^2(\mathcal{M})^\omega_{\mathscr{U}_1} \to L^2(\mathcal{M})^{\omega,k}_{\mathscr{U}_1\otimes\cdots\otimes\mathscr{U}_k}$, defined by

$$\pi_k(\widetilde{x}) = (\pi_k(x)(a_1, \ldots, a_k))^\bullet_{a_1,\ldots,a_k\in\mathbb{N}} = (x(a_k))^\bullet_{a_k\in\mathbb{N}},$$

for every equivalence class $\widetilde{x} = (x(n))^\bullet_{n\in\mathbb{N}}$. If $\widetilde{x} \in L^2(\mathcal{M})^\omega_{\mathscr{U}_1}$ is such that $(|x_k|^2)_{k\in\mathbb{N}}$ is uniformly integrable, then $(|\pi_k(\widetilde{x})(a_1, \ldots, a_k)|^2)_{a_1,\ldots,a_k\in\mathbb{N}}$ is also uniformly integrable, by the definition of the trace on $\mathcal{M}^{\omega,k}_{\mathscr{U}_1\otimes\cdots\otimes\mathscr{U}_k}$ and so π_k also induces a bounded linear map from $L^2(\mathcal{M}^{\omega,1})$ to $L^2(\mathcal{M}^{\omega,k})$.

Given that equimeasurable sequences of operators are uniformly integrable, as per Remark 3.2, it follows that $(|\pi_k(\widetilde{x})(a_1, \ldots, a_k)|^2)_{a_1,\ldots,a_k\in\mathbb{N}}$ is uniformly integrable, which allows us consider $\pi_k(\widetilde{x})$ as an element of $L^2(\mathcal{M}^{\omega,k})$, by Lemma 3.1.

Remark 4.1. If $(w_k)_{k\in\mathbb{N}}$ is a representative sequence of $\widetilde{w} \in L^2(\mathcal{M})^\omega_{\mathscr{U}_1}$, which has a null limit in the weak-L^2 topology, with respect to the limit over the ultrafilter $\mathscr{U}_1$, then it is immediate from the definition of the conditional expectation, (3.2), that $\mathscr{E}_{k-1}(\pi_k(\widetilde{w})) = 0$, for every $k \geq 1$, and so $(\pi_k(\widetilde{w}))_{k\in\mathbb{N}}$ forms a martingale difference sequence.

The following theorem is central to the resolution of Randrianantoanina's question. It shows that given a sequence of operators in a finite von Neumann algebra, which is weakly null, we have that the martingale convergence structure, given by an embedding into the iterated ultrapower, yields a diagonalisation, such that there exists a subsequence of operators, for which all further subsequences generate series which converge bilaterally almost uniformly. Moreover, this may be done for any countable family of weakly null sequences, and so the following result is indeed a specialised diagonalisation argument, which is facilitated through the machinery of the iterated ultrapower.

Theorem 4.2. *Let $\mathcal{M}$ be a finite von Neumann algebra, with a distinguished faithful normal tracial state τ. Then let $(\mathcal{M}^{\omega,n})_{n\in\mathbb{N}}$ denote the increasing filtration of $\mathcal{M}^{\omega,\infty}$, following the construction discussed in Subsection 3.2.*

Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence of equimeasurable operators in $L^1(\mathcal{M})$, which is uniformly bounded in the L^1-norm. Let $\widetilde{x} = (x_n)^\bullet_{n\in\mathbb{N}} \in L^1(\mathcal{M}^\omega)$. Given a sequence $(c_k)_{k\in\mathbb{N}} \in \ell^2$, such that

$$\left(\sum_{k=1}^{n} c_k \pi_k(\widetilde{x})\right)_{n=1}^{\infty}$$

is a $L^2(\mathcal{M}^{\omega,\infty})$-bounded martingale, there exists a sequence $(y_k)_{k\in\mathbb{N}} \subseteq (x_k)_{k\in\mathbb{N}}$, such that for every further subsequence $(z_k)_{k\in\mathbb{N}} \subseteq (y_k)_{k\in\mathbb{N}}$, the series

$$\sum_{k=1}^{\infty} c_k z_k$$

converges bilaterally almost uniformly.

Moreover, let $((x_{k,n}))_{k,n\in\mathbb{N}}$ be a countable family of sequences, such that for every $k \in \mathbb{N}$, $(x_{k,n})_{n\in\mathbb{N}}$ is a weakly null sequence of equimeasurable operators in $L^1(\mathcal{M})$. Let $\widetilde{x}_k = (x_{k,n})^{\bullet}_{n\in\mathbb{N}}$, for each $k \in \mathbb{N}$. Given a sequence $(c_k)_{k\in\mathbb{N}} \in \ell^2$, such that

$$\left(\sum_{k=1}^{n} c_k \pi_k(\widetilde{x}_k)\right)_{n=1}^{\infty}$$

is a $L^2(\mathcal{M}^{\omega,\infty})$-bounded martingale, there exists an increasing sequence of indices, $(s_n)_{n\in\mathbb{N}}$, such that for every subsequence $(t_n)_{n\in\mathbb{N}} \subseteq (s_n)_{n\in\mathbb{N}}$, the series

$$\sum_{k=1}^{\infty} c_k x_{k,t_k}$$

converges bilaterally almost uniformly.

Proof. The proof consists of three parts. Firstly, we must pass from convergence at the level of $L^2(\mathcal{M}^{\omega,\infty})$ to convergence in $L^{\widetilde{p}}(\mathcal{M})$, for some $1 < \widetilde{p} < 2$, and we will do so in a way that gives us a sequence of maximal inequalities, each given by a limit over the product ultrafilters. Secondly, because these maximal inequalities only hold in the limit, we must find a sequence which approximates these inequalities, and we must find such a sequence such that all further subsequences approximate these inequalities. This diagonalisation argument forms the heart of the proof. Finally, we make use of the maximal inequalities in order to find projections which show that the bilateral almost uniform convergence holds. This will follow from Lemma 2.15.

It is clear that the claimed first result for a fixed element $\widetilde{x} \in L^1(\mathcal{M}^\omega)$ follows from the general result for a sequence $(\widetilde{x}_k)_{k\in\mathbb{N}}$, and so we only consider the later case.

Let, for each $n \in \mathbb{N}$,

$$M_n = \sum_{k=1}^{n} c_k \pi_k(\widetilde{x}_k)$$

denote the n-th term in the martingale sequence. As the sequence $(M_n)_{n=1}^{\infty}$ is $L^2(\mathcal{M}^{\omega,\infty})$-strongly convergent, there exists some constant $C > 0$, and some increasing family of indices $(I_k)_{k\in\mathbb{N}}$, with $I_1 = 0$, where $M_0 = 0$, such that

$$\left\|M_{I_{k+1}} - M_{I_k}\right\|_2 \leq 32^{-k} C,$$

for every $k \in \mathbb{N}$. Let $J_k = I_{k+1} - I_k$, for each $k \in \mathbb{N}$.

If we consider only the tail of the series, we see that $(M_n - M_{I_k})_{n>I_k}$ still forms an L^2-bounded martingale, and an easy application of the noncommutative Doob maximal inequality to finite martingales (Theorem 2.18) shows that there exists a constant $C_2 > 0$, such that

$$\left\| \left(\sum_{j=I_k+1}^{n} c_k \pi_j(\widetilde{x}_j) \right)_{n=I_k+1}^{I_{k+1}} \right\|_{L^2(\mathcal{M}^{\omega,I_{k+1}};\ell^\infty_{J_k})} \leq \left\| M_{I_{k+1}} - M_{I_k} \right\|_{L^2(\mathcal{M}^{\omega,I_{k+1}})} \leq 32^{-k} C_2, \tag{4.1}$$

for each $k \in \mathbb{N}$, where we may consider the norm for $L^2(\mathcal{M}^{\omega,I_{k+1}})$ in place of $L^2(\mathcal{M}^{\omega,\infty})$ because of the isometric embedding of these Hilbert spaces [74, Section 1]. Applying Proposition 3.9 to (4.1), we find that

$$\lim_{\substack{(a_1,\ldots,a_{J_k})\in\mathbb{N}^{J_k} \\ \mathscr{U}_{I_k+1}\otimes\cdots\otimes\mathscr{U}_{I_{k+1}}}} \left\| \left(\sum_{j=I_k+1}^{n} c_j x_{j,a_j} \right)_{n=I_k+1}^{I_{k+1}} \right\|_{L^{\tilde{p}}(\mathcal{M};\ell^\infty_{J_k})} \leq 32^{-k} C_2, \tag{4.2}$$

for some fixed $1 < \widetilde{p} < 2$, and each $k \in \mathbb{N}$.

