

Accurate mass estimates from the proper motions of dispersion-supported galaxies

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ABSTRACT

We derive a new mass estimator that relies on internal proper motion measurements of dispersion-supported stellar systems, one that is distinct and complementary to existing estimators for line-of-sight velocities. Starting with the spherical Jeans equation, we show that there exists a radius where the mass enclosed depends only on the projected tangential velocity dispersion, assuming that the anisotropy profile slowly varies. This is well-approximated at the radius where the log-slope of the stellar tracer profile is -2 : r_{-2} . The associated mass is $M(r_{-2}) = 2G^{-1}(\sigma_T^2)^* r_{-2}$ and the circular velocity is $V^2(r_{-2}) = 2(\sigma_T^2)^*$. For a Plummer profile $r_{-2} \simeq 4R_e/5$. Importantly, r_{-2} is smaller than the characteristic radius for line-of-sight velocities derived by Wolf et al. Together, the two estimators can constrain the mass profiles of dispersion-supported galaxies. We illustrate its applicability using published proper motion measurements of dwarf galaxies Draco and Sculptor, and find that they are consistent with inhabiting cuspy NFW subhaloes of the kind predicted in CDM but we cannot rule out a core. We test our combined mass estimators against previously published, non-spherical cosmological dwarf galaxy simulations done in both cold dark matter (CDM; naturally cuspy profile) and self-interacting dark matter (SIDM; cored profile). For CDM, the estimates for the dynamic rotation curves are found to be accurate to 10 per cent while SIDM are accurate to 15 per cent. Unfortunately, this level of accuracy is not good enough to measure slopes at the level required to distinguish between cusps and cores of the type predicted in viable SIDM models without stronger priors. However, we find that this provides good enough accuracy to distinguish between the normalization differences predicted at small radii ($r \simeq r_{-2} < r_{\text{core}}$) for interesting SIDM models. As the number of galaxies with internal proper motions increases, mass estimators of this kind will enable valuable constraints on SIDM and CDM models.

Key words: galaxies: dwarf – galaxies: kinematics and dynamics – cosmology: dark matter.

1 INTRODUCTION

The lambda cold dark matter (Λ CDM) cosmogony, while successful in describing the large-scale structure of our universe, still suffers from potential discrepancies in modelling the properties on small scales, primarily for dark matter haloes that are expected to host the observed dwarf galaxies. For example, Milky Way satellites have significantly lower dark matter densities in the inner regions compared to the corresponding subhaloes in cosmological N -body simulations – this is known as the *Too Big To Fail* problem (Boylan-Kolchin, Bullock & Kaplinghat 2011). A potentially related issue concerns the inner dark matter density profiles inferred from the rotation curves of small disc galaxies, many of which are observed to be cored/flat, while simulated Λ CDM haloes are cusped/rising

– this is the *cusp–core* problem (Flores & Primack 1994; Moore 1994; de Blok 2010). Feedback from star formation can potentially explain this discrepancy in larger dwarf galaxies (Governato et al. 2010; Pontzen & Governato 2012). However, if dark matter cores exist within galaxies that have had too little star formation ($M_* \lesssim 10^6 M_\odot$) to affect the dark matter density slopes (Di Cintio et al. 2014; Chan et al. 2015; Tollet et al. 2016), then this could be an indication that the dark matter is something other than CDM (see Bullock & Boylan-Kolchin 2017, and references there in).

Though particularly important, the question of whether or not the smallest galaxies have cusps or cores is notoriously difficult to answer owing to the fact that they are dispersion supported. While it is possible to quantify the detailed mass profiles of spheroidal galaxies through the use of kinematic measurements of individual stars in 3D (e.g. Wilkinson et al. 2002; Strigari, Bullock & Kaplinghat 2007), until recently we have been limited to data sets that include only 1D velocities along the line of sight. This

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introduces a degeneracy between the inferred mass profile slope and the underlying velocity dispersion anisotropy parameter β , which quantifies the intrinsic difference between the radial and tangential velocity dispersions.

One robust measurement that is possible with line-of-sight velocities is the integrated mass within a single characteristic radius for each galaxy. This idea was first emphasized by Walker et al. (2009), who used spherical Jeans modelling to show that the integrated mass within an effective radius was independent of an assumed β for a wide variety of assumptions for many galaxies. Wolf et al. (2010) extended this idea, also using Jeans modelling, to show that there exists, analytically, an idealized radius within which the mass inferred from line-of-sight velocities is formally insensitive to β . Under mild assumptions, this radius is where the log-slope of the stellar tracer profile is equal to -3 . Both the Walker and Wolf mass estimators do remarkably well when compared to ab initio cosmological simulations of (non-spherical) dwarf galaxies in Campbell et al. (2017) and González-Samaniego et al. (2017). They are also used extensively to interpret observed line-of-sight velocity dispersion measurements (see Simon 2019, and references therein).

We are entering a new era of astrometry, such that the internal proper motions in distant dwarf spheroidal galaxies are now becoming possible to measure with the advent of GAIA (Gaia Collaboration 2016a,b, 2018a,b). Additionally, LSST may provide similar advances (Abell et al. 2009). Measurements of stellar velocities along the plane of the sky promise an important new window into the mass and density structure of dwarf galaxies. The results of Massari et al. (2018, 2019) provide an exciting first look at what we expect to measure in the coming years by providing plane of the sky velocity dispersion measurements for Sculptor and Draco, respectively.

The article is outlined as follows: In Section 2, we briefly introduce the spherical Jeans equation and the coordinate system used as the basis of our analysis. Section 3 derives the mass estimators by combining the Jeans equation and proper motions measured from the plane of the sky, which includes the key assumptions considered therein. Section 4 demonstrates the use of the combined mass estimators to provide an implied mass density slope for currently available proper motions of Draco and Sculptor. Section 5 assesses our estimators with mock observations constructed from high-resolution simulations. In Section 6, we discuss possible biases that might arise due to Jeans modelling, and finally, Section 7 summarizes our results and we provide concluding remarks.

2 PRELIMINARIES

In what follows, lower case r represents the (physical) three-dimensional radius and the upper case R represents the (physical) two-dimensional projected radius.

2.1 The spherical Jeans equation

For a spherically symmetric steady-state system, the first moment of the collisionless Boltzmann equation for a stellar phase-space distribution takes the form of the spherical Jeans equation (Binney & Tremaine 2008):

$$-\frac{d\Phi(r)}{dr} = \frac{1}{n_*(r)} \frac{d}{dr} (n_* \sigma_r^2(r)) + \frac{2\beta \sigma_r^2(r)}{r}, \quad (1)$$

which relates the total gravitational potential, $\Phi(r)$, of the stellar system to its two tracers: the intrinsic radial velocity dispersion,

$\sigma_r^2 := \langle v_r^2 \rangle - \langle v_r \rangle^2$, and the three-dimensional stellar number density, $n_*(r)$. The quantity,

$$\beta(r) := 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2}, \quad (2)$$

is a measure of the velocity dispersion *anisotropy* of the tracer population, where σ_θ and σ_ϕ are the intrinsic velocity dispersions tangential to radius r . We will assume that $\sigma_\theta = \sigma_\phi$. Radially biased systems tend to have $\beta \rightarrow 1$ while $\beta \rightarrow -\infty$ constitutes tangentially biased measurements. In addition, the total intrinsic velocity dispersion follows

$$\sigma_{\text{tot}}^2(r) = \sigma_r^2 + \sigma_\theta^2 + \sigma_\phi^2 = (3 - 2\beta)\sigma_r^2(r). \quad (3)$$

The total mass profile of the dynamical system is an implied quantity of equation (1), such that,

$$M(r|\beta) = \frac{r\sigma_r^2(r)}{G}(\gamma_* + \gamma_\sigma - 2\beta), \quad (4)$$

where G is Newtons gravitational constant and the logarithmic slopes are defined as

$$\begin{aligned} \gamma_* &:= -\frac{d \log n_*}{d \log r}; & \gamma_\sigma &:= -\frac{d \log \sigma_r^2}{d \log r}; \\ \gamma_\beta &:= -\frac{d \log \beta}{d \log r}. \end{aligned} \quad (5)$$

2.2 Coordinate system of measurements

We use the coordinate system discussed in Strigari et al. (2007) such that the three-dimensional components of stars' velocity in a spherically, steady-state systems are comprised of the components radial, v_r , and transverse, v_θ and v_ϕ . The projected proper motions are composed of these three-dimensional velocities, that is, along the measured *line of sight*,

$$v_{\text{los}} = v_r \cos \theta + v_\theta \sin \theta, \quad (6)$$

where $\mathbf{r} \cdot \mathbf{z} = \cos \theta$ and \mathbf{z} is the line-of-sight direction, and along the plane of the sky, the components *parallel* and *transverse* to the projected radius R are

$$v_{\mathcal{R}} = v_r \sin \theta + v_\theta \cos \theta \quad \text{and} \quad v_{\mathcal{T}} = v_\phi, \quad (7)$$

respectively. Here, the variances of the velocity dispersions are given by $\sigma_i^2 := \langle v_i^2 \rangle$ and $\sigma_\phi = \sigma_\theta$ is assumed. The derived mapping of the observable proper motions to the deprojected, three-dimensional tracer profiles are

$$\Sigma_* \sigma_{\text{los}}^2(R) = \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} \left[1 - \frac{R^2}{r^2} \beta \right] n_* \sigma_r^2, \quad (8)$$

$$\Sigma_* \sigma_{\mathcal{R}}^2(R) = \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} \left[1 - \beta + \frac{R^2}{r^2} \beta \right] n_* \sigma_r^2, \quad (9)$$

$$\Sigma_* \sigma_{\mathcal{T}}^2(R) = \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} [1 - \beta] n_* \sigma_r^2. \quad (10)$$

The combination of the proper motions also satisfy $\sigma_{\text{tot}}^2 = \sigma_{\mathcal{R}}^2 + \sigma_{\mathcal{T}}^2 + \sigma_{\text{los}}^2$. For an observed galaxy, $\Sigma_*(R)$ is the projected stellar density, which is related to the three-dimensional $n_*(r)$ via an Abel inversion, equation (A1).

3 MASS ESTIMATORS FROM PROPER MOTIONS

In this section, quantities enclosed in brackets with an asterisk $\langle \dots \rangle^*$ indicates a measurement to be *luminosity weighted*, $r_{1/2}$ is

the three-dimensional, deprojected half-light radius, and R_e is the two-dimensional effective radius.

