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Generalized persistence diagrams

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Abstract We generalize the persistence diagram of Cohen-Steiner, Edelsbrunner, and Harer to the setting of constructible persistence modules valued in a symmetric monoidal category. We call this the *type* $\mathcal{A}$ persistence diagram of a persistence module. If the category is also abelian, then we define a second *type* $\mathcal{B}$ persistence diagram. In addition, we show that both diagrams are stable to all sufficiently small perturbations of the module. The type $\mathcal{B}$ persistence diagram carries less information than the type $\mathcal{A}$ persistence diagram, but it enjoys a stronger stability theorem.

Keywords Persistence diagrams · Möbius Inversion · Abelian category · Symmetric monoidal category

Mathematics Subject Classification 18E10 · 06A07

1 Introduction

Let $f : \mathbb{M} \to \mathbb{R}$ be a Morse function on a compact manifold $\mathbb{M}$. The function f filters $\mathbb{M}$ by sublevel sets $\mathbb{M}_{f \leq r} = \{x \in \mathbb{M} \mid f(x) \leq r\}$. Apply homology with coefficients in a field and we call the resulting object F a *constructible persistence module of vector spaces*. The *persistence diagram* and the *barcode* are two invariants of a persistence module obtained as follows.

- By Images: Edelsbrunner et al. (2002) define the *persistent homology group* F_s^t, for $s < t$, as the image of $\mathsf{F}(s < t)$. Cohen-Steiner et al. (2007) define the

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persistence diagram of F as a finite set of points in the plane above the diagonal satisfying the following property. For each $s < t$, the number of points in the upper-left quadrant defined by (s, t) is the rank of F_s^t.
- By Indecomposables: The module F is isomorphic to a finite direct sum of indecomposable persistence modules $\mathsf{F} \cong \mathsf{F}_1 \oplus \cdots \oplus \mathsf{F}_n$. Any two ways of writing F as a sum of indecomposables are the same up to a reordering of the indecomposables. Furthermore, each indecomposable F_i is an *interval persistence* module. That is, there are a pair of values $r < t$, where t may be infinite, such that $F_i(s)$ is a copy of the field for all values $r \leq s < t$ and zero elsewhere.[1] Zomorodian and Carlsson (2005) define the *barcode* of F as its list of indecomposables. See also Gunnar Carlsson and Vin de Silva (2010).

A barcode translates to a persistence diagram by plotting the left endpoint versus the right endpoint of each interval persistence module. A persistence diagram translates to a barcode by turning each point (s, t) in to an interval persistence module starting at s and ending at t. In this way, the persistence diagram is equivalent to a barcode. However, the two definitions are very different in philosophy.

Suppose the homology of each sublevel set $\mathbb{M}_{f \leq r}$ is calculated using integer coefficients. Then the resulting object F is a *constructible persistence module of finitely generated abelian groups*. However, an indecomposable persistence module of finitely generated abelian groups need not look anything like an interval persistence module. For example, the module in Fig. 4 is indecomposable. Indecomposables are hard to interpret especially under perturbations to the module.

We generalize the persistence diagram of Cohen-Steiner, Edelsbrunner, and Harer to the setting of constructible persistence modules F valued in a symmetric monoidal category C with images. The category of sets, the category of vector spaces, and the category of finitely generated abelian groups are examples of such categories. We call this diagram the *type $\mathcal{A}$ persistence diagram* of F. If C is also abelian, then we define a second *type $\mathcal{B}$ persistence diagram* of F. The category of vector spaces and the category of abelian groups are examples of abelian categories. The type $\mathcal{B}$ persistence diagram of F may contain less information than the type $\mathcal{A}$ persistence diagram of F. However, the advantage of a type $\mathcal{B}$ diagram is a stronger statement of stability. Depending on C, our persistence diagrams may not be a complete invariant of a persistence module.

Persistence is motivated by data analysis and data is noisy. A small perturbation to a persistence module should not result in a drastic change to its persistence diagram. We use the standard *interleaving distance* to measure differences between persistence modules (Chazal et al. 2009). We define a new metric we call *erosion distance* to measure differences between persistence diagrams. In Theorem 8.2, we show that if the interleaving distance between two constructible persistence modules valued in an abelian category C is ε, then the erosion distance between their type $\mathcal{B}$ persistence diagrams is at most ε. We call this *continuity* of type $\mathcal{B}$ persistence diagrams. If C is simply a symmetric monoidal category, then Theorem 8.1 is a weaker one-way statement of continuity for type $\mathcal{A}$ persistence diagrams. We call

[1] The interval persistence module F_i is fully described by the half open interval $[s, t)$.

this *semicontinuity* of type $\mathcal{A}$ persistence diagrams. These theorems show that the information contained in both diagrams is stable to all sufficiently small perturbations of the module.

Cohen-Steiner, Edelsbrunner, and Harer define a stronger metric on the set of persistence diagrams they call *bottleneck distance*. They show that for two Morse functions $f, g : \mathbb{M} \to \mathbb{R}$, the bottleneck distance between their persistence diagrams is at most $\max |f - g|$. They do this by looking at the 1-parameter family of persistence modules obtained from the linear interpolation $h : \mathbb{M} \times [0, 1] \to \mathbb{R}$ taking $h_0 = f$ to $h_1 = g$. Using the Box Lemma, which is a local statement of stability, they track each point in the persistence diagram of h_0 all the way to the persistence diagram of h_1. Theorem 8.1 resembles the Box Lemma and assuming C has colimits, there is a way to construct a 1-parameter 1-Lipschitz family of persistence modules between any two interleaved persistence modules (Peter Bubenik et al. 2017). This suggests that bottleneck stability might extend to type $\mathcal{A}$ persistence diagrams. We leave the issue of bottleneck stability for future investigations.

2 Persistence modules

Let $(\mathsf{C}, \square)$ be an essentially small symmetric monoidal category with images. By essentially small, we mean that the collection of isomorphism classes of objects in C is a set. A symmetric monoidal category is, roughly speaking, a category C with a binary operation $\square$ on its objects and an identity object $e \in \mathsf{C}$ satisfying the following properties:

- (Symmetry) $a \square b \cong b \square a$, for all objects $a, b \in \mathsf{C}$
- (Associativity) $a \square (b \square c) \cong (a \square b) \square c$, for all objects $a, b, c \in \mathsf{C}$
- (Identity) $a \square e \cong a$, for all objects $a \in \mathsf{C}$.

See Weibel (2013, page 114) for a precise definition of a symmetric monoidal category. By images, we mean that for every morphism $f : a \to b$, there is a monomorphism $h : z \to b$ and a morphism $g : a \to z$ such that $f = h \circ g$. Furthermore, for a monomorphim $h' : z' \to b$ and a morphism $g' : a \to z'$ such that $f = h' \circ g'$, there is a unique morphism $u : z \to z'$ such that the following diagram commutes:

$$
\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, "g'"'] & & b \\
& z \arrow[ur, "h"] \arrow[d, "u"] & \\
& z'. \arrow[uur, "h'"'] &
\end{tikzcd}
$$

See Mitchell (1965, page 12) for a discussion of images.

Definition 2.1 A *persistence module* is a functor $\mathsf{F} : (\mathbb{R}, \leq) \to \mathsf{C}$ out of the poset of real numbers.

Let $S = \{s_1 < \cdots < s_n\}$ be a finite set of real numbers. Let $e \in \mathsf{C}$ be an identity object.

Definition 2.2 A persistence module F is *S-constructible* if

- for $p \leq q < s_1$, $\mathsf{F}(p \leq q)$ is the identity on e
- for $s_i \leq p \leq q < s_{i+1}$, $\mathsf{F}(p \leq q)$ is an isomorphism
- for $s_n \leq p \leq q$, $\mathsf{F}(p \leq q)$ is an isomorphism.

We say F is *constructible* if there is a finite set S such that F is S-constructible. If F is S-constructible then it is also T-constructible for any $T \supseteq S$.

We draw examples from the following five essentially small symmetric monoidal categories with images.

Example 2.1 Let FinSet be the category of finite sets. FinSet is a symmetric monoidal category under finite colimits (disjoint unions). A constructible persistence module valued in this category is often called a *merge tree* (Morozov et al. 2013).

The following four categories have more structure: they are abelian (see Weibel 2013, page 124) and Krull-Schmidt (see Appendix). In short, an abelian category is a category that behaves like the category of abelian groups. Finite products and coproducts are the same. Every morphism has a kernel and a cokernel. Every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism. The symmetric monoidal operation $\square$ is the direct sum $\oplus$.

