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Mod *p* and torsion homology growth in nonpositive curvature

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Abstract We compute the mod p homology growth of residual sequences of finite index normal subgroups of right-angled Artin groups. We find examples where this differs from the rational homology growth, which implies the homology of subgroups in the sequence has lots of torsion. More precisely, the homology torsion grows exponentially in the index of the subgroup. For odd primes p, we construct closed locally CAT(0) manifolds with nonzero mod p homology growth outside the middle dimension. These examples show that Singer's conjecture on rational homology growth and Lück's conjecture on torsion homology growth are incompatible with each other, so at least one of them must be wrong.

This paper is about the growth of homology in regular coverings of finite aspherical complexes $X = B\Gamma$. We will content ourselves with the situation when the fundamental group $\Gamma = \pi_1 X$ is residually finite. This means there is a nested sequence of finite index normal subgroups $\Gamma_k \lhd \Gamma$ with $\cap_k \Gamma_k = 1$.

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We fix a choice of such a sequence and will be interested in the normalized limits of Betti numbers with coefficients in a field F

$$b_i^{(2)}(\Gamma; F) := \limsup_k \frac{b_i(B\Gamma_k; F)}{[\Gamma : \Gamma_k]},$$

where F is either $\mathbb{Q}$ or $\mathbb{F}_p$. When $F = \mathbb{Q}$ then Lück's approximation theorem [11] shows this does not depend on the choice of sequence and can be identified with a more analytically defined ith *L^2-Betti number* of the universal cover $E\Gamma$. When $F = \mathbb{F}_p$ we will analogously refer to $b_i^{(2)}(\Gamma; \mathbb{F}_p)$ as the *$\mathbb{F}_p$-L^2-Betti number*, even though it does not (as far as we know) have an analytic interpretation and it is not even known whether the lim sup depends on the choice of sequence (we abuse notation by omitting the sequence from $b_i^{(2)}(\Gamma; \mathbb{F}_p)$). Note that if the lim sup is independent of the sequence, then it becomes an honest limit.

For a finite aspherical complex $B\Gamma$ it is easy to see that the mod p L^2-Betti number is greater or equal to the ordinary L^2-Betti number

$$b_i^{(2)}(\Gamma; \mathbb{F}_p) \geq b_i^{(2)}(\Gamma; \mathbb{Q}),$$

but it might be strictly bigger. We show that this does—in fact—happen for some right-angled Artin groups. This seems to have not been observed previously and contradicts a conjecture of Lück [Conjecture 3.4, [13]] that these numbers are independent of the coefficient field. When Γ is a right-angled Artin group we compute $b_i^{(2)}(\Gamma; F)$ completely for any coefficient field, via a residually finite variant of the argument Davis and Leary [9] used to compute the ordinary L^2-Betti numbers of such groups.

Theorem 1 *Let A_L be a right-angled Artin group with defining flag complex L and F any field (e.g. $\mathbb{Q}$ or $\mathbb{F}_p$). Then*

$$b_i^{(2)}(A_L; F) = \bar{b}_{i-1}(L; F).$$

Here $\bar{b}_{i-1}(L; F)$ denotes the reduced Betti number of L with coefficients in F. In particular, the lim sup is actually a limit, it does not depend on the choice of chain but does depend on the characteristic of the coefficient field.

Corollary 2 *Suppose that L is a flag triangulation of $\mathbb{R}P^2$.*

$$b_3^{(2)}(A_L; \mathbb{Q}) = 0,$$
$$b_3^{(2)}(A_L; \mathbb{F}_2) = 1.$$

The proof of Theorem 1 also shows that F-L^2-Betti numbers of finite index subgroups of RAAG's are multiplicative (see Corollary 10), and we will use this later in Theorem 4. In general, it is not known whether the $\mathbb{F}_p$-L^2-Betti numbers are multiplicative.

