Gaussian Half-Duplex Diamond Networks: Ratio of Capacity the Best Relay can Achieve

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***Abstract*—This paper considers Gaussian half-duplex diamond n-relay networks, where a source communicates with a destination by hopping information through one layer of n non-communicating relays that operate in half-duplex. The main focus consists of investigating the following question: What is the contribution of a single relay on the *approximate capacity* of the entire network? In particular, *approximate capacity* refers to a quantity that approximates the Shannon capacity within an additive gap which only depends on n, and is independent of the channel parameters. This paper answers the above question by providing a fundamental bound on the *ratio* between the approximate capacity of the highest-performing single relay and the approximate capacity of the entire network, for any number n. Surprisingly, it is shown that such a ratio guarantee is $f = 1/(2 + 2\cos(2\pi/(n + 2)))$, that is a sinusoidal function of n, which decreases as n increases. It is also shown that the aforementioned ratio guarantee is *tight*, i.e., there exist Gaussian half-duplex diamond n-relay networks, where the highest-performing relay has an approximate capacity equal to an f fraction of the approximate capacity of the entire network.**

***Index Terms*—Half-duplex, approximate capacity, diamond network, relay selection.**

I. Introduction

Relaying is foreseen to play a key role in the next generation technology, promising performance enhancement of several components of the evolving 5G architecture, such as vehicular communication [1], [2], millimeter wave communication [3], [4] and unmanned aerial vehicles communication [5], [6]. Relays can be classified into two main categories, namely *full-duplex* and *half-duplex*. While a full-duplex relay can simultaneously receive and transmit over the same time/frequency channel, a half-duplex relay has to use different times/bands for transmission and reception. A critical aspect for the implementation of the full-duplex technology consists of the design of proper Self-Interference Cancellation (SIC) techniques [7], [8], [9]. Specifically, SIC techniques can be broadly classified into three main categories, namely: (i) propagation-domain, (ii) analog-domain, and (iii) digital-domain. While the first technique leverages the use of multiple antennas at the transmitter and receiver, the other two techniques are based on the premise that a replica signal – similar to the SI signal – can be created (in the analog or digital domain), and used to subtract the SI signal from the received signal. These techniques work well in some scenarios, but their physical layer robustness is yet to be exhaustively demonstrated in different operating environments. Additionally, the current prototypes are larger and more complicated than practically desirable [10], [11], and the SIC operation might also require a significant energy consumption which cannot be sustained in scenarios where low-cost communication modules are needed and nodes have limited power supply. Given this, it is expected that half-duplex will still be widely used in next generation wireless networks [12].

In wireless networks with relays, several practical challenges arise. For instance, relays must synchronize for reception and transmission, which might result in a highly-complex process. Moreover, operating all the relays might bring to a severe power consumption, which cannot be sustained. With the goal of offering a suitable solution for these practical considerations, in [13] the authors pioneered the so-called *wireless network simplification* problem, which seeks to provide fundamental guarantees on the amount of the entire network capacity that can be retained by operating only a subset of the relays.

In this paper, we investigate the network simplification problem in Gaussian half-duplex diamond n-relay networks, where a source communicates with a destination by hopping information through a layer of n non-communicating half-duplex relays. Our main result consists of deriving a fundamental bound on the amount of the *approximate capacity*[1] of the entire network that can be retained when only one relay is operated. This bound amounts to $f = \frac{1}{2+2\cos(2\pi/(n+2))}$, i.e., a fraction f of the approximate capacity of the entire network can always be retained by operating a single relay. The merit of this result is to provide fundamental trade-off guarantees between network resource utilization and network capacity. For instance, assume a Gaussian half-duplex diamond network with $n = 3$ relays. Our result shows that if one wants to achieve 38% (or less) of the approximate capacity of the entire network, then it suffices to use only one relay, whereas if larger rates are desirable then it might be needed to operate two

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[1]As we will thoroughly explain in Section II, approximate capacity refers to a quantity that approximates the Shannon capacity within an additive gap which only depends on n, and is independent of the channel parameters.

or three relays. We also show that the guarantee f is tight, i.e., there exist Gaussian half-duplex diamond n-relay networks where the highest-performing relay has an approximate capacity equal to f of the entire network approximate capacity. To prove this, we provide two network constructions (one for even and the other for odd values of n) for which this guarantee is tight.

A. Related Work

Characterizing the Shannon capacity for wireless relay networks is a long-standing open problem. In recent years, several approximations for the Shannon capacity have been proposed among which the *constant gap* approach stands out [14], [15], [16], [17], [18]. The main merit of these works is to provide an approximation that is at most an additive gap away from the Shannon capacity; this gap is only a function of the number of relays n, and it is independent of the values of the channel parameters; because of this property, this gap is said to be constant. In the remaining part of the paper, we refer to such an approximation as *approximate capacity*.

In a half-duplex wireless network with n relays, at each point on time, each relay can either receive or transmit, but not both simultaneously. Thus, it follows that the network can be operated in 2^n possible receive/transmit states, depending on the activity of each relay. In [19], the authors proved a surprising result: it suffices to operate any Gaussian half-duplex n-relay network with arbitrary topology in at most $n+1$ states (out of the 2^n possible ones) in order to characterize its approximate capacity. This result generalizes the results in [20], [21] and [22], which were specific to Gaussian half-duplex diamond relay networks with limited number of relays n. This line of work has given rise to the following question: Can these $n+1$ states and the corresponding approximate capacity be found in polynomial time in n? The answer to this question is open in general, and it is known only for *paths*, i.e., the so-called line networks [23], and for a specific class of layered networks [24]. Recently, in [25], the authors derived sufficient conditions for Gaussian half-duplex diamond networks, which guarantee that the approximate capacity and a corresponding set of $n+1$ optimal states can be found in polynomial time in n.

In this work, we are interested in providing fundamental guarantees on the approximate capacity of the entire network that can be retained when only one relay is operated. This problem was first formulated in [13] for Gaussian full-duplex n-relay diamond networks: it was proved that there always exists a sub-network of $k \leq n$ relays that achieves at least a fraction of $k/(k+1)$ of the approximate capacity of the entire network. Moreover, the authors showed that this bound is tight, i.e., there exist Gaussian full-duplex n-relay diamond networks in which the highest-performing sub-network of k relays has an approximate capacity equal to $k/(k+1)$ of the entire network approximate capacity. Recently, in [26] the authors analyzed the guarantee of selecting the highest-performing path in Gaussian full-duplex n-relay networks with arbitrary layered topology. Very few results exist on the network simplification problem in half-duplex networks. In [27], the authors showed that in any Gaussian half-duplex n-relay diamond network, there always exists a 2-relay sub-network that has approximate capacity at least equal to $1/2$ of the approximate capacity of the entire network. Recently, in [28] the authors proved a tight guarantee for Gaussian half-duplex n-relay diamond networks: there always exists an $(n-1)$-relay sub-network that retains at least $(n-1)/n$ of the approximate capacity of the entire network. Moreover, they showed that when $n \gg 1$, then for $k=1$ and $k=2$ this guarantee becomes $1/4$ and $1/2$, respectively, i.e., the fraction guarantee decreases as n increases. These results are fundamentally different from full-duplex [13], where the ratio guarantee is independent of n. The main merit of our work is to provide an answer to a question that was left open in [28], namely: What is the fundamental guarantee (in terms of ratio) when $k=1$ relay is operated, as a function of n?

B. Paper Organization

Section II describes the Gaussian half-duplex diamond relay network, and defines its approximate capacity. Section III presents the main result of the paper, by providing a tight bound on the approximate capacity of the best relay with respect to the entire network approximate capacity. Section IV provides the proof of the bound, and Section V presents some network realizations that satisfy the bound with equality, hence showing that the ratio proved in Section IV is tight. Finally, Section VI concludes the paper. Some of the more technical proofs are in the Appendix.

II. Network Model

Notation. For two integers n_1 and $n_2 \geq n_1$, $[n_1 : n_2]$ indicates the set of integers from n_1 to n_2. For a complex number a, $|a|$ denotes the magnitude of a. Calligraphic letters (e.g., $\mathcal{A}$) denote sets. For two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \subseteq \mathcal{B}$ indicates that $\mathcal{A}$ is a subset of $\mathcal{B}$, and $\mathcal{A} \cap \mathcal{B}$ denotes the intersection between $\mathcal{A}$ and $\mathcal{B}$. The complement of a set $\mathcal{A}$ is indicated as $\mathcal{A}^c$; $\emptyset$ is the empty set. $\mathbb{E}[\cdot]$ denotes the expected value. Finally, $\lfloor x \rfloor$ is the floor of x.

The Gaussian half-duplex diamond n-relay network $\mathcal{N}$ consists of two hops (and three layers of nodes), as shown in Fig. 1: the broadcast hop between the source (node 0) and the set of n relays $\{R_1, R_2, ..., R_n\}$; and the multiple access hop between the relays $\{R_1, R_2, ..., R_n\}$ and the destination (node $n+1$). The n relays are assumed to be non-interfering, and the source can communicate to the destination only by hopping information through the relays, i.e., there is no direct link from the source to the destination. Note that this assumption of absence of direct link models wireless settings where the source and the destination are at a distance greater than the transmission range of the transmitter and hence, the signal sent by the

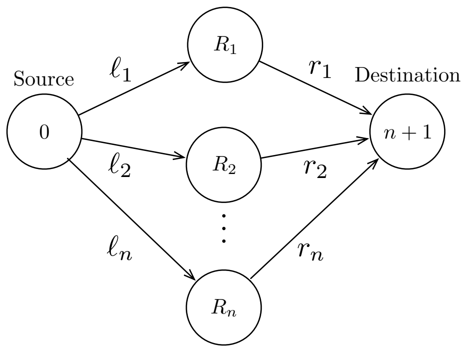


Fig. 1: Gaussian half-duplex diamond network with n relays.

transmitter will be received at or below the noise level at the destination. For instance, consider a scenario where the n relays are placed in the middle between the source and the destination, i.e., for all $i \in [1:n]$ we have that $d_{si} \approx d$ and $d_{id} \approx d$ where d_{si} (respectively, d_{id}) is the distance between the source and relay i (respectively, relay i and the destination). It therefore follows that d_{sd}, namely the distance between the source and the destination is $d_{sd} \approx 2d$. From the simplified path loss model [29], the power received at the destination through the direct link can be written as $P_r^{\rm dl} \approx P_t K (d_0/2d)^\gamma$, where: (i) P_t is the transmit power at the source, (ii) K is a unitless constant that depends on the antenna characteristics and the average channel attenuation, (iii) d_0 is a reference distance for the antenna far field (typically assumed to be 1-10 meters indoors and 10-100 meters outdoors), and (iv) γ is the path loss exponent, which depends on the propagation environment and varies between 2 (free space propagation) and 6 (indoor scenarios such as office building with multiple floors). Thus, the received power at the destination through the direct link is approximately 3γ dB smaller than the one received through each of the n relays, i.e., this attenuation varies between 6 dB (when $\gamma = 2$) and 18 dB (when $\gamma = 6$). Thus, in this range, the signal received through the direct link can be assumed to be received below the noise floor, and hence treated as noise at the destination. Furthermore, it is important to note that the diamond network is the most basic relay network model which captures two inherent aspects of wireless communication, namely: (i) its broadcast nature, and (ii) signal superposition (at the destination). Therefore, understanding the diamond network is a crucial first step towards understanding more complicated models such as those considering a direct link between the source and destination.

Relays are assumed to operate in half-duplex mode, i.e., at any given time they can either receive or transmit, but not both simultaneously. The input/output relationship for the Gaussian half-duplex diamond n-relay network at time t is defined as

$$Y_{i,t} = (1 - S_{i,t})(h_{si} X_{0,t} + Z_{i,t}), \quad \forall i \in [1:n], \tag{1a}$$

$$Y_{n+1,t} = \sum_{i=1}^{n} S_{i,t} h_{id} X_{i,t} + Z_{n+1,t}, \tag{1b}$$

where: (i) $S_{i,t}$ is a binary variable that indicates the state of relay R_i at time t; specifically, $S_{i,t} = 0$ means that relay R_i is in receiving mode at time t, and $S_{i,t} = 1$ means that relay R_i is in transmitting mode at time t; (ii) $X_{i,t}$, $\forall i \in [0:n]$ is the channel input of node i at time t that satisfies the unit average power constraint $\mathbb{E}[|X_{i,t}|^2] \leq 1$; (iii) h_{si} and h_{id} are the *time-invariant*[2] complex channel gains from the source to relay R_i and from relay R_i to the destination, respectively; (iv) $Z_{i,t}$,$i \in [1:n+1]$ is the complex additive white Gaussian noise at node i; noises are independent and identically distributed as $\mathcal{CN}(0,1)$; and finally (v) $Y_{i,t}$, $\forall i \in [1:n+1]$ is the received signal by node i at time instant t.

The Shannon capacity (a.k.a. the maximum amount of information flow) for the Gaussian half-duplex diamond n-relay network in (1) is unknown in general and its computation is notoriously an open problem (even for the case of one relay). However, it is known that the cut-set bound provides an upper bound on the network capacity [30]. Moreover, several relaying schemes, such as quantize-map-and-forward [15] and noisy network coding [16] have been shown to achieve rates within a *constant additive gap* from the Shannon capacity. We continue with the following definition.

