Ergod. Th. & Dynam. Sys. (2021), **41**, 1656–1705
doi:10.1017/etds.2020.14

Asymptotic escape rates and limiting distributions for multimodal maps

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(*Received* 23 *July* 2019 *and accepted in revised form* 14 *January* 2020)

Abstract. We consider multimodal maps with holes and study the evolution of the open systems with respect to equilibrium states for both geometric and Hölder potentials. For small holes, we show that a large class of initial distributions share the same escape rate and converge to a unique absolutely continuous conditionally invariant measure; we also prove a variational principle connecting the escape rate to the pressure on the survivor set, with no conditions on the placement of the hole. Finally, introducing a weak condition on the centre of the hole, we prove scaling limits for the escape rate for holes centred at both periodic and non-periodic points, as the diameter of the hole goes to zero.

Key words: multimodal maps, Hofbauer extension, open systems, escape rate, transfer operator
2010 Mathematics Subject Classification: 37C30, 37D35, 37E05 (Primary)

1. *Introduction*

Dynamical systems with holes arise naturally in the study of systems whose domain is not invariant under the dynamics. They have been studied in connection with absorbing states in Markov chains [**V**, **FKMP**], metastable states in deterministic systems [**DoW**, **BV1**, **GHW**] and neighbourhoods of non-attracting invariant sets [**Y**], as well as in components of large systems of interacting components in non-equilibrium statistical mechanics [**DGKK**].

In the present paper, for a class of multimodal maps with holes in the form of intervals, we study the escape rates and limiting behaviours of the open systems with respect to equilibrium states and conformal measures for broad classes of potentials. The systems in question have exponential rates of escape†, in which the escape rate and limiting behaviour of the open system are expressed through the existence and properties of a physical conditionally invariant measure, absolutely continuous with respect to a given conformal measure. In this setting, given a map $f : I \circlearrowleft$ and identifying a set $H \subset I$ as a hole, one defines the open system by $\mathring{f} = f|_{\mathring{I}^1}$, where $\mathring{I}^1 = (I \setminus H) \cap f^{-1}(I \setminus H)$. A *conditionally invariant measure* μ is a Borel probability measure satisfying

$$\mu(A) = \frac{\mathring{f}_* \mu(A)}{\mathring{f}_* \mu(I)} \quad \text{for all Borel } A \subset I.$$

The evolution of measures in the open system is described by the sequence $\mathring{f}^n_* \mu_0 / \mathring{f}^n_* \mu_0(I)$ for initial distributions μ_0. If the limit of such a sequence exists and is independent of μ_0 for a reasonable class of initial distributions, we call the resulting measure a limiting (or physical) conditionally invariant measure. For open systems with exponential rates of escape, the typical agenda of strong dynamical properties includes a common rate of escape for natural classes of densities, the convergence of such densities to a limiting conditionally invariant measure under iteration of the dynamics, and a variational principle connecting the escape rate to the pressure of the open system on the survivor set, the (singular) set of points which never enters the hole.

Such results have been obtained primarily for uniformly hyperbolic systems, beginning with expanding maps [**PY**, **CMS**, **LM**], Anosov diffeomorphisms [**CM**, **CMT**], finite [**FP**] and countable [**DIMMY**] state topological Markov chains, and dispersing billiards [**DWY**, **D2**]. Their extension to non-uniformly hyperbolic systems has been primarily restricted to unimodal and multimodal interval maps [**BDM**, **DT1**, **PU**] and intermittent maps [**DT2**].

The purpose of the present paper is to prove strong hyperbolic properties for open systems associated with multimodal maps in greater generality, removing many of the technical assumptions made in previous works. As such, the present paper represents a significant simplification and extension of results available in the context of non-uniformly hyperbolic open systems. Previous works in the setting of unimodal maps with holes have required strong conditions both on the map (Misiurewicz maps in [**D1**]; a Benedicks–Carleson condition in [**BDM**, **DT1**]; a topologically tame condition in [**PU**]) and on the placement of the hole (slow approach to (see [**BDM**, **DT1**]) or complete avoidance of (see [**PU**]) the post-critical set by the boundary or centre of the hole).

The principal innovation we introduce to the study of open systems in this paper is the use of Hofbauer extensions, a type of Markov extension of the original system. Introduced in [**H**], they have been used extensively in the study of interval maps. However, to date, they have not been implemented for systems with holes. In this paper we construct Hofbauer extensions of our open system, with additional cuts added to our partition depending on the boundary and centre of our hole. Doing so enables us to consider the lift of the hole as a union of 1-cylinders in the extension. Leveraging recent estimates on complexity from [**DoT**], we proceed to build an induced map and related Young tower

† For systems with subexponential rates of escape, the results are qualitatively different since there can be no conditionally invariant limiting distribution [**DF**]. See [**DG**, **FMS**, **APT**, **DR**, **DT2**, **BDT**] for examples of studies in the subexponential regime.

over the Hofbauer extension in order to apply the framework developed in [**DT2**] for Young towers with holes.

This two-step approach (rather than simply constructing a Young tower for the open system directly) allows us to remove many of the technical assumptions needed in previous works for interval maps with holes, as described above. Indeed, we establish the standard suite of strong hyperbolic properties for the open system, assuming only that the hole is a finite union of small intervals (Theorem 3.1), entirely eliminating the need for previous assumptions on its placement or on the orbits of its boundary points. We also prove the scaling limits for the escape rate as the hole shrinks to a point under much weaker assumptions than used previously (Theorems 3.5 and 3.7). In addition, we greatly broaden the class of potentials we are able to treat in this setting: we treat all Hölder continuous potentials, as well as the geometric potentials $\phi = -t \log |Df|$ for an interval of t containing $[0, 1)$; if the map satisfies a Collet–Eckmann condition, we treat $t \geqslant 1$ as well. This is in contrast to [**DT1**] which restricted t to a small interval around 1, and [**PU**] which treated only Hölder potentials with bounded variation.

The paper is organized as follows. In §2 we define the class of maps and potentials we shall study, and recall important definitions regarding pressure and open systems. In §3 we state our main results, and in §4 we carry out our main construction of the Hofbauer extensions and associated induced maps, proving that they enjoy tail bounds and mixing properties that are uniform in the size of the hole. In §5 we prove the key spectral properties for the induced open system, which are then leveraged in §6 for Young towers, and in §7 to establish the small hole asymptotic.

2. *Set-up*

2.1. *Dynamics.* For I denoting the unit interval, let $\mathcal{F}$ denote the class of C^3 maps $f : I \to I$ with:

- all critical points non-flat: there exists a finite set Crit $\subset I$ such that for each $c \in$ Crit there is a C^3 diffeomorphism ϕ in a neighbourhood of c with $\phi(c) = 0$ such that $f(x) = \pm|\phi(x)|^d + f(c)$ for some $d > 1$, the *order* of c;
- negative Schwarzian derivative, that is, $D^3 f/Df - \frac{3}{2}(D^2 f/Df)^2 \leqslant 0$;
- the *locally eventually onto (leo)*/topologically exact condition: for any open set $U \subset I$ there exists $n \in \mathbb{N}$ such that $f^n(U) = I$, a form of topological transitivity;
- for each c,
$$|Df^n(f(c))| \to \infty.$$

Note that it is possible to weaken the conditions listed here, but this would lead to a significantly more complex exposition.

Sometimes we will require a stronger condition: we say that f satisfies the *Collet–Eckmann* condition if there exist $C, \gamma > 0$ such that for each $c \in$ Crit, and all $n \in \mathbb{N}$,

$$|Df^n(f(c))| \geqslant Ce^{\gamma n}. \tag{CE}$$

2.2. *Potentials, pressure and equilibrium states.* Given $f \in \mathcal{F}$, we let $\mathcal{M}$ denote the set of f-invariant probability measures. Then, for a potential $\phi :\to [-\infty, \infty]$, we define the *pressure* by

$$P(\phi) := \sup\left\{h_\mu(f) + \int \phi \, d\mu : \mu \in \mathcal{M} \text{ and } \mu(-\phi) < \infty\right\}. \tag{2.1}$$

A measure $\mu \in \mathcal{M}$ is called an *equilibrium state* for ϕ if $h_\mu(f) + \int \phi \, d\mu = P(\phi)$.

Given $\phi : I \to [-\infty, \infty]$, we say that a sigma-finite measure m_ϕ is ϕ*-conformal* if whenever U is a Borel set and $f : U \to f(U)$ is a bijection then

$$m_\phi(f(U)) = \int_U e^{-\phi} \, dm_\phi.$$

(For example, Lebesgue measure is $-\log|Df|$-conformal.) Notice that we can iterate this relation: if $f^n : U \to f(U)$ is a bijection, then

$$m_\phi(f^n(U)) = \int_U e^{-S_n\phi} \, dm_\phi, \tag{2.2}$$

where $S_n\phi = \sum_{i=0}^{n-1} \phi \circ f^i$. We will also be interested in functions $\psi : I \to [-\infty, \infty]$ *cohomologous* to ϕ; namely, there exists a function h such that $\phi = \psi + h - h \circ f$. These functions share equilibrium states, though they may produce different, but equivalent, conformal measures.

We will consider equilibrium states for two types of potentials: Hölder continuous potentials and geometric potentials.

(i) Hölder continuous potentials. In [**LR-L**] it was shown that any Hölder potential ϕ is cohomologous to a Hölder potential $\tilde{\phi}$ with $\tilde{\phi} < P(\tilde{\phi})$ on I (note that there can be many such potentials). It is therefore no loss of generality to assume, as we will throughout, that for our Hölder potentials, $\phi < P(\phi)$.

(ii) Geometric potentials. We set $\phi = -\log|Df|$ and consider the family $\{t\phi\}_{t\in\mathbb{R}}$. We let $p_t := P(t\phi)$ and denote $m_t = m_{t\phi - p_t}$ if this measure exists. For a p-periodic point x, define its Lyapunov exponent by $\lambda(x) := (1/p)\log|Df^p(x)|$. As in [**PR-L**, Appendix A], for $f \in \mathcal{F}$ and $x \in I$, it is always the case that $\lambda(x) > 0$. Then define

$$\lambda_{\min} = \inf\{\lambda(x) : x \text{ is periodic}\} \quad \text{and} \quad \lambda_{\max} := \sup\{\lambda(x) : x \text{ is periodic}\}.$$

For $\mu \in \mathcal{M}$, let its Lyapunov exponent be defined by $\lambda(\mu) := \int \log|Df| \, d\mu$. By [**PR-L**, Proposition 4.7], if $f \in \mathcal{F}$ then

$$\inf\{\lambda(\mu) : \mu \in \mathcal{M}\} = \lambda_{\min} \quad \text{and} \quad \sup\{\lambda(\mu) : \mu \in \mathcal{M}\} = \lambda_{\max}.$$

Noting from the definition of pressure that $p_t \geqslant -t\lambda_{\min}$, we define

$$t^+ := \sup\{t \in \mathbb{R} : p_t > -t\lambda_{\min}\} \quad \text{and} \quad t^- := \inf\{t \in \mathbb{R} : p_t > -t\lambda_{\max}\}.$$

These are referred to as the *freezing point* and the *condensation point* of f, respectively. It is immediate that $t^- < 0$. For $f \in \mathcal{F}$, there is always an absolutely continuous invariant probability measure, which implies that $p_1 = 0$ and $t^+ \geqslant 1$. As in [**PR-L**], (CE) implies $t^+ > 1$.

Definition 2.1. We shall call a potential ϕ *admissible* if either (a) ϕ is Hölder continuous and $\phi < P(\phi)$ on I, or (b) $\phi = -t\log|Df|$ with $t \in (t^-, t^+)$.

For each admissible ϕ,

$$P(\phi) = \sup\left\{h_\mu(f) + \int \phi \, d\mu : \mu \in \mathcal{M}, \ \mu(-\phi) < \infty \text{ and } h_\mu(f) > 0\right\}, \tag{2.3}$$

and there is a unique equilibrium state which is exponentially mixing: for geometric potentials with $t \in (t^-, t^+)$, this follows for example by [**IT1**, Theorem A]; in the Hölder case this follows from [**LR-L**, Theorem A]. Moreover, each equilibrium state is absolutely continuous with respect to a unique conformal measure, which is shown to exist in, for example, [**IT2**, Appendix B]. Throughout, we will denote the normalized potential by $\varphi = \phi - P(\phi)$, and say that φ is admissible whenever ϕ is. Moreover, we let m_φ and μ_φ denote the φ-conformal measure and the equilibrium state, respectively. We may drop the φ when the potential is clear.

2.3. *Puncturing the system.* Choose $z \in I$, and let $H_\varepsilon = (z - \varepsilon, z + \varepsilon) \subset I$ be an interval. Denote by $\mathring{I} = I \setminus H_\varepsilon$, and in general by $\mathring{I}^n = \bigcap_{i=0}^n f^{-i}\mathring{I}$, the set of points that do not enter H_ε in the first n iterates. The sequence of maps $\mathring{f}^n := f^n|_{\mathring{I}^n}$ defines the corresponding open system.

We define the upper and lower escape rates through H_ε by

$$\log \overline{\lambda}_\varepsilon := \limsup_{n\to\infty} \frac{1}{n} \log \mu_\varphi(\mathring{I}^n) \quad \text{and} \quad \log \underline{\lambda}_\varepsilon := \liminf_{n\to\infty} \frac{1}{n} \log \mu_\varphi(\mathring{I}^n).$$

When the two quantities coincide, we denote them by $\log \lambda_\varepsilon$, and call $-\log \lambda_\varepsilon$ the *escape rate* with respect to μ_φ.

Given a potential ϕ, once a hole H_ε is introduced, the *punctured potential* is defined by $\phi^{H_\varepsilon} = \phi$ on $\mathring{I}$ and $\phi^{H_\varepsilon} = -\infty$ on H_ε. $P(\phi^{H_\varepsilon})$ denotes the pressure of the punctured potential, and it follows from the requirement $\mu(-\phi^{H_\varepsilon}) < \infty$ that the supremum for this pressure is restricted to f-invariant measures that are supported on the *survivor set* $\mathring{I}^\infty := \bigcap_{n=0}^\infty \mathring{I}^n$.

We will be interested in establishing convergence for limits of the form $\mathring{f}_*^n\mu/\mathring{f}_*^n\mu(I)$ for measures μ which are absolutely continuous with respect to the conformal measure m_φ. To this end, define the transfer operator corresponding to the potential φ by

$$\mathcal{L}_\varphi \psi(x) = \sum_{y \in f^{-1}x} \psi(y) e^{\varphi(y)} \quad \text{for } \psi \in L^1(m_\varphi).$$

Similarly, the punctured transfer operator for the open system is defined by

$$\mathring{\mathcal{L}}_{\varphi^{H_\varepsilon}} \psi(x) = \mathcal{L}_\varphi(1_{\mathring{I}^1}\psi)(x) = \sum_{y \in \mathring{f}^{-1}x} \psi(y) e^{\varphi(y)}.$$

Due to the conformality of m_φ, we have

$$\int_I \mathring{\mathcal{L}}^n_{\varphi^{H_\varepsilon}} \psi \, dm_\varphi = \int_{\mathring{I}^n} \psi \, dm_\varphi \quad \text{for all } n \geq 1,$$

which relates the escape rate with respect to the measure $\psi \, dm_\varphi$ to the spectral radius of $\mathring{\mathcal{L}}_{\varphi^{H_\varepsilon}}$.

3. *Results*

3.1. *Small hole, general placement.*

Theorem 3.1 proves the standard suite of strong hyperbolic properties for the open system. As noted in the introduction, it is a significant improvement over [**BDM**, **DT1**] and [**PU**] which had similar results under much more restrictive assumptions on the map, the potential and the hole.

THEOREM 3.1. *Let $f \in \mathcal{F}$ and ϕ be an admissible potential, with normalized version $\varphi = \phi - P(\phi)$. Let $z \in I$ and, for $\varepsilon > 0$, set $H_\varepsilon(z) = (z - \varepsilon, z + \varepsilon)$. Suppose that $\varepsilon^* > 0$ is sufficiently small so that $-\log \overline{\lambda}_{\varepsilon^*} < \alpha$, where $\alpha > 0$ is the tail decay rate from Theorem 4.10. Then the following hold for all $0 < \varepsilon \leqslant \min\{\varepsilon_1^*, \varepsilon^*\}$, where ε_1^* is from Lemma 6.2.*

(a) *The escape rate $-\log \lambda_\varepsilon$ exists, and $\lambda_\varepsilon < 1$ is the spectral radius of the punctured transfer operator on the associated Young tower. The associated eigenvector projects to a non-negative function $\mathring{g}_\varepsilon$, which is bounded away from zero on $I \setminus H_\varepsilon$ and satisfies $\mathring{\mathcal{L}}_{\varphi^{H_\varepsilon}} \mathring{g}_\varepsilon = \lambda_\varepsilon \mathring{g}_\varepsilon$.*

(b) *There is a unique $(\phi^{H_\varepsilon} - P(\phi^{H_\varepsilon}))$-conformal measure m_{H_ε}. This is singular with respect to m_φ and supported on $\mathring{I}^\infty$.*

(c) *The measure $\nu_{H_\varepsilon} := \mathring{g}_\varepsilon m_{H_\varepsilon}$ is the unique equilibrium state for $\phi^{H_\varepsilon} - P(\phi^{H_\varepsilon})$; in particular,*

$$\log \lambda_\varepsilon = P(\phi^{H_\varepsilon}) - P(\phi) = P(\varphi^{H_\varepsilon}) = h_{\nu_{H_\varepsilon}}(f) + \int \varphi^{H_\varepsilon} \, d\nu_{H_\varepsilon}.$$

Moreover, ν_{H_ε} is supported on $\mathring{I}^\infty$ and can be realized as the limit

$$\nu_{H_\varepsilon}(\psi) = \lim_{n \to \infty} \lambda_\varepsilon^{-n} \int_{\mathring{I}^n} \psi \mathring{g}_\varepsilon \, dm_\varphi \quad \textit{for all } \psi \in C^0(I).$$

(d) *The measure $\mu_\varphi^{H_\varepsilon} := \mathring{g}_\varepsilon m_\varphi$ is a conditionally invariant measure supported on $I \setminus H_\varepsilon$ with eigenvalue λ_ε and is a limiting distribution in the following sense. Fix $\varsigma > 0$ and let $\psi \in C^\varsigma(I)$ satisfy $\psi \geqslant 0$, with $\nu_{H_\varepsilon}(\psi) > 0$. Then*

$$\left| \frac{\mathring{\mathcal{L}}^n_{\varphi^{H_\varepsilon}} \psi}{|\mathring{\mathcal{L}}^n_{\varphi^{H_\varepsilon}} \psi|_{L^1(m_\varphi)}} - \mathring{g}_\varepsilon \right|_{L^1(m_\varphi)} \leqslant C \vartheta^n |\psi|_{C^\varsigma(I)} \tag{3.1}$$

for some $C > 0$ independent of ψ, and $\vartheta < 1$ depending only on ς.

The techniques also imply that $\nu_{H_\varepsilon}, \mu_\varphi^{H_\varepsilon} \to \mu_\phi$ as $\varepsilon \to 0$. Note that the techniques of the proof also extend to holes comprised of finitely many intervals as the only condition required on the hole in [**DT2**] is $-\log \overline{\lambda}_\varepsilon < \alpha$. We prove this theorem in §6.2.

Remark 3.2. In fact, we prove convergence to the conditionally invariant measure $\mu_\varphi^{H_\varepsilon}$ for a larger class of initial densities than $C^\varsigma(I)$. It only matters that ψ satisfies $\nu_{H_\varepsilon}(\psi) > 0$ and that it can be realized as the projection of an element in a certain function space on the related Young tower. So, for example, any function of the form $\psi = \tilde{\psi} g_\varphi$ also satisfies (3.1), where $\tilde{\psi} \in C^\varsigma(I)$ and $g_\varphi = d\mu_\varphi / dm_\varphi$.

The following lemma shows that we can always choose $\varepsilon^* > 0$ small enough so that $-\log \overline{\lambda}_{\varepsilon^*} < \alpha$, and hence the theorem applies to all small holes.

LEMMA 3.3. *Suppose ϕ is an admissible potential and $f \in \mathcal{F}$. For any $z \in I$, and hole $H_\varepsilon(z) = (z - \varepsilon, z + \varepsilon)$, we have that* $\lim_{\varepsilon \to 0} \overline{\lambda}_\varepsilon = 1$.

Proof. This is a simple consequence of Corollary 5.5, since the escape rate for the related induced system, $-\log \Lambda_\varepsilon$, is continuous in ε, and by monotonicity, $\overline{\lambda}_\varepsilon \geqslant \Lambda_\varepsilon$. For details, see the verification of property (P2) in §6.1. □

Remark 3.4. (Bowen formula for Hausdorff dimension of $\mathring{I}^\infty$) If we take $\phi = -\log |Df|$, then under the assumptions of Theorem 3.1 and for ε sufficiently small, $\mathrm{Hdim}(\mathring{I}^\infty) = t^*$, where t^* is the unique value of t such that $P(t\phi^{H_\varepsilon}) = 0$. This follows as in [**DT1**, Theorem 8.1], using the uniform bounds for t close to 1 on the tail of the return time function from Theorem 4.10 to show that any set of Hausdorff dimension greater than some constant $D < 1$ lifts to our inducing scheme. Then Theorem 3.1(c) implies that the dimension of the equilibrium measure $\nu^1_{H_\varepsilon}$, corresponding to $t = 1$, equals $1 + \log \lambda_\varepsilon$, and so is greater than D for ε small. Thus the Hausdorff dimension of $\mathring{I}^\infty$ equals that of the survivor set in our inducing scheme.

3.2. *Zero-hole limits.* Here we consider the asymptotic scaling limit for the escape rate, $-\log \lambda_\varepsilon / \mu_\varphi(H_\varepsilon)$ as $\varepsilon \to 0$. This limit was first computed in the context of escape rates for full shifts in [**BY**], then extended to (piecewise) uniformly expanding systems in [**KL2**] and to more general potentials in the symbolic setting in [**FP**] (see also [**AB**, **BV2**, **FFT2**]). Its extension to unimodal and multimodal maps followed with added assumptions on the centre of the hole z, assuming that the post-critical orbits either approach z slowly [**BDM**, **DT1**], or are bounded away from z [**PU**].

By contrast, for Hölder continuous potentials, we prove our results for all non-periodic $z \in I$, with an additional assumption required only if z is periodic and lies in the post critical orbit. For geometric potentials, we require a (generic) slow approach condition to z, and present an example (§3.4) to show that the scaling limit can fail for geometric potentials if no condition on z is imposed. The proofs of Theorems 3.5 and 3.7 are in §7.

3.2.1. *Hölder potentials.* The asymptotic escape rate depends on whether the chosen centre z is periodic or not.

THEOREM 3.5. *Let $f \in \mathcal{F}$, ϕ be Hölder continuous and $z \in I$.*

(a) *If z is not periodic, then* $\lim_{\varepsilon \to 0} (-\log \lambda_\varepsilon / \mu_\varphi(H_\varepsilon)) = 1$.

(b) *If z is periodic with prime period p and $\{f^n(c) : c \in \mathrm{Crit}, n \geqslant 1\} \cap \{z\} = \emptyset$, then* $\lim_{\varepsilon \to 0} (-\log \lambda_\varepsilon / \mu_\varphi(H_\varepsilon)) = 1 - e^{S_p\varphi(z)}$.

(c) *Suppose z is periodic with prime period p and $\{f^n(c) : c \in \mathrm{Crit}, n \geqslant 1\} \cap \{z\} \neq \emptyset$. If, in addition, either f^p is orientation preserving in a neighbourhood of z, or* $\lim_{\varepsilon \to 0} (m_\varphi(z + \varepsilon, z)/m_\varphi(z, z - \varepsilon)) = 1$, *then* $\lim_{\varepsilon \to 0} (-\log \lambda_\varepsilon / \mu_\varphi(H_\varepsilon)) = 1 - e^{S_p\varphi(z)}$.

Remark 3.6. Even when both conditions in part (c) of Theorem 3.5 fail, we can still find a subsequence of $\varepsilon \to 0$ so that the scaling limit converges to $1 - e^{S_p\varphi(z)}$. Thus we expect that the scaling limit holds for all periodic points in the case of Hölder continuous potentials.

3.2.2. *Geometric potentials.* For the remainder of this section we let $\phi = -\log |Df|$. The geometric case requires a condition on slow approach to the critical set as well as a polynomial rate of growth of the derivative along the post-critical orbit. For simplicity, for a given $d > 1$, we will consider the set $\mathcal{F}_d \subset \mathcal{F}$ with the defining property that for each $f \in \mathcal{F}_d$ all critical points have order d.

For $t \in (t^-, t^+)$, let $s_t := t + p_t/\lambda(\mu_t) \in (0, 1]$ denote the local scaling exponent for $m_{t\phi - p_t}$; see [**DT1**, Lemma 9.5]. Define

$$D_n(c) = |Df^n(f(c))| \quad \text{for each } c \in \text{Crit}.$$

We assume that for each $c \in \text{Crit}$,

$$D_n(c) \geqslant \text{const}.n^q \quad \text{for some } q > d + \frac{d-1}{s_t} \text{ and all } n \geqslant 1. \tag{3.2}$$

With q given as above, we choose $\theta \in (0, 1)$ and $r \in \big(1/(1-\theta)s_t, (q-d)/(d-1)\big)$, and define a sequence $\gamma_n = n^{-r}$, $n \geqslant 1$. We make the following assumption on the centre of the hole, z, in terms of this sequence:

$$\text{there exists } \delta_z > 0 \text{ s.t. } \min_{c \in \text{Crit}} d(f^n(c), z) \geqslant \delta_z \gamma_n^{1-\theta} \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

In particular, we have $\sum_n \gamma_n^{(1-\theta)(s_t - \epsilon)} < \infty$ for some $\epsilon > 0$, so that condition (3.3) is generic with respect to the measures m_φ, $\varphi = t\phi - p_t$, as proved in [**DT1**, Lemma 9.3].

The value of s_t varies continuously with t, and is greater than 0 for each $t \in (t^-, t^+)$, with $s_1 = 1$, but may tend to zero as t tends to the boundary of (t^-, t^+). This means that, in particular when the map f satisfies the (CE) condition, we will restrict to a subinterval (t^-, t_1) where $t_1 \in (1, t^+]$ is determined by (7.28); if f does not satisfy (CE), we let $t_1 = 1$.

THEOREM 3.7. *For $d > 1$, let $f \in \mathcal{F}_d$ and $t \in (t^-, t_1)$. Suppose (3.2) is satisfied and $z \in I$ satisfies (3.3). Then, for $\varphi = -t \log |Df| - p_t$:*
(a) *if z is not periodic then* $\lim_{\varepsilon \to 0} (-\log \lambda_\varepsilon / \mu_\varphi(H_\varepsilon)) = 1$*;*
(b) *if z is periodic with (prime) period p, then* $\lim_{\varepsilon \to 0} (-\log \lambda_\varepsilon / \mu_\varphi(H_\varepsilon)) = 1 - e^{S_p \varphi(z)}$.

Remark 3.8. [**FFT1**, §6] shows that there are examples of maps $f \in \mathcal{F}_d$ and periodic points z satisfying (3.3).

Remark 3.9. It is not clear what the optimal condition on z is so that the scaling limits of Theorem 3.7 hold, but it is clear that the limits can fail without some assumption on z in the case of geometric potentials. To illustrate this point, we present an example in §3.4 using the map $f(x) = 4x(1-x)$ for which (3.3) does not hold, and the relevant scaling limit fails.

3.3. *Escape rate function.* The asymptotics in the previous subsection can be seen as a type of derivative of the escape rate at $\varepsilon = 0$. Our next result addresses the regularity of the escape rate $-\log \lambda_\varepsilon$ from Theorem 3.1 for $\varepsilon > 0$.

THEOREM 3.10. *Let $f \in \mathcal{F}$ and ϕ be an admissible potential. Suppose $z \in I$ and let $\varepsilon^* > 0$ be from Theorem 3.1. Then $\varepsilon \mapsto -\log \lambda_\varepsilon$ is continuous on $[0, \varepsilon^*]$ and forms a devil's staircase: that is, $(d \log \lambda_\varepsilon)/d\varepsilon$ exists and equals 0 on an open and full-measure subset of $[0, \varepsilon^*]$.*

That the escape rate function forms a devil's staircase has been shown in uniformly hyperbolic settings, namely, for expanding systems in [**KL2**], and for Anosov diffeomorphisms in [**DW**]. The present result is the first in the setting of non-uniformly hyperbolic maps. It stands in contrast to Theorems 3.5 and 3.7, which prove that $(d \log \lambda_\varepsilon/d\varepsilon)|_{\varepsilon=0}$ exists and is non-zero. Once Theorem 3.1 is established, it is a direct consequence of the continuity of f and the ergodicity of the measure μ_φ, so we give this short proof immediately.

