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Complete List of Authors	Teng Wu, Runmin Wang, Hao Yan and Xiaofeng Shao
Corresponding Author	Teng Wu
E-mail	tengwu2@illinois.edu

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Adaptive Change Point Monitoring for High-Dimensional Data

Teng Wu, Runmin Wang, Hao Yan, Xiaofeng Shao

Department of Statistics, University of Illinois at Urbana-Champaign

Department of Statistical Science, Southern Methodist University

School of Computing Informatics & Decision Systems Engineering, Arizona State University

Abstract: In this paper, we propose a class of monitoring statistics for a mean shift in a sequence of high-dimensional observations. Inspired by recent U-statistic based retrospective tests, we extend the U-statistic-based approach to the sequential monitoring problem by developing a new adaptive monitoring procedure that can detect both dense and sparse changes in real time. Unlike existing methods in retrospective testing that use self-normalization, we introduce a class of estimators for the q-norm of the covariance matrix and prove their ratio consistency. To facilitate fast computation, we further develop recursive algorithms to improve the computational efficiency of the monitoring procedure. The advantages of the proposed methodology are demonstrated using simulation studies and real-data illustrations.

Key words and phrases: Change point detection, Sequential monitoring, Sequential testing, U-statistics.

1. Introduction

Change-point detection problems have been studied extensively in areas, such as statistics, econometrics, and engineering, and there are wide applications in the fields of physical science and engineering. The literature on this topic is extensive, and growing rapidly. For

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low-dimensional data, early works include those of Page (1954), MacNeill (1974), and Brown et al. (1975), among others. More recent works on change-point problems for low-/fixed-dimensional multivariate time series data include those of Shao and Zhang (2010), Matteson and James (2014), Kirch et al. (2015), Bücher et al. (2019), among others. Refer to Perron (2006), Aue and Horváth (2013), and Aminikhanghahi and Cook (2017) for excellent reviews on this topic.

The literature on change-point detection can be roughly divided into two categories: retrospective testing and the estimation of change points based on a complete data sequence offline, and online sequential monitoring for change points based on some training data and data that arrive sequentially. This study focuses on the sequential monitoring problem for temporally independent, but cross-sectionally dependent high-dimensional data. There are two major lines of research for sequential change-point detection/monitoring. The first follows the paradigm set by pioneers in the field, such as Wald (1945), Page (1954), and Lorden (1971); see Lai (1995, 2001) and Polunchenko and Tartakovsky (2012) for comprehensive reviews. Most sequential detection methods along this line are optimized to have a minimal detection delay, controlling the average run length under the null. Furthermore, most existing procedures are developed for low-dimensional data. These methods often require us to make some parametric assumptions about the pre-change and post-change distributions. In the second line, Chu et al. (1996) assume there is a set of training data (without any change points), and apply sequential monitoring to test the data that arrives sequentially. They employ the invariance principle to control the type-I error, and their framework has been adopted by many other researchers in both parametric and nonparametric contexts;

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see Horváth et al. (2004), Aue et al. (2012), Wied and Galeano (2013), Fremdt (2015), and Dette and Gösmann (2019). Here, it is typical to use the size and power (plus average detection delay) to describe and compare the operating characteristics of competing procedures. Our procedure falls into the second category. It seems to us that these two frameworks are, in general, difficult to compare, because they differ in terms of the model assumptions and evaluation criteria.

Today, with the rapid improvement of data acquisition technology, high-dimensional data streams involving continuous sequential observations appear frequently in modern manufacturing and service industries, and the demand for efficient online monitoring tools for such data has never been higher. For example, Yan et al. (2018) proposed a method for monitoring a multi-channel tonnage profile used for the forging process, which has thousands of attributes. Furthermore, image-based monitoring [Yan et al. (2014)] has become popular in the literature, which includes thousands of pixels per image. Lévy-Leduc and Roueff (2009) considered the problem of monitoring thousands of Internet traffic metrics provided by a French Internet service provider. This kind of high-dimensional data poses significant new challenges to traditional multivariate statistical process control and monitoring, because when the dimension p is high and is comparable to the sample size n, most existing sequential monitoring methods constructed based on fixed-dimension assumptions become invalid.

In this article, we propose a new class of sequential monitoring methodology to detect the change in the mean of independent high-dimensional data based on (sequential) retrospective testing. Our proposal is inspired by recent works on the retrospective testing of mean

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changes in high-dimensional data by Wang et al. (2019) and Zhang et al. (2021). In Wang et al. (2019), the authors propose a U-statistic-based approach to target the L_2-norm of the mean difference by extending the U-statistic idea of Chen and Qin (2010) from two-sample testing to the change-point testing problem. Zhang et al. (2021) further extend the test of Wang et al. (2019) to an L_q-norm-based test mimicking that of He et al. (2018), where $q \in 2\mathbb{N}$, to capture the sparse alternative. By combining the L_2-norm-based test and the L_q-norm-based test, the adaptive test statistic they propose is shown to achieve high power for both dense and sparse alternatives. However, one of the limitations of these works is that the methods are designed for offline analysis, which is not suitable in real-time online monitoring systems. Building on the works of Wang et al. (2019) and Zhang et al. (2021), we propose a new adaptive sequential monitoring procedure that can capture both sparse and dense alternatives. Instead of using the self-normalization scheme [Shao (2010); Shao and Zhang (2010); Shao (2015)], as in Wang et al. (2019) and Zhang et al. (2021), we use ratio-consistent estimators for $\|\Sigma\|_q^q$ based on the training data, where Σ is the common covariance matrix of the sequence of random vectors, and provide a rigorous proof for ratio consistency. Furthermore, we develop recursive algorithms for fast implementation so that at each time, the monitoring statistics can be computed efficiently. Finally, theory and simulations show that the resulting adaptive monitoring procedure using a combination of sequential tests based on L_2 and L_q (say $q = 6$) is powerful against both dense and sparse alternatives.

There is a growing body of literature on high-dimensional change-point detection in the retrospective setting; see Horváth and Hušková (2012),Cho and Fryzlewicz (2015), Jirak

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(2015), Yu and Chen (2017), Wang and Samworth (2018), Yu and Chen (2019), Wang et al. (2019), Zhang et al. (2021), and Wang and Shao (2020), among others. Note that Enikeeva and Harchaoui (2019) developed a test based on a combination of a linear statistic and a scan statistic, and their test can be adaptive to both sparse and dense alternatives. However, their Gaussian and independent component assumptions are too restrictive. In addition, works on the online monitoring of high-dimensional data streams have been growing steadily in the literature on statistics and quality control. In particular, Mei (2010) proposed a global monitoring scheme based on the sum of the cumulative sum monitoring statistics from each individual data stream. His method aims to minimize the delay time and control the global false alarm rate, which is based on the average run length under the null. This is different from the size and power analysis in our work. Note that the assumptions in Mei (2010) are quite restrictive, in the sense that he assumed that no data streams have cross-sectional dependence, and that both the pre-change and the post-change distributions are known. See Wang and Mei (2015), Zou et al. (2015), Liu et al. (2019), and Li (2020) for several variants on how to aggregate the local monitoring statistics. Xie and Siegmund (2013) proposed a mixture detection procedure based on a likelihood ratio statistic that takes into account the fraction of data streams being affected. They argue that the performance is good when the fraction of affected data streams is known, and do not require a complete specification of the post-change distribution. However, the mixture global log-likelihood they specify relies on the hypothesized affected fraction p_0, and they show the robustness of different choices of p_0 using numerical studies only. The results they derive hold for data generated from a normal distribution or from other exponential families of distributions. A common feature of all

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these works is that they assume the data streams do not have cross-sectional dependence, which may be violated in practice. In fact, our theory for the proposed monitoring statistic demonstrates the impact of the correlation/covariance structure of multiple data streams, which is lacking in the above-mentioned literature.