Example 2.2 Let Vec be the category of finite dimensional k-vector spaces, for some fixed field k. Each vector space $a \in \mathsf{Vec}$ is isomorphic to $\mathsf{k}_1 \oplus \mathsf{k}_2 \oplus \cdots \oplus \mathsf{k}_n$, where n is the dimension of a. Note that every short exact sequence $0 \to a \to b \to c \to 0$ splits. That is, $b \cong a \oplus c$.

Example 2.3 Let Ab be the category of finitely generated abelian groups. An indecomposable of Ab is isomorphic to the infinite cyclic group $\mathbb{Z}$ or to a primary cyclic group $\mathbb{Z}/p^m\mathbb{Z}$, for a prime p and a positive integer m. By the fundamental theorem of finitely generated abelian groups, each object is uniquely isomorphic to

$$\mathbb{Z}^n \oplus \frac{\mathbb{Z}}{p_1^{m_1}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_2^{m_2}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{m_k}\mathbb{Z}},$$

for some $n \geq 0$ and primary cyclic groups $\mathbb{Z}/p_i^{m_i}\mathbb{Z}$. Not every short exact sequence in this category splits. Consider the following short exact sequence

$$0 \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \xrightarrow{\times 2} \frac{\mathbb{Z}}{4\mathbb{Z}} \xrightarrow{/} \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow 0.$$

Of course $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A finitely generated abelian group is simple iff it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for p prime. That is, $\mathbb{Z}/p\mathbb{Z}$ has no subgroups other than 0 and itself.

Example 2.4 Let FinAb be the category of finite abelian groups. An indecomposable of FinAb is isomorphic to a primary cyclic group $\mathbb{Z}/p^m\mathbb{Z}$, for prime p and a positive integer m. By the fundamental theorem of finitely generated abelian groups, each object is uniquely isomorphic to

$$\frac{\mathbb{Z}}{p_1^{m_1}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_2^{m_2}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{m_k}\mathbb{Z}}.$$

As shown in the previous example, not every short exact sequence in this category splits.

Example 2.5 Let $\mathsf{Rep}(\mathbb{N})$ be the category of functors from the commutative monoid of natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ to Vec. We think of $\mathbb{N}$ as a category with a single object and an endomorphism for each $n \in \mathbb{N}$ where $n \circ m$ is $n + m$. A functor in $\mathsf{Rep}(\mathbb{N})$ is completely determined by where it sends 1. $\mathsf{Rep}(\mathbb{N})$ is therefore equivalent to the category whose objects are endomorphisms $A : a \to a$ in Vec and whose morphisms $f : A \to B$ are maps $\hat{f} : a \to b$ such that the following diagram commutes:

$$\begin{tikzcd} a \arrow[r, "\hat{f}"] \arrow[d, "A"'] & b \arrow[d, "B"] \\ a \arrow[r, "\hat{f}"'] & b. \end{tikzcd}$$

We represent each object of $\mathsf{Rep}(\mathbb{N})$ by a square matrix of elements in k. Suppose k is algebraically closed. Then such a matrix decomposes into a Jordan normal form

$$\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{pmatrix}$$

where each Jordan block is of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

The indecomposables of $\mathsf{Rep}(\mathbb{N})$ are Jordan blocks. An object of $\mathsf{Rep}(\mathbb{N})$ is simple iff its a Jordan block of dimension one.

Not every short exact sequence in $\mathsf{Rep}(\mathbb{N})$ splits. Let $A : \mathsf{k} \to \mathsf{k}$ be given by (λ), let $B : k^2 \to k^2$ be given by $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, and let $f : A \to B$ be given by $\hat{f}(x) = (x, 0)$. The quotient $C = B/\mathrm{im} f$ is isomorphic to A. This gives us a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{/} C \longrightarrow 0$$

that does not split because B is not isomorphic to $(\lambda) \oplus (\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Let $\mathsf{PMod}(\mathsf{C})$ be the full subcategory of the functor category $[(\mathbb{R}, \leq), \mathsf{C}]$ consisting of constructible persistence modules. Henceforth, all persistence modules are constructible.

3 Interleaving distance

There is a natural distance between persistence modules. For $\varepsilon \in \mathbb{R}$, let

$$\mathsf{Shift}^{\varepsilon} : (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$$

be the poset map that sends r to $r + \varepsilon$. If $\mathsf{F} \in \mathsf{PMod}$ is S-constructible, then $\mathsf{F} \circ \mathsf{Shift}^{\varepsilon}$ is $(S + \varepsilon)$-constructible. Thus $\mathsf{Shift}^{\varepsilon}$ gives rise to a functor

$$\Delta^{\varepsilon} : \mathsf{PMod}(\mathsf{C}) \to \mathsf{PMod}(\mathsf{C}).$$

For each $\varepsilon \geq 0$, there is a canonical morphism $\sigma^{\varepsilon}_{\mathsf{F}} : \mathsf{F} \to \Delta^{\varepsilon}(\mathsf{F})$ given by $\sigma^{\varepsilon}_{\mathsf{F}}(r) = \mathsf{F}(r \leq r + \varepsilon)$.

Definition 3.1 Two modules $\mathsf{F}, \mathsf{G} \in \mathsf{PMod}(\mathsf{C})$ are *ε-interleaved* if there are morphisms $\phi : \mathsf{F} \to \Delta^{\varepsilon}(\mathsf{G})$ and $\psi : \mathsf{G} \to \Delta^{\varepsilon}(\mathsf{F})$ such that $\sigma^{2\varepsilon}_{\mathsf{F}} = \Delta^{\varepsilon}(\psi) \circ \phi$ and $\sigma^{2\varepsilon}_{\mathsf{G}} = \Delta^{\varepsilon}(\phi) \circ \psi$.

Any two persistence modules F an G are constructible with respect to a common set $T = \{t_1 < \cdots < t_m\}$. Both F and G are therefore constant over the half-open intervals $[t_i, t_{i+1})$ and $[t_m, \infty)$. As a consequence, if there is an interleaving between F and G, then there is a minimum interleaving between F and G.

Definition 3.2 The *interleaving distance* $\mathsf{d}_I(\mathsf{F}, \mathsf{G})$ between two persistence modules is the minimum over all $\varepsilon \geq 0$ such that F and G are ε-interleaved. If F and G are not interleaved, let $\mathsf{d}_I(\mathsf{F}, \mathsf{G}) = \infty$.

Example 3.1 Let $f : \mathbb{M} \to \mathbb{R}$ be a Morse function on a compact manifold $\mathbb{M}$. The function f filters $\mathbb{M}$ by sublevel sets $\mathbb{M}_{f \leq r}$. Apply homology with coefficients in k and the resulting object is in $\mathsf{PMod}(\mathsf{Vec})$. Apply homology with integer coefficients and the resulting object is in $\mathsf{PMod}(\mathsf{Ab})$. Apply homology with coefficients in a finite abelian group G and the resulting object is in $\mathsf{PMod}(\mathsf{FinAb})$. Suppose $\varepsilon > |f - g|$. Then $\mathbb{M}_{f \leq r} \subseteq \mathbb{M}_{g \leq r+\varepsilon} \subseteq \mathbb{M}_{f \leq r+2\varepsilon}$ implying, by functoriality of homology, an ε-interleaving between the two persistence modules.

Remark 3.1 The idea of interleavings appears in Cohen-Steiner et al. (2007) but it is not named until (Chazal et al. 2009). Since then, interleavings have been abstracted to other settings (Morozov et al. 2013; Peter Bubenik and Jonathan Scott 2014; Justin Curry 2014; Peter Bubenik et al. 2015; Lesnick 2015; De Silva et al. 2016).

4 Persistence diagrams

We now generalize the persistence diagram of Cohen-Steiner, Edelsbrunner, and Harer.

Definition 4.1 Define $(\mathsf{Dgm}, \supseteq)$ as the poset of all half-open intervals $[q, r) \subset \mathbb{R}$, for $q < r$, and all half-infinite intervals $[q, \infty) \subset \mathbb{R}$. The poset relation is the containment relation.

Let $S = \{s_1 < \cdots < s_n\}$ be a finite set of real numbers and $\mathcal{G}$ an abelian group. In the setting of Cohen-Steiner, Edelsbrunner, and Harer, the group $\mathcal{G}$ is the integers. From this we shall construct the persistence diagram.

Definition 4.2 A map $X : \mathsf{Dgm} \to \mathcal{G}$ is *S-constructible* if for every $J \supseteq I$ such that $J \cap S = I \cap S$, $X(I) = X(J)$. We say a map $X : \mathsf{Dgm} \to \mathcal{G}$ is *constructible* if it is S-constructible for some set S.