Definition 1. *For the Gaussian half-duplex diamond n-relay network $\mathcal{N}$ described in* (1)*, define*

$$\begin{aligned} \mathsf{C}_n(\mathcal{N}) &= \max_{\boldsymbol{\lambda}} \ t \\ \textit{s.t.} \ \ t &\leq \sum_{\mathcal{S} \subseteq [1:n]} \lambda_{\mathcal{S}} \left(\max_{i \in \mathcal{S}^c \cap \Omega^c} \ell_i + \max_{i \in \mathcal{S} \cap \Omega} r_i \right), \\ \sum_{\mathcal{S} \subseteq [1:n]} \lambda_{\mathcal{S}} &= 1, \ \lambda_{\mathcal{S}} \geq 0, \quad \forall \mathcal{S} \subseteq [1:n], \end{aligned} \tag{2}$$

where the first set of constraints hold for all $\Omega \subseteq [1:n]$, *and* $\forall i \in [1:n]$ *we have that*

$$\ell_i = \log(1 + |h_{si}|^2), \ \ r_i = \log(1 + |h_{id}|^2).$$

In the above definition, ℓ_i and r_i are the point-to-point capacities of the link from the source to relay R_i and of the link from relay R_i to the destination, respectively. Moreover, in (2) we have that: (i) $\mathcal{S} \subseteq [1:n]$ corresponds to the state of the network in which the relays $R_i, i \in \mathcal{S}$, are in transmitting mode, while the rest of the relays are in receiving mode; (ii) $\lambda_{\mathcal{S}}$ denotes the fraction of time that the network operates in state $\mathcal{S}$; (iii) $\boldsymbol{\lambda}$ is the vector obtained by stacking together $\lambda_{\mathcal{S}}, \forall \mathcal{S} \subseteq [1:n]$, and is referred to as a *schedule* of the network; (iv) $\Omega \subseteq [1:n]$ is used to denote a partition of the relays in the 'side of the source', i.e., $\{0\} \cup \Omega$ is a cut of the network; similarly, $\Omega^c = [1:n] \setminus \Omega$ denotes a partition of the relays in the 'side of the destination'; note that, for a relay $R_i, i \in \Omega$, to contribute to the flow of information we also need $i \in \mathcal{S}$; similarly, for a relay $R_i, i \in \Omega^c$, to contribute to

[2]The channel coefficients are assumed to remain constant for the entire transmission duration and hence, they are known to all the nodes in the network.

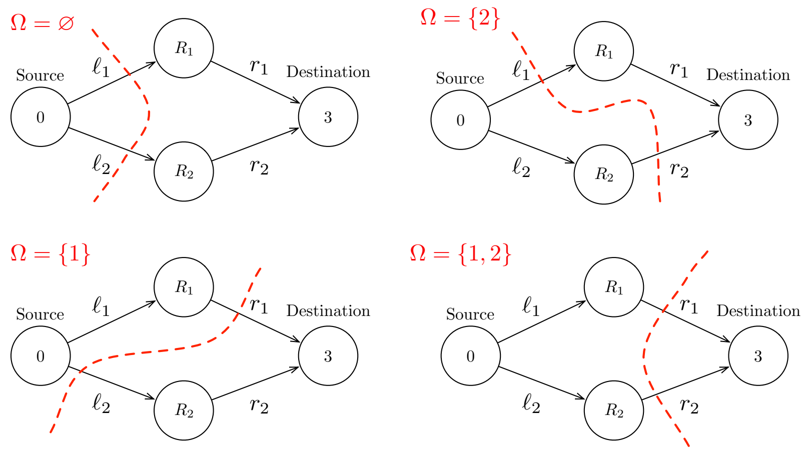


Fig. 2: The 4 possible cuts in Gaussian half-duplex diamond networks with $n = 2$ relays.

$$\begin{aligned}
&\text{For } \Omega = \varnothing : && t \le \max(\ell_1, \ell_2)\lambda_\varnothing + \ell_2\lambda_{\{1\}} + \ell_1\lambda_{\{2\}} + 0\lambda_{\{1,2\}},\\
&\text{For } \Omega = \{1\} : && t \le \ell_2\lambda_\varnothing + (\ell_2 + r_1)\lambda_{\{1\}} + 0\ \lambda_{\{2\}} + r_1\ \lambda_{\{1,2\}},\\
&\text{For } \Omega = \{2\} : && t \le \ell_1\lambda_\varnothing + 0\lambda_{\{1\}} + (\ell_1 + r_2)\ \lambda_{\{2\}} + r_2\ \lambda_{\{1,2\}},\\
&\text{For } \Omega = \{1,2\} : && t \le 0\lambda_\varnothing + r_1\lambda_{\{1\}} + r_2\ \lambda_{\{2\}} + \max(r_1, r_2)\ \lambda_{\{1,2\}},\\
&\text{Sum of } \boldsymbol{\lambda} : && 1 = \lambda_\varnothing + \lambda_{\{1\}} + \lambda_{\{2\}} + \lambda_{\{1,2\}},\\
&\text{Non-negativity of } \boldsymbol{\lambda} : && \boldsymbol{\lambda} \ge 0.
\end{aligned} \tag{3}$$

the flow of information we also need $i \in \mathcal{S}^c$. In what follows, we illustrate this through a simple example.

Example. Consider a Gaussian half-duplex diamond network with $n = 2$. Then, for this network there are $2^2 = 4$ possible cuts (as shown in Fig. 2), each of which is a function of $2^2 = 4$ possible receive/transmit states (i.e., R_1 and R_2 are in receiving mode, R_1 and R_2 are in transmitting mode, one among R_1 and R_2 is in receiving mode and the other is in transmitting mode). Then, the optimization problem in (2) will have the constraints given in (3), at the top of this page. □

The following proposition is a consequence of [13], [15], [16], and shows that $\mathsf{C}_n(\mathcal{N})$ in Definition 1 is within a constant additive gap from the Shannon capacity. Because of this property, in the remaining of the paper we refer to $\mathsf{C}_n(\mathcal{N})$ as *approximate capacity*. Specifically, in [13] the authors have shown that the cut-set bound of a Gaussian n-relay *full-duplex* diamond network can be approximated within a constant gap that is logarithmic in n, in terms of its individual link capacities, i.e., $(\ell_i, r_i), i \in [1:n]$. In *half-duplex* this gap becomes linear in n to account for the fact that there are 2^n network states (and hence the entropy term will be upper bounded by n – see also [18]). In [15], the authors have designed a scheme based on quantize-map-and-forward [14] (which is a network generalization of compress-and-forward) and proved that it approximates the cut-set bound of *any* (i.e., not necessarily diamond) Gaussian half-duplex relay network within a constant gap that is linear in n. Thus, the results in [13] and [15] lead to the result in Proposition 1. It is also worth noting that the noisy network coding scheme [16] (which is also a network generalization of compress-and-forward) can be used as an alternative scheme to quantize-map-and-forward for claiming Proposition 1.

Proposition 1. *Let $\mathsf{C}_n^G(\mathcal{N})$ be the capacity of the Gaussian half-duplex diamond n-relay network $\mathcal{N}$ in* (1), *and $\mathsf{C}_n(\mathcal{N})$ be the quantity defined in Definition 1. Then,*

$$\left|\mathsf{C}_n^G(\mathcal{N}) - \mathsf{C}_n(\mathcal{N})\right| \le \kappa_n,$$

where $\kappa_n = O(n)$ only depends on the number of relays n, and is independent of the channel coefficients.

The optimization problem in (2) seeks to maximize the source-destination information flow. This can be computed as the minimum flow across all the network cuts. Moreover, each relay can be scheduled for reception/transmission so as to maximize the information flow. Thus, the problem in (2) is a linear optimization problem with $O(2^n)$ constraints (corresponding to the 2^n network relay partitions $\Omega \subseteq [1:n]$), and $O(2^n)$ variables (corresponding to the 2^n network states $\mathcal{S} \subseteq [1:n]$).

III. Problem Statement and Main Result

An important problem in wireless communication is to characterize the fraction of the network (approximate) capacity that can be achieved by using only a subset of the relays in the network, while the remaining relays remain silent. In this work, we address this question for a single relay case in a Gaussian half-duplex diamond n-relay network. More precisely, we characterize fundamental guarantees on the approximate capacity of the *best* single relay sub-network, as a fraction of the approximate capacity of the entire network $\mathcal{N}$.

We note that the approximate capacity $\mathsf{C}_n(\mathcal{N})$ in (2) is a function of the network $\mathcal{N}$ only through the point-to-point link capacities $(\ell_i, r_i), i \in [1:n]$. Thus, with a slight abuse of notation, in what follows we let $\mathcal{N} = \{(\ell_i, r_i), i \in [1:n]\}$. We also use $\mathcal{N}_i = \{(\ell_i, r_i)\}$ to denote a half-duplex network consisting of the source, relay R_i and destination. By solving the problem in (2) for the single relay $R_i, i \in [1:n]$, we obtain that the approximate capacity of $\mathcal{N}_i$ is given by

$$\mathsf{C}_1(\mathcal{N}_i) = \frac{\ell_i r_i}{\ell_i + r_i}. \quad (4)$$

We also define the *best single relay* approximate capacity of the network as the maximum approximate capacity among the single relay sub-networks, that is,

$$\mathsf{C}_1(\mathcal{N}) = \max_{i \in [1:n]} \mathsf{C}_1(\mathcal{N}_i).$$

Our goal is to find universal bounds on $\mathsf{C}_1(\mathcal{N})/\mathsf{C}_n(\mathcal{N})$, which hold independently of the actual value of the channel coefficients. In particular, our main result is given in the next theorem, the proof of which is provided in Sections IV and V.

Theorem 1. *For any Gaussian half-duplex diamond network $\mathcal{N}$ with n relays and approximate capacity $\mathsf{C}_n(\mathcal{N})$, the best relay has an approximate capacity $\mathsf{C}_1(\mathcal{N})$ such that*

$$\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})} \geq \frac{1}{2 + 2\cos\left(\frac{2\pi}{n+2}\right)}. \quad (5)$$

Moreover, the bound in (5) *is tight, i.e., for any positive integer n, there exist Gaussian half-duplex diamond n-relay networks for which the best relay has an approximate capacity that satisfies the bound in* (5) *with equality.*

Fig. 3 provides a graphical representation of the bound in (5) as a function of the number of relays n. Before concluding this section, we state a few remarks.

Remark 1. *The bound in* (5) *for $n = 2$ and $n \to \infty$ reduces to*

$$\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})} \geq \begin{cases} 1/2 & n = 2, \\ 1/4 & n \to \infty, \end{cases}$$

which subsumes the result of [28]. However, the bound in (5) *provides a tight and non-asymptotic guarantee for any n, which was left as an open problem in [28].*

Remark 2. *The bound in* (5) *has a surprising behavior, which depends on the cosine of a function of n. This is also fundamentally different from the result in full-duplex [13], where it was shown that the best relay has always a capacity that is at least* $1/2$ *of the approximate capacity of the entire network, independent of n.*

Remark 3. *Fig. 4 shows some of the statistics of the ratio* $\mathsf{C}_1(\mathcal{N})/\mathsf{C}_n(\mathcal{N})$ *for networks with randomly generated*

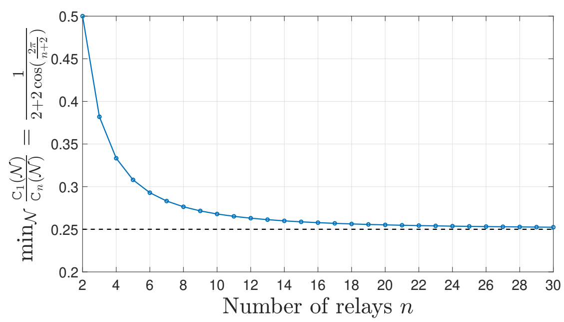


Fig. 3: Analytical ratio in (5) as a function of n.

$(\ell_i, r_i), i \in [1:n]$, *where* $(|h_{si}|, |h_{id}|)$ *follow the Rayleigh distribution with scale parameter* $\sigma = 1$. *For each* $n \in [1:10]$, *1000 sample networks were generated. The ratio* $\mathsf{C}_1(\mathcal{N})/\mathsf{C}_n(\mathcal{N})$ *for these 1000 networks is plotted as a box-plot, wherein on each box: (i) the central mark indicates the median; (ii) the top and bottom edges of the box indicate the* 75^{th} *and* 25^{th} *percentile, respectively. Any point which is at a distance of more than 1.5 times the length of the box from the top or bottom edge is an outlier (represented by a plus sign). Whiskers are drawn from the edges of the box to the furthest observations, which are not outliers. The circular dots indicate the worst case ratio in* (5). *From Fig. 4, we observe that networks with Rayleigh faded channels have a larger ratio on average than the worst case. For example, consider $n = 3$: we have* $\mathsf{C}_1(\mathcal{N}) \geq 66\%$ *of* $\mathsf{C}_3(\mathcal{N})$ *for* 50% *of the networks and* $\mathsf{C}_1(\mathcal{N}) \geq 72\%$ *of* $\mathsf{C}_3(\mathcal{N})$ *for* 25% *of the networks, while the worst case ratio is only* 38.2%.