The rest of the paper is structured as follows. In Section 2, we specify our change point monitoring framework and propose a monitoring statistic that targets the L_q-norm of the mean change. An adaptive monitoring scheme can be derived by combining the test statistic for different q, for $q \in 2\mathbb{N}$. Section 3 provides a ratio-consistent estimator for $\|\Sigma\|_q^q$, which is crucial when constructing the monitoring statistics. Section 4 provides simulation studies that examine the finite-sample performance of the adaptive monitoring statistic. In Section 5, we apply the adaptive monitoring scheme to two real data sets. Section 6 concludes the paper. All technical details can be found in the Appendix.

2. Monitoring Statistics

In this section, we specify the general framework we use to perform change-point monitoring. We consider a closed-end change-point monitoring scenario, following Chu et al. (1996). Assume that we observe a sequence of temporally independent high-dimensional observations $X_1, \ldots, X_n \in \mathbb{R}^p$, which are ordered in time and have constant mean $\boldsymbol{\mu}$ and covariance matrix Σ. We start the monitoring procedure from time $(n+1)$ to detect whether the mean vector changes in the future. Throughout the analysis, we assume that all data X_t are independent over time. A decision is made at each of the time points, and we signal an alarm when the

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monitoring statistic exceeds a certain boundary. The process ends at time nT, regardless of whether a change point is detected, where T is a prespecified number. The type-I error of the monitoring procedure is controlled at α, which means the probability of signaling an alarm when there is no change within the period $[n+1, nT]$ is at most α.

Under the null hypothesis, no change occurs within the monitoring period, and we have

$$E(X_t) = \mu \text{ for } t = 1, \ldots, nT.$$

Under the alternatives, the mean function changes at some time $t_0 > n$, and remains at the same level for the following observations. That is,

$$E(X_t) = \begin{cases} \mu & 1 < t < t_0 \\ \mu + \Delta & t_0 \leq t \leq nT. \end{cases}$$

We propose a family of test statistics $T_{n,q}(k)$, which serves as the monitoring statistic targeting $\|\Delta\|_q$. The case $q = 2$ corresponds to dense alternatives, and larger values of q correspond to sparser alternatives. We discuss the formulation of our monitoring statistic for $q = 2$, and then extend this to general q in the subsequent subsections.

2.1 L_2-norm-based monitoring statistics

In this section, we first develop the L_2-norm-based monitoring statistic, which is especially useful for detecting the dense alternative. Furthermore, we discuss the asymptotic properties of the L_2-norm-based statistic. Finally, the recursive computational algorithm is developed to allow for efficient implementation.

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2.1.1 Monitoring statistics

For a given time $k > n$, suppose we know a change point happens at location m, where $n < m < k$. We can separate the observations into two independent samples: pre-break $X_1, \ldots, X_m$, and post-break $X_{m+1}, \ldots, X_k$. Consider using a two-sample U-statistic with kernel

$$h((X, Y), (X', Y')) = (X - Y)^T (X' - Y'),$$

where (X', Y') is an independent copy of (X, Y). Then, we have

$$E[h((X, Y), (X', Y'))] = \|E(X) - E(Y)\|_2^2,$$

which estimates the squared L_2-norm of the mean difference. Indeed, Wang et al. (2019) constructed an L_2-norm-based retrospective change-point detection statistic by scanning over all possible m. For the online monitoring problem, we combine this idea with the approach in Dette and Gösmann (2019) to propose a monitoring statistic. Specifically, at each time point k, we scan through all possible change-point locations m ($n < m < k - 2$), and perform a change-point testing. We take the maximum of these U-statistics over m as our test statistics at time k. Suppose we get a ratio-consistent estimator of $||\Sigma||_F$ learned from the training sample $\{X_1, \ldots, X_n\}$, denoted by $\widehat{\|\Sigma\|_F}$. Then, our monitoring statistic at time $k = n + 3, \ldots, nT$ is

$$\begin{aligned} T_{n,2}(k) &= \frac{1}{n^3 \widehat{\|\Sigma\|_F}} \max_{m=n+1\ldots,k-2} \sum_{l=1}^{p} \sum_{1 \le i_1, i_2 \le m}^{*} \sum_{m+1 \le j_1, j_2 \le k}^{*} (X_{i_1,l} - X_{j_1,l})(X_{i_2,l} - X_{j_2,l}) \\ &= \frac{1}{n^3 \widehat{\|\Sigma\|_F}} \max_{m=n+1\ldots,k-2} G_k(m). \end{aligned}$$

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2.1.2 Asymptotic properties

To calibrate the size of the testing procedure, we need to obtain the asymptotic distribution of the test statistic under the null. The following conditions are imposed in Wang et al. (2019) to ensure the process convergence results.

Assumption 1. $tr(\Sigma^4) = o(\|\Sigma\|_F^4)$.

Assumption 2. Let $Cum(h) = \sum_{l_1,\ldots,l_h=1}^{p} cum^2(X_{1,l_1},\ldots,X_{1,l_h}) \leq C||\Sigma||_F^h$, for $h = 2,3,4,5,6$ and some constant C. Here $cum(\cdot)$ is the joint cumulant. In general, for a sequence of random variables $Y_1,\ldots,Y_n$, their joint cumulant is defined as

$$cum(Y_1,\ldots,Y_n) = \sum_{\pi}(|\pi|-1)!(-1)^{|\pi|-1}\prod_{B\in\pi}E\left(\prod_{i\in B}Y_i\right),$$

where π runs through the list of all partitions of $\{1,\ldots,n\}$, B runs through the list of all blocks of partition π, and π is the number of parts in the partition.

Assumption 1 was also imposed in Chen and Qin (2010), who pioneered the use of the U-statistic approach in the two-sample testing problem for high-dimensional data, and can be satisfied by a wide range of covariance models. Assumption 2 can be viewed as restrictions on the dependence structure, which holds under uniform bounds on the moments and "short-range" dependence-type conditions on the entries of the vector $(X_{0,1},...,X_{0,p})$. See Wang et al. (2019) for discussions about these two assumptions. Finally, under the null hypothesis and these assumptions, we provide the limiting distribution of the proposed monitoring statistic in Theorem 1.

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Theorem 1. *Under Assumptions 1 and 2, we have*

$$\max_{k=n+3,\ldots,nT} T_{n,2}(k) \xrightarrow{D} \sup_{t\in[1,T]} \sup_{s\in[1,t]} G(s,t),$$

where

$$G(s,t) = t(t-s)Q(0,s) + stQ(s,t) - s(t-s)Q(0,t),$$

and Q is a Gaussian process with the following covariance structure:

$$Cov(Q(a_1,b_1),Q(a_2,b_2)) = \begin{cases} (\min(b_1,b_2) - \max(a_1,a_2))^2 & if \quad \max(a_1,a_2) \leq \min(b_1,b_2) \\ 0 & otherwise. \end{cases}$$

In general, we can also consider some nonconstant boundary function $w(t)$, that is,

$$\max_{k=n+3,\ldots,nT} \frac{T_{n,2}(k)}{w(k/n-1)} \xrightarrow{D} \sup_{t\in[1,T]} \sup_{s\in[1,t]} \frac{G(s,t)}{w(t-1)}.$$

We take the double supremums here to control the familywise error rate. Therefore, we reject the null hypothesis if $T_{n,2}(k) > c_\alpha w(k/n-1)$, for some $k \in \{n+3,\ldots,nT\}$. The size can be calibrated by choosing c_α such that

$$P\left(\sup_{t\in[1,T]} \sup_{s\in[1,t]} \frac{G(s,t)}{w(t-1)} > c_\alpha\right) = \alpha.$$

Different choices of $w(t)$ are considered in Dette and Gösmann (2019).

- (T1) $w(t) = 1$,
- (T2) $w(t) = (t+1)^2$,
- (T3) $w(t) = (t+1)^2 \cdot \max\left\{\left(\frac{t}{t+1}\right)^{1/2}, 10^{-10}\right\}$.