This leads us to the second part of the proof, where we will use a diagonalisation argument to find sequences which approximate the limit (4.2). In particular, for any fixed index $k \in \mathbb{N}$, and any fixed finite sequence $(a_{I_k+1}, \ldots, a_{I_{k+1}}) \in \mathbb{N}^{J_k}$, let us say that the sequence satisfies condition (C_k) if the inequality

$$\left\| \left(\sum_{j=I_k+1}^{n} c_j x_{j,a_j} \right)_{n=I_k+1}^{I_{k+1}} \right\|_{L^{\tilde{p}}(\mathcal{M};\ell^\infty_{J_k})} \leq 2C_2 \cdot 32^{-k} \tag{C_k}$$

holds true. Let us say that an infinite sequence $(a_j)_{j\in\mathbb{N}}$ satisfies condition (C_k) if the finite subsequence $(a_{I_k+1}, \ldots, a_{I_{k+1}})$ satisfies condition (C_k).

We wish to find an infinite sequence which satisfies condition (C_k) for every $k \in \mathbb{N}$, and such that every further subsequence satisfies condition (C_k) for every $k \in \mathbb{N}$. To do so, we will iterate an induction argument to find a suitable sequence for each condition (C_k). This iterated induction will give us a suitable diagonalised sequence.

Let us start by considering condition (C_1). It is immediate from (4.2) that there must exist some set $A \in \mathscr{U}_1 \otimes \cdots \otimes \mathscr{U}_{I_2}$, such that every sequence in A satisfies condition (C_1), as otherwise the limit in equation (4.2) cannot be achieved.

We now wish to find a sequence $(a_j)_{j\in\mathbb{N}}$, such that every finite subsequence of length J_1 satisfies condition (C_1). For a fixed finite sequence $b = (b_1, \ldots, b_m)$, of length $m < J_1$, and a fixed set $A \in \mathscr{U}_1 \otimes \cdots \otimes \mathscr{U}_{I_2}$, let A_b denote the set of all finite sequences $c =$

$(c_{m+1}, \dots, c_{J_1})$, such that $(b_1, \dots, b_m, c_{m+1}, \dots, c_{J_1}) \in A$. By construction of the product ultrafilter, $A_b \in \mathscr{U}_{m+1} \otimes \cdots \otimes U_{I_2}$. Moreover, as the ultrafilter product is associative, the stripe A_b^+, defined by

$$A_b^+ = \{c_{m+1} : \exists (c_{m+1}, \dots, c_{J_1}) \in A_b\},$$

must be an element of $\mathscr{U}_{m+1}$. This follows as $\mathscr{U}_1 \otimes \cdots \otimes \mathscr{U}_{I_2} = \mathscr{U}_1 \otimes (\mathscr{U}_2 \otimes \cdots (\mathscr{U}_{I_2-1} \otimes \mathscr{U}_{I_2}))$, by associativity, and so A_b^+ is the set of admissible first elements of sequences in $\mathscr{U}_{m+1} \otimes \cdots \mathscr{U}_{I_2}$, which must be an element of $\mathscr{U}_{m+1}$, by the definition of tensor products of ultrafilters. In order to find a sequence such that all further subsequences satisfy (C_1), we must consider the intersection of these stripes.

Let us assume that $\mathscr{V}$ is a fixed non-principal ultrafilter, and that $\mathscr{U}_1 = \mathscr{U}_2 = \cdots = \mathscr{U}_{I_2} = \mathscr{V}$. Now let us fix a_1 as any element of $A_\varnothing^+$, the stripe of all possible first elements of sequences in A. Then consider the stripe $A_{a_1}^+$. By closure under finite intersection, the set $A_\varnothing^+ \cap A_{a_1}^+ \cap (a_1, \infty)$ is infinite, and contained in $\mathscr{V}$, and so we may choose any element in this set to be a_2.

To find a_3, we must consider the intersection of $A_{a_1}^+$, $A_{a_2}^+$, $A_{(a_1,a_2)}^+$, and (a_2, ∞), such that $a_3 > a_2$, and such that (a_1, a_2, a_3), (a_1, a_3), and (a_2, a_3) are all admissible heads for sequences which satisfy condition (C_1).

Given a sequence $(a_1, \dots, a_m)$, for $m < I_2$, we may choose a_{m+1} by taking the intersection of (a_m, ∞) and each of the stripes A_c^+, where c is any finite subsequence of $(a_1, \dots, a_m)$.

If we have a sequence $(a_1, \dots, a_m)$, with $m \geq I_2$, for which every subsequence of length J_1 satisfies condition (C_1), then we may extend the sequence another term by choosing any element a_{m+1} of the intersection

$$(a_m, \infty) \cap \left(\bigcap_{\substack{c \subseteq (a_1, \dots, a_m) \\ |c| < J_1}} A_c^+ \right),$$

where $|c|$ denotes the length of the subsequence c. The set of all possible subsequences c is finite, and so this is still a finite intersection of elements of $\mathscr{V}$, and is therefore also an element of the ultrafilter $\mathscr{V}$.

By induction, we have constructed a sequence $(a_j)_{j \in \mathbb{N}}$, which satisfies condition (C_1). Note that at this point, $\mathscr{V}$ is still an arbitrary non-principal ultrafilter. By the ultrafilter lemma [20, Theorem 7.1], there exists a non-principal ultrafilter on $\mathbb{N}$, containing the set $\{a_j : j > a_{I_2}\}$. Then let $\mathscr{V}_2$ be any such ultrafilter, and set $\mathscr{U}_{I_2+1} = \mathscr{U}_{I_2+2} = \cdots = \mathscr{U}_{I_3} = \mathscr{V}_2$. Fix $(t_1, t_2, \dots, t_{I_2}) = (a_1, a_2, \dots, a_{I_2})$.

Repeating the construction of the sequence $(a_j)_{j \in \mathbb{N}}$, now for condition (C_2), we may construct a sequence $(a_j^{(2)})_{j \in \mathbb{N}}$, such that every subsequence of length J_2 satisfies condition (C_2). Moreover, if we take the intersection with $\{a_j : j \in \mathbb{N}\} \in \mathscr{V}_2$ at each stage, we may assume that $(a_j^{(2)})_{j\in\mathbb{N}} \subseteq (a_j)_{j\in\mathbb{N}}$. Then set $(t_{I_2+1},\dots,t_{I_3}) = (a_1^{(2)},\dots,a_{J_2}^{(2)})$, and let $\mathscr{V}_3$ be any non-principal ultrafilter containing the set $\{a_j^{(2)} : j > J_2\}$.

Let $(a_j^{(k)})_{j\in\mathbb{N}}$ be a given sequence, such that every subsequence satisfies condition (C_j) for all $j \le k$, where k is fixed. We may reiterate this prior diagonalisation construction over some new non-principal ultrafilter $\mathscr{V}_{k+1}$ containing $\left\{a_j^{(k)} : j > J_k\right\}$, giving us a new sequence $(a_j^{(k+1)})_{j\in\mathbb{N}}$, such that every subsequence satisfies condition (C_j), for all $j \le k+1$. Then set $(t_{I_{k+1}+1},\dots,t_{I_{k+2}}) = (a_1^{(k+1)},\dots,a_{J_{k+1}}^{(k+1)})$.