Remark 4. *The result in Theorem 1 provides a guarantee in terms of the approximate capacities. However, Proposition 1 readily allows us to obtain a similar guarantee in terms of the Shannon capacities. From Proposition 1, we in fact have the following relationships on the Shannon capacities* $\mathsf{C}_1^G(\mathcal{N})$ *and* $\mathsf{C}_n^G(\mathcal{N})$

$$\mathsf{C}_n^G(\mathcal{N}) \leq \mathsf{C}_n(\mathcal{N}) + \kappa_n, \quad (6a)$$
$$\mathsf{C}_1^G(\mathcal{N}) \geq \mathsf{C}_1(\mathcal{N}) - \kappa_1, \quad (6b)$$

where $\kappa_1 = 1$ *and* $\kappa_n = n + 3\log(n)$. *Thus, we have that*

$$\frac{\mathsf{C}_1^G(\mathcal{N})}{\mathsf{C}_n^G(\mathcal{N})} \overset{(6)}{\geq} \frac{\mathsf{C}_1(\mathcal{N}) - \kappa_1}{\mathsf{C}_n(\mathcal{N}) + \kappa_n} \geq \frac{\mathsf{C}_1(\mathcal{N}) - \kappa_1 - \kappa_n}{\mathsf{C}_n(\mathcal{N})} \geq f - \frac{\kappa_1 + \kappa_n}{\mathsf{C}_n(\mathcal{N})}. \quad (7)$$

It is clear that when $\mathsf{C}_n(\mathcal{N})$ *is large, e.g., when the network is operated at high SNR, then the term* $\frac{\kappa_1+\kappa_n}{\mathsf{C}_n(\mathcal{N})}$ *in* (7) *vanishes leading to* $\frac{\mathsf{C}_1^G(\mathcal{N})}{\mathsf{C}_n^G(\mathcal{N})} \geq f$. *This implication is the main motivation for our study of ratio guarantees in terms of the approximate capacity.*

IV. Proof of the Bound in Theorem 1

In this section, we formally prove that the bound given in Theorem 1 is satisfied for any Gaussian half-duplex diamond network. Towards this end, we first provide

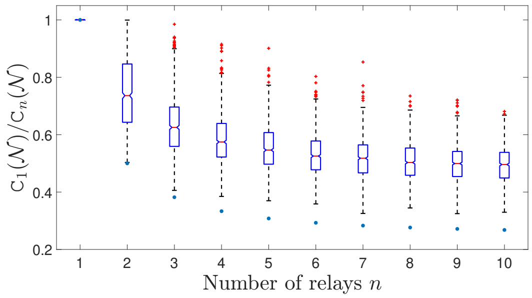


Fig. 4: Numerical ratio from 1000 networks with random link coefficients generated from the Rayleigh distribution with parameter $\sigma = 1$.

a few properties on the approximate capacity and the general theory of optimization in Section IV-A. Then, in Section IV-B, we provide a sketch of the proof of the fraction guarantee in (5), and finally in Section IV-C we use the properties of Section IV-A to prove in detail (5).

A. Properties on the Approximate Capacity

Here, we derive some properties on the approximate capacity of a Gaussian half-duplex diamond n-relay network that we will leverage to prove the fractional guarantee in (5). In particular, we start by stating the following three properties, which directly follow by inspection of the optimization problem in (2). We have,

(P1) The approximate capacity $\mathsf{C}_n(\mathcal{N})$ is a non-decreasing function of each point-to-point link capacity; that is, $\mathsf{C}_n(\mathcal{N}+\boldsymbol{\epsilon}) \geq \mathsf{C}_n(\mathcal{N})$, for any $2n$-vector $\boldsymbol{\epsilon}$ of non-negative entries.

(P2) The ratio $\mathsf{C}_1(\mathcal{N})/\mathsf{C}_n(\mathcal{N})$ is invariant to scaling all the point-to-point link capacities by a constant factor; that is, $\mathsf{C}_1(\mathcal{N})/\mathsf{C}_n(\mathcal{N}) = \mathsf{C}_1(\alpha\mathcal{N})/\mathsf{C}_n(\alpha\mathcal{N})$.

(P3) The ratio $\mathsf{C}_1(\mathcal{N})/\mathsf{C}_n(\mathcal{N})$ is invariant to a relabelling of the relay nodes.

Using the above properties, we have the following lemma.

Lemma 1. *Let $\mathcal{N}^\star$ be the collection of half-duplex diamond n-relay networks for which the ratio $\mathsf{C}_1(\cdot)/\mathsf{C}_n(\cdot)$ is minimum. Then, there exists $\mathcal{N} \in \mathcal{N}^\star$ that satisfies the following three properties:*

$$1 \leq \ell_1 \leq \ell_2 \leq \ldots \leq \ell_{n-1} \leq \ell_n \leq \infty, \tag{8a}$$
$$1 \leq r_n \leq r_{n-1} \leq \ldots \leq r_2 \leq r_1 \leq \infty, \tag{8b}$$
$$\frac{\ell_i r_i}{\ell_i + r_i} = 1, \quad \forall i \in [1:n]. \tag{8c}$$

Proof. We first prove that there exists $\mathcal{N} \in \mathcal{N}^\star$ for which all the n single relay approximate capacities are identical. Consider $\mathcal{N} \in \mathcal{N}^\star$ with approximate capacity $\mathsf{C}_n(\mathcal{N})$ and $\mathsf{C}_1(\mathcal{N}) = \mathsf{C}_1(\mathcal{N}_k)$, i.e., relay R_k has maximum single-relay approximate capacity among all the relays. Thus, $\forall j \in [1:n]$ we have

$$\mathsf{C}_1(\mathcal{N}_k) = \frac{\ell_k r_k}{\ell_k + r_k} \geq \frac{\ell_j r_j}{\ell_j + r_j} = \mathsf{C}_1(\mathcal{N}_j). \tag{9}$$

Now, we can create a new network $\mathcal{N}' = \{(\ell'_i, r'_i), i \in [1:n]\}$, where

$$\ell'_i = \frac{\mathsf{C}_1(\mathcal{N}_k)}{\mathsf{C}_1(\mathcal{N}_i)}\ell_i, \qquad r'_i = \frac{\mathsf{C}_1(\mathcal{N}_k)}{\mathsf{C}_1(\mathcal{N}_i)} r_i, \qquad i \in [1:n].$$

Note that since $\mathsf{C}_1(\mathcal{N}_k) \geq \mathsf{C}_1(\mathcal{N}_i)$, we have $\ell'_i \geq \ell_i$ and $r'_i \geq r_i$. Hence, Property (P1) implies

$$\mathsf{C}_n(\mathcal{N}') \geq \mathsf{C}_n(\mathcal{N}). \tag{10}$$

Moreover, for every $i \in [1:n]$, we have

$$\begin{aligned}
\mathsf{C}_1(\mathcal{N}'_i) &= \frac{\ell'_i r'_i}{\ell'_i + r'_i} = \frac{\left(\frac{\mathsf{C}_1(\mathcal{N}_k)}{\mathsf{C}_1(\mathcal{N}_i)}\right)^2 \ell_i r_i}{\frac{\mathsf{C}_1(\mathcal{N}_k)}{\mathsf{C}_1(\mathcal{N}_i)}(\ell_i + r_i)} \\
&= \frac{\mathsf{C}_1(\mathcal{N}_k)}{\mathsf{C}_1(\mathcal{N}_i)}\mathsf{C}_1(\mathcal{N}_i) = \mathsf{C}_1(\mathcal{N}_k), \\
&\Longrightarrow \mathsf{C}_1(\mathcal{N}') = \max_{i \in [1:n]} \mathsf{C}_1(\mathcal{N}'_i) = \mathsf{C}_1(\mathcal{N}_k).
\end{aligned} \tag{11}$$

This together with (10) yield to $\frac{\mathsf{C}_1(\mathcal{N}')}{\mathsf{C}_n(\mathcal{N}')} \leq \frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})}$, which implies $\mathcal{N}' \in \mathcal{N}^\star$. Now, we can consider $\mathcal{N}'' = \frac{1}{\mathsf{C}_1(\mathcal{N}_k)}\mathcal{N}'$. Property (P2) implies that $\frac{\mathsf{C}_1(\mathcal{N}'')}{\mathsf{C}_n(\mathcal{N}'')} = \frac{\mathsf{C}_1(\mathcal{N}')}{\mathsf{C}_n(\mathcal{N}')} \leq \frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})}$, and hence $\mathcal{N}'' \in \mathcal{N}^\star$. Moreover, it is easy to show that in $\mathcal{N}''$ we have $\mathsf{C}_1(\mathcal{N}''_i) = 1$ for every $i \in [1:n]$. This proves (8c) for the network $\mathcal{N}''$. Next, we can relabel the relay nodes such that they will be sorted in ascending order according to their left-hand link capacities ℓ''_i, and hence satisfy (8a). Note that Property (P3) guarantees that the ratio $\mathsf{C}_1(\mathcal{N}'')/\mathsf{C}_n(\mathcal{N}'')$ is invariant. Finally, combining (8a) and (8c) readily proves (8b), and concludes the proof of Lemma 1. □

Next, we present a lemma, that we will use in the proof of Theorem 1.

Lemma 2. *Let $\mathcal{A}$ be any set, and $\{f_i(\cdot), i \in [1:t]\}$ be any set of functions. Then, the two optimization problems given below have identical solutions:*

$$\begin{aligned}
&\max_{\mathbf{x} \in \mathcal{A}} \ y \\
&\text{s.t.} \quad y \leq f_i(\mathbf{x}),\ i \in [1:t],
\end{aligned} \tag{12}$$

$$\begin{aligned}
\text{and,} \quad &\min_{\boldsymbol{\mu}} \max_{\mathbf{x} \in \mathcal{A}} \sum_{i=1}^{t} \mu_i f_i(\mathbf{x}) \\
&\text{s.t.} \quad \mu_i \geq 0,\ i \in [1:t],\ \sum_{i=1}^{t} \mu_i = 1.
\end{aligned} \tag{13}$$

Proof. We prove Lemma 2 by showing that an optimal solution for (12) is a feasible solution for (13), and an optimal solution for (13) is a feasible solution for (12).

Let $\mathbf{x}^\star$ be an optimal solution for (12) and assume $j \in [1:t]$ to be such that $f_j(\mathbf{x}^\star) \leq f_i(\mathbf{x}^\star), \forall i = [1:t]$. Then, the optimal value of (12) is equal to $f_j(\mathbf{x}^\star)$. Now, letting $\mu_j = 1, \mu_i = 0, \forall i \in [1:t], i \neq j$, and $\mathbf{x} = \mathbf{x}^\star$ in (13), we see that $f_j(\mathbf{x}^\star)$ is a feasible solution for (13). Similarly, let $\mathbf{x}'$ be an optimal solution for (13) and assume $k \in [1:t]$ such that $f_k(\mathbf{x}') \leq f_i(\mathbf{x}'), \forall i = [1:t]$. Then, it is easy to see that the optimal $\boldsymbol{\mu}'$ in (13) is given by $\mu'_k = 1, \mu'_i = 0, \forall i \in [1:t], i \neq k$; moreover, the optimal value for (13) is equal to $f_k(\mathbf{x}')$. Since $\mathbf{x}' \in \mathcal{A}$ and $f_k(\mathbf{x}') \leq f_i(\mathbf{x}'), \forall i = [1:t]$, then $f_k(\mathbf{x}')$ is also a feasible solution for (12). This concludes the proof of Lemma 2. □

B. Sketch of Proof of the Fraction Guarantee in (5)

To prove the ratio guarantee in (5), we use Lemma 1 and Lemma 2. We start by noting that the result in Lemma 1 implies that there always exists an optimal network $\mathcal{N}$ such that $\mathsf{C}_1(\mathcal{N}_i) = 1, \forall i \in [1:n]$, and hence also $\mathsf{C}_1(\mathcal{N}) = 1$. Thus, proving (5) reduces to proving that, for any Gaussian half-duplex diamond n-relay network $\mathcal{N}$ with $\mathsf{C}(\mathcal{N}_i) = 1, \forall i \in [1:n]$, we always have $\mathsf{C}_n(\mathcal{N}) \leq \sigma_n + 2$, where $\sigma_n = 2\cos(\frac{2\pi}{n+2})$, or equivalently,

$$\max_{\mathcal{N}:\mathsf{C}_1(\mathcal{N}_i)=1, \forall i \in [1:n]} \mathsf{C}_n(\mathcal{N}) \leq \sigma_n + 2. \quad (14)$$

At first glance, the optimization problem in (14) is very hard to solve because it has $O(2^n)$ variables and $O(2^n)$ constraints. Also, it is not convex and does not appear to follow any nice property. Therefore, in order to simplify this problem, we construct a sequence of five optimization problems, namely $\text{OPT}_0 - \text{OPT}_4$, where each optimization problem is either a relaxation of or is equivalent to the previous optimization problem. In particular, we show the following flow of simplifications in Section IV-C:

$$\begin{aligned} \max_{\mathcal{N}:\mathsf{C}(\mathcal{N}_i)=1, \forall i \in [1:n]} \mathsf{C}_n(\mathcal{N}) &\overset{\text{(a)}}{=} \text{OPT}_0 \overset{\text{(b)}}{\leq} \text{OPT}_1 \\ &\overset{\text{(c)}}{=} \text{OPT}_2 \overset{\text{(d)}}{\leq} \text{OPT}_3 \overset{\text{(e)}}{=} \text{OPT}_4 \overset{\text{(f)}}{=} \sigma_n + 2, \end{aligned} \quad (15)$$

where, broadly speaking, the labeled equalities and inequalities follow from: (a) using $\mathsf{C}(\mathcal{N}_i) = 1, \forall i \in [1:n]$ to express (ℓ_i, r_i) using a single variable z_i; (b) considering only a subset of the cut constraints and hence, enlarging the set over which a feasible solution can be found; moreover, in this step, the optimization over the network schedule (which has 2^n entries) is relaxed to an optimization over n variables $\alpha_i \in [0,1], i \in [1:n]$ indicating the total fraction of time relay i is in receiving mode; (c) using Lemma 2; (d) observing that in the optimal solution of OPT_2 some of the variables appear in a repeated manner and hence, can be grouped together by also leveraging the fact that the objective function is convex in these variables; in this step, some of the constraints are also removed, hence leading to an increase of the optimum cost function; (e) using Lemma 2; and (f) using properties of linear homogeneous recurrence relations of order 2.