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These $w(t)$ are motivated by the law of the iterated logarithm, and are used to reduce the stopping delay under the alternative. Based on our simulation results and real-data applications, the choice of $w(t)$ from the above three candidates does not seem to have a big impact on the power and detection delay. Thus, in practice, for a closed-end procedure, any choice would work. Detailed comparisons are shown in the simulation studies in Section 4.

Remark The current method can be generalized to an open-end framework. For an open-end monitoring procedure, we are interested in testing

$$E(X_t) = \mu \text{ for } t = 1, 2, \ldots$$

against the alternative

$$E(X_t) = \begin{cases} \mu & 1 < t < t_0 \\ \mu + \Delta & t > t_0, \end{cases}$$

for some $t_0 > n$. Suppose we use the same L_2-norm-based monitoring statistic at time $k = n + 3, \ldots$, that is,

$$T_{n,2}(k) = \frac{1}{n^3 \widehat{\|\Sigma\|_F}} \max_{m=n+1\ldots,k-2} G_k(m).$$

For a suitably chosen boundary function $w(\cdot)$, we expect that

$$\max_{k=n+3,\ldots,\infty} \frac{T_{n,2}(k)}{w(k/n-1)} \xrightarrow{D} \sup_{t\in[1,\infty)} \sup_{s\in[1,t]} \frac{G(s,t)}{w(t-1)},$$

as $n \to \infty$. The critical value can be determined by

$$P\left(\sup_{t\in[1,\infty)} \sup_{s\in[1,t]} \frac{G(s,t)}{w(t-1)} > c_\alpha \right) = \alpha.$$

We reject the null hypothesis if $T_{n,2}(k) > c_\alpha w(k/n-1)$, for some $k \in \{n+1, \ldots\}$. In practice, we can approximate the critical values c_α using the procedure for simulating the

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critical values in the closed-end procedure, using a large T, say $T = 200$. Note that the boundary function used for open-end monitoring needs to satisfy certain smoothness and decay rate assumptions, and the above three we used for the closed-end procedure are no longer applicable; see Assumption 2.4 in Gösmann et al. (2020) and the related discussion.

The following theorem provides a theoretical analysis of the power of the L_2-norm-based monitoring procedure.

Theorem 2. *Suppose that Assumptions 1 and 2 hold. Assume further that the change point location is at $\lfloor nr \rfloor$, for some $r \in (1, T)$. Then, we have*

1. *When $\frac{n\Delta^T\Delta}{||\Sigma||_F} \to 0$,*

$$\max_{k=n+3,\ldots nT} T_{n,2}(k) \xrightarrow{D} \sup_{t\in[1,T]} \sup_{s\in[1,t]} G(s,t).$$

2. *When $\frac{n\Delta^T\Delta}{||\Sigma||_F} \to b \in (0+\infty)$,*

$$\max_{k=n+3,\ldots nT} T_{n,2}(k) \xrightarrow{D} \tilde{T}_2 = \sup_{t\in[1,T]} \sup_{s\in[1,t]} \left[G(s,t) + b\Lambda(s,t)\right],$$

where

$$\Lambda(s,t) = \begin{cases} (t-r)^2 s^2 & s \le r \\ r^2(t-s)^2 & s > r \\ 0 & otherwise \end{cases}.$$

3. *When $\frac{n\Delta^T\Delta}{||\Sigma||_F} \to \infty$,*

$$\max_{k=n+3,\ldots nT} T_{n,2}(k) \xrightarrow{D} \infty.$$

Theorem 2 implies that, under the local alternative where $\frac{n\Delta^T\Delta}{||\Sigma||_F} \to 0$, the proposed monitoring procedure has trivial power. For the diverging alternative where $\frac{n\Delta^T\Delta}{||\Sigma||_F} \to +\infty$,

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the test has power converging to one. When the strength corresponding to the change falls in between, the test has power in the range $(\alpha, 1)$.

2.1.3 Recursive computation

One challenge for the proposed monitoring statistic $T_{n,2}(k)$ is that it needs to be recomputed at each given time k. The brute force calculation of the test statistics has $O(n^4 p)$ time complexity and $O(np)$ space complexity. In this section, we develop a recursive algorithm to efficiently update the monitoring statistic, which greatly improves the computational efficiency for online monitoring. More specifically, we propose a recursive algorithm to update $G_k(m)$, which is a major component of computing the monitoring statistic $T_{n,2}(k)$, as follows:

$$G_k(m) = (k-m)(k-m-1)\sum_{1\le i<j\le m} X_i^T X_j + m(m-1)\sum_{m+1\le i<j\le k} X_i^T X_j$$
$$- (m-1)(k-m-1)\sum_{i=1}^{m}\sum_{j=m+1}^{k} X_i^T X_j.$$

To compute $G_k(m)$, we need to keep track of two CUSUM processes

$$B_t = \sum_{i=1}^{t} X_i \text{ and } C_t = \sum_{i=1}^{t} X_i^T X_i,$$

where B_t are still p-dimensional. The partial sum process $S(a,b) = \sum_{a\le i<j\le b} X_i^T X_j$ in $G_k(m)$ can be expressed in terms of functions of B_t and C_t,

$$S(a,b) = \sum_{a\le i<j\le b} X_i^T X_j = \frac{1}{2}[(B_b - B_{a-1})^T(B_b - B_{a-1}) - (C_b - C_{a-1})].$$

The detailed algorithm is stated as follows:

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1. **Initialization**: Start with the first pair $(m, k) = (n+1, n+3)$. Record the following quantities:

$$B_{n+1}, B_{n+2}, B_{n+3}, C_{n+1}, C_{n+2}, C_{n+3}.$$

The first statistic is calculated based on

$$\begin{aligned} G_{n+3}(n+1) = 2 \cdot (B_{n+1}^T B_{n+1} - C_{n+1})/2 \\ + (n+1)n[(B_{n+3} - B_{n+1})^T (B_{n+3} - B_{n+1}) \\ - (C_{n+3} - C_{n+1})]/2 - nB_{n+1}^T (B_{n+3} - B_{n+1}). \end{aligned}$$

2. **Increase index from** k **to** $k+1$: Fix index m, and compute B_{k+1} and C_{k+1}:

$$B_{k+1} = B_k + X_{k+1}, C_{k+1} = C_k + X_{k+1}^T X_{k+1}.$$

The statistic for the pair $(m, k+1)$ is

$$\begin{aligned} G_{k+1}(m) = (k-m+1)(k-m)(B_m^T B_m - C_m)/2 \\ + m(m-1)[(B_{k+1} - B_m)^T (B_{k+1} - B_m)) \\ - (C_{k+1} - C_m)]/2 - (m-1)(k-m) \sum_{i=1}^{m} B_m^T (B_{k+1} - B_m). \end{aligned}$$

3. **Increase index from** m **to** $m+1$: For fixed index k, all B_i and C_i, for $i = n \ldots, k$, are already recorded. The statistic for the pair $(m+1, k)$ is

$$\begin{aligned} G_k(m+1) = (k-m-1)(k-m-2)(B_{m+1}^T B_{m+1} - C_{m+1})/2 \\ + (m+1)m[(B_k - B_{m+1})^T (B_k - B_{m+1})) - (C_k - C_{m+1})]/2 \\ - (k-m-2)mB_{m+1}^T (B_k - B_{m+1}). \end{aligned}$$

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The algorithm should start with $(m, k) = (n + 1, n + 3)$, increase the second index k first, and then increase along the first index m. This recursive formulation reduces the time complexity to $O(n^2 p)$, with additional space complexity $O(np)$.

2.2 L_q-norm-based monitoring statistics

In this section, we generalize the monitoring statistic from the L_2-norm to the L_q-norm. As shown in the previous analysis, the power of the L_2-norm-based monitoring statistic depends on the quantity $\|\Delta\|_2$, which is sensitive to dense alternatives. However, in real applications, we usually do not know a priori if the mean change is dense or not. As an approximation, we consider a similar test statistic targeting $\|\Delta\|_q$, for $q \in 2\mathbb{N}$. When q is large, we are essentially testing against sparse alternatives. As a special case, if we let $q \to \infty$, $\lim_{q\to\infty} \|\Delta\|_q = \|\Delta\|_\infty$, we only target on the largest element (in absolute value) of Δ.