In the setting of Cohen-Steiner, Edelsbrunner, and Harer, X is the rank function.

Definition 4.3 A map $Y : \mathsf{Dgm} \to \mathcal{G}$ is *S-finite* if $Y(I) \neq e$ implies $I = [s_i, s_j)$ or $I = [s_i, \infty)$. We say a map $Y : \mathsf{Dgm} \to \mathcal{G}$ is *finite* if it is T-finite for some set T.

Definition 4.4 A *persistence diagram* is a finite map $Y : \mathsf{Dgm} \to \mathcal{G}$.

We visualize the poset Dgm as the set of points in the extended plane $\mathbb{R} \times \mathbb{R} \cup \{\infty\}$ above the diagonal. We visualize a persistence diagram Y by marking each $I \in \mathsf{Dgm}$ for which $Y(I) \neq [e]$ with the group element $Y(I)$. See Figs. 2, 3, 4, 5, and 6.

Theorem 4.1 (Möbius Inversion Formula) *For any S-constructible map $X : \mathsf{Dgm} \to \mathcal{G}$, there is an S-finite map $Y : \mathsf{Dgm} \to \mathcal{G}$ satisfying the Möbius inversion formula*

$$X(I) = \sum_{J \in \mathsf{Dgm} : J \supseteq I} Y(J),$$

for each $I \in \mathsf{Dgm}$.

Proof Let $S = \{s_1 < \cdots < s_n\}$. Define

$$Y\big([s_i, s_j)\big) = X\big([s_i, s_j)\big) - X\big([s_i, s_{j+1})\big) + X\big([s_{i-1}, s_{j+1})\big) - X\big([s_{i-1}, s_j)\big) \tag{1}$$

$$Y([s_i, \infty)) = X([s_i, \infty)) - X([s_{i-1}, \infty)). \tag{2}$$

Here we interpret s_0 as any value less than s_1 and s_{n+1} as any value greater than s_n. Define $Y(I) = e$ for all other $I \in \mathsf{Dgm}$. Let us check that Y satisfies the Möbius inversion formula. Fix an interval $I \in \mathsf{Dgm}$. Suppose $I = [s_i, s_j)$. We have

$$\begin{aligned}\sum_{J\in \mathsf{Dgm}:J\supseteq I} Y(J) &= \sum_{k=j}^{n}\sum_{h=1}^{i} Y([s_h,s_k)) + \sum_{h=1}^{i} Y([s_h,\infty)) \\ &= \sum_{k=j}^{n}\sum_{h=1}^{i}[X([s_h,s_k)) - X([s_h,s_{k+1})) \\ &\quad + X([s_{h-1},s_{k+1})) - X([s_{h-1},s_k))] \\ &\quad + \sum_{h=1}^{i}[X([s_h,\infty)) - X([s_{h-1},\infty))] \\ &= \sum_{k=j}^{n}[X([s_i,s_k)) - X([s_i,s_{k+1}))] + X([s_i,\infty)) \\ &= X\big([s_i,s_j)\big).\end{aligned}$$

Suppose I is of the form $[s_i,\infty)$. We have

$$\begin{aligned}\sum_{J\in \mathsf{Dgm}:J\supseteq I} Y(J) &= \sum_{h=1}^{i} Y([s_h,\infty)) \\ &= \sum_{h=1}^{i}[X([s_h,\infty)) - X([s_{h-1},\infty))] \\ &= X([s_i,\infty)).\end{aligned}$$

Suppose I is not of the form $[s_i,s_j)$. Then there is an $I' \in \mathsf{Dgm}$ of the form $[s_i,s_j)$ or $[s_i,\infty)$ such that $I' \cap S = I \cap S$. We have

$$\sum_{J\in \mathsf{Dgm}:J\supseteq I} Y(J) = \sum_{J\in \mathsf{Dgm}:J\supseteq I'} Y(J) = X(I') = X(I).$$

$\square$

The persistence diagram Y of Cohen-Steiner, Edelsbrunner, and Harer is the Möbius inversion of the rank function X.

Remark 4.1 The Möbius inversion formula applies to any constructible map from a poset to an abelian group. See Rota (1964), Bender and Goldman (1975) and Leinster (2012). This suggests a notion of a persistence diagram for constructible persistence modules not just over $(\mathbb{R},\leq)$ but over more general posets. See Peter Bubenik and Jonathan Scott (2014) and Peter Bubenik et al. (2015).

5 Erosion distance

The interleaving distance suggests a natural metric between persistence diagrams. But first, we need a notion of a morphism between persistence diagrams.

Let $(\mathcal{G}, \preceq)$ be an abelian group with a translation invariant partial ordering on its elements. That is if $a \preceq b$, then $a + c \preceq b + c$ for any $c \in \mathcal{G}$. Let $e \in \mathcal{G}$ be the additive identity.

Definition 5.1 Let $Y_1, Y_2 : \mathsf{Dgm} \to (\mathcal{G}, \preceq)$ be two persistence diagrams. A *morphism* $Y_1 \to Y_2$ of persistence diagrams is the relation

$$\sum_{J \in \mathsf{Dgm} : J \supseteq I} Y_1(J) \preceq \sum_{J \in \mathsf{Dgm} : J \supseteq I} Y_2(J),$$

for each $I \in \mathsf{Dgm}$. Let $\mathsf{PDgm}(\mathcal{G})$ be the poset of persistence diagrams valued in $(\mathcal{G}, \preceq)$.

For any $\varepsilon \geq 0$, let $\mathsf{Grow}^{\varepsilon} : \mathsf{Dgm} \to \mathsf{Dgm}$ be the poset map that sends each $[p, q)$ to $[p - \varepsilon, q + \varepsilon)$ and each $[p, \infty)$ to $[p - \varepsilon, \infty)$. For a morphism $Y_1 \to Y_2$ in $\mathsf{PDgm}(\mathcal{G})$, we have $Y_1 \circ \mathsf{Grow}^{\varepsilon} \to Y_2 \circ \mathsf{Grow}^{\varepsilon}$. Thus $\mathsf{Grow}^{\varepsilon}$ gives rise to a functor

$$\nabla^{\varepsilon} : \mathsf{PDgm}(\mathcal{G}) \to \mathsf{PDgm}(\mathcal{G})$$

given by precomposition with $\mathsf{Grow}^{\varepsilon}$. For each $\varepsilon \geq 0$, we have $\nabla^{\varepsilon}(Y) \to Y$. The persistence diagram $\nabla^{\varepsilon}(Y)$ is visualized as the persistence diagram Y with all its points shifted towards the diagonal by a distance $\sqrt{2}\varepsilon$. See Fig. 1.

Definition 5.2 An *ε-erosion* between two persistence diagrams $Y_1, Y_2, \in \mathsf{PDgm}(\mathcal{G})$ is a pair of morphisms $\nabla^{\varepsilon}(Y_2) \to Y_1$ and $\nabla^{\varepsilon}(Y_1) \to Y_2$.

Any two persistence diagrams are finite with respect to a common set $T = \{t_1 < \cdots < t_n\}$. As a consequence, if there is an ε-erosion between Y_1 and Y_2, then there is a minimum ε for which there is an ε-erosion.

Definition 5.3 The *erosion distance* $\mathsf{d}_E(Y_1, Y_2)$ is the minimum over all $\varepsilon \geq 0$ such that there is an ε-erosion between Y_1 and Y_2. If there is no ε-erosion, let $\mathsf{d}_E(Y_1, Y_2) = \infty$.

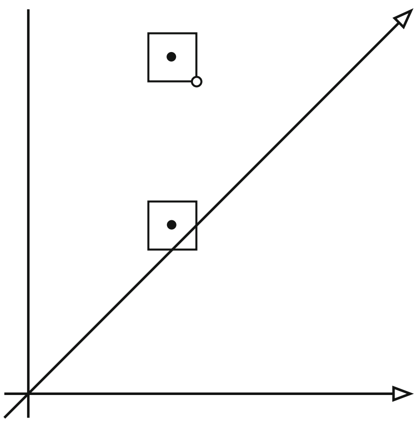


Fig. 1 The ε-erosion $\nabla^{\varepsilon}(Y)$ (circle) of a persistence diagram Y (dots) slides each point of Y to the lower-right corner of the square of side length 2ε centered at that point. Points close to the diagonal disappear into the diagonal. Note that $\nabla^{\varepsilon}(Y) \to Y$

Proposition 5.1 *Let $X : \mathsf{Dgm} \to \mathcal{G}$ be a constructible map and let $Y : \mathsf{Dgm} \to \mathcal{G}$ be a finite map that satisfies the Möbius inversion formula*

$$X(I) = \sum_{J \in \mathsf{Dgm} : J \supseteq I} Y(J),$$

for each $I \in \mathsf{Dgm}$. Then

$$X \circ \mathsf{Grow}^{\varepsilon}(I) = \sum_{J \in \mathsf{Dgm} : J \supseteq I} \nabla^{\varepsilon}(Y)(J),$$

for each $I \in \mathsf{Dgm}$. In other words, $\mathsf{Grow}^{\varepsilon}$ commutes with the Möbius inversion formula.