C. Proof of the Fraction Guarantee in (5)

We use Lemma 1 to rewrite the constraints in the optimization problem in (14). Towards this end, we define

$$z_i \triangleq \ell_i - 1, \qquad i \in [1:n]. \quad (16)$$

Recall that $\mathsf{C}_1(\mathcal{N}_i) = \frac{\ell_i r_i}{\ell_i + r_i} = 1$. This implies that $r_i = \frac{1}{z_i} + 1$. Therefore, the class of networks of interest can be parameterized by $\mathbf{z} = [z_1, z_2, \ldots, z_n]$. Note that the condition in (8a) implies that $0 \leq z_1 \leq z_2 \leq \ldots \leq z_n \leq \infty$. Rewriting our optimization problem in (14) in terms of z_i's, and using the definition of the approximate capacity in (2), we arrive at

$$\begin{aligned} &\text{OPT}_0 = \max_{\mathbf{z}} \max_{\boldsymbol{\lambda}} \ \Gamma \\ &\text{s.t.} \\ &\Gamma \leq \sum_{\mathcal{S} \subseteq [1:n]} \lambda_{\mathcal{S}} \left(\max_{i \in \mathcal{S}^c \cap \Omega^c} \ell_i + \max_{i \in \mathcal{S} \cap \Omega} r_i \right), \ \forall \Omega \subseteq [1:n], \\ &\sum_{\mathcal{S} \subseteq [1:n]} \lambda_{\mathcal{S}} = 1, \ \lambda_{\mathcal{S}} \geq 0, \quad \forall \mathcal{S} \subseteq [1:n], \\ &\ell_i = 1 + z_i, \quad r_i = 1 + \frac{1}{z_i}, \quad i \in [1:n], \\ &0 \leq z_1 \leq z_2 \leq \cdots \leq z_n \leq \infty. \end{aligned} \quad (17)$$

Reducing the Number of Constraints. Note that the optimization problem in (17) has one constraint for each possible partition of the relays $\Omega \subseteq [1:n]$. Instead of considering all relay partitions, we can focus on a small class of them parameterized as $\Omega_t, \forall t \in [0:n]$, where

$$\Omega_t = [t+1:n], \qquad \text{and} \qquad \Omega_t^c = [1:t]. \quad (18)$$

That is, Ω_t partitions all the relays into two groups, namely $\{t+1, \ldots, n-1, n\}$ on the 'source side', and $\{1, 2, \ldots, t\}$ on the 'destination side'. With this, the right-hand side of the cut constraint corresponding to Ω_t in (17) can be simplified as in (19), at the top of the next page, where the inequality in (a) follows from the fact that, in the first summation $t \notin \mathcal{S}^c$ implies $\mathcal{S}^c \cap \Omega_t^c \subseteq [1:t-1]$, which together with $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n$ (according to (8a)) yields $\max_{i \in \mathcal{S}^c \cap \Omega_t^c} \ell_i \leq \max_{i \in [1:t-1]} \ell_i = \ell_{t-1}$. A similar argument holds for the other three summations in (a) of (19). The equality in (b) of (19) follows by letting $\alpha_t = \sum_{\mathcal{S}:t \notin \mathcal{S}} \lambda_{\mathcal{S}}$ and $\bar{\alpha}_t = (1 - \alpha_t) = \sum_{\mathcal{S}:t \in \mathcal{S}} \lambda_{\mathcal{S}}$ for $t \in [1:n]$. Finally, in (c) we replaced ℓ_t by $1 + z_t$ and r_t by $1 + \frac{1}{z_t}$ for $t \in [1:n]$, according to the constraints in (17). Note that, we define $z_i = -1$ for $i \notin [1:n]$. For instance, for $t = 0$, the function $g_0(\mathbf{z}, \boldsymbol{\alpha})$ reduces to

$$g_0(\mathbf{z}, \boldsymbol{\alpha}) = \bar{\alpha}_1 \left(\frac{1}{z_1} + 1 \right) + \alpha_1 \left(\frac{1}{z_2} + 1 \right).$$

Now, by ignoring all the cut constraints except those in $\{\Omega_t : t \in [0:n]\}$, we obtain

$$\begin{aligned} \text{OPT}_1 = \max_{\mathbf{z}, \boldsymbol{\alpha}} \ &\Gamma \\ \text{s.t.} \ &\Gamma \leq g_t(\mathbf{z}, \boldsymbol{\alpha}), \quad \forall t \in [0:n], \\ &\alpha_i \in [0,1], \quad \forall i \in [0:n+1], \\ &0 \leq z_1 \leq z_2 \leq \ldots \leq z_n, \\ &z_{-1} = z_0 = z_{n+1} = z_{n+2} = -1. \end{aligned} \quad (20)$$

It is clear that $\text{OPT}_0 \leq \text{OPT}_1$, where OPT_0 and OPT_1 are the solutions of the optimization problems in (17) and in (20), respectively. This follows since in (20) we only considered a subset of the cut constraints that we have for solving (17) and hence, we enlarged the set over which a feasible solution can be found. Moreover,

$$\begin{aligned}
&\sum_{\mathcal{S}\subseteq[1:n]} \lambda_{\mathcal{S}}\left(\max_{i\in\mathcal{S}^c\cap\Omega_t^c} \ell_i + \max_{i\in\mathcal{S}\cap\Omega_t} r_i\right)\\
&= \sum_{\mathcal{S}:t\in\mathcal{S}} \lambda_{\mathcal{S}} \max_{i\in\mathcal{S}^c\cap\Omega_t^c} \ell_i + \sum_{\mathcal{S}:t\notin\mathcal{S}} \lambda_{\mathcal{S}} \max_{i\in\mathcal{S}^c\cap\Omega_t^c} \ell_i + \sum_{\mathcal{S}:t+1\in\mathcal{S}} \lambda_{\mathcal{S}} \max_{i\in\mathcal{S}\cap\Omega_t} r_i + \sum_{\mathcal{S}:t+1\notin\mathcal{S}} \lambda_{\mathcal{S}} \max_{i\in\mathcal{S}\cap\Omega_t} r_i\\
&\overset{(a)}{\leq} \sum_{\mathcal{S}:t\in\mathcal{S}} \lambda_{\mathcal{S}}\ell_{t-1} + \sum_{\mathcal{S}:t\notin\mathcal{S}} \lambda_{\mathcal{S}}\ell_t + \sum_{\mathcal{S}:t+1\in\mathcal{S}} \lambda_{\mathcal{S}} r_{t+1} + \sum_{\mathcal{S}:t+1\notin\mathcal{S}} \lambda_{\mathcal{S}} r_{t+2}\\
&\overset{(b)}{=} (1-\alpha_t)\ell_{t-1} + \alpha_t\ell_t + (1-\alpha_{t+1})r_{t+1} + \alpha_{t+1}r_{t+2}\\
&\overset{(c)}{=} \bar{\alpha}_t(z_{t-1}+1) + \alpha_t(z_t+1) + \bar{\alpha}_{t+1}\left(\frac{1}{z_{t+1}}+1\right) + \alpha_{t+1}\left(\frac{1}{z_{t+2}}+1\right)\\
&\triangleq g_t(\mathbf{z},\boldsymbol{\alpha}),
\end{aligned} \tag{19}$$

variables α's can be uniquely determined from λ's, but the opposite does not necessarily hold.

Now, using Lemma 2, we can rewrite (20) as the following optimization problem]

$$\begin{aligned}
\mathrm{OPT}_2 = \min_{\boldsymbol{\mu}} \max_{\boldsymbol{z},\boldsymbol{\alpha}} \ & h(\boldsymbol{\mu},\boldsymbol{z},\boldsymbol{\alpha})\\
\text{s.t.} \ & \mu_t \geq 0, \quad \forall t\in[0:n],\\
& \textstyle\sum_{t=0}^{n} \mu_t = 1,\\
& \alpha_i \in [0,1], \quad \forall i \in [0:n+1],\\
& 0 \leq z_1 \leq z_2 \leq \ldots \leq z_n,\\
& z_{-1} = z_0 = z_{n+1} = z_{n+2} = -1,
\end{aligned} \tag{21a}$$

where

$$h(\boldsymbol{\mu},\boldsymbol{z},\boldsymbol{\alpha}) = \sum_{t=0}^{n} \mu_t g_t(\boldsymbol{z},\boldsymbol{\alpha}). \tag{21b}$$

Thus, by means of Lemma 2, we have $\mathrm{OPT}_2 = \mathrm{OPT}_1$.

Optimum $z_t^\star$'s Are Grouped. Our next step towards solving the optimization problem of interest is to show that in the optimum solution of (21), $z_t^\star$ will appear in a repeated manner, i.e., except possibly for $z_1^\star$ and $z_n^\star$, each $z_t^\star$ equals either $z_{t-1}^\star$ or $z_{t+1}^\star$.

We start by taking the derivative of the function $h(\boldsymbol{\mu},\boldsymbol{z},\boldsymbol{\alpha})$ defined in (21b) with respect to each variable z_t, and we obtain

$$\begin{aligned}
\frac{\partial}{\partial z_t} h(\boldsymbol{\mu},\boldsymbol{z},\boldsymbol{\alpha}) &= (\mu_t\alpha_t + \mu_{t-1}\bar{\alpha}_{t+1})\\
&\quad - (\mu_{t-2}\alpha_{t-1} + \mu_{t-1}\bar{\alpha}_t)\frac{1}{z_t^2},\\
\frac{\partial^2}{\partial z_t^2} h(\boldsymbol{\mu},\boldsymbol{z},\boldsymbol{\alpha}) &= 2(\mu_{t-2}\alpha_{t-1} + \mu_{t-1}\bar{\alpha}_t)\frac{1}{z_t^3} \geq 0.
\end{aligned}$$

Therefore, since α_t's and μ_t's are non-negative variables, $h(\boldsymbol{\mu},\boldsymbol{z},\boldsymbol{\alpha})$ is a convex function of z_t for any fixed coefficient vectors $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$. Hence, at the optimum point $(\boldsymbol{\mu}^\star,\boldsymbol{z}^\star,\boldsymbol{\alpha}^\star)$ for (21), each z_t should take one of its extreme values. However, recall that z_t's are sorted, i.e., $z_{t-1} \leq z_t \leq z_{t+1}$. This implies that for the optimum vector $\boldsymbol{z}^\star = [z_1^\star, z_2^\star, \cdots, z_n^\star]$ we have[3] $z_t^\star \in \{z_{t-1}^\star, z_{t+1}^\star\}$ for $t\in[2:n-1]$. Moreover, $0 \leq z_1 \leq z_2$ implies $z_1^\star \in \{0, z_2^\star\}$, and similarly, $z_{n-1} \leq z_n \leq \infty$ implies $z_n^\star \in \{z_{n-1}^\star, \infty\}$. More precisely, the parameters $(z_1^\star, z_2^\star, \cdots, z_n^\star)$ can be grouped into

$$\begin{aligned}
z_1^\star = \ldots = z_{t_1}^\star &= \beta_1,\\
z_{t_1+1}^\star = \ldots = z_{t_2}^\star &= \beta_2,\\
&\vdots\\
z_{t_{m-1}+1}^\star = \ldots = z_{t_m}^\star &= \beta_m,
\end{aligned} \tag{22}$$

where $0 \leq \beta_1 < \beta_2 < \cdots < \beta_{m-1} < \beta_m \leq \infty$. Note that $t_j - t_{j-1}$ (with $t_0 = 0$) is the number of z_i's whose optimum value equals β_j. Also note that m is the number of *distinct* values that the collection of $z_t^\star$'s take. Note that except for possibly β_1 and β_m, each other β_j should be taken by at least two consecutive $z_t^\star$ and $z_{t+1}^\star$, that is $t_j - t_{j-1} \geq 2$ for $j \in [2:m-1]$. This implies that the number of distinct β's cannot exceed $\frac{n+2}{2}$. This together with the fact that m is a non-negative integer, imply $1 \leq m \leq \lfloor \frac{n+2}{2} \rfloor$. Moreover, if $\beta_1 > 0$, then $z_1^\star = z_2^\star = \beta_1$, and hence $t_1 \geq 2$. Similarly, if $z_n^\star < \infty$, we have $z_n^\star = z_{n-1}^\star$, and thus $t_m - t_{m-1} \geq 2$. In summary, we have

$$\begin{cases}
t_1 \geq 1 & \text{if } \beta_1 = 0,\\
t_1 \geq 2 & \text{if } \beta_1 > 0,\\
t_i - t_{i-1} \geq 2 & \text{for } i \in [2:m-1],\\
t_m - t_{m-1} \geq 1 & \text{if } \beta_m = \infty,\\
t_m - t_{m-1} \geq 2 & \text{if } \beta_m < \infty.
\end{cases} \tag{23}$$

