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Closed-loop stabilization of nonlinear systems using Koopman Lyapunov-based model predictive control

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Abstract— **This work considers the problem of stabilizing feedback control design for nonlinear systems. To achieve this, we integrate Koopman operator theory with Lyapunov-based model predictive control (LMPC). A bilinear representation of the nonlinear dynamics is determined using Koopman eigenfunctions. Then, a predictive controller is formulated in the space of Koopman eigenfunctions using an auxiliary Control Lyapunov Function (CLF) based bounded controller as a constraint which enables the characterization of stability of the Koopman bilinear system. Unlike previous studies, we show via an inverse mapping - realized by continuously differentiable functions - that the designed controller translates the stability of the Koopman bilinear system to the original closed-loop system. Remarkably, the feedback control design proposed in this work remains completely data-driven and does not require any explicit knowledge of the original system. Moreover, in contrast to standard LMPC, seeking a CLF for the bilinear system is computationally favorable compared to the original nonlinear system. The application of the proposed method is illustrated on a numerical example.**

I. INTRODUCTION

Nonlinear systems abound in nature. Yet, a universal feedback design for stabilizing nonlinear dynamics remains a daunting challenge unlike its linear counterpart. Existing approaches such as optimization-based Sum of Squares (SoS) [1], geometric-based feedback linearization [2], sliding mode control [3], etc. use state-space description for nonlinear stabilizing control. An alternative to the state-space description is the operator-theoretic viewpoint where we are interested in the evolution of observables (functions of states) and not the states. One example is the Koopman operator which, when acted upon an observable, governs its evolution along the original system trajectory [4]. Hence, the operator-theoretic description provides global insight into the system dynamics which is appropriate for controller design.

The most attractive feature of the Koopman operator theory is that it is a linear operator even when associated with nonlinear dynamics. This means that the spectral properties of the linear operator (i.e., eigenvalues and eigenfunctions) encode global information that allows future state prediction and scalable reconstruction of the underlying dynamics [5]. Several data-driven methods have been developed to approximate the spectrum of the Koopman operator from time-series data of the system such as Dynamic Mode Decomposition [6], Extended Dynamic Mode Decomposition (EDMD) [7], Laplace analysis [8], and machine learning [9].

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These advanced data-driven algorithms have sparked increased research activity in the analysis and control of nonlinear dynamical systems [10]–[15]. However, ensuring stability of the resulting closed-loop system has proven to be difficult as the predictive capability of the Koopman operator can be significantly impacted unless the role of actuation is appropriately accounted. To deal with this, [16] redefined the Koopman operator as a function of both states and the inputs. In [17], a modification of EDMD was presented that compensates for the effect of inputs. In [18], a bilinear representation was provided in the Koopman space that is tight and theoretically justified. Using this representation, the authors in [19] proposed a stabilizing feedback controller which relies on control Lyapunov function (CLF) and achieves stabilization of the Koopman bilinear system.

However, [19] does not consider an optimal control problem accounting for explicit state and input constraints and the stability analysis of the original nonlinear system was not provided. To address this, CLFs were employed in [20] where a feedback controller was designed for the Koopman space (i.e., lifted domain) using Lyapunov constraints within a model predictive control (MPC) formulation. Such a design allowed for explicit characterization of stability properties of the original nonlinear system. However, the method presented in [20] uses CLFs derived for the original system which requires explicit mathematical expression of the original nonlinear dynamics; it is particularly challenging when we have limited a priori knowledge of the original nonlinear system. Additionally, even though we have a good understanding of a general nonlinear system, it is in practice computationally demanding to determine the CLFs.

To address these issues, this work seeks to derive a stabilizing feedback controller based on the Koopman bilinear representation of the original nonlinear system. To do so, a CLF is determined for the bilinear system in the Koopman eigenfunction space which is employed in the Lyapunov-based MPC (LMPC) formulation. Then, a stability criterion is presented that guarantees stability of the original closed-loop system in the $\epsilon - \delta$ sense from the stability of the Koopman bilinear system. Unlike [20], the feedback control design proposed in this work is completely data-driven and does not require any a priori knowledge of the original system. Moreover, deriving CLFs for the Koopman bilinear system is much more computationally affordable than the original nonlinear system. In fact, the search for CLFs can be focused on a class of quadratic functions which are known to effectively characterize the stability region of simpler systems like the (Koopman) bilinear systems.

II. Preliminaries

A. Overview of Koopman Operator

Let us consider a continuous-time nonlinear dynamical system given by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{1}$$

where $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ is the vector of state variables and $\mathbf{F} : \mathcal{X} \to \mathcal{X}$ is the evolution operator that represents the dynamics which map the system states forward in time. It is assumed that the vector field $\mathbf{F}$ is continuously differentiable. Let us denote the solution of (1) by $\mathbf{\Phi}^t(\mathbf{x})$.