3.1 Measurements along the line-of-sight

Here, we rederive the main results from Wolf et al. (2010) using the assumptions discussed therein: Consider a velocity dispersion-supported stellar system that is well studied, such that $\Sigma_*(R)$ and $\sigma_{\text{los}}(R)$ are determined accurately by observations. In this system, all of the stars are assumed to be bound with no dynamical interlopers. If we model the system's mass profile using the Jeans equation, any viable solution will keep the combination of $\Sigma_*\sigma_{\text{los}}^2(R)$ fixed to within allowable errors. We start with the mapping of σ_{los} to σ_r . To utilize equation (1), equation (8) is massaged to an invertible form that is applicable to that of an Abel inversion

$$\Sigma_*\sigma_{\text{los}}^2(R) = \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} \left[n_*\sigma_r^2(1 - \beta) + \int_{r^2}^{\infty} d\tilde{r}^2 \frac{\beta n_*\sigma_r^2}{2\tilde{r}^2} \right].$$

The term in the brackets on the right-hand side has to be a well-defined quantity, as the left-hand side is an accurate, observed quantity ignorant of the form of β . Therefore, we are allowed to compare different forms of β with one another; we equate the isotropic form of the integrand, $\beta = 0$, with an integrand that is dependent on some arbitrary β , as this is the simplest case one can consider as a comparison:

$$n_*\sigma_r^2|_{\beta=0} = n_*\sigma_r^2(1 - \beta) + \int_{r^2}^{\infty} d\tilde{r}^2 \frac{\beta n_*\sigma_r^2}{2\tilde{r}^2}. \quad (11)$$

By then taking a derivative in respect to $\log r$ and introducing a factor of $r\sigma_r^2/G$ on both sides, we can massage the left-hand and right-hand side into their respective forms of equation (4) and evaluate the difference:

$$M(r|\beta) - M(r|0) = \frac{r\sigma_r^2\beta}{G}(\gamma_* + \gamma_\sigma + \gamma_\beta - 3). \quad (12)$$

From this expression, we see that there can exist a radius, r_{eq} , where the term in the parentheses vanishes based off of mapping projected line-of-sight measurements to the intrinsic quantities of the system, that is,

$$\gamma_*(r_{\text{eq}}) = 3 - \gamma_\sigma(r_{\text{eq}}) - \gamma_\beta(r_{\text{eq}}). \quad (13)$$

Moreover, if $\sigma_r^2(r)$ and $\beta(r)$ are slowly varying, such that the log-slope profiles are approximately zero, i.e. $\gamma_\sigma(r_{\text{eq}}) \simeq 0$ and $\gamma_\beta(r_{\text{eq}}) \simeq 0$, then the degeneracy of the mass profile written in equation (4) is effectively minimized. This would then have the right-hand side of equation (12) to be subsequently null. Furthermore, if $\gamma_*(r_{\text{eq}}) \simeq 3$, then this equates the radius of minimized anisotropy as $r_{\text{eq}} \simeq r_{-3}$, which is the radius where the differential log-gradient of the stellar tracer profile is -3 .

To determine the value of $M(r_{\text{eq}})$, equation (11) is deprojected via an Abel inversion to isolate out the combination of $n_*\sigma_r^2$ (equation A5; Wolf et al. 2010). This is then hit with a derivative in respect to $\log r$ and is inserted into equation (4) to obtain

$$\begin{aligned} \frac{GM(r)}{r} &= (3 - 2\beta)\sigma_r^2(r) + (\gamma_* + \gamma_\sigma - 3)\sigma_r^2(r) \\ &= \sigma_{\text{tot}}^2(r) + (\gamma_* + \gamma_\sigma - 3)\sigma_r^2(r), \end{aligned} \quad (14)$$

where we have related the total intrinsic velocity dispersion using equation (3). From the assumptions prior, if $\gamma_\sigma(r_{\text{eq}}) \ll 3$, then the parenthetical term vanishes and $r_{\text{eq}} \simeq r_{-3}$, giving us

$$M(r_{-3}) \simeq \frac{\sigma_{\text{tot}}^2(r_{-3})r_{-3}}{G}. \quad (15)$$

Wolf et al. (2010) showed that to a good approximation, $\sigma_{\text{tot}}^2(r_{-3}) \simeq \langle \sigma_{\text{tot}}^2 \rangle^*$ for models that match observations. Furthermore, spherical symmetry demands that the line-of-sight dispersion obeys $\langle \sigma_{\text{tot}}^2 \rangle^* = 3\langle \sigma_{\text{los}}^2 \rangle^*$ (see Section 6.1). This will lead us to obtain an *idealized* estimator at r_{-3} :¹

$$M_{-3}^{\text{ideal}} \equiv M(r_{-3}) = \frac{3\langle \sigma_{\text{los}}^2 \rangle^* r_{-3}}{G}. \quad (16)$$

Additionally, with the foundations of spherical symmetry, the implied circular velocity at r_{-3} is particularly simple

$$V_{\text{circ}}(r_{-3}) = \sqrt{3\langle \sigma_{\text{los}}^2 \rangle^*}. \quad (17)$$

Wolf et al. (2010) showed that for a variety of analytical stellar profiles, r_{-3} is close to $r_{1/2} \simeq 4R_e/3$, giving

$$M(r_{-3}) \simeq \frac{3\langle \sigma_{\text{los}}^2 \rangle^* r_{-3}}{G}; \quad M(4R_e/3) \simeq \frac{4\langle \sigma_{\text{los}}^2 \rangle^* R_e}{G}. \quad (18)$$

In the coming sections, we utilize the arguments stipulated here in the derivation of equation (16), where we seek to determine if another radius, one that is also independent of the anisotropy, exists for the two other proper motion mappings, such that it is independent of r_{eq} found previously. From here-on, we will refer equation (16) as M_{-3} and equation (17) as V_{-3} .

3.2 Plane of the sky: Measurements parallel to R

Consider a dispersion-supported stellar system that is well studied, such that $\sigma_{\mathcal{R}}(R)$ is determined accurately through observations. We begin by relating the projected measurement of $\sigma_{\mathcal{R}}(R)$ to the intrinsic quantities via equation (9). This is then rewritten to an invertible form (see Appendix A),

$$\Sigma_*\sigma_{\mathcal{R}}^2(R) = \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} \left[n_*\sigma_r^2 - \int_{r^2}^{\infty} d\tilde{r}^2 \frac{\beta n_*\sigma_r^2}{2\tilde{r}^2} \right]. \quad (19)$$

From its invertible form, the left-hand side is an accurate, observable quantity that is ignorant of the form of β . Therefore, the term in the brackets must be a well-defined quantity regardless of the form of β chosen. Therefore, we are allowed to consider the simple case of equating the isotropic integrand with an integrand that is dependent on some arbitrary anisotropy

$$n_*\sigma_r^2|_{\beta=0} = n_*\sigma_r^2 - \int_{r^2}^{\infty} d\tilde{r}^2 \frac{\beta n_*\sigma_r^2}{2\tilde{r}^2}. \quad (20)$$

By then taking the derivative in respect to $\log r$ and introducing a factor of $r\sigma_r^2/G$ to both sides, the left-hand and right-hand side are allowed to be rewritten in the form of the integrated Jeans masses, allowing use to express the difference

$$M(r|\beta) - M(r|0) = \frac{r\sigma_r^2\beta}{G}. \quad (21)$$

Importantly, this expression lacks the parenthetical term seen in equation (12). We conclude that a radius that minimizes the anisotropy, at least, for the assumptions we considered in the Jeans modelled measurements of $\sigma_{\mathcal{R}}(R)$, *does not* exist in whatever

¹Throughout, we refer to an *idealized* solution as one that considers the quintessential case of $\gamma_\beta = \gamma_\sigma = 0$ at the radius that minimizes the anisotropy. We do not expect physical results to perfectly match this behaviour, but we instead presume the profiles to be relatively small enough at the expected radius where this is prominent. We will remain agnostic on this point until later in the article.

limiting case of β we were to impose, since the anisotropy is a dependent quantity throughout the mass profile.

3.3 Plane of the sky: Measurements transverse to R

Consider a dispersion-supported stellar system that is well studied, such that $\sigma_T(R)$ is determined accurately through observations, in which all stars are bounded inside the system. We begin by relating of $\sigma_T(R)$ to $\sigma_r(r)$, given by equation (10). Fortunately, this is already in an invertible form; we now equate its isotropic and general anisotropic form to one another

$$n_* \sigma_r^2|_{\beta=0} = n_* \sigma_r^2(1 - \beta), \quad (22)$$

differentiate it with respect to $\log r$, and algebraically manipulate to acquire the expression

$$M(r|\beta) - M(r|0) = \frac{r \sigma_r^2 \beta}{G} (\gamma_* + \gamma_\sigma + \gamma_\beta - 2). \quad (23)$$

We see that *there can* exist a radius, that we shall denote as \tilde{r}_{eq} ,² where the parenthesis vanishes. The possible existence of \tilde{r}_{eq} therefore minimizes the dependency of β around the region \tilde{r}_{eq} for measurements solely based off of $\sigma_T(R)$, such that,

$$\gamma_*(\tilde{r}_{\text{eq}}) = 2 - \gamma_\sigma(\tilde{r}_{\text{eq}}) - \gamma_\beta(\tilde{r}_{\text{eq}}). \quad (24)$$

Moreover unless galaxies have large variation in σ_r^2 and in β with radius, we may expect $\gamma_\sigma(\tilde{r}_{\text{eq}}) + \gamma_\beta(\tilde{r}_{\text{eq}}) \ll 2$, as least for radii in the vicinity of $\tilde{r}_{\text{eq}} < r_{-3} \simeq r_{1/2}$ for commonly assumed stellar density profiles. Therefore, we can expect that to a good approximation, $\tilde{r}_{\text{eq}} \simeq r_{-2}$, where r_{-2} is the radius at which the log-slope of the tracer profile is equivalent to -2 .

Like before, we now consider the integrated Jeans mass. The dependence of β can be absorbed into the definition of the intrinsic total velocity dispersion. Moreover, the formulation of equation (3) allows, $(1 - \beta)\sigma_r^2 = \sigma_\theta^2 = \sigma_T^2$ with the assumption of spherical symmetry. The Jeans equation becomes

$$\frac{GM(r)}{r} = 2\sigma_\theta^2(r) + (\gamma_* - \gamma_\sigma - 2)\sigma_r^2(r). \quad (25)$$

If in fact that $\gamma_* + \gamma_\sigma \approx \gamma_* \simeq 2$, the term in parenthesis vanishes at the radius r_{-2} . The remaining term on the right-hand side depends only on the intrinsic transverse component, $\sigma_\theta = \sigma_T$, which is an observable.³ Finally, we obtain an idealized estimator

$$M_{-2}^{\text{ideal}} \equiv M(r_{-2}) = \frac{2 \langle \sigma_T^2 \rangle^* r_{-2}}{G}, \quad (26)$$

where we have assumed $\sigma_\theta(r_{-2}) \simeq \sigma_T^2(r_{-2}) \simeq \langle \sigma_T^2 \rangle^*$. The implied circular velocity at r_{-2} is particularly succinct

$$V_{\text{circ}}(r_{-2}) = \sqrt{2 \langle \sigma_T^2 \rangle^*}. \quad (27)$$

From here-on, we will refer equation (26) as M_{-2} and equation (27) as V_{-2} .

²This is not to be associated with the radius, r_{eq} , seen in the derivation of M_{-3} , as that r_{eq} is constrained to measurements of σ_{los} . Simply, the r_{eq} of equation (13) and \tilde{r}_{eq} of equation (24) are taken to be nonequivalent.

³Note that the term in brackets in equation (10) is constrained by observables. Specifically, the intrinsic transverse dispersion, σ_θ , it is related to the transverse component along the plane of the sky via $(1 - \beta)\sigma_r^2 = \sigma_\theta^2 = \sigma_T^2$. This is what allows proper motion measurements to constrain the anisotropy (Strigari et al. 2007).