2.2.1 Monitoring statistics

To define the monitoring statistics, we adopt the idea used in Zhang et al. (2021) without applying self-normalization. Self-normalization requires more extensive computation, and can be avoided by using the Phase-I data to obtain a ratio-consistent estimator of $\|\Sigma\|_q$. Furthermore, as pointed out by Shao (2015), self-normalization can result in a slight loss of power. Essentially, we can construct an L_q-norm-based test statistic at time $k = n + q +$

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$1, \ldots, nT$,

$$T_{n,q}(k) = \frac{1}{\sqrt{n^{3q}\widehat{\|\Sigma\|_q^q}}} \max_{m=n+1,\ldots,k-q} \sum_{l=1}^{p} \sum_{1\le i_1,\ldots,i_q\le m}^{*} \sum_{m+1\le j_1,\ldots,j_q\le k}^{*} (X_{i_1,l} - X_{j_1,l}) \cdots (X_{i_q,l} - X_{j_q,l})$$
$$= \frac{1}{\sqrt{n^{3q}\widehat{\|\Sigma\|_q^q}}} \max_{m=n+1,\ldots,k-q} U_{n,q}(k,m),$$

where $\widehat{\|\Sigma\|_q^q}$ is a ratio-consistent estimator of $\|\Sigma\|_q^q$.

2.2.2 Asymptotic properties

In this subsection, we study the asymptotic properties of the L_q-norm-based test statistics. First, we impose the following conditions in Zhang et al. (2021) to facilitate the asymptotic analysis.

Assumption 3. Let $X_t = \mu + Z_t$. Suppose Z_t are independent and identically distributed (i.i.d.) copies of Z_0 with mean zero and covariance matrix Σ. There exists $c_0 > 0$ independent of n such that $\inf_{i=1\ldots,p} Var(Z_{0,i}) \ge c_0$.

Assumption 4. Z_0 has up to eighth moments, with $\sup_{1\le j\le p} E[Z_{0,j}^8] \le C$, and for $h = 2\ldots 8$, there exist constants C_h depending on h only and a constant $r > 2$ such that

$$|cum(Z_{0,l_1} \ldots, Z_{0,l_h})| \le C_h(1 \vee \max_{1\le i<j\le h} |l_i - l_j|)^{-r}.$$

These assumptions appeared in Zhang et al. (2021), and Wang et al. (2019) showed that they imply Assumptions 1 and 2 for the case $q = 2$. Assumption 4 can be implied by the geometric moment contraction [cf. Proposition 2 of Wu and Shao (2004)], the physical dependence measure proposed by Wu (2005) [cf. Section 4 of Shao and Wu (2007)], or

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α-mixing. It essentially requires weak cross-sectional dependence among the p components in the data.

Under the null hypothesis, to obtain the limiting distribution of the monitoring statistic $T_{n,q}$, we use the limiting process in Zhang et al. (2021). Thus, we have the following theorem.

Theorem 3. *Under Assumptions 3 and 4,*

$$\max_{k=n+q+1,\ldots,nT} T_{n,q}(k) \xrightarrow{d} \tilde{T}_q := \sup_{t\in[1,T]} \sup_{s\in[1,t]} G_q(s,t),$$

where

$$G_q(s,t) = \sum_{c=0}^{q} (-1)^{q-c} \binom{q}{c} s^{q-c}(t-s)^c Q_{q,c}(s;[0,t]),$$

and $Q_{q,c}(r;[a,b])$ is a Gaussian process with covariance structure

$$cov(Q_{q,c_1}(r_1;[a_1,b_1]), Q_{q,c_2}(r_2;[a_2,b_2])) = \binom{C}{c} c!(q-c)!(r-A)^c(R-r)^{C-c}(b-R)^{q-C},$$

where $A = \max(a_1,a_2)$, $c = \min(c_1,c_2)$, $C = \max(c_1,c_2)$, and $b = \min(b_1,b_2)$. Two processes Q_{q_1,c_1} and Q_{q_2,c_2} are mutually independent if $q_1 \neq q_2 \in 2\mathbb{N}$.

The limiting null distribution is pivotal, and its critical values can be simulated based on the following equation:

$$P\left(\sup_{t\in[1,T]} \sup_{s\in[1,t]} \frac{G_q(s,t)}{w(t-1)} > c_\alpha\right) = \alpha.$$

We reject H_0 when $T_{n,q}(k) > c_\alpha w(k/n-1)$, for $k = n+q+1,\ldots,nT$. We tabulate the critical values for $T = 2$, $q = 2, 6$, and different boundary functions in Table 1. Critical values under other settings are available upon request.

Finally, we study the power of the L_q-norm-based monitoring procedure in Theorem 4.

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Table 1: Simulated critical values for L_q-norm-based test, $T = 2$

Boundary	T1		T2		T3	
Quantile	L_2	L_6	L_2	L_6	L_2	L_6
90%	0.756	3.235	0.204	0.867	0.141	0.592
95%	1.264	3.711	0.331	0.973	0.232	0.676
99%	2.715	4.635	0.706	1.196	0.485	0.837

Theorem 4. *Suppose that Assumptions 3 and 4 hold and the change-point location is at* $\lfloor nr \rfloor$*, for some* $r \in (1, T)$*,*

1. *When* $\frac{n^{q/2}\|\Delta\|_q^q}{\|\Sigma\|_q^{q/2}} \to 0$, $\max_{k=n+q+1,\ldots,nT} T_{n,q}(k) \xrightarrow{D} \tilde{T}_q$*;*

2. *When* $\frac{n^{q/2}\|\Delta\|_q^q}{\|\Sigma\|_q^{q/2}} \to \gamma \in (0, +\infty)$*,*

$$\max_{k=n+q+1,\ldots,nT} T_{n,q}(k) \xrightarrow{D} \sup_{t\in[1,T]} \sup_{s\in[1,t]} [G_q(s,t) + \gamma J_q(s;[0,t])],$$

where

$$J_q(s;[0,t]) = \begin{cases} r^q(t-s)^q & r \le s < t \\ s^q(t-r)^q & s \le r < t \\ 0 & \textit{otherwise} \end{cases};$$

3. *When* $\frac{n^{q/2}\|\Delta\|_q^q}{\|\Sigma\|_q^{q/2}} \to \infty$, $\max_{k=n+q+1,\ldots,nT} T_{n,q}(k) \xrightarrow{D} \infty$.

Analogous to the $q = 2$ case, the power of the test depends on $\|\Delta\|_q$. Therefore, for large q, the proposed test is sensitive to sparse alternatives.

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2.2.3 Recursive computation

Similarly to the L_2-based-test statistics, we would like to extend the recursive formulation to the L_q-norm-based test statistic. According to Zhang et al. (2021), under the null, the process $U_{n,q}(k,m)$ can be simplified as

$$U_{n,q}(k,m) = \sum_{c=0}^{q} (-1)^{q-c} \binom{q}{c} P_{q-c}^{m-1-c} P_c^{k-m-q+c} S_{n,q,c}(m;1,k),$$

where $P_l^k = k!/(k-l)!$ and

$$S_{n,q,c}(m;s,k) = \sum_{l=1}^{p} \sum_{s\le i_1,\ldots,i_c\le m}^{*} \sum_{m+1\le j_1,\ldots,j_{q-c}\le k}^{*} \prod_{t=1}^{c} X_{i_t,l} \prod_{g=1}^{q-c} X_{j_g,l}.$$

Because $S_{n,q,c}(m;1,k)$ are the major building blocks of our final test statistic, and need to be computed at each time k, we need to find efficient ways of calculating them recursively. A key element is the sum of product terms such as

$$B(c,m,l) := \sum_{1\le i_1,\ldots,i_c\le m}^{*} \prod_{t=1}^{c} X_{i_t,l}, \qquad \text{and}$$

$$M(c,m,k,l) := \sum_{m\le j_1,\ldots,j_c\le k}^{*} \prod_{g=1}^{c} X_{j_g,l}.$$

When we increase from m to $m+1$,

$$\sum_{1\le i_1,\ldots,i_c\le m+1}^{*} \prod_{t=1}^{c} X_{i_t,l} = \sum_{1\le i_1,\ldots,i_c\le m}^{*} \prod_{t=1}^{c} X_{i_t,l} + X_{m+1,l} \cdot \sum_{1\le i_1,\ldots,i_{c-1}\le m}^{*} \prod_{t=1}^{c-1} X_{i_t,l}.$$

We can derive the following recursive relationship for $B(c,k,l)$:

$$B(c,m+1,l) = B(c,m,l) + B(c-1,m,l) \cdot X_{k+1,l}. \tag{2.1}$$

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There is a similar recursive relationship for $M(c, m, k, l)$,

$$M(c, m+1, k, l) = M(c, m, k, l) + X_{m+1,l} M(c-1, m, k, l). \tag{2.2}$$

To enable the recursive computation, for each $c = 0, \ldots, q$, we maintain a matrix to store the cumulative sums.