Definition 1 (Koopman operator): For a given space $\mathcal{G}$ of observables $g(\mathbf{x})$ with $g \in \mathcal{G} : \mathcal{X} \to \mathbb{C}$, the Koopman (semi)group of operators $\mathcal{K}^t : \mathcal{G} \to \mathcal{G}$ associated with system (1) is defined by

$$[\mathcal{K}^t g](\mathbf{x}) = g \circ \mathbf{\Phi}^t(\mathbf{x}) \tag{2}$$

By definition, the Koopman operator is linear even though the underlying dynamical system is nonlinear and therefore can be characterized by its eigenvalues and eigenfunctions. An eigenfunction $\psi \in \mathcal{G} : \mathcal{X} \to \mathbb{C}$ of the Koopman operator is defined to satisfy

$$\begin{aligned} &[\mathcal{K}^t \psi](\mathbf{x}) = e^{\lambda t}\psi(\mathbf{x}) \\ &\frac{d}{dt}\psi(\mathbf{x}) = \lambda \psi(\mathbf{x}) \end{aligned} \tag{3}$$

where $\lambda \in \mathbb{C}$ is the associated eigenvalue.

B. Koopman bilinear system identification

The Koopman operator theory has been conceptually developed for uncontrolled systems. Here, we adopt it by considering a control affine system as follows:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{G}_i(\mathbf{x}) u_i \tag{4}$$

where $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$, $u_i \in \mathcal{U}$ for $i = 1, \cdots, m$ and $\mathbf{G}_i : \mathcal{X} \to \mathcal{X}$ denotes the control vector fields that dictate the effect of input on the system. Now, the evolution of the observable functions for the controlled system of (4) is given by

$$\frac{\partial}{\partial t}\tilde{g} = L_{\mathbf{F}}\tilde{g} + \sum_{i=1}^{m} u_i L_{\mathbf{G}_i}\tilde{g}; \quad \tilde{g}(0, \mathbf{x}) = g(\mathbf{x}) \tag{5}$$

where $L_{\mathbf{F}}$ and $L_{\mathbf{G}_i}$ denote the Lie derivatives with respect to the vector fields $\mathbf{F}$ and $\mathbf{G}_i$ for $i = 1, \cdots, m$, respectively. The system (5) is analogous to a bilinear system except that the operators $L_{\mathbf{F}}$ and $L_{\mathbf{G}_i}$ are infinite dimensional, operating on the function space $\mathcal{G}$. For practical implementation, we can determine a finite-dimensional approximation [18]. Let

$$\begin{aligned} &\mathbf{z} = \mathbf{\Psi}(\mathbf{x}) = [\psi_1(\mathbf{x}), \cdots, \psi_N(\mathbf{x})]^T \\ &\psi_j(\mathbf{x}) = \tilde{\psi}_j(\mathbf{x}), \quad \text{if } \tilde{\psi}_j(\mathbf{x}) : \mathcal{X} \to \mathbb{R} \\ &[\psi_j(\mathbf{x}),\ \psi_{j+1}(\mathbf{x})]^T = [2Re(\tilde{\psi}_j(\mathbf{x})),\ -2Im(\tilde{\psi}_j(\mathbf{x}))]^T, \\ &\qquad\qquad \text{if } \tilde{\psi}_j,\ \tilde{\psi}_{j+1}(\mathbf{x}) : \mathcal{X} \to \mathbb{C} \end{aligned} \tag{6}$$

Applying the above transformation to (4) yields

$$\dot{\mathbf{z}} = \Lambda \mathbf{z} + \sum_{i=1}^{m} u_i L_{\mathbf{G}_i}\mathbf{\Psi} \tag{7}$$

Assumption 1: $\exists\ \psi_j, j = 1, \cdots, N$ such that

$$L_{\mathbf{G}_i}\mathbf{\Psi} = \sum_{j=1}^{N} b_j^{\mathbf{G}_i}\psi_j(\mathbf{x}) = B_i \mathbf{\Psi}$$

where $b_j^{\mathbf{G}_i} \in \mathbb{R}^n$ and $\psi_j(\mathbf{x})$ are defined in (6). In other words, it is assumed that $L_{\mathbf{G}_i}\mathbf{\Psi}$ lies in the span of the eigenfunctions $\psi_j,\ j = 1, \cdots, N$ so that it can be represented using a constant matrix, $B_i \in \mathbb{R}^{N \times N}$.

Based on this assumption, the system (7) becomes the following bilinear control system in the Koopman space,

$$\dot{\mathbf{z}} = \Lambda \mathbf{z} + \sum_{i=1}^{m} u_i B_i \mathbf{z} \tag{8}$$

where Λ is a block-diagonal matrix constructed using the Koopman eigenvalues $\lambda_j, j = 1, \cdots, N$ in a manner similar to Koopman eigenfunctions shown in (6), i.e.,

$$\begin{aligned} &\Lambda_{j,j} = \lambda_j, \quad \text{if } \lambda_j \in \mathbb{R} \\ &\begin{bmatrix} \Lambda_{j,j} & \Lambda_{j,j+1} \\ \Lambda_{j+1,j} & \Lambda_{j+1,j+1} \end{bmatrix} = |\lambda_j| \begin{bmatrix} cos(\angle\lambda_j) & sin(\angle\lambda_j) \\ -sin(\angle\lambda_j) & cos(\angle\lambda_j) \end{bmatrix}, \\ &\qquad \text{if } \lambda_j, \lambda_{j+1} \in \mathbb{C} \end{aligned} \tag{9}$$

To determine the above continuous bilinear system using time-series data generated by (4), the EDMD algorithm [7] is utilized in this work. Please refer to [19] for the detailed algorithm computing system matrices Λ and B_i. Please note that EDMD cannot directly approximate systems whose Koopman operator exhibit continuous spectrum. This is a challenging issue that remains a subject of research in the field of operator theory [9].

