Nonplanar graphs in boundaries of CAT(0) groups

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Abstract. Croke and Kleiner constructed two homeomorphic locally CAT(0) complexes whose universal covers have visual boundaries that are not homeomorphic. We construct two homeomorphic locally CAT(0) complexes so that the visual boundary of one universal cover contains a nonplanar graph, while the visual boundary of the other does not. In contrast, we prove for any two locally CAT(0) metrics on the Croke–Kleiner complex, if a finite graph embeds in the visual boundary of one universal cover, then the graph embeds in the visual boundary of the other.

1. Introduction

If X is a Gromov hyperbolic space, there is a naturally defined boundary at infinity $\partial_\infty X$, and any quasi-isometry of X induces a self-homeomorphism of $\partial_\infty X$. Moreover, if G is a word-hyperbolic group, then any two boundaries of G are G-equivariantly homeomorphic. Croke and Kleiner [7] showed the same phenomena does not occur for CAT(0) groups; i.e. there is a group which acts geometrically (properly and cocompactly by isometries) on two CAT(0) complexes with non-homeomorphic visual boundaries. Later, Wilson [18] showed that the Croke–Kleiner examples admit uncountably many non-homeomorphic boundaries; see also [3, 8, 17].

One can still ask what properties of the visual boundary are well-defined invariants of a CAT(0) group. For example, the topological dimension of a CAT(0)-group is a quasi-isometry invariant [2, 9]. In a different direction, Guilbault and Mooney [10] proved that all boundaries of the Croke–Kleiner examples are G-equivariantly cell-like equivalent.

In this paper, we show that the existence of a nonplanar graph in a CAT(0) boundary is not a well-defined invariant for a CAT(0) group.

Theorem 1.1. *There exist two homeomorphic locally* CAT(0) *complexes X and X' so that the visual boundary of $\widetilde{X}$ contains a nonplanar graph and the visual boundary of $\widetilde{X}'$ does not.*

On the other hand, we prove that the homeomorphism type of the boundaries for the Croke–Kleiner examples cannot be detected by finite graphs. These complexes, denoted X_{CK}, are constructed by gluing two flat tori T_1 and T_2 onto a third flat torus T_0 along simple closed curves which generate $\pi_1(T_0)$.

Theorem 1.2. *Suppose X_1 and X_2 are locally* CAT(0) *complexes homeomorphic to X_{CK}. If Γ is a finite graph contained in $\partial_\infty \widetilde{X}_1$, then there is an embedding of the graph Γ into $\partial_\infty \widetilde{X}_2$.*

Our interest in (non)-planarity of the visual boundary is motivated by the following two questions, neither of which we can answer.

Question 1.3. *Suppose a group G acts geometrically on a* CAT(0) *space X so that $\partial_\infty X$ is planar. Does G have a finite-index subgroup which is a 3-manifold group?*

Question 1.4. *Can a group G act geometrically on two* CAT(0) *spaces X and X' so that $\partial_\infty X$ is planar and $\partial_\infty X'$ is nonplanar?*

Question 1.3 was asked by Haïssinsky for hyperbolic groups. A positive answer implies the Cannon Conjecture [4, Conjecture 11.34]. Haïssinsky proved that the answer to Question 1.3 is positive if G is hyperbolic and cubulated [11]. It is also necessary to ask for a finite-index subgroup, as there are torsion-free hyperbolic and CAT(0) groups with planar boundary which are not 3-manifold groups but have 3-manifold groups as finite-index subgroups [13, 12].

Regarding Question 1.4, Papasoglu and Swenson [16] showed that the boundary of a one-ended CAT(0) group has no global cut points. In this case, if the boundary is also locally connected, the existence of an embedded nonplanar graph in the boundary is equivalent to nonplanarity of the boundary by a theorem of Claytor [5]. Our groups are torsion-free, but the boundaries in Theorem 1.1 that we construct are not locally connected, and we prove in Theorem 4.1 the boundaries that do not contain nonplanar graphs are also nonplanar.

One relation between planarity of the boundary and 3-manifold groups comes from a paper of Bestvina, Kapovich, and Kleiner [1]. A very special case of their main theorem implies that if G acts geometrically on a CAT(0) space X and $\partial_\infty X$ contains a nonplanar graph, then G is not even quasi-isometric to a 3-manifold group. It follows that our examples are not quasi-isometric to 3-manifold groups.

This paper is organized as follows. In Section 2 we introduce and illustrate our main example; see Figure 1. Our example is formed by gluing tori to the torus boundary components of a 3-manifold that is the product of a surface with