3.4 Overview of assumptions

We have made a few assumptions in the derivation of M_{-3} and M_{-2} . In addition to the strong assumption that galaxies are spherical, we have assumed that the velocity dispersions are relatively flat such that $\sigma_{\text{tot}}^2(r_{-3}) \simeq \langle \sigma_{\text{tot}}^2 \rangle^* = 3 \langle \sigma_{\text{los}}^2 \rangle^*$ and $\sigma_\theta^2(r_{-2}) = \sigma_T^2(r_{-2}) \simeq \langle \sigma_T^2 \rangle^*$. Wolf et al. (2010) showed that the assumption for the line-of-sight component is excellent for a variety of models that match line-of-sight data, yet, for the transverse component, not enough data are available to test this assumption. Some justification comes from Section 5, where we use a set of cosmological simulations of dwarf galaxies in mock observations and find that these assumptions are good to better than 10 per cent.

Secondly, we have assumed that the intrinsic radial velocity dispersion varies minimally with radius compared to the tracer profile out to r_{-3} . More specifically we assume that the log-slopes of the tracer velocity dispersion profiles are small compared to -3 and -2 at r_{-3} and r_{-2} , respectively. Thirdly, we assume that the velocity dispersion anisotropy varies slowly with radius compared to the light profile. If $\beta(r)$ and $\sigma_r(r)$ vary quickly as a function of radius r , then the mass estimators will break down.

In order to map M_{-3} and M_{-2} to observables measured in two dimensions, the characteristic radii of the tracer profile, r_{-2} and r_{-3} , must be mapped to the projected tracer profile that is observed. If we assume that the three-dimensional profile is well described by a Plummer (1911) profile, then $r_{-3} \simeq 4R_e/3$ and $r_{-2} \simeq 4R_e/5$. That is

$$M(3r_{1/2}/5) \simeq \frac{6 \langle \sigma_T^2 \rangle^* r_{1/2}}{5G}; \quad M(4R_e/5) \simeq \frac{8 \langle \sigma_T^2 \rangle^* R_e}{5G}. \quad (28)$$

To clarify, *if the underlying three-dimensional tracer profile is not well-described by a Plummer profile, then this mapping will fonder*. Ultimately, the mapping between the three-dimensional characteristic radii and observed two-dimensional radii will obey another relationship that depends on the underlying profile.

Fig. 1 provides a test and illustration of the derivation presented above using a full mass profile analysis derived in Appendix A. Shown are the mass profiles implied by various choices of constant velocity dispersion anisotropy constrained by dispersion components along the line of sight (equation A2; top panel), parallel (equation A10; middle panel), and tangential (equation A12; bottom panel) under the assumption of constant β (denoted by β_0). We assume that the velocity dispersions for each component are constant with R , and set them equal to the luminosity-weighted median values observed for Sculptor (9, 11.5, and 8.5 km s⁻¹, respectively). We also assume that the tracer profile follows a Plummer, again matched to the median value for Sculptor given in Table 1. The white circles show the estimators M_{-3} and M_{-2} in the top and bottom panels, respectively. Encouragingly, they intersect the regions where all of the varying β_0 mass profiles converge. As anticipated in Section 3.2, constraints imposed by the parallel component, σ_R , show no convergence point. This figure shows that the mass estimators we have derived work under reasonable, but idealized assumptions. In the last last section of Appendix A, we show a similar analysis that allows for parametric forms of $\beta(r)$ commonly used in Jeans modelling analyses. We show the idealized mass estimators work well unless $\beta(r)$ varies rapidly with radius (as expected).

Of course, real galaxies will not obey these assumptions with absolute precision. Perhaps most importantly, no galaxy is perfectly spherical. We expect real galaxies to have velocity dispersion profiles that vary with radius to some degree. Galaxies also have light

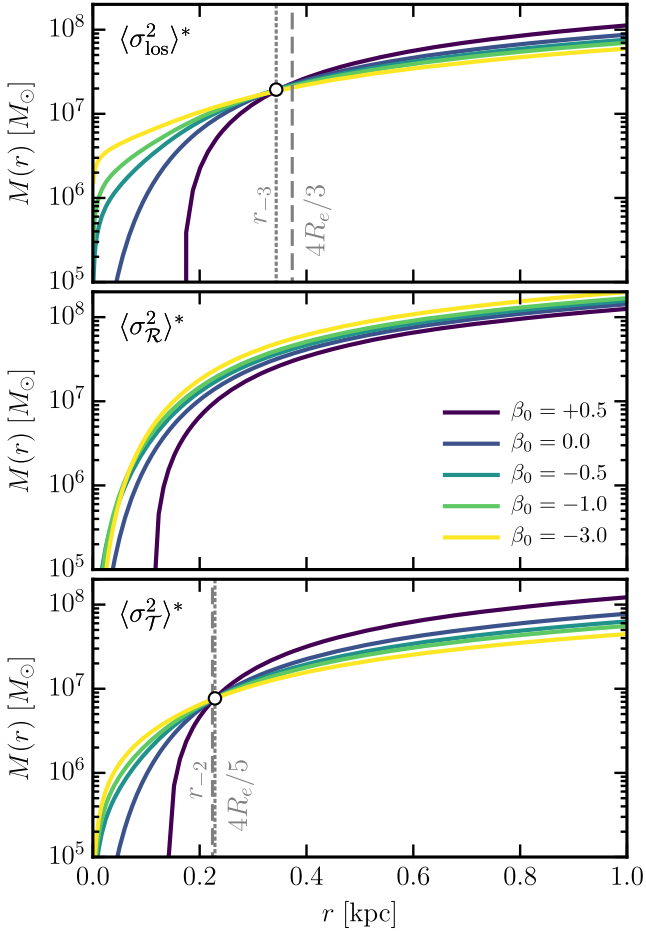


Figure 1. Radii of minimized uncertainty for idealized models. Curves depict the cumulative mass profiles derived in Appendix A based on fixed line-of-sight (top panel), parallel (middle panel), and transverse (bottom panel) velocity dispersions, all of which use the median values for Sculptor from Table 1. The results assume that the observed dispersion profile for each component is constant with R . The lines correspond to several choices of constant intrinsic anisotropy $\beta(r) = \beta_0$ as indicated by the colours. We also assume that the underlying tracer profile follows a Plummer model. The small white circles in the top plot and bottom plot show masses predicted by the M_{-3} and M_{-2} estimators, respectively. These points intersect the region of mass that is independent of the anisotropy. Note that the parallel component has no such intersection, as anticipated in Section 3.2. The dotted lines give the characteristic log-slope radii of the tracer profile while the dashed shows the standard mapping to the projected observable, R_e .

tracer profiles that will not necessarily obey convenient functional characterizations such as the Plummer model in three dimensions, which will make determining r_{-2} and r_{-3} more difficult. We test these assumptions along with our estimator in Section 5 using cosmological dwarf galaxy simulations.

4 MODELLING FROM OBSERVATIONS

We now apply our mass estimator using kinematic measurements for the spheroidal galaxies, Draco and Sculptor. Table 1 lists the observed properties that we adopt. We assume that each galaxies stellar distribution obeys a Plummer (1911) profile in deprojection and in projection. We used the radial conversions for a Plummer profile given in Wolf et al. (2010).

4.1 The internal structure of Draco and Sculptor

Fig. 2 plots the implied mass of Draco (squares) and Sculptor (circles) using both M_{-3} (magenta colored) and M_{-2} (cyan colored). With the current data today, masses implied from well studied, line-of-sight measurements have smaller error bars while the implied masses from the tangential along the plane of the sky have relatively larger error bars. Also plotted are the NFW (Navarro, Frenk & White 1997) mass profiles at fixed halo mass, $M_{\text{vir}} = 2 \times 10^{10}$ and $3 \times 10^9 M_\odot$. Concentrations are set to 16.3 and 20.2, respectively, based on the median values for subhaloes of this mass in the $z = 0$ dark matter only physics results of the Phat-ELVIS simulations (Kelley et al. 2019). The subhalo masses were chosen so that at median value of the concentration for the profiles intersect the line-of-sight mass points. In principle, by comparing the location of the tangentially derived masses to the extrapolated NFW curves can allow us to determine if the predictions are consistent with a cuspy profile. Both galaxies appear consistent with sitting within a typical CDM halo. Note that this result for Draco is in agreement with results by Read, Walker & Steger (2018), who find Draco to be cusped around the same radial range.

Fig. 3 provides an alternative view by plotting observed circular velocities using now V_{-3} and V_{-2} . The rotation curves for NFW profiles at fixed values of $V_{\text{max}} = 19$ and 34 km s^{-1} are also plotted, with median values of $r_{\text{max}} = 1.67$ and 4.71 kpc , respectively, for the same subhaloes of Phat-ELVIS. As seen previously in Fig. 2, both measurements are consistent with the expectations for an NFW. Sculptor’s median does fall below the extrapolated NFW, though it is easily consistent within error. If Sculptor has a cored inner density it could have interesting implications. With a stellar mass of $M_\star \simeq 4 \times 10^6 M_\odot$, this galaxy lies near the low-mass edge of where feedback may be able to produce significant cores (Bullock & Boylan-Kolchin 2017). This motivates the acquisition of additional data to provide a more precise measure of $\langle \sigma_T^2 \rangle^*$.

5 MOCK OBSERVATIONS

We are now interested in testing the mass estimators discuss previously, including the one derived here for the first time. We use simulations that have been previously published with data kindly supplied by the authors (Fitts et al. 2017; Robles et al. 2017). The simulations were run as part of the feedback in realistic environments (FIRE) project and include galaxies simulated in both cold dark matter (CDM) and self-interacting dark matter (SIDM). Table 2 lists the global parameters of the galaxies considered herein as well as the references the reader can refer to with the specific physics used when running the FIRE-2 algorithm (Hopkins et al. 2014; Hopkins 2015; Hopkins et al. 2018).

We specifically have chosen low-mass galaxies that are dispersion supported that resemble dwarf spheroidals. The values of M_\star/M_{vir} for the CDM galaxies do not produce enough energy to transform cusps to cores and thus provide a good test for ‘cuspy’ underlying profiles (Di Cintio et al. 2014; Chan et al. 2015; Tollet et al. 2016; Bose et al. 2019), while SIDM haloes are naturally core-like. Their stellar masses are low enough that episodic gas outflows do not bias estimates from equilibrium when using Jeans modelling (El-Badry et al. 2016, 2017). In summary, we consider two types of simulations:

CDM: Dark matter is considered to be collisionless. The sample of galaxies simulated in CDM are m10b, m10c, m10d, and m10e, which were first presented in Fitts et al. (2017) and explored further in Fitts et al. (2018, 2019). The fiducial CDM simulations have

Table 1. Observational measurements considered in this analysis.