III. Stabilization using Koopman Lyapunov MPC

LMPC provides a powerful tool for the design of an optimal stabilizing feedback controller for nonlinear dynamical systems [21]. Essentially, LMPC is a control strategy that is designed based on an explicit, stable (albeit not optimal) control law $\mathbf{h}(\cdot)$ and a Lyapunov constraint by virtue of which the controller is able to stabilize the closed-loop system. In the proposed method, LMPC is applied in the Koopman eigenspace to determine a stabilizing input for the resulting closed-loop system.

For simplicity, let us consider the Koopman bilinear system of (8) with $i = 1$, i.e., a single input. This system is assumed to be stabilizable (and controllable), which implies the existence of a feedback control law $\mathbf{u}(t) = \mathbf{h}(\mathbf{z})$ that satisfies input constraints for all $\mathbf{z}$ inside a given stability region and renders the origin of the closed-loop system asymptotically stable. This is equivalent to assuming that there exists a CLF for the system of (8). Due to the bilinear structure of the system, the CLF can be limited to a class of quadratic functions, i.e., $V(\mathbf{z}) = \mathbf{z}^T P \mathbf{z}$. The necessary and sufficient conditions for the symmetric positive definite matrix P such that the system of (8) is stabilizable are provided in [19]. The theorem is stated below.

Proposition 1 (see [19], Theorem 2): The bilinear system of (8) is stabilizable if and only if there exists an

$N \times N$ symmetric positive definite matrix P such that for all $\mathbf{z} \neq 0 \in \mathbb{R}^N$ with $\mathbf{z}^T(P\Lambda + \Lambda^T P)\mathbf{z} \geq 0$, we have $\mathbf{z}^T(PB + B^T P)\mathbf{z} \neq 0$.

In other words, for $\dot{V}(\mathbf{z}) = \mathbf{z}^T(P\Lambda + \Lambda^T P)\mathbf{z} + u(\mathbf{z}^T(PB + B^T P)\mathbf{z})$ to be negative, if the first term on the right hand side is positive then the second term cannot be zero so that the control action u can render $\dot{V} < 0$. Once the conditions of *Proposition 1* are satisfied, one way to determine the explicit control law $h(\mathbf{z})$, required to stabilize the bilinear system, is provided by Sontag's formula [22] as below:

$$b(\mathbf{z}) = \begin{cases} -\frac{L_\Lambda V + \sqrt{L_\Lambda V^2 + L_B V^4}}{L_B V}, & \text{if } L_B V \neq 0 \\ 0, & \text{if } L_B V = 0 \end{cases}$$

$$h(\mathbf{z}) = \begin{cases} u_{min}, & \text{if } b(\mathbf{z}) < u_{min} \\ b(\mathbf{z}), & \text{if } u_{min} \leq b(\mathbf{z}) \leq u_{max} \\ u_{max}, & \text{if } b(\mathbf{z}) > u_{max} \end{cases} \tag{10}$$

where $L_\Lambda V = \mathbf{z}^T(P\Lambda + \Lambda^T P)\mathbf{z}$, $L_B V = \mathbf{z}^T(PB + B^T P)\mathbf{z}$, and $h(\mathbf{z})$ represents the saturated control law that accounts for the input constraints $u_{min} \leq u(t) \leq u_{max} \in \mathcal{U}$. Let the largest level set of V be given by $\Omega_r = \{\mathbf{z} \in \mathbb{R}^N : V(\mathbf{z}) \leq r\}$ where r is the largest number for which $\Omega_r \subseteq \Omega$, Ω is the complete stability region, starting from which the origin of the closed-loop system under (10) is guaranteed to be stable.

Now that we have the explicit control law, the idea is to stabilize the bilinear system using the LMPC scheme as below:

$$\min_{u \in \mathcal{S}(\Delta)} \int_{t_k}^{t_{k+N_p}} [\mathbf{z}^T(\tau) W \mathbf{z}(\tau) + u^T(\tau) R u(\tau)] d\tau, \tag{11a}$$

$$\text{s.t} \quad \dot{\mathbf{z}}(t) = \Lambda \mathbf{z}(t) + u(t) B \mathbf{z}(t) \tag{11b}$$

$$\mathbf{z}(t_k) = \mathbf{\Psi}(\mathbf{x}(t_k)) \tag{11c}$$

$$u_{min} \leq u(t) \leq u_{max}, \ \forall t \in [t_k, t_{k+N_p}) \tag{11d}$$

$$V(\mathbf{z}(t)) \leq \hat{r}, \quad \forall t \in [t_k, t_{k+N_p}] \quad \text{if } \mathbf{z}(t_k) \in \Omega_{\hat{r}} \tag{11e}$$

$$\dot{V}(\mathbf{z}(t_k), \mathbf{u}(t_k)) \leq \dot{V}(\mathbf{z}(t_k), h(\mathbf{z}(t_k))), \quad \text{if } \mathbf{z}(t_k) \in \Omega_r / \Omega_{\hat{r}} \tag{11f}$$

where $\mathcal{S}(\Delta)$ is the family of piece-wise constant functions with sampling period $\Delta = t_{k+1} - t_k$, N_p is the prediction horizon, and $W \in \mathbb{R}^{N \times N}$ and $R \in \mathbb{R}$ are positive definite weighting matrices.