Example. Consider a diamond network with $n = 5$ relays. For the optimum vector $\boldsymbol{z}^\star = [z_1^\star, z_2^\star, z_3^\star, z_4^\star, z_5^\star]$ we have

$$\begin{aligned}
&z_1^\star \in \{0, z_2^\star\}, \quad z_2^\star \in \{z_1^\star, z_3^\star\}, \quad z_3^\star \in \{z_2^\star, z_4^\star\},\\
&z_4^\star \in \{z_3^\star, z_5^\star\}, \quad z_5^\star \in \{z_4^\star, \infty\}.
\end{aligned}$$

There are several possible solutions that satisfy the conditions above. One possibility could be

$$z_1^\star = z_2^\star = z_3^\star = z_4^\star = z_5^\star = \beta_1,$$

in which case, with reference to (22), we have $m = 1$ and $t_1 = 5$. Alternatively, we may have

$$z_1^\star = 0 = \beta_1, \qquad z_2^\star = z_3^\star = \beta_2, \qquad z_4^\star = z_5^\star = \beta_3,$$

[3] Otherwise if $z_{t-1}^\star < z_t^\star < z_{t+1}^\star$, the convexity of the function $h(\boldsymbol{\mu},\boldsymbol{z},\boldsymbol{\alpha})$ implies that it can be further increased by either decreasing $z_t^\star$ to $z_{t-1}^\star$ or increasing it to $z_{t+1}^\star$.

in which case, with reference to (22), we have $m = 3$, $t_1 = 1$ (since $\beta_1 = 0$), $t_2 = 3$ and $t_3 = 5$. □

We now leverage (22) to rewrite $g_t(\boldsymbol{z}, \boldsymbol{\alpha})$ in (19) in terms of the optimum values of $z_t^\star$. In particular, we focus on functions $g_t(\boldsymbol{z}, \boldsymbol{\alpha})$ for $t \in \{t_0 = 0, t_1, t_2, \dots, t_m = n\}$. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_m)$. First, for $t = t_0 = 0$, noting that $z_{-1} = z_0 = -1$, we have

$$\begin{aligned} g_0(\boldsymbol{z}^\star, \boldsymbol{\alpha}) &= \bar{\alpha}_1 \left(\frac{1}{z_1^\star} + 1 \right) + \alpha_1 \left(\frac{1}{z_2^\star} + 1 \right) \\ &\overset{\text{(a)}}{\leq} \bar{\alpha}_1 \left(\frac{1}{z_1^\star} + 1 \right) + \alpha_1 \left(\frac{1}{z_1^\star} + 1 \right) \\ &= 1 + \frac{1}{\beta_1} \triangleq G_0(\boldsymbol{\beta}), \end{aligned} \tag{24}$$

where the inequality in (a) follows from $z_1^\star \leq z_2^\star$. Next, for all $t \in \{t_1, t_2, \dots, t_{m-1}\}$, we obtain (25), at the top of the next page. Note that (b) in (25) follows from the fact that $t_i - t_{i-1} \geq 2$, which implies $z_{t_i-1}^\star = z_{t_i}^\star = \beta_i$, and similarly $z_{t_i+1}^\star = z_{t_i+2}^\star = \beta_{i+1}$. However, for $t_1 = 1$ we have $z_0^\star = -1$, and hence (b) is an inequality, and similarly for $t_m - t_{m-1} = 1$ we have $z_{t_{m+1}}^\star = z_{n+1}^\star = -1$ and hence (b) is also an inequality. Finally, since $z_{n+1} = z_{n+2} = -1$ for $t = t_m = n$, we can write

$$\begin{aligned} g_n(\boldsymbol{z}^\star, \boldsymbol{\alpha}) &= \alpha_n (z_n^\star + 1) + \bar{\alpha}_n (z_{n-1}^\star + 1) \\ &\overset{\text{(c)}}{\leq} \alpha_n (z_n^\star + 1) + \bar{\alpha}_n (z_n^\star + 1) \\ &= 1 + \beta_m \triangleq G_m(\boldsymbol{\beta}), \end{aligned} \tag{26}$$

where the inequality in (c) holds since $z_n^\star \geq z_{n-1}^\star$. Therefore, using (24)-(26) we can upper bound the objective function of the optimization problem in (21) as

$$\begin{aligned} h(\boldsymbol{\mu}, \boldsymbol{z}, \boldsymbol{\alpha}) &= \sum_{i=0}^{n} \mu_i g_i(\boldsymbol{z}^\star, \boldsymbol{\alpha}) \\ &= \sum_{i \in \{t_0, \dots, t_m\}} \mu_i g_i(\boldsymbol{z}^\star, \boldsymbol{\alpha}) + \sum_{i \notin \{t_0, \dots, t_m\}} \mu_i g_i(\boldsymbol{z}^\star, \boldsymbol{\alpha}) \\ &\leq \sum_{i=0}^{m} \mu_{t_i} G_i(\boldsymbol{\beta}) + \sum_{i \notin \{t_0, \dots, t_m\}} \mu_i g_i(\boldsymbol{z}^\star, \boldsymbol{\alpha}). \end{aligned} \tag{27}$$

Further Reduction of the Constraints. Recall that the optimization problem in (21) includes a minimization with respect to $\boldsymbol{\mu}$. Hence, setting more restrictions on the variable $\boldsymbol{\mu}$ can only increase the optimum cost function. Let us set $\mu_t = 0$ for $t \notin \{t_0 = 0, t_1, t_2, \dots, t_m = n\}$, and $\mu_{t_i} = \tilde{\mu}_i$ for $i \in [0 : m]$. Here $\tilde{\mu}_i$'s are arbitrary non-negative variables that sum up to 1. Incorporating this and the bound in (27) into the optimization problem in (21) leads us to

$$\begin{aligned} \mathrm{OPT}_3 = \min_{\tilde{\boldsymbol{\mu}}} \max_{m \in \left[1 : \lfloor \frac{n+2}{2} \rfloor\right]} \max_{\boldsymbol{\beta}} & \sum_{t=0}^{m} \tilde{\mu}_t G_t(\boldsymbol{\beta}) \\ \text{s.t.} \quad & \tilde{\mu}_t \geq 0, \quad \forall t \in [0 : m], \\ & \sum_{t=0}^{m} \tilde{\mu}_t = 1, \\ & 0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \infty. \end{aligned} \tag{28}$$

Note that $\mathrm{OPT}_2 \leq \mathrm{OPT}_3$ since: (i) the objective function in (28) is an upper bound for that of (21), and (ii) the feasible set for $\boldsymbol{\mu}$ in (21) is a super-set of that of $\tilde{\boldsymbol{\mu}}$ in (28). Finally, we can again apply Lemma 2 on the optimization problem in (28) and rewrite it as

$$\begin{aligned} \mathrm{OPT}_4 = \max_{m \in \left[1 : \lfloor \frac{n+2}{2} \rfloor\right]} \max_{\boldsymbol{\beta}} \ & \Phi \\ \text{s.t.} \quad & \Phi \leq G_i(\boldsymbol{\beta}), \quad \forall i \in [0 : m], \\ & 0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \infty, \end{aligned} \tag{29}$$

where $G_i(\boldsymbol{\beta})$'s are defined in (24)-(26). Note that Lemma 2 implies that $\mathrm{OPT}_3 = \mathrm{OPT}_4$.

Analysis of the Inner Optimization Problem. Let us fix m in the optimization problem in (29), and further analyze the inner optimization problem. This yields

$$\begin{aligned} \mathrm{OPT}_5(m) = \max_{\boldsymbol{\beta}} \ & \Phi \\ \text{s.t.} \quad & \Phi \leq G_i(\boldsymbol{\beta}), \quad \forall i \in [0 : m], \\ & 0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \infty, \end{aligned} \tag{30}$$

for every fixed $m \in \left[1 : \lfloor \frac{n+2}{2} \rfloor\right]$. The following lemma highlights some important properties of the optimum solution of the optimization problem defined in (30).

Lemma 3. *For every integer m, there exists some solution $(\boldsymbol{\beta}^\star, \Phi^\star)$ for the optimization problem in* (30) *that satisfies*

$$G_i(\boldsymbol{\beta}^\star) = \Phi^\star, \qquad \forall i \in [1 : m-1].$$

Moreover, if $\beta_1^\star > 0$, we have $G_0(\boldsymbol{\beta}^\star) = \Phi^\star$, and similarly, if $\beta_m^\star < \infty$, then $G_m(\boldsymbol{\beta}^*) = \Phi^\star$.

Proof. We use contradiction to formally prove the claim in Lemma 3. Let $\Phi^\star$ be the optimum value of the objective function, which can be attained for each $\boldsymbol{\beta} \in \boldsymbol{B}$, where $\boldsymbol{B}$ denotes the feasible set of $\boldsymbol{\beta}$ i.e.,

$$\min_{i \in [0:m]} G_i(\boldsymbol{\beta}) = \Phi^\star, \qquad \forall \boldsymbol{\beta} \in \boldsymbol{B}.$$

If the first claim in Lemma 3 does not hold, then for every $\boldsymbol{\beta} \in \boldsymbol{B}$ there exists some minimum $q(\boldsymbol{\beta}) \in [1 : m-1]$ such that $G_{q(\boldsymbol{\beta})}(\boldsymbol{\beta}) > \Phi^\star$, i.e., $G_j(\boldsymbol{\beta}) = \Phi^\star$ for every $j < q(\boldsymbol{\beta})$. Among all optimum points $\boldsymbol{\beta} \in \boldsymbol{B}$, let $\boldsymbol{\beta}^\star$ be the one with minimum $q(\boldsymbol{\beta}^\star)$, that is, $q(\boldsymbol{\beta}) \geq q(\boldsymbol{\beta}^\star) \triangleq q$.

We have

$$\begin{aligned} 2 + \beta_q^\star + \frac{1}{\beta_{q+1}^\star} &= G_q(\boldsymbol{\beta}^\star) \\ &> G_{q-1}(\boldsymbol{\beta}^\star) = 2 + \beta_{q-1}^\star + \frac{1}{\beta_q^\star} = \Phi^\star. \end{aligned}$$

It is straight-forward to see that there exists some $\hat{\beta}_q$ such that $\beta_{q-1}^\star < \hat{\beta}_q < \beta_q^\star$ and

$$2 + \hat{\beta}_q + \frac{1}{\beta_{q+1}^\star} = 2 + \beta_{q-1}^\star + \frac{1}{\hat{\beta}_q}.$$

Thus, for the vector $\hat{\boldsymbol{\beta}} = [\beta_1^\star, \cdots, \beta_{q-1}^\star, \hat{\beta}_q, \beta_{q+1}^\star, \cdots, \beta_m^\star]$ we have

$$\begin{aligned} & G_q(\boldsymbol{\beta}^\star) > G_q(\hat{\boldsymbol{\beta}}) = G_{q-1}(\hat{\boldsymbol{\beta}}) > G_{q-1}(\boldsymbol{\beta}^\star) = \Phi^\star, \\ & G_j(\hat{\boldsymbol{\beta}}) = G_j(\boldsymbol{\beta}^\star) \geq \Phi^\star, \quad j \in [0 : m] \setminus \{q, q-1\}. \end{aligned} \tag{31}$$

$$g_{t_i}(\boldsymbol{z}^\star, \boldsymbol{\alpha}) = \alpha_{t_i}(z^\star_{t_i}+1) + \bar{\alpha}_{t_i}(z^\star_{t_i-1}+1) + \bar{\alpha}_{t_i+1}\left(\frac{1}{z^\star_{t_i+1}}+1\right) + \alpha_{t_i+1}\left(\frac{1}{z^\star_{t_i+2}}+1\right)$$
$$\overset{(b)}{\le} (\alpha_{t_i}+\bar{\alpha}_{t_i})(\beta_i+1) + (\alpha_{t_i+1}+\bar{\alpha}_{t_i+1})\left(\frac{1}{\beta_{i+1}}+1\right) = 2+\beta_i+\frac{1}{\beta_{i+1}} \triangleq G_i(\boldsymbol{\beta}). \quad (25)$$

Therefore $(\hat{\boldsymbol{\beta}}, \Phi^\star)$ is an optimum solution of the optimization problem, and we have $\hat{\boldsymbol{\beta}} \in \boldsymbol{B}$. However, from (31) we have $q(\hat{\boldsymbol{\beta}}) \le q-1 = q(\boldsymbol{\beta}^\star)-1$, which is in contradiction with the definition of $q = q(\boldsymbol{\beta}^\star)$ and $\boldsymbol{\beta}^\star$. Similarly, we can show that if $\beta_1^\star > 0$ then $G_0(\boldsymbol{\beta}^\star) = \Phi^\star$, and if $\beta_m^\star < \infty$ then $G_m(\boldsymbol{\beta}^\star) = \Phi^\star$. This concludes the proof of Lemma 3. □

We now analyze the structure of $\text{OPT}_5(m)$. In particular, for a given m, we will find the optimum $\boldsymbol{\beta}^\star$ that satisfies Lemma 3. We distinguish the following two cases.