This iterated induction gives us a strictly increasing sequence $(t_j)_{j\in\mathbb{N}}$ such that every subsequence satisfies condition (C_k), for every $k \in \mathbb{N}$.

We are now able to complete the proof, and show that for every subsequence $(s_k)_{k\in\mathbb{N}} \subseteq (t_k)_{k\in\mathbb{N}}$, the series

$$\sum_{k\in\mathbb{N}} c_k x_{k,s_k}$$

converges bilaterally almost uniformly. To complete this part of the proof, we will use the maximal inequality given by condition (C_k) to induce a bound on the maximal rearrangement, which when combined with a noncommutative Chebyshev type inequality will allow us to construct projections which verify the bilateral almost uniform convergence.

For the remainder of the proof, let us fix some subsequence $(s_k)_{k\in\mathbb{N}} \subseteq (t_k)_{k\in\mathbb{N}}$. To see that $\sum_{k=1}^{\infty} c_k x_{k,s_k}$ converges bilaterally almost uniformly, let us consider the blocks $\sum_{j=I_k+1}^{I_{k+1}} c_j x_{j,s_j}$, for each $k \in \mathbb{N}$. As $(s_j)_{j\in\mathbb{N}}$ satisfies condition (C_k), we may apply Lemma 2.15, to find that

$$\mu\left(\varepsilon^{-1}4^{-k}; \left(\sum_{j=I_k+1}^{n} c_j x_{j,s_j}\right)_{n=I_k+1}^{I_{k+1}}\right) \le (4^k\varepsilon)^{2\widetilde{p}} \cdot \left(2C_2 \cdot 32^{-k}\right)^{\widetilde{p}},$$

for any fixed $\varepsilon > 0$, where $\widetilde{p}$ is the fixed index in the open interval $(1,2)$, chosen when we applied Proposition 3.9.

By definition of the maximal rearrangement, Definition 2.14, there must then exist a projection $e_k \in \mathcal{M}$, for each $k \in \mathbb{N}$, such that

$$\tau(1-e_k) \le \left(\frac{\varepsilon^2 \cdot 4C_2 \cdot 32^{-k}}{4^{-2k}}\right)^{\widetilde{p}} = \left(\varepsilon^2 4C_2 \cdot 2^{-k}\right)^{\widetilde{p}},$$

and

$$\sup_{I_k < n \le I_{k+1}} \left\| \sum_{j=I_k+1}^{n} e_k c_j x_{j,s_j} e_k \right\|_\infty \le \varepsilon^{-1}4^{-k}, \tag{4.3}$$

where the estimate on the trace is up to a factor of 2, as an approximation of the infimum which defines the maximal rearrangement.

Let $e_\infty = \bigwedge_{k\in\mathbb{N}} e_k$. On the one hand, we have that

$$\begin{aligned}\tau(1-e_\infty) &\leq \sum_{n\in\mathbb{N}} \tau(1-e_n)\\ &= \sum_{n\in\mathbb{N}} \left(\varepsilon^2 \cdot 4C \cdot 2^{-n}\right)^{\widetilde{p}}\\ &\leq (\varepsilon^2 \cdot 4C)^{\widetilde{p}}.\end{aligned}$$

We may then choose ϵ, such that $\tau(1-e_\infty)$ is arbitrarily small.

On the other hand, we may write

$$\sum_{k=n}^{\infty} c_k x_{k,s_k} = \sum_{k=n}^{I_{k(n)}} c_k x_{k,s_k} + \sum_{j=k(n)}^{\infty} \sum_{l=I_j+1}^{I_{j+1}} c_k x_{l,s_l},$$

for each $n \in \mathbb{N}$, where $k(n)$ is the least integer k such that $n \leq I_k$, which gives us that

$$\begin{aligned}&\left\| e_\infty \left(\sum_{k=n}^{\infty} c_k x_{k,s_k} \right) e_\infty \right\|_\infty\\ &\leq \left\| e_\infty \left(\sum_{k=n}^{I_{k(n)}} c_k x_{k,s_k} \right) e_\infty \right\|_\infty + \sum_{j=k(n)}^{\infty} \left\| e_\infty \left(\sum_{l=I_j+1}^{I_{j+1}} c_k x_{l,s_l} \right) e_\infty \right\|_\infty\\ &\leq 2\varepsilon^{-1} 4^{-(k(n)-1)} + \varepsilon^{-1} \sum_{j=k(n)}^{\infty} 4^{-j},\end{aligned}$$

where the second inequality follows by application of equation (4.3).

It follows that

$$\lim_{n\to\infty} \left\| e_\infty \left(\sum_{k=n}^{\infty} c_k x_{k,s_k} \right) e_\infty \right\|_\infty \leq \lim_{n\to\infty} 2\varepsilon^{-1} 4^{-(k(n)-1)} + \varepsilon^{-1} \sum_{j=k(n)}^{\infty} 4^{-j} = 0,$$

and so the series

$$\sum_{k=1}^{\infty} c_k x_{k,s_k}$$

converges bilaterally almost uniformly for every subsequence $(s_k)_{k\in\mathbb{N}}$ of $(t_k)_{k\in\mathbb{N}}$. The necessary bilateral almost uniform convergence then holds. $\square$

5. The noncommutative Révész theorem

Theorem 5.1. *Let $\mathcal{M}$ be a finite von Neumann algebra, with distinguished faithful normal tracial state τ. For any sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^2(\mathcal{M},\tau)$, such that $\sup_{n\in\mathbb{N}} \|f_n\|_2 < \infty$, and any fixed sequence $(c_n)_{n\in\mathbb{N}}$, there exists a subsequence $(g_n)_{n\in\mathbb{N}} \subseteq (f_n)_{n\in\mathbb{N}}$, and an operator $f \in L^2(\mathcal{M},\tau)$, such that for every further subsequence $(h_n)_{n\in\mathbb{N}} \subseteq (g_n)_{n\in\mathbb{N}}$, the series*

$$\sum_{n\in\mathbb{N}} c_k(h_k - f)$$

converges bilaterally almost uniformly.

Proof. To start, let us consider the real and imaginary decomposition of each operator f_n, such that $f_n = a_n + ib_n$, with every a_n and b_n self-adjoint. By Lemma 2.8, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge bilaterally almost uniformly, to a and b, respectively, then the sequence $(a_n + ib_n)_{n\in\mathbb{N}}$ converges bilaterally almost uniformly to $a + ib$. We may then assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence of self-adjoint operators, without loss of generality. As an L^2-bounded sequence of self-adjoint operators, there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ which is weakly convergent to some self-adjoint $f \in L^2(\mathcal{M})$.

By relabelling the sequence $(f_n)_{n\in\mathbb{N}}$, we may assume that the sequence is weakly convergent to f. As we only wish to determine a suitable subsequence of $(f_n)_{n\in\mathbb{N}}$, we may do so without loss of generality, and we will repeat this technique in the sequel.

In order to pass from weak to strong convergence, we will pass to the finite dimensional setting. Let $L^2_\infty = \overline{\text{span}}\,\{f_n - f : n \in \mathbb{N}\}$ denote the L^2-closure of the space generated by $(f_n - f)_{n\in\mathbb{N}}$. L^2_∞ is then a separable Hilbert space, and admits some orthonormal basis $(x_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let $L^2_n = \text{span}\,\{x_k : 1 \le k \le n\}$, and let $\mathcal{P}_n : L^2(\mathcal{M}) \to L^2_n$ denote the orthogonal projection onto L^2_n.

We claim that there exist two increasing sequences $(n_k)_{k=1}^\infty$ and $(v_k)_{k=1}^\infty$ of positive integers, such that for every $k \ge 2$,

$$\left\|\mathcal{P}_{v_{k-1}}(f_{n_k} - f)\right\|_2 < 4^{-k}, \tag{5.1}$$

and

$$\|(f_{n_k} - f) - \mathcal{P}_{v_k}(f_{n_k} - f)\|_2 < 4^{-k}. \tag{5.2}$$

We show that such sequences exist by an inductive diagonalisation argument. For every operator $x \in L^2_\infty$, $\lim_{n\to\infty} \|x - P_n(x)\|_2 = 0$. There then exists some $v_1 \ge 1$, such that $\|(f_1 - f) - P_{v_1}(f_1 - 1)\|_2 < 4^{-1}$.