The constraints (11e) and (11f) in the LMPC formulation above correspond to the Lyapunov constraints. To explicitly deal with the sampled system, we consider a region $\Omega_{\hat{r}} \subset \Omega_r$, where $\hat{r} < r$ as a 'safe' zone to make Ω_r invariant (details given below in *Proposition 2*). When $\mathbf{z}(t_k)$ is received at a sampling time t_k, (11e) is active only when $\mathbf{z}(t_k) \in \Omega_{\hat{r}}$ and ensures that the sampled state is maintained in the region $\Omega_{\hat{r}}$ (so that the actual state of the closed-loop system is in the stability region Ω_r). The constraint (11f) is only active when $\hat{r} < V(\mathbf{z}(t_k)) \leq r$ and ensures the rate of change of the Lyapunov function is smaller than or equal to that of the value obtained if the explicit control law $h(\mathbf{z})$ is applied to the closed-loop system in a sample-and-hold fashion. These constraints allow the LMPC controller to inherit the stability properties of $h(\mathbf{z})$, i.e., it possesses at least the same stability region Ω_r as the controller $h(\mathbf{z})$. This implies that the (equilibrium point of) closed-loop system of (11a)-(11f) is guaranteed to be stable for any initial state inside the region Ω_r provided the sampling time Δ is sufficiently small.

Proposition 2: Consider the system of (8) under the MPC control law of (11a)-(11f), which is designed using a CLF V that has a stability region Ω_r under continuous implementation of the explicit controller $h(\mathbf{z})$. Then, given any positive real number d, $\exists$ positive real numbers $\Delta^\star$ such that if $\mathbf{z}(0) \in \Omega_r$ and $\Delta \in (0, \Delta^\star]$, then $\mathbf{z}(t) \in \Omega_r, \forall t \geq 0$ and $\lim_{t \to \infty} \|\mathbf{z}(t)\| \leq d$.

Proof: The proof is divided into three parts. In *Part 1*, the robustness of the explicit controller under the sample-and-hold implementation is shown. In *Part 2*, the controller of (11a)-(11f) is shown to be feasible for all $\mathbf{z}(0) \in \Omega_r$. Subsequently, in *Part 3*, it is shown that the stability region Ω_r is invariant under the predictive controller of (11a)-(11f).

Part 1: To prove the robustness of the explicit controller, we need to show the existence of a positive real number $\Delta^\star$ such that all state trajectories originating in Ω_r converge to the level set $\Omega_{\hat{r}}$ for any value of $\Delta \in (0, \Delta^\star]$. To achieve this, we need to consider different cases for $\mathbf{z}(0)$ inside the stability region. Figure 1 represents a schematic of the different cases considered in the following proof.

First, consider a small region close to the boundary of the stability region denoted as $\mathcal{Z} := \{\mathbf{z} : (r - r') \leq V(\mathbf{z}) \leq r\}$, for some $0 < r' < r$. Now, let $h(0) = h_0$ be computed for $\mathbf{z}(0) = \mathbf{z}_0 \in \mathcal{Z}$ and held constant until a time $\hat{\Delta}$ such that $h(t) := h_0 \ \forall t \in (0, \hat{\Delta}]$. Then,

$$\begin{aligned} \dot{V}(\mathbf{z}(t)) &= L_\Lambda V(\mathbf{z}(t)) + L_B V(\mathbf{z}(t)) h_0 \\ &= L_\Lambda V(\mathbf{z}_0) + L_B V(\mathbf{z}_0) h_0 \\ &\quad + (L_\Lambda V(\mathbf{z}(t)) - L_\Lambda V(\mathbf{z}_0)) \\ &\quad + (L_B V(\mathbf{z}(t)) h_0 - L_B V(\mathbf{z}_0) h_0). \end{aligned} \tag{12}$$

Since the initial state $\mathbf{z}_0 \in \mathcal{Z} \subseteq \Omega_r$ and h_0 is computed based on the stabilizing control law (10), it follows that $\dot{V}(\mathbf{z}_0) := L_\Lambda V(\mathbf{z}_0) + L_B V(\mathbf{z}_0) h_0 \leq -\rho V(\mathbf{z}_0)$ (this can be shown by substituting (10) in $\dot{V}$). Combining this with the definition of $\mathcal{Z}$, we have $L_\Lambda V(\mathbf{z}_0) + L_B V(\mathbf{z}_0) h_0 \leq -\rho(r - r')$.

We also need the following properties corresponding to Lipschitz continuity to complete the proof.

Property 1: Since the evolution of $\mathbf{z}$ is continuous, $\|u\| \leq u_{max}$ and $\mathcal{Z}$ is bounded, one can find, for all $\mathbf{z}_0 \in \mathcal{Z}$ and a fixed $\hat{\Delta}$, a positive real number k_1 such that $\|\mathbf{z}(t) - \mathbf{z}_0\| \leq k_1 \hat{\Delta}$ for all $t \leq \hat{\Delta}$.