(I) If $\beta_1^\star > 0$, then we define

$$b_0 = 1, \qquad b_i = \frac{1}{\prod_{k=1}^{i}\beta_k^\star}, \ \forall i \in [1:m]. \quad (32)$$

(II) If $\beta_1^\star = 0$, then we define

$$b_0 = 0, \qquad b_1 = 1, \qquad b_i = \frac{1}{\prod_{k=2}^{i}\beta_k^\star}, \quad (33)$$

where $i \in [2:m]$ in (33). Under both cases we have

$$\beta_i^\star = \frac{b_{i-1}}{b_i}, \qquad \forall i \in [1:m].$$

Using the change of variables above and $G_i(\boldsymbol{\beta}^\star) = \text{OPT}_5(m), i \in [1:m-1]$ (see Lemma 3), we get that

$$G_i(\boldsymbol{\beta}^\star) = 2 + \beta_i^\star + \frac{1}{\beta_{i+1}^\star} = 2 + \frac{b_{i-1}}{b_i} + \frac{b_{i+1}}{b_i},$$

where $i \in [1:m-1]$. Then, for a given n (number of relays in the network) and m (number of relays with distinct channel gains in the network), we define

$$\sigma_{n,m} \triangleq \text{OPT}_5(m) - 2 = \frac{b_{i-1}}{b_i} + \frac{b_{i+1}}{b_i}, \quad (34)$$

where $i \in [1:m-1]$, which implies

$$b_{i+1} - \sigma_{n,m} b_i + b_{i-1} = 0, \quad \forall i \in [1:m-1]. \quad (35)$$

The expression in (35) is a linear homogeneous recurrence relation of order 2 and its solution can be written as [31]

$$b_i = uU^i + vV^i, \qquad i \in [0:m], \quad (36)$$

where U and V are the roots[4] of the characteristic equation of the recurrence relation in (35), that is,

$$X^2 - \sigma_{n,m}X + 1 = 0. \quad (37)$$

[4]The solution in (36) holds only if the characteristic equation in (37) has simple (non-repeated) roots. Note that if $\sigma_{n,m} = 2$ then $U = V = 1$ and hence, the solution of the recurrence relation would be $b_i = u + vi$. This is, however, a monotonic function of i, and cannot satisfy both the initial and final conditions of the recurrence relation.

Moreover, u and v in (36) can be found from the initial conditions of the recurrence relation. In particular, under case (I) and $\beta_1^\star > 0$ we have $b_0 = 1$ and $b_1 = \frac{1}{\beta_1^\star} = G_0(\boldsymbol{\beta}^\star) - 1 = \text{OPT}_5(m) - 1 = \sigma_{n,m} + 1$. Similarly, under case (II) and $\beta_1^\star = 0$ we have $b_0 = 0$ and $b_1 = 1$.

Once u and v are found, we can fully express b_i as a function of $\sigma_{n,m}$, for $i \in [0:m]$. Then, we can use the final condition for $\beta_m^\star$ to identify the value of $\sigma_{n,m}$. More precisely, if $\beta_m^\star = \infty$ then $b_m = 0$. Otherwise, if $\beta_m^\star < \infty$, from Lemma 3 we have $\sigma_{n,m} + 2 = \text{OPT}_5(m) = G_m(\boldsymbol{\beta}^\star) = 1 + \beta_m^\star$, which implies $1 + \sigma_{n,m} = \beta_m^\star = \frac{b_{m-1}}{b_m}$. The optimum value of $\sigma_{n,m}$ is given in the following proposition, the proof of which is in Appendix A.

Proposition 2. *The optimal value $\sigma_{n,m}$ defined in* (34) *is given by*

$$\sigma_{n,m} = \begin{cases} 2\cos\left(\frac{2\pi}{2m+2}\right) & \text{if } \beta_1^\star > 0 \text{ and } \beta_m^\star < \infty, \\ 2\cos\left(\frac{2\pi}{2m+1}\right) & \text{if } \beta_1^\star > 0 \text{ and } \beta_m^\star = \infty, \\ 2\cos\left(\frac{2\pi}{2m+1}\right) & \text{if } \beta_1^\star = 0 \text{ and } \beta_m^\star < \infty, \\ 2\cos\left(\frac{2\pi}{2m}\right) & \text{if } \beta_1^\star = 0 \text{ and } \beta_m^\star = \infty. \end{cases} \quad (38)$$

Optimizing Over m. Recall from (34) that $\text{OPT}_5(m) = \sigma_{n,m} + 2$. Therefore, Proposition 2 fully characterizes the optimum solution of the maximization problem in (30). The last step of the proof of the ratio guarantee in Theorem 1 consists of finding the optimal solution for the optimization problem in (29). Recall from (29) that

$$\text{OPT}_4 = \max_{m \in \left[1:\lfloor\frac{n+2}{2}\rfloor\right]} \text{OPT}_5(m) = 2 + \max_{m \in \left[1:\lfloor\frac{n+2}{2}\rfloor\right]} \sigma_{n,m}, \quad (39)$$

where $\sigma_{n,m}$ is given in (38). The following proposition provides the optimum m and hence, the optimum solution for the optimization problem in (29).

Proposition 3. *The optimal solution for the optimization problem in* (39) *is given by*

$$\text{OPT}_4 = 2 + 2\cos\left(\frac{2\pi}{n+2}\right).$$

Proof. To find the optimal solution OPT_4 for the optimization problem in (39), we need to compute the maximum value of $\sigma_{n,m}$ over m for the four different cases in Proposition 2. Note that all the four expressions in Proposition 2 are increasing functions of m. Hence, we only need to find the maximum value of m in each case. We can analyze the next four cases, separately.

1) $\beta_1^\star > 0$ and $\beta_m^\star < \infty$. For this case, from (23) we have $t_1 \geq 2$ and $t_i - t_{i-1} \geq 2$ for $i \in [2:m]$. Thus, since $t_m = n$, we get

$$n = t_m = \sum_{i=2}^{m}(t_i - t_{i-1}) + t_1 \geq 2(m-1) + 2 = 2m,$$

which implies $m \leq \frac{n}{2}$, and hence

$$\begin{aligned}\mathrm{OPT}_4 = 2 + \max_{m \leq \frac{n}{2}} \sigma_{n,m} &= 2 + \max_{m \leq \frac{n}{2}} 2\cos\left(\frac{2\pi}{2m+2}\right)\\ &= 2 + 2\cos\left(\frac{2\pi}{n+2}\right).\end{aligned}$$

2) $\beta_1^\star > 0$ and $\beta_m^\star = \infty$. For this case, from (23) we obtain $t_1 \geq 2$, $t_m - t_{m-1} \geq 1$ and $t_i - t_{i-1} \geq 2$ for $i \in [2:m-1]$. Therefore,

$$\begin{aligned}n = t_m &= (t_m - t_{m-1}) + \sum_{i=2}^{m-1}(t_i - t_{i-1}) + t_1\\ &\geq 1 + 2(m-2) + 2 = 2m-1,\end{aligned}$$

which implies $m \leq \frac{n+1}{2}$. Therefore,

$$\begin{aligned}\mathrm{OPT}_4 = 2 + \max_{m \leq \frac{n+1}{2}} \sigma_{n,m} &= 2 + \max_{m \leq \frac{n+1}{2}} 2\cos\left(\frac{2\pi}{2m+1}\right)\\ &= 2 + 2\cos\left(\frac{2\pi}{n+2}\right).\end{aligned}$$

3) $\beta_1^\star = 0$ and $\beta_m^\star < \infty$. For this case, from (23) we have $t_1 \geq 1$ and $t_i - t_{i-1} \geq 2$ for $i \in [2:m]$. Thus,

$$\begin{aligned}n = t_m &= (t_m - t_{m-1}) + \sum_{i=2}^{m-1}(t_i - t_{i-1}) + t_1\\ &\geq 2(m-1) + 1 = 2m-1,\end{aligned}$$

which implies $m \leq \frac{n+1}{2}$. Therefore, we obtain

$$\begin{aligned}\mathrm{OPT}_4 = 2 + \max_{m \leq \frac{n+1}{2}} \sigma_{n,m} &= 2 + \max_{m \leq \frac{n+1}{2}} 2\cos\left(\frac{2\pi}{2m+1}\right)\\ &= 2 + 2\cos\left(\frac{2\pi}{n+2}\right).\end{aligned}$$

4) $\beta_1^\star = 0$ and $\beta_m^\star = \infty$. Finally, for this case, from (23) we can write $t_1 \geq 1$, $t_m - t_{m-1} \geq 1$ and $t_i - t_{i-1} \geq 2$ for $i \in [2:m-1]$. Hence,

$$\begin{aligned}n = t_m &= (t_m - t_{m-1}) + \sum_{i=2}^{m-1}(t_i - t_{i-1}) + t_1\\ &\geq 1 + 2(m-2) + 1 = 2m-2,\end{aligned}$$

which implies $m \leq \frac{n+2}{2}$. Therefore, we obtain

$$\begin{aligned}\mathrm{OPT}_4 = 2 + \max_{m \leq \frac{n+2}{2}} \sigma_{n,m} &= 2 + \max_{m \leq \frac{n+2}{2}} 2\cos\left(\frac{2\pi}{2m}\right)\\ &= 2 + 2\cos\left(\frac{2\pi}{n+2}\right).\end{aligned}$$

Therefore, for all four cases we obtain $\mathrm{OPT}_4 = 2 + 2\cos\left(\frac{2\pi}{n+2}\right)$, which proves our claim in Proposition 3. This concludes the proof of Proposition 3. □

In summary, by collecting all the results above together, we have proved that for any Gaussian half-duplex diamond n-relay network $\mathcal{N}$, the sequence of simplifications in (15) is always satisfied. This proves the inequality in (14), and concludes the proof of (5) in Theorem 1.

V. The Worst Networks: Proof of the Tightness of Theorem 1

We here prove that the bound in (5) is tight, that is, for any number of relays, there exists some networks for which $\mathsf{C}(\mathcal{N}_1)/\mathsf{C}_n(\mathcal{N}) = 1/(2 + 2\cos(2\pi/(n+2)))$. Towards this end, for every integer n we provide some constructions of half-duplex diamond n-relay networks for which the best relay has an approximate capacity that satisfies the bound in (5) with equality.

Our constructions are inspired by the discussion and results in Section IV-C. More precisely, we need to satisfy all the bounds in (15) with equality.

Case A.1: Let $n = 2k$ be an even integer, and consider a half-duplex diamond n-relay network $\mathcal{N}$ with $\theta = 2\pi/(n+2)$ and for $i \in [1:k]$ let

$$\begin{aligned}\ell_{2i} = \ell_{2i-1} &= \frac{2\sin(\theta)\sin(i\theta)}{\cos(i\theta) - \cos((i+1)\theta)},\\ r_{2i} = r_{2i-1} &= \frac{2\sin(\theta)\sin(i\theta)}{\cos((i-1)\theta) - \cos(i\theta)}.\end{aligned} \tag{40}$$

It is not difficult to see that, for the network in (40), we have $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_n$, $r_1 \geq r_2 \geq \ldots \geq r_n$. Moreover, for every relay $t \in [1:n]$ with $i = \lfloor \frac{t+1}{2} \rfloor$, we have

$$\begin{aligned}\mathsf{C}_1(\mathcal{N}_t) &= \frac{\ell_t r_t}{\ell_t + r_t} = \left(\frac{1}{\ell_t} + \frac{1}{r_t}\right)^{-1}\\ &= \frac{2\sin(\theta)\sin(i\theta)}{\cos((i-1)\theta) - \cos((i+1)\theta)} = 1\\ &\implies \mathsf{C}_1(\mathcal{N}) = 1,\end{aligned} \tag{41}$$

i.e., the best relay in $\mathcal{N}$ has an approximate capacity of 1. Finally, $\forall t \in [0:n-1]$, with $i = \lfloor \frac{t+1}{2} \rfloor$

$$\begin{aligned}&\ell_t + r_{t+2}\\ &= \frac{2\sin(\theta)\sin(i\theta)}{\cos(i\theta) - \cos((i+1)\theta)} + \frac{2\sin(\theta)\sin((i+1)\theta)}{\cos(i\theta) - \cos((i+1)\theta)}\\ &= 2\sin(\theta)\frac{2\sin\left(\frac{(2i+1)\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{(2i+1)\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}\\ &= 4\cos^2\left(\frac{\theta}{2}\right) = 2\cos(\theta) + 2,\end{aligned} \tag{42}$$

where we let $\ell_0 = r_{n+1} = 0$.