Proof of Theorem 3.10. The continuity of $\varepsilon \mapsto -\log \lambda_\varepsilon$ follows from Corollary 5.5 and (7.2). We proceed to prove the statement about the derivative of this map. Denote the survivor set by $\mathring{I}_\varepsilon^\infty = \bigcap_{n=0}^\infty f^{-n}(I \setminus H_\varepsilon(z))$.

If $\mathring{I}_\varepsilon^\infty \cap \partial H_\varepsilon = \emptyset$, then $\mathrm{dist}(\mathring{I}_\varepsilon^\infty, \partial H_\varepsilon) > 0$. This follows from the continuity of f and the fact that $H_\varepsilon(z)$ is open: If $\mathring{I}_\varepsilon^\infty \cap \partial H_\varepsilon = \emptyset$ then there exists $n > 0$ such that $f^n(z+\varepsilon) \in H_\varepsilon(z)$; by the continuity of f, there exists a neighbourhood of $z+\varepsilon$, $N_\delta(z+\varepsilon)$, such that $f^n(N_\delta(z+\varepsilon)) \subset H_\varepsilon(z)$. A similar argument holds for $z-\varepsilon$.

Thus if $\mathring{I}_\varepsilon^\infty \cap \partial H_\varepsilon = \emptyset$, then $\mathring{I}_{\varepsilon'}^\infty \cap \partial H_{\varepsilon'} = \emptyset$ for all $\varepsilon' \in (\varepsilon - \delta', \varepsilon + \delta')$ for some $\delta' > 0$, that is, the fact that the boundary of the hole falls into the hole is an open condition. It follows from this that $\mathring{I}_\varepsilon^\infty = \mathring{I}_{\varepsilon'}^\infty$ for all $\varepsilon' \in (\varepsilon - \delta', \varepsilon + \delta')$, and thus that $P(\varphi^{H_\varepsilon}) = P(\varphi^{H_{\varepsilon'}})$ and, by Theorem 3.1(c), $\lambda_\varepsilon = \lambda_{\varepsilon'}$ for all $\varepsilon' \in (\varepsilon - \delta', \varepsilon + \delta')$.

Thus $\log \lambda_\varepsilon$ is locally constant whenever $\mathring{I}_\varepsilon^\infty \cap \partial H_\varepsilon = \emptyset$.

Finally, since $\mu_\varphi(H_\varepsilon) > 0$, ergodicity implies that generic $z \pm \varepsilon$ fall in the hole, so the condition $\mathring{I}_\varepsilon^\infty \cap \partial H_\varepsilon = \emptyset$ is generic. Therefore,

$$\mu_\varphi\left\{x = z + \varepsilon \in I : \varepsilon \in (0, \varepsilon^*) \text{ and } \frac{d \log \lambda_\varepsilon}{d\varepsilon} \neq 0\right\} = 0,$$

as required. □

3.4. *An example of scaling limit failure.* In this subsection we present an example of a map in our class $\mathcal{F}$ and choice of z such that condition (3.3) is violated and the conclusion of Theorem 3.7 fails.

Let $f : I \circlearrowleft$ be defined by $f(x) = 4x(1-x)$. Let X also denote the unit interval, and $T : X \circlearrowleft$ be the tent map $T(x) = 2x$, $x \in [0, 1/2]$, and $T(x) = 2(1-x)$, $x \in [1/2, 1]$.

The well-known conjugacy between f and T is $g : X \to I$, $g(x) = \sin^2(\pi x/2)$, so that $f \circ g(x) = g \circ T(x)$ for all $x \in X$.

Let m denote Lebesgue measure on X, which is T-invariant and the equilibrium state for the potential $-\log |DT|$. The absolutely continuous invariant probability measure for f can then be written as $\mu = g_* m$, which is the equilibrium state for the potential $-\log |Df|$.

We choose $z = 0$, a fixed point for f, and define $H_\varepsilon = [0, \varepsilon)$. It is clear that (3.3) fails, since $\mathrm{Crit} = \{\frac{1}{2}\}$ and $f^2(\frac{1}{2}) = 0$.

Now $g^{-1}(H_\varepsilon) = [0, \varepsilon')$, where $\varepsilon' = (2/\pi)\sin^{-1}(\sqrt{\varepsilon})$. Note that since $X = g^{-1}(I)$, we have

$$m(\mathring{X}^n) := m\bigg(\bigcap_{i=0}^{n} T^{-i}(X \setminus g^{-1}(H_\varepsilon))\bigg) = m\bigg(\bigcap_{i=0}^{n} T^{-i}(g^{-1}(I \setminus H_\varepsilon))\bigg)$$
$$= m\bigg(\bigcap_{i=0}^{n} g^{-1} \circ f^{-i}(I \setminus H_\varepsilon)\bigg) = m\bigg(g^{-1}\bigg(\bigcap_{i=0}^{n} f^{-i}(I \setminus H_\varepsilon)\bigg)\bigg) = \mu(\mathring{I}^n),$$

where $\mathring{X}^n$ and $\mathring{I}^n$ denote the n-step survivor sets for T and f, respectively.

Thus the escape rate $-\log \lambda_\varepsilon$ for (f, μ, H_ε) is the same as the escape rate for $(T, m, g^{-1}(H_\varepsilon))$.

Now applying [**BY**, Theorem 4.6.1 and §5] (see also [**KL2**, Theorem 2.1 and §3.1]) to T, we compute the scaling limit,

$$\lim_{\varepsilon\to 0} \frac{-\log \lambda_\varepsilon}{\mu(H_\varepsilon)} = \lim_{\varepsilon\to 0} \frac{-\log \lambda_\varepsilon}{m(g^{-1}(H_\varepsilon))} = 1 - \frac{1}{DT(0)} = \frac{1}{2}.$$

Yet $Df(0) = 4$, so that the expected scaling limit for f would be $1 - 1/Df(0) = \frac{3}{4} \neq \frac{1}{2}$.

Remark 3.11. Although the scaling limit of Theorem 3.7 fails in this case, we note that an alternate formulation is possible. Indeed, the invariant density for f with respect to Lebesgue measure has a spike of order $x^{-1/2}$ at $z = 0$. So the limit of $\frac{1}{2}$ that we compute is compatible with the formula

$$\lim_{\varepsilon\to 0} \frac{-\log \lambda_\varepsilon}{\mu(H_\varepsilon)} = 1 - \bigg(\frac{1}{Df(0)}\bigg)^{1/2} = 1 - \bigg(\frac{1}{4}\bigg)^{1/2} = 1 - \frac{1}{2} = \frac{1}{2},$$

where the scaling exponent of 1/2 matches the exponent in the spike of the invariant density. Such relations follow from O'Brien's formula for the extremal index (see [**FFT2**, (2.6)] for a dynamical setting of this), and, given the connection between extremal indices and scaling limits for escape rates established in [**BDT**], we conjecture that it holds in greater generality for scaling limits.

4. *Construction of extensions and preliminary results*

4.1. *Distortion and contraction.* As is standard in this field, we wish to recover some uniform expansion and uniform distortion from a system which is non-uniformly hyperbolic. We will often use versions of the Koebe lemma, so state it here (see [**MS**, Theorem IV.1.2]), recalling that elements of $\mathcal{F}$ have negative Schwarzian derivative.

LEMMA 4.1. (Koebe Lemma) *For any $\epsilon > 0$, there exists $K(\epsilon) \geqslant 1$ such that the following hold. If $f \in \mathcal{F}$ and $U \Subset U'$ is such that $U' \setminus U$ consists of two intervals length at least $\epsilon|U|$ and $f^n : U' \to f^n(U')$ is a diffeomorphism, then:*

(a) *for $x, y \in U$,*

$$\frac{Df^n(x)}{Df^n(y)} \leqslant K(\epsilon);$$

(b) *for $x, y \in U$,*

$$\left|\frac{Df^n(x)}{Df^n(y)} - 1\right| \leqslant K(\epsilon)\frac{|x-y|}{|U|}.$$

For expansion/backward contraction we use 'polynomial shrinking'. That is, for $\beta > 0$, we have the following property.

- $(\mathrm{PolShr})_\beta$: there are constants $\delta, C > 0$ such that for each $x \in I$ and every integer $n \geqslant 1$, any connected component W of $f^{-n}(B_\delta(x))$ has $|W| \leqslant Cn^{-\beta}$.

Combining [**R-LS**, Theorem A] and [**BRSS**, Theorem 1], for each $f \in \mathcal{F}$ this holds for any $\beta > 0$†. Notice that for intervals of size larger than δ in our setting, we can simply chop these up into smaller intervals at the cost of adding a multiplicative constant.

4.2. *Hofbauer extensions.* Hofbauer extensions are Markov extensions of $f : I \circlearrowleft$ usually defined by introducing cuts at (images of) critical points, but in fact we can cut at arbitrary points: in §4.4 we will give a definition of our 'extended critical set'. So we let $\mathcal{C}r \subset I$ be a finite set of points with $\mathrm{Crit} \subset \mathcal{C}r$. Set $\mathcal{P}_0 := I$, let $\mathcal{P}_1$ be the partition defined by $\mathcal{C}r$, and define n-cylinders by

$$\mathcal{P}_n := \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}_1.$$

We will denote the n-cylinder in which $x \in I$ lies by $Z_n[x]$ (note that if there are two, then we can make an arbitrary choice). Now define $\mathcal{D} := \{f^k(Z) : Z \in \mathcal{P}_k, k \geqslant 0\}$. As $\mathcal{D}$ is a set, each element $D \in \mathcal{D}$ appears once (i.e. if $f^k(Z) = f^j(Z')$ then these elements are naturally identified as the same set). The *Hofbauer extension* is defined as the disjoint union

$$\hat{I} = \hat{I}(\mathcal{C}r) := \bigsqcup_{D \in \mathcal{D}} D.$$

We call each D a *domain* of $\hat{I}$. There is a natural projection map $\pi : \hat{I} \to I$, so each point $\hat{x} \in \hat{I}$ can be represented as (x, D) where $x = \pi\hat{x}$. The map $\hat{f} : \hat{I} \circlearrowleft$ is defined by $\hat{f}(\hat{x}) = \hat{f}(x, D) = (f(x), D')$ if there are cylinder sets $Z' \subset Z$ with $Z' \in \mathcal{P}_{k+1}$ and $Z \in \mathcal{P}_k$ such that

$$x \in f^k(Z') \subset f^k(Z) = D \quad \text{and} \quad D' = f^{k+1}(Z'). \tag{4.1}$$

In this case we write $D \to D'$, so $(\mathcal{D}, \to)$ has the structure of a directed graph. With this set-up, π acts as a semiconjugacy between $\hat{f}$ and f:

$$\pi \circ \hat{f} = f \circ \pi.$$

We can think of points in $\mathcal{C}r$ as 'cut points' since if an open interval $\hat{A} = (A, D) \subset \hat{I}$ and $\#\{A \cap \mathcal{C}r\} = k \geqslant 1$, then $\hat{A}$ gets cut at each element of $\mathcal{C}r$ (strictly speaking, of $\pi^{-1}(\mathcal{C}r)$) so that $\hat{f}(\hat{A})$ lies in $k+1$ different elements of $\mathcal{D}$.

Let D_0 be the *base* of $\hat{I}$, that is, the copy of I in the extension. Define ι to be the natural inclusion map sending I to D_0. For $D \in \mathcal{D}$, we let $\mathrm{level}(D)$ be the length of the shortest path $D_0 \to \cdots \to D$ in $(\mathcal{D}, \to)$. Then, for $L \in \mathbb{N}$, the truncated extension at level L is

$$\hat{I}(L) := \bigsqcup \{D \in \mathcal{D} : \mathrm{level}(D) \leqslant L\}.$$

The following lemma and proof are well known in the area, but we include them for illustrative purposes and for use later.

† In fact, these results imply that to obtain $(\mathrm{PolShr})_\beta$ for a particular β, one does not need $|Df^n(f(c))| \to \infty$ for all $c \in \mathrm{Crit}$, but rather a specific lower bound for $|Df^n(f(c))|$ depending on β suffices.

LEMMA 4.2. *Suppose that $\hat{x}$, $\hat{y} \in \hat{I} \setminus \partial \hat{I}$ have $\pi \hat{x} = \pi \hat{y}$. Then there exists $n \in \mathbb{N}$ such that $\hat{f}^n(\hat{x}) = \hat{f}^n(\hat{y})$.*

Proof. Let $w = \pi \hat{x}$. Observe that since π is a semiconjugacy, $\hat{f}^k(\hat{x}), \hat{f}^k(\hat{y}) \in \pi^{-1}(f^k(w))$ for all $k \geqslant 0$. Let $D_{\hat{x}}$ and $D_{\hat{y}}$ denote the domains of $\hat{I}$ which contain $\hat{x}$ and $\hat{y}$, respectively. Then choose n so large that $(\pi|_{D_{\hat{x}}})^{-1}(Z_n[w])$ and $(\pi|_{D_{\hat{y}}})^{-1}(Z_n[w])$ are both compactly contained inside $D_{\hat{x}}$ and $D_{\hat{y}}$ respectively, where $Z_n[w]$ denotes the element of $\mathcal{P}_n$ containing w. Now notice that $f^n(Z_n[w])$ is a domain of the Hofbauer extension, and indeed it follows from the construction in (4.1) that $\hat{f}^n(\hat{x})$ and $\hat{f}^n(\hat{y})$ must lie in $f^n(Z_n[w])$. Since these iterates must also both lie on the fibre $\pi^{-1}(f^n(w))$ by the conjugacy property, the points must coincide, as required. □

In general, Hofbauer extensions split into a collection of transitive components and a non-transitive set (see [**HR**]), but the above lemma and the leo property imply that there is a unique transitive component. Since any points outside this must map into it and stay there forever, we will adopt the convention that $\hat{I}(L)$ is always restricted to the transitive component.

Given a set $A \subset I$, the set $\hat{A} = \pi^{-1}(A)$ is called the *lift* of A. We now consider how to lift measures to $\hat{I}$. Suppose that μ is an ergodic f-invariant probability measure. Set $\hat{\mu}^{(0)} := \mu \circ \iota^{-1}$, and for $n \in \mathbb{N}$,

$$\hat{\mu}^{(n)} := \frac{1}{n} \sum_{k=0}^{n-1} \hat{\mu}^{(0)} \circ \hat{f}^{-k}.$$

As in [**K**], if $h_\mu(f) > 0$, then $\hat{\mu}^{(n)}$ converges in the vague topology† to $\hat{\mu}$, which is an $\hat{f}$-invariant ergodic measure with

$$\hat{\mu} \circ \pi^{-1} = \mu.$$

Also, [**K**] shows that $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$.

We will also be interested in lifting conformal measures. Given a conformal measure m_ϕ on I, define $\hat{m}_\phi := m_\phi \circ \pi^{-1}$. Clearly $\hat{m}_\phi$ is $\hat{\phi}$-conformal for $\hat{\phi} := \phi \circ \pi$ on $\hat{I}$. Note that, in general, it could be the case that $\hat{m}_\phi(\hat{I}) = \infty$.

Remark 4.3. We can define pressure $P(\hat{\phi})$ analogously to (2.1). As in (2.3), for admissible potentials we need only consider measures with positive entropy, so we deduce that $P(\hat{\phi}) = P(\phi)$. This implies that when we lift the normalized potential, $\hat{\varphi} := \varphi \circ \pi$, the relation $\hat{\varphi} = \hat{\phi} - P(\hat{\phi})$ continues to hold.

4.3. *Inducing schemes.* We wish to define inducing schemes via first return maps to truncated domains in the Hofbauer extension, whose partition we will refine further below: it will also be useful to set this up for our punctured systems, though there will be a small difference in the structure there. To this end, let $\hat{\mathcal{P}}_n$ be the set of intervals

† Recall that $\hat{\mu}^{(n)}$ converges to $\hat{\mu}$ vaguely if $\hat{\mu}^{(n)}(\psi)$ converges to $\hat{\mu}(\psi)$ for all continuous ψ with compact support in $\hat{I}$.

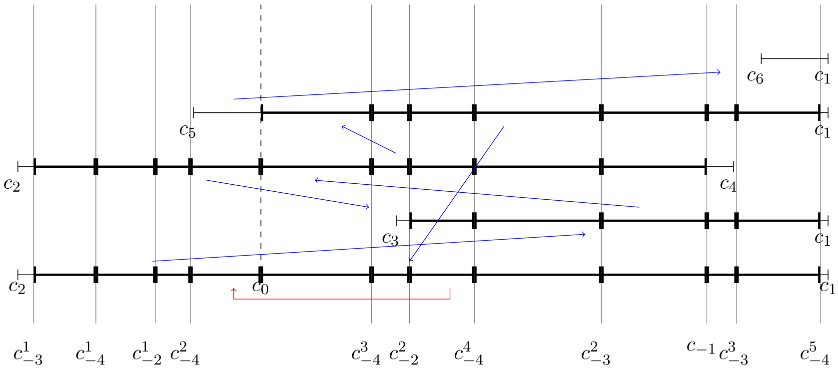

FIGURE 1. A sketch of the first few levels of a Hofbauer extension for a unimodal map. The dashed line shows where we cut at the critical point c_0, a blue arrow shows movement between domains in different levels and a red arrow shows movement between domains in the same level (the colouring will be most useful when we have extra cuts, as we will later; see online version for colours). We denote $c_n = f^n(c_0)$. We also indicate the boundaries of the cylinder sets, denoted by c^i_{-j}, in $\mathcal{P}_4$, and draw vertical lines to indicate how this lifts to $\hat{\mathcal{P}}_4$. Thick vertical lines imply these points are doubled. These endpoints are then used to determine $\hat{I}'(4)$, as well as the domains Q of Y, which are drawn with thick black lines.

$\{(\pi|_D)^{-1}(Z) : D \in \mathcal{D},\ Z \in \mathcal{P}_n\}$. For a domain $D \in \mathcal{D}$, let D^L_ℓ be the leftmost interval of $\hat{\mathcal{P}}_L$ in D and D^L_r be the rightmost. Then

$$\hat{I}'(L) := \hat{I}(L) \cap \Bigg[\bigsqcup_{D\in\mathcal{D}} (D \setminus (D^L_\ell \cup D^L_r))\Bigg]. \tag{4.2}$$

It follows, for example from [**DoT**, Lemma 8.2] that, so long as $\hat{I}$ has more than one domain, then for all $\epsilon > 0$ there exists $L \in \mathbb{N}$ such that if $h_\mu(f) > \epsilon$ then $\hat{\mu}(\hat{I}'(L)) > 0$.

We further partition $\hat{I}'(L)$ into the elements of $\hat{\mathcal{P}}_L$ intersecting it and denote this collection by $\mathcal{Q}$, (i.e. $\mathcal{Q} = \{Q \in \hat{\mathcal{P}}_L : Q \subset \hat{I}'(L)\}$); see Figure 1. Letting R be the first return time to $Y := \hat{I}'(L)$, the map $F = \hat{f}^R$ is the first return map. We denote the domains of F by $\{Y_i\}_i$. These are the maximal sets U such that $U \subset Q$ and $F(U) \subset Q'$ for some $Q,\ Q' \in \mathcal{Q}$, so that F is monotonic and R is constant on U. We set $R_i = R|_{Y_i}$. The cylinder structure of $\mathcal{Q}$ ensures that the $\{Y_i\}_i$ are disjoint and the Markov structure ensures that the image of such a domain is an interval Q of $\mathcal{Q}$; see [**DoT**, Lemma 4.9]. We give a short proof of this fact to explain how the changes we make later will not affect this structure.

LEMMA 4.4. (Markov property of F) *If Y_i is a domain of F with $F(Y_i) \subset Q \in \mathcal{Q}$ then $F(Y_i) = Q$.*

Proof. Let $D \in \mathcal{D}$ denote the domain in which Y_i lies and suppose $R_i = n$. By the Markov structure of the Hofbauer extension there must exist $Y_i' \subset D$ such that $\hat{f}^n(Y_i') = Q$. If $Y_i \neq Y_i'$ then the only constraint that Y_i must satisfy which Y_i' need not is that Y_i must be contained in some $Q' \in \hat{\mathcal{P}}_L$. This means that Y_i must have an element of $\partial\hat{\mathcal{P}}_L$ as a boundary point: indeed, it must be adjacent to some D^L_ℓ or D^L_r. Denote such a point by a_{-j}, where $\pi(a_{-j}) \in f^{-j}\mathcal{C}r$. In particular, $j \leqslant L$. So if $n > j$ then in fact $\hat{f}^n(a_{-j})$ must be a boundary point of some $D \in \mathcal{D}$, which is a contradiction. On the other hand, if

$n \leqslant j$ then $\hat{f}^n(a_{-j})$ is a boundary point of an element of $\hat{\mathcal{P}}_L$ so in fact $\hat{f}^n(Y_i) = Q$ and $Y_i = Y_i'$. □

Remark 4.5. In the construction above, we used $\hat{\mathcal{P}}_L = \hat{\mathcal{P}}_L(\mathcal{C}r)$ to firstly arrange for $\hat{I}(L)$ to be trimmed to $\hat{I}'(L)$ and then secondly to partition the domains of $\hat{I}'(L)$ into Q. We observe here that if a subset $\mathcal{C}r' \subset \mathcal{C}r$ is instead used to produce $\hat{\mathcal{P}}_L(\mathcal{C}r')$ and this set used in place of $\hat{\mathcal{P}}_L(\mathcal{C}r)$, the set-up above, and in particular the conclusion of Lemma 4.4, still holds. We will employ such a construction in §4.4.

Note that the set of domains generate a cylinder structure for F, which we will denote by $\{Y_i^{(n)}\}_i$ for the collection of n-cylinders. The Markov structure of the Hofbauer extension implies for that each domain of F, if it maps onto $D(L) \in \hat{I}'(L)$, where $D(L) \subset D \in \mathcal{D}$, then there is an extension so that F extends to a map onto D. As in Lemma 4.1, this extension property gives us bounded distortion for F: there exists $K \geqslant 1$ such that for Y_i a domain of F, if $x, y \in Y_i$ then

$$\frac{|DF(x)|}{|DF(y)|} \leqslant K$$

(we improve on this estimate in Lemma 4.6). Note that K depends on L since L determines the constant ϵ in Lemma 4.1.

We also note that by [**DoT**, Lemma 10.7], F is uniformly hyperbolic, that is, there exist $C_F > 0$ and $\sigma_F > 1$ such that, for $x \in Y$ and any $n \geqslant 1$,

$$|DF^n| \geqslant C_F \sigma_F^n. \tag{4.3}$$

Given a potential $\phi : I \to [-\infty, \infty]$, and its normalized lift $\hat{\varphi} = \varphi \circ \pi$ as in Remark 4.3, we define the *induced potential*

$$\Phi(x) = \hat{\varphi}(x) + \hat{\varphi}(\hat{f}(x)) + \cdots + \hat{\varphi}(\hat{f}^{R(x)}(x)), \quad x \in \hat{I}'(L).$$

As in (2.2), if $\hat{m}_\varphi$ is $\hat{\varphi}$-conformal for $\hat{f}$, then it is also Φ-conformal for F.

By Kac's lemma, since F is a first return map to Y, if $\hat{\mu}$ is a $\hat{f}$-invariant probability measure then

$$\hat{\mu}_Y = \frac{\hat{\mu}|_Y}{\hat{\mu}(Y)} \text{ is an } F\text{-invariant probability measure and } \hat{\mu}(Y) = \frac{1}{\int R \, d\hat{\mu}_Y}. \tag{4.4}$$

We also note that

$$\hat{\mu}(A) = \hat{\mu}(Y) \sum_i \sum_{j=0}^{R_i - 1} \hat{\mu}_Y(Y_i \cap \hat{f}^{-j} A) = \sum_i \sum_{j=0}^{R_i - 1} \hat{\mu}(Y_i \cap \hat{f}^{-j} A) \quad \text{for any Borel } A \subset \hat{I}, \tag{4.5}$$

where the sum over i is taken over all 1-cylinders Y_i for F, and $R_i = R|_{Y_i}$.

We close this subsection with the following distortion result, which is primarily due to Lemma 4.1.

LEMMA 4.6.

(a) *Suppose that $\phi : I \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\eta \leqslant 1$. Then there exists $K_{F,\phi} > 0$ such that, for any n-cylinder $Y_i^{(n)}$ and all $x, y \in Y_i^{(n)}$,*

$$|S_n\Phi(x) - S_n\Phi(y)| \leqslant K_{F,\phi} |F^n(x) - F^n(y)|^\eta.$$

(b) *There exists $K_F > 0$ such that, for any n-cylinder of the scheme $Y_i^{(n)}$ and all $x, y \in Y_i^{(n)}$,*

$$\left|\frac{DF^n(x)}{DF^n(y)} - 1\right| \leqslant K_F|F^n(x) - F^n(y)|.$$

Proof. We prove (a) first. We begin by taking a 1-cylinder Y_i and $x, y \in Y_i$. Then

$$\begin{aligned}|\Phi(x) - \Phi(y)| &\leqslant \sum_{k=0}^{R_i-1} |\phi \circ f^k(x) - \phi \circ f^k(y)| \leqslant C \sum_{k=0}^{R_i-1} |f^k(x) - f^k(y)|^\eta \\ &= C \sum_{k=0}^{R_i-1} \left(\frac{|f^k(x) - f^k(y)|}{|F(x) - F(y)|}\right)^\eta |F(x) - F(y)|^\eta \\ &\leqslant KC|F(x) - F(y)|^\eta \sum_{k=0}^{R_i-1} \left(\frac{|f^k(Y_i)|}{|f^{R_i}(Y_i)|}\right)^\eta,\end{aligned}$$

where K is a distortion constant coming from Lemma 4.1. So for a Hölder condition on the induced potential it suffices to have a bound on $\sum_{k=0}^{R_i-1}(|f^k(Y_i)|/|f^R(Y_i)|)^\eta$, which follows from (PolShr)$_\beta$ for $\beta > 1/\eta$.

Note that since F is uniformly hyperbolic as in (4.3), this result passes to n-cylinders, proving (a).

Part (b) is an immediate consequence of Lemma 4.1(b). Note that when considering a cylinder $Y_i^{(n)}$, the switch from $|x - y|/|Y_i^{(n)}|$ to $|F^n(x) - F^n(y)|$ follows by Lemma 4.1(a) and that $|F^n(Y_i^{(n)})| \asymp 1$. □

Remark 4.7. The above lemma, Remark 4.3 and the proof of [**DT2**, Propostion 1.6] imply that for admissible normalized potentials φ, the induced potential Φ has $P(\Phi) = 0$, where pressure for the induced system is defined analogously to (2.1).

4.4. *Punctured extensions with uniform images and uniform tails.* In order to study open systems via the Hofbauer extension, once we fix a point $z \in I$ to be the centre of our hole, we will introduce extra cuts during the construction of the extension. Indeed, in order to compare Hofbauer extensions with different sets of cuts in a neighbourhood of z, we will construct extensions with uniform images for the induced maps that are independent of these extra cuts.

Our notation is as follows. For $\varepsilon_0 > 0$ to be chosen below and $0 < \varepsilon < \varepsilon_0$, we will construct two related Hofbauer extensions: $\hat{I}_{z,\varepsilon_0}$ introducing cuts at z and $z \pm \varepsilon_0$; and $\hat{I}_{z,\varepsilon_0,\varepsilon}$ introducing cuts at z, $z \pm \varepsilon_0$ and $z \pm \varepsilon$. In particular, this means that we will add $f^{-1}(z)$, $f^{-1}(z \pm \varepsilon_0)$ and $f^{-1}(z \pm \varepsilon)$ to our critical set. The corresponding dynamics is denoted by $\hat{f}_{z,\varepsilon_0}$ and $\hat{f}_{z,\varepsilon_0,\varepsilon}$, respectively. A simplified diagram is presented in Figure 2.

We fix z, and at the beginning of §4.5 we will choose the relevant quantities in the following order. First, we will choose L according to Theorem 4.10, which will provide uniform control on the complexity of the tail of the Hofbauer extension and will depend only on the cardinality of the critical set plus 5 deg(f). Next, we will choose ε_0^* according to (4.6), then finally we choose $\varepsilon_0 \leqslant \varepsilon_0^*$, which will fix the return domain Y, and work with $0 < \varepsilon < \varepsilon_0$ as the variable size of the hole.

