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BOTTLENECK STABILITY FOR GENERALIZED PERSISTENCE DIAGRAMS

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Abstract. In this paper, we extend bottleneck stability to the setting of one dimensional constructible persistence modules valued in any skeletally small abelian category.

1. Introduction

Persistent homology is a way of quantifying the topology of a function. Given a function $f : X \to \mathbb{R}$, persistence scans the homology of the sublevel sets $f^{-1}(-\infty, r]$ as r varies from $-\infty$ to ∞. As it scans, homology appears and homology disappears. This history of births and deaths is recorded as a *persistence diagram* [CSEH07] or a *barcode* [ZC05]. What makes persistence special is that the persistence diagram of f is stable to arbitrary perturbations to f. This is the celebrated *bottleneck stability* of Cohen-Steiner, Edelsbrunner, and Harer [CSEH07]. Bottleneck stability makes persistent homology a useful tool in data analysis and in pure mathematics. All of this is in the setting of vector spaces where each homology group is computed using coefficients in a field.

Fix a field k and let Vec be the category of k-vector spaces. As persistence scans the sublevel sets of f, it records its homology as a functor $\mathsf{F} : (\mathbb{R}, \leq) \to \mathsf{Vec}$ where $\mathsf{F}(r) := \mathsf{H}_*\big(f^{-1}(-\infty, r]; \mathsf{k}\big)$ and $\mathsf{F}(r \leq s) : \mathsf{F}(r) \to \mathsf{F}(s)$ is the map induced by the inclusion of the sublevel set at r into the sublevel set at s. The functor F is called the *persistence module* of f. Assuming some tameness conditions on f, the persistence diagram of F is equivalent to its barcode, but the two definitions are very different. The *barcode* of F is its list of indecomposables. This list is unique up to a permutation and furthermore, each indecomposable is an *interval persistence module* [ZC05, CdS10, CB15]. The barcode model is how most people now think about persistence. However in [CSEH07] where bottleneck stability was first proved, the persistence diagram is defined as a purely combinatorial object. The *rank function* of F assigns to each pair of values $r \leq s$ the rank of the map $\mathsf{F}(r \leq s)$. The Möbius inversion of the rank function is the *persistence diagram* of F. Remarkably, these two very different approaches to persistence give equivalent answers.

The persistence diagram of [CSEH07] generalizes to the setting of constructible persistence modules valued in any skeletally small abelian category $\mathcal{C}$ [Pat18]. The

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rank function of such a persistence module records the image of each $\mathsf{F}(r \leq s)$ as an element of the Grothendieck group of $\mathcal{C}$. Here we are using the Grothendieck group of an abelian category: this is the abelian group with one generator for each isomorphism class of objects and one relation for each short exact sequence. The persistence diagram of F is then the Möbius inversion of this rank function. A weak form of stability was shown in [Pat18]. In this paper, we prove bottleneck stability. Our proof is an adaptation of the proofs of [CSEH07] and [CdSGO16].

2. Persistence modules

Fix a skeletally small abelian category $\mathcal{C}$. By skeletally small, we mean that the collection of isomorphism classes of objects in $\mathcal{C}$ is a set. For example, $\mathcal{C}$ may be the category of finite dimensional k-vector spaces, the category of finite abelian groups, or the category of finite length R-modules. Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be the totally ordered set of real numbers with the point ∞ satisfying $p < \infty$ for all $p \in \mathbb{R}$. For any $p \in \bar{\mathbb{R}}$, we let $\infty + p = \infty$.

Definition 2.1. A *persistence module* is a functor $\mathsf{F} : \mathbb{R} \to \mathcal{C}$. Let

$$S = \{s_1 < s_2 < \cdots < s_k < \infty\} \subseteq \bar{\mathbb{R}}$$

be a finite subset. A persistence module F is *S-constructible* if it satisfies the following conditions:

- For $p \leq q < s_1$, $\mathsf{F}(p \leq q) : 0 \to 0$ is the zero map.
- For $s_i \leq p \leq q < s_{i+1}$, $\mathsf{F}(p \leq q)$ is an isomorphism.
- For $s_k \leq p \leq q < \infty$, $\mathsf{F}(p \leq q)$ is an isomorphism.

For example, let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold M. The function f filters M by sublevel sets $M^f_{\leq r} := \{p \in M \mid f(p) \leq r\}$. For every $r \leq s$, $M^f_{\leq r} \subseteq M^f_{\leq s}$. Now apply homology with coefficients in a finite abelian group. The result is a persistence module of finite abelian groups that is constructible with respect to the set of critical values of f union $\{\infty\}$. If one applies homology with coefficients in a field k, then the result is a constructible persistence module of finite dimensional k-vector spaces. In topological data analysis, one usually starts with a constructible filtration of a finite simplicial complex.

There is a natural distance between persistence modules called the *interleaving distance* [CCSG+09]. For any $\varepsilon \geq 0$, let $\mathbb{R} \times_\varepsilon \{0,1\}$ be the poset $\big(\mathbb{R} \times \{0\}\big) \cup \big(\mathbb{R} \times \{1\}\big)$ where $(p,t) \leq (q,s)$ if

- $t = s$ and $p \leq q$, or
- $t \neq s$ and $p + \varepsilon \leq q$.

Let $\iota_0, \iota_1 : \mathbb{R} \hookrightarrow \mathbb{R} \times_\varepsilon \{0,1\}$ be the poset maps $\iota_0 : p \mapsto (p,0)$ and $\iota_1 : p \mapsto (p,1)$.

Definition 2.2. An *ε-interleaving* between two constructible persistence modules F and G is a functor Φ that makes the following diagram commute up to a natural

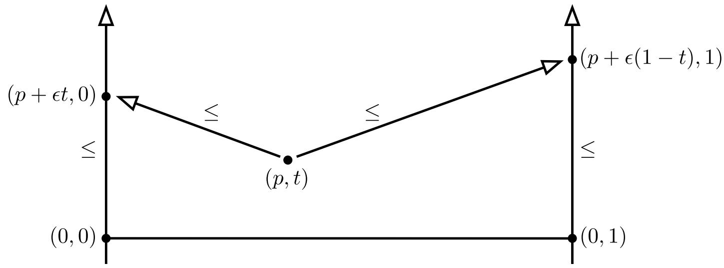
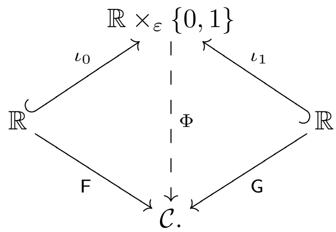


FIGURE 1. An illustration of the poset relation on $\mathbb{R} \times_\varepsilon [0,1]$.

isomorphism:

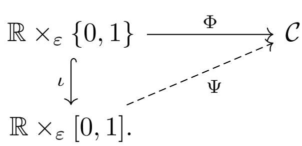
(1)



Two constructible persistence modules F and G are *ε-interleaved* if there is an ε-interleaving between them. The *interleaving distance* $d_I(\mathsf{F}, \mathsf{G})$ between F and G is the infimum over all $\varepsilon \geq 0$ such that F and G are ε-interleaved. This infimum is attained since both F and G are constructible. If F and G are not interleaved, then we let $d_I(\mathsf{F}, \mathsf{G}) = \infty$.