1. **Initialization**: Starting with $c = 0$ and $c = 1$, for all $l = 1, \ldots, p$, initialize $B(0, k+1, l), \ldots, B(0, k+q, l) = 0$ and calculate

$$B(1, k+1, l) = \sum_{i=1}^{k+1} X_{i,l}, \ldots, B(1, k+q, l) = \sum_{i=1}^{k+q} X_{i,l}.$$

   Then, recursively calculate $B(c, i, l)$, for all $c = 0, \ldots, q$ and $i \leq k+q$, based on Equation 2.1.

2. **Update from** $B(c, k, l)$ **to** $B(c, k+1, l)$: Let $B(0, k+1, l) = B(0, k, l) + X_{k+1,l}$, and obtain the result for other $B(c, k+1, l)$ $(c \leq q)$ using Equation 2.1.

3. **Update from** $M(c, m, k, l)$ **to** $M(c, m+1, k, l)$: Fix index k, for any $n+1 \leq m \leq k-q$, $l = 1, \ldots, p$, let $M(0, m, k, l) = 0$, and calculate

$$M(1, m, k, l) = \sum_{i=m}^{k} X_{i,l}.$$

   All other $M(c, m, k, l)$, where $c \leq q$ and $n+1 \leq m \leq k-q$, can be obtained using Equation 2.2. Construct the test statistic $T_{n,q}(k+1)$ using $B(c, k, l)$ and $M(c, m, k, l)$ and repeat from step 2.

The time complexity of the recursive formulation is $O(n^2 pq)$, with space complexity $O(npq)$.

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2.3 Combining multiple L_q-norm-based statistics

In this section, we propose combining multiple L_q statistics to detect both dense and sparse alternatives. Specifically, based on the theoretical results in Zhang et al. (2021), the U-process for different q are asymptotically independent, which implies that $\{T_{n,q}\}_{k=n+q+1}^{nT}$ are asymptotically independent for $q \in 2\mathbb{N}$. Therefore, $\max_{k=n+q+1}^{nT} T_{n,q}(k)$ are asymptotically independent for $q \in I$, where $I \subset 2\mathbb{N}$, say $I = \{2, 6\}$. Thus, we can combine the monitoring procedure for different q and adjust for the asymptotic size. In general, if we want to combine a set of $q \in I$, we can adjust the size of each individual test to be $1 - (1 - \alpha)^{1/|I|}$, given the asymptotic independence, and reject the null if any of the monitoring statistics exceeds its critical value. Zhang et al. (2021) provide power analysis for the identity covariance matrix case,showing that the adaptive test enjoys good overall power.

In practice, there is this issue of which q to use. Based on the recommendation in Zhang et al. (2021), we set $q = 6$. As mentioned in the latter paper, using larger q leads to more trimming and more computational cost. As we demonstrate in the simulations, using $q = 6$ and combining with $q = 2$ show a very promising performance; see Section 4 for more details.

3. Ratio-consistent estimator of $\|\Sigma\|_q^q$

Note that the test statistic $T_n(k)$ requires a ratio-consistent estimator of $\|\Sigma\|_q^q$. For example, when $q = 2$, this can be simplified to $\|\Sigma\|_F^2$. A ratio-consistent estimator of $\|\Sigma\|_F^2$ is proposed in Chen and Qin (2010), but it seems difficult to generalize to $\|\Sigma\|_q^q$. In this section, we introduce a new class of ratio-consistent estimators of $\|\Sigma\|_q^q$ based on U-statistics. We first

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show the result when $q = 2$, and generalize it to $q \in 2\mathbb{N}$ later.

Assume $\{X_t\}_{t=1}^n \in \mathbb{R}^p$ are i.i.d. random vectors with mean $\boldsymbol{\mu}$ and variance Σ. Define

$$\widehat{\|\Sigma\|_F^2} = \frac{1}{4\binom{n}{4}} \sum_{1\le j_1<j_2<j_3<j_4\le n} tr\left((X_{j_1} - X_{j_2})(X_{j_1} - X_{j_2})^T(X_{j_3} - X_{j_4})(X_{j_3} - X_{j_4})^T\right) \quad (3.1)$$

as an estimator of $\|\Sigma\|_F^2$.

Theorem 5. *Under Assumption 1 and Cum(4)$\le C\|\Sigma\|_F^4$ in Assumption 2, $\widehat{\|\Sigma\|_F^2}$ is a ratio-consistent estimator of $\|\Sigma\|_F^2$, that is $\widehat{\|\Sigma\|_F^2}/\|\Sigma\|_F^2 \xrightarrow{p} 1$.*

Now, we extend this idea to general $q \in 2\mathbb{N}$. We let

$$\widehat{\|\Sigma\|_q^q} = \frac{1}{2^q\binom{n}{2q}} \sum_{l_1,l_2=1}^{p} \sum_{1\le i_1<\cdots<i_q<j_1<\cdots<j_q\le n} \prod_{k=1}^{q} (X_{i_k,l_1} - X_{j_k,l_1})(X_{i_k,l_2} - X_{j_k,l_2}),$$

as an estimator for $\|\Sigma\|_q^q$, for any finite positive even number q. The following proposition states that the proposed estimator is unbiased.

Proposition 1. *$\widehat{\|\Sigma\|_q^q}$ is an unbiased estimator of $\|\Sigma\|_q^q$.*

Proof of Proposition 1. Because $\{X_t\}_{t=1}^n$ are i.i.d.,

$$\begin{aligned}
\mathbb{E}[\widehat{\|\Sigma\|_q^q}] &= \frac{1}{2^q\binom{n}{2q}} \sum_{l_1,l_2=1}^{p} \sum_{1\le i_1<\cdots<i_q<j_1<\cdots<j_q\le n} \prod_{k=1}^{q} \mathbb{E}[(X_{i_k,l_1} - X_{j_k,l_1})(X_{i_k,l_2} - X_{j_k,l_2})] \\
&= \frac{1}{2^q\binom{n}{2q}} \sum_{l_1,l_2=1}^{p} \sum_{1\le i_1<\cdots<i_q<j_1<\cdots<j_q\le n} \prod_{k=1}^{q} (2\Sigma_{l_1,l_2}) \\
&= \frac{1}{2^q\binom{n}{2q}} \sum_{l_1,l_2=1}^{p} \binom{n}{2q} 2^q \Sigma_{l_1,l_2}^q = \|\Sigma\|_q^q.
\end{aligned}$$

This completes the proof. □

The ratio consistency can be shown under the following assumption.