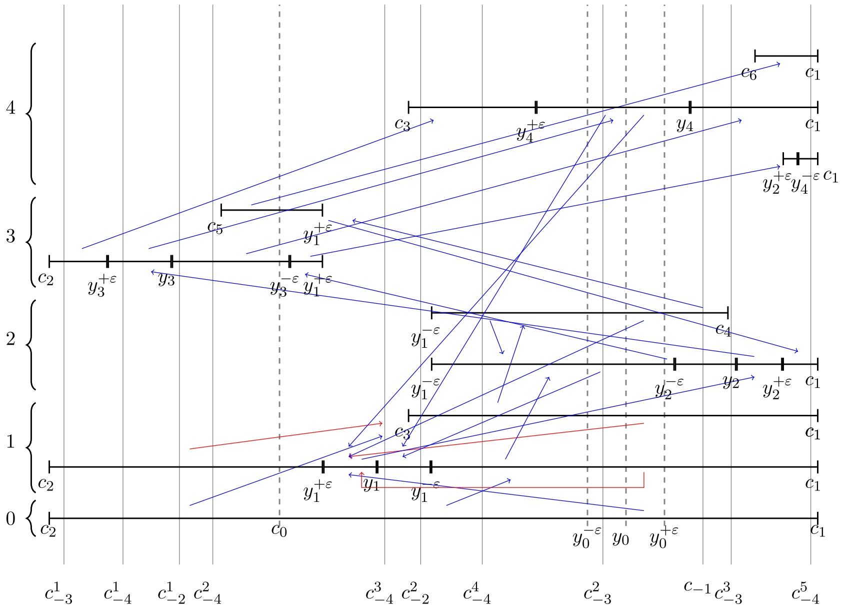

FIGURE 2. The Hofbauer extension based on the same unimodal map as in Figure 1, but with new cuts at points $y_0^{+\varepsilon}$, y_0 and $y_0^{-\varepsilon}$. These represent one set of preimages of $f^{-1}(z-\varepsilon_0)$, $f^{-1}(z)$ and $f^{-1}(z+\varepsilon_0)$: adding in all preimages adds to the complexity of the diagram significantly. Similarly, for simplicity we include only the preimages of c_0 at the bottom of the diagram, and omit the preimages of $y_0^{+\varepsilon}$, y_0 and $y_0^{-\varepsilon}$ (therefore, unlike in Figure 1, we do not mark out the domains Q of Y here). In levels above 0 any marked point is a boundary point of $\mathcal{D}$: thicker markers imply that these points are doubled. Since, in contrast to Figure 1, the number of domains of a given level can be greater than one, we add in the numbers on the left to clarify the level of each domain. Note that with ε_0 fixed less than ε_0^*, additional cuts can be introduced at $z \pm \varepsilon$ for $\varepsilon < \varepsilon_0$ which do not affect the structure of the cylinders outside the intervals $(y_n, y_n^{+\varepsilon})$ and $(y_n^{-\varepsilon}, y_n)$.

$\hat{I}_z = \hat{I}(\mathrm{Crit}_z)$. Let Crit_z denote the expanded critical set, that is, $\mathrm{Crit} \cup \{f^{-1}z\}$. Next, consider the partition $\mathcal{P}_L = \mathcal{P}_L(\mathrm{Crit}_z)$ of I into L-cylinders with endpoints at $\{f^{-j}(y) : y \in \mathrm{Crit}_z, 0 \leqslant j \leqslant L\}$. We choose

$$\varepsilon_0^* < \frac{1}{|Df^L|_\infty} \min\{|x-y| : x \neq y,\ x \in \partial\mathcal{P}_L,\ y \in f^j(\mathrm{Crit}_z),\ 0 \leqslant j \leqslant L\}. \tag{4.6}$$

$\hat{I}_{z,\varepsilon_0} = \hat{I}(\mathrm{Crit}_{z,\varepsilon_0})$. For $0 < \varepsilon_0 \leqslant \varepsilon_0^*$, we define $\hat{I}_{z,\varepsilon_0} = \hat{I}(\mathrm{Crit}_{z,\varepsilon_0})$ as above, where $\mathrm{Crit}_{z,\varepsilon_0}$ has $f^{-1}(z \pm \varepsilon_0)$ added to Crit_z. Let $\hat{I}_{z,\varepsilon_0}(L)$ denote the first L levels of $\hat{I}_{z,\varepsilon_0}$, and let $\hat{I}'_{z,\varepsilon_0}(L)$ denote $\hat{I}_{z,\varepsilon_0}(L)$ minus the elements of $\hat{\mathcal{P}}_L(\mathrm{Crit}_{z,\varepsilon_0})$ adjacent to each boundary point in $\hat{I}_{z,\varepsilon_0}(L)$, as in (4.2), so that the new boundary points are of the form $f^{-j}(y)$ for some $y \in \mathrm{Crit}_{z,\varepsilon_0}$ and $0 \leqslant j \leqslant L$. Note that by choice of ε_0^*, we completely remove elements of the form $[f^k(z), f^k(z+\varepsilon_0)]$ for $0 \leqslant k \leqslant L$, and analogues, in going from $\hat{I}_{z,\varepsilon_0}(L)$ to $\hat{I}'_{z,\varepsilon_0}(L)$.

$\hat{I}_{z,\varepsilon_0,\varepsilon} = \hat{I}(\mathrm{Crit}_{z,\varepsilon_0,\varepsilon})$. For any $\varepsilon \in (0, \varepsilon_0)$, we define $\mathrm{Crit}_{z,\varepsilon_0,\varepsilon}$ to be $\mathrm{Crit}_{z,\varepsilon_0}$ with $f^{-1}(z \pm \varepsilon)$ added. Let $\hat{I}_{z,\varepsilon_0,\varepsilon} = \hat{I}(\mathrm{Crit}_{z,\varepsilon_0,\varepsilon})$ and define $\hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$ to be the first L levels, $\hat{I}_{z,\varepsilon_0,\varepsilon}(L)$, minus the elements of $\hat{\mathcal{P}}_L(\mathrm{Crit}_{z,\varepsilon_0,\varepsilon})$ adjacent to each boundary point in $\hat{I}_{z,\varepsilon_0,\varepsilon}(L)$ so that the new boundary points are of the form $f^{-j}(y)$ for some $y \in \mathrm{Crit}_{z,\varepsilon_0,\varepsilon}$

and $0 \leqslant j \leqslant L$. As above, we completely remove elements of the form $[f^k(z), f^k(z+\varepsilon_0)]$ for $0 \leqslant k \leqslant L$, and analogues, in going from $\hat{I}_{z,\varepsilon_0,\varepsilon}(L)$ to $\hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$.

As can be seen from this construction, the domains of $\hat{I}'_{z,\varepsilon_0}(L)$ and $\hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$ are the same. We choose $\mathcal{Q}$ to be the domains of $\hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$ further partitioned by $\hat{\mathcal{P}}_L(\mathrm{Crit}_{z,\varepsilon_0})$. We choose this partition rather than $\hat{\mathcal{P}}_L(\mathrm{Crit}_{z,\varepsilon_0,\varepsilon})$ to ensure our F-images have size independent of ε and because, as in Remark 4.5, this does not affect the Markov structure for F since the extra cuts due to ε fall within intervals of the form $[f^k(z), f^k(z+\varepsilon_0)]$ for $0 \leqslant k \leqslant L$, which have already been removed from $\hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$.

Remark 4.8. Here we explain how cutting at $f^{-1}(z)$ and our choice of ε_0^* ensure that the representatives of the holes in the Hofbauer extension are disjoint from our inducing domains.

(a) If $\hat{f}: U \to D$, $D \in \mathcal{D}$, is a homeomorphism, then since we cut at $f^{-1}(z)$, the interior of U cannot intersect $\pi^{-1}(f^{-1}(z))$, which also implies that the interior of D cannot intersect $\pi^{-1}(z)$. Therefore, this fact must be true for any D in the transitive part of $\hat{I}_z$. So we conclude that $\pi^{-1}(z) \cap \hat{I}'_z(L) = \emptyset$ due to trimming of L-cylinders.

(b) Suppose that $\hat{J}_{\varepsilon_0}(z) \subset D \in \mathcal{D}$, where $\mathrm{level}(D) = k \in \{0, \ldots, L\}$ and $\pi(\hat{J}_{\varepsilon_0}(z)) \subset (z-\varepsilon_0, z+\varepsilon_0)$. By (4.6), $(\hat{f}^j_{z,\varepsilon_0}(\hat{J}_{\varepsilon_0}(z))) \cap \hat{I}'_{z,\varepsilon_0}(L) = \emptyset$, for all $j = 0, \ldots, L-k$. As a consequence $\pi^{-1}((z-\varepsilon_0, z+\varepsilon_0)) \cap \hat{I}'_{z,\varepsilon_0}(L) = \emptyset$ and there is a one-to-one correspondence between elements of $\hat{I}'_{z,\varepsilon_0}(L)$ and $\hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$; indeed, precisely the same domains appear on each level. Abusing notation slightly, we write $\hat{I}'_{z,\varepsilon_0}(L) = \hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$, and once L is fixed, simply refer to the common set of domains as

$$Y = \bigsqcup_{Q \in \hat{\mathcal{P}}_L(\mathrm{Crit}_{z,\varepsilon_0})} Q.$$

As a result of this construction, $Y \cap \hat{H}_\varepsilon = \emptyset$ for all $\varepsilon < \varepsilon_0$, where $\hat{H}_\varepsilon = \pi^{-1}(H_\varepsilon)$.

Remark 4.9. (Role of ε and ε_0) The cuts at $z \pm \varepsilon$ form the boundary of the hole H_ε, and defining $\hat{I}_{z,\varepsilon_0,\varepsilon}$ with respect to these cuts guarantees that the Markov structure will respect the hole. The extra cuts at $z \pm \varepsilon_0$ are used to guarantee uniform images and tails for returns to Y as $\varepsilon \to 0$. Without loss of generality on H_ε, we may always choose ε_0 to satisfy

$$\{f^\ell(z \pm \varepsilon_0)\}_{\ell \geqslant 0} \cap \mathrm{Crit}_z = \emptyset \quad \text{and} \quad \{f^\ell(z)\}_{\ell \geqslant 0} \cap \{z \pm \varepsilon_0\} = \emptyset. \tag{4.7}$$

In fact, we will only need to invoke (4.7) in §7 to prove convergence to the asymptotic escape rate in the case where z is periodic (see Lemmas 7.6 and 7.11). All results in §§4.5, 5 and 6 hold for all $\varepsilon_0 \leqslant \varepsilon_0^*$.

The size of $\varepsilon > 0$ will be further reduced in Corollaries 4.13 and 5.3 and Lemma 6.2 to satisfy $\varepsilon < \varepsilon_1^*$, where $\varepsilon_1^* < \varepsilon_0$ guarantees that the corresponding induced maps are uniformly mixing and the associated transfer operators have a uniform spectral gap.

As defined above, $\mathcal{Q}$ denotes the finite partition of Y into its domains. Define the induced maps $F_{z,\varepsilon_0} = \hat{f}_{z,\varepsilon_0}^{R_{z,\varepsilon_0}}$ and $F_{z,\varepsilon_0,\varepsilon} = \hat{f}_{z,\varepsilon_0,\varepsilon}^{R_{z,\varepsilon_0,\varepsilon}}$ acting on the domain Y, where $R_\varkappa$ denotes the first return time to Y in the extension $(\hat{f}_\varkappa, \hat{I}_\varkappa)$, and $\varkappa$ stands for either of the indices (z, ε_0) or $(z, \varepsilon_0, \varepsilon)$. By construction, all images of elements of $\mathcal{Q}$ under $F_\varkappa$ are unions of elements of $\mathcal{Q}$. Thus $F_\varkappa$ has the finite images property.

We have a natural projection $\pi_\varkappa : \hat{I}_\varkappa \to I$ which commutes with the dynamics, $\pi_\varkappa \circ \hat{f}_\varkappa = f \circ \pi_\varkappa$. Note that from here on we will fix $\hat{m} = \hat{m}_\varphi$ for the relevant $\varphi = \phi - P(\phi)$. As in Remark 4.8, $\hat{m}$ is the same for all $Y = \hat{I}'_\varkappa(L)$; moreover, $\hat{m}$ is always conformal for $\Phi_\varkappa$ under $F_\varkappa$ and we obtain $F_\varkappa$-invariant measures as in (4.4) and (4.5).

Define $\mathfrak{d} = \#\mathcal{C}r + 5\deg(f) \geqslant \#\mathrm{Crit}_{z,\varepsilon_0,\varepsilon}$, and note that, by definition, $\mathfrak{d}$ does not depend on ε and ε_0, just on the fact that we have introduced extra cuts at the preimages of the five points z, $z \pm \varepsilon_0$ and $z \pm \varepsilon$. Our first result provides uniform bounds on the tail of the return time functions R_{z,ε_0} and $R_{z,\varepsilon_0,\varepsilon}$.

THEOREM 4.10. *Suppose that*
(a) *either $\phi = -t\log|Df|$ for $t \in (t^-, t^+)$,*
(b) *or $\phi : I \to \mathbb{R}$ is Hölder continuous.*
Then there exist $L \in \mathbb{N}$, $C > 0$ and $\alpha > 0$ such that for all $0 < \varepsilon < \varepsilon_0 \leqslant \varepsilon_0^*$, $F_\varkappa$, the first return map to $\hat{I}'_\varkappa(L)$, has tails $\hat{m}_{\phi-P(\phi)}(R_\varkappa > n) \leqslant Ce^{-\alpha n}$, where $\varkappa = \{z, \varepsilon_0\}$ or $\{z, \varepsilon_0, \varepsilon\}$. Here L, C, α depend only on $(f, \phi, \mathfrak{d})$.

Proof of Theorem 4.10. For ease of notation, we will drop the subscript $\varkappa$ in the proof, but all statements apply equally well to F_{z,ε_0} and $F_{z,\varepsilon_0,\varepsilon}$.

As shown in [**DoT**, Lemma 4.15], for each $\xi > 0$ there exist $L = L(\xi)$ and $n_0 = n_0(L)$ such that $\#\{i : R(Y_i) = n\} \leqslant e^{\xi n}$ for all $n \geqslant n_0$. Crucially, these numbers only depend on $\mathfrak{d}$, so are independent of the actual values of ε and ε_0. Thus to prove the theorem, it suffices to show that there exists some $\bar{\alpha} > \xi$ such that, for any 1-cylinder Y_i of F, $\hat{m}(Y_i) \leqslant e^{-\bar{\alpha}R_i}$, where $R_i := R|_{Y_i}$.

In the geometric case (case (a)) we will set

$$\bar{\alpha} = p_t + t\lambda_{\min}.$$

The fact that $\bar{\alpha} > 0$ follows immediately from our having set $t < t^+$. In the Hölder case (b) we obtain an analogous $\bar{\alpha} > 0$ using the assumed pressure gap, $\phi < P(\phi)$. In both cases we now can select $\xi < \bar{\alpha}$, which then fixes $L(\xi)$ and $\mathcal{Q}$. We will see below that our estimates on the measures of the domains Y_i yield $\alpha = \bar{\alpha} - \xi$.

We will use the expansion on periodic orbits to estimate the measure of the domains Y_i. The proof of this theorem would be simpler if we had $Y_i \subset F(Y_i)$ for all i, since then each Y_i would contain a point of period R_i, allowing us to connect $\bar{\alpha}$ and the measure of Y_i. To overcome this issue, we will first prove that F is transitive on elements of $\mathcal{Q}$. Recall that by Lemma 4.2, if $O_1, O_2 \subset \hat{I}'_{z,\varepsilon_0}(L)$ are two open sets such that $\pi(O_1) \cap \pi(O_2) \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that $F^n(O_1) \cap F^n(O_2) \neq \emptyset$.

Now let $Q_1, Q_2 \in \mathcal{Q}$. Since f is leo, there exists $n_1 \in \mathbb{N}$ such that $\pi(\hat{f}^{n_1} Q_1) \supset I \supset \pi(Q_2)$. By Lemma 4.2, there exists $n_2 \in \mathbb{N}$ such that $\hat{f}^{n_1+n_2}(Q_1) \cap \hat{f}^{n_2}(Q_2) \neq \emptyset$. Since Q_2 is a recurrent element of $\mathcal{Q}$, there exists $n \in \mathbb{N}$ such that $F^n(Q_1) \cap Q_2 \neq \emptyset$. Then the Markov property of F implies that $F^n(Q_1) \supset Q_2$, and the claimed transitivity follows.

Since $\mathcal{Q}$ is finite, there exist $N \geqslant 1$ and $C > 0$ such that, for each pair $Q_1, Q_2 \in \mathcal{Q}$, there are $J \subset Q_1$ and $n \leqslant N$ such that $\hat{f}^n : J \to Q_2$ is a diffeomorphism with $|D\hat{f}^n|_J| \geqslant C$. Therefore, each domain Y_i of the inducing scheme contains a periodic point y_i with period

$R_i \leqslant p \leqslant R_i + N$ for $\hat{f}$. Then $|DF(y_i)| \geqslant C^{-1}|D\hat{f}^p(y_i)| \gtrsim e^{\lambda_{\min} R_i}$. Throughout we will treat $\hat{f}^{R_i}(Y_i)$ as having uniform size, that is, independent of i.

In case (a), Lemma 4.1 implies

$$\hat{m}(\hat{f}^{R_i}(Y_i)) = \int_{Y_i} e^{-S_{R_i}(t\hat{\phi}-p_t)}\,d\hat{m} = \int_{Y_i} |DF|^t e^{R_i p_t}\,d\hat{m} \asymp |DF(y_i)|^t \hat{m}(Y_i) e^{R_i p_t}. \quad (4.8)$$

Therefore, $\hat{m}(Y_i) \lesssim e^{-R_i(t\lambda_{\min}+p_t)} = e^{-\bar{\alpha} R_i}$.

For the Hölder case (b), recall that we have assumed that $\phi < P(\phi)$, and thus, by Remark 4.3, $\hat{\phi} < P(\hat{\phi})$ on $\hat{I}$. Our value of $\bar{\alpha}$ here is $\inf\{P(\phi) - (S_p\phi(x)/p) : f^p(x) = x\} \geqslant \inf\{P(\phi) - \phi(x) : x \in I\} > 0$. So again, using a slightly more elementary version of the estimate in (4.8) in conjunction with Lemma 4.6, to give us our requisite distortion property, the result follows. □

4.5. *Uniform mixing for $F_{z,\varepsilon_0,\varepsilon}$.* Now we choose L large enough so that the conclusion of Theorem 4.10 is satisfied. Furthermore, we enlarge L if necessary so that

(a) $\kappa = \max\{\hat{m}(\hat{I}_{z,\varepsilon_0,\varepsilon} \setminus \hat{I}'_{z,\varepsilon_0}(L)), \hat{m}(\hat{I}_{z,\varepsilon_0} \setminus \hat{I}'_{z,\varepsilon_0}(L))\} < 1/3$; and

(b) any ergodic invariant measure ν with entropy $h_\nu(f) > (\log \overline{\lambda}_{\varepsilon^*} + \alpha)/2$ lifts to our inducing scheme on $\hat{I}'_{z,\varepsilon_0}(L)$, where ε^* is from Theorem 3.1.

Item (b) is possible due to [**DoT**, Lemma 8.2], and the fact that α does not decrease as L increases.

With L fixed, we define ε_0^* as in (4.6), and for $\varepsilon_0 \leqslant \varepsilon_0^*$, we let $Y = \hat{I}'_{z,\varepsilon_0}(L)$ as in Remark 4.8.

Our next result proves a necessary mixing property for our return maps.

LEMMA 4.11. *For all $\varepsilon_0 \leqslant \varepsilon_0^*$ and $\varepsilon < \varepsilon_0$, the induced maps F_{z,ε_0} and $F_{z,\varepsilon_0,\varepsilon}$ are topologically mixing on Y.*

Proof. We write our arguments for $F_{z,\varepsilon_0,\varepsilon}$, but the same proof holds for F_{z,ε_0}.

By the proof of Theorem 4.10, $F_{z,\varepsilon_0,\varepsilon}$ is transitive on the finitely many elements of $\mathcal{Q}$. The only way it can fail to be mixing is if the images decompose into a periodic cycle. Let $Q \in \mathcal{Q}$. Since f is leo, there exists n_Q such that $f^n(\pi(Q)) \supset I$ for all $n \geqslant n_Q$. By our choice of L, $\kappa = \hat{m}(\hat{I}_{z,\varepsilon_0,\varepsilon} \setminus Y) < 1/3$. Then, since $f^n \circ \pi = \pi \circ \hat{f}^n_{z,\varepsilon_0,\varepsilon}$, we have $m(\pi(\hat{f}^n_{z,\varepsilon_0,\varepsilon}(Q) \cap Y)) \geqslant 1 - \kappa$, for $n \geqslant n_Q$.

Applying this to $n = n_Q$ and $n = n_Q + 1$, we conclude

$$m(\pi(Y \cap \hat{f}^{n_Q}_{z,\varepsilon_0,\varepsilon}(Q)) \cap \pi(Y \cap \hat{f}^{n_Q+1}_{z,\varepsilon_0,\varepsilon}(Q))) \geqslant 1 - 2\kappa > 0.$$

Thus there must exist intervals $O_1 \subset Y \cap \hat{f}^{n_Q}_{z,\varepsilon_0,\varepsilon}(Q)$ and $O_2 \subset Y \cap \hat{f}^{n_Q+1}_{z,\varepsilon_0,\varepsilon}(Q)$ such that $\pi(O_1) \cap \pi(O_2) \neq \emptyset$. By Lemma 4.2, there exists $n_1 \in \mathbb{N}$ such that $F^{n_1}_{z,\varepsilon_0,\varepsilon}(O_1) \cap F^{n_1}_{z,\varepsilon_0,\varepsilon}(O_2) \neq \emptyset$. Since $Q, O_1, O_2 \subset Y$, there exists $k_Q \in \mathbb{N}$ such that $F^{k_Q}_{z,\varepsilon_0,\varepsilon}(Q) \cap F^{k_Q+1}_{z,\varepsilon_0,\varepsilon}(Q) \neq \emptyset$, so the period of Q under $F_{z,\varepsilon_0,\varepsilon}$ is 1. Thus $F_{z,\varepsilon_0,\varepsilon}$ is aperiodic and therefore mixing. □

Our next two lemmas show that the mixing established in Lemma 4.11 is in fact uniform in ε.

LEMMA 4.12. *Fix $\varepsilon_0 \leqslant \varepsilon_0^*$ and suppose there exist $Q_1, Q_2 \in \mathcal{Q}$ and an interval $J \subset Q_1$ such that $F_{z,\varepsilon_0}^{n_J}(J) = Q_2$ for some $n_J \in \mathbb{N}$. Then there exists $\varepsilon_1 < \varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_1)$, $F_{z,\varepsilon_0,\varepsilon}^{n_J}(J) = Q_2$.*

Proof. Fix ε_0. Suppose there exist $Q_1, Q_2 \in \mathcal{Q}$ and an interval $J \subset Q_1$ and $n_J \in \mathbb{N}$ such that $F_{z,\varepsilon_0}^{n_J}(J) = Q_2$ as in the statement of the lemma. Let $n_1 \in \mathbb{N}$ be such that $\hat{f}_{z,\varepsilon_0}^{n_1}(J) = Q_2$.

A key property of our construction of $Y = \hat{I}'_{z,\varepsilon_0}(L)$ is that we have 'trimmed' the edges of the domains at returns; that is, the endpoints of Q_1 and Q_2 are elements of $\partial\mathcal{P}_L = \partial\mathcal{P}_L(\mathrm{Crit}_{z,\varepsilon_0})$ and the Markov property of F_{z,ε_0} (Lemma 4.4) implies that there exist domains $Q'_1 \supsetneq Q_1$ and $Q'_2 \supsetneq Q_2$ in the extension $\hat{I}_{z,\varepsilon_0}$ (note that Q'_2 is an element of $\mathcal{D}$) and an interval J' with $J \subsetneq J' \subset Q'_1$ such that $\hat{f}_{z,\varepsilon_0}^{n_1}(J') = Q'_2$.

Let $\widehat{z}$ and $\widehat{z \pm \varepsilon_0}$ denote the fibres above z and $z \pm \varepsilon_0$, respectively. Due to the Markov property and because we have treated $f^{-1}(z)$ and $f^{-1}(z \pm \varepsilon_0)$ as cut points during our construction of $\hat{I}_{z,\varepsilon_0}$ and $\hat{I}_{z,\varepsilon_0,\varepsilon}$, it follows that $\partial(\hat{f}_{z,\varepsilon_0}^{j}(J)) \cap \{\widehat{z}, \widehat{z \pm \varepsilon_0}\} = \emptyset$, for all $0 \leqslant j \leqslant n_1$.

Case 1: $\bigcup_{j=0}^{n_1} \hat{f}_{z,\varepsilon_0}^{j}(J) \cap (\widehat{z - \varepsilon_0}, \widehat{z + \varepsilon_0}) = \emptyset$. Then introducing new cuts at $f^{-1}(z \pm \varepsilon)$ in the construction of $\hat{I}_{z,\varepsilon_0,\varepsilon}$ does not affect the endpoints of either J' or Q'_2, and the lemma holds with $\varepsilon_1 = \varepsilon_0$.

Case 2: $\bigcup_{j=0}^{n_1} \hat{f}_{z,\varepsilon_0}^{j}(J) \cap (\widehat{z - \varepsilon_0}, \widehat{z + \varepsilon_0}) \neq \emptyset$. Choose

$$\varepsilon_1 < \min\{d(\partial(\hat{f}_{z,\varepsilon_0}^{j}(J)), \widehat{z}) : 0 \leqslant j \leqslant n_1\}.$$

It follows that for all $\varepsilon < \varepsilon_1$, $\hat{f}_{z,\varepsilon_0,\varepsilon}^{n_1}(J) = Q_2$. Moreover, there exist an interval $J'_\varepsilon \supsetneq J$ and a domain $Q'_{2,\varepsilon} \supsetneq Q_2$ in $\hat{I}_{z,\varepsilon_0,\varepsilon}$ such that $\hat{f}_{z,\varepsilon_0,\varepsilon}^{n_1}(J'_\varepsilon) = Q'_{2,\varepsilon}$. Then, since $F_{z,\varepsilon_0,\varepsilon}$ is the first return map to Y, and Y is independent of ε, it follows that $F_{z,\varepsilon_0,\varepsilon}^{n_J}(J) = Q_2$. □

COROLLARY 4.13. *For all $\delta > 0$ there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for all $\varepsilon \in (0, \varepsilon_1)$,*

$$\hat{m}(\hat{x} \in Y : F_{z,\varepsilon_0}(\hat{x}) \neq F_{z,\varepsilon_0,\varepsilon}(\hat{x})) \leqslant \delta.$$

Proof. Fix $\delta > 0$. By Theorem 4.10, we may choose N such that $\hat{m}(R_{z,\varepsilon_0} > N) \leqslant Ce^{-\alpha N} \leqslant \delta$. Considering the 1-cylinders for F_{z,ε_0}, there are only finitely many with $R \leqslant N$.

For each 1-cylinder Y_i, Lemma 4.12 yields an $\varepsilon_1(i) > 0$ such that for all $\varepsilon \in (0, \varepsilon_1(i))$, Y_i is also a 1-cylinder for $F_{z,\varepsilon_0,\varepsilon}$; moreover, $F_{z,\varepsilon_0}(Y_i) = F_{z,\varepsilon_0,\varepsilon}(Y_i)$ and $R_{z,\varepsilon_0}(Y_i) = R_{z,\varepsilon_0,\varepsilon}(Y_i)$.

Taking $\varepsilon_1 = \min\{\varepsilon_1(i) : R_{z,\varepsilon_0}(Y_i) \leqslant N\} > 0$ completes the proof of the corollary. □

5. *A spectral gap for the induced punctured transfer operators*

In this section we work with the induced maps F_{z,ε_0} and $F_{z,\varepsilon_0,\varepsilon}$ defined on the common domain $Y = \hat{I}'_{z,\varepsilon_0}(L)$. Since z and $\varepsilon_0 \leqslant \varepsilon_0^*$ are fixed throughout this section, for brevity, we will denote these maps simply by $F_\varepsilon := F_{z,\varepsilon_0,\varepsilon}$ and $F_0 := F_{z,\varepsilon_0}$. Related objects will also be denoted by the subscript ε or 0. One of the main points of this section is to show that certain key properties are uniform for $\varepsilon \in [0, \varepsilon_0)$, where $\varepsilon = 0$ is understood to correspond to the map F_{z,ε_0} whose Hofbauer extension is defined by introducing cuts only at z and $z \pm \varepsilon_0$.

For $\varepsilon \in [0, \varepsilon_0)$, let $\mathcal{Y}_\varepsilon = \{Y_i\}_i$ denote the set of 1-cylinders for F_ε on which $R_\varepsilon = R_{z,\varepsilon_0,\varepsilon}$ is constant. As before, denote by $\mathcal{Q}$ the finite partition of Y into intervals which comprise the finite images of $\mathcal{Y}_\varepsilon$ under F_ε. It is important that Y and $\mathcal{Q}$ are independent of ε. Indeed, the uniformity of $\mathcal{Q}$ and L allows us to take the constants in (4.3) and Lemma 4.6 uniformly in ε. This is formalized in properties (GM2) and (GM3) below.