Proof We have

$$\begin{aligned} \sum_{J \in \mathsf{Dgm} : J \supseteq I} \nabla^{\varepsilon}(Y)(J) &= \sum_{J \in \mathsf{Dgm} : J \supseteq I} Y \circ \mathsf{Grow}^{\varepsilon}(J) \\ &= X \circ \mathsf{Grow}^{\varepsilon}(I) \end{aligned}$$

□

Remark 5.1 The erosion distance first appears in Edelsbrunner et al. (2011) which is an early attempt to develop a theory of persistence for maps from a surface to the Euclidean plane.

6 Grothendieck groups

We are interested in two abelian groups: the Grothendieck group $\mathcal{A}$ of an essentially small symmetric monoidal category and the Grothendieck group $\mathcal{B}$ of an essentially small abelian category. See Weibel (2013) for an introduction to the two Grothendieck groups. Note that every abelian category is a symmetric monoidal category under the direct sum $\oplus$ and the additivity identity is the zero object.

6.1 Symmetric monoidal category

Let C be an essentially small monoidal category. The set $\mathcal{I}(\mathsf{C})$ of isomorphism classes in C is a commutative monoid under $\Box$. We write the isomorphism class of an object $a \in \mathsf{C}$ as $[a] \in \mathcal{I}(\mathsf{C})$, the binary operation in $\mathcal{I}(\mathsf{C})$ as $[a] + [b] = [a \Box b]$, and the additive identity of $\mathcal{I}(\mathsf{C})$ as $[e]$.

Definition 6.1.1 The *Grothendieck group* $\mathcal{A}(\mathsf{C})$ of C is the group completion of the commutative monoid $\mathcal{I}(\mathsf{C})$.

Explicitly, an element of $\mathcal{A}(\mathsf{C})$ is of the form $[a] - [b]$ with addition coordinatewise, and $[a] = [c]$ iff $[a] + [d] = [c] + [d]$, for some element $[d] \in \mathcal{I}(\mathsf{C})$. If C is additive and Krull-Schmidt (see Appendix), then each object in C is isomorphic to a unique direct sum of indecomposables. This means $\mathcal{A}(\mathsf{C})$ is the free abelian group generated by the set of isomorphism classes of indecomposables. The Grothendieck group $\mathcal{A}(\mathsf{C})$ has a natural translation-invariant partial ordering. We define $[a] \preceq [b]$ iff $[b] - [a] \in \mathcal{I}(\mathsf{C})$. If $[a] \preceq [b]$, then $[a] + [c] \preceq [b] + [c]$ for any $[c] \in \mathcal{A}(\mathsf{C})$. See Weibel (2013, page 72) for an introduction to translation-invariant partial orderings on Grothendieck groups.

Example 6.1.1 Every finite set is a finite disjoint union of the singleton set. We have

$$\mathcal{A}(\mathsf{FinSet}) \cong \mathbb{Z}.$$

Example 6.1.2 Every finite dimensional vector space is isomorphic to a finite direct sum of k. We have

$$\mathcal{A}(\mathsf{Vec}) \cong \mathbb{Z}.$$

Example 6.1.3 An indecomposable of Ab is the free cyclic group or a primary cyclic group. We have

$$\mathcal{A}(\mathsf{Ab}) \cong \mathbb{Z} \oplus \bigoplus\nolimits_{(m,p)} \mathbb{Z},$$

over all primes p and positive integers m.

Example 6.1.4 An indecomposable of FinAb is a primary cyclic group. We have

$$\mathcal{A}(\mathsf{FinAb}) \cong \bigoplus\nolimits_{(m,p)} \mathbb{Z}$$

over all primes p and positive integers m.

Example 6.1.5 An indecomposable of $\mathsf{Rep}(\mathbb{N})$ is a Jordan block. We have

$$\mathcal{A}(\mathsf{Rep}(\mathbb{N})) \cong \bigoplus\nolimits_{(m,\lambda)} \mathbb{Z},$$

over all positive integers m and elements λ in the field k.

6.2 Abelian category

Suppose C is an essentially small abelian category. We say two elements $[b]$ and $[a] + [c]$ in $\mathcal{A}(\mathsf{C})$ are related, written $[b] \sim [a] + [c]$, if there is a short exact sequence $0 \to a \to b \to c \to 0$.

Definition 6.2.1 The *Grothendieck group* $\mathcal{B}(\mathsf{C})$ of C is the quotient group $\mathcal{A}(\mathsf{C})/\sim$. That is, $\mathcal{B}(\mathsf{C})$ is the abelian group with one generator for each isomorphism classes $[a]$ in C and one relation $[b] \sim [a] + [c]$ for each short exact sequence $0 \to a \to b \to c \to 0$.

Let $\pi : \mathcal{A}(\mathsf{C}) \to \mathcal{B}(\mathsf{C})$ be the quotient map. Note that $\pi(\mathcal{I}(\mathsf{C}))$ is a commutative monoid that generates $\mathcal{B}(\mathsf{C})$. This allows us to define a translation-invariant partial ordering on $\mathcal{B}(\mathsf{C})$ as follows. We define $[a] \preceq [b]$ iff $[b]-[a] \in \pi(\mathcal{I}(\mathsf{C}))$. If $[a] \preceq [b]$, then $[a]+[c] \preceq [b]+[c]$ for any $[c] \in \mathcal{B}(\mathsf{C})$. The quotient map π is a poset map.

Example 6.2.1 Every short exact sequence in Vec splits. We have

$$\mathcal{B}(\mathsf{Vec}) \cong \mathbb{Z}.$$

The quotient map $\pi : \mathcal{A}(\mathsf{Vec}) \to \mathcal{B}(\mathsf{Vec})$ is the identity.

Example 6.2.2 Every primary cyclic group $\mathbb{Z}/p^m\mathbb{Z}$ fits into a short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \frac{\mathbb{Z}}{p^m\mathbb{Z}} \to 0.$$

This means $[\mathbb{Z}] \sim [\mathbb{Z}] + \left[\frac{\mathbb{Z}}{p^m\mathbb{Z}}\right]$ and therefore $0 \sim \left[\frac{\mathbb{Z}}{p^m\mathbb{Z}}\right]$. We have

$$\mathcal{B}(\mathsf{Ab}) \cong \mathbb{Z}.$$

The quotient map $\pi : \mathcal{A}(\mathsf{Ab}) \to \mathcal{B}(\mathsf{Ab})$ forgets the torsion part of every finitely generated abelian group.

Example 6.2.3 Every primary cyclic group $\mathbb{Z}/p^m\mathbb{Z}$ fits into a short exact sequence

$$0 \to \frac{\mathbb{Z}}{p\mathbb{Z}} \to \frac{\mathbb{Z}}{p^m\mathbb{Z}} \to \frac{\mathbb{Z}}{p^{m-1}\mathbb{Z}} \to 0.$$

This means

$$\left[\frac{\mathbb{Z}}{p^m\mathbb{Z}}\right] \sim m\left[\frac{\mathbb{Z}}{p\mathbb{Z}}\right].$$

Furthermore, $\frac{\mathbb{Z}}{p\mathbb{Z}}$ is a simple object so it can not be broken by a short exact sequence. We have

$$\mathcal{B}(\mathsf{FinAb}) \cong \bigoplus\nolimits_p \mathbb{Z}$$

over all p prime. The quotient map $\pi : \mathcal{A}(\mathsf{FinAb}) \to \mathcal{B}(\mathsf{FinAb})$ takes each primary cyclic group $\left[\frac{\mathbb{Z}}{p^m\mathbb{Z}}\right]$ to m in the p factor of $\mathcal{B}(\mathsf{FinAb})$.

Example 6.2.4 Every Jordan block fits into a short exact sequence. For example,

$$0 \to (\lambda) \to \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \to \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \to 0$$

and

$$0 \to (\lambda) \to \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \to (\lambda) \to 0.$$

This means

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \sim 3(\lambda).$$

Furthermore, each one-dimensional Jordan block (λ) is simple so it can not be broken by a short exact sequence. We have

$$\mathcal{B}(\mathsf{Rep}(\mathbb{N})) \cong \bigoplus_{\lambda \in \mathsf{k}} \mathbb{Z}.$$

The quotient map $\pi : \mathcal{A}(\mathsf{Rep}(\mathbb{N})) \to \mathcal{B}(\mathsf{Rep}(\mathbb{N}))$ takes each Jordan block of dimension $m \in \mathbb{N}$ with eigenvalue $\lambda \in \mathsf{k}$ to m in the λ factor of $\mathcal{B}(\mathsf{Rep}(\mathbb{N}))$.