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Assumption 5. We assume that

1. there exists $c > 0$ such that $\inf_{i=1,\ldots,p} \Sigma_{i,i} > c$;

2. there exists $C > 0$ and $r > 2$ such that for $h = 2, 3, 4$ and $1 \leq l_1 \leq \cdots \leq l_h \leq p$,

$$|cum(X_{0,l_1}, ..., X_{0,l_h})| \leq C(1 \vee (l_h - l_1))^{-r}.$$

Note that Assumption 5(2) is required for the ratio consistency, which is weaker than Assumption 4. The Assumptions 1–5 required for our theory do not state the explicit relationship between n and p. For example, when $\Sigma = I_p$, which means there is no cross-sectional dependence, all the assumptions are satisfied and (n, p) can go to infinity freely without any restrictions. When there is cross-sectional dependence, our assumptions may implicitly restrict the relative scale of n and p. In general, a larger p is a blessing in our setting, and it makes the asymptotic approximation more accurate. Furthermore, a larger n is always preferred, owing to the large-sample approximation. On the other hand, the computational cost increases when both the dimension and the sample size get large.

Theorem 6. *Under Assumption 5,* $\widehat{\|\Sigma\|_q^q}$ *is a ratio-consistent estimator of* $\|\Sigma\|_q^q$*, that is,* $\widehat{\|\Sigma\|_q^q}/\|\Sigma\|_q^q \xrightarrow{p} 1$.

Note that implementing the above estimator may be time-consuming for large q. In practice, we can always take a random sample of all possible indices and form an incomplete U-statistic to approximate. The consistency of the incomplete U-statistic can also be established, but is not pursued here for simplicity.

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4. Simulation Results

We compare the monitoring procedures for $q = 2, q = 6$, and $q = (2, 6)$ combined. We consider $(n, p) = (100, 50)$ with $T = 2$, where the observations $X_i \sim N(\mu_i, \Sigma)$ are generated independently over time. We consider four possible choices of Σ,

$$\Sigma_{ij} = \rho^{|i-j|} \text{ for } \rho = 0, 0.2, 0.5, 0.8,$$

to evaluate the performance of the monitoring scheme for the independent-components setting or under weak and strong dependence between components. All tests have nominal size $\alpha = 0.1$.

Under the null H_0, there is no change point: $\mu_i = 0$, for all i. For the alternative, we consider $\mu_i = \sqrt{\delta/r}(\mathbf{1}_r, \mathbf{0}_{p-r})$, for $i = (\lfloor 1.25n \rfloor + 1), \ldots, nT$. Under the dense alternative, we set $(\delta, r) = (1, p), (2, p)$. Under the sparse alternative, we set $(\delta, r) = (1, 3), (1, 1)$.

To illustrate the finite-sample performance of our monitoring statistics, we compare our results with those of Mei (2010) (denoted as Mei) and Liu et al. (2019) (denoted as LZM), which are similar to the open-end scenario in Chu et al. (1996). Neither method require Phase-I data, and both were originally designed to minimize the average run length. Therefore, they do not explicitly control the type-I error. To make a fair comparison with the current methods, which are proposed under the closed-end monitoring framework, we generate n independent Gaussian samples from $N(\mathbf{0}, \boldsymbol{I}_{p\times p})$, and calculate the Mei and LZM monitoring statistics. We empirically determine the critical value such that the empirical rejection rate is 10%, based on 2500 simulated data sets. For Mei's methods, we need to specify the distribution after the change point, which we set as the distribution under the

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alternative $(\delta, r) = (1, p)$. For LZM's method, we use the same setting in Liu et al. (2019), and set $b = \log(10), \rho = 0.25, t = 4$, and $s = 1$.

Table 2 shows the size of the monitoring procedure for the benchmark methods and the proposed methods for the three boundary functions T1, T2, and T3 introduced in Section 2.1 under different correlation coefficients ρ. Note that the size is noticeably worse for $\rho = 0.8$. This is partially due to the poor performance of the ratio-consistent estimator, because its variance increases as the cross-sectional dependence increases. Furthermore, note that the size seems to go in different directions for $q = 2$ and $q = 6$ as the correlation increases. The combined test, on the other hand, balances out such distortions. To make sure this is only a finite-sample behavior, we increase (n, p) from $(100, 50)$ to $(200, 200)$, showing that the size distortion for all tests improved noticeably for almost all settings. The additional results are available in the Supplementary Material. In contrast, Mei and LZM only achieved the correct size for the independent-component case, because we select the threshold under the same setting. However, when there is cross-sectional dependence between data streams, the size is no longer controlled, and the size distortion is much more severe than it is in the L_q-based tests.

Table 3 provides the power result (left column) and average delay time (ADT) (right column) for different tests under dense alternatives. As expected, the L_2-based test demonstrates higher power than that of the L_6-based test. The power of the combined test lies between and is closer to the power of the L_2-based test. As the correlation increases, the power of each test decreases, owing to the reduced signal. Of the three different boundary functions, T2 seems to have the shortest ADT, with a slight sacrifice in power. Mei's method

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Table 2: Sizes of different monitoring procedures

			T1			T2			T3		
$\alpha = 0.1$	Mei	LZM	L_2	L_6	Comb	L_2	L_6	Comb	L_2	L_6	Comb
$\rho = 0$	0.094	0.105	0.086	0.048	0.067	0.093	0.045	0.071	0.097	0.045	0.070
$\rho = 0.2$	0.058	0.125	0.083	0.048	0.057	0.082	0.045	0.055	0.083	0.046	0.051
$\rho = 0.5$	0.002	0.176	0.103	0.048	0.084	0.104	0.048	0.082	0.108	0.048	0.080
$\rho = 0.8$	0.000	0.409	0.135	0.028	0.085	0.145	0.027	0.093	0.137	0.026	0.086

is only better than the L_6-based test when there is no strong cross-sectional dependence, and is generally worse than the other methods and has a relatively longer delay, even when the distribution under the alternative is correctly specified. Note that when $\rho = 0.8$, Mei's method loses power completely. LZM, in general, has a slightly shorter detection delay, but at the cost of much lower power compared with that of the L_2-based test and the combined test. This means the LZM is quicker in signaling an alarm when a change point is detected. Although LZM showed good power for the strong cross-sectional dependence case compared with the combined test, it comes at the price of a much distorted size. This is because LZM assumes all data streams are independent.

Table 4 provides the power of different tests under sparse alternatives. The L_6-based test and the combined test are comparable in terms of power, and the L_2-based test exhibits inferior power in most settings, as expected. An interesting observation is that for the case $(\delta, r) = (1, 3)$, the L_2-based test still shows slightly higher power than the L_6-based test when

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Table 3: Power under dense alternatives

Power		Mei		LZM			L_2		L_6		Comb	
$\alpha = 0.1$	(δ, r)	power	ADT	power	ADT	$w(t)$	power	ADT	power	ADT	power	ADT
$\rho = 0.0$	$(1, p)$	0.852	72.9	0.628	38.0	T1	0.958	51.9	0.295	64.6	0.926	55.0
						T2	0.951	44.3	0.284	63.0	0.921	47.7
						T3	0.953	46.8	0.286	63.4	0.921	50.2
	$(2, p)$	0.999	69.3	1.000	15.1	T1	1.000	27.5	0.919	56.2	1.000	29.5
						T2	1.000	20.4	0.919	54.3	1.000	21.9
						T3	1.000	22.9	0.920	54.9	1.000	24.7
$\rho = 0.2$	$(1, p)$	0.740	73.3	0.675	38.2	T1	0.935	51.8	0.302	64.4	0.907	54.9
						T2	0.930	44.2	0.291	62.9	0.906	47.7
						T3	0.933	46.7	0.294	63.5	0.903	50.3
	$(2, p)$	1.000	69.9	1.000	15.6	T1	1.000	28.0	0.884	56.6	1.000	30.0
						T2	1.000	20.8	0.884	54.8	1.000	22.3
						T3	1.000	23.4	0.883	55.3	1.000	25.2
$\rho = 0.5$	$(1, p)$	0.243	74.1	0.715	34.3	T1	0.844	52.9	0.274	63.3	0.796	55.8
						T2	0.843	45.2	0.267	61.5	0.787	47.9
						T3	0.847	47.9	0.267	62.0	0.792	50.7
	$(2, p)$	0.932	72.2	1.000	15.7	T1	1.000	30.7	0.864	55.9	1.000	33.0
						T2	1.000	23.1	0.861	54.2	1.000	24.8
						T3	1.000	25.7	0.861	54.8	1.000	27.8
$\rho = 0.8$	$(1, p)$	0.000	NA	0.803	29.0	T1	0.632	54.6	0.165	62.5	0.560	56.8
						T2	0.637	46.4	0.162	60.9	0.575	48.6
						T3	0.642	49.4	0.162	61.4	0.568	51.8
	$(2, p)$	0.001	74.0	0.997	16.1	T1	0.990	38.3	0.666	56.0	0.984	40.8
						T2	0.990	30.1	0.663	54.2	0.983	32.1
						T3	0.990	32.7	0.666	54.9	0.983	35.4