Let $\Phi_\varepsilon = S_{R_\varepsilon}\varphi$ be the induced version of φ on Y. Note that as in, for example, [**DoT**, Lemma 14.9], the fact that $\hat{\mu}_\varepsilon(\hat{I}'(L)) > 0$ guarantees that $P(\Phi_\varepsilon) = 0$. Also, the conformal measure m_φ lifted to $\hat{I}_{z,\varepsilon_0,\varepsilon}$, and denoted by $\hat{m}_{\varphi,\varepsilon}$, depends on both ε and φ. However, $\hat{m}_{\varphi,\varepsilon}$ restricted to Y is independent of ε since Y is independent of ε. Since we will work exclusively in Y in this section, we suppress the dependence on φ and refer to this measure on Y as simply $\hat{m}$. For each $\varepsilon \in [0, \varepsilon_0)$, it is a conformal measure for F_ε with respect to the potential Φ_ε.

The key properties of the Gibbs–Markov maps F_ε, $\varepsilon \in [0, \varepsilon_0)$, are as follows.

- **(GM1)** $F_\varepsilon(Y_i) \in \mathcal{Q}$ for each $Y_i \in \mathcal{Y}_\varepsilon$.
- **(GM2)** There exist $\sigma > 1$ and $C_e \in (0, 1]$ (an expansion constant) such that for all $n \in \mathbb{N}$, if $Y_i^{(n)}$ is an n-cylinder for F_ε and $x, y \in Y_i^{(n)}$, then $d(F_\varepsilon^n x, F_\varepsilon^n y) \geqslant C_e \sigma^n d(x, y)$, where $d(\cdot, \cdot)$ is the distance on each interval in $\hat{I}$ induced by the Euclidean metric on I.
- **(GM3)** There exists $C_d > 0$ (a distortion constant) such that for all $n \in \mathbb{N}$, if $Y_i^{(n)}$ is an n-cylinder for F_ε and $x, y \in Y_i^{(n)}$, then
$$|e^{S_n\Phi_\varepsilon(x) - S_n\Phi_\varepsilon(y)} - 1| \leqslant C_d d(F_\varepsilon^n x, F_\varepsilon^n y)^\eta,$$
for some $\eta > 0$.

Note that (GM3) follows from Lemma 4.6, and that the constants in (GM2) and (GM3) are independent of ε by construction of Y. Due to (GM3), conformality and large images,

$$e^{S_n\Phi_\varepsilon(x)} \leqslant (1 + C_d)\frac{\hat{m}(Y_i^{(n)})}{\hat{m}(F_\varepsilon^n(Y_i^{(n)}))} \leqslant \frac{1 + C_d}{q}\hat{m}(Y_i^{(n)}) \quad \text{for all } x \in Y_i^{(n)}, \tag{5.1}$$

where $q := \min_{Q\in\mathcal{Q}} \hat{m}(Q) > 0$.

Let $\mathcal{C}^\eta(\mathcal{Q})$ denote the set of Hölder continuous functions on elements of $\mathcal{Q}$, equipped with the norm

$$\|\psi\|_{\mathcal{C}^\eta} = \sup_{Q\in\mathcal{Q}}\left(|\psi|_{\mathcal{C}^0(Q)} + \sup_{x,y\in Q}|\psi(x) - \psi(y)|d(x, y)^{-\eta}\right) =: \sup_{Q\in\mathcal{Q}}(|\psi|_{\mathcal{C}^0(Q)} + H_Q^\eta(\psi)).$$

We define the transfer operator $\mathcal{L}_\varepsilon = \mathcal{L}_{\Phi_\varepsilon}$ acting on $L^1(\hat{m}_\varepsilon)$ by

$$\mathcal{L}_\varepsilon^n\psi(x) = \sum_{y\in F_\varepsilon^{-n}(x)} \psi(y)e^{S_n\Phi_\varepsilon(y)} \quad \text{for each } n \geqslant 1.$$

Analogously, define $\mathcal{L}_0$ to be the transfer operator corresponding to the map $F_0 = F_{z,\varepsilon_0}$.

Given a hole $H_\varepsilon = (z - \varepsilon, z + \varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$, as in Remark 4.8, its lift $\hat{H}_\varepsilon$ is disjoint from Y due to our choice of ε_0^*. We denote by $\hat{H}'_\varepsilon \subset Y$ the *pre-hole*, the set of points in Y

which do not return to Y before entering $\hat{H}_\varepsilon$. Due to our construction, $\hat{H}'_\varepsilon$ is a (countable) union of 1-cylinders for F_ε,

$$\hat{H}'_\varepsilon = \{Y_i \in \mathcal{Y}_\varepsilon : \hat{f}^n_\varepsilon(Y_i) \subset \hat{H}_\varepsilon \text{ for some } n < R_\varepsilon(Y_i)\}.$$

We will treat $\hat{H}'_\varepsilon$ as our effective hole for F_ε. Let $\mathring{Y}_\varepsilon = Y \setminus \hat{H}'_\varepsilon$, and for $n \geqslant 0$ define

$$\mathring{Y}^n_\varepsilon = \bigcap_{i=0}^{n} F_\varepsilon^{-i}(\mathring{Y}_\varepsilon)$$

to be the set of points which do not enter $\hat{H}'_\varepsilon$ in the first n iterates of F_ε. The dynamics of the induced open system is defined by $\mathring{F}^n_\varepsilon = F^n_\varepsilon|_{\mathring{Y}^{n-1}_\varepsilon}$. Since $\hat{H}'_\varepsilon$ is a union of 1-cylinders for F_ε, the punctured map $\mathring{F}_\varepsilon$ has the same finite image property: $\mathring{F}_\varepsilon(Y_i) \in \mathcal{Q}$ for each $Y_i \subset \mathring{Y}_\varepsilon$. The punctured transfer operator for the open system is defined for $n \geqslant 1$ by

$$\mathring{\mathcal{L}}^n_\varepsilon \psi(x) = \mathcal{L}^n_\varepsilon(\psi 1_{\mathring{Y}^{n-1}_\varepsilon})(x) = \sum_{y \in \mathring{F}^{-n}_\varepsilon(x)} \psi(y) e^{S_n \Phi_\varepsilon(y)}. \tag{5.2}$$

The punctured transfer operator is defined only for $\varepsilon > 0$. There is no analogous object for $\mathcal{L}_0$.

5.1. *Spectral properties of* $\mathcal{L}_\varepsilon$. In this subsection we prove that for sufficiently small ε, all the operators $\mathcal{L}_\varepsilon$ have a uniform spectral gap.

PROPOSITION 5.1. *There exists $C > 0$ such that for all $n \geqslant 0$,*

$$\|\mathring{\mathcal{L}}^n_\varepsilon \psi\|_{\mathcal{C}^\eta} \leqslant C\sigma^{-\eta n} \|\psi\|_{\mathcal{C}^\eta} \hat{m}(\mathring{Y}^{n-1}_\varepsilon) + C \int_{\mathring{Y}^{n-1}_\varepsilon} |\psi| \, d\hat{m} \quad \textit{for all } \psi \in \mathcal{C}^\eta(\mathcal{Q}), \tag{5.3}$$

$$|\mathring{\mathcal{L}}^n_\varepsilon \psi|_{L^1(\hat{m})} \leqslant \int_{\mathring{Y}^{n-1}_\varepsilon} |\psi| \, d\hat{m} \quad \textit{for all } \psi \in L^1(\hat{m}). \tag{5.4}$$

The analogous inequalities hold for $\mathcal{L}^n_\varepsilon \psi$ and $\mathcal{L}^n_0 \psi$ with $\mathring{Y}^{n-1}_\varepsilon$ replaced by Y.

Proof. Due to definition (5.2), $\int_Y \mathring{\mathcal{L}}^n_\varepsilon \psi \, d\hat{m} = \int_{\mathring{Y}^{n-1}_\varepsilon} \psi \, d\hat{m}$, so that (5.4) is immediate. We focus on verifying (5.3) for $\psi \in \mathcal{C}^\eta(\mathcal{Q})$.

First, we estimate the Hölder constant of $\mathring{\mathcal{L}}^n_\varepsilon \psi$. Let $Q \in \mathcal{Q}$ and $x, y \in Q$. For $n \geqslant 0$, notice that each $u \in \mathring{F}^{-n}_\varepsilon(x)$ has a (unique) corresponding $v \in \mathring{F}^{-n}_\varepsilon(y)$ lying in the same n-cylinder $Y^{(n)}_i(u)$ as u. Thus,

$$\begin{aligned}
\mathring{\mathcal{L}}^n_\varepsilon \psi(x) - \mathring{\mathcal{L}}^n_\varepsilon \psi(y) &= \sum_{u \in \mathring{F}^{-n}_\varepsilon(x)} (\psi(u) - \psi(v)) e^{S_n \Phi_\varepsilon(u)} \\
&\quad + \sum_{v \in \mathring{F}^{-n}_\varepsilon(y)} \psi(v)(e^{S_n \Phi_\varepsilon(u)} - e^{S_n \Phi_\varepsilon(v)}) \\
&\leqslant \sum_{u \in \mathring{F}^{-n}_\varepsilon(x)} H^\eta(\psi) d(u, v)^\eta \frac{1 + C_d}{q} \hat{m}(Y^{(n)}_i(u)) \\
&\quad + \sum_{v \in \mathring{F}^{-n}_\varepsilon(y)} |\psi(v)| \, \hat{m}(Y^{(n)}_i(v)) \frac{1 + C_d}{q} C_d d(x, y)^\eta,
\end{aligned}$$

where we have used the bounded distortion property (GM3) as well as (5.1). Now using the regularity of ψ as well as the expanding property (GM2), for any $v \in \mathring{F}_\varepsilon^{-n}(y)$,

$$\left| |\psi(v)| - \frac{1}{\hat{m}(Y_i^{(n)}(v))} \int_{Y_i^{(n)}(v)} |\psi| \, d\hat{m} \right| \leqslant H^\eta(\psi) \operatorname{diam}(Y_i^{(n)}(v))^\eta \leqslant H^\eta(\psi) C_e^{-\eta} \sigma^{-\eta n}. \tag{5.5}$$

Putting these estimates together, we obtain

$$|\mathring{\mathcal{L}}_\varepsilon^n \psi(x) - \mathring{\mathcal{L}}_\varepsilon^n \psi(y)| \leqslant C_e^{-\eta} \sigma^{-\eta n} H^\eta(\psi) d(x, y)^\eta \sum_{u \in \mathring{F}_\varepsilon^{-n}(x)} \frac{1 + C_d}{q} (1 + C_d) \hat{m}(Y_i^{(n)}(u))$$
$$+ \sum_{v \in \mathring{F}_\varepsilon^{-n}(y)} \int_{Y_i^{(n)}(v)} |\psi| \, d\hat{m} \frac{1 + C_d}{q} C_d d(x, y)^\eta.$$

Due to the fact that the hole respects the Markov structure of our inducing scheme, it follows that $\bigcup_{u \in \mathring{F}_\varepsilon^{-n}(x)} Y_i^{(n)}(u) \subset \mathring{Y}_\varepsilon^{n-1}$, allowing us to evaluate both sums. Now dividing through by $d(x, y)^\eta$ and taking the appropriate suprema yields the required inequality in (5.3) for the Hölder constant of $\mathring{\mathcal{L}}_\varepsilon^n \psi$ with $C = C_e^{-\eta}((1 + C_d)^2/q)$.

Next, we estimate $|\mathring{\mathcal{L}}_\varepsilon^n \psi|_\infty$. Let $Q \in \mathcal{Q}$ and $x \in Q$. Now

$$|\mathring{\mathcal{L}}_\varepsilon^n \psi(x)| \leqslant \sum_{u \in \mathring{F}_\varepsilon^{-n}(x)} |\psi(u)| e^{S_n \Phi_\varepsilon(u)} \leqslant \sum_{u \in \mathring{F}_\varepsilon^{-n}(x)} \frac{1 + C_d}{q} |\psi(u)| \, \hat{m}(Y_i^{(n)}(u)), \tag{5.6}$$

where we have used (5.1) for the second inequality. Using (5.5), we estimate

$$|\mathring{\mathcal{L}}_\varepsilon^n \psi(x)| \leqslant \frac{1 + C_d}{q} \sum_{u \in \mathring{F}_\varepsilon^{-n}(x)} H^\eta(\psi) C_e^{-\eta} \sigma^{-\eta n} \hat{m}(Y_i^{(n)}(u)) + \int_{Y_i^{(n)}(u)} |\psi| \, d\hat{m}$$
$$\leqslant C' \sigma^{-\eta n} H^\eta(\psi) \hat{m}(\mathring{Y}_\varepsilon^{n-1}) + \int_{\mathring{Y}_\varepsilon^{n-1}} |\psi| \, d\hat{m},$$

so (5.3) holds with $C = 2C_e^{-\eta}((1 + C_d)^2/q)$, completing the proof of the proposition. □

Define the norm for $\mathcal{L}_\varepsilon : \mathcal{C}^\eta(\mathcal{Q}) \to L^1(\hat{m})$ by

$$|||\mathcal{L}_\varepsilon||| = \sup\{|\mathcal{L}_\varepsilon \psi|_{L^1(\hat{m})} : \|\psi\|_{\mathcal{C}^\eta} \leqslant 1\}.$$

LEMMA 5.2. *For any* $\delta > 0$, *there exists* $\varepsilon_\delta > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_\delta)$, $|||\mathcal{L}_0 - \mathcal{L}_\varepsilon||| \leqslant \delta$.

Proof. Fix $\delta > 0$. Define $\mathcal{G}_\varepsilon = \{Y_i \in \mathcal{Y}_\varepsilon : \hat{f}_{z,\varepsilon_0}^k(Y_i) = \hat{f}_{z,\varepsilon_0,\varepsilon}^k(Y_i), \forall k = 1, \ldots, R_\varepsilon(Y_i)\}$. Note that if $Y_i \in \mathcal{G}_\varepsilon$, then $\Phi_0 = \Phi_\varepsilon$ on Y_i.

Next define $B_\varepsilon = \{y \in Y : Y_{i,\varepsilon}(y) \notin \mathcal{G}_\varepsilon\}$, where $Y_{i,\varepsilon}(y)$ is the 1-cylinder with respect to F_ε containing y. For $\psi \in \mathcal{C}^\eta(\mathcal{Q})$ and $x \in Y$,

$$|(\mathcal{L}_0 - \mathcal{L}_\varepsilon)\psi(x)| = \left| \sum_{y \in F_0^{-1}x} \psi(y) e^{\Phi_0(y)} - \sum_{y \in F_\varepsilon^{-1}x} \psi(y) e^{\Phi_\varepsilon(y)} \right|$$
$$\leqslant |\psi|_\infty \sum_{\substack{y \in F_0^{-1}x \\ y \in B_\varepsilon}} \frac{1 + C_d}{q} \hat{m}(Y_i(y)) + |\psi|_\infty \sum_{\substack{y \in F_\varepsilon^{-1}x \\ y \in B_\varepsilon}} \frac{1 + C_d}{q} \hat{m}(Y_i(y)). \tag{5.7}$$

By the proof of Corollary 4.13, the total mass of 1-cylinders where F_0 and F_ε do not agree can be made arbitrarily small.

Let $\delta' = \delta(q/2(1 + C_d)\hat{m}(Y))$. Choose $\varepsilon_\delta > 0$ such that $\hat{m}(B_{\varepsilon_\delta}) \leqslant \delta'$ by Corollary 4.13. Then

$$|(\mathcal{L}_0 - \mathcal{L}_\varepsilon)\psi(x)| \leqslant |\psi|_\infty 2\frac{1 + C_d}{q}\hat{m}(B_\varepsilon) \leqslant |\psi|_\infty \delta/\hat{m}(Y). \tag{5.8}$$

Integrating over $x \in Y$ proves the lemma: $\int |(\mathcal{L}_0 - \mathcal{L}_\varepsilon)\psi|\, d\hat{m} \leqslant |\psi|_\infty \delta$. □

COROLLARY 5.3. *There exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that the family of operators $\mathcal{L}_\varepsilon$, $\varepsilon \in [0, \varepsilon_1)$, acting on $\mathcal{C}^\eta(\mathcal{Q})$ have a uniform spectral gap. There exists $\beta > 0$ such that $\mathcal{L}_\varepsilon$ admits the following spectral decomposition for all $\varepsilon \in [0, \varepsilon_1)$. There exist $G_\varepsilon \in \mathcal{C}^\eta(\mathcal{Q})$, a linear functional $e_\varepsilon : \mathcal{C}^\eta(\mathcal{Q}) \to \mathbb{R}$ and an operator $\mathcal{R}_\varepsilon : \mathcal{C}^\eta(\mathcal{Q}) \circlearrowleft$ such that*

$$\mathcal{L}_\varepsilon = G_\varepsilon \otimes e_\varepsilon + \mathcal{R}_\varepsilon \quad \textit{and} \quad \mathcal{R}_\varepsilon G_\varepsilon = 0.$$

The spectral radius of $\mathcal{R}_\varepsilon$ is at most $e^{-\beta}$ and $e_\varepsilon(\psi) = \int_Y \psi\, d\hat{m}$ for all $\psi \in \mathcal{C}^\eta(\mathcal{Q})$.

Moreover, $G_\varepsilon \to G_0$ in $L^1(\hat{m})$ and $|||\mathcal{R}_\varepsilon - \mathcal{R}_0||| \to 0$ as $\varepsilon \to 0$.

We may normalize the above so that $\hat{m}(G_\varepsilon) = 1$, so $\hat{\mu}_{Y,\varepsilon} = G_\varepsilon \hat{m}$ is the corresponding invariant probability measure for F_ε.

Proof. The fact that all the operators $\mathcal{L}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0)$, are quasi-compact on $\mathcal{C}^\eta(\mathcal{Q})$ with essential spectral radius bounded by σ^{-1} follows from Proposition 5.1 and the fact that the unit ball of $\mathcal{C}^\eta(\mathcal{Q})$ is compactly embedded in $L^1(\hat{m})$. Moreover, the spectrum of $\mathcal{L}_\varepsilon$ on the unit circle is finite-dimensional and forms a cyclic group.

Since F_0 is mixing by Lemma 4.11, $\mathcal{L}_0$ has a single simple eigenvalue at 1 and the rest of the spectrum of $\mathcal{L}_0$ is contained in a disk of radius $e^{-2\beta} \geqslant \sigma^{-1}$ for some $\beta > 0$. Next, by Lemma 5.2 and [**KL1**, Corollary 1], the spectrum of $\mathcal{L}_\varepsilon$ outside the disk of radius σ^{-1} can be made arbitrarily close to that of $\mathcal{L}_0$ by choosing ε sufficiently small. Thus we may choose $\varepsilon_1 \in (0, \varepsilon_0]$ such that the spectrum of $\mathcal{L}_\varepsilon$ outside the disk of radius $e^{-\beta}$ consists only of a simple eigenvalue at 1, for all $\varepsilon \in (0, \varepsilon_1)$. The closeness of G_ε and $\mathcal{R}_\varepsilon$ to G_0 and $\mathcal{R}_0$ follows similarly from [**KL1**, Corollary 1]. Finally, the fact that $e_\varepsilon(\psi) = \hat{m}(\psi)$ for all $\psi \in \mathcal{C}^\eta(\mathcal{Q})$ follows from the conformality of $\hat{m}$. □

5.2. *Spectral properties of the punctured operators $\mathring{\mathcal{L}}_\varepsilon$.* Due to the uniform Lasota–Yorke inequalities provided by Proposition 5.1, it only remains to show that $\mathcal{L}_\varepsilon$ and $\mathring{\mathcal{L}}_\varepsilon$ are close in the $|||\cdot|||$-norm.

LEMMA 5.4. *For any $\varepsilon \in (0, \varepsilon_0)$, $|||\mathcal{L}_\varepsilon - \mathring{\mathcal{L}}_\varepsilon||| \leqslant \hat{m}(\hat{H}'_\varepsilon)$.*

Proof. The proof is immediate using the definition of $\mathring{\mathcal{L}}_\varepsilon$ and the conformality of $\hat{m}$,

$$\int (\mathcal{L}_\varepsilon - \mathring{\mathcal{L}}_\varepsilon)\psi\, d\hat{m} = \int \mathcal{L}_\varepsilon(1_{Y\setminus \mathring{Y}^1_\varepsilon}\psi)\, d\hat{m} = \int_{\hat{H}'_\varepsilon} \psi\, d\hat{m} \leqslant |\psi|_\infty \hat{m}(\hat{H}'_\varepsilon), \tag{5.9}$$

since $\hat{H}'_\varepsilon = Y \setminus \mathring{Y}_\varepsilon$. □

COROLLARY 5.5. *There exists* $\varepsilon_2 \leqslant \varepsilon_1$ *such that for all* $\varepsilon \in (0, \varepsilon_2)$, *the operators* $\mathring{\mathcal{L}}_\varepsilon$ *have a uniform spectral gap: there exist* $\Lambda_\varepsilon \in (e^{-\beta/3}, 1)$, $\mathring{G}_\varepsilon \in \mathcal{C}^\eta(\mathcal{Q})$, *a functional* $\mathring{e}_\varepsilon : \mathcal{C}^\eta(\mathcal{Q}) \to \mathbb{R}$, *and an operator* $\mathring{\mathcal{R}}_\varepsilon : \mathcal{C}^\eta(\mathcal{Q}) \circlearrowleft$ *such that*

$$\mathring{\mathcal{L}}_\varepsilon = \Lambda_\varepsilon \mathring{G}_\varepsilon \otimes \mathring{e}_\varepsilon + \mathring{\mathcal{R}}_\varepsilon \quad \textit{and} \quad \mathring{\mathcal{R}}_\varepsilon \mathring{G}_\varepsilon = 0. \tag{5.10}$$

The spectral radius of $\mathring{\mathcal{R}}_\varepsilon$ *is at most* $e^{-2\beta/3} < \Lambda_\varepsilon$.

Moreover, $\Lambda_\varepsilon \to 1$, $\mathring{G}_\varepsilon \to G_0$ *in* $L^1(\hat{m})$ *and* $|||\mathring{\mathcal{R}}_\varepsilon - \mathcal{R}_0||| \to 0$ *as* $\varepsilon \to 0$.

Proof. Lemmas 5.2 and 5.4 together with the triangle inequality show that $\mathring{\mathcal{L}}_\varepsilon$ and $\mathcal{L}_0$ are close in the $|||\cdot|||$-norm. The uniform Lasota–Yorke inequalities given by Proposition 5.1 together with [**KL1**, Corollary 1] imply that the spectrum (and corresponding spectral projectors) of $\mathring{\mathcal{L}}_\varepsilon$ outside the disk of radius $e^{-\beta}$ are close to those of $\mathcal{L}_0$. Without requiring a rate of approach, we may choose $\varepsilon_2 > 0$ with the stated properties. □

We may normalize $\mathring{G}_\varepsilon$ and $\mathring{e}_\varepsilon$ so that $\hat{m}(\mathring{G}_\varepsilon) = 1$ and $\mathring{e}_\varepsilon(\mathring{G}_\varepsilon) = 1$, so that $\mathring{\mathcal{L}}_\varepsilon \mathring{G}_\varepsilon = \Lambda_\varepsilon \mathring{G}_\varepsilon$.

6. *Young towers and proof of Theorem 3.1*

The Markov structure of the return map $F_\varepsilon = F_{z,\varepsilon_0,\varepsilon}$ to Y immediately implies the existence of another, related extension, called a Young tower. These have been well studied in the context of open systems, so we will recall their structure in order to apply some results in our setting.

As in §5, let $R_\varepsilon = R_{z,\varepsilon_0,\varepsilon}$. Define the Young tower over Y with return time R_ε by

$$\Delta := \{(y, \ell) \in Y \times \mathbb{N} : \ell < R_\varepsilon(y)\}.$$

We view Δ as a tower with $\Delta_\ell = \{(y, n) \in Y \times \mathbb{N} : n = \ell\}$ as the ℓth level. The dynamics on the tower is defined by $f_\Delta(y, \ell) = (y, \ell + 1)$ when $\ell + 1 < R_\varepsilon(y)$, and $f_\Delta(y, \ell) = (F_\varepsilon(y), 0)$ otherwise. Thus Δ_0 corresponds to Y and $F_\varepsilon = f_\Delta^{R_\varepsilon}$ can be viewed as the first return map to Δ_0. With this definition, there is a natural projection $\hat{\pi}_\Delta : \Delta \to \hat{I}$ satisfying $\hat{\pi}_\Delta \circ f_\Delta = \hat{f}_\varepsilon \circ \hat{\pi}_\Delta$. Then also defining $\pi_\Delta = \pi \circ \hat{\pi}_\Delta : \Delta \to I$, we have

$$\pi_\Delta \circ f_\Delta = f \circ \pi_\Delta.$$

Clearly, $\Delta = \Delta(z, \varepsilon_0, \varepsilon)$ depends on z, ε_0 and ε through the construction of $\hat{I}_{z,\varepsilon_0,\varepsilon}$, $Y = \hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$ and F_ε. However, since we fix these three parameters in this section, we will drop explicit mention of this dependence in the notation we use for objects associated with Δ.

The map f_Δ inherits a Markov structure as follows. On Δ_0, we use the elements of the finite partition $\mathcal{Q}$ as our partition elements, labelling them by $\Delta_{0,i}$. On Δ_ℓ, $\ell \geqslant 1$, we define $\Delta_{\ell,i} = f_\Delta^\ell(Y_i)$, $Y_i \in \mathcal{Y}_\varepsilon$. The collection $\{\Delta_{\ell,i}\}_{\ell,i\geqslant 0}$ forms a countable Markov partition for f_Δ. Since, at return times to Δ_0, f_Δ maps the image of each 1-cylinder Y_i to an element of the finite partition $\mathcal{Q}$ of $Y = \Delta_0$, we will view (f_Δ, Δ) as a Young tower with finitely many bases. The partition $\{\Delta_{\ell,i}\}$ is generating since $\{Y_i\}_i$ is a generating partition for F_ε. Moreover, the first return time R_ε to Δ_0 under f_Δ is the same as the first return to Y under $\hat{f}_\varepsilon$.

We make Δ into a metric space by defining a symbolic metric based on the Markov partition. Let $R^n_\varepsilon(x)$ denote the nth return time of x to Δ_0. Define the *separation time* on Δ_0 by

$$s(x, y) = \min\{n \geqslant 0 : f_\Delta^{R^n_\varepsilon}(x) \text{ and } f_\Delta^{R^n_\varepsilon}(y) \text{ lie in different elements of } \mathcal{Y}_\varepsilon\}.$$

We extend the separation time to all of Δ by setting $s(x, y) = s(f_\Delta^{-\ell}x, f_\Delta^{-\ell}y)$ for $x, y \in \Delta_\ell$. It follows that $s(x, y)$ is finite almost everywhere since $\{\Delta_{i,j}\}$ is a generating partition. For $\theta > 0$, define a metric on Δ by $d_\theta(x, y) = e^{-\theta s(x,y)}$. We will choose θ according to property (P3) in §6.1.

Given our (normalized) potential $\hat{\varphi}$ on $\hat{I}$, and $\hat{\varphi}$-conformal measure $\hat{m} = \hat{m}_\varphi$, we define a reference measure m_Δ on Δ by setting $m_\Delta = \hat{m}$ on Δ_0, and $m_\Delta|_{\Delta_\ell} := (f_\Delta)_*(m_\Delta|_{\Delta_{\ell-1}\cap f_\Delta^{-1}\Delta_\ell})$.

Similarly, we lift the potential $\hat{\varphi}$ to a potential φ_Δ on Δ as follows. For $x \in \Delta_\ell$, let $x^- = f_\Delta^{-\ell}(x)$ denote the pullback of x to Δ_0. Then

$$\varphi_\Delta(x) := S_{R_\varepsilon}\hat{\varphi}(x^-) \quad \text{for } x \in f_\Delta^{-1}(\Delta_0), \quad \text{and} \quad \varphi_\Delta = 0 \quad \text{on } \Delta \setminus f_\Delta^{-1}(\Delta_0).$$

With this definition, m_Δ is a φ_Δ-conformal measure.

We may also define a related invariant measure on Δ. Let $G_\varepsilon \in \mathcal{C}^\eta(\mathcal{Q})$ be the invariant density from Corollary 5.3. Define

$$g_\Delta = G_\varepsilon \quad \text{on } \Delta_0 \quad \text{and} \quad g_\Delta(x) = G_\varepsilon(x^-) \quad \text{for } x \in \Delta_\ell, \ell \geqslant 1, \tag{6.1}$$

where x^- is defined as above.

It follows that the measure $d\mu_\Delta = g_\Delta dm_\Delta / \int_\Delta g_\Delta dm_\Delta$ is an invariant probability measure for f_Δ. Moreover, we have $(\hat{\pi}_\Delta)_*\mu_\Delta = \hat{\mu}_\varepsilon$. And since $\pi_*\hat{\mu}_\varepsilon = \mu_\varphi$, we have also that $(\pi_\Delta)_*\mu_\Delta = \mu_\varphi$. Note that here $\hat{\mu}_\varepsilon$ is defined on $\hat{I}_{z,\varepsilon_0,\varepsilon}$ and depends on ε, while μ_φ does not.