Proposition 2.3 (Interpolation [BdSN17]). *Let F and G be two ε-interleaved constructible persistence modules. Then there exists a one-parameter family of constructible persistence modules $\{\mathsf{K}_t\}_{t\in[0,1]}$ such that $\mathsf{K}_0 \cong \mathsf{F}$, $\mathsf{K}_1 \cong \mathsf{G}$, and $d_I(\mathsf{K}_t, \mathsf{K}_s) \leq \varepsilon|t-s|$.*

Proof. Let F and G be ε-interleaved by Φ as in Definition 2.2. Define $\mathbb{R}\times_\varepsilon[0,1]$ as the poset with the underlying set $\mathbb{R}\times[0,1]$ and $(p,t) \leq (q,s)$ whenever $p+\varepsilon|t-s| \leq q$. Note that $\mathbb{R}\times_\varepsilon\{0,1\}$ naturally embeds into $\mathbb{R}\times_\varepsilon[0,1]$ via $\iota : (p,t) \mapsto (p,t)$. See Figure 1. Finding $\{\mathsf{K}_t\}_{t\in[0,1]}$ is equivalent to finding a functor Ψ that makes the following diagram commute up to a natural isomorphism:



This functor Ψ is the right Kan extension of Φ along ι for which we now give an explicit construction. For convenience, let $\mathsf{P} := \mathbb{R}\times_\varepsilon\{0,1\}$ and $\mathsf{Q} := \mathbb{R}\times_\varepsilon[0,1]$. For $(p,t) \in \mathsf{Q}$, let $\mathsf{P}\uparrow(p,t)$ be the subposet of P consisting of all elements $(p',t') \in \mathsf{P}$ such that $(p,t) \leq (p',t')$. The poset $\mathsf{P}\uparrow(p,t)$, for any $p \in \mathbb{R}$ and $t \notin \{0,1\}$, has two minimal elements: $(p+\varepsilon t, 0)$ and $\big(p+\varepsilon(1-t), 1\big)$. For $t \in \{0,1\}$, the poset $\mathsf{P}\uparrow(p,t)$ has one minimal element, namely (p,t). Let $\Psi\big((p,t)\big) := \lim \Phi|_{\mathsf{P}\uparrow(p,t)}$. For $(p,t) \leq (q,s)$, the poset $\mathsf{P}\uparrow(q,s)$ is a subposet of $\mathsf{P}\uparrow(p,t)$. This subposet relation

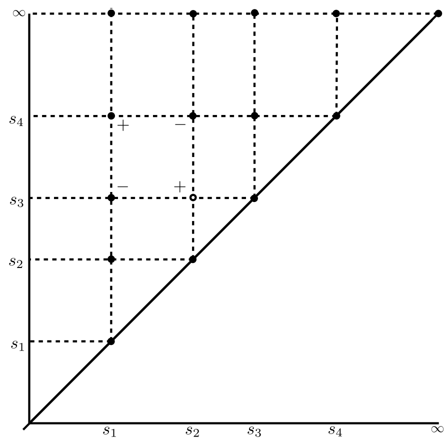


FIGURE 2. An interval $I = [p, q)$ is visualized as the point (p, q) in the plane. The poset Dgm is therefore the set of points in the plane on and above the diagonal. In this example, $S = \{s_1 < s_2 < s_3 < s_4 < \infty\}$ and $\mathsf{Dgm}(S)$ is its set of grid points. Given an S-constructible persistence module F and an interval $[s_2, s_3)$, $\tilde{\mathsf{F}}\big([s_2, s_3)\big) = d\mathsf{F}\big([s_2, s_3)\big) - d\mathsf{F}\big([s_2, s_4)\big) + d\mathsf{F}\big([s_1, s_4)\big) - d\mathsf{F}\big([s_1, s_3)\big)$.

allows us to define the morphism $\Psi\big((p,t) \leq (q,s)\big)$ as the universal morphism between the two limits. Note that $\Psi\big((p,0)\big)$ is isomorphic to $\mathsf{F}(p)$ and $\Psi\big((p,1)\big)$ is isomorphic to $\mathsf{G}(p)$.

We now argue that each persistence module $\mathsf{K}_t := \Psi(\cdot, t)$ is constructible. As we increase p while keeping t fixed, the limit $\mathsf{K}_t(p)$ changes only when one of the two minimal objects of $\mathsf{P} \uparrow (p,t)$ changes isomorphism type. Since F and G are constructible, there are only finitely many such changes to the isomorphism type of $\mathsf{K}_t(p)$. □

3. Persistence diagrams

Fix an abelian group $\mathcal{G}$ with a translation invariant partial ordering $\preceq$. That is, for all $a, b, c \in \mathcal{G}$, if $a \leq b$, then $a + c \leq b + c$. Roughly speaking, a persistence diagram is the assignment to each interval of the real line an element of $\mathcal{G}$. In our setting, only finitely many intervals will have a nonzero value.

Definition 3.1. Let Dgm be the *poset of intervals* consisting of the following data:

- The objects of Dgm are intervals $[p, q) \subseteq \bar{\mathbb{R}}$ where $p \leq q$.
- The ordering is inclusion $[p_2, q_2) \subseteq [p_1, q_1)$.

Given a finite set $S = \{s_1 < s_2 < \cdots < s_k < \infty\} \subseteq \bar{\mathbb{R}}$, we use $\mathsf{Dgm}(S)$ to denote the subposet of Dgm consisting of all intervals $[p, q)$ with $p, q \in S$. The *diagonal* $\Delta \subseteq \mathsf{Dgm}$ is the subset of intervals of the form $[p, p)$. See Figure 2.

Definition 3.2. A *persistence diagram* is a nonnegative map $Y : \mathsf{Dgm} \to \mathcal{G}$ with finite support. That is, $0 \preceq Y(I)$ for all $I \in \mathsf{Dgm}$ and $Y(I) \neq 0$ for finitely many $I \in \mathsf{Dgm}$.