Property 2: Additionally, since $L_\Lambda V(\cdot), L_B V(\cdot)$ are continuous functions, the following properties hold:

$$\begin{aligned} \|L_\Lambda V(\mathbf{z}(t)) - L_\Lambda V(\mathbf{z}_0)\| &\leq k_2 \|\mathbf{z}(t) - \mathbf{z}_0\| \leq k_1 k_2 \hat{\Delta} \\ \|L_B V(\mathbf{z}(t)) h_0 - L_B V(\mathbf{z}_0) h_0\| &\leq k_3 \|\mathbf{z}(t) - \mathbf{z}_0\| \leq k_1 k_3 \hat{\Delta} \end{aligned} \tag{13}$$

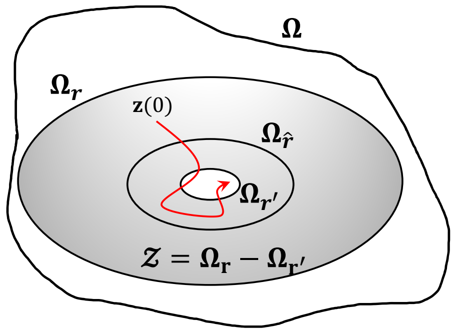


Fig. 1. A schematic representing the stability region of the bounded controller Ω_r, together with the sample-and-hold constrained set, $\Omega_{\hat{r}}$, and the overall stability region of the system, Ω. The grey shaded part represents the ring, $\mathcal{Z}$, close to the boundary of the stability region, Ω_r.

Using all the above inequalities in (12),

$$\dot{V}(\mathbf{z}(t)) \leq -\rho(r-r') + (k_1k_3 + k_2k_3)\hat{\Delta} \tag{14}$$

Now, if we choose $\hat{\Delta} < (\rho(r-r')-c)/(k_1k_3+k_2k_3)$ where $c < \rho(r-r')$ is a positive number, we get $\dot{V}(\mathbf{z}(t)) \leq -c < 0$ for all $t \leq \hat{\Delta}$ ensuring that the state does not escape Ω_r.

Now, we need to show the existence of a Δ' such that for all $\mathbf{z}_0 \in \Omega_{r'} := \{\mathbf{z}_0 : V(\mathbf{z}_0) \leq r - r'\}$ we have $\mathbf{z}_0 \in \Omega_{\hat{r}} := \{\mathbf{z}_0 : V(\mathbf{z}_0) \leq \hat{r}\}$. Consider Δ' such that

$$\hat{r} = \max_{\mathbf{z}_0 \in \Omega'_r, h \in \mathcal{U}, t \in [0,\Delta']} V(\mathbf{z}(t)) \tag{15}$$

This is possible because both V and $\mathbf{z}$ are continuous functions, and therefore for any $r' < r$, one can find a sufficiently small Δ' such that (15) holds. All that remains now is to show that for all $\mathbf{z}_0 \in \Omega_{\hat{r}}$ if $\Delta \in (0, \Delta^\star]$ where $\Delta^\star = \min\{\hat{\Delta}, \Delta'\}$, then $\mathbf{z}(t) \in \Omega_{\hat{r}}\ \forall t \geq 0$.

Consider all $\mathbf{z}_0 \in \Omega_{\hat{r}} \cap \Omega_{r'}$. Then by definition, $\mathbf{z}(t) \in \Omega_{\hat{r}}$ for $t \in [0, \Delta^\star]$ since $\Delta^\star \leq \Delta'$. On the other hand, for all $\mathbf{z}_0 \in \Omega_{\hat{r}}/\Omega_{r'}$, i.e., $\mathbf{z}_0 \in \mathcal{Z}$, it was shown that $\dot{V} < 0$ for $t \in [0, \Delta^\star]$ since $\Delta^\star \leq \hat{\Delta}$. Therefore, $\Omega_{\hat{r}}$ is an invariant set under the control law of (10). Hence, all trajectories originating in Ω_r converge to $\Omega_{\hat{r}}$ with a hold time less than $\Delta^\star$. That is, for all $\mathbf{z}_0 \in \Omega_r, \lim\sup_{t\to\infty} V(\mathbf{z}(t)) \leq \hat{r}$. Since, $V(\cdot)$ is a continuous function, one can always find a finite, positive number d such that $V(\mathbf{z}) \leq \hat{r} \implies \|\mathbf{z}\| \leq d$. Therefore, $\lim\sup_{t\to\infty} V(\mathbf{z}(t)) \leq \hat{r} \implies \lim\sup_{t\to\infty}\|\mathbf{z}(t)\| \leq d$.

Part 2: Let us consider some $\mathbf{z}(0) \in \Omega_r$ under the predictive controller of (11a)-(11f) with a prediction horizon N_p denoting the number of prediction steps such that $t_{k+N_p} = t_k + N_p\Delta$. There are two cases. If $\mathbf{z}_0 \in \Omega_r/\Omega_{\hat{r}}$, the feasibility of constraint (11f) is guaranteed by the control law of (10) as shown in *Part 1*. Additionally, if $V(\mathbf{z}(0)) \leq \hat{r}$, once again the control input trajectory under the explicit controller of (10), given by $u(t) = h(\mathbf{z}(t)),\ \forall t \in [t_k, t_{k+N_p}]$, provides a feasible initial guess to constraint (11e) because it was designed to stabilize the system, i.e., $V(\mathbf{z}(t)) \leq \hat{r}$. This shows that for all $\mathbf{z}(0) \in \Omega_r$, (11a)-(11f) is feasible.