We lift the hole $H = H(z, \varepsilon)$ to Δ by setting $H_\Delta := \pi_\Delta^{-1}H = \hat{\pi}_\Delta^{-1}\hat{H}$. Due to the construction of $\hat{I}_{z,\varepsilon_0,\varepsilon}$, H_Δ comprises a countable collection of elements of the Markov partition $\Delta_{\ell,j}$, which we shall denote by $H_{\ell,j}$. Set $\mathring{\Delta} = \Delta \setminus H_\Delta$, and define the open system $\mathring{f}_\Delta = f_\Delta|_{\mathring{\Delta}}$.

LEMMA 6.1. *Define* $\mathring{\Delta}^{(n)} = \bigcap_{i=0}^n f_\Delta^{-1}(\mathring{\Delta})$. *Then*

$$\log \overline{\lambda}_\varepsilon := \limsup_{n\to\infty} \frac{1}{n} \log \mu_\varphi(\mathring{I}^n) = \limsup_{n\to\infty} \frac{1}{n} \log \mu_\Delta(\mathring{\Delta}^{(n)}) = \limsup_{n\to\infty} \frac{1}{n} \log m_\Delta(\mathring{\Delta}^{(n)}).$$

Proof. The first equality follows immediately from the fact that $(\pi_\Delta)_*\mu_\Delta = \mu_\varphi$ and $\pi_\Delta \circ f_\Delta = f \circ \pi_\Delta$, so that $\mu_\Delta(\mathring{\Delta}^{(n)}) = \mu_\varphi(\mathring{I}^n)$ for each n. The second equality follows from the fact that $\mu_\Delta = g_\Delta m_\Delta$, and g_Δ is bounded (uniformly in ε) away from 0 and ∞ on Δ by (6.1) and Lemma 7.1 below. □

Our next lemma says that the open system $\mathring{f}_\Delta$ is mixing† on partition elements under our assumptions on f and our construction of $\hat{f}_{z,\varepsilon_0,\varepsilon}$.

Let $\varepsilon_1^* = \min\{\varepsilon_1(Q_1, Q_2) : Q_1, Q_2 \in \mathcal{Q}\} > 0$, where $\varepsilon_1(Q_1, Q_2)$ is from Lemma 4.12.

† Mixing for an open system is not generally defined, and topologically transitivity does not hold unless we restrict to the survivor set $\mathring{\Delta}^{(\infty)} = \bigcap_{n=0}^\infty \mathring{\Delta}^{(n)}$. In the open systems context, a mixing property can be formulated in terms of transitions between elements of the Markov partition $\{\Delta_{\ell,j}\}$, after removing those elements which lie above components of H_Δ in Δ.

LEMMA 6.2. *For all $\varepsilon \in (0, \varepsilon_1^*)$, the open system $(\mathring{f}_\Delta, \mathring{\Delta})$ is transitive and aperiodic on elements of $\{\Delta_{\ell,j}\}$ that do not lie above a component of H_Δ.*

Proof. Transitivity of f_Δ on elements of the Markov partition is guaranteed by the transitivity of $F_{z,\varepsilon_0,\varepsilon}$, proved in Lemma 4.11. That this property carries over to the open map $\mathring{f}_\Delta$ follows from Lemma 4.12. Considering case 2 in the proof of that lemma, we see that for $\varepsilon \in (0, \varepsilon_1)$, the orbit of the desired interval J connecting Q_1 to Q_2 is disjoint from H_ε. Thus the connection holds for the open system $(\mathring{f}_\Delta, \mathring{\Delta})$.

Next, we show that $\mathring{f}_\Delta$ is aperiodic. Due to the structure of the tower map, it suffices to show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $\mathring{f}_\Delta^n(\Delta_0) \supset \Delta_0$. Since returns to Δ_0 must be to one of the finitely many elements of the partition $\mathcal{Q}$, this property is in turn implied by the following claim: for all $Q \in \mathcal{Q}$, there exists $n_Q \in \mathbb{N}$ such that $\mathring{f}_\Delta^{n_Q}(Q) \supset Q$ and $\mathring{f}_\Delta^{n_Q+1}(Q) \supset Q$. We proceed to prove the claim, which is a refinement of the proof of Lemma 4.11.

Let $Q \in \mathcal{Q}$. Since f is leo, there exists $\bar{n} \in \mathbb{N}$ such that $f^n(\pi(Q)) \supset I$ for all $n \geqslant \bar{n}$. Thus, as in the proof of Lemma 4.11, $m_\varphi(\pi_\Delta(f_\Delta^n(Q) \cap \Delta_0)) \geqslant 1 - \kappa$, by choice of L. Applying this to $n = \bar{n}$ and $n = \bar{n} + 1$, and recalling that we identify $Y = \hat{I}'_{z,\varepsilon_0}$ with Δ_0, we obtain

$$m_\varphi(\pi_\Delta(\Delta_0 \cap f_\Delta^{\bar{n}}(Q)) \cap \pi_\Delta(\Delta_0 \cap f_\Delta^{\bar{n}+1}(Q))) \geqslant 1 - 2\kappa > 0.$$

Thus there must exist intervals $O_1 \subset \Delta_0 \cap f_\Delta^{\bar{n}}(Q)$ and $O_2 \subset \Delta_0 \cap f_\Delta^{\bar{n}+1}(Q)$ such that $\pi(O_1) \cap \pi(O_2) \neq \emptyset$. By Lemma 4.2, there exists $n_1 \in \mathbb{N}$ such that $\hat{f}_\varepsilon^{n_1}(O_1) \cap \hat{f}_\varepsilon^{n_1}(O_2) \neq \emptyset$, and we can choose this time n_1 so that this intersection occurs in $Y = \Delta_0$. This implies that also $f_\Delta^{n_1}(O_1) \cap f_\Delta^{n_1}(O_2) \neq \emptyset$.

Now using the transitivity of F_ε, as well as its Markov property, there exists $k \in \mathbb{N}$ such that $F_\varepsilon^k(f_\Delta^{n_1}(O_1) \cap f_\Delta^{n_1}(O_2)) \supset Q$. Let r_k denote the number of iterates of f_Δ contained in F_ε^k on this set. This implies that both $f_\Delta^{r_k+n_1+\bar{n}}(Q) \supset Q$, and $f_\Delta^{r_k+n_1+\bar{n}+1}(Q) \supset Q$.

As a final step, we invoke Lemma 4.12 as earlier. We have constructed two times k_1 and k_2 for which $F_\varepsilon^{k_j}(Q) \supset Q$, $j = 1, 2$. By case 2 of the proof of Lemma 4.12, for $\varepsilon < \varepsilon_1(Q, Q)$, these connections still occur in the open system. Thus we conclude that both $\mathring{f}_\Delta^{r_k+n_1+\bar{n}}(Q) \supset Q$, and $\mathring{f}_\Delta^{r_k+n_1+\bar{n}+1}(Q) \supset Q$, as required. □

6.1. *Transfer operator on Δ and a spectral gap.* In order to study the dynamics on the open tower, we define the transfer operator associated with the potential φ_Δ,

$$\mathcal{L}_\Delta \psi = \sum_{y \in f_\Delta^{-1} x} \psi(y) e^{\varphi_\Delta(y)},$$

and its usual punctured counterpart for the open system, $\mathring{\mathcal{L}}_\Delta \psi = \mathcal{L}_\Delta(\psi \cdot 1_{\mathring{\Delta}^{(1)}})$. We also define the corresponding punctured potential on the tower by $\varphi_\Delta^{H_\varepsilon} = \varphi_\Delta$ on $\mathring{\Delta}$ and $\varphi_\Delta^{H_\varepsilon} = -\infty$ on H_Δ.

We will prove that for sufficiently small holes H_ε, the transfer operator $\mathring{\mathcal{L}}_\Delta$ has a spectral gap on a certain Banach space $\mathcal{B}$, using the abstract result [**DT2**, Theorem 4.12]. Note that this result is not perturbative, but rather relies on checking four explicit conditions (P1)–(P4) from [**DT2**, §4.2]. They are as follows.

(P1) *Exponential tails.* This follows from Theorem 4.10, since by definition of m_Δ,

$$m_\Delta(\Delta_n) = m_\Delta(\Delta_0 \cap \{R_\varepsilon > n\}) = \hat{m}_\varphi(Y \cap \{R_\varepsilon > n\}) \leqslant Ce^{-\alpha n},$$

where C and α are uniform for $\varepsilon < \varepsilon_0$.

(P2) *Slow escape:* $-\log \overline{\lambda}_\varepsilon < \alpha$. This can be guaranteed by noting that $\overline{\lambda}_\varepsilon \geqslant \Lambda_\varepsilon$, where $\Lambda_\varepsilon < 1$ is from Corollary 5.5. This inequality is due to the fact that the escape from the induced system cannot be slower than the escape from the uninduced system. The requirement on the upper escape rate in [**DT2**] is defined in terms of m_Δ, which in our case is equal to $\log \overline{\lambda}_\varepsilon$ by Lemma 6.1. Again using Corollary 5.5, there exists $\varepsilon^* > 0$ such that $\Lambda_\varepsilon > e^{-\alpha}$ for all $\varepsilon \in (0, \varepsilon^*)$. This guarantees (P2).

(P3) *Bounded distortion and Lipschitz property for* e^{φ_Δ}. The potential $\varphi_\Delta = 0$ on $\Delta \setminus f_\Delta^{-1}(\Delta_0)$ so we need only verify this property at return times. This follows from Lemma 4.6 and the following estimate linking the Euclidean metric on I with the separation time metric on Δ. If $s(x, y) = n$, then $F_\varepsilon^i(\hat{\pi}_\Delta(x))$ and $F_\varepsilon^i(\hat{\pi}_\Delta(y))$ lie in the same element of $\mathcal{Y}_\varepsilon$ for each $i < n$, and $F_\varepsilon^n(\hat{\pi}_\Delta(x))$ and $F_\varepsilon^n(\hat{\pi}_\Delta(y))$ lie in the same element of $\mathcal{Q}$. Then, since $DF_\varepsilon^n \geqslant C_e\sigma^n > 1$,

$$\frac{|\hat{\pi}_\Delta(x) - \hat{\pi}_\Delta(y)|^\eta}{d_\theta(x, y)} = \frac{|\hat{\pi}_\Delta(x) - \hat{\pi}_\Delta(y)|^\eta}{e^{-\theta s(x,y)}} \leqslant \frac{C_e^\eta \sigma^{-\eta n}}{e^{-\theta n}}. \tag{6.2}$$

Choosing $\theta < \eta \log \sigma$ guarantees that a η-Hölder continuous function on I (and $\hat{I}_{z,\varepsilon_0,\varepsilon}$) lifts to a Lipschitz function on Δ. Then Lemma 4.6(a) implies the required bounded distortion for φ_Δ.

(P4) *Subexponential growth of potential.* For each $\delta > 0$, there exists $C > 0$ such that

$$|S_{R_\varepsilon}\varphi_\Delta(x)| \leqslant Ce^{\delta R_\varepsilon(x)} \quad \text{for all } x \in \Delta_0.$$

This is immediate for Hölder continuous potentials since φ is bounded so

$$|S_{R_\varepsilon}\varphi_\Delta(x)| \leqslant R_\varepsilon(x)|\varphi|_\infty \quad \text{for all } x \in \Delta_0.$$

For geometric potentials, $\varphi = -t \log |Df| - p_t$, (P4) is guaranteed by the uniform expansion of F_ε at return times, noting that

$$S_{R_\varepsilon}\varphi_\Delta(x) = S_{R_\varepsilon}\varphi(\hat{\pi}_\Delta(x)) = -t \log |DF_\varepsilon(\hat{\pi}_\Delta(x))| - R_\varepsilon(x)p_t.$$

By (GM2), $C_e\sigma \leqslant |DF_\varepsilon| \leqslant (\sup |Df|)^{R_\varepsilon}$, and since $\sup |Df| < \infty$, we have

$$|S_{R_\varepsilon}\varphi_\Delta(x)| \leqslant R_\varepsilon(x)|p_t| + t \max\{|\log(C_e\sigma)|, R_\varepsilon(x) \log |Df|_\infty\}, \quad \text{for all } x \in \Delta_0.$$

With (P1)–(P4) verified, we are in a position to study the action of $\mathring{\mathcal{L}}_\Delta$ on an appropriate function space. Using (P2), choose β such that $-\log \overline{\lambda}_\varepsilon < \beta < \alpha$. Define a weighted L^∞ norm on Δ by

$$\|\psi\|_\infty = \sup_\ell e^{-\beta\ell} \sup\{|\psi(x)| : x \in \Delta_\ell\},$$

as well as the weighted Lipschitz norm,

$$|\psi|_{\text{Lip}} = \sup_\ell e^{-\beta\ell} \sup\{e^{-\theta s(x,y)}|\psi(x) - \psi(y)| : x, y \in \Delta_\ell\}.$$

Then define $\mathcal{B} = \{\psi \in L^1(m_\Delta) : \|\psi\|_{\mathcal{B}} < \infty\}$, where $\|\psi\|_{\mathcal{B}} = \|\psi\|_\infty + |\psi|_{\mathrm{Lip}}$. We define $\mathcal{B}_0 \subset \mathcal{B}$ to be the set of bounded functions on Δ whose Lipschitz constant is also bounded, that is, $\mathcal{B}_0$ uses the same definition as $\mathcal{B}$, but with $\beta = 0$. Recall $\varepsilon_1^* > 0$ from Lemma 6.2 and $\varepsilon^* > 0$ from the verification of (P2).

THEOREM 6.3. [**DT2**, Theorem 4.12] *Since the open system $(f_\Delta, \Delta; H_\Delta)$ is mixing on partition elements and satisfies properties (P1)–(P4), we conclude that $\mathring{\mathcal{L}}_\Delta$ has a spectral gap on $\mathcal{B}$ for all $\varepsilon < \min\{\varepsilon^*, \varepsilon_1^*\}$. Let λ_ε denote the largest eigenvalue of $\mathring{\mathcal{L}}_\Delta$ and let $\mathring{g}_\Delta$ denote the corresponding normalized eigenfunction.*

(a) *The escape rate with respect to m_Δ exists and equals $-\log \lambda_\varepsilon$.*

(b) $\log \lambda_\varepsilon = \sup\{h_\vartheta(f_\Delta) + \int_{\mathring{\Delta}} \varphi_\Delta^{H_\varepsilon}\, d\vartheta : \vartheta \in \mathcal{M}_{f_\Delta}, \vartheta(-\varphi_\Delta^H) < \infty\}$*, where $\mathcal{M}_{f_\Delta}$ is the set of f_Δ-invariant probability measures on Δ.*

(c) *The following limit defines a probability measure ν_Δ, supported on $\bigcap_{n=0}^\infty \mathring{\Delta}^{(n)}$:*

$$\nu_\Delta(\varphi) = \lim_{n\to\infty} \lambda_\varepsilon^{-n} \int_{\mathring{\Delta}^{(n)}} \psi\, \mathring{g}_\Delta\, dm_\Delta \quad \text{for all } \psi \in \mathcal{B}_0.$$

Moreover, the measure ν_Δ is the unique measure in $\mathcal{M}_{f_\Delta}$ that attains the supremum in (b), that is, it is the unique equilibrium state for $\varphi_\Delta^{H_\varepsilon}$.

(d) *There exist constants $D > 0$ and $\sigma_0 < 1$ such that for all $\psi \in \mathcal{B}$,*

$$\|\lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\Delta^n \psi - d(\psi)\mathring{g}_\Delta\|_{\mathcal{B}} \le D\|\psi\|_{\mathcal{B}} \sigma_0^n,$$
$$\text{where } d(\psi) = \lim_{n\to\infty} \lambda_\varepsilon^{-n} \int_{\mathring{\Delta}^{(n)}} \psi\, dm_\Delta < \infty.$$

Also, for any $\psi \in \mathcal{B}$ with $d(\psi) > 0$,

$$\left| \frac{\mathring{\mathcal{L}}_\Delta^n \psi}{|\mathring{\mathcal{L}}_\Delta^n \psi|_{L^1(m_\Delta)}} - \mathring{g}_\Delta \right|_{L^1(m_\Delta)} \leqslant D\|\psi\|_{\mathcal{B}} \sigma_0^n.$$

6.2. *Proof of Theorem 3.1.* In this subsection we will prove the items of Theorem 3.1 using Theorem 6.3. The following lemma will allow us to lift Hölder continuous functions on I to Lipschitz functions on Δ.

LEMMA 6.4. *Suppose $\theta/\log\sigma \leqslant \varsigma \leqslant 1$, where $\sigma > 1$ is from (GM2). Let $\psi \in C^\varsigma(I)$ and define $\tilde{\psi}$ on Δ by $\tilde{\psi} = \psi \circ \pi_\Delta$. Then $|\tilde{\psi}|_\infty \leqslant |\psi|_\infty$ and $\mathrm{Lip}(\tilde{\psi}) \leqslant C|\psi|_{C^\varsigma(I)}$ for some constant C depending on the minimum length of elements of $\mathcal{Q}$.*

Proof. The bound $|\tilde{\psi}|_\infty \leqslant |\psi|_\infty$ is immediate. To prove the bound on the Lipschitz constant of $\tilde{\psi}$, suppose $x, y \in \Delta_{\ell,j}$ and estimate

$$\frac{|\tilde{\psi}(x) - \tilde{\psi}(y)|}{d_\theta(x, y)} = \frac{|\psi(\pi_\Delta(x)) - \psi(\pi_\Delta(y))|}{|\pi_\Delta(x) - \pi_\Delta(y)|^\varsigma} \cdot \frac{|\pi_\Delta(x) - \pi_\Delta(y)|^\varsigma}{|\pi_\Delta(f_\Delta^{R_\varepsilon} x) - \pi_\Delta(f_\Delta^{R_\varepsilon} y)|^\varsigma} \cdot \frac{|\pi_\Delta(f_\Delta^{R_\varepsilon} x) - \pi_\Delta(f_\Delta^{R_\varepsilon} y)|^\varsigma}{e^{-\theta s(x,y)}}.$$

The first ratio above is bounded by $|\psi|_{C^\varsigma(I)}$. The second ratio is bounded due to bounded distortion and the backward contraction condition $(\mathrm{PolShr})_\beta$ at return times to Y. For the third ratio, we use (6.2), recalling that the separation time only counts returns to Δ_0, and that $\theta \leqslant \varsigma \log\sigma$. □

In order to project densities from Δ to I, for $\psi \in L^1(m_\Delta)$ and $x \in I$, define

$$\mathcal{P}_\Delta \psi(x) = \sum_{y \in \pi_\Delta^{-1} x} \frac{\psi(y)}{J\pi_\Delta(y)}, \tag{6.3}$$

where $J\pi_\Delta$ is the Jacobian of π_Δ with respect to the measures m_φ and m_Δ. Note that for $y \in \Delta_\ell$, with $y = f_\Delta^\ell(z)$ for $z \in \Delta_0$, the conformality of m_φ implies

$$\frac{1}{J\pi_\Delta(y)} = \frac{dm_\Delta(y)}{dm_\varphi(\pi_\Delta y)} = \frac{dm_\Delta(y)}{dm_\varphi(\pi_\Delta(f_\Delta^\ell z))} = \frac{dm_\varphi(\pi_\Delta z)}{dm_\varphi(f^\ell(\pi_\Delta z))} = e^{S_\ell \varphi(\pi_\Delta z)}. \tag{6.4}$$

Then the proof of Lemma 6.4 implies that $1/J\pi_\Delta$ is Lipschitz continuous on each $\Delta_{\ell,j}$ with Lipschitz constant depending only on the level ℓ.

It follows from the definition of m_Δ that $\mathcal{P}_\Delta \psi \in L^1(m_\varphi)$ and $\int_I \mathcal{P}_\Delta \psi \, dm_\varphi = \int_\Delta \psi \, dm_\Delta$. Moreover,

$$\mathring{\mathcal{L}}^n_{\varphi H_\varepsilon}(\mathcal{P}_\Delta \psi) = \mathcal{P}_\Delta(\mathring{\mathcal{L}}^n_\Delta \psi) \quad \text{for each } n \in \mathbb{N}. \tag{6.5}$$

The final step in translating Theorem 6.3 to Theorem 3.1 is the following lemma.

LEMMA 6.5. *$C^\varsigma(\mathring{I}) \subset \mathcal{P}_\Delta \mathcal{B}_0$ for all $\varsigma \geqslant \theta/\log\sigma$.*

Proof. Let $Q \in \mathcal{Q}$ and note that by the leo property there exists $N \in \mathbb{N}$ such that $f^N(\pi(Q)) = I$. This implies that $\pi(\hat{I}_{z,\varepsilon_0,\varepsilon}(N)) = I$, where $\hat{I}_{z,\varepsilon_0,\varepsilon}(N)$ denotes the first N levels of $\hat{I}_{z,\varepsilon_0,\varepsilon}$ as in §4.4. This in turn implies that $\pi_\Delta(\bigcup_{\ell \leqslant N} \Delta_\ell) = I$ (mod 0 with respect to m_φ).

Next, we select a collection $\mathcal{K}$ of $\Delta_{\ell,i}$, $\ell \leqslant N$, such that $\pi_\Delta(\bigcup_{\Delta_{\ell,i} \in \mathcal{K}} \Delta_{\ell,i}) = I$ and $\pi_\Delta(\Delta_{\ell,i}) \cap \pi_\Delta(\Delta_{\ell',j}) = \emptyset$ except for at most finitely many pairs $\Delta_{\ell,i}$, $\Delta_{\ell',j} \in \mathcal{K}$. Such a collection exists since f^N has at most finitely many intervals of monotonicity, so that when the images of two branches overlap, we may eliminate all the $\Delta_{\ell,i}$ in one branch from our set $\mathcal{K}$. The only time when we may be forced to retain two overlapping $\Delta_{\ell,i}$ occurs at the end of one of the branches of monotonicity. In this way, we are guaranteed the existence of a set $\mathcal{K}$ with the property that only finitely many elements have projections that overlap.

With the set $\mathcal{K}$ established, the rest of the proof follows along the lines of [**BDM**, Proposition 4.2]. Essentially, it amounts to inverting the projection operator $\mathcal{P}_\Delta$ defined in (6.3).

Let $\psi \in C^\varsigma(I)$ be given. Define $\tilde{\psi} \equiv 0$ on $\Delta \setminus \bigcup_{\Delta_{\ell,i} \in \mathcal{K}} \Delta_{\ell,i}$. Next, if $\Delta_{\ell,i} \in \mathcal{K}$ and $\pi_\Delta(\Delta_{\ell,i})$ does not overlap the projection of any other $\Delta_{\ell',j} \in \mathcal{K}$, then for $x \in \Delta_{\ell,i}$ we may define $\tilde{\psi}(x) = \psi(\pi_\Delta x) J\pi_\Delta(x)$. It follows that $\mathcal{P}_\Delta \tilde{\psi}(x) = \psi(\pi x)$ for $x \in \Delta_{\ell,i}$, and by (6.4) and Lemma 6.4, $\tilde{\psi}$ is Lipschitz with norm depending on the level ℓ.

Finally, for elements of $\mathcal{K}$ whose projections overlap, we proceed as follows. Suppose $\pi_\Delta(\Delta_{\ell,i}) \cap \pi_\Delta(\Delta_{\ell',j}) \neq \emptyset$. Let $A = \pi_\Delta(\Delta_{\ell,i}) \cup \pi_\Delta(\Delta_{\ell',j})$ and choose a partition of unity $\{\rho_1, \rho_2\}$ for the interval A such that $\rho_1, \rho_2 \in C^\varsigma(A)$, and $\rho_1 = 1$ on $\pi_\Delta(\Delta_{\ell,i}) \setminus \pi_\Delta(\Delta_{\ell',j})$, while $\rho_2 = 1$ on $\pi_\Delta(\Delta_{\ell',j}) \setminus \pi_\Delta(\Delta_{\ell,i})$.

Define $\tilde{\psi}$ for $x \in \Delta_{\ell,i}$ by

$$\tilde{\psi}(x) = \psi(\pi_\Delta x) J\pi_\Delta(x) \rho_1(\pi x),$$

and similarly define $\tilde{\psi}$ on $\Delta_{\ell',j}$ using ρ_2. It is clear that $\mathcal{P}_\Delta\tilde{\psi}(y) = \psi(y)$ for $y \in A$. This construction using partitions of unity ρ_i can be modified to account for finitely may overlaps in $\pi(\Delta_{\ell,i})$, $\Delta_{\ell,i} \in \mathcal{K}$, while keeping a uniform bound on the C^ς-norm of ρ_i.

In this way, we define $\tilde{\psi}$ on $\Delta_{\ell,i}$ for all $\Delta_{\ell,i} \in \mathcal{K}$. Since $\pi_\Delta(\bigcup_{\Delta_{\ell,i}\in\mathcal{K}} \Delta_{\ell,i}) = I$, we have $\mathcal{P}_\Delta\tilde{\psi} = \psi$ (mod 0). And since $\mathcal{K}$ contains only elements on level at most N, by (6.4) and Lemma 6.4, $\tilde{\psi} \in \mathcal{B}_0$. □

We proceed to prove the items of Theorem 3.1.

Recall that $\eta \in (0, 1]$ is the relevant Hölder exponent for φ. For geometric potentials, we take $\eta = 1$ due to Lemma 4.6(b). Fix $\varsigma \in (0, \eta]$. Then we may choose $\theta \leqslant \varsigma \log \sigma$, so that Lemma 6.5 holds. Then also $\theta \leqslant \eta \log \sigma$ as required by (P3). Choosing β such that $-\log \overline{\lambda}_\varepsilon < \beta < \alpha$ then fixes the appropriate Banach space $\mathcal{B}$ for Theorem 6.3. In what follows, we assume $\varepsilon < \min\{\varepsilon^*, \varepsilon_1^*\}$.

(a) The existence of the escape rate $-\log \lambda_\varepsilon$ follows from Theorem 6.3(a) and Lemma 6.1. Define

$$\mathring{g}_\varepsilon = \mathcal{P}_\Delta \mathring{g}_\Delta.$$

By (6.5), we have $\mathring{g}_\varepsilon \in L^1(m_\varphi)$ and, for each n,

$$\mathring{\mathcal{L}}^n_{\varphi H_\varepsilon} \mathring{g}_\varepsilon = \mathcal{P}_\Delta(\mathring{\mathcal{L}}^n_\Delta \mathring{g}_\Delta) = \mathcal{P}_\Delta(\lambda^n_\varepsilon \mathring{g}_\Delta) = \lambda^n_\varepsilon \mathring{g}_\varepsilon,$$

so that $\mathring{g}_\varepsilon dm_\varphi$ defines a conditionally invariant probability measure on I with eigenvalue λ_ε.

(b) We define the required conformal measure m_{H_ε}, using the by now standard procedure,

$$m_{H_\varepsilon}(\psi) := \lim_{n\to\infty} \lambda_\varepsilon^{-n} \int_{\mathring{I}^n} \psi \, dm_\varphi \quad \text{for } \psi \in C^\varsigma(I). \tag{6.6}$$

Using Lemma 6.5, we find $\tilde{\psi} \in \mathcal{B}_0$ such that $\mathcal{P}_\Delta\tilde{\psi} = \psi$. Then, by (6.5),

$$\int_{\mathring{I}^n} \psi \, dm_\varphi = \int_I \mathring{\mathcal{L}}^n_{\varphi H_\varepsilon} \psi \, dm_\varphi = \int_\Delta \mathring{\mathcal{L}}^n_\Delta \tilde{\psi} \, dm_\Delta = \int_{\mathring{\Delta}^{(n)}} \tilde{\psi} \, dm_\Delta,$$

so that the limit in (6.6) exists by Theorem 6.3(d), using the spectral gap enjoyed by $\mathring{\mathcal{L}}_\Delta$. Indeed, $d(\tilde{\psi}) = m_{H_\varepsilon}(\psi)$. The fact that m_{H_ε} defined in this way is φ-conformal follows from the same calculation as in the proof of [**DT2**, Theorem 1.7]. The fact that m_{H_ε} is supported on $\mathring{I}^\infty$ follows from its definition in (6.6).