boundary and a circle. In Section 3 we determine the local path components of the visual boundary; we prove these depend on the locally CAT(0) metric on the complex. In Section 4 we show for any locally CAT(0) metric on the complex, the visual boundary is non-planar. In Section 5 we show the existence of nonplanar graphs depends on the locally CAT(0) metric. In Section 6 we analyze the finite subgraphs of the Croke–Kleiner boundaries.
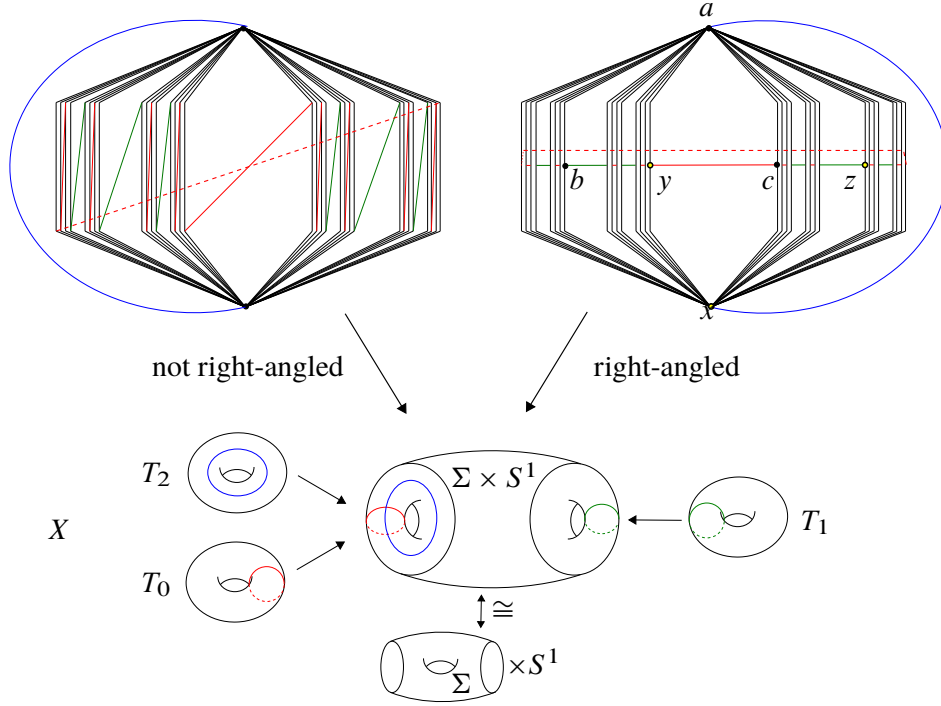
Figure 1. The main example X and a subset of the boundary of $\widetilde{X}$. The space X is *right-angled* if the tori T_0 and T_2 are glued to geodesics in a boundary torus of $\Sigma \times S^1$ that meet at a right angle. On the right X is right-angled, and on the left X is not. The suspension of the Cantor set in black is the boundary of a sub-block which covers $\Sigma \times S^1$. The red and green paths are subsets of the visual boundary $\widetilde{T}_i$. On the right these paths yield an embedded circle in $\partial_\infty X$, and on the left they do not. Using this embedded circle, it is simple to find a $K_{3,3}$ subgraph of the right-angled boundary; the vertex sets are $\{a, b, c\}$ and $\{x, y, z\}$. Most of the work in this paper is to show that there are no $K_{3,3}$ subgraphs of the non right-angled boundary.

2. Main example

Let Σ be a genus one surface with two boundary components. Let α_0 and α_1 denote the boundary curves of Σ. Form the product $\Sigma \times S^1$. Let $b_i \in \Sigma$ be a point on the boundary curve α_i for $i \in \{0, 1\}$, and let $\beta_i = b_i \times S^1$.

Form the space X by gluing three tori, T_0, T_1, and T_2, onto the curves α_0, α_1, and β_0, respectively. Note that after gluing on any two of these tori, the resulting space is homotopy equivalent to a 3-manifold.

Notation. Let $M = (\Sigma \times S^1) \cup_{\beta_0} T_2$, let $W_i = \alpha_i \times \beta_i$ for $i = 0, 1$, and let $Y_i = W_i \cup_{\alpha_i} T_i$. Thus $\partial(\Sigma \times S^1) = W_0 \cup W_1$.

Lemma 2.1. *We collect a few easy facts about $\pi_1(X)$.*

- *The subgroups $\pi_1(M)$ and $\pi_1(Y_i)$ are isomorphic to the direct product of a free group with $\mathbb{Z}$.*
- *$\pi_1(X)$ has cohomological dimension 2, and hence $\partial_\infty X$ is 1-dimensional by [2].*
- *$\pi_1(X)$ splits as an amalgamated product over each of the $\pi_1(\alpha_i)$.*
- *The subgroup $\pi_1(Y_0 \cup_{\beta_0} T_2)$ is isomorphic to the Croke–Kleiner group.*

We define the *angle* between α_0 and β_0 to be the minimal distance in the Tits metric between the endpoints of their lifts in $\partial_\infty \widetilde{W}_0$.

Theorem 2.2. *Given any $\delta \in (0, \frac{\pi}{2}]$, there is a locally CAT(0) metric on $\Sigma \times S^1$ so that the angle between α_0 and β_0 is δ.*

Proof. Choose generators a, b, c for $\pi_1(\Sigma)$ so that c represents α_0, and let d be the generator of $\pi_1(S^1)$. Assume that Σ has a hyperbolic metric; so, there is an discrete and faithful representation $\psi : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^2)$.

Let T be a nontrivial homomorphism from $\mathbb{Z} \to \mathbb{R}$, and skew the standard product action of $\pi_1(\Sigma) \times \mathbb{Z}$ on $\mathbb{H}^2 \times \mathbb{R}$ by

$$
\begin{aligned}
a &: (x, y) \longmapsto (\psi(a)x, y), \\
b &: (x, y) \longmapsto (\psi(b)x, y), \\
c &: (x, y) \longmapsto (\psi(c)x, T(c) + y), \\
d &: (x, y) \longmapsto (x, y + 1).
\end{aligned}
$$

Since we can choose any homomorphism T, the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by this skewed action defines a locally CAT(0) metric with a desired angle. $\square$

If the angle between β and α_0 is $\pi/2$ (so that $T = 0$), then we say that X is *right-angled*. We will always choose simple closed curves on the T_i and flat metrics so that the T_i are glued onto $\Sigma \times S^1$ by isometries. The resulting complex is then locally CAT(0).

Remark. We were originally motivated by an example of Kapovich and Kleiner [13, Section 9]. Roughly speaking, their example is obtained by gluing together two graph manifolds with boundary along simple closed curves in the interior. They use coarse Alexander duality to show that their examples are not virtually 3-manifold groups. Their examples also have visual boundaries which are nonplanar, and the existence of nonplanar graphs depends on how the Seifert-fibered pieces are glued together.

Having two boundary components in our example is a little artificial; there are similar constructions formed by starting with the product of a circle and a genus one surface with one boundary component, and then gluing on two tori along the boundary torus. The difference here is that one cannot skew the metric in the same way; any locally CAT(0) metric on this space restricts to the right-angled flat metric on the boundary torus. Therefore, to obtain different boundaries one has to vary the fundamental groups of these examples by gluing tori onto different simple closed curves on the boundary tori. One can show using work of Kapovich–Leeb [14, Proposition 2.2] that all these groups are quasi-isometric. We wanted an example where this phenomena occurs for the same group, hence used two boundary components.