Galaxy	$\sqrt{\langle\sigma_{\text{los}}^2\rangle}^*$ (km s ⁻¹)	$\sqrt{\langle\sigma_R^2\rangle}^*$ (km s ⁻¹)	$\sqrt{\langle\sigma_T^2\rangle}^*$ (km s ⁻¹)	R_e (pc)	$r_{1/2}$ (pc)	M_{-3} (M_\odot)	V_{-3} (km s ⁻¹)	M_{-2} (M_\odot)	V_{-2} (km s ⁻¹)
Draco	(a,b) $10.1^{+0.5}_{-0.5}$	(e) $11.0^{+2.1}_{-1.5}$	(e) $9.9^{+2.3}_{-3.1}$	(d,f) 214^{+2}_{-2}	279^{+2}_{-2}	$2.03^{+0.20}_{-0.20} \times 10^7$	$17.5^{+0.9}_{-0.9}$	$7.80^{+4.89}_{-3.63} \times 10^6$	$14.0^{+3.3}_{-4.4}$
Sculptor	(a,b) $9.0^{+0.2}_{-0.2}$	(c) $11.5^{+4.3}_{-4.3}$	(c) $8.5^{+3.2}_{-3.2}$	(d,f) 280^{+1}_{-1}	365^{+1}_{-1}	$2.11^{+0.09}_{-0.09} \times 10^7$	$15.6^{+0.3}_{-0.3}$	$7.53^{+5.67}_{-5.67} \times 10^6$	$12.0^{+4.5}_{-4.5}$

Notes. Here, M_{-3} and V_{-3} are computed using equations (16) and (17), respectively, while M_{-2} and V_{-2} used equations (26) and (27), respectively.

References – (a): Walker et al. (2009), (b): Wolf et al. (2010), (c): Massari et al. (2018), (d): Muñoz et al. (2018), (e): Massari et al. (2019), (f): Simon (2019).

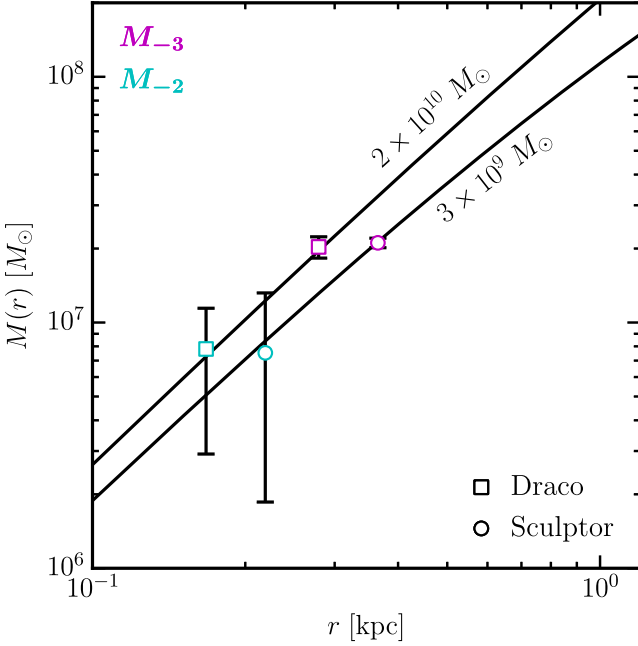


Figure 2. Mass measurements for Draco and Sculptor. For each galaxy, points correspond to the line-of-sight mass (magenta), M_{-3} , and the projected tangential mass (cyan), M_{-2} , at two characteristic radii. Lines show representative NFW mass profiles of fixed M_{vir} with median concentration set by subhaloes in the Phat-ELVIS simulations.

a baryonic particle mass of $m_b = 500 M_\odot$ with force resolution $\epsilon_b = 2$ pc and a dark matter mass $m_{\text{DM}} = 2500 M_\odot$ with softening $\epsilon_{\text{DM}} = 35$ pc. This sample of galaxies have their dark haloes forming cusps $z = 0$.

SIDM: This considers the CDM power spectrum but with a imposed self-interaction cross-section of $\sigma/m = 1 \text{ cm}^2 \text{ g}^{-1}$ that is velocity independent. The sample of galaxies considered here are the analogues of the CDM galaxies: m10b, m10c, m10d, and m10e. SIDM analogues of m10d and m10b were first presented in Robles et al. (2017) and further explored with m10c and m10e in Fitts et al. (2019). A key result is that all haloes have formed appreciable cores at $z = 0$.

5.1 Methodology

5.1.1 Properties in three dimensions

The centre position of the galaxies is determined by using an iterative ‘shrinking spheres’ method (Power et al. 2003; Navarro et al. 2004). That is, the centre of mass of star particles is successively computed in a sphere and then has its radius reduced by 50 per cent,

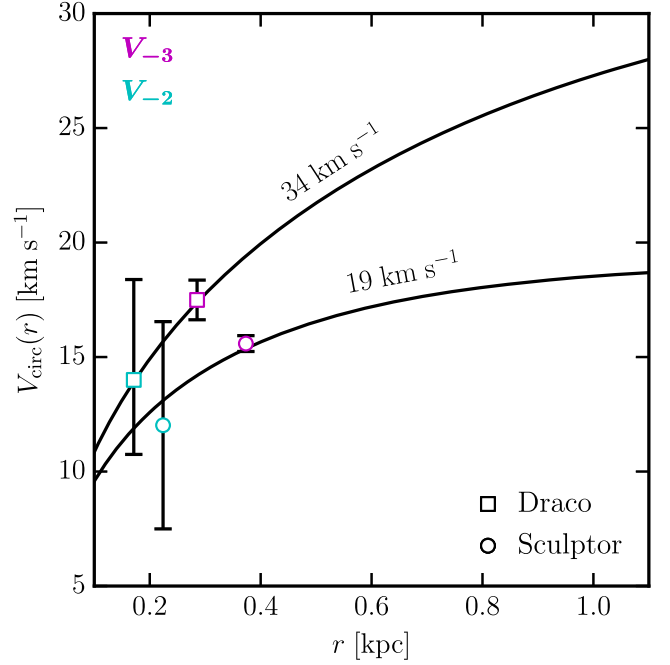


Figure 3. Observed circular velocities of Draco and Sculptor. Circular velocity curves for NFW subhaloes of a given V_{max} are shown for the two characteristic radii. Each assumes a median r_{max} as derived from the Phat-ELVIS simulations.

Table 2. Global properties of the $10^{10} M_\odot$ galaxies at $z = 0$.

Halo Name	M_{vir} (M_\odot) (1)	M_\star (M_\odot) (2)	$r_{1/2}$ (pc) (3)	DM Core? (4)	References
Cold dark matter					
m10b	9.29×10^9	4.65×10^5	340	✗	a, b, c
m10c	8.92×10^9	5.75×10^5	350	✗	a, b, c
m10d	8.43×10^9	1.53×10^6	530	✗	a, c
m10e	1.02×10^{10}	1.98×10^6	620	✗	a, c
Self-interacting dark matter; $\sigma/m = 1 \text{ cm}^2 \text{ g}^{-1}$					
m10b	8.13×10^9	1.05×10^6	504	✓	b, c
m10c	8.71×10^9	7.48×10^5	430	✓	c
m10d	8.10×10^9	1.37×10^6	591	✓	b, c
m10e	9.95×10^9	1.63×10^6	572	✓	c

Notes. (1): The mass of the dark matter halo defined by the background virial overdensity (Bryan & Norman 1998).

(2): The stellar mass of the galaxy; $M_\star := M_{\text{vir}}(< 10 \text{ per cent } r_{\text{vir}})$.

(3): The deprojected radius that contains half of M_{star} .

(4): Verification that a dark matter core has formed.

References – (a): Fitts et al. (2017), (b): Robles et al. (2017), (c): Fitts et al. (2019).

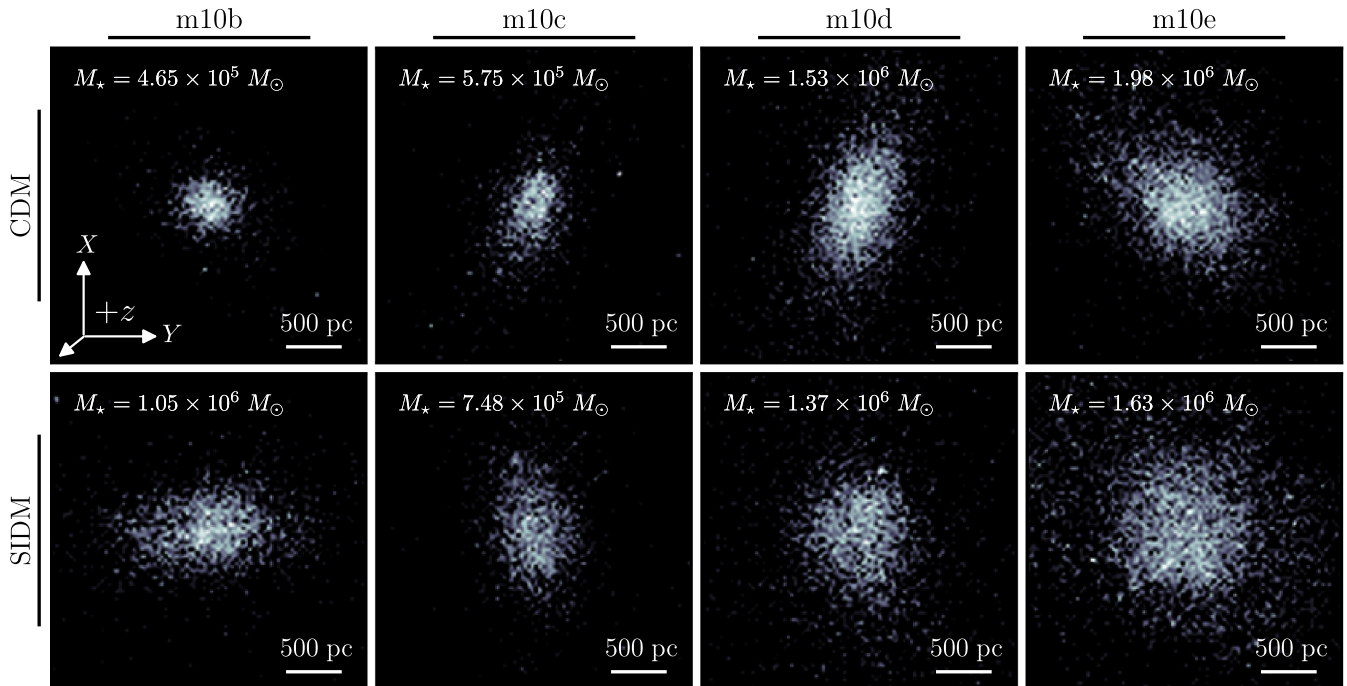


Figure 4. Mock observations of galaxies in the plane of the sky: The stellar surface density of the stars for our dwarf galaxies (given in columns) actualized along a random orientation of the plane (X, Y) looking along the line of sight, z , in CDM (top row) and SIDM (bottom row). The centre of mass of the galaxy determined from this plane of observation is centred at the origin. This shows how elongated several galaxies can appear projections when viewed in the plane of the sky for observations.

which is then re-centred on the new centre of mass. This is done iteratively until a thousand particles enclose the minimized sphere. The centre of mass velocity is then computed using all of the star particles enclosing the final minimized radius. Three-dimensional positions and velocities of all the star particles, associated with that galaxy, are then transformed to be relative to the centre of mass position and velocity, respectively.