By the universal coefficient theorem, $H_n(X, \mathbb{F}_p)$ is determined by $H_n(X, \mathbb{Q})$ and $\mathbb{Z}/p$-summands in $H_n(X, \mathbb{Z})$ and $H_{n-1}(X, \mathbb{Z})$. In this case, if A_L is as in Corollary 2, then since A_L has a 3-dimensional model for BA_L, $H_3(B\Gamma_k; \mathbb{Z})$ is torsion-free. Therefore, this discrepancy between $\mathbb{Q}$ and $\mathbb{F}_2$ homology leads to exponentially growing torsion in homology in degree 2.

Corollary 3 *The group A_L as in Corollary 2 has exponential H_2-torsion growth:*

$$\limsup_k \frac{\log |H_2(B\Gamma_k; \mathbb{Z})_{tors}|}{[A_L : \Gamma_k]} > 0.$$

Furthermore, the rank of the 2-torsion subgroup of $H_2(B\Gamma_k; \mathbb{Z})$ grows linearly in $[A_L : \Gamma_k]$.

While it is conjectured that for arithmetic hyperbolic 3-manifold groups the torsion in homology grows exponentially in residual chains of congruence covers, this is the first example of a finitely presented group of any sort where one can prove that homology torsion grows exponentially in a residual chain, answering a query of Bergeron for such a group. By contrast, Abert, Gelander, and Nikolov showed that if L is connected then H_1-torsion of A_L grows slower than exponentially [1].

For other groups Γ the computation of L^2-Betti numbers and homology torsion growth is a difficult problem. A basic vanishing principle which can make computations of L^2-Betti numbers simpler is the following conjecture often attributed to Singer.

Singer Conjecture Let M^n be a closed aspherical manifold. Then

$$b_i^{(2)}(\pi_1(M^n); \mathbb{Q}) = 0 \quad \text{for } i \neq \frac{n}{2}.$$

So in the residually finite setting, the free part of homology should grow sublinearly outside the middle dimension. A more recent vanishing principle regarding torsion growth, motivated by considerations in number theory, is the following conjecture made by Bergeron and Venkatesh in the context of arithmetic locally symmetric spaces [4] (see also [3]).

Bergeron–Venkatesh Conjecture Let G be a semisimple Lie group, Γ a cocompact arithmetic lattice in G, and Γ_k a sequence of congruence subgroups with $\cap_k \Gamma_k = 1$. Then

$$\limsup_k \frac{\log |H_i(B\Gamma_k; \mathbb{Z})_{tors}|}{[\Gamma : \Gamma_k]} = 0$$

unless $i = \frac{\dim(G/K)-1}{2}$.

Remark The conjecture is actually more precise and predicts that the limit is positive in some cases, e.g. when G is $SL(3, \mathbb{R})$, $SL(4, \mathbb{R})$ or $SO(m, n)$ for mn odd.

Partially motivated by this conjecture, in [12] Lück suggested such a vanishing principle could hold quite generally for arbitrary closed aspherical manifolds.

Lück Conjecture (1.12(2), [12]) Let M^n be a closed aspherical n-manifold with residually finite fundamental group. Let $\Gamma_k \lhd \pi_1(M^n)$ be any normal chain with $\bigcap_k \Gamma_k = 1$. If $i \neq (n-1)/2$ then

$$\limsup_k \frac{\log |H_i(B\Gamma_k; \mathbb{Z})_{tors}|}{[\pi_1(M^n) : \Gamma_k]} = 0.$$

It is interesting to note that the Singer and Lück Conjectures together imply an $\mathbb{F}_p$-version of the Singer conjecture.
$\mathbb{F}_p$–**Singer Conjecture** Let M^n be a closed aspherical n-manifold with residually finite fundamental group. Then

$$b_i^{(2)}(\pi_1(M^n); \mathbb{F}_p) = 0 \quad \text{for } i \neq \frac{n}{2}.$$

To see this, suppose we have an n-manifold M^n with $b_i^{(2)}(\pi_1(M^n); \mathbb{F}_p) \neq 0$ for $i \neq n/2$. By Poincaré duality, we can assume $i > n/2$. The Künneth formula implies that $M^n \times M^n \times M^n$ has nontrivial $\mathbb{F}_p$-L^2-Betti numbers in dimension $3i$. Since the Singer Conjecture predicts that $b_{3i}^{(2)}(\pi_1((M^n)^3); \mathbb{Q}) = 0$, the universal coefficient theorem implies exponential homological torsion growth in dimension $3i$ or $3i - 1$, which lies above the middle dimension, contradicting Lück's Conjecture.