2.1. Structure of the universal cover of X. Let $\pi \colon \widetilde{X} \to X$ denote the universal covering. A *block* is a connected component of $\pi^{-1}(M)$ or $\pi^{-1}(Y_i)$. Each block is convex in $\widetilde{X}$. A *sub-block* is a connected component of $\pi^{-1}(\Sigma \times S^1)$. A *wall* is a connected component of $\pi^{-1}(W_i)$. Note that each wall is contained in exactly two blocks, and two blocks either intersect in a wall or are disjoint. In the first case we say that the blocks are *adjacent*.

Each block and sub-block is quasi-isometric to the product of a tree with $\mathbb{R}$, so the boundary of each block and sub-block is homeomorphic to the suspension of a Cantor set. There is a natural homeomorphism from the boundary of $\widetilde{\Sigma}$ to the Cantor set that maps the endpoints of copies of the universal cover of the curves α_i inside $\widetilde{\Sigma}$ to endpoints of removed intervals. A *pole* of a block boundary is one of the suspension points. A *longitude* is an embedded arc in the block boundary connecting the two poles. If the longitude of a sub-block covering $\Sigma \times S^1$ contains a pole of an adjacent block in its interior, we say it is a *boundary longitude*, and if a pair of such longitudes contains the two poles of an adjacent block, we say the longitudes form a *boundary pair*. The boundary pairs are precisely the pairs of longitudes which contain the endpoints of lifts of the curves α_i.

We will also need the notion of an itinerary for a geodesic ray in X. Choose a basepoint $p \in \widetilde{X}$ not contained in any wall. For a point ψ in $\partial_\infty \widetilde{X}$, the *p-itinerary of ψ*, denoted $\mathrm{Itin}_p(\psi)$, is the ordered sequence of blocks $\{B_i\}_{i=1}^\infty$, where the geodesic ray $p\psi$ intersects B_i in a point not contained in a wall. A boundary point has finite itinerary (for any basepoint p) if and only if it is contained in the boundary of some block.

3. Paths in the boundary

In this section, we study the local path components of points in $\partial_\infty \widetilde{X}$ and show that they depend on whether X is right-angled. In the next sections, we will show that this changes what possible finite graphs embed into $\partial_\infty \widetilde{X}$. The main difference between the cases comes from studying paths between points on different longitudes of $\partial_\infty SB \subset \partial_\infty \widetilde{X}$, where SB is a sub-block covering $\Sigma \times S^1$. If X is right-angled, it turns out that there is a path between points on any two longitudes that misses the two poles of $\partial_\infty SB$ (see Lemma 3.1). If not, these paths only exist between points on boundary pairs. See Figure 1.

3.1. Horizontal paths in $\partial_\infty \widetilde{X}$

Lemma 3.1. *Suppose that X is right-angled, and SB is a sub-block of $\widetilde{X}$. Then there is a circle inside $\partial_\infty \widetilde{X}$ containing $\partial_\infty \widetilde{\Sigma} \subset \partial_\infty SB$ that avoids the poles $\{\psi_+, \psi_-\}$ of $\partial_\infty SB$. In particular, if x and y are two points in $\partial_\infty SB$, then x and y are in the same path component of $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$.*

Proof. Choose a basepoint p in the interior of $\widetilde{\Sigma}$, for some lift $\widetilde{\Sigma}$ of Σ in SB, and choose a circle in the tangent space $T_p(\widetilde{\Sigma}) \subset T_p(SB)$. We will define an embedding from this circle to $\partial_\infty \widetilde{X}$. For a vector $v \in T_p(\widetilde{\Sigma})$, if the horizontal geodesic ρ in the direction of v stays in SB for all time, then map v to the endpoint of that geodesic. Otherwise, suppose that ρ hits a wall W between SB and a block Y in a geodesic line ℓ, where $\pi(Y) = Y_i$ for $i \in \{0, 1\}$. The line ℓ is a lift of a closed curve in the torus T_i, and hence ℓ bounds two half-flats meeting W in ℓ. Given ℓ, choose one of the two half-flats; this choice determines a way to extend ρ to a geodesic ray ρ' contained in Y. Map the vector v to the endpoint of ρ'. □

We call such a circle in $\partial_\infty \widetilde{X}$ a *horizontal circle* for the sub-block boundary $\partial_\infty SB$. Note that a path connecting two points on $\partial_\infty SB$ which lies on this horizontal circle will have infinite Tits length unless x and y are on the same longitude, or are on longitudes that form a boundary pair.

3.2. Local path components in $\partial_\infty \widetilde{X}$ if X is not right-angled

Theorem 3.2. *Suppose X is not right-angled. Suppose that ψ is a point on a block boundary $\partial_\infty B$ that is not a pole of any other block. Then, there is a neighborhood Ω of ψ in $\partial_\infty \widetilde{X}$ so that the path component of ψ in Ω is contained in the block boundary $\partial_\infty B$. Furthermore, this path component is contained in the set of longitudes of $\partial_\infty B$ that ψ lies on.*

Proof. Our argument is the same as in [7, Lemma 4], though we have to additionally argue that the horizontal paths constructed in Lemma 3.1 cannot exist when X is not right-angled.

Case 1: ψ is a pole of $\partial_\infty B$ where B covers M. Choose a basepoint p in B that is not contained in a wall, and suppose the minimum Tits angle based at p between ψ and the poles of adjacent blocks is equal to $\epsilon > 0$. Let

$$\Omega = \{\psi' \in \partial_\infty \widetilde{X} \mid \angle_p(\psi, \psi') < \epsilon/2\}.$$

For $i = 0$ or 1, choose a lift $\tilde{\alpha}_i$ of α_i in B, and let $\Omega_{\tilde{\alpha}_i}$ denote the geodesics in Ω that exit from this lift. We claim that these sets are open and closed in Ω.

Open. If ψ' in $\Omega_{\tilde{\alpha}_i}$, note that any sufficiently close ray ψ'' to ψ' must exit B at a point close to where ψ' exits. Since the lifts of α_i in B are discrete, ψ'' and ψ must leave from the same lift.