Part 3: Since constraint (11f) is feasible, upon implementation it ensures that the value of the Lyapunov function under the predictive controller $u(t)$ decreases at each sampling time. Since Ω_r is a level set of V, and $\dot{V}$ decreases, the state trajectories cannot escape Ω_r. Additionally, satisfying constraint (11e) means that $\Omega_{\hat{r}}$ continues to remain invariant under the implementation of the predictive controller of (11a)-(11f). The recursive feasibility of (11d)-(11f) implies that $V \leq r$ and $\dot{V} < 0$ for all $\mathbf{z}(t)$ under the controller given by (11a)-(11f). However, since it is implemented in a sample-and-hold fashion there exists a maximum sampling time $\Delta^\star$, given in *Part 1* such that when $\Delta \in (0, \Delta^\star)$ it is guaranteed that for all $\mathbf{z}(\mathbf{0}) \in \Omega_r$ implies $\lim_{t\to\infty}\|\mathbf{z}(t)\| \leq d$.
This completes the proof. ■

In order to extend these results to the original nonlinear system of (4), we make the following assumption.

Assumption 2: Let the inverse mapping from the Koopman space, $\mathbf{z}$ to the original state space, $\mathbf{x}$ be continuously differentiable, i.e., $\exists\ \boldsymbol{\xi}(\mathbf{z}) = [\xi_1(\mathbf{z}), \cdots, \xi_n(\mathbf{z})]^T \in C^1 : \mathbb{R}^N \to \mathbb{R}^n$ such that $\hat{x}_i = \xi_i(\mathbf{z}), i = 1, \cdots, n$ where $\hat{\mathbf{x}} = [\hat{x}_1, \cdots, \hat{x}_n]$ is the predicted state vector obtained from the inverse mapping defined above.

Theorem 1: Suppose that system (4) satisfies *Assumptions 1-2*. Let $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ denote the original state and the predicted state values, respectively. The solutions for $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ are given by the following dynamic equations:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{16}$$

$$\hat{\mathbf{x}}(t) = \xi(\mathbf{z}(t)), \qquad \hat{\mathbf{x}}(0) = \mathbf{x}_0 \tag{17}$$

$$\dot{\mathbf{z}}(t) = \Lambda\mathbf{z}(t) + u(t)B\mathbf{z}(t), \quad \mathbf{z}(0) = \boldsymbol{\phi}(\hat{\mathbf{x}}(0)) \tag{18}$$

Then, the difference between $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ is bounded by

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \leq \frac{\nu}{l_x}(e^{l_x t} - 1) \tag{19}$$

where ν denotes the modeling error which bounds the difference between

$$\|\mathbf{f}(\hat{\mathbf{x}}, u) - \hat{\mathbf{f}}(\hat{\mathbf{x}}, u)\| \leq \nu \tag{20}$$

where $\mathbf{f}(\cdot) = \mathbf{F}(\cdot) + \mathbf{G}(\cdot)u$ is the original nonlinear dynamical system, and $\hat{\mathbf{f}}(\hat{\mathbf{x}}, u) = \frac{\partial \xi}{\partial \mathbf{z}}\dot{\mathbf{z}}$ denotes the solution to $\dot{\hat{\mathbf{x}}}(t)$. Under this condition, the stabilizing feedback control input $u^\star(t)$ obtained from the control law of (11a)-(11f) for the Koopman bilinear system of (6) also stabilizes the original system of (4), i.e., the origin of the closed-loop system of (4) is Lyapunov stable.

Proof: The proof is divided into two parts. First, we show that the predicted state $\hat{\mathbf{x}}(t)$ is stable under the application of the Koopman LMPC controller of (11a)-(11f) to the Koopman bilinear system. In the second part, we show that the evolution of the error between the original state and the predicted state is bounded under *Assumption 2* and the Lipschitz property of the vector fields $\mathbf{F}, \mathbf{G}$.

Part 1: Let us consider any initial condition $\mathbf{x}(0)$ such that $\mathbf{x}(0) = \hat{\mathbf{x}}(0) = \mathbf{x}_0$ and $\|\mathbf{x}_0\| \leq \delta$. Recall from *Proposition 2* that the predictive controller of (11a)-(11f) ensures that the lifted states do not escape the stability region Ω_r, i.e., $V(\mathbf{z}(t)) \leq r, \dot{V} < 0\ \forall t$. Therefore, $\lim\sup_{t\to\infty}\|\mathbf{z}(t)\| \leq d_r$. Now, from *Assumption 2*, since the inverse mapping $\boldsymbol{\xi}(\mathbf{z})$

is assumed to be continuous (differentiable), the following holds true:

$$\begin{aligned} &\|\boldsymbol{\xi}(\mathbf{z}(t))\| := \|\hat{\mathbf{x}}(t)\| \leq \epsilon_z \|\mathbf{z}(t)\| \\ &\lim\ \sup_{t\to\infty} \|\hat{\mathbf{x}}(t)\| \leq \hat{d} \end{aligned} \tag{21}$$