The $\mathbb{F}_p$-Singer Conjecture is open even for $n = 3$ (but see [4,6]). But in high enough dimensions, we show this conjecture is not true for any odd prime p.

Theorem 4 *For any odd prime p, the $\mathbb{F}_p$-Singer Conjecture fails in all odd dimensions ≥ 7 and all even dimensions ≥ 14.*

Our examples are manifolds constructed via right-angled Coxeter groups; in particular they are locally CAT(0), so it follows that the rational homology and torsion homology growth conjectures are incompatible in the CAT(0) setting. On the other hand, our examples are not locally symmetric so even though the Singer conjecture is known for locally symmetric spaces the Bergeron–Venkatesh conjecture remains open.

Here is a brief outline of our construction. In [14], it was shown that if a finite type group Γ acts properly on a contractible n-manifold and $b_i^{(2)}(\Gamma;\mathbb{Q}) \neq 0$ for $i > \frac{n}{2}$, then there is a counterexample to the Singer Conjecture (in some dimension possibly different from n.) We employ a similar strategy here. Our group is a finite index subgroup of a right-angled Artin group with $b_4^{(2)}(A_L;\mathbb{F}_p) \neq 0$, and the 7-manifold is going to be the Davis complex corresponding to a right-angled Coxeter group associated to a flag triangulation of a S^6.

This uses Theorem 1 and the main result of [2]. More precisely, suppose $L = S^2 \cup_p D^3$ is a flag triangulation of a complex obtained by gluing a 3-disk to a 2-sphere along a degree p map. Theorem 1 shows that $b_4^{(2)}(A_L;\mathbb{F}_p) = 1$. Since $H_3(L;\mathbb{F}_2) = 0$, [2, Theorem 5.1] shows that a related flag complex OL (the link of a vertex in the Salvetti complex of A_L) embeds into a flag triangulation T of S^6. This is where we need $p \neq 2$; interestingly this goes back to the fact that van Kampen's obstruction to embedding d-dimensional simplicial complexes into $\mathbb{R}^{2d}$ is an order two invariant.

Now, A_L is commensurable to the right-angled Coxeter group W_{OL} by [8], and W_{OL} is a subgroup of W_T. This acts properly on the associated Davis complex, a contractible 7-manifold, so we obtain the desired proper action for a finite index subgroup of A_L.

We then show that the $\mathbb{F}_p$-Singer Conjecture fails for either the right-angled Coxeter group associated to T or to a link of an odd-dimensional simplex in T. In other words, there must be a right-angled Coxeter group counterexample in one of the dimensions 3, 5, or 7.

Taking cartesian products of counterexamples and surface groups, we get counterexamples in all the dimensions stated in the theorem. In this way, we also get a single closed aspherical manifold contradicting $\mathbb{F}_p$-Singer for a finite collection of primes. This suggests the following.

Question 5 *Given a closed aspherical manifold M^n, is there a number N so that for all primes $p > N$, the $\mathbb{F}_p$-Singer Conjecture holds for M^n?*

Of course, this is at least as difficult as the ordinary Singer conjecture, but it seems interesting (and open) in many cases where the ordinary Singer conjecture is known. Along the same lines, one can modify the Lück conjecture by ignoring the contributions to torsion coming from a finite collection of exceptional primes which should be determined by the geometry of the manifold M^n (akin to how the exceptional primes for right-angled Artin groups are determined by the complex L.) This seems particularly interesting for locally symmetric spaces and might be easier than the Bergeron–Venkatesh conjecture.