Closed. Let E denote the set of exit points in $\tilde{\alpha}_i$ for elements of $\Omega_{\tilde{\alpha}_i}$. Then E is bounded, for otherwise we could find a sequence of exit points on $\tilde{\alpha}_i$ diverging from p, so we get a limit geodesic originating at p and ending at ψ' in $\partial_\infty \tilde{\alpha}_i$. However, this point is a pole of an adjacent block, and hence the Tits angle at p is $\geq \epsilon$, contradiction. Therefore, the set E is bounded.

Now, suppose we have a sequence $\psi'_k \to \psi'$, where $\psi'_k \in \Omega_{\tilde{\alpha}_i}$ and $\psi' \in \Omega$. After passing to a subsequence, we can assume the geodesic segments $p\bar{e}_k$ converge to a segment $p\bar{e}_\infty$. Therefore, the geodesic $p\psi'$ exits B through $p\bar{e}_\infty$, and is therefore in $\Omega_{\tilde{\alpha}_i}$.

It follows that the path component of ψ in Ω consists of geodesics which never leave through lifts of the α_i, since any subset $C \subset \Omega$ containing ψ and intersecting $\Omega_{\tilde{\alpha}_i}$ for some lift admits a separation into open subsets of C, and any $\psi' \in \Omega - \partial_\infty B$ lies in $\Omega_{\tilde{\alpha}_i}$ for some lift.

Case 2: ψ is contained in the boundary of a wall W and not a pole. Let ε denote the minimum Tits angle based at p between ψ and the poles of blocks which contain W, and let Ω be defined as above. A similar argument to the above shows that a path of geodesics starting at ψ does not exit the sub-block containing W through a lift of β or α_i. Therefore, the path component in Ω is contained in the boundary of that sub-block. Since the boundary of this sub-block is homeomorphic to a suspension of a Cantor set, the path component in Ω is contained in $\partial_\infty W$.

Case 3: ψ is not in the boundary of an adjacent block. Choose a basepoint p in B and not contained in a wall. Let $H \subset B$ be the halfplane in $\widetilde{X}$ containing the ray $p\psi$ and the vertical geodesic through p between the poles of B. For any wall W, the space $W \cap H$ is either empty, a vertical geodesic, or a flat strip bounded by vertical geodesics. See Figure 2. After removing these subsets, we get either an infinite collection of open strips or a finite collection of strips and an open half
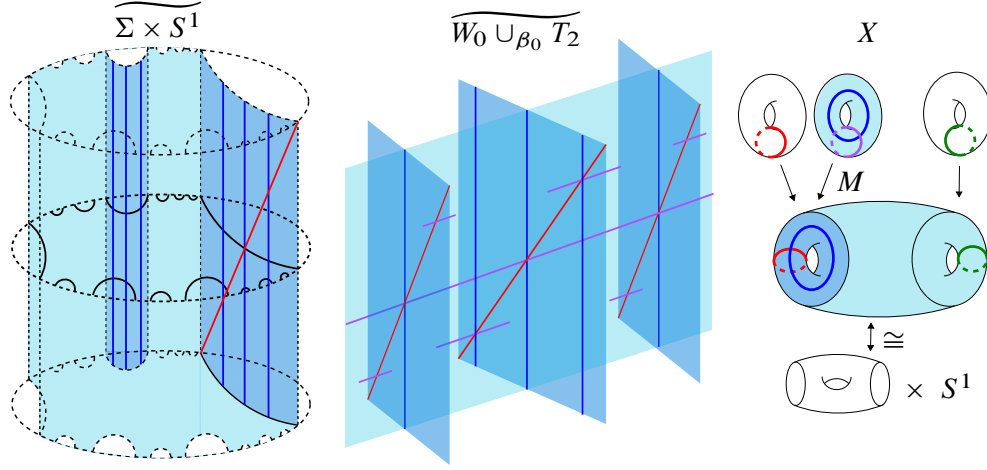
Figure 2. The proof of Theorem 3.2 analyzes the boundary of the universal cover of $M \subset X$, which is shaded in blue. The universal cover consists of infinitely many copies of sub-blocks $\widetilde{\Sigma \times S^1}$, shown on the left, and infinitely many copies of $\widetilde{W_0 \cup_{\beta_0} T_2}$, shown in the middle, which are glued to each other along walls, drawn in darker blue. The vertical strips in Case (3) are unions of vertical strips $[-1, 1] \times \mathbb{R}$ in the above two spaces with boundary $\pm 1 \times \mathbb{R}$ vertical lines in the walls.

plane. Note that the second case occurs only if the ray $p\psi$ is eventually contained in a single sub-block, and hence ψ is contained in the boundary of that sub-block.

Let ϵ be the minimal Tits angle between ψ and the poles of B. We may have that the minimum Tits angle between ψ and the poles of blocks adjacent to B is zero. However, since X is not right-angled we can choose $\varepsilon' < \varepsilon/2$ so that for each adjacent block, one of the poles has Tits distance with ψ greater than ε'.

First suppose there is no half-plane. If S is such an open strip in H, then let Ω_S be the points in $\partial_\infty \widetilde{X}$ whose representative rays originating at p nontrivially intersect S. Again, this set is closed and open in Ω. Therefore, since $p\psi$ intersects all strips, each point in a path starting at p must intersect them all as well. Therefore, the path is contained in $\partial_\infty H$.