Let $n_1 = 1$. As $(f_n)_{n=1}^\infty$ converges weakly to f in $L^2(\mathcal{M},\tau)$, we have that the sequence $(P_k(f_n - f))_{n=1}^\infty$ is weakly null, for every $k \ge 1$. As L^2_k is finite dimensional, the weak and strong topologies on L^2_k coincide. As the strong topology on L^2_k is the subspace topology from $L^2(\mathcal{M},\tau)$, it follows that for any fixed $k \geq 1$, the sequence $(P_k(f_n - f))_{n=1}^{\infty}$ converges to zero in $L^2(\mathcal{M},\tau)$. There then exists some $n_2 > n_1 = 1$, such that $\|P_{v_1}(f_{n_2} - f)\|_2 < 4^{-2}$. We may then choose some $v_2 > v_1$, such that $\|(f_{n_2} - f) - P_{v_2}(f_{n_2} - f)\|_2 < 4^{-2}$, following much the same argument that let us choose v_1.

By induction, it follows that sequences $(n_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$ which satisfy (5.1) and (5.2) for all $k \geq 2$ exist.

Now, for each $k \geq 2$, define

$$\mathcal{D}_k(x) = \mathcal{P}_{v_k}(x) - \mathcal{P}_{v_{k-1}}(x).$$

For any $k > j \geq 2$, and any $x \in L^2(\mathcal{M},\tau)$, it follows by construction that $\mathcal{D}_k(x)$ and $\mathcal{D}_j(x)$ are orthogonal with respect to the $L^2(\mathcal{M},\tau)$-inner product. Moreover, by orthogonality of the basis $(x_n)_{n=1}^{\infty}$ of L^2_{∞}, we have that for any operators $x, y \in L^2(\mathcal{M},\tau)$, and any $k > j \geq 2$, the operators $\mathcal{D}_k(x)$ and $\mathcal{D}_j(y)$ are also orthogonal.

The essential difficulty of the proof is to show that for a fixed sequence $(c_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$, the series

$$\sum_{k=2}^{\infty} c_k \mathcal{D}_{j_k}(f_{n_{j_k}} - f) \tag{5.3}$$

converges bilaterally almost uniformly, for all subsequences $(j_k)_{k\in\mathbb{N}}$ of some increasing sequence $(i_k)_{k\in\mathbb{N}}$. We will address this difficulty before we consider how to recover the series $\sum_{k\in\mathbb{N}} c_k(f_k - f)$.

Before we address bilateral almost uniform convergence, let us check that the series

$$\sum_{k=2}^{\infty} \mathcal{D}_{j_k}(f_{n_{j_k}} - f)$$

converges in the L^2-strong topology, for any increasing sequence $(j_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$. It is sufficient to note that the operators $\mathcal{D}_{j_k}(f_{n_{j_k}} - f)$ are pairwise orthogonal, and that $\left\|\mathcal{D}_{j_k}(f_{n_{j_k}} - f)\right\|_2$ is bounded, uniformly in k, by $4\sup_{n\in\mathbb{N}} \|f_n\|_2$. The convergence of (5.3), in the L^2-strong topology, then follows by use of the triangle inequality, and from the fact that $(|c_k|)_{k\in\mathbb{N}} \in \ell^2$.

We may now focus on the heart of the proof, at which point we must make a substantial departure from classical techniques. We will show that there is a natural embedding of the partial sums of (5.3), for all sequences $(j_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, into the L^2-space associated to the iterated ultrapower $\mathcal{M}^{\omega,\infty}$, and then using the noncommutative maximal L^p-spaces, we will be able to show that there exist suitable projections, such that the series converges bilaterally almost uniformly.

Following the constructions in Section 3 let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ denote a sequence of non-principal ultrafilters on $\mathbb{N}$. For each $n \geq 1$, let $\mathcal{M}^{\omega,n}$ denote the n-th iterated ultrapower of $\mathcal{M}$, that is $(((\mathcal{M}^{\omega}_{\mathscr{U}_1})^{\omega}_{\mathscr{U}_2})\cdots)^{\omega}_{\mathscr{U}_n}$, such that

$$\mathcal{M}^{\omega,n} = (\mathcal{M}^{\omega,n-1})^{\omega}_{\mathscr{U}_n}.$$

Then let $\mathcal{M}^{\omega,\infty}$ denote the w*-completion of $\cup_{n\in\mathbb{N}}\mathcal{M}^{\omega,n}$, with the union taken over the natural embeddings.

The algebra $\mathcal{M}^{\omega,\infty}$ is again a finite von Neumann algebra (see Subsection 3.2), and the sequence $(\mathcal{M}^{\omega,n})_{n\in\mathbb{N}}$, under the natural embeddings into $\mathcal{M}^{\omega,\infty}$, forms a filtration. For each $n \in \mathbb{N}$, let $\mathcal{E}_n : \mathcal{M}^{\omega,\infty} \to \mathcal{M}^{\omega,n}$ denote the normal conditional expectation. As there exists a normal conditional expectation $\mathbb{E}_0 : \mathcal{M}^{\omega,1} \to \mathcal{M}$, then by composition there also exists a normal conditional expectation $\mathcal{E}_0 : \mathcal{M}^{\omega,\infty} \to \mathcal{M}$.

In order to consider an equivalence class in $L^2(\mathcal{M}^{\omega})$, we must find a sequence which is not only L^2-bounded, but is equimeasurable, and which we will also require to be weakly null, for later calculation. For each $k \in \mathbb{N}$, let $x_k = \mathcal{D}_k(f_{n_k} - f)$. Let us pass to a subsequence of $(x_k)_{k\in\mathbb{N}}$ which is weakly convergent, with limit x, and relabel $(n_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$, such that $(x_k)_{k\in\mathbb{N}}$ is weakly convergent to x.

Let $w_k = x_k - x$, for each $k \in \mathbb{N}$. By Lemma 2.13, there exists a subsequence $(w'_k)_{k\in\mathbb{N}}$ of $(w_k)_{k\in\mathbb{N}}$, such that for each k, $w_k = y_k + z_k$, where $(z_k)_{k\in\mathbb{N}}$ vanishes bilaterally almost uniformly, and $(y_k)_{k\in\mathbb{N}}$ is equimeasurable. Without loss of generality, let us relabel the sequence $(n_k)_{k\in\mathbb{N}}$, $(v_k)_{k\in\mathbb{N}}$, such that w_k is the sequence given by Lemma 2.13. We may also choose this sequence to be such that $(c_k z_k)_{z\in\mathbb{N}}$ vanishes sufficiently quickly, in the sense that

$$\sum_{k\in\mathbb{N}} c_k z_k$$

converges bilaterally almost uniformly. We then simply assume that $(w_k)_{k\in\mathbb{N}}$ is equimeasurable, and can again do so without loss of generality.

As $(w_k)_{k\in\mathbb{N}}$ is an L^2-bounded sequence, $\widetilde{w} = (w_k)^{\bullet}_{k\in\mathbb{N}}$ defines an equivalence class in $L^2(\mathcal{M})^{\omega}$, and as $(w_k)_{k\in\mathbb{N}}$ is equimeasurable, we have that $(|w_k|^2)_{k\in\mathbb{N}}$ is uniformly integrable (see Remark 3.2), and so $\widetilde{w} \in L^2(\mathcal{M}^{\omega,1})$.