7 Diagram of a module

Fix an essentially small symmetric monoidal category C with images. We now assign to each persistence module $\mathsf{F} \in \mathsf{PMod}(\mathsf{C})$ a persistence diagram $\mathsf{F}_{\mathcal{A}} \in \mathsf{PDgm}(\mathcal{A}(\mathsf{C}))$. If C is also abelian, then we assign to F a second persistence diagram $\mathsf{F}_{\mathcal{B}} \in \mathsf{PDgm}(\mathcal{B}(\mathsf{C}))$.

We start by constructing a map

$$d\mathsf{F}_{\mathcal{I}} : \mathsf{Dgm} \to \mathcal{I}(\mathsf{C}).$$

Recall $\mathcal{I}(\mathsf{C})$ is the commutative monoid of isomorphism classes of objects in C. Suppose F is $S = \{s_1 < \cdots < s_n\}$-constructible. Then there is a $\delta > 0$ such that $s_{i-1} < s_i - \delta$, for each $1 < i \leq n$. Choose a value $s' > s_n$. Define

$$d\mathsf{F}_{\mathcal{I}}(I) = \begin{cases} [\mathrm{im}\, \mathsf{F}(p < s_i - \delta)] & \text{for } \mathrm{I} = [\mathrm{p}, \mathrm{s_i}) \\ [\mathrm{im}\, \mathsf{F}(p < s')] & \text{for } \mathrm{I} = [\mathrm{p}, \infty) \\ [\mathrm{im}\, \mathsf{F}(p < q)] & \text{for all other } \mathrm{I} = [\mathrm{p}, \mathrm{q}). \end{cases}$$

Note that if F is also T-constructible, then $d\mathsf{F}_{\mathcal{I}}$ constructed using T is the same as $d\mathsf{F}_{\mathcal{I}}$ constructed using S. Now compose with the inclusion map $\mathcal{I}(\mathsf{C}) \hookrightarrow \mathcal{A}(\mathsf{C})$ and we have an S-constructible map

$$d\mathsf{F}_{\mathcal{A}} : \mathsf{Dgm} \to \mathcal{A}(\mathsf{C}).$$

Suppose C is abelian. Then by composing with the quotient map $\pi : \mathcal{A}(\mathsf{C}) \to \mathcal{B}(\mathsf{C})$, we have an S-constructible map

$$d\mathsf{F}_{\mathcal{B}} : \mathsf{Dgm} \to \mathcal{B}(\mathsf{C}).$$

Definition 7.1 The *type $\mathcal{A}$ persistence diagram* of F is the Möbius inversion

$$\mathsf{F}_{\mathcal{A}} : \mathsf{Dgm} \to \mathcal{A}(\mathsf{C})$$

of $d\mathsf{F}_{\mathcal{A}} : \mathsf{Dgm} \to \mathcal{A}(\mathsf{C})$.

Definition 7.2 The *type $\mathcal{B}$ persistence diagram* of F is the Möbius inversion

$$\mathsf{F}_{\mathcal{B}} : \mathsf{Dgm} \to \mathcal{B}(\mathsf{C})$$

of $d\mathsf{F}_{\mathcal{B}} : \mathsf{Dgm} \to \mathcal{B}(\mathsf{C})$.

Note that if F is S-constructible, then both $\mathsf{F}_{\mathcal{A}}$ and $\mathsf{F}_{\mathcal{B}}$ are S-finite persistence diagrams.

Proposition 7.1 (Positivity) *For each $I \in \mathsf{Dgm}$, $[e] \preceq \mathsf{F}_{\mathcal{B}}(I)$.*

Proof Suppose F is $S = \{s_1 < \cdots < s_n\}$-constructible. We need only show the inequality for intervals I of the form $[s_i, s_j)$ and $[s_i, \infty)$. For all other I, $\mathsf{F}_{\mathcal{B}}(I) = [e]$.

Suppose $I = [s_i, s_j)$. Consider the following subdiagram of F, for a sufficiently small $\delta > 0$:

$$\begin{tikzcd}
\mathsf{F}(s_{i-1}) \arrow[r, "\mathsf{F}(s_{i-1} < s_i)"] \arrow[d] & \mathsf{F}(s_i) \arrow[d, "\mathsf{F}(s_i < s_j - \delta)"] \\
\mathsf{F}(s_{j+1} - \delta) & \mathsf{F}(s_j - \delta). \arrow[l, "\mathsf{F}(s_j - \delta < s_{j+1} - \delta)"]
\end{tikzcd}$$

Here we interpret s_0 as any value less than s_1 and s_{n+1} as any value greater than s_n. By Eq. 1,

$$\mathsf{F}_{\mathcal{B}}\big([s_i, s_j)\big) = d\mathsf{F}_{\mathcal{B}}\big([s_i, s_j)\big) - d\mathsf{F}_{\mathcal{B}}\big([s_i, s_{j+1})\big) + d\mathsf{F}_{\mathcal{B}}\big([s_{i-1}, s_{j+1})\big) - d\mathsf{F}_{\mathcal{B}}\big([s_{i-1}, s_j)\big)$$

Observe

$$\begin{aligned}
d\mathsf{F}_{\mathcal{B}}\big([s_i, s_j)\big) - d\mathsf{F}_{\mathcal{B}}\big([s_i, s_{j+1})\big) &= \big[\mathrm{im}\ \mathsf{F}(s_i < s_j - \delta)\big] \\
&- \left[\frac{\mathrm{im}\ \mathsf{F}(s_i < s_j - \delta)}{\mathrm{im}\ \mathsf{F}(s_i < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta)}\right] \\
&= \big[\mathrm{im}\ \mathsf{F}(s_i < s_j - \delta)\big] - \big[\mathrm{im}\ \mathsf{F}(s_i < s_j - \delta)\big] \\
&+ \big[\mathrm{im}\ \mathsf{F}(s_i < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta)\big] \\
&= \big[\mathrm{im}\ \mathsf{F}(s_i < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta)\big].
\end{aligned}$$

Here the intersection is interpreted as the pullback of the two subobjects. By a similar argument,

$$\begin{aligned}
&d\mathsf{F}_{\mathcal{B}}\big([s_{i-1}, s_{j+1})\big) - d\mathsf{F}_{\mathcal{B}}\big([s_{i-1}, s_j)\big) \\
&\quad = -\big[\mathrm{im}\ \mathsf{F}(s_{i-1} < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta)\big].
\end{aligned}$$

Note that

$$\operatorname{im}\ \mathsf{F}(s_{i-1} < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta)$$

is a subobject of

$$\operatorname{im}\ \mathsf{F}(s_i < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta).$$

Therefore

$$\mathsf{F}_{\mathcal{B}}\big([s_i, s_j)\big) = \left[\frac{\operatorname{im}\ \mathsf{F}(s_i < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta)}{\operatorname{im}\ \mathsf{F}(s_{i-1} < s_j - \delta) \cap \ker\ \mathsf{F}(s_j - \delta < s_{j+1} - \delta)}\right] \succeq [e].$$

Suppose $I = [s_i, \infty)$. Then by a similar argument using Eq. 2, we have

$$\mathsf{F}_{\mathcal{B}}([s_i, \infty)) = \left[\frac{\operatorname{im}\ \mathsf{F}(s_i < s_{n+1})}{\operatorname{im}\ \mathsf{F}(s_{i-1} < s_{n+1})}\right] \succeq [e].$$

$\square$

Example 7.1 See Fig. 2 for an example of a persistence module in $\mathsf{PMod}(\mathsf{FinSet})$ and its type $\mathcal{A}$ persistence diagram. Note that FinSet is not an abelian category so it does not have a type $\mathcal{B}$ persistence diagram.