The stellar profiles are assembled using 25, log-spaced radial bins of starting from the centre of mass of the stars out to $4 \times r_{1/2}$. In quantifying the characteristic radii of r_{-2} and r_{-3} , the stellar profiles are smoothed using a third ordered spline fit as profiles tend to be noisy. From there, r_{-2} and r_{-3} are interpolated from the log-gradients of the resultant fits. In the construction of the intrinsic dispersion profiles, the Cartesian velocities relative to the centre of mass are transformed to spherical coordinates and are evaluate using the same spherical bin spacing. In each bin shell of r , the relative velocities are weighted by their associated stellar particle mass. This includes both the random motions and streaming motions.

We also compared between a sample containing *only* star particles bound to the dynamical system and another sample containing *both* bound and unbound star particles to the dynamical system. Results for these two population samples were found to be indistinguishable, as unbound star particles only comprised 1 per cent of the galaxies' stellar population. Final results presented here include both bound stars and unbound stars.

5.1.2 Idealized mock observations

For each galaxy, we construct 1000 mock observations. That is, mock observations are done in 1000 random orientations with each orientation evaluated as follows: the relative Cartesian positions and velocities of the galaxies' stellar particles are rotated into a new

orientation denoted by prime coordinates, such that the star particles along the new line-of-sight axis, z' with velocity $v_{z'} \equiv v_{\text{los}}$, are stacked along the projected $x'-y'$ plane. From the galaxy projected on this plane, the centre position is determined by re-implementing an iterative 'shrinking spheres' method. This again determines the centre-of-mass position and velocities of the stars found on the $x'-y'$ plane. We define this as the centre of the galaxy when analysing its projection in two dimensions, where we now label the centre position and velocity as $X = (X, Y)$ and $V = (V_X, V_Y)$, respectively. Hereafter, we drop the prime notation for the line-of-sight axis.

Fig. 4 illustrates a single mock observations by projecting the stellar particles of each galaxy using the method discussed in the previous paragraph. These images have been made after the transformation of coordinates and placing the origin at the centre of mass from the projected distribution of stars. Note that for both CDM and SIDM, the galaxies are not spherical but do appear to have morphologies comparable to actual observed dwarf spheroidals. That is, dwarf galaxies can appear elongated in the plane of sky (plane $X-Y$ in the figure).

The stellar surface profile is then assembled using spherical bins of $R = \sqrt{X^2 + Y^2}$, where we used 25, log-spaced concentric bins starting from the projected centre of mass. From this profile, we fit a projected Plummer profile out to $R = 4 \times r_{1/2}$ in order to obtain the value of the effective radius, R_e . That is, the best-fitting parameters are determined by adjusting the parameters of a projected Plummer in order to minimize a figure-of-merit function. The dispersion profiles are evaluated using the same bin spacing in spherical shells. The relative velocities found in projection are transformed to cylindrical values in correspondence to the coordinate system used in Section 2. That is, the velocity components parallel and transverse to radius R follows $v_R = (X \cdot V)/R$ and $v_T = |X \wedge V|/R$, respectively. In each bin shell of R , the relative velocities in projection are weighted

Table 3. Simulated properties of our galaxies in relation to the assumptions summarized at the end of Section 3. Uncertainties are quoted as the 1σ dispersion from the median. Values measured based off of the characteristic radii are given by columns (1–8) and measured values using only R_e are given in columns (9–12).

Halo	$\gamma_\sigma(r_{-2})$ (1)	$\gamma_\sigma(r_{-3})$ (2)	$\tilde{\sigma}_{\mathcal{T},-2}$ (3)	$\tilde{\sigma}_{\text{tot},-3}$ (4)	V_{-2}/V_{true} (5)	V_{-3}/V_{true} (6)	ξ_{est} (7)	ξ_{true} (8)	V'_{-2}/V_{true} (9)	V'_{-3}/V_{true} (10)	$\tilde{R}_{e,-2}$ (11)	$\tilde{R}_{e,-3}$ (12)
Cold dark matter; (Cusps)												
m10b	0.11	0.16	$0.96^{+0.04}_{-0.04}$	$0.98^{+0.03}_{-0.03}$	$0.96^{+0.01}_{-0.01}$	$0.92^{+0.03}_{-0.03}$	$0.45^{+0.08}_{-0.08}$	0.54	$1.09^{+0.13}_{-0.12}$	$1.02^{+0.14}_{-0.14}$	$0.83^{+0.11}_{-0.10}$	$0.80^{+0.11}_{-0.10}$
m10c	0.09	0.74	$0.97^{+0.04}_{-0.04}$	$1.02^{+0.06}_{-0.04}$	$0.99^{+0.01}_{-0.01}$	$0.96^{+0.04}_{-0.05}$	$0.51^{+0.12}_{-0.10}$	0.57	$1.10^{+0.05}_{-0.06}$	$1.05^{+0.08}_{-0.09}$	$0.86^{+0.04}_{-0.05}$	$0.84^{+0.04}_{-0.04}$
m10d	0.02	0.56	$1.01^{+0.04}_{-0.04}$	$0.95^{+0.04}_{-0.04}$	$1.00^{+0.02}_{-0.02}$	$1.01^{+0.05}_{-0.04}$	$0.47^{+0.11}_{-0.11}$	0.46	$1.00^{+0.04}_{-0.04}$	$1.01^{+0.07}_{-0.07}$	$1.01^{+0.03}_{-0.03}$	$0.98^{+0.03}_{-0.03}$
m10e	0.21	0.48	$0.97^{+0.04}_{-0.03}$	$0.93^{+0.06}_{-0.06}$	$1.01^{+0.04}_{-0.04}$	$1.02^{+0.07}_{-0.07}$	$0.64^{+0.19}_{-0.19}$	0.63	$1.09^{+0.11}_{-0.09}$	$1.12^{+0.12}_{-0.12}$	$0.89^{+0.06}_{-0.05}$	$0.86^{+0.06}_{-0.05}$
Self-interacting dark matter; (Cores)												
m10b	0.34	0.44	$1.01^{+0.05}_{-0.06}$	$1.04^{+0.08}_{-0.06}$	$1.02^{+0.04}_{-0.04}$	$0.93^{+0.04}_{-0.04}$	$0.46^{+0.08}_{-0.08}$	0.75	$1.14^{+0.09}_{-0.10}$	$1.04^{+0.12}_{-0.12}$	$0.87^{+0.05}_{-0.05}$	$0.84^{+0.05}_{-0.04}$
m10c	0.20	0.26	$0.96^{+0.05}_{-0.05}$	$0.91^{+0.08}_{-0.06}$	$1.15^{+0.04}_{-0.06}$	$1.02^{+0.09}_{-0.08}$	$0.50^{+0.12}_{-0.10}$	0.84	$1.36^{+0.09}_{-0.10}$	$1.25^{+0.13}_{-0.14}$	$0.82^{+0.04}_{-0.04}$	$0.79^{+0.04}_{-0.04}$
m10d	0.50	0.76	$0.94^{+0.03}_{-0.03}$	$1.13^{+0.03}_{-0.04}$	$1.08^{+0.02}_{-0.02}$	$0.93^{+0.03}_{-0.03}$	$0.46^{+0.11}_{-0.11}$	0.77	$1.27^{+0.05}_{-0.05}$	$1.18^{+0.06}_{-0.05}$	$0.82^{+0.02}_{-0.02}$	$0.79^{+0.02}_{-0.02}$
m10e	0.17	0.49	$0.97^{+0.03}_{-0.03}$	$1.02^{+0.08}_{-0.08}$	$0.99^{+0.05}_{-0.03}$	$1.03^{+0.07}_{-0.08}$	$0.64^{+0.19}_{-0.19}$	0.62	$1.02^{+0.11}_{-0.08}$	$1.06^{+0.14}_{-0.13}$	$0.97^{+0.05}_{-0.05}$	$0.93^{+0.04}_{-0.05}$

Notes. (1): The value of the log-slope of the radial dispersion profile at r_{-2} .

(2): The value of the log-slope of the radial dispersion profile at r_{-3} .

(3): The ratio of the stellar transverse velocity dispersion at r_{-2} normalized by the weighted median measurement:

$$\tilde{\sigma}_{\mathcal{T},-2} := \sigma_{\mathcal{T}}(r_{-2}) / \langle \sigma_{\mathcal{T}} \rangle^*.$$

(4): The ratio of the stellar total velocity dispersion at r_{-3} normalized by the weighted line-of-sight measurement:

$$\tilde{\sigma}_{\text{tot},-3} := \sigma_{\text{tot}}(r_{-3}) / (\sqrt{3} \langle \sigma_{\text{los}} \rangle^*).$$

(5): Ratio between the value of V_{-2} , equation (27) at r_{-2} , to the true dynamical circular velocity, V_{circ} .

(6): Ratio between the value of V_{-3} , equation (17) at r_{-3} , to the true value of V_{circ} .

(7): The implied power-law slope of the circular velocity profile using the mass estimators at the two characteristic radii, $\xi_{\text{est}} := \Delta \log V_{\text{est}} / \Delta \log r$.

(8): The true slope of the dynamical component of the circular velocity curve at these two characteristic radii assuming a power law.

(9): Ratio between the estimator value of V that approximates $r_{-2} = 4R_e/5$ from a forced Plummer fit, to the true value of V_{circ} .

(10): Ratio between the estimator value of V that approximates $r_{-3} = 4R_e/3$ from a forced Plummer fit, to the true value of V_{circ} .

(11): Ratio between the value of r_{-2} and fitted value of R_e forcing a Plummer profile: $\tilde{R}_{e,-2} := (4R_e/5)/r_{-2}$.

(12): Ratio between the value of r_{-3} and fitted value of R_e forcing a Plummer profile: $\tilde{R}_{e,-3} := (4R_e/3)/r_{-3}$.

by their associated stellar particle mass. Finally, the stellar mass-weighted velocity dispersions of the entire galaxy is measured within $4 \times R_e$ for the value of R_e determined from the surface density fit. We consider both random and streaming motions.

5.2 Results

Our key results are presented in both Table 3 and Fig. 5. In the table, we first list quantities measured to test the assumptions discussed in Section 3.4. We start with columns 1 and 2, which give the log-gradient slope of the intrinsic radial velocity dispersion, γ_σ , at r_{-2} and r_{-3} , respectively. These values are not precisely zero (as we have assumed in our idealized estimator) but they are small compared to the log-slope of the tracer profile (-3 and -2) at these radii and therefore are roughly in line with our assumptions. This behaviour is found to be present for all of our galaxies, regardless of dark matter cores and cusps lying dormant. The radial anisotropy is similarly slowly varying though we have not summarized it here.

Columns 3 and 4 show ratios that measure the flatness of observable velocity profiles as the ratios $\tilde{\sigma}_{\mathcal{T},-2} := \sigma_{\mathcal{T}}(r_{-2}) / \langle \sigma_{\mathcal{T}} \rangle^*$ and $\tilde{\sigma}_{\text{tot},-3} := \sigma_{\text{tot}}(r_{-3}) / (\sqrt{3} \langle \sigma_{\text{los}} \rangle^*)$, respectively. For the component transverse to the projected radius R , the median results are found to be well approximated by $\sigma_{\mathcal{T}}(r_{-2}) \simeq \langle \sigma_{\mathcal{T}} \rangle^*$ within 10 per cent even when considering the 68 per cent dispersion for galaxies either with cusps and cores. Interestingly, uncertainties are well constrained for all of the galaxies in our sample when just considering binned unit circles of projected radius R . Looking at the relation argued in Wolf et al. (2010) and here for the total intrinsic velocity dispersion (referring to column 4), the median results are found to be well approximated by $\sigma_{\text{tot}}(r_{-3}) \simeq \sqrt{3} \langle \sigma_{\text{los}} \rangle^*$ to better than

about 10 per cent for the cusped galaxies. The galaxies with cores have this approximation accurate to 15–20 per cent when including the 1σ deviations.