1 Right-angled Artin and Coxeter groups

We collect some facts about right-angled Artin groups (RAAG's), right-angled Coxeter groups (RACG's) and relations between one and the other which we will need later. The philosophy to keep in mind is that RAAGs are the things we can compute, RACGs are the things related to closed aspherical manifolds, and translating from the former to the later involves a bit of (classical) embedding theory.

Let L be a flag complex with vertex set V. The one-skeleton of L determines two group presentations. A presentation for the RAAG A_L has generators $\{g_v\}_{v\in V}$; there are relations $[g_v, g_{v'}] = 1$ (i.e., g_v and $g_{v'}$ commute) whenever $\{v, v'\} \in L^{(1)}$. The RACG W_L is the quotient of A_L formed by adding the relations $(g_v)^2 = 1$, for all $v \in V$.

We now describe a standard classifying space for a RAAG A_L. More precisely, let T^V denote the product $(S^1)^V$. Each copy of S^1 is given a cell structure with one vertex e_0 and one edge. For each simplex $\sigma \in L$, $T(\sigma)$ denotes the subset of T^V consisting of points $(x_v)_{v\in V}$ such that $x_v = e_0$ whenever v is not a vertex of σ. So, $T(\sigma)$ is a $(\dim\sigma + 1)$-dimensional standard subtorus of T^V. The *Salvetti complex* for A_L is the subcomplex BA_L of T^V defined as the union of the subtori $T(\sigma)$ over all simplices σ in L:

$$BA_L := \bigcup_{\sigma\subset L} T(\sigma).$$

The link of the unique vertex is a flag complex of the same dimension as L, and is usually denoted OL (and called the *octahedralization* of L.)

We now give a similar construction of a classifying space for the commutator subgroup C_L of W_L (which is torsion-free and finite index in W_L.) Let I^V denote the product $([-1, 1])^V$. Each copy of $[-1, 1]$ is given a cell structure with two vertices and one edge. For each simplex $\sigma \subset L$, $I(\sigma)$ denotes the subset of I^V consisting of those points $(x_v)_{v\in V}$ such that $x_v \in \{\pm 1\}$ whenever v is not a vertex of σ. So, $I(\sigma)$ is a disjoint union of parallel faces of I^V of dimension $\dim\sigma + 1$. The *standard classifying space* for C_L is the subcomplex BC_L of I^V defined as the union of the $I(\sigma)$ over all simplices σ in L:

$$BC_L := \bigcup_{\sigma\in L} I(\sigma).$$

The link of each vertex of BC_L is a copy of L. The universal cover of BC_L is denoted Σ_L and called the *Davis complex* of W_L. $(\mathbb{Z}/2)^V$ acts on I^V and preserves the subcomplex BC_L. The lifts of this induced action to Σ_L are precisely W_L and we have the exact sequence

$$1 \to C_L \to W_L \to (\mathbb{Z}/2)^V \to 1.$$

Lemma 6 *Let L be a flag complex.*

(1) *A_L is commensurable to W_{OL}* [8].
(2) *W_L is linear, and hence residually finite. Therefore, so is A_L.*
(3) *If L is a triangulation of S^{n-1}, then Σ_L is a contractible n-manifold* [7].

With Davis in [2], we studied the minimal dimension of aspherical manifolds with right-angled Artin fundamental groups. Constructing such manifolds involves embedding right-angled Artin groups A_L into manifold Coxeter groups $W_{S^{n-1}}$. This boils down to finding PL-embeddings of OL into spheres. The complexes OL have "join-like" properties which make them difficult to embed directly but one can compute when the van Kampen embedding obstruction vanishes for these complexes. It is a complete obstruction to PL-embedding d-complexes in S^{2d}, except when $d = 2$, and gives the following embedding criterion.