We may now consider convergence of the sums

$$M_n = \sum_{k=1}^{n} c_k \pi_k(\widetilde{w}) \in L^2(\mathcal{M}^{\omega,n}) \subseteq L^2(\mathcal{M}^{\omega,\infty}),$$

defined for each $n \in \mathbb{N}$. Note that, as per the discussion in Section 4, $(M_n)_{n\in\mathbb{N}}$ forms a martingale, as $\widetilde{w}$ is weakly null. We claim that $(M_n)_{n\in\mathbb{N}}$ is an L^2-bounded martingale in $L^2(\mathcal{M}^{\omega,\infty})$, under the natural inclusion of each M_n into $L^2(\mathcal{M}^{\omega,\infty})$, with limit

$$M_\infty = \sum_{k=1}^{\infty} c_k \pi_k(\widetilde{w}).$$

To see this, let us apply Pisier and Xu's characterisation of L^p-bounded martingales, Theorem 2.17. Note that

$$\begin{aligned}\|\pi_k(\widetilde{w})\|_{L^2(\mathcal{M}^{\omega,\infty})} &= \|\pi_k(\widetilde{w})\|_{L^2(\mathcal{M}^{\omega,k})}\\ &\le \sup_{n\in\mathbb{N}} \|w_n\|_2\\ &\le \sup_{n\in\mathbb{N}} \|x_n - x\|_2\\ &\le 2\sup_{j\in\mathbb{N}} \left\|\mathcal{D}_j(f_{n_j} - f)\right\|_2\\ &\le 8\sup_{j\in\mathbb{N}} \|f_j\|_2 < \infty.\end{aligned}$$

It is then immediate, as $(c_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$, that

$$\|(\pi_k(\widetilde{w}))_{k\in\mathbb{N}}\|_{L^2(\mathcal{M}^{\omega,\infty};\ell^2_C)} = \left\|\left(\sum_{k\in\mathbb{N}} |c_k|^2 \, |\pi_k(\widetilde{w})|^2\right)^{1/2}\right\|_2 < \infty,$$

with the row estimate given by much the same calculation, and so $(M_n)_{n\in\mathbb{N}}$ is an L^2-bounded martingale, which must then converge strongly to M_∞. We may then apply Theorem 4.2, to see that there exists an increasing sequence $(s_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, such that for every subsequence $(t_k)_{k\in\mathbb{N}} \subseteq (s_k)_{k\in\mathbb{N}}$, the series

$$\sum_{k=1}^{\infty} c_k w_{t_k}$$

converges bilaterally almost uniformly.

We chose to denote $x_k = \mathcal{D}_k(f_{n_k} - f)$, such that $(x_k)_{k\in\mathbb{N}}$ is weakly convergent to x. We then set $w_k = x_k - x$, for every k, and proved that there exists a sequence $(i_k)_{k\in\mathbb{N}}$, such that for every further subsequence $(j_k)_{k\in\mathbb{N}}$, the series

$$\sum_{k=1}^{\infty} c_k w_{j_k} \tag{5.4}$$

converges bilaterally almost uniformly. That is then to say, by adding the term $\sum_{k=1}^{\infty} c_k x$, that the series

$$\sum_{k=1}^{\infty} c_k \mathcal{D}_{j_k}(f_{n_{j_k}} - f)$$

converges bilaterally almost uniformly.

For each $k \in \mathbb{N}$, let

$$S_k = \mathcal{P}_{v_{k-1}}(f_{n_k} - f),$$

and

$$R_k = (f_{n_k} - f) - \mathcal{P}_{v_k}(f_{n_k} - f),$$

such that by the norm estimates (5.1) and (5.2),

$$\|S_k\|_2 < 4^{-k},$$

and

$$\|R_k\|_2 < 4^{-k}.$$

Given the growth conditions from (5.1) and (5.2), we can find a relabelling, by passing to a subsequence, of $(i_k)_{k\in\mathbb{N}}$, such that

$$\sum_{k=1}^{\infty} \|c_k S_{j_k}\|_2 < \infty,$$

and

$$\sum_{k=1}^{\infty} \|c_k R_{j_k}\|_2 < \infty,$$

for every subsequence $(j_k)_{k\in\mathbb{N}} \subseteq (i_k)_{k\in\mathbb{N}}$.

It follows that the series

$$\sum_{k=1}^{\infty} c_k S_{j_k}, \tag{5.5}$$

and

$$\sum_{k=1}^{\infty} c_k R_{j_k} \tag{5.6}$$

must converge bilaterally almost uniformly, by Lemma 2.10.

Finally, as we have that $f_{n_{j_k}} - f = w_{j_k} + S_{j_k} + R_{j_k}$, we may add together the three series, (5.4), (5.5), and (5.6), to see that

$$\sum_{k=1}^{\infty} c_k (f_{n_{j_k}} - f)$$

converges bilaterally almost uniformly, for all subsequences $(j_k) \in \mathbb{N}$ of $(i_k)_{k\in\mathbb{N}}$. Labelling $g_k = f_{n_{i_k}}$, for each $k \in \mathbb{N}$, we have that the result holds. $\square$

6. The noncommutative Komlós theorem

Finally, we may answer Randrianantoanina's question, and show that the Komlós theorem extends to arbitrary finite von Neumann algebras. For the most part, the proof is again classical. However, as for the proof of Theorem 5.1, a substantial appeal to the techniques of iterated ultrapowers is once again necessary, so as to compensate for the lack of an intrinsic martingale structure in a non-hyperfinite algebra.

Theorem 6.1. *Let $\mathcal{M}$ be a finite von Neumann algebra, with a distinguished faithful normal tracial state τ. If $(f_n)_{n=1}^\infty \subseteq L^1(\mathcal{M},\tau)$ is an L^1-bounded sequence, then there exists a subsequence $(g_n)_{n=1}^\infty \subseteq (f_n)_{n=1}^\infty$, and an operator $f \in L^1(\mathcal{M},\tau)$, such that for every further subsequence $(h_n)_{n=1}^\infty \subseteq (g_n)_{n=1}^\infty$, the sequence*

$$\left(\frac{1}{n}\sum_{k=1}^{n} h_k\right)_{n=1}^{\infty} \tag{6.1}$$

converges bilaterally almost uniformly to f.

Proof. Without loss of generality, we may assume that the sequence $(f_n)_{n=1}^\infty$ consists of only self-adjoint operators, by use of Lemma 2.8.

In order to show that the result holds, we will consider three approximations of the sequence $(f_n)_{n=1}^\infty$. We will show that the sequence is equimeasurable, up to a sequence which converges bilaterally almost uniformly to zero, approximate the operators by truncations of their height, and use a finite dimensional approximation.

These three approximations will allow us to decompose the sequence of Cesàro averages into three components, each of which we may show is bilaterally almost uniformly convergent, such that the assertion holds.

To start, we appeal to Lemma 2.13. There exists some subsequence $(a_n)_{n=1}^\infty \subseteq (f_n)_{n=1}^\infty$ such that for each $n \geq 1$, $a_n = b_n + c_n$, where $(b_n)_{n=1}^\infty$ and $(c_n)_{n=1}^\infty$ are L^1-bounded sequences, $(b_n)_{n=1}^\infty$ is a sequence of equimeasurable operators, and $(c_n)_{n=1}^\infty$ converges to zero in the measure topology.

Applying [31, Proposition 1], there exists a subsequence $(c_{n_k})_{k=1}^\infty$ which converges bilaterally almost uniformly to zero. Let $(g_n)_{n=1}^\infty = (b_{n_k})_{n=1}^\infty$. By Lemma 2.9, the Cesàro averages of $(c_{n_k})_{k=1}^\infty$ converge bilaterally almost uniformly to zero, and so if the Cesàro averages of $(g_n)_{n=1}^\infty$ converge bilaterally almost uniformly, then by Lemma 2.8, the Cesàro averages of $(a_{n_k})_{k=1}^\infty$ must also converge bilaterally almost uniformly, and to the same limit as for $(g_n)_{n=1}^\infty$.

We may then work with the sequence $(g_n)_{n=1}^\infty$ in place of $(a_{n_k})_{k=1}^\infty \subseteq (f_n)_{n=1}^\infty$.