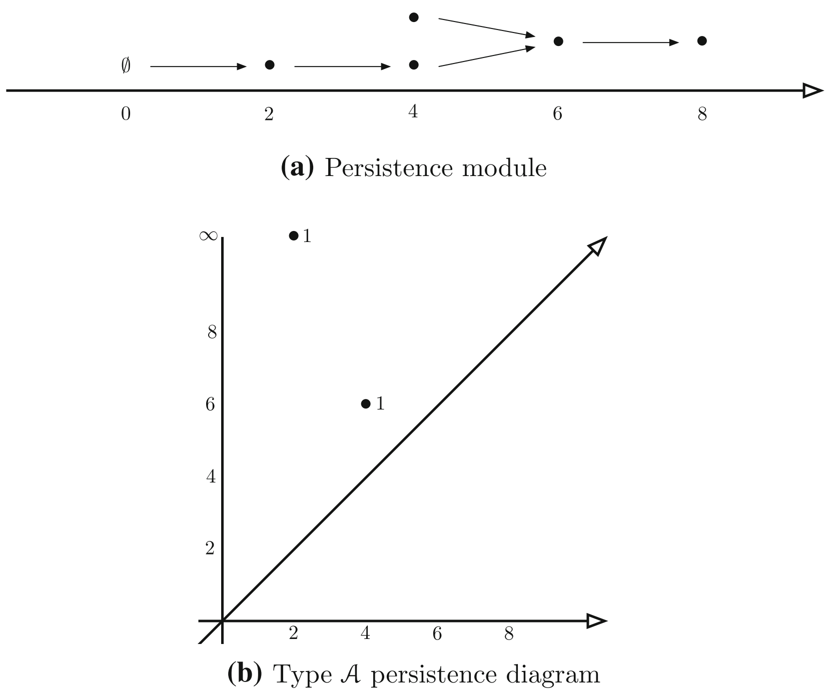


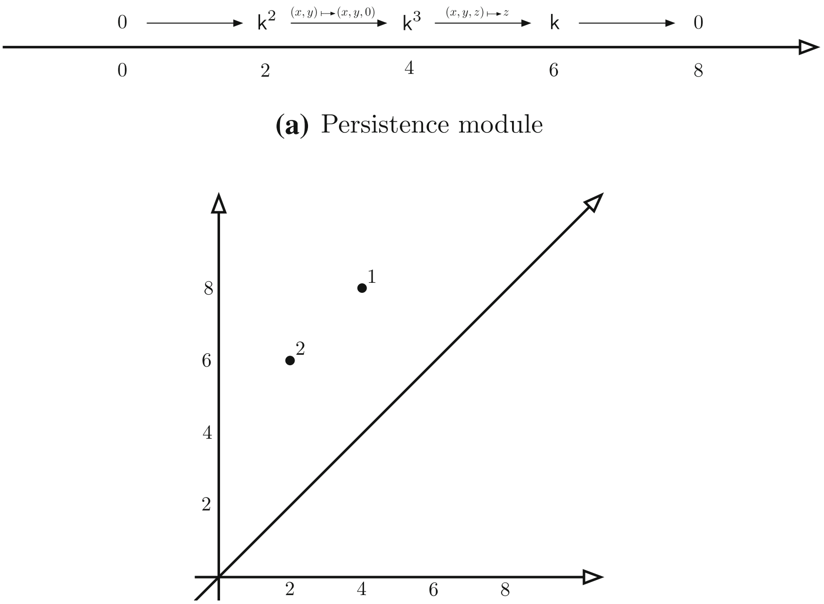
Fig. 2 Here we have an example of a persistence module in $\mathsf{PMod}(\mathsf{FinSet})$ and its type $\mathcal{A}$ persistence diagram. The type $\mathcal{B}$ persistence diagram is not defined

Example 7.2 See Fig. 3 for an example of a persistence module in $\mathsf{PMod}(\mathsf{Vec})$ and its type $\mathcal{A}$ and type $\mathcal{B}$ persistence diagrams. Note that the quotient map $\pi : \mathcal{A}(\mathsf{Vec}) \to \mathcal{B}(\mathsf{Vec})$ is an isomorphism and therefore the two diagrams are the same.

Example 7.3 See Fig. 4 for an example of a persistence module in $\mathsf{PMod}(\mathsf{Ab})$ and its type $\mathcal{A}$ persistence diagram. Note that the quotient map $\pi : \mathcal{A}(\mathsf{C}) \to \mathcal{B}(\mathsf{C})$ forgets torsion and therefore the type $\mathcal{B}$ persistence diagram is, for this example, zero.

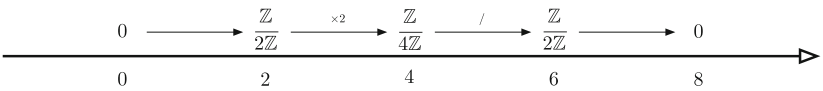
Example 7.4 See Fig. 5 for an example of a persistence module in $\mathsf{PMod}(\mathsf{FinAb})$ and its type $\mathcal{A}$ and type $\mathcal{B}$ persistence diagrams.

Example 7.5 See Fig. 6 for an example of a persistence module in $\mathsf{PMod}(\mathsf{Rep}(\mathbb{N}))$ and its type $\mathcal{A}$ and type $\mathcal{B}$ persistence diagrams.

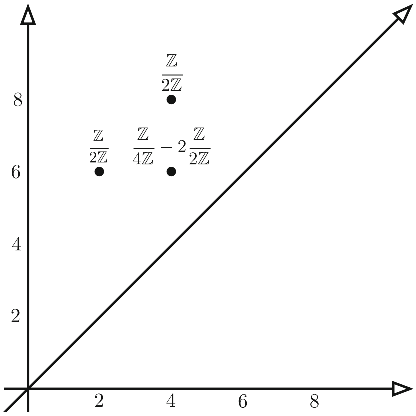


(b) Type $\mathcal{A}$ and $\mathcal{B}$ persistence diagrams

Fig. 3 Here we have an example of a persistence module in $\mathsf{PMod}(\mathsf{Vec})$ and its type $\mathcal{A}$ and $\mathcal{B}$ persistence diagrams



(a) Persistence module



(b) Type $\mathcal{A}$ persistence diagram

Fig. 4 Here we have an example of a persistence module in $\mathsf{PMod}(\mathsf{Ab})$ and its type $\mathcal{A}$ persistence diagram. The map from 4 to 6 is the quotient of $\mathbb{Z}/4\mathbb{Z}$ by the image of the previous map. The type $\mathcal{B}$ persistence diagram is zero

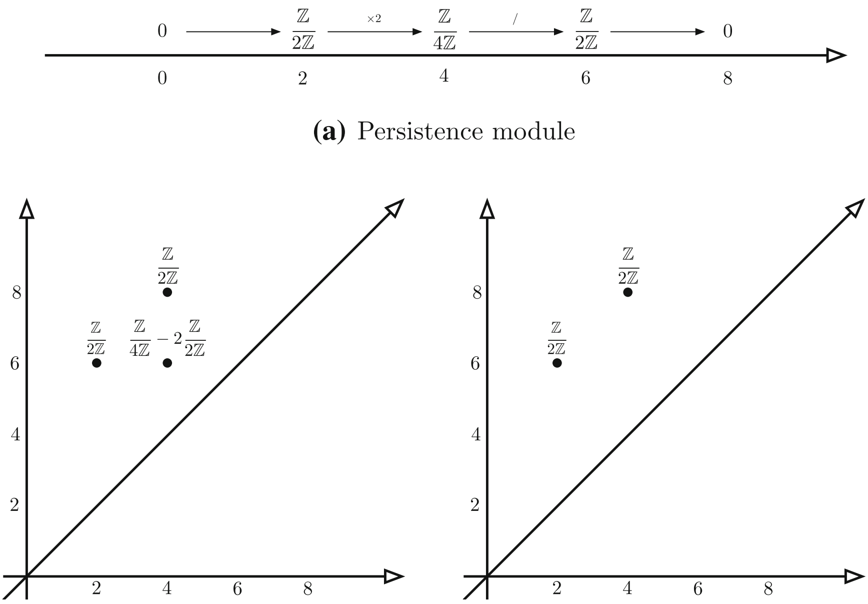
8 Stability

We now relate the interleaving distance between persistence modules to the erosion distance between their persistence diagrams.

For the first theorem, we make a simplifying assumption on C that makes it possible to chase diagrams. We assume that C is concrete and that its images are concrete. That is, C embeds into the category Set and an image of a morphism in C is the image of the corresponding set map. Note that all our examples satisfy this criteria. By the Freyd–Mitchell embedding theorem (Weibel 1995, page 28), an essentially small abelian category C embeds into the category of R-modules, for some ring R, and the image of a morphism in C is the image under the corresponding set map. Therefore, all essentially small abelian categories satisfy our criteria.