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$\rho = 0.2$, which means that for this particular setting, the change is not "sparse" enough. As the correlation increases, we observe a noticeable drop in power, which is similar to the dense alternative setting and is again attributed to the reduced signal size. Of the three boundary functions, T2 still has the shortest ADT with a slight power loss compared to the other two boundary functions. Mei's method has worse power because it is designed for dense signals and the distribution under the alternative is misspecified. By comparison, LZM gives consistently good power and short ADTs across all settings. However, the good power under strong cross-sectional dependence is still offset by the severe size distortion under the null.

In addition to evaluating the size and power of the monitoring procedure, we compare the computational cost of the recursive formulation versus that of the brute force approach. For the case of $(n, p) = (100, 50)$, the average run-time of the brute force approach is 12.89 times that of the recursive algorithm under H_0, and is 13.07 times that of the recursive algorithm under the alternative. The code is implemented in R. This demonstrates the substantial efficiency gain from the recursive computational algorithm.

5. Data Illustration

5.1 Tonnage dat aset

We first apply the proposed methodology to monitor the multi-channel tonnage profile collected in a forging process in (Lei et al., 2010), where four different strain gauge sensors are mounted at each column of the forging machine, measuring the exerted force of the press.

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Table 4: Power under sparse alternatives

Power		Mei		LZM			L_2		L_6		Comb	
$\alpha = 0.1$	(δ, r)	power	ADT	power	ADT	$w(t)$	power	ADT	power	ADT	power	ADT
$\rho = 0.0$	$(1,3)$	0.422	74.0	0.990	27.4	T1	0.976	51.5	0.999	37.8	0.999	40.5
						T2	0.967	43.8	0.999	35.9	0.999	38.0
						T3	0.972	46.4	0.999	36.6	0.999	39.0
	$(1,1)$	0.400	73.9	1.000	23.4	T1	0.961	51.5	0.951	51.0	0.976	52.2
						T2	0.958	44.1	0.953	49.5	0.974	46.3
						T3	0.959	46.4	0.953	50.0	0.976	48.7
$\rho = 0.2$	$(1,3)$	0.274	74.1	0.990	29.1	T1	0.946	52.2	0.937	51.6	0.955	52.6
						T2	0.939	44.6	0.935	50.0	0.955	47.1
						T3	0.943	47.1	0.936	50.5	0.954	49.2
	$(1,1)$	0.268	74.1	1.000	23.9	T1	0.961	52.6	0.998	37.3	0.999	40.2
						T2	0.951	45.3	0.998	35.4	0.999	37.7
						T3	0.957	47.6	0.998	36.0	0.999	38.6
$\rho = 0.5$	$(1,3)$	0.048	74.5	0.972	28.2	T1	0.871	54.7	0.881	51.5	0.887	53.4
						T2	0.856	47.1	0.878	49.8	0.884	48.7
						T3	0.860	49.9	0.880	50.4	0.886	50.6
	$(1,1)$	0.036	74.3	1.000	23.2	T1	0.880	55.9	0.997	38.0	0.997	40.7
						T2	0.871	49.1	0.997	36.1	0.997	38.2
						T3	0.879	51.2	0.997	36.8	0.997	39.2
$\rho = 0.8$	$(1,3)$	0.000	NA	0.971	24.7	T1	0.621	58.9	0.800	52.9	0.808	55.3
						T2	0.610	50.6	0.801	51.3	0.802	51.5
						T3	0.614	53.7	0.803	51.9	0.807	53.2
	$(1,1)$	0.000	NA	1.000	21.5	T1	0.602	61.1	0.998	38.8	0.997	41.7
						T2	0.588	53.6	0.998	36.8	0.997	39.3
						T3	0.601	56.8	0.998	37.5	0.997	40.2

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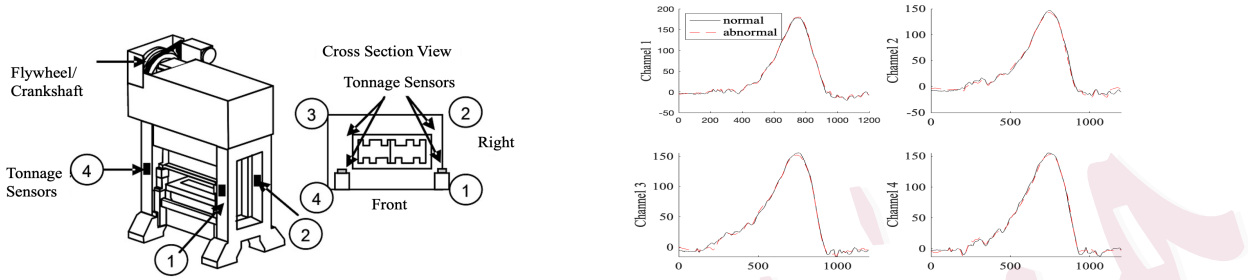

(a) The setup of the forging machine

(b) Data collected in the forging machine

Figure 1: Forging machine setup and the collected tonnage data set

The setup of the process is shown in Figure 1a. The four strain gauge sensors represent the signature of the product and are used for process monitoring and change detection in Lei et al. (2010). For example, 10 examples of the signals before the changes and after the changes are shown in Figure 1b. As mentioned by Lei et al. (2010) and Yan et al. (2018), the missing part affects only a small region of the signals, making it difficult to detect, as shown in Figure 1b.

We select a subset of the data with $n = 200$ observations, where the first 130 observations are from the normal tonnage sample, and the last 70 observations are abnormal. We project the data onto a 20-dimensional space by training the anomaly basis on a holdout sample, as in Yan et al. (2018). The first 100 observations are treated as a Phase-I stage without any changes, and we learn the 2-norm and q-norm of the covariance matrix from them. The monitoring scheme started at observation 107 (trimming due to $q = 6$). The L_6-based test stopped at time 137, and estimated the possible change-point location at time 128 by performing a retrospective test at time 137. The L_2-based test signaled slightly earlier at

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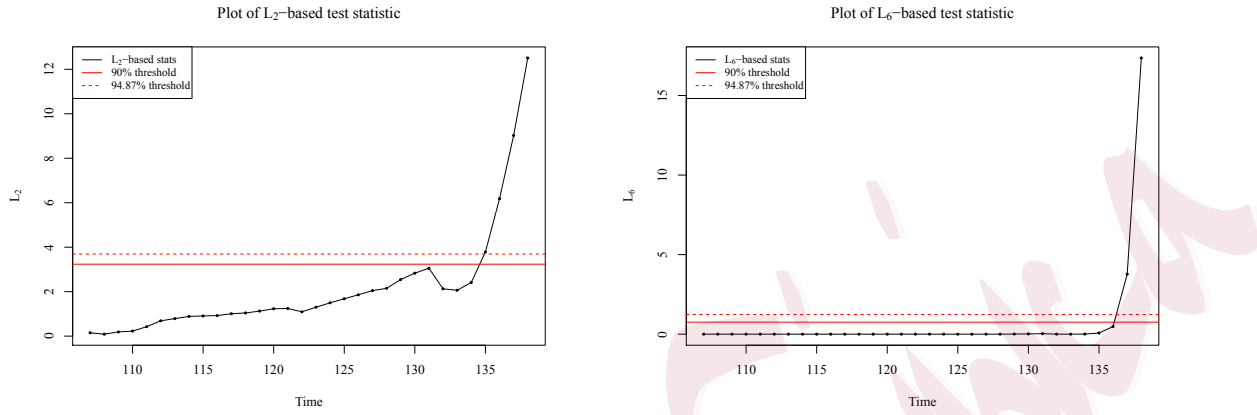

(a) L_2 based test for tonnage data (b) L_6 based test for tonnage data

Figure 2: Testing Statistics for tonnage data

time 135, and also estimated the change at 128. The combined test signaled an alarm at time 135, with the same estimated location. The trajectory of the L_2 and the L_6 test statistics are shown in Figure 2a and 2b, respectively. Note that when we set the size of the individual test to $\alpha^* = (1-0.1)^{1/2} = 5.13\%$, the size of the combined test is $\alpha = 1-\alpha^{*2} = 0.1$. We signal an alarm when at least one test statistic exceeds the corresponding threshold.