We now introduce the bottleneck distance between persistence diagrams.

Definition 3.3. A *matching* between two persistence diagrams $Y_1, Y_2 : \mathsf{Dgm} \to \mathcal{G}$ is a nonnegative map $\gamma : \mathsf{Dgm} \times \mathsf{Dgm} \to \mathcal{G}$ satisfying

$$Y_1(I) = \sum_{J \in \mathsf{Dgm}} \gamma(I, J) \text{ for all } I \in \mathsf{Dgm} \backslash \Delta,$$

$$Y_2(J) = \sum_{I \in \mathsf{Dgm}} \gamma(I, J) \text{ for all } J \in \mathsf{Dgm} \backslash \Delta.$$

The *norm* of a matching γ is

$$||\gamma|| := \max_{\left\{ I=[p_1,q_1), J=[p_2,q_2) \,\middle|\, \gamma(I,J) \neq 0 \right\}} \left\{ |p_1 - p_2|, |q_1 - q_2| \right\}.$$

If both $q_1 = q_2 = \infty$, then $|q_1 - q_2| = 0$. If just one of them is ∞, then $|q_1 - q_2| = \infty$. The *bottleneck distance* between two persistence diagrams $Y_1, Y_2 : \mathsf{Dgm} \to \mathcal{G}$ is

$$d_B(Y_1, Y_2) := \inf_\gamma ||\gamma||$$

over all matchings γ between Y_1 and Y_2. This infimum is attained since persistence diagrams have finite support.

Example 3.4. Let $Y_1 : \mathsf{Dgm} \to \mathbb{Z}$ be the persistence diagram $Y_1(0, 6) = 2$, $Y_1(4, 6) = 1$, and zero elsewhere. Let $Y_2 : \mathsf{Dgm} \to \mathbb{Z}$ be the persistence digram $Y_2(1, 5) = 1$, $Y_2(1, 4) = 1$, and zero elsewhere. In this case, there is just one matching γ that minimizes the norm: $\gamma\big((0, 6), (1, 5)\big) = 1$, $\gamma\big((0, 6), (1, 4)\big) = 1$, $\gamma\big((4, 6), (5, 5)\big) = 1$, and zero elsewhere. The norm of γ is 2 and therefore $d_B(Y_1, Y_2) = 2$.

4. Diagram of a module

We now describe the construction of a persistence diagram from a constructible persistence module. Fix a skeletally small abelian category $\mathcal{C}$.

Definition 4.1. The *Grothendieck group* $\mathcal{G}(\mathcal{C})$ of $\mathcal{C}$ is the abelian group with one generator for each isomorphism class $[a]$ of objects $a \in \mathsf{ob}\ \mathcal{C}$ and one relation $[b] = [a] + [c]$ for each short exact sequence $0 \to a \to b \to c \to 0$. The Grothendieck group has a natural translation invariant partial ordering where $[a] \preceq [b]$ if there is an object $c \in \mathcal{C}$ such that $[b] - [a] = [c]$. For each $a \hookrightarrow b$, we have $a \oplus c \hookrightarrow b \oplus c$ for any object c in $\mathcal{C}$. This makes $\preceq$ a translation invariant partial ordering.

Example 4.2. Here are three examples of $\mathcal{C}$ with their Grothendieck groups.

- Let Vec be the category of finite dimensional k-vector spaces for some fixed field k. Every finite dimensional k-vector space is isomorphic to k^n for some natural number $n \geq 0$. This means that the free abelian group generated by the set of isomorphism classes in Vec is $\bigoplus_n \mathbb{Z}$ over all $n \geq 0$. Since every short exact sequence in Vec splits, the only relations are of the form $[A] + [B] = [C]$ whenever $A \oplus B \cong C$. Therefore $\mathcal{G}(\mathsf{Vec}) \cong \mathbb{Z}$ where the translation invariant partial ordering $\preceq$ is the usual total ordering on the integers.
- Let FinAb be the category of finite abelian groups. A finite abelian group is isomorphic to

$$\frac{\mathbb{Z}}{p_1^{n_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{n_k}\mathbb{Z}}$$

where each p_i is prime. The free abelian group generated by the set of isomorphism classes in FinAb is $\bigoplus_{(p,n)} \mathbb{Z}$ over all pairs (p,n) where p is prime and $n \geq 0$ a natural number. Every primary cyclic group $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$ fits into a short exact sequence

$$0 \longrightarrow \frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}} \xrightarrow{\times p} \frac{\mathbb{Z}}{p^n\mathbb{Z}} \xrightarrow{/} \frac{\mathbb{Z}}{p\mathbb{Z}} \longrightarrow 0$$

giving rise to a relation $\left[\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right] = \left[\frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}}\right] + \left[\frac{\mathbb{Z}}{p\mathbb{Z}}\right]$. By induction, $\left[\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right] = n\left[\frac{\mathbb{Z}}{p\mathbb{Z}}\right]$. Therefore $\mathcal{G}(\mathsf{FinAb}) \cong \bigoplus_p \mathbb{Z}$ where p is prime. For two elements $[a],[b] \in \mathcal{G}(\mathsf{FinAb})$, $[a] \preceq [b]$ if the multiplicity of each prime factor of $[a]$ is at most the multiplicity of each prime factor of $[b]$.

- Let Ab be the category of finitely generated abelian groups. A finitely generated abelian group is isomorphic to

$$\mathbb{Z}^m \oplus \frac{\mathbb{Z}}{p_1^{n_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{n_k}\mathbb{Z}}$$

where each p_i is prime. The free abelian group generated by the set of isomorphism classes in Ab is $\bigoplus_m \mathbb{Z} \oplus \bigoplus_{(p,n)} \mathbb{Z}$ over all natural numbers $m \geq 0$ and over all pairs (p,n) where p is prime and $n \geq 0$ a natural number. In addition to the short exact sequences in FinAb, we have

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{/} \frac{\mathbb{Z}}{p\mathbb{Z}} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}^m \longrightarrow \mathbb{Z}^{m+n} \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

giving rise to the relations $\left[\frac{\mathbb{Z}}{p\mathbb{Z}}\right] = [0]$ and $[\mathbb{Z}^m] + [\mathbb{Z}^n] = [\mathbb{Z}^{m+n}]$. Therefore $\mathcal{G}(\mathsf{Ab}) \cong \mathbb{Z}$ where $\preceq$ is the usual total ordering on the integers. Unfortunately all torsion is lost.