Theorem 7 (2.2, 5.1 and 5.4, [2]) *Suppose L is a d-dimensional flag complex, $d \neq 2$. Then OL embeds as a full subcomplex into a flag PL-triangulation of S^{2d} if and only if $H_d(L; \mathbb{F}_2) = 0$.*

The prime 2 plays a special role in this theorem because van Kampen's obstruction looks at what happens to pairs of distinct points under a generic map $OL \to S^{2d}$, which leads to an order two (co)-homological invariant. Therefore, if $H_d(L; \mathbb{F}_2) = 0$ we get an embedding $OL \hookrightarrow S^{2d}$ irrespective of the $\mathbb{F}_p$-homology of L for odd primes p. This observation is key to the proof of Theorem 4.

2 Proof of Theorem 1

Let L be a flag complex, $\Gamma = A_L$ the right-angled Artin group defined by this complex, $B\Gamma$ its Salvetti complex, and F any field. By Lemma 6, we can choose a chain $\Gamma_k \lhd \Gamma$ of normal, finite index subgroups with $\bigcap_k \Gamma_k = 1$.

Consider the cover of the Salvetti complex $B\Gamma$ by the standard maximal tori T_α. Its nerve is a simplex Δ since all the tori intersect at the base-point. For a simplex σ in Δ, we denote the intersection of the corresponding tori by $T_\sigma = \bigcap_{\alpha\in\sigma} T_\alpha$. We look at the finite cover $B\Gamma_k$ or equivalently, look at the coefficient module $V = F[\Gamma/\Gamma_k]$. The Mayer–Vietoris spectral sequence (see VII.4 [5]) corresponding to the cover $\{T_\alpha\}$ has E^1 term

$$E^1_{i,j} := C_i(\Delta; H_j(T_\sigma; V)) \implies H_{i+j}(B\Gamma; V) = H_{i+j}(B\Gamma_k; F).$$

The following lemma is crucial.

Lemma 8

$$\lim_{k\to\infty} \frac{\dim_F H_j(T_\sigma; F[\Gamma/\Gamma_k])}{[\Gamma : \Gamma_k]} = \begin{cases} 1 & \textit{if } T_\sigma = pt, \textit{ and } j = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof Since covers of tori are tori, the only way the homology of covers of T_σ can grow linearly is if the number of components of the preimage of T_σ in $B\Gamma_k$ grows linearly in the index. Since Γ_k is a residual sequence of normal covers, the number of components grows linearly if and only if T_σ is a point. In more detail, since the cover is normal, the number of components is the ratio of indices $\frac{[\Gamma:\Gamma_k]}{|\pi_1 T_\sigma:\pi_1 T_\sigma \cap \Gamma_k|}$, and since the sequence Γ_k is residual, the denominator grows with k as long as $\pi_1 T_\sigma$ is infinite. □

Therefore, up to an error whose dimension is sublinear in the index $[\Gamma : \Gamma_k]$, the spectral sequence is concentrated on the $E^1_{i,0}$ line. This implies

$$\limsup_k \frac{\dim_F E^2_{i,0}}{[\Gamma : \Gamma_k]} = \limsup_k \frac{b_i(B\Gamma_k; F)}{[\Gamma : \Gamma_k]}. \tag{1}$$

Next, we approximate the chain complex $E^1_{i,0}$ by something that we will be able to compute exactly. For this, set

$$V_\sigma := \begin{cases} F[\Gamma/\Gamma_k] & \text{if } T_\sigma = pt, \\ 0 & \text{otherwise.} \end{cases}$$

Then the projection $E^1_{i,0} \to C_i(\Delta; V_\sigma)$ is a chain map and its kernel has dimension that is sublinear in the index $[\Gamma : \Gamma_k]$. Therefore

$$\limsup_k \frac{\dim_F E^2_{i,0}}{[\Gamma : \Gamma_k]} = \limsup_k \frac{\dim_F H_i(\Delta; V_\sigma)}{[\Gamma : \Gamma_k]}. \tag{2}$$

The quantity in the limit on the right can be computed exactly, just in terms of the topology of L.