Shown in Fig. 5 are the actual circular velocity curves compared to the combined measurements of the estimators at their characteristic radii, V_{-2} at r_{-2} (cyan points) and V_{-3} at r_{-3} (magenta points). The vertical error bars of the estimators depict the 1σ dispersion from all 1000 mock projections. The total circular velocity profile is given by the black curves for the CDM and grey curves for SIDM. Columns 5 and 6 list the ratio between these velocity estimators to the true value of the galaxies dynamical mass at the respective characteristic radii. The CDM galaxies perform remarkably well in predicting both the actual circular velocity measurement at r_{-2} and r_{-3} within 10 per cent including uncertainties. The SIDM galaxies are as good to 20 per cent when including 1σ dispersions. By examining the outliers, we see the worst offsets stem from difficulties in determining r_{-2} and r_{-3} of the simulated stellar density profile, as these profiles are, in essence, noisy, which makes the measurements of the log-gradient profiles without smoothing the density profile problematic.

Since the idealized estimators, V_{-2} and V_{-3} , predict the values of the dynamical profile to acceptable accuracy, we now see established predictions are when using characterizations modelled from the Plummer profile. Columns 9–12 in Table 3 give the results for performing a fit using a Plummer profile on the projected surface density in each mock observation. The resulting values of R_e are used to measure the stellar mass-weighted median dispersion, which have been depicted in Fig. 5. Columns 9 and 10 are the ratios of using the estimators with R_e while columns 11 and 12 are the comparisons of the characteristic radii to the predicted mapping. We see that for

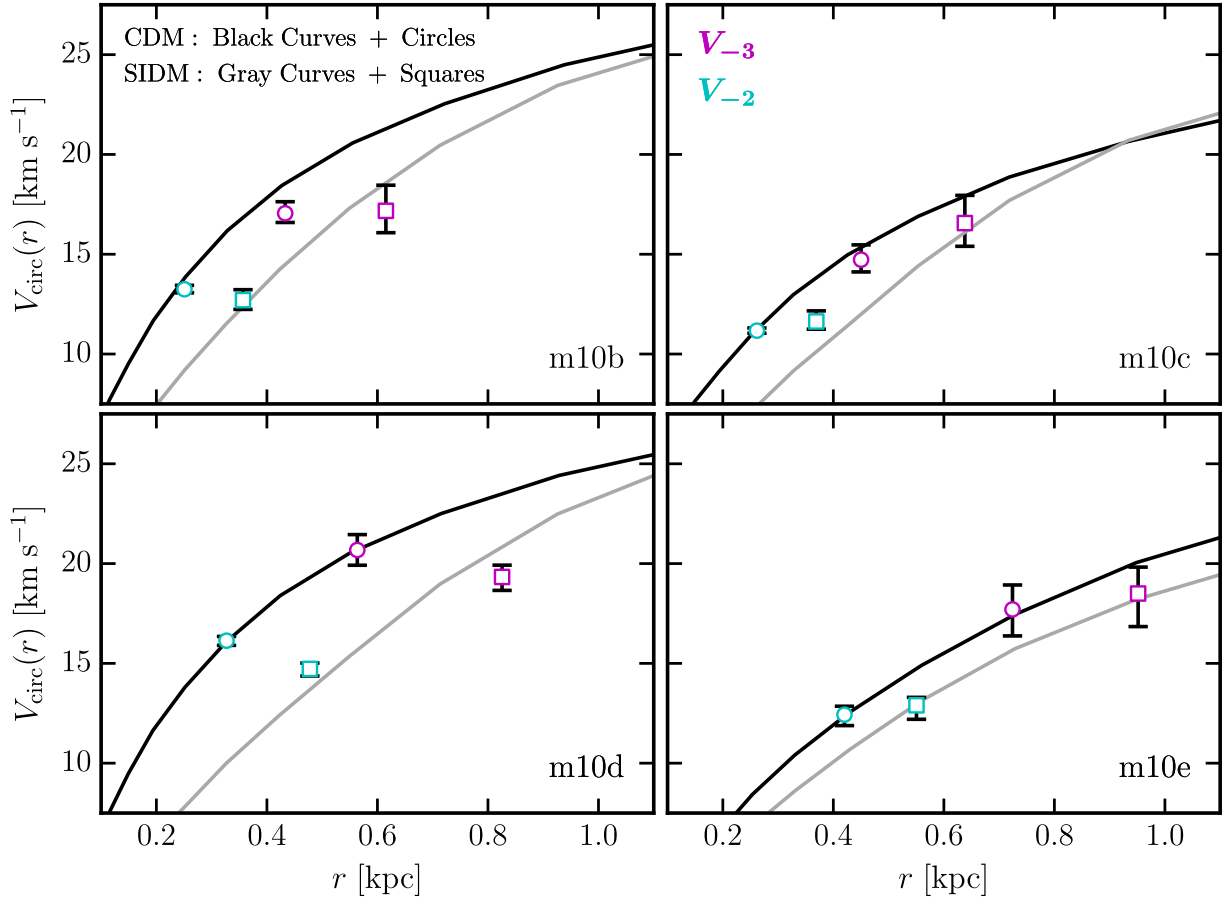


Figure 5. *Measurements from mock observations.* The rotation curves of our galaxies compared to the estimators at the characteristic radii r_{-2} and r_{-3} of the stellar density profile. Black and grey lines show the rotation curves for each system simulated in CDM (cusps) and SIDM (cores). The estimators V_{-2} (cyan) and V_{-3} (magenta) are plotted at r_{-2} and r_{-3} , respectively, where circles denote the estimators for the CDM galaxies and squares are for galaxies in SIDM. Error bars are the 1σ dispersion over all 1000 projections. Note that while the estimators are not perfect, they are accurate enough to discriminate between SIDM and CDM models in each case, especially when the two estimators are combined. Estimates of the shapes of the rotation curves will be more uncertain than the overall circular velocity normalization at each radius. Given large enough galaxy samples, measurements should enable a strong discriminant between CDM and SIDM based on normalization alone.

many galaxies, the Plummer fits do not provide precise enough characterizations to infer the values of r_{-2} and r_{-3} to better than ~ 20 per cent.

As for modelling the slope of the underlying profile, we expect that the local inner density behaves like a power law, $\rho \propto r^{-\alpha}$ such that the integrated mass scales as $M \propto r^{3-\alpha}$. This leads us the expected behaviour of circular velocity in relation to the local dark matter density: $V_{\text{circ}}^2 \propto r^{2-\alpha}$. We derive the implied slope given by the estimators by relating the inner density of the circular velocities as a power law that is defined like $\xi := \Delta \log V_{\text{circ}} / \Delta \log r$. This allows then to relate the power laws for the density profile, i.e. $\alpha = 2(1 - \xi)$.⁴ The implied slope of the circular velocity estimators is given by the dashed red line in Fig. 5. In columns 7 and 8, we give the implied slope of the combined estimators, ξ_{est} , and the true slope of the *dynamical* profile found at r_{-2} and r_{-3} , ξ_{true} , respectively. Without considering the 1σ dispersion of measurements, estimates from galaxies in CDM are predicted within 20 per cent while the SIDM analogues are off by almost 50 per cent. While the cuspy

profiles are reasonably well measured, the SIDM core profiles estimated to be too cuspy via this method. This is unfortunate, as this precision is not enough to distinguish between a cusp and core. However, the accuracy on the normalization (V_{-2} at r_{-2}) is good enough to discriminate between absolute core densities expected for CDM versus SIDM. With large enough data sets, this will provide important constraints on models of this kind.

6 DISCUSSION

We have used the spherical Jeans equation to infer two idealized mass estimators that depend on the stellar proper motions measured in observations. Specifically, we there are two radii, independent from one another, that potentially minimizes the anisotropy of the mass profile: one radius based off of measurements of the velocity dispersions *along* the line of sight and another radius from measurements for dispersions transverse along the plane of the sky.

6.1 Constraints from the virial theorem

The scalar virial theorem has been historically utilized to provide approximate mass constraints for spheroidal galaxies (e.g.

⁴Di Cintio et al. (2014) and Tollet et al. (2016) define cusps as $\alpha \simeq [1 - 1.5]$, which maps to $\xi \simeq 0.5$, and define cores as $\alpha \simeq [0 - 0.5]$, which maps to $\xi \simeq 0.75$.

Tully & Fisher 1977; Busarello et al. 1997). That is, the scalar virial theorem is observationally applicable, such that dispersion-supported systems can probe the integrated mass profile within the stellar extent without the degeneracies provided by the anisotropy. It is constructed from the diagonalized components of the velocity dispersion tensor, which describes the *local* distribution of velocities at each point in space. The trace of diagonal components provides an extended scalar virial theorem (Errani, Peñarrubia & Walker 2018)

$$\begin{aligned} \langle \sigma_\alpha^2 \rangle^* + \langle \sigma_\delta^2 \rangle^* + \langle \sigma_{\text{los}}^2 \rangle^* &= 4\pi G \int_0^\infty dr \, r n_\star(r) M(r) \\ &\equiv \langle \sigma_{\text{tot}}^2 \rangle^* \end{aligned} \quad (29)$$

where $\langle \sigma_\alpha^2 \rangle^*$ and $\langle \sigma_\delta^2 \rangle^*$ are defined as the luminosity-averaged velocity dispersions of the two velocity components tangential to the line of sight. By design, equation (29) provides a good integral constraint on the dynamical mass, as the entire expression is independent of the anisotropy. The line-of-sight component can be utilized as a constraint via the projected virial theorem (e.g. Agnello & Evans 2012; Errani et al. 2018). Adding dispersions in the α and δ directions would enable a tighter constraint on $\langle \sigma_{\text{tot}}^2 \rangle^*$. Note however, that when written this way we do not provide any additional constraint on β .

Working in a Cartesian coordinate system, such that $\text{los} \rightarrow z$, $\alpha \rightarrow x$, and $\delta \rightarrow y$, then spherical symmetry would demand each component of velocity dispersion to be equal: $\langle \sigma_x^2 \rangle^* = \langle \sigma_y^2 \rangle^* = \langle \sigma_z^2 \rangle^*$. The coordinate system introduced in Section 2 does not force this equality and allows for separate components of the luminosity-averaged velocity dispersion to constrain the velocity dispersion anisotropy β (Strigari et al. 2007). The two components, σ_R and σ_T , depend on β differently and are not necessarily equal.⁵ Note that when equations (8)–(10) are added together, we find $\langle \sigma_T^2 \rangle^* + \langle \sigma_R^2 \rangle^* + \langle \sigma_{\text{los}}^2 \rangle^* = \langle \sigma_{\text{tot}}^2 \rangle^*$ such that equation (29) can be satisfied. By examining the components separately, we can have mass estimators that provide information at a different radius than the one enabled from line-of-sight motions alone.