(c) Defining $\nu_{H_\varepsilon} := \mathring{g}_\varepsilon m_{H_\varepsilon}$, we see that

$$\nu_{H_\varepsilon}(\psi) = \lim_{n\to\infty} \lambda_\varepsilon^{-n} \int_{\mathring{I}^n} \mathring{g}_\varepsilon \psi \, dm_\varphi \quad \text{for } \psi \in C^\varsigma(I), \tag{6.7}$$

since $\mathcal{P}_\Delta(\mathring{g}_\Delta\tilde{\psi}) = \mathring{g}_\varepsilon\psi$, and $\mathring{g}_\Delta\tilde{\psi} \in \mathcal{B}$ by Lemma 6.5. This extends to $\psi \in C^0(I)$ by approximation: for each $\epsilon > 0$, we may choose $\psi_\delta \in C^\varsigma(I)$ such that $|\psi - \psi_\delta|_{C^0(I)} \leqslant \epsilon$ and $|\psi_\delta|_{C^\varsigma(I)} \leqslant \delta^{-\varsigma}$. (This can be accomplished, for example, through convolution of ψ with a C^∞ mollifier.) Then $\nu_{H_\varepsilon}(\psi_\delta - \psi_{\delta'}) \leqslant 2\epsilon$ for each $\delta' < \delta$, so that $(\nu_{H_\varepsilon}(\psi_\delta))_{\delta>0}$ forms a Cauchy family as $\delta \to 0$. Moreover,

$$\lim_{n\to\infty} \lambda_\varepsilon^{-n} \int_{\mathring{I}^n} \mathring{g}_\varepsilon \psi \, dm_\varphi = \lim_{n\to\infty} \lambda_\varepsilon^{-n} \int_{\mathring{I}^n} \mathring{g}_\varepsilon (\psi - \psi_\delta) \, dm_\varphi + \nu_{H_\varepsilon}(\psi_\delta) = \nu_{H_\varepsilon}(\psi_\delta) + \mathcal{O}(\epsilon),$$

since $\lambda_\varepsilon^{-n} \int_{\mathring{I}^n} \mathring{g}_\varepsilon \, dm_\varphi = 1$ for each $n \in \mathbb{N}$. Since $\epsilon > 0$ was arbitrary, $\nu_{H_\varepsilon}(\psi)$ exists and is given by the limit in (6.7).

Next, again using the commutativity given by (6.5), we see that $\nu_{H_\varepsilon} = (\pi_\Delta)_* \nu_\Delta$, where ν_Δ is from Theorem 6.3(c). It follows that

$$\log \lambda_\varepsilon = h_{\nu_\Delta}(f_\Delta) + \int \varphi_\Delta \, d\nu_\Delta = h_{\nu_{H_\varepsilon}}(f) + \int \varphi \, d\nu_{H_\varepsilon}, \tag{6.8}$$

since $\pi_\Delta : \Delta \to I$ is at most countable-to-one, so that ν_{H_ε} achieves the supremum in the variational principle among all invariant probability measures on $\mathring{I}^\infty$ that lift to an invariant probability measure on Δ, and ν_{H_ε} is unique in this class.

In order to conclude that in fact ν_{H_ε} achieves the supremum over all invariant probability measures ν with $\nu(-\phi^{H_\varepsilon}) < \infty$, that is, that are supported on $\mathring{I}^\infty$, we note the following inequality, taking our notation from Theorem 4.10:

$$P(\phi) - \int \phi \, d\nu = \int (P(\phi) - \phi) \, d\nu \geqslant \bar{\alpha} = \alpha + \xi \geqslant \alpha, \tag{6.9}$$

for any such measure ν, which follows from the proof of Theorem 4.10 for all classes of our admissible potentials. Note also that $\int \phi^{H_\varepsilon} \, d\nu = \int \phi \, d\nu$ whenever $\nu(-\phi^{H_\varepsilon}) < \infty$.

By choice of L in §4.5, any ergodic invariant measure ν with entropy $h_\nu(f) > (\log \lambda_{\varepsilon^*} + \alpha)/2$ lifts to our inducing scheme. For an f-invariant measure ν with $\nu(-\phi^{H_\varepsilon}) < \infty$, define the pressure of ν to be $P_\nu(\phi^{H_\varepsilon}) = h_\nu(f) + \int \phi \, d\nu$. Now if $P_\nu(\phi^{H_\varepsilon}) \geqslant P_{\nu_{H_\varepsilon}}(\phi^{H_\varepsilon})$, then

$$h_\nu(f) + \int \phi \, d\nu - P(\phi) \geqslant h_{\nu_{H_\varepsilon}}(f) + \int \phi \, d\nu_{H_\varepsilon} - P(\phi) = \log \lambda_\varepsilon$$

by (6.8), so that $h_\nu(f) \geqslant \log \lambda_\varepsilon + \alpha \geqslant \log \lambda_{\varepsilon^*} + \alpha$, using (6.9), and so ν lifts to our inducing scheme by our choice of L. Thus $P_\nu(\phi^{H_\varepsilon}) \leqslant P_{\nu_{H_\varepsilon}}(\phi^{H_\varepsilon})$, and ν_{H_ε} achieves the supremum among all invariant measures ν satisfying $\nu(-\phi^{H_\varepsilon}) < \infty$ (so in fact $\nu = \nu_{H_\varepsilon}$). Thus, ν_{H_ε} is the unique equilibrium state for ϕ^{H_ε}, proving item (c) of the theorem.

(d) The characterization of the limit proving item (d) now follows from Theorem 6.3(d), again using Lemma 6.5 to lift any $\psi \in C^\varsigma(\mathring{I})$ to a function $\tilde{\psi} \in B_0$, and then evolving that function according to (6.5). The convergence extends to any $\psi \in C^\varsigma(I)$ since in one iterate, $\mathring{\mathcal{L}}_{\varphi^{H_\varepsilon}} \psi$ is supported on $\mathring{I}$ so the values of ψ on $H_\varepsilon = I \setminus \mathring{I}$ are irrelevant to the value of the limit.

To justify Remark 3.2, note that the convergence in (d) holds for any $\psi \in \mathcal{P}_\Delta \mathcal{B}_0$ with $\nu_{H_\varepsilon}(\psi) > 0$, due to (6.5). In particular, since the invariant density $g_\varphi = d\mu_\varphi / dm_\varphi$ satisfies $g_\varphi = \mathcal{P}_\Delta g_\Delta$ for some $g_\Delta \in \mathcal{B}_0$, for any $\psi \in C^\varsigma(I)$ we may define $\tilde{\psi} = \psi \circ \pi_\Delta$, and then conclude that $\tilde{\psi} g_\Delta \in \mathcal{B}_0$ by Lemma 6.4. Thus $\psi g_\varphi \in \mathcal{P}_\Delta \mathcal{B}_0$, and so $\mathring{\mathcal{L}}^n_{\varphi^{H_\varepsilon}}(\psi g_\varphi) / |\mathring{\mathcal{L}}^n_{\varphi^{H_\varepsilon}}(\psi g_\varphi)|_{L^1(m_\varphi)}$ converges to $\mathring{g}_\varepsilon$ as $n \to \infty$.

7. *Zero-hole limit*

In this section we will focus on the limit $\lim_{\varepsilon \to 0} -\log \lambda_\varepsilon / \mu_\varphi(H_\varepsilon)$, the content of Theorems 3.5 and 3.7. We assume throughout that $\varepsilon \in (0, \varepsilon_2)$, so that the conclusions of Corollary 5.5 hold. Indeed, we will use the spectral gap for $\mathring{\mathcal{L}}_\varepsilon$ to construct a canonical invariant measure $\hat{\nu}_\varepsilon$ for $\mathring{F}_\varepsilon$, supported on the survivor set, $\mathring{Y}_\varepsilon^\infty = \bigcap_{n=0}^\infty \mathring{F}_\varepsilon^{-n}(Y)$.

For $\psi \in \mathcal{C}^\eta(\mathcal{Q})$, define

$$\hat{\nu}_\varepsilon(\psi) := \lim_{n\to\infty} \Lambda_\varepsilon^{-n} \int_{\mathring{Y}_\varepsilon^{n-1}} \psi \, \mathring{G}_\varepsilon \, d\hat{m}. \tag{7.1}$$

The limit exists since

$$\Lambda_\varepsilon^{-n} \int_{\mathring{Y}_\varepsilon^{n-1}} \psi \, \mathring{G}_\varepsilon \, d\hat{m} = \int_Y \Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon^n(\psi \, \mathring{G}_\varepsilon) \, d\hat{m} \xrightarrow[n\to\infty]{} \mathring{e}_\varepsilon(\psi \, \mathring{G}_\varepsilon),$$

where $\mathring{e}_\varepsilon$ is from Corollary 5.5. Since $|\hat{\nu}_\varepsilon(\psi)| \leqslant \hat{\nu}_\varepsilon(1)|\psi|_\infty$, $\hat{\nu}_\varepsilon$ extends to a bounded linear functional on $\mathcal{C}^0(\mathcal{Q})$, that is, $\hat{\nu}_\varepsilon$ is a Borel measure. Moreover, $\hat{\nu}_\varepsilon(1) = 1$, so $\hat{\nu}_\varepsilon$ is a probability measure, clearly supported on $\mathring{Y}_\varepsilon^\infty$.

Let $\mathring{\Phi}_\varepsilon$ denote the punctured version of the induced potential Φ_ε, that is, $\mathring{\Phi}_\varepsilon = \Phi_\varepsilon$ on $Y \setminus \hat{H}'_\varepsilon$, and $\mathring{\Phi}_\varepsilon = -\infty$ on $\hat{H}'_\varepsilon$. Recall $P(\Phi_\varepsilon) = 0$ by Remark 4.7. According to [**DT2**, §6.4.1], $\hat{\nu}_\varepsilon$ is an equilibrium state for the potential $\mathring{\Phi}_\varepsilon - \log \Lambda_\varepsilon$; on the other hand, by [**BDM**, Lemma 5.3], $\hat{\nu}_\varepsilon$ is a Gibbs measure for the potential $\mathring{\Phi}_\varepsilon - R_\varepsilon \log \lambda_\varepsilon$, with pressure $P_{\hat{\nu}_\varepsilon}(\mathring{\Phi}_\varepsilon - R_\varepsilon \log \Lambda_\varepsilon) = 0$. We conclude

$$\log \Lambda_\varepsilon = \left(\int R_\varepsilon \, d\hat{\nu}_\varepsilon \right) \log \lambda_\varepsilon. \tag{7.2}$$

Recalling that $\hat{\mu}_{Y,\varepsilon} = G_\varepsilon \hat{m}$ is the invariant probability measure for $F_{z,\varepsilon_0,\varepsilon}$, supported on Y, Kac's lemma in (4.4) implies $\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon) = \hat{\mu}_{Y,\varepsilon}(\hat{H}'_\varepsilon)/\int R_\varepsilon \, d\hat{\mu}_{Y,\varepsilon}$. So putting these together yields

$$\frac{\log \lambda_\varepsilon}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} = \frac{\log \lambda_\varepsilon}{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)} \cdot \frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} = \frac{\log \Lambda_\varepsilon}{\hat{\mu}_{Y,\varepsilon}(\hat{H}'_\varepsilon)} \cdot \frac{\int R_\varepsilon \, d\hat{\mu}_{Y,\varepsilon}}{\int R_\varepsilon \, d\hat{\nu}_\varepsilon} \cdot \frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)}. \tag{7.3}$$

Therefore to prove Theorems 3.5 and 3.7 we must show that as $\varepsilon \to 0$,

$$-\frac{\log \Lambda_\varepsilon}{\hat{\mu}_{Y,\varepsilon}(\hat{H}'_\varepsilon)} \to 1, \quad \frac{\int R_\varepsilon \, d\hat{\mu}_{Y,\varepsilon}}{\int R_\varepsilon \, d\hat{\nu}_\varepsilon} \to 1, \quad \text{and} \quad \frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} \to 1 - e^{-S_p\varphi(z)} \tag{7.4}$$

(we take $e^{-S_p\varphi(z)} = 0$ when z is aperiodic). These are Theorem 7.2, Proposition 7.3 and then Lemmas 7.5 and 7.6 in the Hölder case and Lemmas 7.10 and 7.11 in the geometric case.

7.1. *An asymptotic for* Λ_ε. In this subsection we obtain a precise asymptotic for Λ_ε in terms of the quantity $\hat{\mu}_{Y,\varepsilon}(\hat{H}'_\varepsilon)$, proving the first limit in (7.4).

We remark that we are not able to apply the results of [**KL2**] in our setting since it does not fit into the assumptions of that paper. In [**KL2**], it is assumed that there is a sequence of operators P_ε, with a decomposition similar to that given by Corollary 5.5 and having largest eigenvalue ρ_ε. These operators approach a fixed operator P_0 with eigenvalue 1, and the derivative of $\log \rho_\varepsilon$ is expressed in terms of the size of the perturbation $P_0 - P_\varepsilon$.

In our setting, the only candidate for P_0 is our transfer operator $\mathcal{L}_0 = \mathcal{L}_{z,\varepsilon_0}$, the transfer operator corresponding to F_{z,ε_0}, which does not depend on ε. However, the relation between δ and ε given by Lemma 5.2 is not explicit, so that a good asymptotic expression for Λ_ε is not available starting from $\mathcal{L}_0$ (indeed, the relation between ε and δ depends in part on the rate of approach of the orbit of z to itself, which is not guaranteed to be proportional to the measure of $\hat{H}'_\varepsilon$). Instead, as suggested by Lemma 5.4, the difference between $\mathcal{L}_\varepsilon$ and $\mathring{\mathcal{L}}_\varepsilon$ has the correct order for the asymptotic we want. In order to exploit this, we consider then two sequences of operators, $(\mathcal{L}_\varepsilon)_{\varepsilon>0}$ and $(\mathring{\mathcal{L}}_\varepsilon)_{\varepsilon>0}$, and use their uniform spectral properties to prove the required asymptotic for the maximal eigenvalues Λ_ε of the latter sequence in terms of the maximal eigenfunctions of the former sequence.

We begin by establishing the following improved regularity for the functions G_ε and $\mathring{G}_\varepsilon$.

LEMMA 7.1. *For all $\varepsilon \in (0, \varepsilon_2)$, where $\varepsilon_2 > 0$ is from Corollary 5.5,*

$$H^\eta(\log G_\varepsilon) \leqslant C_d \quad \textit{and} \quad H^\eta(\log \mathring{G}_\varepsilon) \leqslant C_d. \tag{7.5}$$

As a consequence, there exists $c_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$,

$$c_0 \leqslant \inf_Y \mathring{G}_\varepsilon \leqslant \|\mathring{G}_\varepsilon\|_{\mathcal{C}^\eta} \leqslant c_0^{-1}, \tag{7.6}$$

and similar bounds hold for G_ε.

Proof. Suppose $\psi \in \mathcal{C}^\eta$ satisfies $H^\eta(\log\psi) \leqslant K$. Then $\psi(x)/\psi(y) \leqslant e^{Kd(x,y)^\eta}$, for any x, y belonging to the same element of $\mathcal{Q}$.

We follow the notation in the proof of Proposition 5.1. Let $x, y \in Q \in \mathcal{Q}$. For $n \geqslant 0$ and $u \in \mathring{F}_\varepsilon^{-n}(x)$, let $Y_i^{(n)}(u)$ denote the n-cylinder containing u. For each u, there is a unique $v \in \mathring{F}_\varepsilon^{-n}(y) \cap Y_i^{(n)}(u)$.

Using the log-Hölder regularity of ψ as well as the bounded distortion property (GM3), we estimate

$$\begin{aligned}\mathring{\mathcal{L}}_\varepsilon^n \psi(x) &= \sum_{u \in \mathring{F}_\varepsilon^{-n}(x)} \psi(u) e^{S_n\Phi_\varepsilon(u)} \leqslant \sum_{u \in \mathring{F}_\varepsilon^{-n}(x)} \psi(v) e^{Kd(u,v)^\eta} e^{S_n\Phi_\varepsilon(v)} (1 + C_d d(x, y)^\eta) \\ &\leqslant \mathring{\mathcal{L}}_\varepsilon^n \psi(y) e^{KC_e^{-\eta}\sigma^{-n\eta} d(x,y)^\eta} (1 + C_d d(x, y)^\eta),\end{aligned}$$

where, for the last inequality, we have used property (GM2). Now taking logs, and using the inequality $\log(1+t) \leqslant t$ for all $t \geqslant 0$, we have

$$H^\eta(\log \mathring{\mathcal{L}}_\varepsilon^n \psi) \leqslant KC_e^{-\eta}\sigma^{-n\eta} H^\eta(\log\psi) + C_d, \quad \text{for all } n \geqslant 1. \tag{7.7}$$

This implies that for n large enough, $\mathring{\mathcal{L}}_\varepsilon^n$ preserves the set of functions $\{\psi \in C^\eta(\mathcal{Q}) : H^\eta(\log\psi) \leqslant 1 + C_d\}$. Thus $\mathring{G}_\varepsilon$ must belong to this set. Since $\mathring{\mathcal{L}}_\varepsilon \mathring{G}_\varepsilon = \Lambda_\varepsilon \mathring{G}_\varepsilon$, substituting $\mathring{G}_\varepsilon$ into (7.7) and taking $n \to \infty$ implies that $H^\eta(\log \mathring{G}_\varepsilon) \leqslant C_d$, proving (7.5).

By a nearly identical argument, (7.7) applies to $\mathcal{L}_\varepsilon$ as well, and so its fixed point G_ε satisfies (7.5).

Finally, we show how (7.5) implies (7.6). The uniform upper bounds on $|\mathring{G}_\varepsilon|_{\mathcal{C}^\eta}$ and $|G_\varepsilon|_{\mathcal{C}^\eta}$ follow immediately from Proposition 5.1; we can set $c_1^{-1} = C$ from that proposition, so we focus on the lower bounds.

Since $\int \mathring{G}_\varepsilon \, d\hat{m} = 1$, there exists $Q_0 \in \mathcal{Q}$ such that $\sup_{x\in Q_0} \mathring{G}_\varepsilon(x) \geqslant 1$. By (7.5), $\inf_{x\in Q_0} \mathring{G}_\varepsilon(x) \geqslant e^{-C_d}$. Now by the mixing property of F_ε together with Lemma 4.12, there exists $n_0 \in \mathbb{N}$, independent of $\varepsilon \in (0, \varepsilon_1)$, such that $\hat{f}_\varepsilon^{n_0}(Q_0) \supset Y$. Thus, for any $y \in Y$, there exists $n(y) \leqslant n_0$ such that $R^{n(y)}(y) = n_0$. Then

$$\mathring{G}_\varepsilon(y) = \Lambda_\varepsilon^{-n(y)} \mathring{\mathcal{L}}_\varepsilon^{n(y)} \mathring{G}_\varepsilon(y) \geqslant \Lambda_\varepsilon^{-n(y)} e^{-C_d} \inf_{x \in Q_0 \cap \hat{f}_\varepsilon^{-n_0}(Y)} e^{S_{n_0}\varphi(x)} =: c_2.$$

Let $c_0 := \min\{c_1, c_2\}$. Note that, by our assumptions on f, we have

$$\inf_{x\in Q_0\cap \hat{f}_\varepsilon^{-n_0}(Y)} e^{S_{n_0}\varphi(x)} > 0$$

even when φ is of the form $-t\log|Df| - P(-t\log|Df|)$ because the orbit $x, f(x), \ldots, f^{n_0-1}(x)$ avoids a neighbourhood of Crit for any $x \in Q_0 \cap \hat{f}_\varepsilon^{-n_0}(Y)$ since n_0 is a return time to Y on this set. Thus c_0 is strictly positive and is also independent of ε by Lemma 4.12. This proves (7.6) for $\mathring{G}_\varepsilon$, and an identical argument can be used for G_ε. □

THEOREM 7.2.

$$\lim_{\varepsilon\to 0}\frac{1-\Lambda_\varepsilon}{\hat{\mu}_{Y,\varepsilon}(\hat{H}'_\varepsilon)} = 1.$$

Proof. We assume $\varepsilon \in (0, \varepsilon_2)$ since we are interested in the limit $\varepsilon \to 0$. Iterating (5.10) for $n \geqslant 1$,

$$\mathring{\mathcal{L}}_\varepsilon^n G_\varepsilon = \Lambda_\varepsilon^n \mathring{e}_\varepsilon(G_\varepsilon)\,\mathring{G}_\varepsilon + \mathring{\mathcal{R}}_\varepsilon^n G_\varepsilon \implies \mathring{G}_\varepsilon = \frac{1}{\mathring{e}_\varepsilon(G_\varepsilon)}(\Lambda_\varepsilon^{-n}\mathring{\mathcal{L}}_\varepsilon^n G_\varepsilon - \Lambda_\varepsilon^{-n}\mathring{\mathcal{R}}_\varepsilon^n G_\varepsilon).$$

Using this identity and (5.9), we estimate

$$\begin{aligned}
1-\Lambda_\varepsilon &= \int \mathring{G}_\varepsilon\,d\hat{m} - \int \mathring{\mathcal{L}}_\varepsilon \mathring{G}_\varepsilon\,d\hat{m} = \int (\mathcal{L}_\varepsilon - \mathring{\mathcal{L}}_\varepsilon)\mathring{G}_\varepsilon\,d\hat{m} = \int_{\hat{H}'_\varepsilon} \mathring{G}_\varepsilon\,d\hat{m}\\
&= \frac{1}{\mathring{e}_\varepsilon(G_\varepsilon)}\int_{\hat{H}'_\varepsilon}(\Lambda_\varepsilon^{-n}\mathring{\mathcal{L}}_\varepsilon^n G_\varepsilon - \Lambda_\varepsilon^{-n}\mathring{\mathcal{R}}_\varepsilon^n G_\varepsilon)\,d\hat{m}\\
&= \frac{1}{\mathring{e}_\varepsilon(G_\varepsilon)}\bigg(\int_{\hat{H}'_\varepsilon} G_\varepsilon\,d\hat{m} - \int_{\hat{H}'_\varepsilon}(1-\Lambda_\varepsilon^{-n}\mathring{\mathcal{L}}_\varepsilon^n)G_\varepsilon\,d\hat{m} - \int_{\hat{H}'_\varepsilon}\Lambda_\varepsilon^{-n}\mathring{\mathcal{R}}_\varepsilon^n G_\varepsilon\,d\hat{m}\bigg).
\end{aligned} \tag{7.8}$$

Using Corollary 5.5, we estimate the third term on the right-hand side of (7.8) by

$$\|\Lambda_\varepsilon^{-n}\mathring{\mathcal{R}}_\varepsilon^n G_\varepsilon\|_{\mathcal{C}^\eta} \leqslant e^{-2\beta n/3}\Lambda_\varepsilon^{-n}\|G_\varepsilon\|_{\mathcal{C}^\eta} \leqslant e^{-\beta n/3}\|G_\varepsilon\|_{\mathcal{C}^\eta}.$$

Due to (7.6), $\|G_\varepsilon\|_{\mathcal{C}^\eta} \leqslant c_0^{-1}$ and $G_\varepsilon \geqslant c_0$ uniformly in ε. Thus

$$\left|\int_{\hat{H}'_\varepsilon}\Lambda_\varepsilon^{-n}\mathcal{R}_\varepsilon^n G_\varepsilon\,d\hat{m}\right| = \int_{\hat{H}'_\varepsilon} G_\varepsilon\,d\hat{m}\cdot\mathcal{O}(e^{-\beta n/3}). \tag{7.9}$$

Next, the second term on the right-hand side of (7.8) can be rewritten as

$$\int_{\hat{H}'_\varepsilon}(1-\Lambda_\varepsilon^{-n}\mathring{\mathcal{L}}_\varepsilon^n)G_\varepsilon\,d\hat{m} = (1-\Lambda_\varepsilon^{-n})\int_{\hat{H}'_\varepsilon} G_\varepsilon\,d\hat{m} + \Lambda_\varepsilon^{-n}\int_{\hat{H}'_\varepsilon}(\mathcal{L}_0^n - \mathring{\mathcal{L}}_\varepsilon^n)G_\varepsilon\,d\hat{m},$$

recalling that $\mathcal{L}_0$ is the transfer operator corresponding to F_{z,ε_0} which also has $\hat{m}$ as a conformal measure. Now the maps F_{z,ε_0} and $\mathring{F}_{z,\varepsilon_0,\varepsilon}$ differ on the 1-cylinders contained in $B_\varepsilon \cup \hat{H}'_\varepsilon$, where B_ε is defined in the proof of Lemma 5.2. Thus F_{z,ε_0}^n and $\mathring{F}_{z,\varepsilon_0,\varepsilon}^n$ differ on the n-cylinders contained in $B'_{\varepsilon,n} := (\bigcup_{i=0}^{n-1} F_{z,\varepsilon_0}^{-i}(B_\varepsilon \cup \hat{H}'_\varepsilon)) \bigcup (\bigcup_{i=0}^{n-1} F_{z,\varepsilon_0,\varepsilon}^{-i}(B_\varepsilon \cup \hat{H}'_\varepsilon))$. Now following (5.7) and (5.8), we have

$$|(\mathcal{L}_0^n - \mathring{\mathcal{L}}_\varepsilon^n)G_\varepsilon|_\infty \leqslant \frac{1+C_d}{q}2\,\hat{m}(B'_{\varepsilon,n})|G_\varepsilon|_\infty.$$

Then the second term on the right-hand side of (7.8) can be bounded by

$$\left|\int_{\hat{H}'_\varepsilon}(1-\Lambda_\varepsilon^{-n}\mathring{\mathcal{L}}_\varepsilon^n)G_\varepsilon\,d\hat{m}\right| = \int_{\hat{H}'_\varepsilon} G_\varepsilon\,d\hat{m}\cdot\mathcal{O}((1-\Lambda_\varepsilon^{-n})+\Lambda_\varepsilon^{-n}\hat{m}(B'_{\varepsilon,n})), \tag{7.10}$$

using (7.6) again to estimate $\int_{\hat{H}'_\varepsilon}|G_\varepsilon|_\infty\,d\hat{m}\leqslant c_0^{-2}\int_{\hat{H}'_\varepsilon}G_\varepsilon\,d\hat{m}$.

Putting (7.9) and (7.10) together with (7.8) and dividing through by $\int_{\hat{H}'_\varepsilon}G_\varepsilon\,d\hat{m} = \hat{\mu}_{Y,\varepsilon}(\hat{H}'_\varepsilon)$ yields,

$$\frac{1-\Lambda_\varepsilon}{\hat{\mu}_{Y,\varepsilon}(\hat{H}'_\varepsilon)} = \frac{1}{\mathring{e}_\varepsilon(G_\varepsilon)}(1+\mathcal{O}((1-\Lambda_\varepsilon^{-n})+\Lambda_\varepsilon^{-n}\hat{m}(B'_{\varepsilon,n})+e^{-\beta n/3})).$$

The quantity $\mathring{e}_\varepsilon(G_\varepsilon)$ can be made arbitrarily close to $e_0(G_\varepsilon)=1$ by Corollary 5.5.

Now fix $\delta>0$ and first choose n sufficiently large that $e^{-\beta n/3}<\delta$. Next choose ε sufficiently small so that $|\mathring{e}_\varepsilon(G_\varepsilon)-1|<\delta$, $|1-\Lambda_\varepsilon^{-n}|<\delta$ and $\Lambda_\varepsilon^{-n}\leqslant 2$ by Corollary 5.5, and $\hat{m}(B'_{n,\varepsilon})<\delta$ by Corollary 4.13. Then the error term is $\mathcal{O}(\delta)$, and, since δ was arbitrary, the theorem follows. □

7.2. *Convergence of the integral of the return time.* In this subsection we prove the convergence of the second limit in (7.4), regarding the integral of the return time. As before, we assume $\varepsilon\in(0,\varepsilon_2)$, so that the conclusions of Corollary 5.5 hold.

Recall the invariant measure $\hat{\nu}_\varepsilon$ from (7.1) supported on $\mathring{Y}^\infty$, and that $\hat{\mu}_{Y,\varepsilon}=G_\varepsilon\hat{m}$ is the invariant measure for F_ε given by Corollary 5.3. The main result of this subsection is the following proposition.

PROPOSITION 7.3. *Let $R_\varepsilon = R_{z,\varepsilon_0,\varepsilon}$. Then*

$$\lim_{\varepsilon\to 0}\frac{\int R_\varepsilon\,d\hat{\mu}_{Y,\varepsilon}}{\int R_\varepsilon\,d\hat{\nu}_\varepsilon}=1.$$

Proof. First we show that for $\psi\in\mathcal{C}^\eta(\mathcal{Q})$, $|\hat{\nu}_\varepsilon(\psi)-\hat{\mu}_{Y,0}(\psi)|\to 0$ as $\varepsilon\to 0$. Let $\mathring{\Pi}_\varepsilon$ be the projector defined by $\mathring{G}_\varepsilon\otimes\mathring{e}_\varepsilon$, that is,

$$\mathring{\Pi}_\varepsilon(\psi)=\mathring{e}_\varepsilon(\psi)\,\mathring{G}_\varepsilon\quad\text{for all }\psi\in\mathcal{C}^\eta(\mathcal{Q}),$$

and similarly for Π_0. Recall that we have normalized the eigenvectors so that $\hat{m}(\mathring{G}_\varepsilon)=\hat{m}(G_0)=1$.