Given a constructible persistence module, we now record the images of all its maps as elements of the Grothendieck group.

Definition 4.3. Let $S = \{s_1 < \cdots < s_k < \infty\}$ be a finite set and let F be an S-constructible persistence module valued in $\mathcal{C}$. The *rank function* of F is the map $d\mathsf{F} : \mathsf{Dgm} \to \mathcal{G}(\mathcal{C})$ defined as follows:

- For $I = [p, s_i)$ where $p \neq s_i$, let $d\mathsf{F}(I) := \big[\operatorname{im} \mathsf{F}(p \leq s_i - \delta)\big]$ for some sufficiently small $\delta > 0$.
- For $I = [p, \infty)$, let $d\mathsf{F}(I) := \big[\operatorname{im} \mathsf{F}(p \leq s_k)\big]$.
- For all other $I = [p,q)$, let $d\mathsf{F}(I) := \big[\operatorname{im} \mathsf{F}(p \leq q)\big]$.

Note that for any $[p,q) \in \mathsf{Dgm}$, $d\mathsf{F}\big([p,q)\big)$ equals $d\mathsf{F}(I)$ where I is the smallest interval in $\mathsf{Dgm}(S)$ containing $[p,q)$. This means that $d\mathsf{F}$ is uniquely determined by its value on $\mathsf{Dgm}(S)$.

Proposition 4.4. *Let F be a constructible persistence module valued in a skeletally small abelian category $\mathcal{C}$. Then its rank function $d\mathsf{F} : \mathsf{Dgm} \to \mathcal{G}(\mathcal{C})$ is a poset reversing map. That is, for any pair of intervals $[p_2,q_2) \subseteq [p_1,q_1)$, $d\mathsf{F}\big([p_1,q_1)\big) \preceq d\mathsf{F}\big([p_2,q_2)\big)$.*

Proof. Suppose F is $S = \{s_1 < \cdots < s_k < \infty\}$-constructible. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathsf{F}(p_1) & \xrightarrow{e:=\mathsf{F}(p_1 \leq p_2)} & \mathsf{F}(p_2) \\ {\scriptstyle h:=\mathsf{F}(p_1\leq q_1)}\downarrow & & \downarrow{\scriptstyle f:=\mathsf{F}(p_2\leq q_2)} \\ \mathsf{F}(q_1) & \xleftarrow[g:=\mathsf{F}(q_2\leq q_1)]{} & \mathsf{F}(q_2). \end{array}$$

We may assume $q_1, q_2 \notin S$. If this is not the case, replace q_1 and/or q_2 in the above diagram with $q_1 - \delta$ and $q_2 - \delta$ for some sufficiently small $\delta > 0$. We have $d\mathsf{F}\big([p_1, q_1)\big) = [\mathsf{im}\, h]$ and $d\mathsf{F}\big([p_2, q_2)\big) = [\mathsf{im}\, f]$. Let $I := \mathsf{im}\,(f \circ e)$ and $K := I \cap \ker g$. Then $K \hookrightarrow I \hookrightarrow \mathsf{im}\, f$ and $\mathsf{im}\, h \cong {}^{I}/_{K}$. This means $[I] \preceq [\mathsf{im}\ f]$ and $[\mathsf{im}\ h] = [I] - [K]$ implying $[\mathsf{im}\ h] \preceq [I]$. Therefore $d\mathsf{F}\big([p_1, q_1)\big) \preceq d\mathsf{F}\big([p_2, q_2)\big)$. ☐

Given the rank function $d\mathsf{F} : \mathsf{Dgm} \to \mathcal{G}(\mathcal{C})$ of an S-constructible persistence module F, there is a unique map $\tilde{\mathsf{F}} : \mathsf{Dgm} \to \mathcal{G}$ such that

$$d\mathsf{F}(I) = \sum_{J \in \mathsf{Dgm} : J \supseteq I} \tilde{\mathsf{F}}(J) \tag{2}$$

for each $I \in \mathsf{Dgm}$. This equation is the *Möbius inversion formula.* For each $I = [s_i, s_j)$ in $\mathsf{Dgm}(S)$,

$$\tilde{\mathsf{F}}(I) = d\mathsf{F}\big([s_i, s_j)\big) - d\mathsf{F}\big([s_i, s_{j+1})\big) + d\mathsf{F}\big([s_{i-1}, s_{j+1})\big) - d\mathsf{F}\big([s_{i-1}, s_j)\big). \tag{3}$$

For each $I = [s_i, \infty)$ in $\mathsf{Dgm}(S)$,

$$\tilde{\mathsf{F}}(I) = d\mathsf{F}\big([s_i, \infty)\big) - d\mathsf{F}\big([s_{i-1}, \infty)\big). \tag{4}$$

For all other $I \in \mathsf{Dgm}$, $\tilde{\mathsf{F}}(I) = 0$. Here we have to be careful with our indices. It is possible s_{j+1} or s_{i-1} is not in S. If s_{j+1} is not in S, let $s_{j+1} = \infty$. If s_{i-1} is not in S, let s_{i-1} be any value strictly less than s_1. We call $\tilde{\mathsf{F}}$ the *Möbius inversion* of $d\mathsf{F}$.

Definition 4.5. The *persistence diagram of a constructible persistence module* F is the Möbius inversion $\tilde{\mathsf{F}}$ of its rank function $d\mathsf{F}$.

The Grothendieck group of $\mathcal{C}$ has one relation for each short exact sequence in $\mathcal{C}$. These relations ensure that the persistence diagram of a persistence module is positive which plays a key role in the proof of Lemma 5.3.

Proposition 4.6 (Positivity [Pat18])**.** *Let* F *be a constructible persistence module valued in a skeletally small abelian category* $\mathcal{C}$*. Then for any* $I \in \mathsf{Dgm}$*, we have* $[0] \preceq \tilde{\mathsf{F}}(I)$*.*

5. Stability

We now begin the task of proving bottleneck stability. Throughout this section, persistence modules are valued in a fixed skeletally small abelian category $\mathcal{C}$.

Definition 5.1. For an interval $I = [p, q)$ in Dgm and a value $\varepsilon \geq 0$, let

$$\Box_\varepsilon I := \big\{ [r, s) \in \mathsf{Dgm} \,\big|\, p - \varepsilon < r \leq p + \varepsilon \text{ and } q - \varepsilon \leq s < q + \varepsilon \big\}$$

be the subposet of Dgm consisting of intervals ε-close to I. If I is too close to the diagonal, that is, if $q - \varepsilon \leq p + \varepsilon$, then we let $\Box_\varepsilon I$ be empty. We call $\Box_\varepsilon I$ the ε-*box* around I. See Figure 3. Note that if $q = \infty$, then $\Box_\varepsilon I = \big\{ [r, \infty) \,\big|\, p - \varepsilon < r \leq p + \varepsilon \big\}$.