6.2 Possible biases in Jeans modelled mass estimates

Our mass estimates rely on the fact that dispersion-supported systems are approximately in dynamical equilibrium and are accurately modelled by the spherical Jeans equation. Non-steady-state systems, ones that significantly deviate from dynamical equilibrium, can lead to biased estimations of the complete dynamical mass. This can lead to systematically biased mass estimates (e.g. Amorisco & Evans 2011; Errani et al. 2018). For the simulated galaxy sizes considered in our analysis, mass estimates with short time-scale fluctuations of the potential well are non-trivially biased (El-Badry et al. 2017; González-Samaniego et al. 2017). For the largest kind of dispersion-supported systems, ones with a stellar mass of $M_\star \approx 10^8 - 10^{10} M_\odot$, uncertainty is as large 20 per cent of the dynamical mass. To minimize the variability of energetic outflows, mass estimates are best focused on dwarf spheroidals at around the threshold of lowest detectability, i.e. low-mass dwarfs, as this should reduce the likelihood of potential fluctuations biasing the stellar tracers. Using simulated data, we have shown that our estimate at r_{-2} is able to obtain the normalization to better than 20 per cent when using V_{-2} for low-mass dwarf galaxies. As for the applicability to observations, it is important that careful measurements of the highest precision are obtained in order to dissociate between possible models embellished with systematic errors

Although our simulated galaxies are analogous to those in the field, Local Group satellites also experience tidal stripping of the main halo, which can preferentially bias the estimates of the satellites dynamical mass. However, analysis from Klimontowski et al. (2007) has already eluded that velocity dispersions are well modelled by the Jeans equation for even in the case of mildly tidally disrupted dwarf galaxies, as long as unbound, interloping stars are properly accommodated for in the stellar sample. For the case of Draco and Sculptor considered here, they are both satellites of the Milky Way and are therefore, in principle, subjected to tidal forces that could render mass models from the Jeans equation inadequate. However, no sign of strong tidal influence is apparent (Piatek et al. 2002; Coleman, Da Costa & Bland-Hawthorn 2005).

7 CONCLUDING REMARKS

Using the spherical Jeans equation, we have derived a mass estimator that depends on stellar kinematics measured along the plane of the sky, specifically the velocity dispersion tangential to the projected radius R . We have shown that under idealized but reasonable assumptions, equation (26) provides the cumulative mass within a characteristic radius, r_{-2} , independent of the stellar velocity dispersion anisotropy β . This ideal radius is where the log-slope of the underlying tracer profile is -2 . For Plummer profiles $r_{-2} \simeq 4R_c/5 \simeq 3r_{1/2}/5$. We also showed that a β -independent estimator does not exist for the velocity dispersion parallel along the plane of the sky. Fig. 1 summarizes this result. Our derivation followed the approach in Wolf et al. (2010), and relied on similar assumptions: that the stellar velocity dispersion profiles $\sigma_r(r)$ and $\beta(r)$ vary slowly compared to the tracer profile itself out to $r_{1/2}$.

To test our assumptions and our estimators, we employ previously published simulations of dwarf galaxies done for both CDM and SIDM dark matter physics. We find that σ_T is indeed flat in the vicinity of r_{-2} for both dwarf galaxies of CDM and SIDM and found that our mass estimator is accurate in quantifying the enclosed mass at r_{-2} . For CDM, the estimates for the dynamic rotation curves are found to be accurate to 10 per cent for both estimators while SIDM are accurate to 15 per cent. This level of absolute mass accuracy is good enough to discriminate between expected core densities in SIDM and CDM models. Unfortunately, this level of accuracy is *not* good enough to regularly measure slopes at the precision required to differentiate between cusps and cores in real data without deeper prior to help us understand the underlying tracer profile shape in real galaxies. However, the difference in absolute circular velocity predicted between SIDM and CDM at these radii is well within the normalization uncertainties of the estimators (see Fig. 5).

As an example of the applicability of our estimator, we have combined it with the Wolf et al. (2010) estimator at r_{-3} for line-of-sight velocities to explore the mass profiles of Draco and Sculptor. Both galaxies are consistent with inhabiting cuspy NFW subhaloes with densities consistent with those expected in CDM with $V_{\text{max}} \simeq 34$ and 19 km s^{-1} , respectively, though current uncertainties allow for a variety of inner profile slopes and are consistent with SIDM densities given the sparsity of the data. In the coming era of precision-based measurements of stellar proper motions, we expect the internal structure of dwarf galaxies to be revealed with more clarity.

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⁵Though symmetry demands $\langle \sigma_T^2 \rangle^* + \langle \sigma_R^2 \rangle^* = \frac{2}{3} \langle \sigma_{\text{tot}}^2 \rangle^* = 2 \langle \sigma_{\text{los}}^2 \rangle^*$.

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APPENDIX A: MASS PROFILES AS A FUNCTION OF OBSERVABLES

Here, we derive a single expression for the mass profile of spheroidal galaxies as a function of observable combinations found in the plane of the sky. A crucial argument that we have imposed previously is that projected observables can be re-formalized to de-projected quantities, and vice versa. To do this, we make note of the utilization of the Abel inversion (Binney & Tremaine 2008)

$$f(x) = \int_x^\infty \frac{dt}{\sqrt{t-x}} g(t) \iff g(x) = -\frac{1}{\pi} \int_t^\infty \frac{dx}{\sqrt{x-t}} \frac{df}{dt}. \quad (\text{A1})$$

A1 Measurements along the line of sight

While a complete derivation is given in Wolf et al. (2010), we quote the mass profile for completeness: Given measurements along the line of sight and assuming a constant anisotropy model, $\beta(r) = \beta_0$, the mass profile come to be

$$M(r|\beta_0) = \frac{\{\mathcal{K}_1(r, R|\beta_0) + \mathcal{K}_2(r, R|\beta_0)\}}{G\pi(\beta_0 - 1)n_\star(r)}, \quad (\text{A2})$$

where the integral kernels are

$$\begin{aligned} \mathcal{K}_1(r, R|\beta_0) &= \int_{r^2}^\infty dR^2 R^2 \frac{d^2(\Sigma_\star \sigma_{\text{los}}^2)}{(dR^2)^2} \frac{2r^3/R^3}{\sqrt{1-r^2/R^2}} \\ \mathcal{K}_2(r, R|\beta_0) &= \int_{r^2}^\infty dR^2 R^2 \frac{d^2(\Sigma_\star \sigma_{\text{los}}^2)}{(dR^2)^2} \beta_0 \frac{3-2\beta_0}{\beta_0-1} \\ &\quad \times \left(\frac{r}{R}\right)^{\frac{1}{1-\beta_0}} \mathcal{B}_{1-r^2/R^2} \left(\frac{1}{2}, \frac{2-3\beta_0}{2(1-\beta_0)}\right), \end{aligned}$$

For compactness, the lower incomplete beta function is incorporated

$$\mathcal{B}_x(p, q) := \int_0^x dy y^{p-1} (1-y)^{q-1}. \quad (\text{A3})$$

In Fig. 1, we demonstrate the robustness of this mass profile. As an example, we consider the assumptions used in deriving the idealized case of M_{-3} : a constant β model and a constant velocity dispersion measurement $\sigma_{\text{los}}^2(R) \simeq \langle \sigma_{\text{los}}^2 \rangle^*$. For $\langle \sigma_{\text{los}}^2 \rangle^*$, we use the

median observational quantities for Draco and Sculptor given in Table 1. As shown, the different values of β_0 converge at r_{-3} of the stellar density profile, at least for an assumed Plummer profile. The idealized mass estimator, M_{-3} , is shown as the white dot intersecting where the profiles converge.

A2 Plane of the sky: Measurements parallel to R

To further clarify that lack of a radius that minimizes the anisotropy in an idealized case, consider the utilization of the Jeans equation for measurements of σ_R . We start by the massaging the form of equation (9) in order to isolate out the R dependence in the integral kernel:

$$\begin{aligned}\Sigma_* \sigma_R^2(R) &= \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} \left[1 - \beta(r) + \frac{R^2}{r^2} \beta(r) \right] n_* \sigma_r^2(r) \\ &= \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} n_* \sigma_r^2 - \int_{R^2}^{\infty} dr^2 \frac{n_* \sigma_r^2 (r^2 - R^2)}{\sqrt{r^2 - R^2}} \\ &= \int_{R^2}^{\infty} \frac{dr^2}{\sqrt{r^2 - R^2}} \left[n_* \sigma_r^2(r) - \int_{r^2}^{\infty} d\tilde{r}^2 \frac{\beta n_* \sigma_r^2}{2\tilde{r}^2} \right]. \quad (\text{A4})\end{aligned}$$

Here in the second line, we expanded the second term with integration by parts and evaluating the boundary integration to null by motivating that combination of $\beta n_* \sigma_r^2$ falls faster than r^{-1} at large r .

We then use the invertable form and deproject via an Abel inversion to obtain

$$n_* \sigma_r^2(r) - \int_{\log r}^{\infty} d \log \tilde{r} \beta n_* \sigma_r^2 = -\frac{1}{\pi} \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d(\Sigma_* \sigma_R^2)}{dR^2}. \quad (\text{A5})$$

To isolate out $n_* \sigma_r^2$, we differentiate with respect to $\log r$,

$$\frac{d(n_* \sigma_r^2)}{d \log r} + \beta n_* \sigma_r^2 = -\frac{2r^2}{\pi} \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d^2(\Sigma_* \sigma_R^2)}{(dR^2)^2}, \quad (\text{A6})$$

and then deploy the integrating factor

$$h(r) = \exp \left\{ \int_{\log a}^{\log r} d \log \tilde{r} \beta(r) \right\}, \quad (\text{A7})$$

where a is a constant chosen so that the value of the integrand approaches zero at the lower limit. This then gives us

$$\begin{aligned}n_* \sigma_r^2(r|\beta) &= -\frac{h^{-1}}{\pi} \int_{r^2}^{\infty} d\tilde{r}^2 \left[\int_{\tilde{r}^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - \tilde{r}^2}} \frac{d^2(\Sigma_* \sigma_R^2)}{(dR^2)^2} \right] h \\ &= -\frac{h^{-1}}{\pi} \int_{r^2}^{\infty} dR^2 \left[\int_{r^2}^{\tilde{r}^2} \frac{d\tilde{r}^2}{\sqrt{R^2 - \tilde{r}^2}} h \right] \frac{d^2(\Sigma_* \sigma_R^2)}{(dR^2)^2}.\end{aligned} \quad (\text{A8})$$

Here, $n_* \sigma_r^2(r)$ can be modelled by the adoption of a parametric form of $\beta(r)$. This can also be taken and inserted in equation (4) to model the integrated mass.