where $\hat{d} = \epsilon_z d_r$. In other words, since the controller ensures asymptotic stability of the lifted state, it implies that $\|\mathbf{z}(t)\|$ is bounded at all times and eventually converges to d_r. This in turn implies that $\hat{\mathbf{x}}(t)$ is bounded at all times, albeit by different constants at different sampling times. Now, if we choose $\hat{\epsilon}$ to be the maximum of all these bounds, then $\|\hat{\mathbf{x}}(t)\| < \hat{\epsilon},\ \forall t$. Hence, for any initial condition $\|\mathbf{x}_0\| \leq \delta$, the implementation of the predictive controller of (11a)-(11f) guarantees that $\|\hat{\mathbf{x}}(t)\| \leq \hat{\epsilon}, \forall t$. This implies that the predicted states of the original system starting close enough to the equilibrium (at a distance δ) will be maintained close to the equilibrium at all times.

Part 2: Now, it remains to prove that the modeling error between the original state vector and the predicted states is bounded at all times for all $\|\mathbf{x}_0\| \leq \delta$. Let us consider the modeling error $e(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, then the evolution of the error is given as

$$\|\dot{e}(t)\| = \|\dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)\| = \|\mathbf{f}(\mathbf{x}, u) - \hat{\mathbf{f}}(\hat{\mathbf{x}}, u)\| \tag{22}$$

where $\mathbf{f}(\mathbf{x}, u) = \mathbf{F}(\mathbf{x}) + \mathbf{G}(\mathbf{x})u$ is the nonlinear dynamical system, and $\hat{\mathbf{f}}(\hat{\mathbf{x}}, u)$ denotes the evolution of the predicted state $\hat{\mathbf{x}}$ which can be determined from the following Koopman bilinear sytem:

$$\hat{\mathbf{f}}(\hat{\mathbf{x}}, u) = \frac{\partial \xi}{\partial \mathbf{z}} \dot{\mathbf{z}} \tag{23}$$

By adding and subtracting $\mathbf{f}(\hat{\mathbf{x}}, u)$ to (22), we get

$$\begin{aligned} \|\dot{e}(t)\| &= \|\mathbf{f}(\mathbf{x}, u) - \mathbf{f}(\hat{\mathbf{x}}, u) + \mathbf{f}(\hat{\mathbf{x}}, u) - \hat{\mathbf{f}}(\hat{\mathbf{x}}, u)\| \\ &\leq \|\mathbf{f}(\mathbf{x}, u) - \mathbf{f}(\hat{\mathbf{x}}, u)\| + \|\mathbf{f}(\hat{\mathbf{x}}, u) - \hat{\mathbf{f}}(\hat{\mathbf{x}}, u)\| \end{aligned} \tag{24}$$

The Lipschitz property of $\mathbf{f}(\cdot)$, combined with the bounds on u, implies that there exists a positive constant l_x such that the following inequality holds for all $\mathbf{x}, \mathbf{x}' \in \mathcal{X}, u \in \mathcal{U}$:

$$\|\mathbf{f}(\mathbf{x}, u) - \mathbf{f}(\mathbf{x}', u)\| \leq l_x \|\mathbf{x} - \mathbf{x}'\| \tag{25}$$

Additionally, since $\hat{\mathbf{x}}$ is bounded (see *Part 1* in the proof of *Theorem 1*), $\mathbf{f}$ is Lipschitz, and the mapping $\boldsymbol{\xi}$ is continuously differentiable, there exists a positive constant ν such that the second term in (24) is bounded by ν. Combining it with (25) we have

$$\begin{aligned} \|\dot{e}(t)\| &\leq l_x \|\mathbf{x} - \hat{\mathbf{x}}\| + \nu \\ &\leq l_x \|e(t)\| + \nu \end{aligned} \tag{26}$$

Therefore, given the zero initial condition (i.e., $e(0) = 0$), the upper bound for the norm of the error vector can be determined by integrating (26) as

$$\int_0^t \frac{\|\dot{e}(\tau)\|}{l_x \|e(\tau)\| + \nu} \leq t \tag{27}$$

and solving for $\|e(t)\|$

$$\|e(t)\| = \|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \leq \frac{\nu}{l_x}(e^{l_x t} - 1) \tag{28}$$

Finally, since the error between the original and predicted vectors is bounded and that the Koopman LMPC controller of (11a)-(11f) stabilizes the predicted state vector $\|\hat{\mathbf{x}}(t)\| \leq \hat{\epsilon}$, there exists a positive constant ϵ such that $\|\mathbf{x}(t)\| \leq \epsilon$ for all t. Therefore, for all $\|\mathbf{x}_0\| \leq \delta$ the implementation of the predictive controller of (11a)-(11f) ensures that $\|\mathbf{x}(t)\| \leq \epsilon$ for all t, thereby rendering the original nonlinear system stable.

This completes the proof. ■

Remark 1: Assumption 2 seems restrictive in selecting the types of basis functions to determine the Koopman bilinear models. However, in practice, one can numerically obtain a separate mapping from the Koopman space to the original space without actually inverting the eigenfunctions. In this case, the error of the optimization problem must be certified to be bounded to ensure that the proposed controller successfully stabilizes the closed-loop system.