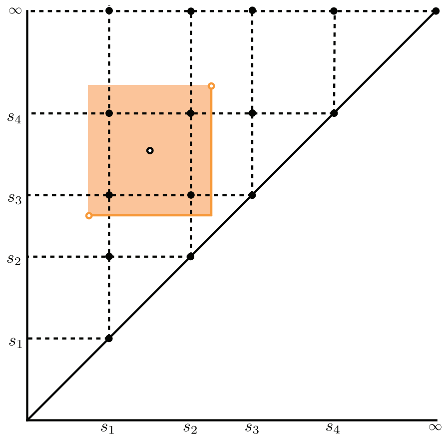


FIGURE 3. The shaded area is the box $\square_\varepsilon I$ where I is the circle. Note that $\square_\varepsilon I$ is closed on the bottom and right, and it is open on the top and left.

Lemma 5.2. *Let* F *be an* S*-constructible persistence module,* $I = [p, q)$*, and* $\varepsilon > 0$*. If* $\square_\varepsilon I$ *is nonempty, then*

$$\sum_{J \in \square_\varepsilon I} \tilde{F}(J) = d\mathsf{F}\big([p+\varepsilon, q-\varepsilon)\big) - d\mathsf{F}\big([p+\varepsilon, q+\varepsilon)\big) + d\mathsf{F}\big([p-\varepsilon, q+\varepsilon)\big) - d\mathsf{F}\big([p-\varepsilon, q-\varepsilon)\big)$$

whenever $q \neq \infty$ *and*

$$\sum_{J \in \square_\varepsilon I} \tilde{F}(J) = d\mathsf{F}\big([p+\varepsilon, \infty)\big) - d\mathsf{F}\big([p-\varepsilon, \infty)\big)$$

whenever $q = \infty$*.*

Proof. Both equalities follow easily from the Möbius inversion formula; see equation (2). If $q \neq \infty$, then

$$\begin{aligned} &\sum_{J \in \square_\varepsilon I} \tilde{\mathsf{F}}(J) \\ &\quad = \sum_{\substack{J \in \mathsf{Dgm}: \\ J \supseteq [p+\varepsilon, q-\varepsilon)}} \tilde{\mathsf{F}}(J) - \sum_{\substack{J \in \mathsf{Dgm}: \\ J \supseteq [p+\varepsilon, q+\varepsilon)}} \tilde{\mathsf{F}}(J) + \sum_{\substack{J \in \mathsf{Dgm}: \\ J \supseteq [p-\varepsilon, q+\varepsilon)}} \tilde{\mathsf{F}}(J) - \sum_{\substack{J \in \mathsf{Dgm}: \\ J \supseteq [p-\varepsilon, q-\varepsilon)}} \tilde{\mathsf{F}}(J) \\ &\quad = d\mathsf{F}\big([p+\varepsilon, q-\varepsilon)\big) - d\mathsf{F}\big([p+\varepsilon, q+\varepsilon)\big) - d\mathsf{F}\big([p-\varepsilon, q+\varepsilon)\big) + d\mathsf{F}\big([p-\varepsilon, q-\varepsilon)\big). \end{aligned}$$

If $q = \infty$, then

$$\begin{aligned} \sum_{J \in \square_\varepsilon I} \tilde{\mathsf{F}}(J) &= \sum_{\substack{J \in \mathsf{Dgm}: \\ J \supseteq [p+\varepsilon, \infty)}} \tilde{\mathsf{F}}(J) - \sum_{\substack{J \in \mathsf{Dgm}: \\ J \supseteq [p-\varepsilon, \infty)}} \tilde{\mathsf{F}}(J) \\ &= d\mathsf{F}\big([p+\varepsilon, \infty)\big) - d\mathsf{F}\big([p-\varepsilon, \infty)\big). \end{aligned}$$

□

Lemma 5.3 (Box lemma)**.** *Let* F *and* G *be two* ε*-interleaved constructible persistence modules,* $I \in \mathsf{Dgm}$*, and* $\mu > 0$*. Then*

$$\sum_{J \in \square_\mu I} \tilde{\mathsf{F}}(J) \preceq \sum_{J \in \square_{\mu+\varepsilon} I} \tilde{\mathsf{G}}(J)$$

whenever $\square_{\mu+\varepsilon} I$ *is nonempty.*

Proof. Suppose F and G are ε-interleaved by Φ in Diagram 1. Define $\varphi_r : \mathsf{F}(r) \to \mathsf{G}(r+\varepsilon)$ as $\Phi\big((r,0) \leq (r+\varepsilon,1)\big)$ and define $\psi_r : \mathsf{G}(r) \to \mathsf{F}(r+\varepsilon)$ as $\Phi\big((r,1) \leq (r+\varepsilon,0)\big)$.

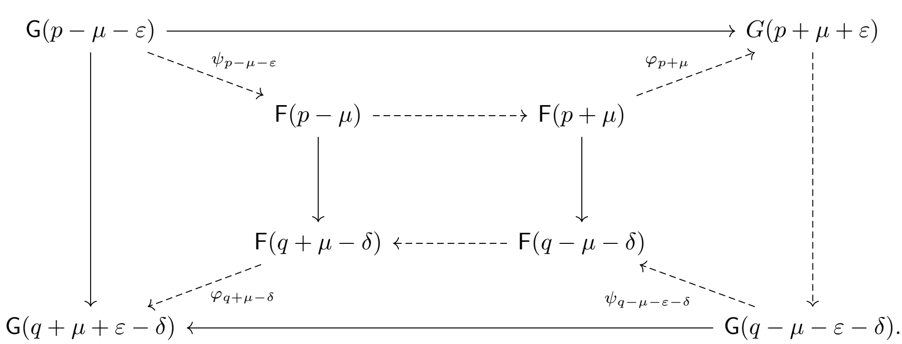
Suppose $I = [p,q)$ where $q \neq \infty$. By Lemma 5.2,

$$\begin{aligned}
\sum_{J \in \square_\mu I} \tilde{F}(J) &= d\mathsf{F}\big([p+\mu, q-\mu)\big) - d\mathsf{F}\big([p+\mu, q+\mu)\big) \\
&\quad + d\mathsf{F}\big([p-\mu, q+\mu)\big) - d\mathsf{F}\big([p-\mu, q-\mu)\big), \\
\sum_{J \in \square_{\mu+\varepsilon} I} \tilde{G}(J) &= d\mathsf{G}\big([p+\mu+\varepsilon, q-\mu-\varepsilon)\big) - d\mathsf{G}\big([p+\mu+\varepsilon, q+\mu+\varepsilon)\big) \\
&\quad + d\mathsf{G}\big([p-\mu-\varepsilon, q+\mu+\varepsilon)\big) - d\mathsf{G}\big([p-\mu-\varepsilon, q-\mu-\varepsilon)\big).
\end{aligned}$$