Now, suppose there is a half-plane. Let SB be the sub-block covering $\Sigma \times S^1$ which contains this half plane, and let ρ be a path starting at ψ. For each geodesic ray $p\psi'$ for ψ' in ρ, project its intersection with SB to the horizontal subsurface $\widetilde{\Sigma}$. See Figure 3. If this projection is a single geodesic in $\widetilde{\Sigma}$ we are done. If not, this path contains geodesic rays arbitrarily close to $p\psi$ whose endpoints are contained in the boundaries of walls and are not poles of adjacent blocks (note that this fails if X is right-angled and ψ lies in $\partial\Sigma$, as ψ is contained in a horizontal path from Lemma 3.1). This leads to a contradiction, since by Case 2, the path components of these points in Ω are contained on a single longitude. Therefore, such a path would have to continue on this longitude until it hits a pole of adjacent block, which by assumption makes the path leave Ω.
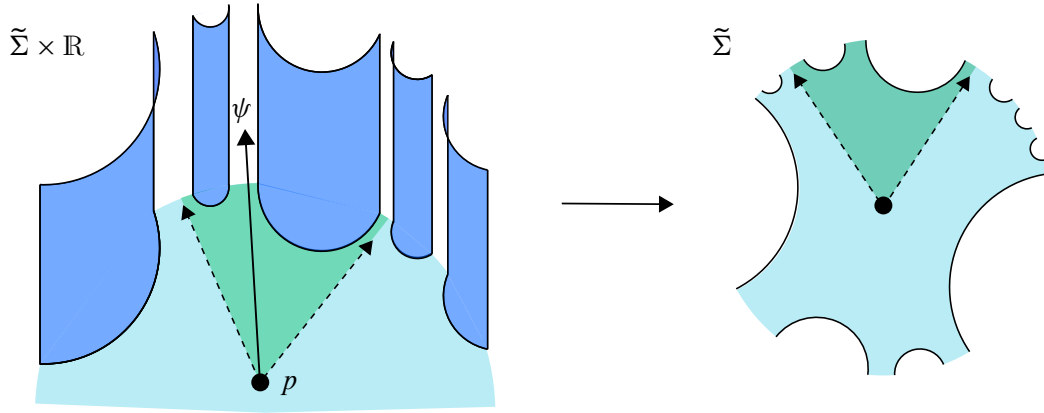
Figure 3. On the left is a geodesic ray $p\psi$ that is contained in a block isometric to $\widetilde{\Sigma} \times \mathbb{R}$. For each geodesic ray $p\psi'$ representing a point in a path in the boundary starting at ψ, the projection to $\widetilde{\Sigma}$ of the intersection of $p\psi'$ with the block $\widetilde{\Sigma} \times \mathbb{R}$ is drawn in green.

Case 4: ψ is a pole of $\partial_\infty Y$ where Y covers Y_i. Choose a basepoint p in a lift $\tilde{\alpha}_i$ of α_i, so that ψ is one of the endpoints of $\tilde{\alpha}_i$. The argument in Case 1 above shows that the geodesics in any path starting at ψ cannot exit Y through a lift of β. The argument in the second half of Case 3 rules out geodesics in this path exiting Y into an adjacent sub-block through a wall. Therefore, the local path component around ψ is contained in $\partial_\infty Y$. □

We now use the analysis of local path components in the non right-angled case to work out the path components of $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$, where ψ_+ and ψ_- are poles of a block boundary.

Lemma 3.3. *Suppose that X is not right-angled and suppose that Y is a block which covers Y_0 or Y_1. Let ψ_+ and ψ_- be the poles of $\partial_\infty Y$, and suppose that x and y are points in $\partial_\infty Y - \{\psi_+, \psi_-\}$. Then x and y are in the same path component of $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$ if and only if they lie on the boundary of an adjacent sub-block SB or are contained in the same longitude.*

Proof. Suppose that $x, y \in \partial_\infty \widetilde{Y}$ are not contained in a single longitude or the boundary of an adjacent block. Suppose an arc ρ starts at x and ends at y. We must show that ρ intersects ψ_+ or ψ_-. If x or y is not contained in the boundary of a wall, then the longitude containing x or y has no poles on it, which by Theorem 3.2 implies the path ρ intersects ψ_+ or ψ_-. Therefore, we can assume that x and y are contained in two different sub-block boundaries SB_1 and SB_2. The group G splits as $\pi_1(M) *_{\langle \alpha_i \rangle} T_i$, and the two sub-blocks SB_1 and SB_2 correspond to subgroups in different conjugates of $\pi_1(M)$. The SB_i are separated by a lift of α_i in Y, which implies that there is no path connecting x and y which misses ψ_+ or ψ_-. □

Lemma 3.4. *Suppose that X is not right-angled and suppose that $\{\psi_+, \psi_-\}$ are the poles of a block B which covers M. Suppose x and y are two points in $\partial_\infty B - \{\psi_+, \psi_-\}$. Then x and y are in different path components of the space $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$ if and only if they do not lie on longitudes which form a boundary pair or are contained in the same longitude.*

Proof. Suppose L_1 and L_2 are longitudes that do not form a boundary pair, and that there is an arc ρ which starts at $x \in L_1$ and ends at $y \in L_2$. We must show that ρ intersects ψ_+ or ψ_-. This is certainly true if neither of the L_i are boundary longitudes by Theorem 3.2, as there are no poles of adjacent blocks on these longitudes. So, we can assume that both of the points lie on boundary longitudes for different lifts of the α_i.

If ρ does not intersect ψ_+ or ψ_-, then ρ must intersect a pole ψ'_+ of an adjacent block boundary $\partial_\infty Y$ covering Y_i for $i \in \{0, 1\}$. If ρ continues in $\partial_\infty B$, it intersects ψ_+ or ψ_-, so we assume the path continues into $\partial_\infty Y$. If ψ'_- is the other pole of $\partial_\infty Y$, then $\{\psi'_+, \psi'_-\}$ is a cut pair for $\partial_\infty \widetilde{X}$ as in Lemma 3.3. Since ρ exits $\partial_\infty B$, it enters a different path component of $\partial_\infty \widetilde{X} - \{\psi'_+, \psi'_-\}$ by Lemma 3.3. Hence, ρ intersects ψ'_-, since x and y are in the same component of $\partial_\infty \widetilde{X} - \{\psi'_+, \psi'_-\}$. Now, ρ must continue in $\partial_\infty B$, since otherwise by repeating this argument ρ would intersect ψ'_+ and not be an embedded path. However, then ρ intersects one of the poles of $\partial_\infty B$ by Theorem 3.2. □

4. Non-planarity

Theorem 4.1. *For any locally* CAT(0) *metric on X, the visual boundary $\partial_\infty \widetilde{X}$ is nonplanar.*

Proof. Suppose that $h \colon \partial_\infty \widetilde{X} \to S^2$ is an embedding. Let SB be a sub-block covering $\Sigma \times S^1$, and let $\{\psi_+, \psi_-\}$ be the poles of $\partial_\infty SB$. Let $\mathcal{P}$ be the collection of poles of adjacent blocks to SB. For each pair of poles in $\mathcal{P}$, the longitudes from this pair to ψ_+ and ψ_- give an embedded circle inside $\partial_\infty SB$.