For our idealized case, suppose the anisotropy is taken to be a constant value for the inner region of the system, $\beta(r) = \beta_0$. We will have $h(r) \rightarrow r^{\beta_0}$ which allows us rewrite the inner integral in terms of the lower incomplete beta function:

$$\begin{aligned}n_* \sigma_r^2(r|\beta_0) &= \frac{r^{-\beta_0}}{\pi} \int_{r^2}^{\infty} dR^2 R^{\beta_0+1} \frac{d^2(\Sigma_* \sigma_R^2)}{(dR^2)^2} \mathcal{B}_{1-r^2/R^2} \\ &\times \left(\frac{1}{2}, \frac{\beta_0 + 2}{2} \right).\end{aligned} \quad (\text{A9})$$

The mass profile is then obtained by hitting the previous expression with a derivative in respect to $\log r$ and insert it into equation (4)

to acquire the implied profile

$$M(r|\beta_0) = \frac{r}{\pi G n_*(r)} \left\{ \tilde{\mathcal{R}}_1(r|\beta_0) + \tilde{\mathcal{R}}_2(r|\beta_0) \right\}, \quad (\text{A10})$$

where

$$\begin{aligned}\tilde{\mathcal{R}}_1(r|\beta_0) &= 2r^2 \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d^2(\Sigma_* \sigma_R^2)}{(dR^2)^2}, \\ \tilde{\mathcal{R}}_2(r|\beta_0) &= -\beta_0 r^{-\beta_0} \int_{r^2}^{\infty} dR^2 R^{\beta_0+1} \frac{d^2(\Sigma_* \sigma_R^2)}{(dR^2)^2} \mathcal{B}_{1-r^2/R^2} \\ &\times \left(\frac{1}{2}, \frac{\beta_0 + 2}{2} \right).\end{aligned}$$

This relation replaces the dependence of deriving the mass of a dispersion-supported system from unknown radial velocity dispersion with the second ordered derivatives of the the observable product, $\Sigma_* \sigma_R^2(R)$. The middle plot in Fig. 1 realizes equation (A10) for various values of β_0 and a constant velocity dispersion, $\sigma_R^2 \simeq \langle \sigma_R^2 \rangle^*$. We use the median parameters of Sculptor given in Table 1 as demonstration. As we predicted previously, none of the constant β_0 dependent Jeans mass profiles converge to a mass value like we have seen in the top plot in Fig. 1.

A3 Plane of the sky: Measurements transverse to R

Consider the application of the Jeans equation for measurements based on the mapping of equation (10). Since this is already in a form that is invertable, we deproject via an Abel inversion in order to isolate out $n_* \sigma_r^2(r)$ combination:

$$n_* \sigma_r^2(r) = \frac{1}{\pi(\beta - 1)} \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d(\Sigma_* \sigma_T^2)}{dR^2}. \quad (\text{A11})$$

We see that the above relation is unique in comparison with what we have seen in the previous sections, as equation (A11) is unembellished and relatively simple in its form. This allows us to write a mass profile that can be dependent on some arbitrary form of $\beta(r)$.

A3.1 Spatially constant velocity dispersion anisotropy

Assume a constant anisotropy model, $\beta(r) = \beta_0$. It is straightforward to differentiate both sides with respect to $\log r$ and massage it to the form that, along with equation (A11), can be inserted into equation (4) to obtain the implied mass profile

$$M(r|\beta_0) = \frac{2r}{G\pi(1 - \beta_0)n_*(r)} \left\{ \tilde{\mathcal{T}}_1(r|\beta_0) + \tilde{\mathcal{T}}_2(r|\beta_0) \right\}, \quad (\text{A12})$$

where

$$\begin{aligned}\tilde{\mathcal{T}}_1(r|\beta_0) &= r^2 \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d^2(\Sigma_* \sigma_T^2)}{(dR^2)^2}, \\ \tilde{\mathcal{T}}_2(r|\beta_0) &= \beta_0 \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d(\Sigma_* \sigma_T^2)}{dR^2}.\end{aligned}$$

With this, we have eliminated the dependency of the unknown dispersion profile, $\sigma_r(r)$, and have the mass profile be dependent only on the form of well-defined observables and an arbitrary value of β_0 . The bottom plot in Fig. 1, actualizes this mass profile by considering the idealized assumptions used in the derivation of M_{-2} : an constant anisotropy model and a constant velocity dispersion $\sigma_T^2 \simeq \langle \sigma_T^2 \rangle^*$. We use the median values of Sculptor from Table 1 for as a demonstration. As shown, we recover the *idealized* results derived previously in the main text, where the of the mass profiles converge at r_{-2} for various values of constant anisotropy, β_0 .

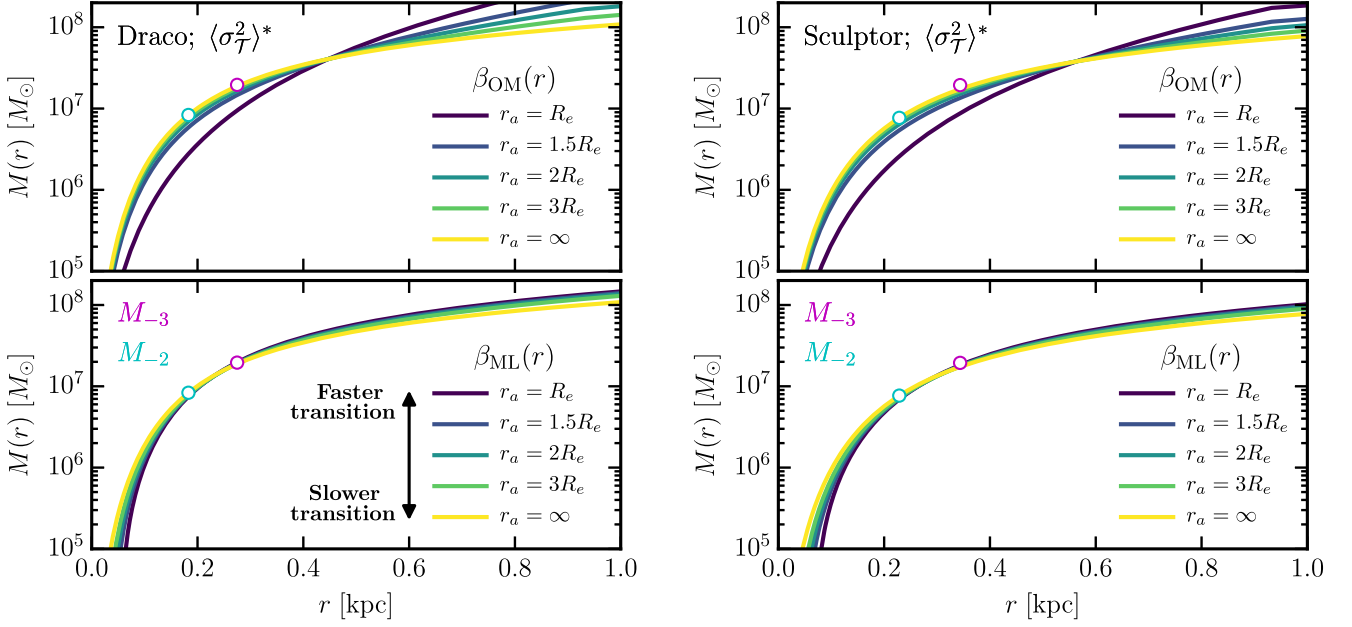


Figure A1. Integrated mass profiles for Draco and Sculptor with radial anisotropy. The lines correspond to several choices of fixed velocity dispersion anisotropy β_0 . Mass profile curves are generated using equation (A14) assuming a Plummer profile and constant σ_{los} . Here, the small white circle indicates the mass predicted by the *idealized* estimator, equation (26), which intersects at the mass that is independent of the value of β_0 .

A3.2 General velocity dispersion anisotropy

Assume now a generalized $\beta(r)$ model, we differentiate both sides of equation (A11) with respect to $\log r$ to have

$$\frac{d(n_* \sigma_r^2)}{d \log r} (\beta - 1) - \gamma_\beta \beta n_* \sigma_r^2(r) = \frac{2r^2}{\pi} \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \times \frac{d^2(\Sigma_* \sigma_T^2)}{(dR^2)^2}. \quad (\text{A13})$$

We then expand out the differential on the left-hand side and substitute its resulting form into equation (4) to obtain the implied mass profile for some generalized $\beta(r)$ model:

$$M(r|\beta) = \frac{2r}{G\pi[1 - \beta(r)]n_*(r)} \left\{ \tilde{\mathcal{T}}_1(r|\beta) + \tilde{\mathcal{T}}_2(r|\beta) \right\}, \quad (\text{A14})$$

where

$$\tilde{\mathcal{T}}_1(r|\beta) = r^2 \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d^2(\Sigma_* \sigma_T^2)}{(dR^2)^2},$$

$$\tilde{\mathcal{T}}_2(r|\beta) = \beta(r) \left[1 - \frac{\gamma_\beta}{2[1 - \beta(r)]} \right] \int_{r^2}^{\infty} \frac{dR^2}{\sqrt{R^2 - r^2}} \frac{d(\Sigma_* \sigma_T^2)}{dR^2}.$$

In the limiting case of a constant β_0 model, we see that we re-obtain equation (A12). In particular, γ_β term, if large enough, could nominally shift the location of r_{-2} of the implied mass profile, as this is true for any anisotropic characterizations that become close to isotropic. However, the behaviour of the $\beta\gamma_\beta$ combination in $\tilde{\mathcal{T}}_2(r)$ is still well behaved.

In Fig. A1, we demonstrate equation (A14) by only assuming a constant velocity dispersion, $\sigma_T^2(R) \simeq \langle \sigma_T^2 \rangle^*$, and a Plummer profile. We consider two functional forms frequently used to model in observational Jeans modelling studies:

(i) The *Osipkov–Merritt* (Osipkov 1979; Merritt 1987, subscript OM) profile is formulated from a one-parameter family of spheri-

cally stellar systems, such that,

$$\beta_{\text{OM}}(r) = \frac{r^2}{r^2 + r_a^2}, \quad (\text{A15})$$

where r_a is the *anisotropy* radius that gives the stellar sub-component of a velocity distribution. Notably, $\beta \rightarrow 0$ as $r \ll r_a$ and $\beta \rightarrow 1$ as $r \gg r_a$. Different values of r_a are given in the top row.

(ii) Another parametrization is introduced by *Mamon–Łokas* (Mamon & Łokas 2005, subscript ML) profile:

$$\beta_{\text{ML}}(r) = \frac{1}{2} \frac{r}{r + r_a}. \quad (\text{A16})$$

Similar limits as $\beta_{\text{OM}}(r)$, but $\beta \rightarrow 1/2$ as $r \gg r_a$. Different values of r_a are given in the bottom row.

The values of r_a are subjugated between two extreme cases, one where $r_a = \infty$, making the profile isotropic at all radii, and another as low as $r_a = R_e$, which allows for a quicker transition of the profile being anisotropic to isotropic around the region of R_e . For values of $r_a \gtrsim 2R_e$, the estimator appears robust for both models of $\beta_{\text{OM}}(r)$ and $\beta_{\text{ML}}(r)$. In the instances where $r_a \simeq R_e$, both M_{-3} (magenta circle) and M_{-2} (cyan circle) fail for the faster transition value of β_{OM} while is still consistent with β_{ML} . Indubitably, as M_{-3} and M_{-2} breaks down as the log-slope of the anisotropy profile is too large. The relative simplicity of equation (A14) allows for a complete modelling of the mass profile based off of the proper motions tangential along the plane of the sky. The caveat here is that realization of the complete dynamical profile requires highly accurate observational data, as first and second derivatives of surface profile is required.

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