Notice that, since $\mathcal{L}_0^*\hat{m}=\hat{m}$, $e_0(\psi)$ is simply $\hat{m}(\psi)$. Thus $\hat{\mu}_{Y,0}(\psi)=e_0(\psi G_0)$. Now

$$\begin{aligned}|\hat{\nu}_\varepsilon(\psi)-\hat{\mu}_{Y,0}(\psi)| &\leqslant |\mathring{e}_\varepsilon(\psi\mathring{G}_\varepsilon)-e_0(\psi\mathring{G}_\varepsilon)|+|e_0(\psi\mathring{G}_\varepsilon)-e_0(\psi g_0)|\\ &\leqslant\left|\int_Y\mathring{\Pi}_\varepsilon(\psi\mathring{G}_\varepsilon)-\Pi_0(\psi\mathring{G}_\varepsilon)\,d\hat{m}\right|+\left|\int_Y\psi(\mathring{G}_\varepsilon-G_0)\,d\hat{m}\right|\\ &\leqslant|||\mathring{\Pi}_\varepsilon-\Pi_0|||\;\|\psi\mathring{G}_\varepsilon\|_{\mathcal{C}^\eta(\mathcal{Q})}+|\psi|_\infty|\mathring{G}_\varepsilon-G_0|_{L^1(\hat{m})},\end{aligned}$$

and both terms go to zero as $\varepsilon\to 0$ by Corollary 5.5 (which in turn uses [**KL1**]).

It also follows from Corollary 5.3 that $\hat{\mu}_{Y,\varepsilon}(\psi)\to\hat{\mu}_{Y,0}(\psi)$ as $\varepsilon\to 0$. Thus, by the triangle inequality, $|\hat{\nu}_\varepsilon(\psi)-\hat{\mu}_{Y,\varepsilon}(\psi)|\to 0$ as $\varepsilon\to 0$, for all $\psi\in\mathcal{C}^\eta(\mathcal{Q})$.

This does not immediately imply the proposition since $R_\varepsilon \notin \mathcal{C}^\eta(\mathcal{Q})$. However, we claim that $\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon) \in \mathcal{C}^\eta(\mathcal{Q})$. First, $\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon)$ is bounded for all $x \in Y$ by

$$\mathring{\mathcal{L}}_\varepsilon R_\varepsilon(x) = \sum_{u \in \mathring{F}_\varepsilon^{-1}(x)} R_\varepsilon(u) e^{\Phi_\varepsilon(u)} \leqslant \frac{1 + C_d}{q} \sum_{u \in \mathring{F}_\varepsilon^{-1}(x)} R_\varepsilon(u) \hat{m}(Y_i(u)), \tag{7.11}$$

by (5.1), where $Y_i(u)$ is the 1-cylinder containing u. The last sum is simply bounded by $\hat{m}(R_\varepsilon) = \hat{m}_\varepsilon(\hat{I}_\varepsilon)$, since F_ε is a first return map to Y in the Hofbauer extension. This is uniformly bounded in ε by Theorem 4.10. Next, since R_ε is constant on 1-cylinders $Y_i \in \mathcal{Y}_\varepsilon$, using (GM3), the Hölder constant of $\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon)$ is bounded by

$$\begin{aligned}
\mathring{\mathcal{L}}_\varepsilon R_\varepsilon(x) - \mathring{\mathcal{L}}_\varepsilon R_\varepsilon(y) &= \sum_{u \in \mathring{F}_\varepsilon^{-1}(x)} R_\varepsilon(u)(e^{\Phi_\varepsilon(u)} - e^{\Phi_\varepsilon(v)}) \\
&\leqslant C_d d(x, y)^\eta \sum_{u \in \mathring{F}_\varepsilon^{-1}(x)} R_\varepsilon(u) e^{\Phi_\varepsilon(u)},
\end{aligned}$$

for all $x, y \in Q \in \mathcal{Q}$, where each $v \in \mathring{F}_\varepsilon^{-1}(y)$ is paired with $u \in \mathring{F}_\varepsilon^{-1}(x)$ lying in the same 1-cylinder. The sum is again uniformly bounded in ε as in (7.11), proving the claim.

It follows that $\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon \mathring{G}_\varepsilon) \in \mathcal{C}^\eta(\mathcal{Q})$ and, by Lemma 7.1, also $\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon \mathring{G}_\varepsilon)/\mathring{G}_\varepsilon \in \mathcal{C}^\eta(\mathcal{Q})$. Now by (7.1),

$$\lim_{n\to\infty} \Lambda_\varepsilon^{-n} \int_{\mathring{Y}_\varepsilon^n} R_\varepsilon\, \mathring{G}_\varepsilon\, d\hat{m} = \lim_{n\to\infty} \Lambda_\varepsilon^{-n} \int_{\mathring{Y}_\varepsilon^{n-1}} \frac{\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon \mathring{G}_\varepsilon)}{\mathring{G}_\varepsilon}\, \mathring{G}_\varepsilon\, d\hat{m} \xrightarrow[n\to\infty]{} \hat{\nu}_\varepsilon\left(\frac{\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon \mathring{G}_\varepsilon)}{\Lambda_\varepsilon \mathring{G}_\varepsilon}\right).$$

Thus, $\hat{\nu}_\varepsilon(R_\varepsilon)$ exists and is defined by (7.1).

For $N \in \mathbb{N}$, define the truncation $R_\varepsilon^{(N)} = \min\{R_\varepsilon, N\}$. For $R_0 = R_{z,\varepsilon_0}$, define $R_0^{(N)}$ similarly. By the above arguments, it follows that $\mathring{\mathcal{L}}_\varepsilon(R_\varepsilon^{(N)} \mathring{G}_\varepsilon) \in \mathcal{C}^\eta(\mathcal{Q})$ and that $\hat{\nu}_\varepsilon(R_\varepsilon^{(N)})$ exists and is defined by (7.1). Similarly, for the complementary function, $\mathring{\mathcal{L}}_\varepsilon(1_{R_\varepsilon > N} \cdot R_\varepsilon) \in \mathcal{C}^\eta(\mathcal{Q})$, and $\hat{\nu}_\varepsilon(1_{R_\varepsilon > N} \cdot R_\varepsilon)$ exists and is defined by (7.1).

Next, we claim that R_ε is uniformly integrable with respect to $\hat{\nu}_\varepsilon$; in particular, $\hat{\nu}_\varepsilon(1_{R_\varepsilon > N} \cdot R_\varepsilon) \to 0$ as $N \to \infty$ uniformly in ε. To see this, note that, by (7.1),

$$\begin{aligned}
\hat{\nu}_\varepsilon(1_{R_\varepsilon > N} \cdot R_\varepsilon) &= \lim_{n\to\infty} \Lambda_\varepsilon^{-n} \int_Y \mathring{\mathcal{L}}_\varepsilon^n(1_{R_\varepsilon > N} \cdot R_\varepsilon\, \mathring{G}_\varepsilon)\, d\hat{m} \\
&\leqslant \lim_{n\to\infty} \left| \Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon^{n-1}(\mathring{\mathcal{L}}_\varepsilon(1_{R_\varepsilon > N} \cdot R_\varepsilon\, \mathring{G}_\varepsilon)) \right|_\infty \\
&\leqslant C |\mathring{\mathcal{L}}_\varepsilon(1_{R_\varepsilon > N} \cdot R_\varepsilon\, \mathring{G}_\varepsilon)|_\infty,
\end{aligned}$$

where we have used (5.6) for the last inequality, together with the fact that $\Lambda_\varepsilon^{-n} \hat{m}(\mathring{Y}_\varepsilon^{n-1})$ is bounded uniformly in ε and n by Corollary 5.3. Then, estimating as in (7.11),

$$|\mathring{\mathcal{L}}_\varepsilon(1_{R_\varepsilon > N} \cdot R_\varepsilon\, \mathring{G}_\varepsilon)(x)| \leqslant C \sum_{\substack{u \in \mathring{F}_\varepsilon^{-1}(x) \\ R_\varepsilon(u) > N}} R_\varepsilon(u) \hat{m}(Y_i(u)) \leqslant C \sum_{k > N} k\, \hat{m}(R_\varepsilon = k) \leqslant C' e^{-\alpha N},$$

by Theorem 4.10, and the claim is proved.

It follows from the proof of Corollary 4.13 that for each $N > 0$ there exists $\varepsilon_N > 0$ such that, for $\varepsilon \leqslant \varepsilon_N$, all 1-cylinders Y_i for F_ε with $R_\varepsilon(Y_i) \leqslant N$ are also 1-cylinders for F_0 with the same return time. This implies that $R_\varepsilon^{(N)} = R_0^{(N)}$ for $\varepsilon \leqslant \varepsilon_N$.

Let $\delta > 0$ be arbitrary. Choose N such that $\hat{\nu}_\varepsilon(R_\varepsilon > N) < \delta$, $\hat{\mu}_{Y,\varepsilon}(R_\varepsilon > N) < \delta$ and $\hat{\mu}_{Y,0}(R_0 > N) < \delta$, for all $\varepsilon < \varepsilon_1$, which is possible by the claim and Theorem 4.10. Then, for $\varepsilon \leqslant \varepsilon_N$, we have

$$\hat{\nu}_\varepsilon(R_\varepsilon) = \hat{\nu}_\varepsilon(R_0^{(N)}) + \mathcal{O}(\delta) \xrightarrow[\varepsilon\to 0]{} \hat{\mu}_{Y,0}(R_0^{(N)}) + \mathcal{O}(\delta) = \hat{\mu}_{Y,0}(R_0) + \mathcal{O}(\delta).$$

Similarly,

$$\hat{\mu}_{Y,\varepsilon}(R_\varepsilon) = \hat{\mu}_{Y,\varepsilon}(R_0^{(N)}) + \mathcal{O}(\delta) \xrightarrow[\varepsilon\to 0]{} \hat{\mu}_{Y,0}(R_0^{(N)}) + \mathcal{O}(\delta) = \hat{\mu}_{Y,0}(R_0) + \mathcal{O}(\delta).$$

Since δ was arbitrary, this proves the proposition. □

7.3. *Final step of the proof of Theorem 3.5: the Hölder continuous case.* In the next two subsections we prove the third limit in (7.4) in both the periodic and non-periodic cases. In the present subsection we address the case where φ is Hölder continuous, and in §7.4 we will address the case where φ is a geometric potential. As a preliminary result, we prove the following lemma.

LEMMA 7.4. *For $f \in \mathcal{F}$ and a Hölder potential φ, we have $\inf_{x\in I}\,(d\mu_\varphi/dm_\varphi)(x) > 0$, where μ_φ and m_φ are the relevant invariant and conformal measures.*

Proof. For simplicity we write $g(x) = (d\mu_\varphi/dm_\varphi)(x)$ and note that m_φ is a φ-conformal measure so $\mathcal{L}_\varphi g = g$, where $\mathcal{L}_\varphi$ is the transfer operator associated to φ and f (not the induced dynamics), defined in §2.3. Since $\pi_*\hat{\mu}_\varepsilon = \mu_\varphi$ (we take any $\varepsilon \in (0, \varepsilon_2)$), Lemma 7.1 implies that there is an open set U such that $\inf_{x\in U}\, g(x) > 0$. By leo, there is some $n \in \mathbb{N}$ such that $f^n(U) = I$. Hence, for any $x \in I$ we can estimate

$$g(x) = \mathcal{L}_\varphi^n g(x) = \sum_{y\in f^{-n}(x)} g(y)e^{S_n\varphi(y)} \geqslant \sum_{y\in\{f^{-n}(x)\}\cap U} g(y)e^{S_n\varphi(y)}. \tag{7.12}$$

So we conclude by noting that $\inf S_n\varphi > -\infty$. □

We first address the case in which z is aperiodic.

LEMMA 7.5. *Let z be an aperiodic point for f and suppose φ is Hölder continuous. Then*

$$\lim_{\varepsilon\to 0} \frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} = 1.$$

Proof. Recall from (4.4) and (4.5) that $\hat{\mu}_{Y,\varepsilon}$ and $\hat{\mu}_\varepsilon$ are related by $\hat{\mu}_{Y,\varepsilon} = \hat{\mu}_\varepsilon|_Y/\hat{\mu}_\varepsilon(Y)$ and

$$\hat{\mu}_\varepsilon(A) = \sum_i \sum_{j=0}^{R_i-1} \hat{\mu}_\varepsilon(Y_i \cap \hat{f}^{-j}A) \quad \text{for any Borel } A \subset \hat{I}. \tag{7.13}$$

We will apply the above expression to $A = \hat{H}_\varepsilon$. Note that, due to our construction of $\hat{I}_{z,\varepsilon_0,\varepsilon}$, for each j, if $\hat{f}^j(Y_i) \cap \hat{H}_\varepsilon \neq \emptyset$, then $\hat{f}^j(Y_i) \subset \hat{H}_\varepsilon$. Thus each term in the above sum is either 0 or $\hat{\mu}_\varepsilon(Y_i)$. Define, for $k \geqslant 1$,

$$\hat{H}'_\varepsilon(k) = \{Y_i \subset \hat{H}'_\varepsilon : Y_i \text{ enters } \hat{H}_\varepsilon \text{ exactly } k \text{ times before time } R_i\}. \tag{7.14}$$

Now using (7.13) and our observation about Y_i,

$$\hat{\mu}_\varepsilon(\hat{H}_\varepsilon) = \sum_{k\geqslant 1} \sum_{Y_i\in \hat{H}'_\varepsilon(k)} k\hat{\mu}_\varepsilon(Y_i) = \hat{\mu}_\varepsilon(\hat{H}'_\varepsilon) + \sum_{k\geqslant 2} \sum_{Y_i\in \hat{H}'_\varepsilon(k)} (k-1)\hat{\mu}_\varepsilon(Y_i). \tag{7.15}$$

We proceed to estimate the double sum over k and Y_i.

By (7.13), since F_ε is the first return map to Y in $\hat{I}$, the invariant density G_ε from Corollary 5.3 is also the density for $\hat{\mu}_\varepsilon$ on Y, up to a normalizing constant. Applying the uniform bounds on G_ε from Lemma 7.1, we replace $\hat{\mu}_\varepsilon(Y_i)$ with $\hat{m}(Y_i)$ in (7.15), up to a uniform constant. For $Y_i \subset \hat{H}'_\varepsilon(k)$, let T_i denote the time of the kth entry of Y_i to $\hat{H}_\varepsilon$ under iteration of $\hat{f}$. By the conformality of $\hat{m} = \hat{m}_{\hat{\varphi}}$,

$$\hat{m}_\varepsilon(Y_i) \leqslant C e^{S_{T_i}\hat{\varphi}(y_i)} \hat{m}_\varepsilon(\hat{f}^{T^i} Y_i) \leqslant C e^{-\bar{\alpha}T_i} \hat{m}_\varepsilon(\hat{f}^{T_i} Y_i), \tag{7.16}$$

for any $y_i \in Y_i$, where $\bar{\alpha} > 0$ is from the proof of Theorem 4.10.

Fixing $Y_i \in \hat{H}'_\varepsilon(k)$, we wish to estimate $\#\{Y_j \in \hat{H}'_\varepsilon(k) : T_j = T_i \text{ and } \hat{f}^{T_i} Y_i \cap \hat{f}^{T_j} Y_j \neq \emptyset\}$. Due to our construction of the Hofbauer extension, such a Y_j is contained in a set $Z_j \in \hat{\mathcal{P}}_{T_j+L}$, such that f^{T_j} maps Z_j injectively into a connected component of $\hat{H}_\varepsilon$. Z_j can be associated with a word of length $T_j + L$, the first symbol of which lies in Y, while the remaining symbols lie in $\hat{I}_{z,\varepsilon_0,\varepsilon} \setminus Y$. We divide this word into blocks of length L, and note there are $\lfloor T_j/L \rfloor$ of them. They are all *external blocks* according to the terminology of [**DoT**]. According to [**DoT**, Lemma 4.6], there are at most $16\mathfrak{d}^2 L^3$ external blocks of length L. In addition, since $f^{T_j} Z_j \subset \hat{H}_\varepsilon$, we may choose ε sufficiently small that any remaining symbols between $\lfloor T_j/L \rfloor L$ and T_j also belong to an external block of length L. Finally, there are at most $(2\mathfrak{d}L)^2$ choices for the first symbol of Z_j since this is an upper bound on the number of elements in $\hat{I}'_{z,\varepsilon_0,\varepsilon}(L)$. Putting these estimates together, we conclude that

$$\#\{Y_j \in \hat{H}'_\varepsilon(k) : T_j = T_i \text{ and } \hat{f}^{T_i} Y_i \cap \hat{f}^{T_j} Y_j \neq \emptyset\} \leqslant (2\mathfrak{d}L)^2 (16\mathfrak{d}^2 L^3)^{T_j/L+1} \leqslant C e^{\xi T_j}, \tag{7.17}$$

where $\xi < \bar{\alpha}$ (by choice of L) is the same as in the proof of Theorem 4.10.

Next, due to the aperiodicity of z and the continuity of f, for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\hat{f}^j \hat{H}_\varepsilon \cap \hat{H}_\varepsilon = \emptyset$ for all $j < N$ and $N(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This implies, in particular, that if $Y_i \in \hat{H}'_\varepsilon(k)$, then $T_i \geqslant (k-1)N$.

We organize our estimate for $Y_i \subset \hat{H}'_\varepsilon(k)$ by considering $\hat{H}'_\varepsilon(k) = \bigcup_{t\geqslant (k-1)N} \{Y_i \in \hat{H}'_\varepsilon(k) : T_i = t\}$. Then, using (7.16) and (7.17),

$$\begin{aligned} \sum_{\substack{Y_i\in \hat{H}'_\varepsilon(k) \\ T_i = t}} (k-1)\hat{\mu}_\varepsilon(Y_i) &\leq \sum_{\substack{Y_i\in \hat{H}'_\varepsilon(k) \\ T_i = t}} C(k-1) e^{-\bar{\alpha}t} \hat{m}_\varepsilon(\hat{f}^t(Y_i)) \\ &\leq C(k-1) e^{-(\bar{\alpha}-\xi)t} (t+L)^3 m_\varphi(H_\varepsilon), \end{aligned}$$

where, for the last inequality, we have used the fact that $\hat{f}^t(Y_i)$ lies in a component of $\hat{H}_\varepsilon$ on level at most $t + L$ in the Hofbauer extension. Since there are at most $\mathfrak{d}\ell^2$ connected components on level ℓ according to the proof of [**DoT**, Lemma 4.6], we obtain that, projecting $\hat{H}_\varepsilon|_{\text{level } \ell}$ down to H_ε, we have $\hat{m}_\varepsilon(\hat{H}_\varepsilon|_{\text{level } \ell}) \leqslant \mathfrak{d}\ell^2 m_\varphi(H_\varepsilon)$ and summing over $\ell \leqslant (t+L)$ yields the required bound.

Using this estimate in the double sum in (7.15), we obtain

$$\begin{aligned}\sum_{k\geqslant 2}\sum_{Y_i\subset \hat{H}'_\varepsilon(k)}(k-1)\hat{\mu}_\varepsilon(Y_i) &\leqslant \sum_{k\geqslant 2}\sum_{t\geqslant N(k-1)}\sum_{\substack{Y_i\in \hat{H}'_\varepsilon(k)\\ T_i=t}} Ce^{-\bar{\alpha}t}(k-1)\hat{m}_\varepsilon(\hat{f}^t Y_i)\\ &\leqslant \sum_{k\geqslant 2}\sum_{t\geqslant N(k-1)} Ce^{-(\bar{\alpha}-\xi)t}(k-1)(t+L)^3 m_\varphi(H_\varepsilon)\\ &\leqslant C' m_\varphi(H_\varepsilon)\sum_{k\geqslant 2} e^{-\alpha N(k-1)}(k-1)\leqslant C''\mu_\varphi(H_\varepsilon)e^{-\alpha N},\end{aligned} \tag{7.18}$$

where in the last step we have used Lemma 7.4. Combining this estimate with (7.15) and dividing through by $\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)$ (using that $\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)=\mu_\varphi(H_\varepsilon)$) yields,

$$\frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} = 1-\mathcal{O}(e^{-\alpha N}).$$

Since $N(\varepsilon)\to\infty$ as $\varepsilon\to 0$, this completes the proof of the lemma. □

Our next lemma addresses the case in which z is periodic with prime period p.

LEMMA 7.6. *Suppose z is a periodic point for f of prime period p, and that φ is Hölder continuous.*

(a) *If* $\{f^n(c) : c\in \text{Crit}, n\geqslant 1\}\cap\{z\}=\emptyset$, *then*

$$\lim_{\varepsilon\to 0}\frac{-\log\lambda_\varepsilon}{\mu_\varphi(H_\varepsilon)} = 1-e^{S_p\varphi(z)}.$$

(b) *Suppose* $\{f^n(c) : c\in \text{Crit}, n\geqslant 1\}\cap\{z\}\neq\emptyset$. *If, in addition, either f^p is orientation preserving in a neighbourhood of z, or* $\lim_{\varepsilon\to 0}(m_\varphi(z+\varepsilon, z)/m_\varphi(z, z-\varepsilon))=1$, *then* $\lim_{\varepsilon\to 0}((-\log\lambda_\varepsilon)/\mu_\varphi(H_\varepsilon)) = 1-e^{S_p\varphi(z)}$.

Proof. Fix N_0 arbitrarily large. Due to (4.7), we may choose $\varepsilon'>0$ sufficiently small so that for all $\varepsilon<\varepsilon'$, the following properties hold.

(i) If $y\in H_\varepsilon(z)$, then for all $j=1,\ldots,pN_0$, $f^j(y)\in H_\varepsilon(z)$ only if $j=kp$ for some $k=1,\ldots N_0$.

(ii) If $y\in H_\varepsilon(z)$ and there exists $k_1\leqslant N_0$ such that $f^{k_1p}(y)\in H_\varepsilon$, then $f^{kp}(y)\in H_\varepsilon(z)$ for all $k=1,\ldots,k_1$.

(iii) Each 1-cylinder $Y_i\subset\hat{H}'_\varepsilon$ whose first entry time ℓ to $\hat{H}_\varepsilon$ is less than N_0 is contained in an interval $Z_j\subset Q\in\mathcal{Q}$ such that $\hat{f}^\ell(Z_j)$ maps injectively onto a connected component of $\hat{H}_\varepsilon$, which we will denote by $\hat{H}_\varepsilon(Z_j)$.

(iv) $\hat{f}^{pN_0}$ is injective and continuous on each connected component of $\hat{H}_\varepsilon\cap\hat{f}^{-pN_0}(\hat{H}_\varepsilon)$ that occurs below level N_0 in $\hat{I}_{z,\varepsilon_0,\varepsilon}$.

Properties (i) and (ii) follow from the periodicity of z and the uniform continuity of f^n for each orbit segment of length $n\leqslant pN_0$. To deduce property (iii), since $f^k(z\pm\varepsilon_0)=z$ is not allowed by choice of ε_0 in (4.7), it suffices to choose

$$\varepsilon'\leqslant\tfrac{1}{2}\min\{d(z, f^k(w)) : w\in\text{Crit}_{z,\varepsilon_0}, f^k(w)\neq z, k\leqslant N_0\}.$$

With this choice of ε', no boundary points of $\hat{I}_{z,\varepsilon_0,\varepsilon}$ for $\varepsilon < \varepsilon'$ may fall in the interior of a connected component of $\hat{H}_\varepsilon$ with a first entry time less than N_0. Finally, property (iv) holds since the orbit of z must be disjoint from Crit; otherwise f would have an attracting periodic orbit, which is forbidden in our class of maps $\mathcal{F}$. Thus, we may choose

$$\varepsilon' \leqslant |Df^{pN_0}|_\infty^{-1} \min\{d(f^k(z), \text{Crit}) : k = 0, \ldots, p-1\},$$

in order to guarantee (iv).

Starting from (7.13), we group the 1-cylinders $Y_i \subset \hat{H}'_\varepsilon$ as follows. Let $\ell_i \in \mathbb{N}$ denote the greatest $\ell \leqslant N_0$ such that $\hat{f}^{\ell p}(Y_i) \subset \hat{H}_\varepsilon$. By (i) and (ii) above, if $j \leqslant pN_0$, then $\hat{f}^j(Y_i) \subset \hat{H}_\varepsilon$ if and only if $j = \ell p$ for some $\ell \leqslant \ell_i$. Recalling (7.14), we let $\hat{H}'_\varepsilon(k, N_0)$ denote the set of $Y_i \subset \hat{H}'_\varepsilon(k)$ such that the first entry of Y_i to $\hat{H}_\varepsilon$ occurs before time N_0, while $\hat{H}'_\varepsilon(k, \sim) = \hat{H}'_\varepsilon(k) \setminus \hat{H}'_\varepsilon(k, N_0)$. Moreover, $\hat{H}'_\varepsilon(*, N_0) := \bigcup_{k \geqslant 1} \hat{H}'_\varepsilon(k, N_0)$. Then

$$\hat{\mu}_\varepsilon(\hat{H}_\varepsilon) = \sum_{\ell=1}^{N_0} \sum_{\substack{Y_i \subset \hat{H}'_\varepsilon(*,N_0) \\ \ell_i = \ell}} \ell \hat{\mu}_\varepsilon(Y_i) + \sum_{k > N_0} \sum_{Y_i \in \hat{H}'_\varepsilon(k,N_0)} (k - \ell_i)\hat{\mu}_\varepsilon(Y_i)$$
$$+ \sum_{k \geqslant 1} \sum_{Y_i \in \hat{H}'_\varepsilon(k,\sim)} k \hat{\mu}_\varepsilon(Y_i). \tag{7.19}$$

Since the entry times to $\hat{H}_\varepsilon$ are greater than N_0 for each of the sets counted in the second and third sums above, we may use (7.16) and (7.18) to estimate that these two sums are of order $\mathcal{O}(e^{-\alpha N_0} \hat{\mu}_\varepsilon(\hat{H}_\varepsilon))$. It remains to estimate the first sum above. We rewrite (7.19) as

$$\hat{\mu}_\varepsilon(\hat{H}_\varepsilon) = \sum_{\ell=1}^{N_0} \sum_{\substack{Y_i \subset \hat{H}'_\varepsilon(*,N_0) \\ \ell_i \geqslant \ell}} \hat{\mu}_\varepsilon(Y_i) + \mathcal{O}(e^{-\alpha N_0} \hat{\mu}_\varepsilon(\hat{H}_\varepsilon)). \tag{7.20}$$

For $\ell = 1$, we have simply

$$\sum_{\substack{Y_i \subset \hat{H}'_\varepsilon(*,N_0) \\ \ell_i \geqslant 1}} \hat{\mu}_\varepsilon(Y_i) = \hat{\mu}_\varepsilon(\hat{H}'_\varepsilon) + \mathcal{O}(e^{-\alpha N_0} \hat{\mu}_\varepsilon(\hat{H}_\varepsilon)),$$

since any $Y_i \subset \hat{H}'_\varepsilon$ not counted in the sum for $\ell = 1$ has first entry time to $\hat{H}_\varepsilon$ greater than N_0.

To estimate the contribution for the terms corresponding to $\ell = 2$, we use property (iii) above. If $Y_i \subset \hat{H}_\varepsilon(*, N_0)$ with $\ell_i \geqslant 2$, then Y_i is contained in an interval Z_j such that $\hat{f}^k(Z_j)$ maps injectively onto a connected component of $\hat{H}_\varepsilon$ (for the first time) at some time $k = k(Z_j) \leqslant N_0$. Let us denote this component of $\hat{H}_\varepsilon$ by $\hat{H}_\varepsilon(Z_j)$. Let A_j denote those indices i for which $Y_i \subset Z_j$. Then

$$\sum_{\substack{Y_i \subset \hat{H}'_\varepsilon(*,N_0) \\ \ell_i \geqslant 2}} \hat{\mu}_\varepsilon(Y_i) = \sum_{Z_j} \hat{\mu}_\varepsilon(Z_j) \sum_{i \in A_j} \frac{\hat{\mu}_\varepsilon(Y_i)}{\hat{\mu}_\varepsilon(Z_j)}.$$

Notice that, since $\ell = 2$, for each $i \in A_j$, $\hat{f}^{k(Z_j)}(Y_i) \subset \hat{H}_\varepsilon(Z_j) \cap \hat{f}^{-p}(\hat{H}_\varepsilon)$. Recalling Lemma 7.1 and the conformality of $\hat{m}_\varepsilon$ (recall that $\hat{m} = \hat{m}_\varepsilon$ depends on ε on $\hat{I}_{z,\varepsilon_0,\varepsilon} \setminus Y$), we estimate†

$$\sum_{i\in A_j}\frac{\hat\mu_\varepsilon(Y_i)}{\hat\mu_\varepsilon(Z_j)}=\sum_{i\in A_j}e^{\pm 2C_d\operatorname{diam}(Z_j)^\eta}\frac{\int_{\hat f^k(Y_i)}e^{S_k\hat\varphi\circ\hat f_j^{-k}}\,d\hat m_\varepsilon}{\int_{\hat f^k(Z_j)}e^{S_k\hat\varphi\circ\hat f_j^{-k}}\,d\hat m_\varepsilon}$$
$$=e^{\pm 2C_d\operatorname{diam}(Z_j)^\eta}P_\varepsilon(S_k\hat\varphi(Z_j))^{\pm 2}\frac{\hat m_\varepsilon(\hat H_\varepsilon(Z_j)\cap\hat f^{-p}(\hat H_\varepsilon))}{\hat m_\varepsilon(\hat H_\varepsilon(Z_j))},\tag{7.21}$$

where $\hat f_j^{-k}$ is the inverse branch of $\hat f^k|_{Z_j}$ and

$$P_\varepsilon(S_k\hat\varphi(Z_j))=\sup_{x,y\in Z_j}e^{S_k\hat\varphi(x)-S_k\hat\varphi(y)}.$$

Recall that, since we cut at $f^{-1}(z)$ and $f^{-1}(z\pm\varepsilon)$, during our construction of $\hat I_{z,\varepsilon_0,\varepsilon}$, $\hat H_\varepsilon(Z_j)$ must satisfy either $\pi(\hat H_\varepsilon(Z_j))=(z,z+\varepsilon)$ or $\pi(\hat H_\varepsilon(Z_j))=(z-\varepsilon,z)$. Let us denote these intervals above half the hole by $\hat H_\varepsilon(Z_j)^+$ or $\hat H_\varepsilon(Z_j)^-$, accordingly. Since $\hat f^p$ is continuous and injective on $\hat H_\varepsilon(Z_j)$ by (iv), $\hat f^p(\hat H_\varepsilon(Z_j)\cap\hat f^{-p}(\hat H_\varepsilon))$ contains a full interval in the fibre above half the hole (possibly different from $\hat H_\varepsilon(Z_j)$), which we can also denote by $+$ or $-$ as appropriate. Note that the conformal measure of all the lifts of the right half hole $(z,z+\varepsilon)$ have the same measure, and so do all the lifts of the left half hole.