Consider now a two-state schedule given by

$$\lambda_{\mathcal{S}} = \begin{cases} \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_o = \{1,3,5,\ldots,2k-1\},\\ \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_e = \{2,4,6,\ldots,2k\},\\ 0 & \text{otherwise.}\end{cases}$$

The rate $\mathsf{R}_n(\mathcal{N})$ achieved by this two-state schedule can be found from (2), and satisfies the set of (in)equalities in (43), at the top of the next page, where in (a) we set $t = \max \mathcal{S}_e \cap \Omega^c$ and $s = \max \mathcal{S}_o \cap \Omega^c$, and (b) is due to the

$$
\begin{aligned}
\mathsf{R}_n(\mathcal{N}) &= \min_{\Omega\subseteq[1:n]} \sum_{\mathcal{S}\subseteq[1:n]} \lambda_{\mathcal{S}} \left(\max_{i\in\mathcal{S}^c\cap\Omega^c} \ell_i + \max_{i\in\mathcal{S}\cap\Omega} r_i \right) \\
&= \min_{\Omega\subseteq[1:n]} \left\{ \frac{1}{2} \left(\max_{i\in\mathcal{S}_e\cap\Omega^c} \ell_i + \max_{i\in\mathcal{S}_o\cap\Omega} r_i \right) + \frac{1}{2} \left(\max_{i\in\mathcal{S}_o\cap\Omega^c} \ell_i + \max_{i\in\mathcal{S}_e\cap\Omega} r_i \right) \right\} \\
&\overset{(a)}{=} \min_{\Omega\subseteq[1:n]} \left\{ \frac{1}{2} \left(\ell_t + \max_{i\in\mathcal{S}_e\cap\Omega} r_i \right) + \frac{1}{2} \left(\ell_s + \max_{i\in\mathcal{S}_o\cap\Omega} r_i \right) \right\} \\
&\overset{(b)}{\geq} \min_{\Omega\subseteq[1:n]} \left\{ \frac{1}{2} \left(\ell_t + r_{t+2} \right) + \frac{1}{2} \left(\ell_s + r_{s+2} \right) \right\} \\
&\overset{(c)}{=} \min_{\Omega\subseteq[1:n]} \left\{ \frac{1}{2} \left(2\cos(\theta) + 2 \right) + \frac{1}{2} \left(2\cos(\theta) + 2 \right) \right\} = 2\cos(\theta) + 2. \qquad (43)
\end{aligned}
$$

fact that if $t = \max \mathcal{S}_e \cap \Omega^c$ then $t+2$ is an even number that belongs to Ω, and similarly $s+2 \in \mathcal{S}_o \cap \Omega$. Finally, in (c) we used the equality derived in (42). Therefore, the rate of $2\cos(\theta)+2$ is achievable for this network. Moreover, note that the approximate capacity $\mathsf{C}_n(\mathcal{N})$ of a Gaussian half-duplex diamond n-relay network is always upper bounded by that of the same network when operated in full-duplex mode (i.e., each relay can transmit and receive simultaneously). Also, note that, for the network in (40), we have that $r_1 = \max_{i\in[1:n]} r_i$. Hence, we have

$$
\mathsf{C}_n(\mathcal{N}) \leq \mathsf{C}_n^{\mathrm{FD}}(\mathcal{N}) \leq r_1 = \frac{2\sin^2(\theta)}{1-\cos(\theta)} = 2\cos(\theta)+2. \qquad (44)
$$

Finally, (43) and (44) imply $\mathsf{C}_n(\mathcal{N}) = 2\cos(\theta)+2$. This together with (41) leads to

$$
\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})} = \frac{1}{2\cos(\theta)+2} = \frac{1}{2\cos\left(\frac{2\pi}{n+2}\right)+2} \qquad (45)
$$

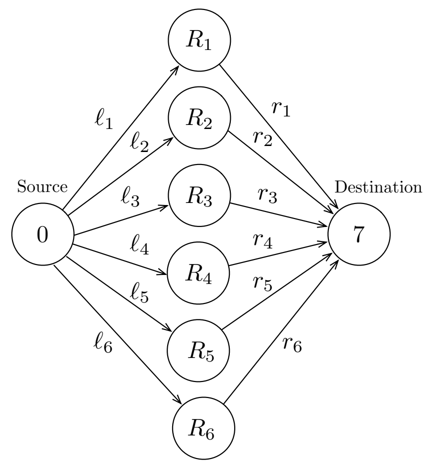
for the network in (40) and hence, proves the tightness of the bound in (5) when n is even. Note that this network corresponds to Case I of the analysis in Appendix A, where $\beta_1^\star > 0$ and $\beta_m^\star < \infty$. An example of the network construction in (40) for $n=6$ is provided in Fig. 5b.

Case A.2: There is also another network for even values of $n=2k$ that achieves the bound in (5). This network is given by

$$
\begin{aligned}
\ell_1 &= r_n = 1, \qquad r_1 = \ell_n = L \to \infty, \\
\ell_{2i} &= \ell_{2i+1} = \frac{\sin(i\theta) + \sin((i+1)\theta)}{\sin((i+1)\theta)}, \qquad (46) \\
r_{2i} &= r_{2i+1} = \frac{\sin(i\theta) + \sin((i+1)\theta)}{\sin(i\theta)},
\end{aligned}
$$

where $\theta = 2\pi/(n+2)$ and $i \in [1:k-1]$. It is easy to check that for this network we also have $\mathsf{C}_1(\mathcal{N}) = 1$ and $\mathsf{C}_n(\mathcal{N}) = 2\cos(\theta)+2$, which can be achieved using the two-state schedule

$$
\lambda_{\mathcal{S}} = \begin{cases} \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_o = \{3,5,\ldots,2k-1,2k\}, \\ \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_e = \{2,4,6,\ldots,2k\}, \\ 0 & \text{otherwise.} \end{cases}
$$



(a)

i	ℓ_i	r_i
1	$\sqrt{2}$	$2+\sqrt{2}$
2	$\sqrt{2}$	$2+\sqrt{2}$
3	2	2
4	2	2
5	$2+\sqrt{2}$	$\sqrt{2}$
6	$2+\sqrt{2}$	$\sqrt{2}$

(b)

i	ℓ_i	r_i
1	1	$L \to \infty$
2	$\frac{2+\sqrt{2}}{2}$	$1+\sqrt{2}$
3	$\frac{2+\sqrt{2}}{2}$	$1+\sqrt{2}$
4	$1+\sqrt{2}$	$\frac{2+\sqrt{2}}{2}$
5	$1+\sqrt{2}$	$\frac{2+\sqrt{2}}{2}$
6	$L \to \infty$	1

(c)

Fig. 5: Gaussian half-duplex diamond networks with $n=6$ relays for which the bound in (5) is tight. The table in (b) shows the link capacities for the network defined in (40) and the table in (c) indicates the link capacities of the network given in (46).

Note that in this schedule relay R_1 is (asymptotically) always in receive mode and relay R_n is always in transmit mode. This leads to

$$
\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})} = \frac{1}{\cos\left(\frac{2\pi}{n+2}\right)+2}.
$$

This network corresponds to Case IV of the network analysis in Appendix A, where $\beta_1^\star = 0$ and $\beta_m^\star = \infty$. The realization of this network configuration for $n=6$ is provided in Fig. 5c.

Remark 5. *We give an example to illustrate that the network in Fig. 5b has the ratio $\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})}$ smaller than or equal to any other $n=6$ relay network. Specifically, consider the same network as in Fig. 5b, but where the* *values of ℓ_3 and r_3 are slightly changed to $\ell_3 = 2.1$ and $r_3 = 1.9091$; note that we still ensure that $\frac{\ell_3 r_3}{\ell_3+r_3} = 1$. The approximate capacity of this new network is 3.3741 and for Fig. 5b is $3.4142 = \cos\left(\frac{2\pi}{6+2}\right) + 2$. Since, $\mathsf{C}_1(\mathcal{N}) = 1$ for both the networks, the ratio $\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})}$ is smaller for the network given by Fig. 5b compared to the slightly modified network.*

Case B.1: Let $n = 2k+1$ be an odd number. We consider a Gaussian half-duplex diamond n-relay network $\mathcal{N}$ with

$$\begin{aligned} &\ell_1 = 1, \qquad r_1 = L \to \infty, \\ &\ell_{2i} = \ell_{2i+1} = \frac{\sin(i\theta) + \sin((i+1)\theta)}{\sin((i+1)\theta)}, \quad i \in [1:k], \\ &r_{2i} = r_{2i+1} = \frac{\sin(i\theta) + \sin((i+1)\theta)}{\sin(i\theta)}, \quad i \in [1:k], \end{aligned} \tag{47}$$

where $\theta = 2/pi/(n+2)$ Similar to Case A.1, the network in (47) satisfies $\ell_1 \le \ell_2 \le \ldots \le \ell_n$ and $r_1 \ge r_2 \ge \ldots \ge r_n$. Also, the single relay approximate capacities satisfy

$$\mathsf{C}_1(\mathcal{N}_i) = \frac{\ell_i r_i}{\ell_i + r_i} = \left(\frac{1}{\ell_i} + \frac{1}{r_i}\right)^{-1} = 1, \tag{48}$$

for $i \in [1:n]$, which implies $\mathsf{C}_1(\mathcal{N}) = 1$, i.e., the best relay in $\mathcal{N}$ has unitary approximate capacity. Furthermore, for any $t \in [1:n]$ with $i = \lfloor t/2 \rfloor$, we have

$$\begin{aligned} &\ell_t + r_{t+2} \\ &= \frac{\sin(i\theta) + \sin((i+1)\theta)}{\sin((i+1)\theta)} + \frac{\sin((i+1)\theta) + \sin((i+2)\theta)}{\sin((i+1)\theta)} \\ &= \frac{2\sin((i+1)\theta) + 2\sin((i+1)\theta)\cos(\theta)}{\sin((i+1)\theta)} \\ &= 2\cos(\theta) + 2, \end{aligned}$$

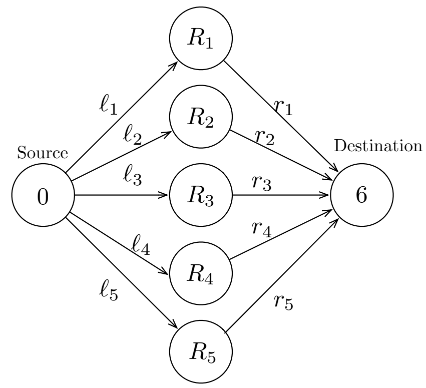
where we let $r_{n+1} = r_{n+2} = 0$. Therefore, similar to (43) we can show that $\mathsf{R}_n(\mathcal{N}) = 2\cos(\theta) + 2$ is achievable for this network, using the two-state schedule given by

$$\lambda_{\mathcal{S}} = \begin{cases} \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_o = \{3, 5, \ldots, 2k+1\}, \\ \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_e = \{2, 4, 6, \ldots, 2k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this schedule, relay R_1 is (asymptotically) always receiving, since its transmit capacity is unboundedly greater than its receive capacity. Moreover, similar to (44), we can argue that $\mathsf{C}_n(\mathcal{N}) \le \ell_n = 2\cos(\theta) + 2$. Therefore, we get

$$\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})} = \frac{1}{\cos\left(\frac{2\pi}{n+2}\right) + 2},$$

which proves the tightness of the bound in (5) when n is odd. Note that this network topology corresponds to Case III of the network analysis in Appendix A. An example of the network construction in (47) for $n = 5$ is provided in Fig. 6b.



(a)

i	ℓ_i	r_i
1	1	$L \to \infty$
2	1.8019	2.2470
3	1.8019	2.2470
4	3.2470	1.4450
5	3.2470	1.4450

(b)

i	ℓ_i	r_i
1	1.4450	3.2470
2	1.4450	3.2470
3	2.2470	1.8019
4	2.2470	1.8019
5	$L \to \infty$	1

(c)

Fig. 6: Gaussian half-duplex diamond networks with $n = 5$ relays for which the bound in (5) is tight. The table in (b) shows the link capacities of the network given in (47) and the table in (c) indicates the link capacities of the network in (49).