For each $j,k \geq 1$, let $T_k(g_j) = g_j\chi_{[0,k)}(|g_j|)$, the k-the truncation of g_j. For any fixed $k \geq 1$, the sequence $(T_k(g_j))_{j=1}^\infty$ is equimeasurable, and thereby necessarily L^2-bounded. Note that the key distinction between the proofs of the Komlós and Révész theorems is that $(T_k(g_j))_{k,j\in\mathbb{N}}$ is now unbounded over k, in the L^2-norm. As $L^2(\mathcal{M},\tau)$ is a

Hilbert space, it is weakly sequentially compact. Using a standard diagonal subsequence argument, there exists some subsequence $(g'_n)_{n=1}^{\infty} \subseteq (g_n)_{n=1}^{\infty}$, such that for any fixed $k \geq 1$, the sequence $(T_k(g'_j))_{j=1}^{\infty}$ is weakly convergent. We may then relabel the sequence $(g'_n)_{n=1}^{\infty}$ as $(g_n)_{n=1}^{\infty}$.

For each $k \geq 1$, let φ_k denote the weak limit of $(T_k(g_j))_{j=1}^{\infty}$.

Before we may show that the Cesàro averages are well behaved, we must find a further subsequence of $(g_n)_{n=1}^{\infty}$, satisfying two key estimates.

We claim that for every subsequence $(h_n)_{n=1}^{\infty} \subseteq (g_n)_{n=1}^{\infty}$,

$$\sum_{n=1}^{\infty} \left\| \frac{T_n(h_n)}{n} \right\|_2^2 < \infty, \tag{6.2}$$

and

$$\sum_{n=1}^{\infty} \tau\left(\chi_{[n,\infty)}(|h_n|)\right) < \infty. \tag{6.3}$$

As the sequence $(g_n)_{n=1}^{\infty}$ is equimeasurable, we have that for any $k \geq 1$, the sequence $(\tau(\chi_{[k-1,k)}(g_j)))_{j=1}^{\infty}$ is constant. Then let $\alpha_k = \tau(\chi_{[k-1,k)}(g_1))$ and $\beta_k = \tau(\chi_{[k-1,k)}(-g_1)$, for each $k \geq 1$.

Key to showing the estimates (6.2) and (6.3) are the bounds

$$\sum_{k=1}^{\infty} k\alpha_k < \infty, \tag{6.4}$$

and

$$\sum_{k=1}^{\infty} k\beta_k < \infty. \tag{6.5}$$

We will show that (6.4) holds, and the proof of the bound (6.5) follows by an almost identical calculation.

To see that (6.4) holds, consider the bound

$$\begin{aligned}\sum_{k=1}^{n}(k-1)\alpha_k &= \sum_{k=1}^{n}(k-1)\tau\left(\chi_{[k-1,k)}(g_1)\right)\\ &\leq \sum_{k=1}^{n}\tau\left(g_1\chi_{[k-1,k)}(g_1)\right)\\ &= \tau\left(g_1\chi_{[0,n)}(g_1)\right).\end{aligned}$$

Taking the limit over n, we have that

$$\sum_{k=1}^{\infty}(k-1)\alpha_k \leq \tau\left(g_1\chi_{[0,\infty)}(g_1)\right) = \|g_1\|_1 < \infty.$$

Then, for each $k \geq 2$, $(k-1)\alpha_k \geq \alpha_k$, and so we also have that $\sum_{k=1}^{\infty}\alpha_k < \infty$. Adding the two series, we see that $\sum_{k=1}^{\infty} k\alpha_k < \infty$, and (6.4) holds.

We may now show that (6.2) and (6.3) hold. Let $(h_n)_{n=1}^{\infty}$ be an arbitrary subsequence of $(g_n)_{n=1}^{\infty}$. By the construction of $(g_n)_{n=1}^{\infty}$, we have that

$$\|T_j(h_j)\|_2^2 \leq \sum_{k=1}^{j} k^2\tau\left(\chi_{[k-1,k)}(|h_j|)\right) = \sum_{k=1}^{j} k^2\left(\alpha_k+\beta_k\right),$$

for any $j \geq 1$.

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left\|\frac{T_n(h_n)}{n}\right\|_2^2 &\leq \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}k^2\alpha_k+\beta_k\right)\\
&= \sum_{k=1}^{\infty}\left(k^2(\alpha_k+\beta_k)\sum_{n=k}^{\infty}\frac{1}{n^2}\right)\\
&\leq \sum_{k=1}^{\infty}\left(\frac{2k^2(\alpha_k+\beta_k)}{k}\right)\\
&= 2\sum_{k=1}^{\infty}k(\alpha_k+\beta_k) < \infty,
\end{aligned}$$

with the estimate on $\sum_{n=k}^{\infty} n^{-2}$ given by

$$\sum_{n=k}^{\infty}\frac{1}{n^2} \leq \int_{x=1}^{\infty}\frac{1}{x^2}dx = \frac{1}{k-1} \leq \frac{2}{k}.$$

Then (6.2) holds.

To show that (6.3) holds, we take a decomposition over the spectrum and apply Chebyshev's inequality,

$$\begin{aligned}
\tau\left(\chi_{[n,\infty)}(h_n)\right) &= \left(\sum_{k=n+1}^{n^2}\tau\left(\chi_{[k-1,k)}(h_n)\right)\right) + \tau\left(\chi_{[n^2,\infty)}(h_n)\right)\\
&= \left(\sum_{k=n+1}^{n^2}\alpha_k\right) + \frac{\|g_1\|_1}{n^2}\\
&\leq \sum_{k=n+1}^{\infty}\alpha_k + \frac{\|g_1\|_1}{n^2}
\end{aligned}$$

Summing over n,

$$\sum_{n=1}^{\infty}\tau\left(\chi_{[n,\infty)}(h_n)\right) \leq \sum_{n=1}^{\infty}\left(\sum_{k=n+1}^{\infty}\alpha_k + \frac{\|g_1\|_1}{n^2}\right)$$
$$= \sum_{k=2}^{\infty}\sum_{n=1}^{k-1}\alpha_k + \sum_{n=1}^{\infty}\frac{\|g_1\|_1}{n^2}$$
$$= \sum_{k=2}^{\infty}(k-1)\alpha_k + \sum_{n=1}^{\infty}\frac{\|g_1\|_1}{n_2} < \infty.$$

The right hand series is finite, and the series $\sum_{k=2}^{\infty}(k-1)\alpha_k$ is finite as per the proof of (6.4). The estimate (6.3) then holds.

Recall that for each $k \geq 1$, φ_k is the L^2-weak limit of the sequence $(T_k(g_j))_{j\in\mathbb{N}}$. We claim that there exists some subsequence $(g_n')_{n=1}^{\infty}$ of $(g_n)_{n=1}^{\infty}$, such that for every further subsequence $(h_n)_{n=1}^{\infty} \subseteq (g_n')_{n=1}^{\infty}$, the sequences of Cesàro averages,

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left(T_{k+1}(h_k)-\varphi_{k+1}\right)\right)_{n=1}^{\infty} \tag{6.6}$$

and

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left(h_k - T_{k+1}(h_k)\right)\right)_{n=1}^{\infty}, \tag{6.7}$$

converge bilaterally almost uniformly to zero, and the series

$$\sum_{n=1}^{\infty}\left(\varphi_{n+1}-\varphi_n\right) \tag{6.8}$$

converges bilaterally almost uniformly to some operator $f' \in L^1(\mathcal{M},\tau)$. Having shown these three convergences, the Komlós theorem will follow easily.

To start, let us show that (6.6) holds. In doing so, we will determine the subsequence $(g_n')_{n=1}^{\infty} \subseteq (g_n)_{n=1}^{\infty}$.

For each $j, k \geq 1$, let

$$f_{k,j} = T_{k+1}(g_j) - \varphi_{k+1}. \tag{6.9}$$

Then let $L^2_\infty = \overline{\operatorname{span}}\{f_{k,j} : j,k \geq 1\}$, where the closure is taken in the L^2-norm. Then L^2_∞ is a closed separable subspace of $L^2(\mathcal{M},\tau)$, and admits some orthonormal basis $(x_n)_{n=1}^{\infty}$. For each $n \geq 1$, let $L^2_n = \operatorname{span}\{x_1,\dots,x_n\}$, and let $P_n : L^2(\mathcal{M},\tau) \to L^2_n$ denote the orthogonal projection onto L^2_n. We will use these subspaces to form finite dimensional approximations of the operators $f_{k,j}$, which will allow us to pass from weak to strong convergence in $L^2(\mathcal{M},\tau)$.