5.2 Rolling data set

Here, we consider process monitoring in a steel rolling manufacturing process. Surface defects, such as seam defects, can result in a stress concentration on the bulk, and may cause failures if the steel bar is used in a product. However, the rolling process is a high-speed process, with the rolling bar moving at around 200 miles per hour. Thus, providing real-time online anomaly detection for the high-speed rolling process is very important to

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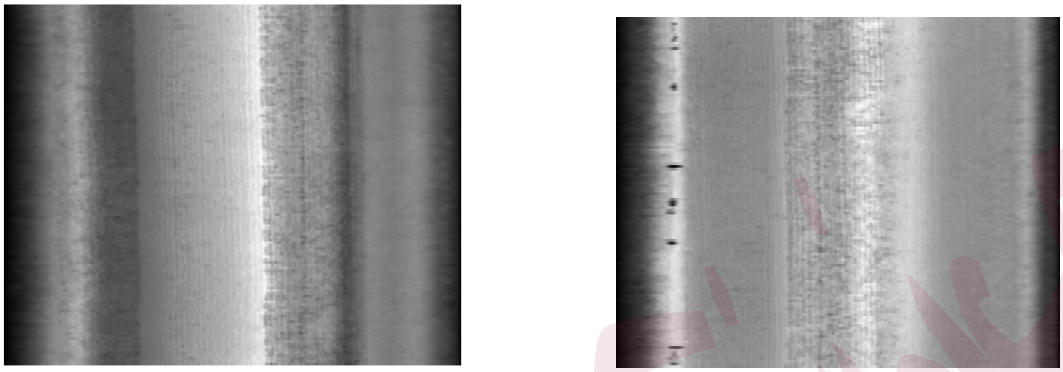

(a) Normal rolling image (b) Abnormal rolling image

Figure 3: Examples of the rolling images

prevent further product damage.

The data set is collected in the high-speed rolling process. Here, we selected a segment near the end of the rolling bar, which contains 100 images of the rolling process. To remove the trend, we have applied a smooth background remover and downsampled the image to 16×64 pixels. An example of the normal image and the abnormal image are shown in Figure 3a and 3b, respectively.

We treated the first 50 observations as the training set and obtained ratio-consistent estimators $\widehat{\|\Sigma\|_q^q}$. After performing the change-point monitoring procedure, the L_6-norm-based test signaled an alarm at time 97, and estimated that the possible change-point location is at time 89, based on the retrospective test. On the other hand, the L_2 based test failed to detect the change within the finite time horizon. The combined test also signaled an alarm at time 97. We present the rolling image at time 91 in Figure 3b. This shows that

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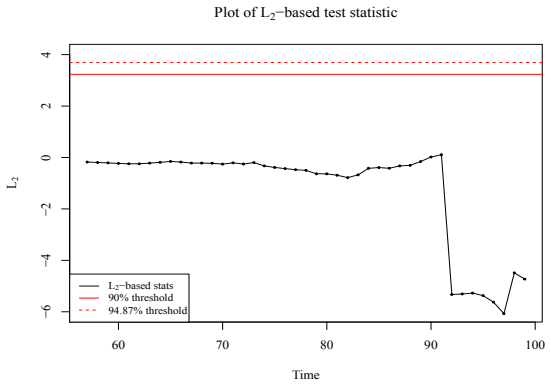
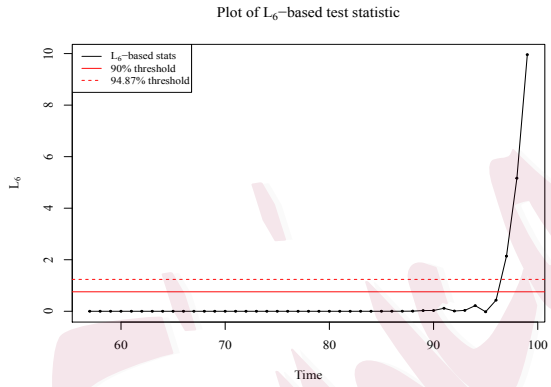


(a) L_2 based test for rolling data



(b) L_6 based test for rolling data

Figure 4: Examples of the rolling images

after downsampling, the change is still quite sparse. The adaptive monitoring procedure is still powerful, as long as one test has power. We also present the trajectory of the test statistic at each time point in Figure 4a and 4b. Note that there is a downshift in the L_2-based monitoring statistic right after the estimated change. This is because the signal is very sparse, and the construction of our proposed statistic may admit negative values for a short period. The negative values here should not be a major concern, because the test statistic should admit positive values in probability under the alternatives. We confirmed this by adding an artificial dense change to the data. On the other hand, the L_6-based monitoring statistics detect the change efficiently, owing to their ability to capture the sparse change in the system.

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6. Conclusion

In this article, we have proposed a new methodology to monitor a mean shift in temporally independent high-dimensional observations. Our change point monitoring method targets the L_q-norm of the mean change for $q = 2, 4, 6, \cdots$ By combining the monitoring statistics for different values of $q \in 2\mathbb{N}$, the adaptive procedure achieves overall satisfactory power against both sparse and dense changes in the mean. This can be desirable from a practitioner's viewpoint, because often we do not have knowledge about the types of alternatives. Compared with the recently developed methods for monitoring large-scale data streams [e.g., Mei (2010), Xie and Siegmund (2013), Liu et al. (2019)], our method is fully nonparametric and does not require strong distributional assumptions. Furthermore, our method allows for certain cross-sectional dependence across data streams, which could arise naturally in many applications.

To conclude, we mention a few interesting directions for future work. First, our focus is on the mean change, and it is natural to ask whether the method can be extended to monitor a change in the covariance matrix. Second, many streaming data have weak dependence over time, owing to their sequential nature. Thus, how to accommodate weak temporal dependence is of interest. Third, in the current implementation, the ratio-consistent estimators are learned from the training data, and do not change as more observations become available. In practice, if the monitoring scheme runs for a long time without signaling an alarm, it might be helpful to periodically update the ratio-consistent estimators to gain efficiency, especially when the initial training sample is short. However, it may be impractical to update this

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estimator for each k, because there seems no easy recursive way to update this estimator and the associated computational cost is high. The user might need to determine how often to update it based on the actual computational resources. Fourth, even though the proposed algorithm can detect a sparse change, in many applications, it is also an important problem to identify which individual data stream actually experiences a change. These issues are left for future research.

Supplementary Material

The online Supplementary Material contains technical proofs for the theoretical results, as well as additional simulation results.

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Teng Wu, Department of Statistics, University of Illinois at Urbana-Champaign

E-mail: tengwu2@illinois.edu

Runmin Wang, Department of Statistical Science, Southern Methodist University

E-mail: runminw@smu.edu

Hao Yan, School of Computing Informatics & Decision Systems Engineering, Arizona State University

E-mail: haoyan@asu.edu

Xiaofeng Shao, Department of Statistics, University of Illinois at Urbana-Champaign

E-mail: xshao@illinois.edu