Choose a sufficiently small $\delta > 0$ so that we have the following equalities:

$$\begin{aligned}
d\mathsf{F}\big([p+\mu, q-\mu)\big) &= \big[\mathsf{im}\ \mathsf{F}(p+\mu < q-\mu-\delta)\big], \\
d\mathsf{F}\big([p+\mu, q+\mu)\big) &= \big[\mathsf{im}\ \mathsf{F}(p+\mu < q+\mu-\delta)\big], \\
d\mathsf{F}\big([p-\mu, q+\mu)\big) &= \big[\mathsf{im}\ \mathsf{F}(p-\mu < q+\mu-\delta)\big], \\
d\mathsf{F}\big([p-\mu, q-\mu)\big) &= \big[\mathsf{im}\ \mathsf{F}(p-\mu < q-\mu-\delta)\big], \\
d\mathsf{G}\big([p+\mu+\varepsilon, q-\mu-\varepsilon)\big) &= \big[\mathsf{im}\ \mathsf{G}(p+\mu+\varepsilon < q-\mu-\varepsilon-\delta)\big], \\
d\mathsf{G}\big([p+\mu+\varepsilon, q+\mu+\varepsilon)\big) &= \big[\mathsf{im}\ \mathsf{G}(p+\mu+\varepsilon < q+\mu+\varepsilon-\delta)\big], \\
d\mathsf{G}\big([p-\mu-\varepsilon, q+\mu+\varepsilon)\big) &= \big[\mathsf{im}\ \mathsf{G}(p-\mu-\varepsilon < q+\mu+\varepsilon-\delta)\big], \\
d\mathsf{G}\big([p-\mu-\varepsilon, q-\mu-\varepsilon)\big) &= \big[\mathsf{im}\ \mathsf{F}(p-\mu-\varepsilon < q-\mu-\varepsilon-\delta)\big].
\end{aligned}$$

Consider the following commutative diagram where the horizontal and vertical arrows are the appropriate morphisms from F and G:



Choose two values $a < b$ such that $a + \mu + \varepsilon < b - \mu - \varepsilon$ and let

$$T := \{a-\mu-\varepsilon < a-\mu < a+\mu < a+\mu+\varepsilon < c < b-\mu-\varepsilon < b-\mu < b+\mu < \infty\} \subseteq \bar{\mathbb{R}}.$$

Let $\mathsf{H} : \mathbb{R} \to \mathcal{C}$ be the T-constructible persistence module determined by the following diagram:

$$\begin{array}{ccccc}
\mathsf{H}(a-\mu-\varepsilon) = \mathsf{G}(p-\mu-\varepsilon) & \longrightarrow & \mathsf{H}(a-\mu) = \mathsf{F}(p-\mu) & \longrightarrow & \mathsf{H}(a+\mu) = \mathsf{F}(p+\mu) \\
 & & & & \downarrow \\
\mathsf{H}(b-\mu-\varepsilon) = \mathsf{F}(q-\mu-\delta) & \longleftarrow & \mathsf{H}(c) = \mathsf{G}(q-\mu-\varepsilon-\delta) & \longleftarrow & \mathsf{H}(a+\mu+\varepsilon) = \mathsf{G}(p+\mu+\varepsilon) \\
\downarrow & & & & \\
\mathsf{H}(b-\mu) = \mathsf{F}(q+\mu-\delta) & \longrightarrow & \mathsf{H}(b+\mu) = \mathsf{G}(q+\mu+\varepsilon-\delta). & &
\end{array}$$

Here the value of H is given on each value in T and morphisms between adjacent objects are the dashed arrows in the above commutative diagram. For example, for all $a+\mu+\varepsilon \leq r < c$, $\mathsf{H}(r) = \mathsf{G}(p+\mu+\varepsilon)$ and $\mathsf{H}(a+\mu+\varepsilon \leq r) = \mathsf{id}$. The morphism $\mathsf{H}(c \leq b-\mu-\varepsilon)$ is $\psi_{q-\mu-\varepsilon-\delta}$. We have the following equalities:

$$\begin{aligned}
\big[\mathrm{im}\ \mathsf{F}(p+\mu < q-\mu-\delta)\big] &= d\mathsf{H}\big([a+\mu, b-\mu)\big),\\
\big[\mathrm{im}\ \mathsf{F}(p+\mu < q+\mu-\delta)\big] &= d\mathsf{H}\big([a+\mu, b+\mu)\big),\\
\big[\mathrm{im}\ \mathsf{F}(p-\mu < q+\mu-\delta)\big] &= d\mathsf{H}\big([a-\mu, b+\mu)\big),\\
\big[\mathrm{im}\ \mathsf{F}(p-\mu < q-\mu-\delta)\big] &= d\mathsf{H}\big([a-\mu, b-\mu)\big),\\
\big[\mathrm{im}\ \mathsf{G}(p+\mu+\varepsilon < q-\mu-\varepsilon-\delta)\big] &= d\mathsf{H}\big([a+\mu+\varepsilon, b-\mu-\varepsilon)\big),\\
\big[\mathrm{im}\ \mathsf{G}(p+\mu+\varepsilon < q+\mu+\varepsilon-\delta)\big] &= d\mathsf{H}\big([a+\mu+\varepsilon, b+\mu+\varepsilon)\big),\\
\big[\mathrm{im}\ \mathsf{G}(p-\mu-\varepsilon < q+\mu+\varepsilon-\delta)\big] &= d\mathsf{H}\big([a-\mu-\varepsilon, b+\mu+\varepsilon)\big),\\
\big[\mathrm{im}\ \mathsf{G}(p-\mu-\varepsilon < q-\mu-\varepsilon-\delta)\big] &= d\mathsf{H}\big([a-\mu-\varepsilon, b-\mu-\varepsilon)\big).
\end{aligned}$$

By Lemma 5.2 along with the above substitutions, we have

$$\sum_{J \in \square_\mu [a,b)} \tilde{\mathsf{H}}(J) = \sum_{J \in \square_\mu I} \tilde{\mathsf{F}}(J),$$

$$\sum_{J \in \square_{\mu+\varepsilon} [a,b)} \tilde{\mathsf{H}}(J) = \sum_{J \in \square_{\mu+\varepsilon} I} \tilde{G}(J).$$

By the inclusion $\square_\mu[a,b) \subseteq \square_{\mu+\varepsilon}[a,b)$ along with Proposition 4.6, we have

$$\sum_{J \in \square_\mu [a,b)} \tilde{\mathsf{H}}(J) \preceq \sum_{J \in \square_{\mu+\varepsilon} [a,b)} \tilde{\mathsf{H}}(J).$$

This proves the statement.