We wish to find two increasing sequences of indices, $(n_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$, such that

$$\|P_{v_k-1}(f_{s,n_k})\|_2 < 4^{-k} \tag{6.10}$$

and

$$\|f_{s,n_k} - P_{v_k}(f_{s,n_k})\|_2 < 4^{-k}, \tag{6.11}$$

for every $k \geq 2$, and any $1 \leq s \leq k$. To do so, we appeal to an inductive construction.

To start, note that for each $f \in L^2_\infty$, $\lim_{n\to\infty} \|f - P_n(f)\|_2 = 0$. Then, if we set $n_1 = 1$, there must exist some $v_1 \geq 1$, such that

$$\|f_{1,1} - P_{v_1}(f_{1,1})\|_2 < 4^{-1}.$$

Moreover, weak and strong convergence coincide over L^2_n, for each $n \geq 1$, as these spaces are finite dimensional. As L^2_n inherits the topology of $L^2(\mathcal{M},\tau)$, and so weak convergence in L^2_n implies strong convergence in $L^2(\mathcal{M},\tau)$.

For any fixed $k \geq 1$, the sequence $(f_{k,j})_{j=1}^{\infty}$ converges weakly to zero, by construction, and so the finite approximation $(P_{v_1}(f_{k,j}))_{j=1}^{\infty}$ also converges weakly to zero with respect to L^2_n, and in turn must converge strongly to zero with respect to $L^2(\mathcal{M},\tau)$. There must exist some index $n_2 > n_1$, such that for each $s \in \{1, 2\}$,

$$P_{v_1}(f_{s,n_2}) < 4^{-1}.$$

Repeating this procedure, we generate the desired sequences $(n_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$, such that the estimates (6.10) and (6.11) are satisfied.

Let $(m_k)_{k=1}^{\infty} \subseteq (n_k)_{k=1}^{\infty}$ be an arbitrary subsequence, such that $m_k = n_{j(k)}$ for each $k \geq 1$. The equations (6.10) and (6.11) then become

$$\left\|P_{v_{j(k)}-1}(f_{s,m_k})\right\|_2 < 4^{-j(k)} \tag{6.12}$$

and

$$\left\|f_{s,m_k} - P_{v_{j(k)}}(f_{s,m_k})\right\|_2 < 4^{-j(k)}, \tag{6.13}$$

for any $k \geq 1$ and $1 \leq s \leq k$. As the sequence $(j(k))_{k=1}^{\infty}$ is necessarily increasing, $j(k-1) \leq j(k) - 1$, for any $k \geq 2$, and so we find from (6.12) that

$$\left\|P_{v_{j(k-1)}}(f_{s,m_k})\right\|_2 < 4^{-j(k)}, \tag{6.14}$$

for any $k \geq 1$ and $1 \leq s \leq k$.

Now, for each $k \geq 2$, let

$$s_k = P_{v_{j(k-1)}}(f_{k,m_k}),$$
$$w_k = \left(P_{v_{j(k)}} - P_{v_{j(k-1)}}\right)(f_{k,m_k}),$$

and

$$r_k = f_{k,m_k} - P_{v_{j(k)}}(f_{k,m_k}).$$

Then

$$f_{k,m_k} = s_k + w_k + r_k \tag{6.15}$$

for each $k \geq 2$. The equations (6.12) and (6.13) respectively show that $\|r_k\|_2 < 4^{-j(k)}$ and $\|s_k\|_2 < 4^{-j(k)}$, and so $\sum_{k=2}^{\infty} \|r_k\|_2^2 < \infty$ and $\sum_{k=2}^{\infty} \|s_k\|_2^2 < \infty$. Applying Lemma 2.10 (ii), we conclude that the sequences $(r_k)_{k=2}^{\infty}$ and $(s_k)_{k=2}^{\infty}$ converge bilaterally almost uniformly to zero.

This leaves the sequence $(w_k)_{k=2}^{\infty}$, which is a pairwise orthogonal sequence in $L^2(\mathcal{M}, \tau)$. As each P_n is a contraction on $L^2(\mathcal{M}, \tau)$, for each $n \geq 1$, $\|w_k\|_2 \leq 2\|f_{k,m_k}\|_2 \leq 4\left\|T_{k+1}(h_{m_{k+1}})\right\|_2$, for every $k \geq 2$. Then

$$\sum_{k=2}^{\infty} \left\|\frac{w_k}{k}\right\|_2^2 \leq 4 \sum_{k=2}^{\infty} \left\|\frac{f_{k,m_k}}{k}\right\|_2^2 \leq 16 \sum_{k=2}^{\infty} \left\|\frac{T_{k+1}(h_{m_{k+1}})}{k}\right\|_2^2 < \infty, \tag{6.16}$$

with finiteness given by (6.2).

Now let us again apply the iterated ultrapower technique to find a sequence $(s_k)_{k\in\mathbb{N}} \subseteq (m_k)_{k\in\mathbb{N}}$, such that for every further subsequence $(t_k)_{k\in\mathbb{N}}$, the series

$$\sum_{k=2}^{\infty} \left(P_{v_{j(k)}} - P_{v_{j(k-1)}}\right)(f_{k,t_k})$$

converges bilaterally almost uniformly. For each pair $k, l \in \mathbb{N}$, let

$$w_{k,l} = \left(P_{v_{j(k)}} - P_{v_{j(k-1)}}\right)(f_{k,n_{j(l)}}).$$

As, for any fixed index k, the sequence $(f_{k,n_{j(l)}})_{l\in\mathbb{N}}$ is L^2-weakly null, it is easy to see that $(w_{k,l})_{l\in\mathbb{N}}$ is also weakly null. Then, as for the proof of Theorem 5.1, let us apply Lemma 2.13, such that we may pass to a subsequence, which we will also label $(w_{k,l})_{l\in\mathbb{N}}$, without loss of generality, which we may again assume is equimeasurable.

Then let us consider the equivalence classes $\widetilde{w}_k = (w_{k,l})_{l\in\mathbb{N}} \in L^1(\mathcal{M}^\omega)$, defined for each $k \in \mathbb{N}$. As each sequence $(w_{k,l})_{l\in\mathbb{N}}$ is weakly null, the partial sums of

$$\sum_{k=1}^{\infty} \frac{\pi_k(\widetilde{w}_k)}{k}$$

forms a martingale in $L^1(\mathcal{M}^{\omega,\infty})$, see the discussion in Remark 4.1. However we must verify that it is an L^2-bounded martingale before we may apply Theorem 4.2.

To see this, let us again apply Theorem 2.17, which we may do so, as

$$\|(\pi_k(\widetilde{w}_k))_{k\in\mathbb{N}}\|_{L^2(\mathcal{M}^{\omega,\infty};\ell^2_C)} = \left\| \left(\sum_{k\in\mathbb{N}} \left(\frac{|\pi_k(\widetilde{w}_k)^*|}{k} \right)^2 \right)^{1/2} \right\|_2 < \infty,$$

with finiteness following from (6.16), which shows that

$$\sum_{k=2}^{\infty} \frac{w_k}{k}$$

is absolutely convergent with respect to the L^2-norm. The necessary estimate for the row norm again follows from an almost identical calculation.

We may then apply Theorem 4.2, to see that there must exist some increasing sequence $(s_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, such that for all further subsequences $(t_k)_{k\in\mathbb{N}}$, the series

$$\sum_{k=2}^{\infty} \frac{w_{k,t_k}}{k} \tag{6.17}$$

converges bilaterally almost uniformly.

Let us again relabel the sequence such that $w_k = w_{k,t_k}$, for each $k \in \mathbb{N}$, for some fixed sequence $(t_k)_{k\in\mathbb{N}}$.