Lemma 9

$$\frac{\dim_F H_i(\Delta; V_\sigma)}{[\Gamma : \Gamma_k]} = \bar{b}_{i-1}(L; F).$$

Proof The complex Δ has a subcomplex $\mathcal{L} \subset \Delta$ whose simplices are those intersections of tori that consist of more than one point. In other words,

$$\mathcal{L} := \{\sigma \subset \Delta \mid V_\sigma = 0\}.$$

This complex $\mathcal{L}$ is precisely the nerve of the cover of L by maximal simplices, so it is homotopy equivalent to L. From the definition of $\mathcal{L}$ and V_σ we get the exact sequence of chain complexes

$$0 \to C_*(\mathcal{L}; F) \otimes V \to C_*(\Delta; F) \otimes V \to C_*(\Delta; V_\sigma) \to 0.$$

Since Δ is a simplex, this implies

$$H_i(\Delta; V_\sigma) \cong \bar{H}_{i-1}(L; F) \otimes V.$$

This finishes the proof since V is a $[\Gamma : \Gamma_k]$-dimensional F-vector space. □

The theorem follows from this lemma, together with (1) and (2). Note that we only used normality of the Γ_k in Lemma 8. The proof goes through for chains Γ_k where the number of lifts in $B\Gamma_k$ of standard n-tori in $B\Gamma$ for $n > 0$ grows sublinearly. For example, this occurs for normal chains in a finite index subgroup of Γ.

Corollary 10 *For a finite index subgroup H of a RAAG Γ we have*

$$b_i^{(2)}(H; F) = [\Gamma : H] b_i^{(2)}(\Gamma; F).$$

Proof Let $\Gamma_k \lhd H$ be a normal chain. The cover BH of $B\Gamma$ has $\leq [\Gamma : H]$ lifts of each torus. For each lifted torus $\tilde{T}_\sigma$ in BH, the number of lifts in $B\Gamma_k$ is given by

$$\frac{[H : \Gamma_k]}{|\pi_1 \tilde{T}_\sigma : \pi_1 \tilde{T}_\sigma \cap \Gamma_k|},$$

which is sublinear. Hence, $b_i^{(2)}(\Gamma; F) = \lim \frac{b_i(B\Gamma_k; F)}{[\Gamma:\Gamma_k]}$, which implies the multiplicativity formula. □

3 Mayer–Vietoris sequences for F-L^2-Betti numbers of Coxeter groups

Let L be a flag complex and W_L the corresponding RACG. Look at a decomposition $L = A \cup_C B$ where A, B and hence C are full subcomplexes of L. The Coxeter group W_L splits as an amalgamated product $W_L = W_A *_{W_C} W_B$, and our goal in this section is to describe relations between F-L^2-Betti numbers that arise from such splittings. Everywhere in this section coefficients are in an arbitrary field F and will be omitted to improve readability.

Let Γ_k be a chain of finite index torsion-free normal subgroups with $\bigcap_k \Gamma_k = 1$ (note that any residual normal chain in W_L is eventually torsion-free.) Let Σ_L be the associated Davis complex for W_L, and let $Y_L^k = \Sigma_L / \Gamma_k$.

Given any full subcomplex A of L, the RACG W_A is a subgroup of W_L, and the corresponding Davis complex Σ_A is naturally a subcomplex of Σ_L. The stabilizer of Σ_A in W_L is precisely W_A. The W_L-orbit of Σ_A in Σ_L is a disjoint union of copies of Σ_A. The intersections of Γ_k with W_A give a corresponding chain of finite index subgroups $W_A \cap \Gamma_k \lhd W_A$.