Suppose $I = [p, \infty)$ and assume F and G are both $\{s_1 < \cdots < s_k < \infty\}$-constructible. By Lemma 5.2,

$$\begin{aligned}
\sum_{J \in \square_\mu I} \tilde{F}(J) &= d\mathsf{F}\big([p+\mu, \infty)\big) - d\mathsf{F}\big([p-\mu, \infty)\big),\\
\sum_{J \in \square_{\mu+\varepsilon} I} \tilde{G}(J) &= d\mathsf{G}\big([p+\mu+\varepsilon, \infty)\big) - d\mathsf{G}\big([p-\mu-\varepsilon, \infty)\big).
\end{aligned}$$

Choose a $z \in \mathbb{R}$ sufficiently large. We have the following equalities:

$$\begin{aligned} d\mathsf{F}\big([p+\mu,\infty)\big) &= \big[\mathsf{im}\ \mathsf{F}(p+\mu \le z)\big], \\ d\mathsf{F}\big([p-\mu,\infty)\big) &= \big[\mathsf{im}\ \mathsf{F}(p-\mu \le z)\big], \\ d\mathsf{G}\big([p+\mu+\varepsilon,\infty)\big) &= \big[\mathsf{im}\ \mathsf{G}(p+\mu+\varepsilon \le z)\big], \\ d\mathsf{G}\big([p-\mu-\varepsilon,\infty)\big) &= \big[\mathsf{im}\ \mathsf{G}(p-\mu-\varepsilon \le z)\big]. \end{aligned}$$

Consider the following commutative diagram where the vertical and horizontal arrows are the appropriate morphisms from F and G:

$$\begin{array}{ccccccc} \mathsf{G}(p-\mu-\varepsilon) & \xrightarrow{\psi_{p-\mu-\varepsilon}} & \mathsf{F}(p-\mu) & \dashrightarrow & \mathsf{F}(p+\mu) & \xrightarrow{\varphi_{p+\mu}} & \mathsf{G}(p+\mu+\varepsilon) \\ & & & & \downarrow & & \downarrow \\ & & & & \mathsf{F}(z+\varepsilon) & \xleftarrow[\psi_z]{\cong} & \mathsf{G}(z). \end{array}$$

Note that ψ_z is an isomorphism. Let

$$T := \{-\mu-\varepsilon < -\mu < \mu < \mu+\varepsilon < z < \infty\} \subseteq \bar{\mathbb{R}}.$$

Let $\mathsf{H} : \mathbb{R} \to \mathcal{C}$ be the T-constructible persistence module determined by the following diagram:

$$\begin{array}{ccccc} \mathsf{H}(-\mu-\varepsilon) = \mathsf{G}(p-\mu-\varepsilon) & \longrightarrow & \mathsf{H}(-\mu) = \mathsf{F}(p-\mu) & \longrightarrow & \mathsf{H}(\mu) = \mathsf{F}(p+\mu) \\ & & & & \downarrow \\ & & \mathsf{H}(z) = \mathsf{F}(z+\varepsilon) & \longleftarrow & \mathsf{H}(\mu+\varepsilon) = \mathsf{G}(p+\mu+\varepsilon). \end{array}$$

Here the value of H is given on each value in T and morphisms between adjacent objects are the dashed arrows in the above commutative diagram. We have the following equalities:

$$\begin{aligned} \big[\mathsf{im}\ \mathsf{F}(p+\mu \le z)\big] &= d\mathsf{H}\big([\mu,\infty)\big), \\ \big[\mathsf{im}\ \mathsf{F}(p-\mu \le z)\big] &= d\mathsf{H}\big([-\mu,\infty)\big), \\ \big[\mathsf{im}\ \mathsf{G}(p+\mu+\varepsilon \le z)\big] &= d\mathsf{H}\big([\mu+\varepsilon,\infty)\big), \\ \big[\mathsf{im}\ \mathsf{G}(p-\mu-\varepsilon \le z)\big] &= d\mathsf{H}\big([-\mu-\varepsilon,\infty)\big). \end{aligned}$$

By Lemma 5.2 along with the above substitutions, we have

$$\sum_{J \in \square_\mu [0,\infty)} \tilde{\mathsf{H}}(J) = \sum_{J \in \square_\mu I} \tilde{\mathsf{F}}(J),$$

$$\sum_{J \in \square_{\mu+\varepsilon} [0,\infty)} \tilde{\mathsf{H}}(J) = \sum_{J \in \square_{\mu+\varepsilon} I} \tilde{G}(J).$$

By the inclusion $\square_\mu[0,\infty) \subseteq \square_{\mu+\varepsilon}[0,\infty)$ along with Proposition 4.6, we have

$$\sum_{J \in \square_\mu [0,\infty)} \tilde{\mathsf{H}}(J) \preceq \sum_{J \in \square_{\mu+\varepsilon} [0,\infty)} \tilde{\mathsf{H}}(J).$$

This proves the statement. $\square$

Definition 5.4. The *injectivity radius* of a finite set $S = \{s_1 < s_2 < \cdots < s_k < \infty\}$ is

$$\rho := \min_{1<i\leq k} \frac{s_i - s_{i-1}}{2}.$$

Note that if a persistence module F is S-constructible and $I \in \mathsf{Dgm}(S)$, then

$$\tilde{\mathsf{F}}(I) = \sum_{J \in \square_\rho I} \tilde{\mathsf{F}}(J).$$

Also if $\tilde{\mathsf{F}}\big([p,q)\big) \neq 0$, then $|p-q| \geq 2\rho$.