As Kronecker's Lemma preserves bilateral almost uniform convergence, see Lemma 2.11, the sequence of Cesàro averages $(n^{-1}\sum_{k=2}^{n+1} w_k)_{n=1}^{\infty}$ must converge bilaterally almost uniformly to zero. By Lemma 2.9, with $p = 1$, the sequences $(n^{-1}\sum_{k=2}^{n+1} r_k)_{n=1}^{\infty}$ and $(n^{-1}\sum_{k=2}^{n+1} s_k)_{n=1}^{\infty}$ also converge bilaterally almost uniformly to zero. By (6.15), we may add these sequences, such that the Cesàro averages

$$\frac{1}{n}\sum_{k=2}^{n+1} (s_k + w_k + r_k) = \frac{1}{n}\sum_{k=2}^{n+1} f_{k,m_k} = \frac{1}{n}\sum_{k=2}^{n+1} (T_{k+1}(g_{m_k}) - \varphi_{k+1})$$

converge bilaterally almost uniformly to zero, by Lemma 2.8. If we relabel $(g_{n_k})_{k=1}^{\infty}$ as $(g_n)_{n=1}^{\infty}$, then we have a sequence $(g_n)_{n=1}^{\infty}$, for which every subsequence thereof satisfies (6.6). This completes the most difficult part of the proof, however it remains to show that (6.7) and (6.8) hold.

To show that (6.7) holds, it is sufficient to use (6.3) and the noncommutative Borel–Cantelli lemma, Lemma 2.3.

For clarity of notation, let $u_n = h_n - T_{n+1}(h_n)$, for each $n \geq 1$, where $(h_n)_{n=1}^\infty$ is an arbitrary subsequence of $(g_n)_{n=1}^\infty$, and let $v_n = \chi_{[n+1,\infty)}(h_n)$, such that $u_n = h_n v_n$. From (6.3), $\sum_{k=1}^\infty \tau(v_n) < \infty$, and so we may apply Lemma 2.3. This gives us that $\lim_{n\to\infty} \tau(\bigvee_{k=n}^\infty v_k) = 0$. To show that (6.7) converges bilaterally almost uniformly to zero, fix $\varepsilon > 0$. There then exists some index N, and a projection $p_\varepsilon = 1 - \bigvee_{k=N}^\infty v_k$, such that $\tau(1 - p_\varepsilon) < \varepsilon$. However, for any $n \geq N$, p_ε is orthogonal to v_n, and so $\|p_\varepsilon u_n p_\varepsilon\|_\infty = 0$. Obviously $\lim\limits_{n\to\infty} \|p_\varepsilon u_n p_\varepsilon\|_\infty = 0$, and so u_n converges bilaterally almost uniformly to zero. Applying Lemma 2.9, the Cesàro averages of the sequence $(u_n)_{n=1}^\infty$ must also converge bilaterally almost uniformly to zero, which is to say that (6.7) holds under the appropriate conditions.

We may now show that (6.8) converges bilaterally almost uniformly to some limit in $L^1(\mathcal{M}, \tau)$. This follows in much the same way as in [73]. Namely, it is sufficient to show that $\|\varphi_{k+1} - \varphi_k\|_1 \leq (k+1)(\alpha_{k+1} + \beta_{k+1})$, for any $k \geq 1$, where $\alpha_k = \tau(\chi_{[k-1,k)}(g_1))$, and φ_k is the L^2-weak limit of $(T_k(g_j))_{j=1}^\infty$, for each $k \geq 1$. As $L^2(\mathcal{M}, \tau)$ has the Banach–Saks property (see, for example [28, Theorem 2.14]), which is to say that every weakly null sequence admits a "Banach–Saks" subsequence, for each $k \geq 1$, there exists an increasing sequence $(n_j)_{j=1}^\infty$, such that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=1}^n \left(T_{k+1}(g_{n_j}) - T_k(g_{n_j}) - (\varphi_{k+1} - \varphi_k) \right) \right\|_2 = 0.$$

Then the sequence

$$\left(\frac{1}{n} \sum_{j=1}^n \left(T_{k+1}(g_{n_j}) - T_k(g_{n_j}) - (\varphi_{k+1} - \varphi_k) \right) \right)_{n=1}^\infty$$

converges to $(\varphi_{k+1} - \varphi_k)$ in measure Lemma 2.1, and so we may use the noncommutative Fatou lemma, Lemma 2.2, to find an estimate for $\|\varphi_{k+1} - \varphi_k\|_1$.

In order to find an estimate on $\|\varphi_{k+1} - \varphi_k\|_1$, let us decompose each φ_k into a positive and negative component, such that $\varphi_k = \phi_k^+ - \phi_k^-$, for each $k \geq 1$. We then see that

$$\begin{aligned}
\left\|\phi_{k+1}^+ - \phi_k^-\right\|_1 &\leq \liminf_{n\to\infty} \left\| \frac{1}{n} \sum_{j=1}^n \left(T_{k+1}(g_{n_j}) - T_k(g_{n_j}) \right) \right\|_1 \\
&= \liminf_{n\to\infty} \left\| \frac{1}{n} \sum_{j=1}^n g_{n_j} \chi_{[k,k+1)}(g_{n_j}) \right\|_1 \\
&\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \left\| g_{n_j} \chi_{[k,k+1)}(g_{n_j}) \right\|_1 \\
&= \liminf_{n\to\infty} \frac{1}{n} \tau\left(g_{n_j} \chi_{[k,k+1)}(g_{n_j}) \right)
\end{aligned}$$

$$\leq \liminf_{n\to\infty} \frac{(k+1)}{n} \sum_{j=1}^{n} \tau\left(\chi_{[k,k+1)}(g_{n_j})\right)$$

$$= (k+1)\alpha_{k+1},$$

where $\alpha_k = \tau(\chi_{[k-1,k)}(g_1))$, for each $k \geq 1$. By (6.4), we have that

$$\sum_{k=1}^{\infty} \left\|\phi_{k+1}^{+} - \phi_k^{+}\right\|_1 \leq \sum_{k=1}^{\infty}(k+1)\alpha_{k+1} < \infty.$$

An almost identical calculation for the negative component, ϕ_k^-, shows that

$$\sum_{k=1}^{\infty} \left\|\phi_{k+1}^{-} - \phi_k^{-}\right\|_1 \leq \sum_{k=1}^{\infty}(k+1)\beta_{k+1} < \infty,$$

where $\beta_k = \tau\left(\chi_{[k-1,k)}(-g_1)\right)$, with finiteness following from (6.5).

Applying Lemma 2.10 (i), we have that the series $(\varphi_n - \varphi_1)_{n=1}^{\infty}$ converges bilaterally almost uniformly to some operator $f' \in L^1(\mathcal{M}, \tau)$. If we let $f = f' + \varphi_1$, then $(\varphi_n)_{n=1}^{\infty}$ converges bilaterally almost uniformly to f. In turn, the Cesàro averages of $(\varphi_n)_{n=1}^{\infty}$ must converge bilaterally almost uniformly to f. Adding these averages to (6.6) and (6.7), we have that the Cesàro averages

$$\frac{1}{n}\sum_{k=1}^{n} h_k$$

converge bilaterally almost uniformly to f, for any choice of subsequence $(h_n)_{n=1}^{\infty} \subseteq (g_n)_{n=1}^{\infty}$. This completes the proof of Theorem 6.1. □

Although this resolves Randrianintoanina's problem, it remains to be seen what can be said for the setting of arbitrary σ-finite von Neumann algebras, and in particular type III factors. The underlying difficulty here is not the ultrapower construction, which has been extensively studied in the setting of arbitrary σ-finite algebras, but the nature of suitable analogues of almost everywhere convergence. While there have been several attempts at defining such an extension, see for example any of the papers [24,32,33,40–42,58,63,66–68], it is not clear that any of these are the correct notion, and all face substantial limitations and difficulties.

Problem 6.2. Can the Komlós and Révész theorems be extended to arbitrary σ-finite von Neumann algebras? What is the correct extension of almost everywhere convergence in this general setting?

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