Theorem 8.1 (Semicontinuity) *Let C be an essentially small symmetric monoidal category with images. Consider an $S = \{s_1 < \cdots < s_n\}$-constructible $\mathsf{F} \in \mathsf{PMod}(\mathsf{C})$ and let*

$$\rho = \frac{1}{4} \min_{1 < i \le n} (s_i - s_{i-1}).$$



(a) Persistence module

(b) Type $\mathcal{A}$ persistence diagram

(c) Type $\mathcal{B}$ persistence diagram

Fig. 5 Here we have an example of a persistence module in $\mathsf{PMod}(\mathsf{FinAb})$ and its type $\mathcal{A}$ and type $\mathcal{B}$ persistence diagrams. This is the same example module as in Fig. 4

Let $\mathsf{G} \in \mathsf{PMod}(\mathsf{C})$ be any persistence module such that $\varepsilon = \mathsf{d}_I(\mathsf{F}, \mathsf{G}) < \rho$. For each interval $[s_i, s_j)$,

$$\mathsf{F}_{\mathcal{A}}\big([s_i, s_j)\big) = \sum_{\substack{J \in \mathrm{Dgm}: \\ [s_{i-1}+\varepsilon, s_{j+1}-\varepsilon) \supseteq J \supseteq [s_i+\varepsilon, s_j-\varepsilon) \\ \text{and } s_{i-1}+\varepsilon, s_{j+1}-\varepsilon \notin J}} G_{\mathcal{A}}(J)$$

If $i = 1$, then we interpret s_0 as any value less than s_1 and if $j = n$, then we interpret s_{n+1} as any value greater than s_n. Similarly, for each interval $[s_i, \infty)$,

$$\mathsf{F}_{\mathcal{A}}([s_i, \infty)) = \sum_{\substack{J \in \mathrm{Dgm}: \\ [s_{i-1}+\varepsilon, \infty) \supseteq J \supseteq [s_i+\varepsilon, \infty) \\ \text{and } s_{i-1}+\varepsilon \notin J}} G_{\mathcal{A}}(J)$$

Proof Let $\phi : \mathsf{F} \to \Delta^{\varepsilon}(\mathsf{G})$ and $\psi : \mathsf{G} \to \Delta^{\varepsilon}(\mathsf{F})$ be an ε-interleaving. Consider the following commutative diagram:

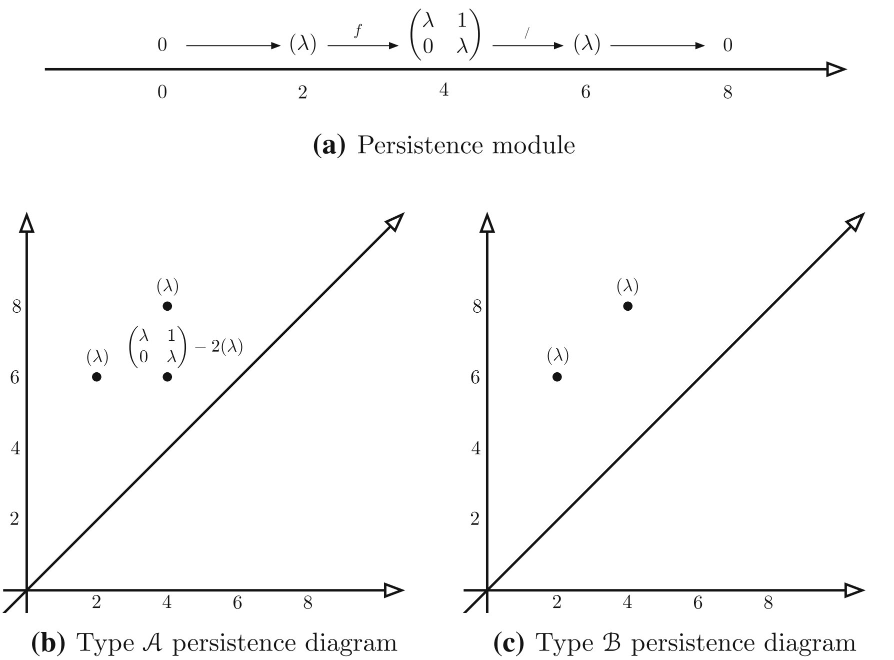


Fig. 6 Here we have an example of a persistence module in $\mathsf{PMod}(\mathsf{Ab})$ and its type $\mathcal{A}$ and type $\mathcal{B}$ persistence diagrams. The map from 4 to 6 is the quotient by the image of f

$$\begin{tikzcd}
\mathsf{F}(s_i) \arrow[r, "\mathsf{F}(s_i<s_j-\delta)"] \arrow[d, "\phi(s_i)"] & \mathsf{F}(s_j-\delta) \\
\mathsf{G}(s_i+\varepsilon) \arrow[r, "\mathsf{G}(s_i+\varepsilon<s_j-\varepsilon-\delta)"] \arrow[d, "\psi(s_i+\varepsilon)"] & \mathsf{G}(s_j-\varepsilon-\delta) \arrow[u, "\psi(s_j-\varepsilon-\delta)"'] \\
\mathsf{F}(s_i+2\varepsilon) \arrow[r, "\mathsf{F}(s_i+2\varepsilon<s_j-2\varepsilon-\delta)"] & \mathsf{F}(s_j-2\varepsilon-\delta). \arrow[u, "\phi(s_j-2\varepsilon-\delta)"']
\end{tikzcd} \tag{3}$$

By S-constructibility of F, the two vertical compositions are isomorphisms. By a diagram chase, we see that

$$d\mathsf{F}_{\mathcal{A}}\big([s_i, s_j)\big) = d\mathsf{G}_{\mathcal{A}}\big([s_i+\varepsilon, s_j-\varepsilon)\big).$$

Thus

$$\begin{aligned}
\mathsf{F}_{\mathcal{A}}([s_i, s_j]) =& d\mathsf{F}_{\mathcal{A}}([s_i, s_j)) - d\mathsf{F}_{\mathcal{A}}([s_i, s_{j+1})) \\
&+ d\mathsf{F}_{\mathcal{A}}([s_{i-1}, s_{j+1})) - d\mathsf{F}_{\mathcal{A}}([s_{i-1}, s_j)) \\
=& d\mathsf{G}_{\mathcal{A}}([s_i + \varepsilon, s_j - \varepsilon)) - d\mathsf{G}_{\mathcal{A}}([s_i + \varepsilon, s_{j+1} - \varepsilon)) \\
&+ d\mathsf{G}_{\mathcal{A}}([s_{i-1} + \varepsilon, s_{j+1} - \varepsilon)) - d\mathsf{G}_{\mathcal{A}}([s_{i-1} + \varepsilon, s_j - \varepsilon)) \\
=& \sum_{\substack{J \in \mathrm{Dgm}: \\ [s_{i-1} + \varepsilon, s_{j+1} - \varepsilon) \supseteq J \supseteq [s_i + \varepsilon, s_j - \varepsilon) \\ \text{and } s_{i-1} + \varepsilon, s_{j+1} - \varepsilon \notin J}} G_{\mathcal{A}}(J).
\end{aligned}$$

The second claim for $[s_i, \infty)$ follows by a similar argument. □

Semicontinuity is saying there is an open neighborhood of F in the metric space of persistence modules such that for each G in this open neighborhood, $\mathsf{F}_{\mathcal{A}}$ lives on in $\mathsf{G}_{\mathcal{A}}$. However, semicontinuity is unsatisfying in two interesting ways. First, the ε must be smaller than ρ which is half the injectivity radius of S in $\mathbb{R}$. Second, the claim is assymetric. The fundamental limitation here is that not all short exact sequences in C split.

Theorem 8.2 (Continuity) *Let* C *be an essentially small, concrete, abelian category. For any two persistence modules* $\mathsf{F}, \mathsf{G} \in \mathsf{PMod}(\mathsf{C})$, *we have*

$$\mathsf{d}_E(\mathsf{F}_{\mathcal{B}}, \mathsf{G}_{\mathcal{B}}) \leq \mathsf{d}_I(\mathsf{F}, \mathsf{G}).$$

Proof Let $\varepsilon = \mathsf{d}_I(\mathsf{F}, \mathsf{G})$. For each $I \in \mathsf{Dgm}$ such that $\mathsf{F}_{\mathcal{A}}(I) \neq [e]$, we must show

$$d\mathsf{F}_{\mathcal{A}} \circ \mathsf{Grow}^{\varepsilon}(I) \preceq d\mathsf{G}_{\mathcal{A}}(I)$$

and for each $I \in \mathsf{Dgm}$ such that $\mathsf{G}_{\mathcal{B}}(I) \neq [e]$, we must show

$$d\mathsf{G}_{\mathcal{A}} \circ \mathsf{Grow}^{\varepsilon}(I) \preceq d\mathsf{F}_{\mathcal{A}}(I).$$

We will prove the first inequality and the second inequality follows by simply interchanging the roles of F and G in the proof.