We let Y_A^k denote the image of this orbit in Y_L^k, so that Y_A^k is a disjoint union of $\frac{[W_L:\Gamma_k]}{[W_A:\Gamma_k \cap W_A]}$ copies of $\Sigma_A / W_A \cap \Gamma_k$. It follows that we can compute $b_i^{(2)}(W_A)$ (with respect to the chain $W_A \cap \Gamma_k$) using Y_A^k:

$$b_i^{(2)}(W_A) = \limsup_k \frac{b_i(Y_A^k)}{[W_L : \Gamma_k]}.$$

Suppose that $L = A \cup_C B$ where A, B and hence C are full subcomplexes of L. We then have a decomposition of spaces:

$$Y_L^k = Y_A^k \cup_{Y_C^k} Y_B^k,$$

and hence a Mayer–Vietoris sequence

$$\cdots \to H_i(Y_C^k) \to H_i(Y_A^k) \oplus H_i(Y_B^k) \to H_i(Y_L^k) \to \ldots$$

By the above discussion taking lim sup of dimensions of the homology groups in this sequence divided by $[W_L : \Gamma_k]$ gives F-L^2-Betti numbers of the corresponding Coxeter groups. Since lim sup is subadditive, it follows that having $b_i^{(2)} = 0$ for one of the terms gives the usual inequalities between the nearby terms.

The decomposition we will use is when $A = \mathrm{St}(v)$ is the star of a vertex v, $B = L - v$ is its complement and $C = \mathrm{Lk}(v)$ is the link of v. In this case, the Mayer–Vietoris sequence leads to the following inequalities.

Lemma 11 (1) $b_i^{(2)}(W_L) \leq b_i^{(2)}(W_{L-v}) \quad$ *if* $\quad b_{i-1}^{(2)}(W_{\mathrm{Lk}(v)}) = 0,$

(2) $b_i^{(2)}(W_L) \geq b_i^{(2)}(W_{L-v}) \quad$ *if* $\quad b_i^{(2)}(W_{\mathrm{Lk}(v)}) = 0.$

Proof Removing the vertex v from L gives a Mayer–Vietoris sequence

$$\cdots \to H_i(Y^k_{\mathrm{Lk}(v)}) \xrightarrow{i_{1*}\oplus i_{2*}} H_i(Y^k_{\mathrm{St}(v))}) \oplus H_i(Y^k_{L-v}) \to H_i(Y^k_L) \to H_{i-1}(Y^k_{\mathrm{Lk}(v)}) \to \dots$$

The map $i_1 : Y^k_{\mathrm{Lk}(v)} \to Y^k_{\mathrm{St}(v)}$ is an inclusion of the form $Y \times \{\pm 1\} \hookrightarrow Y \times [-1, 1]$, so i_{1*} maps $H_i(Y^k_{\mathrm{Lk}(v)})$ onto $H_i(Y^k_{\mathrm{St}(v)})$. The stated inequalities follow from this. □

Iteratively removing vertices leads to the following lemma. It lets us reduce dimension by passing from complexes to their links.

Lemma 12 (1) *If A is a flag complex with $b_i^{(2)}(W_A) \neq 0$, then there exists a vertex $v \in A$ and a full subcomplex B of* $\mathrm{Lk}_A(v)$ *with $b_{i-1}^{(2)}(W_B) \neq 0$.*

(2) *If L is a flag complex with $b_i^{(2)}(W_L) = 0$ and if A is a full subcomplex of L with $b_i^{(2)}(W_A) \neq 0$, then there exists a vertex $v \in L$ and a full subcomplex B of* $\mathrm{Lk}_L(v)$ *with $b_i^{(2)}(W_B) \neq 0$.*

Proof Assume that all the link terms have $b_{i-1}^{(2)} = 0$. Then removing vertices from A one at a time until we are left with a single vertex leads, by the first part of Lemma 11, to

$$b_i^{(2)}(W_A) \leq b_i^{(2)}(W_{A-v}) \leq \cdots \leq b_i^{(2)}(W_{pt}) = 0$$

which contradicts the assumption that $b_i^{(2)}(W_A) > 0$. This proves the first part.