We proceed to prove item (b) of the lemma first. If f^p is orientation preserving at z then, using conformality and bounded distortion, we have, on either half of the hole,

$$\hat m_\varepsilon(\hat H_\varepsilon(Z_j)^\pm\cap\hat f^{-p}(\hat H_\varepsilon))=P_\varepsilon(S_p\varphi(H_\varepsilon(z)))^{\pm 1}e^{S_p\varphi(z)}\hat m_\varepsilon(\hat H_\varepsilon(Z_j)^\pm),\tag{7.22}$$

where we have used the fact that $S_p\hat\varphi(z)=S_p\varphi(z)$. On the other hand, if f^p is orientation reversing at z, then we are left with, for example, the right half hole mapping onto the left half,

$$\hat m_\varepsilon(\hat H_\varepsilon(Z_j)^+\cap\hat f^{-p}(\hat H_\varepsilon))=P_\varepsilon(S_p\varphi(H_\varepsilon(z)))^{\pm 1}e^{S_p\varphi(z)}\hat m_\varepsilon(\hat H_\varepsilon(Z_j)^-),$$

and so, to conclude the desired cancellation in (7.21), we use the assumption $\lim_{\varepsilon\to 0}(m_\varphi(z,z+\varepsilon)/m_\varphi(z-\varepsilon,z))=1$.

Thus, under either alternative in item (b), we combine the estimates in (7.21) to write

$$\sum_{\substack{Y_i\subset\hat H'_\varepsilon(*,N_0)\\ \ell_i\geqslant 2}}\hat\mu_\varepsilon(Y_i)=\sum_{Z_j}\hat\mu_\varepsilon(Z_j)e^{\pm 2C_d\operatorname{diam}(Z_j)^\eta}P_\varepsilon(S_k\hat\varphi(Z_j))^{\pm 2}P_\varepsilon(S_p\varphi(H_\varepsilon(z)))^{\pm 1}e^{S_p\varphi(z)}$$
$$=e^{\pm 2C_d\max_j\operatorname{diam}(Z_j)^\eta}P_\varepsilon(S_k\hat\varphi(Z_j))^{\pm 2}P_\varepsilon(S_p\varphi(H_\varepsilon(z)))^{\pm 1}e^{S_p\varphi(z)}\hat\mu_\varepsilon(H'_\varepsilon)$$
$$+\mathcal O(e^{-\alpha N_0}\hat\mu_\varepsilon(\hat H_\varepsilon)).$$

Analogous estimates follow for each $\ell\geqslant 3$. Then, using that $e^{S_{\ell p}\varphi(z)}=e^{\ell S_p(z)}$, we estimate (7.20):

$$\hat\mu_\varepsilon(\hat H_\varepsilon)=\sum_{\ell=1}^{N_0}e^{\pm 2C_d\max_j\operatorname{diam}(Z_j)^\eta}P_\varepsilon(S_k\hat\varphi(Z_j))^{\pm 2}P_\varepsilon(S_{\ell p}\varphi(H_\varepsilon(z)))^{\pm 1}e^{\ell S_p\varphi(z)}\hat\mu_\varepsilon(H'_\varepsilon)$$
$$+\mathcal O(N_0e^{-\alpha N_0}\hat\mu_\varepsilon(\hat H_\varepsilon)).\tag{7.23}$$

† We use the notation $a=C^{\pm 1}b$ to mean $C^{-1}b\leqslant a\leqslant Cb$ for some constant $C\geqslant 1$.

Since N_0 is fixed, $Z_j \subset Y$ and the first entry of Z_j to $\hat{H}_\varepsilon$ occurs before time N_0, we have $\max_j \operatorname{diam}(Z_j) \to 0$ as $\varepsilon \to 0$. In addition, both $P_\varepsilon(S_k\hat{\varphi}(Z_j))$ and $P_\varepsilon(S_{\ell p}\varphi(H_\varepsilon(z)))$ approach 1 as $\varepsilon \to 0$ since the lengths of the orbit segments are uniformly bounded by pN_0 and $\hat{\varphi}$ is continuous along each orbit segment. Dividing by $\hat{\mu}_\varepsilon(H_\varepsilon)$ and taking the limit $\varepsilon \to 0$ yields, for each $N_0 > 0$,

$$1 = \sum_{\ell=1}^{N_0} e^{\ell S_p\varphi(z)} \lim_{\varepsilon\to 0} \frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} + \mathcal{O}(N_0 e^{-\alpha N_0}). \tag{7.24}$$

Finally, taking $N_0 \to \infty$ proves item (b) of the lemma.

The proof of item (a) proceeds similarly, starting from (7.21). Now, however, since z is disjoint from the post-critical orbit, we may choose $\varepsilon > 0$ sufficiently small that $f^n(c) \notin H_\varepsilon$ for all $n \leqslant pN_0$ and $c \in \text{Crit}$. Then the interval Z_j from (iii) can be chosen so that $\pi(\hat{f}^{k(Z_j)}(Z_j)) = (z-\varepsilon, z+\varepsilon)$, that is, $\hat{f}^{k(Z_j)}(Z_j)$ covers a level of the fibre above the full hole. Thus we may combine the left and right halves of the hole to obtain the analogue of (7.22) in this case,

$$\hat{m}_\varepsilon(\hat{H}_\varepsilon(Z_j) \cap \hat{f}^{-p}(\hat{H}_\varepsilon)) = P_\varepsilon(S_p\varphi(H_\varepsilon(z)))^{\pm 1} e^{S_p\varphi(z)} \hat{m}_\varepsilon(\hat{H}_\varepsilon(Z_j)), \tag{7.25}$$

and the orientation-preserving character of f^p at z is irrelevant. The proof of item (a) of the lemma is then complete, following (7.23) and (7.24) precisely as written. □

Now Lemmas 7.5 and 7.6, together with Theorem 7.2 and Proposition 7.3, complete the proof of Theorem 3.5, via (7.3).

7.4. *Final step of the proof of Theorem 3.7: the geometric case.* In this subsection we prove the third limit in (7.4) in the case where $\varphi = -t \log|Df| - p_t$, $t \in (t^-, t_1)$, where t_1 is defined by (7.28). We assume the slow approach condition (3.3) as well as the polynomial growth condition on the derivative along the post-critical orbit (3.2), formulated in §3.2.2.

We first prove an analogue of Lemma 7.4 in this case.

LEMMA 7.7. *If $f \in \mathcal{F}$, $\varphi = -t\log|Df| - p_t$ and z satisfies (3.3) with $t \in (t^-, t_1)$, then there exists $\zeta > 0$ such that for all $\varepsilon > 0$ sufficiently small, $\inf_{x\in H_\varepsilon(z)} (d\mu_\varphi/dm_\varphi)(x) \geqslant \zeta$, where μ_φ and m_φ are the relevant invariant and conformal measures.*

Proof. The proof is nearly identical to that of Lemma 7.4. While for the geometric potential with $t < 0$ it may be that $\inf_{x\in I} S_n\varphi(x) = -\infty$, the slow approach condition (3.3) ensures that for $x \in H_\varepsilon$ there is a finite lower bound on $S_n\varphi(x)$ that is uniform in ε, since n is fixed and independent of ε in (7.12). □

In order to prove the required convergence for geometric potentials, we will use the set-up and notation of [**BLS**]. It follows from (3.2) and our choice of γ_n that

$$\sum_n \gamma_n < \infty \quad \text{and} \quad \sum_{n\geqslant 1} (\gamma_n^{d-1} D_n(c))^{-1/d} < \infty \quad \text{for all } c \in \text{Crit}.$$

This is precisely the condition† required of f in [**BLS**].

† Indeed, this condition is equivalent to the simpler condition $\sum_{n\geqslant 1} D_n(c)^{-1/(2d-1)} < \infty$ [**BLS**, Lemma 2.1], but we use the formulation above in order to directly apply the results of [**BLS**]. Our condition (3.2) is slightly stronger and generalizes the exponent to values of $s_t < 1$.

We will not need the full strength of the results from [**BLS**]; rather, we will use the estimates on the recovery times for expansion for orbits that pass close to the set Crit. To this end, for a value of $\delta > 0$ to be specified later, we define $B_\delta(\text{Crit}) = \bigcup_{c \in \text{Crit}}(c - \delta, c + \delta)$, for $\delta > 0$. A key estimate of [**BLS**] is given by the following lemma.

LEMMA 7.8. [**BLS**, Lemma 2.4] *For $\delta > 0$ sufficiently small, there exist constants $C_\delta, \beta_\delta > 0$ such that for every orbit segment $\{f^i(x)\}_{i=0}^{k-1}$ such that $\{f^i(x)\}_{i=0}^{k-1} \cap B_\delta(\text{Crit}) = \emptyset$, we have*

$$|Df^k(x)| \geqslant C_\delta e^{\beta_\delta k}.$$

If, in addition, $f^k(x) \in B_\delta(\text{Crit})$, then there exists $\kappa > 0$ independent of δ such that

$$|Df^k(x)| \geqslant \max\{\kappa, C_\delta e^{\beta_\delta k}\}.$$

Next, we define the notion of binding period, recalling the sequence $(\gamma_n)_{n \in \mathbb{N}}$ from (3.2). If $x \in B_\delta(\text{Crit})$, then

$$b(x) = \max\{b \in \mathbb{N} : |f^k(x) - f^k(c)| \leqslant \gamma_k |f^k(c) - \text{Crit}| \ \forall k \leqslant b - 1\},$$

and $b(x) = 0$ if $x \notin B_\delta(\text{Crit})$. Let $I_b = \{x \in I : b(x) = b\}$ denote the level sets of b. The binding period will be useful in estimating the important quantity

$$Df^b_{\min}(c) := \min\{|Df^b(x)| : x \in I_b \cap B_\delta(c)\},$$

defined for each $c \in \text{Crit}$, which governs the minimum rate of growth in expansion along orbit segments.

Note that $b_\delta = \min\{b(x) : x \in B_\delta(\text{Crit})\}$ tends to ∞ as $\delta \to 0$. This fact is used in [**BLS**] to make $Df^{b_\delta}_{\min}(c)$ arbitrarily large by choosing $\delta > 0$ sufficiently small.

Each orbit of length n is assigned an itinerary $(\nu_1, b_1), (\nu_2, b_2), \ldots, (\nu_k, b_k)$, where each $\nu_i = \nu_i(x)$ represents the first time larger than $\nu_{i-1} + b_{i-1}$ such that the orbit of $x \in I$ makes a return to $B_\delta(\text{Crit})$. Each return i is called a deep return and placed in a set $S_d = S_d(x)$ if the orbit enters $B_\delta(\text{Crit})$ at time ν_i; it is called a shallow return and placed in a set $S_s(x)$ if the orbit does not enter $B_\delta(\text{Crit})$ at time ν_i, but is part of a dynamically defined interval that intersects $B_\delta(\text{Crit})$.

The key estimates from [**BLS**] using the information from binding periods are as follows.

LEMMA 7.9. [**BLS**, Lemmas 2.5 and 3.2]

(a) *There exists $C_0 > 0$ independent of $\delta > 0$ such that for all $c \in \text{Crit}$ and $b \geqslant b_\delta$ with $I_b \neq \emptyset$,*

$$Df^b_{\min}(c) \geqslant C_0(\gamma_b^{d-1} D_b(c))^{1/d}.$$

(b) *There exist $K_0 > 0$ and $\rho \in (0, 1)$, independent of δ, such that for an orbit $\{f^i(x)\}_{i=0}^{n-1}$ with a given sequence $(\nu_1, b_1), \ldots, (\nu_k, b_k)$ at time $n \geqslant \nu_k + b_k$, we have*

$$|Df^n(x)| \geqslant \max\left\{C_\delta^{\#S_d} e^{\beta_\delta(n - \sum_{i=0}^k b_i)}, \left(\frac{\kappa}{K_0}\right)^{\#S_d} \rho^{-\#S_s}\right\} \prod_{i \in S_d} Df^{b_i}_{\min}(c_i),$$

where c_i is the critical point associated to the return at time ν_i.

With these key estimates recalled, we are ready to begin our proofs of the relevant limits. As in §7.3, we begin with the aperiodic case.

LEMMA 7.10. *Suppose $f \in \mathcal{F}_d$ and $\phi = -t \log |Df|$ for $t \in (t^-, t_1)$ satisfies (3.2). Let z be an aperiodic point for f satisfying (3.3). Then*

$$\lim_{\varepsilon \to 0} \frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} = 1.$$

Proof. We will follow the strategy of the proof of Lemma 7.5, using the same notation defined there. Following (7.15), we must show as before that

$$\sum_{k \geqslant 2} \sum_{Y_i \subset \hat{H}'_\varepsilon(k)} (k-1)\hat{\mu}_\varepsilon(Y_i) = o(\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)). \tag{7.26}$$

However, the estimate in this case is not so simple since the analogous expression to (7.16) does not enjoy uniform exponential contraction in T_i. Rather, we split $Y_i \subset \hat{H}'_\varepsilon(k)$ into those cylinders which are 'bound' (i.e. in the midst of a binding period) at the time T_i of their kth entry to $\hat{H}_\varepsilon$, and those cylinders which are not bound, which we call 'free'.

As before, we fix $N \in \mathbb{N}$ and choose ε sufficiently small that $\hat{f}^j(\hat{H}_\varepsilon) \cap \hat{H}_\varepsilon = \emptyset$, for all $j < N$.

Estimate on free pieces. To estimate the contribution to (7.26) from cylinders that are free at time T_i, we begin as in (7.16):

$$\hat{\mu}_\varepsilon(Y_i) = C^{\pm 1}\hat{m}_\varepsilon(Y_i) = C^{\pm 1} e^{S_{T_i}\hat{\varphi}(y_i)}\hat{m}_\varepsilon(\hat{f}^{T_i} Y_i) = C^{\pm 1}|Df^{T_i}(y_i)|^{-t} e^{-p_t T_i}\hat{m}_\varepsilon(\hat{f}^{T_i} Y_i). \tag{7.27}$$

We estimate the above expression differently depending on whether $t < 1$ or $t \geqslant 1$. In all cases, we fix $\delta > 0$ sufficiently small that $Df_{\min}^{b_\delta}(c_i) \geqslant 2K_0/\kappa$.

If $t < 1$, then we consider the following two cases, depending on the itinerary $(\nu_1, b_1), \ldots, (\nu_{k_i}, b_{k_i})$ associated to y_i from time 0 until time T_i. Since $\hat{f}^{T_i}(Y_i)$ is free, we have $T_i \geqslant \nu_{k_i} + b_{k_i}$, so we may apply Lemma 7.9. Choose $\epsilon > 0$ sufficiently small that $\epsilon \leqslant \min\{\frac{1}{4}, -\beta_\delta b_\delta/(4 \log C_\delta)\}$.

Case 1. $\sum_{j=0}^{k_i} b_j > \epsilon T_i$. Using the second estimate in Lemma 7.9(b), we have, by choice of δ,

$$|D\hat{f}^{T_i}(y_i)| \geqslant 2^{\#S_d} = 2^{k_i}.$$

Case 2. $\sum_{j=0}^{k_i} b_j < \epsilon T_i$. It follows that $\#S_d(y_i) = k_i \leqslant \epsilon T_i/b_\delta$. Thus, using the first estimate in Lemma 7.9(b), we have, by our choice of ϵ,

$$|D\hat{f}^{T_i}(y_i)| \geqslant C_\delta^{\epsilon T_i/b_\delta} e^{\beta_\delta T_i(1-\epsilon)} \geqslant e^{\beta_\delta T_i/2}.$$

In either case, our estimate in (6.6) for $t < 1$ becomes,

$$\hat{m}(Y_i) \leqslant Ce^{-b_t T_i}\hat{m}(\hat{f}^{T_i} Y_i).$$

In the statement of Theorem 3.7 we only consider the case $t \geqslant 1$ under a (CE) condition along the critical orbits:

$$\text{there exist } C, \gamma > 0 \text{ such that } D_n(c) \geqslant Ce^{\gamma n}.$$

In this case, it suffices that $\gamma_n \geqslant e^{-\gamma n/(2(d-1))}$, so that $Df^b_{\min}(c) \geqslant C_0 e^{\gamma b/(2d)}$ by Lemma 7.9(a). For this range of $t \geqslant 1$, we consider two slightly different cases. Using the same choice of δ as above, we choose $\epsilon = -\bar{\beta}/\log(C_\delta C_0)$, where $\bar{\beta} = \frac{1}{2}\min\{\beta_\delta, \gamma/(2d)\}$.

Case 1. $k_i = \#S_d(y_i) \geqslant \epsilon T_i$. Using the second estimate in Lemma 7.9(b) and our choice of δ,

$$|D\hat{f}^{T_i}(y_i)| \geqslant \left(\frac{\kappa}{K_0}\right)^{k_i} \prod_{j\in S_d(y_i)} Df^{b_j}_{\min}(c_j) \geqslant 2^{\epsilon T_i}.$$

Case 2. $k_i \leqslant \epsilon T_i$. Using the first estimate in Lemma 7.9(b), we have, using our choice of ϵ,

$$|D\hat{f}^{T_i}(y_i)| \geqslant C_\delta^{\epsilon T_i} e^{\beta_\delta (T_i - \sum_{j=0}^{k_i} b_j)} C_0^{\epsilon T_i} e^{\gamma \sum_{j=0}^{k_i} b_j/(2d)} \geqslant e^{\bar{\beta} T_i}.$$

In either case, our estimate in (6.6) for $t \geqslant 1$ becomes

$$\hat{m}(Y_i) \leqslant C e^{-(t\beta_1 + p_t)T_i} \hat{m}(\hat{f}^{T_i} Y_i),$$

where $e^{\beta_1} = \min\{e^{\bar{\beta}}, 2^\epsilon\}$.

To unify notation, set $\hat{\alpha} = p_t$ when $t < 1$ in all cases, and $\hat{\alpha} = t\beta_1 + p_t$ in the (CE) case when $t \geqslant 1$. Recall that we defined $t_1 = 1$ in the non-(CE) case; in the (CE) case set

$$t_1 := \sup\{t \in (1, t^+) : t\beta_1 + p_t > 0\}, \tag{7.28}$$

noting that, since $p_1 = 0$, such a $t_1 > 1$ exists by continuity of p_t.

Now the above estimates in conjunction with the complexity estimate (7.17) yield, by (7.18),

$$\begin{aligned}\sum_{k\geqslant 2} \sum_{\substack{Y_i \subset \hat{H}'_\varepsilon(k)\\ Y_i \text{ free}}} (k-1)\hat{\mu}_\varepsilon(Y_i) &\leqslant \sum_{k\geqslant 2} \sum_{j\geqslant N(k-1)} C e^{-(\hat{\alpha}-\xi)j}(k-1)(j+L)^3 m_\varphi(H_\varepsilon) \\ &\leqslant C' \hat{\mu}_\varepsilon(\hat{H}_\varepsilon) e^{-(\hat{\alpha}-\xi)N},\end{aligned} \tag{7.29}$$

where we may choose L sufficiently large that $\xi < \hat{\alpha}$, and in the second inequality we have used Lemma 7.7 and the fact that $\hat{\mu}_\varepsilon(\hat{H}_\varepsilon) = \mu_\varphi(H_\varepsilon)$.

Estimate on bound pieces. Next we estimate the contribution to (7.26) from cylinders Y_i which are undergoing a bound period at time T_i. Let ν_i denote the time that Y_i enters this bound period. By assumption, $\nu_i \leqslant T_i < \nu_i + b_i$. Let $x \in \hat{f}^{\nu_i}(Y_i) \subset B_\delta(c)$. Then, using the slow approach condition (3.3) and the definition of b,

$$\begin{aligned}\delta_z \gamma_{T_i-\nu_i}^{1-\theta} &\leqslant |\hat{f}^{T_i-\nu_i}(c) - z| \leqslant |\hat{f}^{T_i-\nu_i}(c) - \hat{f}^{T_i-\nu_i}(x)| + |\hat{f}^{T_i-\nu_i}(x) - z| \\ &\leqslant \gamma_{T_i-\nu_i} + \tfrac{1}{2}|H_\varepsilon(z)|.\end{aligned}$$

This implies that

$$\delta_z \leqslant 2\max\{\gamma_{T_i-\nu_i}^\theta, \tfrac{1}{2}|H_\varepsilon(z)|\gamma_{T_i-\nu_i}^{\theta-1}\}.$$

We consider the ways in which this can be satisfied. First,

$$\delta_z \leqslant 2\gamma_{T_i-\nu_i}^\theta \implies \gamma_{T_i-\nu_i} \geqslant (\delta_z/2)^{1/\theta}.$$

Since γ_n is summable, this condition can be satisfied by only finitely many values of $T_i - \nu_i$, that depend only on γ_n, δ_z and θ. Indeed, we can render this set empty since (3.3) implies $\{f^n(c)\}_{n\geqslant 0} \cap \{z\} = \emptyset$. So, by choosing $\varepsilon, \delta > 0$ sufficiently small, we can make $f^k(B_\delta(\mathrm{Crit}))$ disjoint from H_ε for these finitely many iterates.

The second possibility is that

$$\delta_z \leqslant |H_\varepsilon(z)|\gamma_{T_i-\nu_i}^{\theta-1} \implies \gamma_{T_i-\nu_i} \leqslant \left(\frac{|H_\varepsilon(z)|}{\delta_z}\right)^{1/(1-\theta)}. \tag{7.30}$$

Recall from §3.2.2 that we defined $\gamma_n = n^{-r}$ for some $r > 1/s_t(1-\theta)$. Then (7.30) implies

$$T_i - \nu_i \geqslant \left(\frac{\delta_z}{|H_\varepsilon(z)|}\right)^{1/r(1-\theta)}.$$

This implies that the return time to Y for Y_i satisfies

$$R_i \geqslant \max\{(k-1)N, T_i\} \geqslant \frac{1}{2}\left((k-1)N + \left(\frac{\delta_z}{|H_\varepsilon(z)|}\right)^{1/r(1-\theta)}\right) =: \tau_\varepsilon,$$

where the first condition comes from the fact that $Y_i \subset \hat{H}'_\varepsilon(k)$ and N comes from the aperiodicity condition on z. Thus, using Theorem 4.10,

$$\begin{aligned}\sum_{k\geqslant 2} \sum_{\substack{Y_i \subset \hat{H}'_\varepsilon(k) \\ Y_i \text{ bound}}} (k-1)\hat{\mu}_\varepsilon(Y_i) &\leqslant \sum_{k\geqslant 2}\sum_{j\geqslant \tau_\varepsilon}\sum_{R_i=j}(k-1)\hat{\mu}_\varepsilon(Y_i) \\ &\leqslant \sum_{k\geqslant 2}\sum_{j\geqslant\tau_\varepsilon} C(k-1)e^{-\alpha j} \\ &\leqslant \sum_{k\geqslant 2}(k-1)C'e^{-\alpha(k-1)N/2}e^{-(\alpha/2)(\delta_z/|H_\varepsilon(z)|)^{1/r(1-\theta)}} \\ &= o(e^{-\alpha N/2}|H_\varepsilon(z)|^s) = o(\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)),\end{aligned} \tag{7.31}$$

where $s > 0$ represents any positive power, and the switch to $\hat{\mu}_\varepsilon(H_\varepsilon)$ is possible due to the scaling exponent s_t for the conformal measure m_φ as well as Lemma 7.7.

Combining (7.29) and (7.31) proves (7.26), which by (7.15) completes the proof of the lemma. □

Next, we address the case where z is periodic with prime period p. We continue to assume the slow approach condition (3.3).

LEMMA 7.11. *Suppose $f \in \mathcal{F}_d$ and $\phi = -t\log|Df|$ for $t \in (t^-, t_1)$ satisfies (3.2). Let z be a periodic point for f of prime period p satisfying (3.3). Then*

$$\lim_{\varepsilon\to 0}\frac{\hat{\mu}_\varepsilon(\hat{H}'_\varepsilon)}{\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)} = 1 - e^{S_p\varphi(z)}.$$

Proof. We follow the proof of Lemma 7.6, which needs few modifications now that we have recorded the relevant estimates over free and bound pieces.

Fix $N_0 > 0$ and choose $\varepsilon > 0$ sufficiently small that properties (i)–(iv) enumerated at the start of the proof of Lemma 7.6 hold. We expand $\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)$ precisely as in (7.19). First, we must show that the second and third sums in that expression are the error terms in the expansion.

As in the proof of Lemma 7.10, we call each Y_i bound or free depending on whether $\hat{f}^{T_i}(Y_i)$ is undergoing a bound period at time T_i or not. When summing over the free pieces, (7.29) implies that both sums are of order $\mathcal{O}(\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)e^{-(\hat{\alpha}-\xi)N_0})$ since the entry time for each such Y_i at $\hat{H}_\varepsilon$ is greater than N_0. Similarly, we estimate the second and third sums in (7.19) over bound pieces Y_i using the slow approach condition (3.3) so that, by (7.31), these sums are $\mathcal{O}(\hat{\mu}_\varepsilon(\hat{H}_\varepsilon)e^{-\alpha N_0/2})$. We thus arrive at equation (7.20) as before.

Next, we derive (7.21) as before since that uses only property (iii) and the uniform log-Hölder property of the invariant density g_ε (Lemma 7.1); we thus obtain the same expressions with the same definition of $P_\varepsilon(S_k\hat{\varphi}(Z_j))$.

Since the slow approach condition (3.3) implies that z is disjoint from the post-critical orbit, we may choose ε sufficiently small such that $f^k(c) \notin H_\varepsilon(z)$ for $k \leqslant pN_0$ and all $c \in \mathrm{Crit}$. Thus we may follow the proof of the simpler item (a) of Lemma 7.6, without having to consider the left and right halves of the hole separately. We use (7.25) to estimate the ratio in (7.21) and so arrive at (7.23) precisely as before.

Now $\hat{\varphi} = -t \log|Df| \circ \hat{\pi} - P(-t \log|Df|)$. Although $\hat{\varphi}$ is not continuous on $\hat{I}_{z,\varepsilon_0,\varepsilon}$, it is still true on each Y_i and for each orbit segment of length at most pN_0, that $S_k\hat{\varphi}$ is continuous with bounded ratio on Y_i and each component of $\hat{H}'_\varepsilon$ on level at most pN_0. This follows since we have trimmed L-cylinders in our construction of $Y = \hat{I}'(L)$. This extends to Z_j since $\hat{f}^{-pN_0}(\hat{c}) \cap Z_j = \emptyset$ for each Z_j by choice of ε, and so $P_\varepsilon(S_k\hat{\varphi}(Z_j)) \to 1$ as $\varepsilon \to 0$.

We thus arrive at (7.24) with error term $\mathcal{O}(N_0 e^{-\tilde{\alpha}N_0})$ and $\tilde{\alpha} = \min\{\hat{\alpha} - \xi, \alpha/2\}$, and taking $N_0 \to \infty$ completes the proof of the lemma. □

Finally, Lemmas 7.10 and 7.11 together with Theorem 7.2 and Proposition 7.3 complete the proof of Theorem 3.7, using (7.3).

Acknowledgements. Part of this work was completed at CIRM, Luminy, in 2017 and 2018, at ICMS, Edinburgh, UK in 2018, and during visits of MT to Fairfield University in 2017, 2018 and 2019. MD is partially supported by NSF grant DMS 1800321.

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