We first claim that the images of each of these circles under the embedding h must bound a disc inside $S^2 - h(\partial_\infty SB)$ if h extends to an embedding of $\partial_\infty \widetilde{X}$. To see this, let $\{\psi'_+, \psi'_-\}$ be a pair of poles in $\mathcal{P}$ such that the corresponding circle C does not bound a disc in $S^2 - h(\partial_\infty SB)$. This implies there are longitudes ℓ_1 and ℓ_2 in $\partial_\infty SB$ which connect ψ_+ to ψ_- and map to different components of $S^2 - C$. In $\partial_\infty \widetilde{X}$, we can connect ψ'_+ to ψ'_- with a longitude of an adjacent block. The image of this longitude under h must intersect one of the $h(\ell_i)$, which is a contradiction.

We now claim there is no embedding of

$$\partial_\infty SB \cup \ell$$

into S^2 such that each embedded circle as above bounds a disc, where ℓ is a longitude of the block B containing SB and not contained in $\partial_\infty SB$. Note that under the usual identification of $\partial_\infty \widetilde{\Sigma}$ with the Cantor set, the pairs in $\mathcal{P}$ are identified with endpoints of deleted intervals. If there was such an embedding h, then by attaching discs to the domain along these circles we could extend $\bar{h}$ to an embedding $S^2 \cup \ell \to S^2$, which is a contradiction. $\qquad\square$

If X is right-angled, we can do better and find a nonplanar graph inside of $\partial_\infty \widetilde{X}$.

Theorem 4.2. *If X is right-angled, then there is an embedded $K_{3,3}$ graph inside $\partial_\infty \widetilde{X}$.*

Proof. An illustration of the $K_{3,3}$ subgraph on vertex sets $\{a, b, c\}$ and $\{x, y, z\}$ appears in Figure 1. Choose any block covering M, and let b, y, c, z be points that lie, in this order, on the horizontal circle containing a copy of $\partial_\infty \widetilde{\Sigma}$ guaranteed by Lemma 3.1. Let a and x be the suspension points of $\partial_\infty M$. Then, there are longitudes in $\partial_\infty (\widetilde{\Sigma \times S^1})$ connecting a to y and z, and longitudes connecting x to b and c. Furthermore, the horizontal circle gives paths from b to y and z and from c to y and z. Finally, in $\partial_\infty \widetilde{M}$, there is a longitude connecting a and x that does not intersect the horizontal circle. Thus, $\partial_\infty \widetilde{X}$ contains an embedded copy of $K_{3,3}$. $\qquad\square$

5. No nonplanar graphs if X is not right-angled

In this section, we use our analysis in Section 3 of the paths in $\partial_\infty \widetilde{X}$ to show the following theorem.

Theorem 5.1. *If X is not right-angled, then there is no embedded nonplanar graph inside $\partial_\infty \widetilde{X}$.*

In order to deal with the points in $\partial_\infty \widetilde{X}$ that are not contained in a block boundary, we need the following lemma, which is a slightly refined version of [7, Lemma 7].

Lemma 5.2. *Choose a basepoint p in X as above and ψ a point in $\partial_\infty X$ with infinite p-itinerary $(B_0, B_1, B_2, \dots)$. Let $\rho \colon [0, 1] \to \partial_\infty X$ be a path in $\partial_\infty X$ with $\rho(0) = \psi$ so that there exists points in ρ with different p-itinerary than ψ. Then there is $N \in \mathbb{N}$ so that for each $n > N$, there is a point in ρ with finite p-itinerary $(B_0, B_1, \dots B_n)$. In particular, these points in ρ are contained in $\partial_\infty B_n$.*

Proof. Since all of the points in the path ρ do not have the same p-itinerary, there is a block $B_N \in \mathrm{Itin}_p(\psi)$ so that B_N is not in the itinerary of $\rho(t)$ for some $t \in [0, 1]$.

Now, change p to a basepoint p' which lies on $p\psi$ and is contained in B_{N-1}. Then the p'-itinerary of ψ is the same as the p-itinerary, but starting at B_{N-1}. Suppose B_N is in the p'-itinerary of $\rho(t)$ for all t. Choose $\psi' \in \rho(t)$ so that B_N is not in the p-itinerary of ψ. The rays $p\psi'$ and $p'\psi'$ are bounded Hausdorff distance, and at some point are separated by the wall between B_{N-1} and B_N. Therefore, the rays are both bounded Hausdoff distance from this wall, which implies ψ' is in the boundary of this wall, and we are done.

Therefore, we can assume B_N is not in the p'-itinerary of $\rho(t)$ for some t. Now, let

$$t_0 = \inf\{t \mid B_N \notin \mathrm{Itin}_p(\rho(t))\}.$$

The proof of [7, Lemma 7] goes through to show that the p'-itinerary of $\rho(t_0)$ just consists of the block B_{n-1}. Therefore, we have found a point in $\partial_\infty B_{n-1}$. We can repeat this argument for any block B_n for $n > N$, since all of these blocks are not contained in ρ for all t. □

Proof of Theorem 5.1. A graph H is a *topological minor* of a graph G if G contains a subdivision of H as a subgraph. Kuratowski's theorem [15] states that a finite graph is nonplanar if and only if it does not have $K_{3,3}$ or a K_5 as a minor, so we have to rule out these two graphs. We give the proof for $K_{3,3}$; the proof for K_5 is similar.

First assume that a vertex of $K_{3,3}$ is mapped into a block boundary. In this case, the vertex is mapped to a pole of a block by Theorem 3.2.