Lemma 5.5 (Easy bijection)**.** *Let* F *be an* S*-constructible persistence module and let* $\rho > 0$ *be the injectivity radius of* S*. If* G *is a second constructible persistence module such that* $d_I(\mathsf{F},\mathsf{G}) < \rho/2$*, then* $d_B(\tilde{\mathsf{F}},\tilde{\mathsf{G}}) \leq d_I(\mathsf{F},\mathsf{G})$*.*

Proof. Let $\varepsilon = d_I(\mathsf{F},\mathsf{G})$. Choose a sufficiently small $\mu > 0$ such that $2\mu + 2\varepsilon < \rho$. We construct a matching $\gamma_\mu : \mathsf{Dgm} \times \mathsf{Dgm} \to \mathcal{G}(\mathcal{C})$ such that

$$\tilde{\mathsf{F}}(I) = \sum_{J \in \mathsf{Dgm}} \gamma_\mu(I,J) \text{ for all } I \in \mathsf{Dgm}\backslash\Delta, \tag{5}$$

$$\tilde{\mathsf{G}}(J) = \sum_{I \in \mathsf{Dgm}} \gamma_\mu(I,J) \text{ for all } J \in \mathsf{Dgm}\backslash\Delta. \tag{6}$$

Fix an $I \in \mathsf{Dgm}(S) \setminus \Delta$. By Lemma 5.3,

$$\tilde{\mathsf{F}}(I) = \sum_{J \in \square_\mu I} \tilde{\mathsf{F}}(J) \preceq \sum_{J \in \square_{\mu+\varepsilon} I} \tilde{\mathsf{G}}(J) \preceq \sum_{J \in \square_{\mu+2\varepsilon} I} \tilde{\mathsf{F}}(J) = \tilde{\mathsf{F}}(I).$$

Let $\gamma_\mu(I,J) := \tilde{\mathsf{G}}(J)$ for all $J \in \square_{\mu+\varepsilon}(I)$. Repeat for all $I \in \mathsf{Dgm}(S)$. Equation (5) is satisfied.

We now check that γ_μ satisfies equation (6). Fix an interval $J = [p,q)$ and suppose $\tilde{\mathsf{G}}(J) \neq 0$. If $\frac{q-p}{2} > \mu + \varepsilon$, then by Lemma 5.3

$$\tilde{\mathsf{G}}(J) \preceq \sum_{I \in \square_\mu J} \tilde{\mathsf{G}}(I) \preceq \sum_{I \in \square_{\mu+\varepsilon} J} \tilde{\mathsf{F}}(I).$$

This means $\gamma_\mu(I,J) \neq 0$ for some $I \in \square_{\mu+\varepsilon} J$. If $\frac{q-p}{2} \leq \mu + \varepsilon$, then it must be that $\gamma_\mu\big(I,[p,q)\big) = 0$ for all $I \in \mathsf{Dgm} \setminus \Delta$ for the following reason. Suppose $I = [p',q')$, where $p' \neq q'$, and $\gamma_\mu\big([p',q'),[p,q)\big) \neq 0$. Then $\max\{|p'-p|,|q'-q|\} \leq \mu + \varepsilon$ and therefore $|p'-q'| \leq 3\mu + 3\varepsilon$ which is less than twice the injectivity radius ρ. This means J is unmatched and we may match it to the diagonal. That is, we let $\gamma_\mu\big(\big[\frac{q-p}{2},\frac{q-p}{2}\big),J\big) := \tilde{\mathsf{G}}(J)$.

By construction, $||\gamma_\mu|| \leq \mu + \varepsilon$ for all $\mu > 0$ sufficiently small. Therefore $d_B(\tilde{\mathsf{F}},\tilde{\mathsf{G}}) \leq \varepsilon = d_I(\tilde{\mathsf{F}},\tilde{\mathsf{G}})$. □

We are now ready to prove our main result.

Theorem 5.6 (Bottleneck stability)**.** *Let* $\mathcal{C}$ *be a skeletally small abelian category and let* $\mathsf{F},\mathsf{G} : \mathbb{R} \to \mathcal{C}$ *be two constructible persistence modules. Then* $d_B\big(\tilde{\mathsf{F}},\tilde{\mathsf{G}}\big) \leq d_I(\mathsf{F},\mathsf{G})$ *where* $\tilde{\mathsf{F}}$ *and* $\tilde{\mathsf{G}}$ *are the persistence diagrams of* F *and* G*, respectively.*

Proof. Let $\varepsilon = d_I(\mathsf{F},\mathsf{G})$. By Proposition 2.3, there is a one-parameter family of constructible persistence modules $\{\mathsf{K}_t\}_{t\in[0,1]}$ such that $d_I(\mathsf{K}_t,\mathsf{K}_s) \leq \varepsilon|t-s|$, $\mathsf{K}_0 \cong \mathsf{F}$, and $\mathsf{K}_1 \cong \mathsf{G}$. Each K_t is constructible with respect to some set S_t, and each set S_t has an injectivity radius $\rho_t > 0$. For each time $t \in [0,1]$, consider the open interval

$$U(t) = (t - \rho_t/4\varepsilon, t + \rho_t/4\varepsilon) \cap [0,1].$$

By compactness of $[0,1]$, there is a finite set $Q = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ such that $\bigcup_{i=0}^{n} U(t_i) = [0,1]$. We assume that Q is minimal, that is, there does not exist a pair $t_i, t_j \in Q$ such that $U(t_i) \subseteq U(t_j)$. If this is not the case, simply throw away $U(t_i)$ and we still have a covering of $[0,1]$. As a consequence, for any consecutive pair $t_i < t_{i+1}$, we have $U(t_i) \cap U(t_{i+1}) \neq \emptyset$. This means

$$t_{i+1} - t_i \leq \frac{1}{4\varepsilon}(\rho_{t_{i+1}} + \rho_{t_i}) \leq \frac{1}{2\varepsilon} \max\{\rho_{t_{i+1}}, \rho_{t_i}\}$$

and therefore $d_I(\mathsf{K}_{t_i}, \mathsf{K}_{t_{i+1}}) \leq \frac{1}{2} \max\{\rho_{t_i}, \rho_{t_{i+1}}\}$. By Lemma 5.5,

$$d_B\big(\tilde{\mathsf{K}}_{t_i}, \tilde{\mathsf{K}}_{t_{i+1}}\big) \leq d_I\big(\mathsf{K}_{t_i}, \mathsf{K}_{t_{i+1}}\big)$$

for all $0 \leq i \leq n-1$. Therefore

$$d_B(\tilde{\mathsf{F}}, \tilde{\mathsf{G}}) \leq \sum_{i=0}^{n-1} d_B\big(\tilde{\mathsf{K}}_{t_i}, \tilde{\mathsf{K}}_{t_{i+1}}\big) \leq \sum_{i=0}^{n-1} d_I\big(\mathsf{K}_{t_i}, \mathsf{K}_{t_{i+1}}\big) \leq \varepsilon.$$

□

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