Case B.2: The second network configuration that satisfies the bound in (5) with equality for an odd number of relays, i.e., $n = 2k+1$, is given by

$$\begin{aligned} \ell_{2i-1} = \ell_{2i} &= \frac{2\sin(\theta)\sin(i\theta)}{\cos(i\theta) - \cos((i+1)\theta)}, \\ r_{2i-1} = r_{2i} &= \frac{2\sin(\theta)\sin(i\theta)}{\cos((i-1)\theta) - \cos(i\theta)}, \\ \ell_n = L \to \infty, &\qquad r_n = 1, \end{aligned} \tag{49}$$

where $\theta = 2\pi/(n+2)$ and $i \in [1:k]$. It is easy to see that this network also satisfies $\ell_1 \le \ell_2 \le \ldots \le \ell_n$ and $r_1 \ge r_2 \ge \ldots \ge r_n$. Moreover, the approximate single relay capacities equal one, and hence $\mathsf{C}_1(\mathcal{N}) = 1$. Furthermore, the approximate capacity of the entire network is $\mathsf{C}_n(\mathcal{N}) = 2\cos(\theta) + 2$, which can be achieved using the two-state schedule given by

$$\lambda_{\mathcal{S}} = \begin{cases} \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_o = \{1, 3, 5, \ldots, 2k+1\}, \\ \frac{1}{2} & \text{if } \mathcal{S} = \mathcal{S}_e = \{2, 4, 6, \ldots, 2k, 2k+1\}, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., the relay node R_n always transmits. This leads to

$$\frac{\mathsf{C}_1(\mathcal{N})}{\mathsf{C}_n(\mathcal{N})} = \frac{1}{\cos\left(\frac{2\pi}{n+2}\right) + 2},$$

which shows that the network in (49) satisfies the bound in (5) with equality. Note that this network topology corresponds to Case II of the analysis in Appendix A. An example of such network for $n = 5$ relay nodes is shown in Fig. 6c. It is worth noting that the two network topologies introduced for an odd number of relays are indeed identical up to flipping of the left and right point-to-point link capacities, and relabeling of the relays.

VI. Conclusion

In this paper, we have shown that in any n-relay half-duplex Gaussian diamond wireless network, operating the best relay always guarantees that at least an $f = \frac{1}{2+2\cos\left(\frac{2\pi}{n+2}\right)}$ fraction of the entire network approximate capacity can be retained. We have also proved that this bound is tight, i.e., we have shown that there exist n-relay half-duplex Gaussian diamond networks for which the best relay's approximate capacity is exactly equal to an f fraction of the entire network approximate capacity.

Future directions would consist of extending the presented ratio guarantees to: (i) cases where $k > 1$ relays are selected for operation; (ii) networks where the source can communicate to the destination through a direct link; and (iii) networks with multi-antenna nodes. Specifically, the second and third research directions on networks with a direct link between the source and destination, and on multi-antenna nodes might require a novel proof technique from the one in this paper for the case of no direct link and single-antenna nodes. When the source can directly communicate with the destination, the expressions for the approximate capacity become more complicated to be handled analytically than in the case of a diamond topology. For instance, the single relay approximate capacity for relay $R_i, i \in [1:n]$ is given by

$$\mathsf{C}_1(\mathcal{N}_i) = \begin{cases} c_{ds} & \text{if } c_{ds} \geq \min\{\ell_i, r_i\}, \\ \frac{\ell_i r_i - c_{ds}^2}{\ell_i + r_i - 2c_{ds}} & \text{if } c_{ds} < \min\{\ell_i, r_i\}, \end{cases} \tag{50}$$

where c_{ds} is the point-to-point capacity of the link between the source and the destination. We note that the expression in (50) is more involved than the expression in (4) for the single relay approximate capacity in diamond networks and hence, obtaining a tight ratio guarantee in this setting might require a different proof technique. For the multi-antenna nodes, although the constant gap approximations in [15] and [16] extend to this case, it is not anymore possible to approximate the Shannon capacity in terms of the individual link capacities as we have done in Definition 1. In the multi-antenna case, in fact, the phase of the channels plays a critical role and might lead to ill-conditioned matrices inside the $\log\det(\cdot)$ expressions. To the best of our knowledge, finding techniques to properly deal with such $\log\det(\cdot)$ expressions in a way that leads to tight ratio guarantees in multi-antenna networks is an open problem.

Appendix A
Proof of Proposition 2

We consider the four possible cases, depending on the values of $\beta_1^\star$ and $\beta_m^\star$.

Case I: $\beta_1^\star > 0$ and $\beta_m^\star < \infty$. Since $\beta_1^\star > 0$, then from Lemma 3, we know that

$$1 + \frac{1}{\beta_1^\star} = G_0(\boldsymbol{\beta}^\star) = \text{OPT}_5(m) \stackrel{(34)}{=} \sigma_{n,m} + 2$$
$$\Rightarrow \frac{1}{\beta_1^\star} = \sigma_{n,m} + 1.$$

Moreover, using (32) inside (36), we obtain

$$\begin{cases} uU^0 + vV^0 = b_0 = 1, \\ uU^1 + vV^1 = b_1 = \frac{1}{\beta_1^\star} = \sigma_{n,m} + 1, \end{cases}$$
$$\Rightarrow \begin{cases} u = \frac{U-1}{\sigma_{n,m}-2}, \\ v = \frac{V-1}{\sigma_{n,m}-2}. \end{cases} \tag{51}$$

Then, since $\beta_m^\star < \infty$, Lemma 3 implies that

$$1 + \beta_m^\star = G_m(\boldsymbol{\beta}^\star) = \text{OPT}_5(m) = 2 + \sigma_{n,m},$$

or equivalently,

$$\sigma_{n,m} + 1 = \beta_m^\star = \frac{b_{m-1}}{b_m} = \frac{uU^{m-1} + vV^{m-1}}{uU^m + vV^m}.$$

Therefore, we have

$$\begin{aligned} 0 &= u\left(U^m(\sigma_{n,m}+1) - U^{m-1}\right) \\ &\quad + v\left(V^m(\sigma_{n,m}+1) - V^{m-1}\right) \\ &= uU^m(U+1) + vV^m(V+1), \end{aligned} \tag{52}$$

where the last equality follows since we have

$$\begin{aligned} U^m(\sigma_{n,m}+1) - U^{m-1} &= U^{m-1}(U\sigma_{n,m} + U - 1) \\ &\stackrel{(a)}{=} U^{m-1}(U^2 + U) \\ &= U^m(U+1), \end{aligned}$$

and (a) follows from the characteristic function in (37). Thus, since $UV = 1$, from (52) we obtain

$$\begin{aligned} U^{2m} = \left(\frac{U}{V}\right)^m &= -\frac{v}{u}\frac{V+1}{U+1} \stackrel{(b)}{=} -\frac{V-1}{U-1}\frac{V+1}{U+1} \\ &\stackrel{(c)}{=} \frac{1}{U^2} \Rightarrow U^{2m+2} = 1, \end{aligned}$$

where the equality in (b) follows by using the values in (51) for u and v, and the equality in (c) follows by substituting $V = 1/U$. Thus, we get $2m+2$ pairs of (U, V), enumerated by a parameter $k \in [0 : 2m+1]$, given by

$$U(k) = \exp\left(\frac{2k\pi i}{2m+2}\right), \quad V(k) = \exp\left(-\frac{2k\pi i}{2m+2}\right).$$

Therefore, we have

$$\begin{aligned} \sigma_{n,m}(k) = U(k) + V(k) &= \exp\left(\frac{2k\pi j}{2m+2}\right) + \exp\left(-\frac{2k\pi j}{2m+2}\right) \\ &= 2\cos\left(\frac{2k\pi}{2m+2}\right). \end{aligned}$$

Note that $\sigma_{n,m}$ above is a function of k. However, the choice of $k = 0$ leads to $U = V = 1$ and $\sigma_{n,m} = 2$ which is an invalid choice (see Footnote 4). Other than that, for every given m we have

$$\sigma_{n,m} = \max_{\substack{k \in [0:2m+1] \\ k \neq 0}} \sigma_{n,m}(k) = \sigma_{n,m}(1) = 2\cos\left(\frac{2\pi}{2m+2}\right),$$

which proves Proposition 2 when $\beta_1^\star > 0$ and $\beta_m^\star < \infty$.

Case II: $\beta_1^\star > 0$ and $\beta_m^\star = \infty$. The initial condition of the recurrence relation are identical to that of Case I. Hence, we get $b_i = uU^i + vV^i$, where u and v are given in (51). Moreover, $\beta_m^\star = \infty$ implies $b_m = 0$. Substituting this in (36) for $i = m$ leads to

$$0 = b_m = uU^m + vV^m,$$

which implies

$$U^{2m} \stackrel{(a)}{=} \left(\frac{U}{V}\right)^m = -\frac{v}{u} \stackrel{(b)}{=} -\frac{V-1}{U-1} \stackrel{(a)}{=} \frac{1}{U}$$
$$\Rightarrow U^{2m+1} = 1,$$

where (a) is due to the fact that $V = 1/U$, and (b) follows from (51). Thus,

$$U(k) = \exp\left(\frac{2k\pi j}{2m+1}\right), \quad V(k) = \exp\left(-\frac{2k\pi j}{2m+1}\right),$$

and hence,

$$\begin{aligned}\sigma_{n,m}(k) &= U(k) + V(k)\\ &= \exp\left(\frac{2k\pi j}{2m+1}\right) + \exp\left(-\frac{2k\pi j}{2m+1}\right)\\ &= 2\cos\left(\frac{2k\pi}{2m+1}\right),\end{aligned}$$

for $k \in [0 : 2m]$. Maximizing $\sigma_{n,m}(k)$ we get

$$\sigma_{n,m} = \max_{\substack{k\in[0:2m]\\ k\neq 0}} \sigma_{n,m}(k) = \sigma_{n,m}(1) = 2\cos\left(\frac{2\pi}{2m+1}\right),$$

as claimed in Proposition 2.

Case III: $\beta_1^\star = 0$ and $\beta_m^\star < \infty$. When $\beta_1^\star = 0$, the initial conditions of the recurrence equation are given in (33). We have

$$\begin{cases} uU^0 + vV^0 = b_0 = 0,\\ uU^1 + vV^1 = b_1 = 1,\end{cases}$$
$$\Rightarrow \begin{cases} u = \frac{1}{\sqrt{\sigma_{n,m}^2 - 4}},\\ v = -\frac{1}{\sqrt{\sigma_{n,m}^2-4}}.\end{cases} \tag{53}$$

Moreover, Lemma 3 for $\beta_m^\star < \infty$ implies

$$\begin{aligned} 1 + \beta_m^\star &= G_m(\boldsymbol{\beta}^\star) = \mathrm{OPT}_5(m) = 2 + \sigma_{n,m}\\ \Rightarrow \quad 1 + \sigma_{n,m} &= \beta_m^\star = \frac{b_{m-1}}{b_m}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned} uU^{m-1} + vV^{m-1} &= b_{m-1} = (1+\sigma_{n,m})b_m\\ &= (1+\sigma_{n,m})(uU^m + vV^m)\end{aligned}$$

or equivalently,

$$\begin{aligned}&uU^{m-1}(U + \sigma_{n,m}U - 1) + vV^{m-1}(V + \sigma_{n,m}V - 1)\\ &\stackrel{(a)}{=} uU^{m-1}(U + U^2) + vV^{m-1}(V + V^2) = 0,\end{aligned}$$

where (a) follows since U and V are the roots of the characteristic function in (37). Thus,

$$U^{2m} = \left(\frac{U}{V}\right)^m = -\frac{v}{u}\frac{V+1}{U+1} \stackrel{(a)}{=} \frac{1}{U} \Rightarrow U^{2m+1} = 1,$$

where the equality in (a) follows from (53). Therefore, similar to Case II, we get

$$\sigma_{n,m} = 2\cos\left(\frac{2\pi}{2m+1}\right),$$

which proves our claim in Proposition 2.

Case IV: $\beta_1^\star = 0$ and $\beta_m^\star = \infty$. Since $\beta_1^\star = 0$, the initial conditions of this case are identical to those of Case III given in 53. However, from $\beta_m^\star = \infty$ we have $b_m = 0$, which implies

$$0 = b_m = uU^m + vV^m.$$

This leads to

$$U^{2m} = \left(\frac{U}{V}\right)^m = -\frac{v}{u} \stackrel{(a)}{=} 1 \Rightarrow U^{2m} = 1, \tag{54}$$

where the equality in (a) follows by using (53). Thus,

$$\begin{aligned} U(k) &= \exp\left(\frac{2k\pi j}{2m}\right), \quad V(k) = \exp\left(-\frac{2k\pi j}{2m}\right),\\ \sigma_{n,m}(k) &= U(k) + V(k) = 2\cos\left(\frac{2k\pi}{2m}\right),\end{aligned}$$

for some $k \in [0 : 2m-1]$. Maximizing $\sigma_{n,m}(k)$ over $k \neq 0$ we get

$$\sigma_{n,m} = \max_{\substack{k\in[0:2m-1]\\ k\neq 0}} \sigma_{n,m}(k) = \sigma_{n,m}(1) = 2\cos\left(\frac{2\pi}{2m}\right). \tag{55}$$

This proves our claim in Proposition 2, for the forth case when $\beta_1^\star = 0$ and $\beta_m^\star = \infty$.

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