Suppose that a vertex v is mapped to a pole ψ_+ in the boundary of a block Y which covers Y_i for some $i \in \{0, 1\}$. Let e_1, e_2, and e_3 denote the edges containing v in $K_{3,3}$, and let ψ_- denote the other pole of $\partial_\infty Y$. If each e_i maps to different components of $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$, then any loop λ_{ij} in $K_{3,3}$ which contains e_i and e_j passes through ψ_-. However, for each pair e_i and e_j, there is a loop λ_{ij} in $K_{3,3}$ so that λ_{ij} connects e_i with e_j. The total intersection of such λ_{ij} is v, which is a contradiction, see Figure 4.
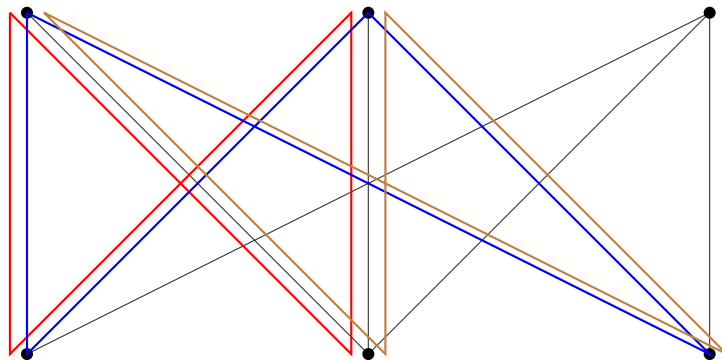

Figure 4. The red, blue and brown loops are the three loops λ_{ij} in the $K_{3,3}$ graph. Each connects two of the edges emanating from the upper left vertex, and the total intersection of all three loops is the upper left vertex.

Therefore, we can assume e_2 and e_3 are contained in the same component of $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$ which, by Lemma 3.3, means that e_2 and e_3 lie on the boundary $\partial_\infty B$ of an adjacent sub-block B. Since e_2 and e_3 emanate from v, they are both contained in the boundary of a wall. Since e_1 cannot be contained in the boundary of this wall as it is disjoint from e_2 and e_3, this implies that e_1 is contained in a different component of $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$. As above, the images of the loops λ_{12} and λ_{13} have to intersect at ψ_-, which implies that ψ_- lies on e_1. Note that the images of e_2 and e_3 contain the poles of $\partial_\infty B$.

Assume without loss of generality that the other endpoint w of e_1 maps to ψ_-. There are paths in $K_{3,3}$ connecting w to the endpoints of e_2 and e_3 that are disjoint from v, which implies by Lemma 3.3 that the other edges of $K_{3,3}$ containing w other than e_1 must map into $\partial_\infty B$. The images of these edges must contain the poles of $\partial_\infty B$, but in $K_{3,3}$ these edges are disjoint from e_2 and e_3, which is a contradiction.

Now suppose a vertex v is mapped to a pole ψ in the boundary of a block B which covers M. Again, let e_1, e_2, and e_3 denote the edges leaving v, and ψ_- the other pole. In this case, no pair of edges can be mapped to a boundary pair. If this occurred, there would be two points on these edges that are mapped to the poles of an adjacent block. This pair $\{\psi'_+, \psi'_-\}$ is a cut pair for $\partial_\infty X - \{\psi, \psi_-\}$, and no pair of points on these edges is a cut pair for $K_{3,3}$. Therefore, the paths would have to continue in the same component, which implies by Lemma 3.3 that they continue on $\partial_\infty B$. This implies that they eventually intersect at ψ_-, which is a contradiction.

Therefore, these paths must map to longitudes that are not boundary pairs. By Lemma 3.4 these edges lie in different path components of the space $\partial_\infty \widetilde{X} - \{\psi_+, \psi_-\}$, so any loop between them has to pass through ψ_-. This again is a contradiction.

We now suppose that all vertices in $K_{3,3}$ map to points in $\partial_\infty \widetilde{X}$ that are not contained in a block boundary, and hence have infinite p-itinerary for some (any) choice of basepoint $p \in \widetilde{X}$.

We first claim that given a point $x \in \partial_\infty \widetilde{X}$ with infinite p-itinerary, there cannot exist three disjoint paths γ_i starting at x such that each point in the γ_i has the same p-itinerary as x. Suppose that there are, and let Y denote their union. The union of the geodesics from p to points in Y yields a proper map from $\mathrm{Cone}_\infty(Y) := (Y \times [0, \infty))/(Y \times 0) \to \widetilde{X}$. Furthermore, there exists $t > 0$ so that

$$(v \times t) \cap (e \times [t, \infty)) = \emptyset$$

for every pair of disjoint vertex and edge $v, e \subset Y$.

Choose a block in the p-itinerary of these points and a lift $\tilde{\alpha}_i$ of α_i so that all geodesics in this cone exit the block through $\tilde{\alpha}_i$. We can choose $\tilde{\alpha}_i$ far enough away from p so that for each edge $e \subset T$, the sector $e \times [t, \infty)$ intersects $\tilde{\alpha}_i$ in an interval containing the exit point for the geodesic $\overline{px}$. At least one of the exit points

of the geodesics $\overline{pv}$ must be contained in the interior of these intervals, which is a contradiction.

We now see the itineraries of the vertices of $K_{3,3}$ are restricted. Suppose x is a vertex of $K_{3,3}$ with fixed p-itinerary. The previous paragraphs imply there are three edges in $K_{3,3}$ where one vertex of the edge has the same p-itinerary as x and the edge does not consist entirely of points with the same itinerary p-itinerary as x. Lemma 5.2 implies there is a block B and a vertex in the interior of each of these edges mapping to $\partial_\infty B$. The endpoints of these edges are not in $\partial_\infty B$, so each of these edges must enter and exit $\partial_\infty B$, intersecting a pole of B. Since there are three edges and only two poles, this is a contradiction. $\qquad\square$

6. Finite graphs in Croke–Kleiner boundaries

We now prove Theorem 1.2 from the Introduction. We first briefly recall some of the details of the Croke–Kleiner construction. The 2-complex X_{CK} they consider is precisely $Y_0 \cup_\beta T^2$. Again, the homeomorphism type of $\partial_\infty \widetilde{X}_{\mathrm{CK}}$ depends on the angle between α_0 and β.