Suppose F is $S = \{s_1 < \cdots < s_n\}$-constructible. By constructibility, it is sufficient to show the first inequality for I of the form $[s_i + \varepsilon, s_j - \varepsilon)$ and $[s_i + \varepsilon, \infty)$. Suppose $I = [s_i + \varepsilon, s_j - \varepsilon)$. Let $\phi : \mathsf{F} \to \Delta^{\varepsilon}(\mathsf{G})$ and $\psi : \mathsf{G} \to \Delta^{\varepsilon}(\mathsf{F})$ be an ε-interleaving. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathsf{F}(s_i) \arrow[r, "\mathsf{F}(s_i < s_j - \delta)"] \arrow[d, "\phi(s_i)"] & \mathsf{F}(s_j - \delta) \\
\mathsf{G}(s_i + \varepsilon) \arrow[r, "\mathsf{G}(s_i + \varepsilon < s_j - \varepsilon - \delta)"] & \mathsf{G}(s_j - \varepsilon - \delta). \arrow[u, "\psi(s_j - \varepsilon - \delta)"]
\end{tikzcd} \tag{4}$$

By commutativity,

$$\text{im } \mathsf{F}(s_i < s_j - \delta) \cong \frac{\text{im } \mathsf{G}(s_i + \varepsilon < s_j - \varepsilon - \delta)}{\text{im } \mathsf{G}(s_i + \varepsilon < s_j - \varepsilon - \delta) \cap \ker \ \psi(s_j - \varepsilon - \delta)}.$$

Therefore

$$\begin{aligned} d\mathsf{F}_{\mathcal{B}}\big([s_i < s_j)\big) &= d\mathsf{G}(s_i + \varepsilon < s_j - \varepsilon) - [\ker \ \psi(s_j - \varepsilon - \delta)] \\ &\preceq d\mathsf{G}_{\mathcal{B}}\big([s_i + \varepsilon < s_j - \varepsilon)\big) \end{aligned}$$

This proves the claim. Suppose $I = [s_i, \infty)$. Then

$$d\mathsf{F}_{\mathcal{B}}([s_i < \infty)) \preceq d\mathsf{G}_{\mathcal{B}}([s_i + \varepsilon < \infty)).$$

by a similar commutative diagram. □

9 Concluding remarks

Torsion in data We hope our theory will allow for the study of torsion in data. For example, let $P \subset \mathbb{R}^n$ be a finite set of points. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function dependent on P, for example $f(x) = \min_{p \in P} ||x - p||_2$. Apply homology with integer coefficients to the sublevel set filtration induced by f and we have a constructible persistence module $\mathsf{F} \in \mathsf{PMod}(\mathsf{Ab})$. Its type $\mathcal{A}$ persistence diagram is measuring torsion in data and semicontinuity applies. If continuity is required, then we may look at the type $\mathcal{B}$ persistence diagram of F. However, the type $\mathcal{B}$ persistence diagram forgets all torsion. Perhaps a better approach is to apply homology with coefficients in a finite abelian group. Then the resulting persistence module is in $\mathsf{PMod}(\mathsf{FinAb})$ and its type $\mathcal{B}$ diagram encodes simple torsion.

Time series The flexibility we offer in choosing C should allow for the encoding of more structure in data. Consider time series data. Suppose $P = \{p_1, \ldots, p_k\}$ is a finite sequence of points in $\mathbb{R}^n$. There is more to P than its shape. The forward shift $p_i \to p_{i+1}$ along the sequence should induce dynamics on the shape of P at each scale. The algebraic object of study is not clear, but it will certainly have more structure than a vector space or an abelian group.

Non-constructible modules Suppose we are given an infinite set of points $P \subset \mathbb{R}^n$. Then the resulting persistence module, as constructed above, is not constructible. Is there a persistence diagram for a non-constructible persistence module?

This question is addressed by Chazal et al. (2016) for $\mathsf{C} = \mathsf{Vec}$. They define a persistence diagram for a non-constructible persistence module as a *rectangular measure* $\mu : \mathsf{Rect} \to \mathbb{N}$, where Rect is the poset of all pairs $J \supset I$ in Dgm, satisfying a certain additivity condition. Our type $\mathcal{B}$ diagram should generalize to a rectangular measure. For C abelian, we may use an argument similar to the one in the proof of Proposition 7.1 to assign an element of $\mathcal{B}(\mathsf{C})$ to each $J \supset I$ without making use of constructibility. Is this assignment a rectangular measure?

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Appendix: Krull-Schmidt

We now provide a compact treatment of Krull-Schmidt categories. The following ideas are classical and may be found in many books, for example Anderson and Fuller (1992).

A category C is *additive* if all its hom-sets are abelian, composition is bilinear, and finite products and finite coproducts are the same. The (co)product of the empty set is the *zero object* of C. Suppose C is additive.

Definition A.1 A non-zero object $a \in \mathsf{C}$ is *indecomposable* if it is not the direct sum of two non-zero objects.

Definition A.2 An additive category C is *Krull-Schmidt* if each object $a \in \mathsf{C}$ is isomorphic to a finite direct sum $a \cong a_1 \oplus a_2 \oplus \cdots \oplus a_n$ and each ring of endomorphisms $\mathsf{End}_{\mathsf{C}}(a_i)$ is *local*. That is, $0 \neq 1$ and if $f_1 + f_2 = 1$, then f_1 or f_2 is invertible.

Suppose C is Krull-Schmidt.

Proposition 9.1 *An object $a \in \mathsf{C}$ is indecomposable iff its endomorphism ring $\mathsf{End}(a)$ is local.*

Proof Suppose $a \in \mathsf{C}$ is decomposable. That is, there is an isomorphism $i : a \to a_1 \oplus a_2$ such that $a_1, a_2 \neq 0$. Define $\pi_1 : a_1 \oplus a_2 \to a_1 \oplus a_2$ as the endomorphism that sends the first factor to zero and $\pi_2 : a_1 \oplus a_2 \to a_1 \oplus a_2$ as the endomorphism that sends the second factor to zero. Then the two maps $\rho_1, \rho_2 : a \to a$, where $\rho_1 = i^{-1} \circ \pi_1 \circ i$ and $\rho_2 = i^{-1} \circ \pi_2 \circ i$, are both non-isomorphisms in $\mathsf{End}_{\mathsf{C}}(a)$. However, $\rho_0 + \rho_1 : a \to a$ is an isomorphism. We have a contradiction of locality.

Suppose $a \in \mathsf{C}$ is indecomposable. Then, by definition of a Krull-Schmidt category, $\mathsf{End}_{\mathsf{C}}(a)$ is a local ring. □

Proposition 9.2 *Each object $a \in \mathsf{C}$ is isomorphic to a finite direct sum of indecomposables.*

Proof By definition of a Krull-Schmidt category, $a \cong a_1 \oplus a_2 \oplus \cdots \oplus a_n$ where each $\mathsf{End}_{\mathsf{C}}(a_i)$ is a local ring. By Proposition 9.1, each a_i is indecomposable. □

Theorem 9.1 (Krull-Schmidt) *Suppose an object $c \in \mathsf{C}$ is isomorphic to $a_1 \oplus a_2 \oplus \cdots \oplus a_m$ and $b_1 \oplus b_2 \oplus \cdots \oplus b_n$, where each a_i and b_j are indecomposable. Then $m = n$, and there is a permutation $p : [m] \to [n]$ such that $a_i \cong b_{p(i)}$.*

Proof By definition of an additive category, we have canonical projections $\pi_i : \oplus_i a_i \to a_i$ and $\rho_j : \oplus_j b_j \to b_j$ and canonical inclusions $\mu_i : a_i \to \oplus_i a_i$ and $\nu_j : b_j \to \oplus_j b_j$. Furthermore $\mu_j \circ \pi_i$ and $\nu_j \circ \rho_i$ are the identity on a_i and b_i, respectively, iff $i = j$. Let $f : a_1 \oplus a_2 \oplus \cdots \oplus a_m \to b_1 \oplus b_2 \oplus \cdots \oplus b_n$ be an isomorphism.

Define $h_j : a_1 \to a_1$ as $h_j = \pi_1 \circ f^{-1} \circ \nu_j \circ \rho_j \circ f \circ \mu_1$. Let $h = \sum_j h_j : a_1 \to a_1$. Observe h is an isomorphism. By locality, there is an index j such that h_j is an isomorphism. This means $a_1 \cong b_j$ and we specify $p(1) = j$. Quotient by a_1 and b_j. Repeat. □

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