Now, assume that all the link terms have $b_i^{(2)} = 0$. Then removing vertices from L one at a time until we are left with A leads, by the second part of Lemma 11, to

$$b_i^{(2)}(W_L) \geq b_i^{(2)}(W_{L-v}) \geq \cdots \geq b_i^{(2)}(W_A)$$

which contradicts the assumption $b_i^{(2)}(W_L) = 0 < b_i^{(2)}(W_A)$. This proves the second part. □

4 Proof of Theorem 4

We are now ready for the proof of Theorem 4. Fix an odd prime p. Let $L = S^2 \cup_p D^3$ be a flag triangulation of a complex obtained by gluing a 3-disk to a 2-sphere via a degree p map. Since $H_3(L; \mathbb{F}_2) = 0$, by Theorem 7 the octahedralization OL embeds as a full subcomplex of a flag PL-triangulation T of S^6.

Using commensurability we choose a common finite index subgroup N of A_L and W_{OL}, which is normal in W_{OL}. We fix a torsion-free normal residual chain in W_T which intersects W_{OL} inside N. These are abundant as there is an obvious retraction $r : W_T \to W_{OL}$ so we can intersect any residual chain with $r^{-1}(N)$.

Since Σ_T is a 7-manifold, the $\mathbb{F}_p$-Singer Conjecture predicts vanishing of $b_*^{(2)}(W_T; \mathbb{F}_p)$, and similarly, vanishing for the links of odd-dimensional simplices.

Proposition 13 *The $\mathbb{F}_p$-Singer Conjecture fails either for T, or for one of the links of* 1 *or* 3*-dimensional simplices.*

Proof Suppose the $\mathbb{F}_p$-Singer Conjecture holds for T, and in particular $b_4^{(2)}(W_T; \mathbb{F}_p) = 0$. Since the chain in W_{OL} is contained in N and N has finite index in A_L, Corollary 10 implies $b_4^{(2)}(W_{OL}; \mathbb{F}_p) \neq 0$. By the second part of Lemma 12 applied to OL and T, there is a full subcomplex B of $\mathrm{Lk}_T(v)$ with $b_4^{(2)}(W_B; \mathbb{F}_p) \neq 0$. Now we apply the first part of Lemma 12 to B, to get a full subcomplex C of $\mathrm{Lk}_B(u)$ with $b_3^{(2)}(W_C; \mathbb{F}_p) \neq 0$. Note that $\mathrm{Lk}_B(u)$, and therefore C, is a full subcomplex of $\mathrm{Lk}_{\mathrm{Lk}_T(v)}(u) = \mathrm{Lk}_T(uv) \approx S^4$.

If the $\mathbb{F}_p$-Singer Conjecture still holds for $\mathrm{Lk}_T(uv)$, we can repeat this argument to get get a full subcomplex D of $\mathrm{Lk}_T(\sigma^3) \approx S^2$ with $b_2^{(2)}(W_D; \mathbb{F}_p) \neq 0$. Now, the $\mathbb{F}_p$-Singer Conjecture must fail for $\mathrm{Lk}_T(\sigma^3)$, because repeating this argument once more produces a subcomplex of S^0 with $b_1^{(2)} \neq 0$, which is clearly impossible. $\square$

It follows that the $\mathbb{F}_p$-Singer Conjecture must fail in at least one of the dimensions 3, 5, or 7. Since a closed surface S_g with $g \geq 2$ has $b_1^{(2)}(\pi_1(S_g); \mathbb{F}_p) \neq 0$, taking cartesian products between surface groups and our counterexamples gives, via the Künneth formula, counterexamples in all odd dimensions ≥ 7 and all even dimensions ≥ 14.

Remark The reason why we used a 3-dimensional complex $S^2 \cup_p D^3$ instead of a 2-dimensional complex $S^1 \cup_p D^2$ above is twofold. First, for 2-dimensional complexes there are other obstructions to embedding in S^4 besides the classical van Kampen obstruction [10]. Second, in codimension 2 there is a problem of extending a given triangulation on the complex to a triangulation of S^4 (the embedding might be locally knotted.) If for a flag triangulation L of $S^1 \cup_p D^2$ one can exhibit its octahedralization OL as a subcomplex of S^4, then our method would yield a 5-dimensional counterexample to the $\mathbb{F}_p$-Singer Conjecture.

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