The universal cover $\widetilde{X}_{\mathrm{CK}}$ again decomposes into blocks meeting along walls in a similar way. (The blocks are preimages of Y_0 or $W_0 \cup_\beta T^2$ and the walls are preimages of W_0.)

In $\partial_\infty \widetilde{X}_{\mathrm{CK}}$, the local path components of any pole are contained in the block boundary [7, Lemma 4]. (See also [6].) In particular, there are no paths with the same behavior as our "horizontal circles." This implies the following lemma.

Lemma 6.1. *Suppose an arc $\rho\colon [0, 1] \to \partial_\infty \widetilde{X}_{\mathrm{CK}}$ has infinite Tits length and $\rho(0)$ is contained in the union of the block boundaries. Then ρ contains a point not contained in the union of the block boundaries.*

Proof. Suppose $\rho \subset \cup_{B \in \mathcal{B}} \partial_\infty B$, where $\mathcal{B}$ is the set of all blocks in $\widetilde{X}_{\mathrm{CK}}$. For any $t \in [0, 1]$ there is a neighborhood U of $\rho(t)$ and a block $B_0 \in \mathcal{B}$ such that $\rho \cap U \subset \partial_\infty B_0$ by [7, Lemma 4]. These neighborhoods cover ρ, and taking a finite subcover guarantees that ρ is contained in a finite union of block boundaries, and therefore has finite length in the Tits metric. $\qquad\square$

Theorem 6.2. *Suppose X_1 and X_2 are homeomorphic to the Croke–Kleiner complex X_{CK} and equipped with locally CAT(0) metrics. If Γ is a finite graph contained in $\partial_\infty \widetilde{X}_1$, then there is an embedding of the graph Γ into $\partial_\infty \widetilde{X}_2$.*

Proof. A theorem of Xie [19, Theorem 6.1] implies that the *core* of the Tits boundaries of $\widetilde{X}_1$ and $\widetilde{X}_2$ are homeomorphic, where the core of the Tits boundary is the union of all topological circles in the boundary. Thus, if the graph $\Gamma \subset \partial_\infty \widetilde{X}_1$ is contained in the union of the block boundaries, then the theorem follows.

We prove that if a graph Γ embeds in $\partial_\infty \widetilde{X}_1$, then Γ embeds in the union of the block boundaries in $\partial_\infty \widetilde{X}_1$.

The coning argument in the proof of Theorem 5.1 implies that there are no circles in the boundary so that each point on the circle has infinite p-itinerary for some basepoint p. Moreover, any vertex of the graph Γ of valence greater than two must be contained in the union of the block boundaries. Thus, we may assume the graph $\Gamma \subset \partial_\infty \widetilde{X}_1$ contains a point contained in the union of the block boundaries.

We show that any edge of Γ of infinite Tits length can be replaced with a nearby edge of finite Tits length. We may assume the edge contains a point contained in the union of the block boundaries. If the edge has infinite Tits length, then the edge must contain a point v with infinite itinerary by Lemma 6.1. By [17, Proposition 3.14], there are points arbitrarily close to v with finite itinerary. The proof of Lemma 5.2 goes through to show that there is a block B of $\widetilde{X}_1$ so that each edge of Γ containing v contains points in $\partial_\infty B$. Thus, by the same reasoning as in the proof of Theorem 5.1, v is a vertex of degree ≤ 2 in Γ or contained in an edge of Γ. If the vertex v has degree 1, shorten the edge containing v to an edge ending at a pole of $\partial_\infty B$, where B is a block as above.

If the vertex v has degree 2 or is contained in an edge of Γ, choose a neighborhood U of v in $\partial_\infty \widetilde{X}_1$ so that $U \cap \Gamma$ only intersects a neighborhood of v in Γ. Choose a closed neighborhood of v in Γ homeomorphic to an interval so that the endpoints of the neighborhood are contained in U and map to poles of the same block. Replace this path with a longitude in this neighborhood which connects these poles.

Replace paths in this way for each vertex v with infinite itinerary in the edge e. The collection of these points is closed, so these replacements can be made for a finite number of points and will cover the edge (or at least the part of the edge which contains points with infinite itinerary). Each pair of endpoints of the replacement paths form a cut pair in $\partial_\infty \widetilde{X}_{\mathrm{CK}}$. Thus, if two replacements overlap, then they contain the same pole and can be combined, see Figure 5. Therefore, these replacements can be glued together to form a path with finite Tits length. Since the path contained a point in the union of the block boundaries, the path is contained in the union of the block boundaries as desired. □
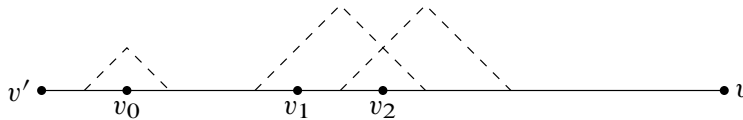

Figure 5. Modifying the path between v and v' to have finite Tits length. The dashed lines represent the new paths constructed between poles of the same block.

Remark 6.3. Note that the above proof works for embedding graphs in the boundaries of our main example, as long as X is not right-angled. Therefore,

any two of these non-right-angled boundaries contain the same set of finite graphs. We expect that no two of these boundaries are homeomorphic.

Remark 6.4. Once one knows that the finite graphs inside $\partial_\infty \widetilde{X}_{\mathrm{CK}}$ can be "pushed" into the union of block boundaries, it is easy to describe precisely which finite graphs live inside $\partial_\infty \widetilde{X}_{\mathrm{CK}}$. A concise description is the following: given a graph Γ, let $D(\Gamma)$ denote the graph obtained by "doubling the vertices," i.e. replacing each vertex $v \in \Gamma$ with v^+ and v^-, and letting $[v^{+/-}, w^{+/-}]$ be an edge in $D(\Gamma)$ if and only if $[vw]$ is an edge in Γ. For example, an edge in Γ becomes a 4-cycle in $D(\Gamma)$. If Γ is a tree, then $D(\Gamma)$ is planar. The finite subgraphs of $\partial_\infty \widetilde{X}_{\mathrm{CK}}$ are precisely the finite subgraphs of some subdivision of $D(\Gamma)$ for Γ a finite tree.

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