IV. NUMERICAL EXPERIMENTS

To demonstrate the performance of Koopman-based nonlinear stabilization presented in Section III, it is applied to a canonical nonlinear system, the Van der Pol oscillator. The Van der Pol oscillator is described by the following equations:

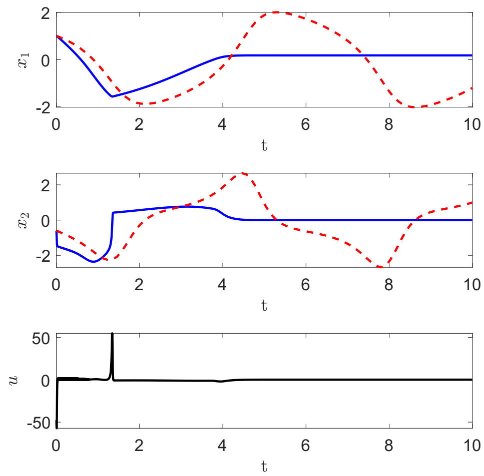
$$\dot{x}_1 = x_2; \quad \dot{x}_2 = (1 - x_1^2)x_2 - x_1 + u \tag{29}$$

At $u = 0$, the phase plot of the Van der Pol oscillator is characterized by a limit cycle and an unstable equilibrium point at the origin. The data required is generated using simulations initialized uniformly over a circle around the origin and 10 s long trajectories were collected with a sampling time of $\Delta = 0.01$ s, i.e., 10^3 time-series samples per trajectory. In the next step, the states were lifted to the high-dimensional space by using monomials of degree 5 as the dictionary functions $\boldsymbol{\phi}(\mathbf{x}(t))$, i.e., $\boldsymbol{\phi}(\mathbf{x}(t)) = [1, x_1, x_2, x_1^2, x_1x_2, \cdots, x_2^5]^T$. This results in a lifted system of dimension $\mathbf{z} \in \mathbb{R}^{21}$ and the system matrices Λ and B were constructed using the EDMD algorithm [19].

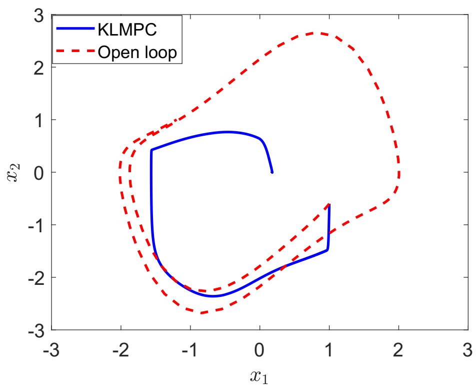
Next, the Koopman LMPC developed in Section III was applied to control the system (29) with $N = 21$ eigenfunctions as the new states, $\mathbf{z}$, in the transformed space. The initial condition was chosen randomly around the unstable equilibrium and the control objective was to stabilize the system at the origin. The CLF used to define the explicit stable controller $h(\mathbf{z})$ was obtained by solving the following optimization problem [19]:

$$\begin{aligned} \min_{\sigma>0, P=P^T} \quad & \sigma - \gamma \text{trace}(PB) \\ \text{s.t} \quad & \sigma I - (PA + A^T P) \geq 0 \\ & c^L I \leq P \leq c^U I \end{aligned} \tag{30}$$

where σ represents the epigraph form of the largest singular value of $(PA + A^T P)$, and $c^L, c^U > 0$ are two positive scalars used to bound the eigenvalues of P. The weighting parameter $\gamma > 0$ is chosen as 2 in this example. The explicit controller, $h(\mathbf{z})$, was determined by using the obtained CLF, $V = \mathbf{z}^T P \mathbf{z}$, within the Sontag's formula as shown in (10).



(a)



(b)

Fig. 2. Comparison of open-loop and closed-loop trajectories for the Van der Pol oscillator with u from the control law of (11a) - (11f)

The matrices Q and R in (11a) were chosen to be $Q = I \in \mathbb{R}^{21\times 21}$ and $R = 1$, respectively, and the input was bounded, i.e., $u \in [-50, 50]$. The prediction horizon was set to 1 s, i.e., $N_p = 1/\Delta = 100$. At every sampling time, the KLMPC problem of (11a)-(11f) was solved using the IPOPT solver (interfaced with MATLAB). Figure 2 shows the comparison between open and closed loop results. It can be observed from Figure 2 that the system was stabilized at the origin as desired.

V. CONCLUSIONS

This manuscript presented a new approach for the design of stabilizing controllers for nonlinear dynamical systems using operator theory. Bilinear models that are valid on the entire basin of attraction are computed using finite-dimensional approximations to the Koopman operator. A feedback controller is then designed using LMPC to obtain closed-loop stability of the bilinear system. Due to the bilinearity of the Koopman model, a quadratic CLF can be obtained easily via an optimization problem. Furthermore, based on the stability of the Koopman model, the original nonlinear system under the proposed controller can be guaranteed to be stable, provided that a continuously differentiable inverse mapping exists. The proposed method was applied to a nonlinear Van der Pol example and the theoretical analysis of the paper was verified.

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