UNIQUENESS AND EXPONENTIAL MIXING FOR THE MEASURE OF MAXIMAL ENTROPY FOR PIECEWISE HYPERBOLIC MAPS

MARK F. DEMERS

Abstract. For a class of piecewise hyperbolic maps in two dimensions, we propose a combinatorial definition of topological entropy by counting the maximal, open, connected components of the phase space on which iterates of the map are smooth. We prove that this quantity dominates the measure theoretic entropies of all invariant probability measures of the system, and then construct an invariant measure whose entropy equals the proposed topological entropy. We prove that our measure is the unique measure of maximal entropy, that it is ergodic, gives positive measure to every open set, and has exponential decay of correlations against Hölder continuous functions. As a consequence, we also prove a lower bound on the rate of growth of periodic orbits. The main tool used in the paper is the construction of anisotropic Banach spaces of distributions on which the relevant weighted transfer operator has a spectral gap. We then construct our measure of maximal entropy by taking a product of left and right maximal eigenvectors of this operator.

1. Introduction

There has been a flurry of recent activity in establishing the existence and uniqueness of equilibrium states for broad classes of potentials and systems outside the uniformly hyperbolic setting. This topic traces back to the work of Margulis [Ma1], who proved that the number of periodic orbits of length at most L for the geodesic flow on a compact manifold of strictly negative curvature grow at an exponential rate determined by the topological entropy $h_{\rm top}$ of the flow. To prove this result, Margulis constructed an invariant measure $\mu_{\rm top}$ via conditional measures on the local stable and unstable manifolds of the flow which scaled by $e^{\pm t h_{\rm top}}$. An important feature of the measure $\mu_{\rm top}$ is that it is the unique measure of maximal entropy for the flow: its measure-theoretic entropy equals the topological entropy $h_{\rm top}$.

These results were generalized and further developed for broader classes of Anosov and Axiom A flows and diffeomorphisms through the work of Sinai, Bowen, Ruelle and many others using thermodynamic formalism [S, BR, Ru1], topological techniques [Bo1, Bo2, Bo3, Bo4], and dynamical zeta functions [PaP, Ru2]. Later, Dolgopyat's proof of exponential decay of correlations for some geodesic flows [Do] led to more precise asymptotics for counting periodic orbits [PS].

Recent attempts to extend proofs of existence and uniqueness of equilibrium states in general, and measures of maximal entropy in particular, to the nonuniformly hyperbolic setting have employed symbolic dynamics [Sa1, Sa2, LiM, BS], as well as adapting the approach of Bowen via a notion of non-uniform specification [BCFT, CFT, CKW, CPZ]. These works have greatly broadened the classes of systems for which one can prove the existence and uniqueness of equilibrium states, yet they do not usually provide rates of mixing for these measures.

Simultaneously, there have been advances made in the study of the transfer operator associated with hyperbolic systems with singularities, first to piecewise hyperbolic maps (with bounded derivatives) [DL, BG1, BG2], and then to dispersing and other hyperbolic billiards [DZ1, DZ2, DZ3]. This approach, which avoids the coding associated with Markov partitions or extensions, exploits the hyperbolicity of the system to prove that the action of the transfer operator on appropriately defined Banach spaces has good spectral properties. It was used recently to prove exponential decay

Date: March 9, 2020.

MD was partly supported by NSF grant DMS 1800321.

of correlations for the finite horizon Sinai billiard flow [BDL], adapting ideas of Dolgopyat [Do] and Liverani [L2]. It was then applied to prove the existence and uniqueness of a measure of maximal entropy for finite horizon Sinai billiard maps [BD], establishing a variational principle for this class of billiards.

For hyperbolic systems with discontinuities, a priori results that guarantee the existence of an invariant measure maximizing the entropy, or even a simple definition of topological entropy, are not available as they are for continuous maps and flows. Indeed, in order to overcome this shortcoming, one approach is to redefine the map as a continuous map on a noncompact space, and then apply generalized definitions of topological entropy in this setting. Yet such definitions can be cumbersome to work with, and the resulting entropy can depend on the choice of metric in the reduced space.

To simplify matters, the first step in [BD] is to define an intuitive notion of growth in complexity given by the number of domains of continuity $\mathcal{M}_0^n$ for the map T^n. This leads to an asymptotic quantity h_*, which plays the role of topological entropy [BD, Definition 2.1] (see also Definition 2.5 below). This quantity is proved to equal the supremum of the measure-theoretic entropies of the invariant measures for the billiard map, and the unique measure μ_* whose entropy achieves this maximum is constructed by taking a product of left and right maximal eigenvectors of an associated weighted transfer operator $\mathcal{L}$, following the methods in [GL] which generalize the classical Parry construction.

Despite this success, the weight in the relevant transfer operator in [BD] is unbounded due to the unbounded expansion and contraction that occur near grazing collisions in dispersing billiards. The presence of this weight forced significant changes in the Banach spaces from [DZ1] on which the operator acted, and it was not possible to establish a spectral gap in this context. Indeed, the rate of mixing for the measure of maximal entropy is an open question for billiards.

The purpose of the present paper is to demonstrate that under the additional assumption that the derivative of the map is bounded, the techniques employed in [BD] are sufficient to prove the existence and uniqueness of a measure of maximal entropy that is exponentially mixing. To this end, we study a class of piecewise hyperbolic maps, defined in Section 2. The existence and statistical properties of Sinai-Ruelle-Bowen (SRB) measures[1] for this class of maps has been studied via a variety of techniques [P, L1, Y, DL, BG2]. Transfer operators with more general potentials were studied in [BG2] and a bound on the essential spectral radius was obtained; however, lower bounds on the spectral radius of the transfer operator were not obtained, so that no spectral gap was established and the related invariant measures were not constructed. Currently there are no results regarding measures of maximal entropy, nor more general equilibrium states for this class of maps.

In structure, this paper mainly follows the approach in [BD]. Yet there are several key differences between the class of piecewise hyperbolic maps studied here and dispersing billiards. The primary simplification is that our maps have bounded derivatives, as mentioned above, and this fact permits us to prove a spectral gap for the relevant transfer operator, which leads to exponential decay of correlations for the measure of maximal entropy μ_*. However, there are two additional difficulties in the current setting that are not present in Sinai billiards.

(i) We do not assume that the singularity curves for our map T satisfy the *continuation of singularities* property enjoyed by billiards.
(ii) We do not assume the map is associated with a continuous flow.

Point (i) creates significant complications in the study of the rate of growth of $\#\mathcal{M}_0^n$, the number of maximal, connected domains of continuity of T^n. In particular, the submultiplicative property of $\#\mathcal{M}_0^n$ proved in [BD, Lemma 3.3], and often exploited in that work, may fail in the present context due to the fact that dynamical refinements of $\#\mathcal{M}_0^n$ may have elements that are not simply connected. Indeed, the uniform exponential upper and lower bounds on $\#\mathcal{M}_0^n$ proved in

[1]Recall that an SRB measure for a hyperbolic system is an invariant probability measure whose conditional measures on local unstable manifolds are absolutely continuous with respect to the Riemannian volume.

Proposition 2.12 are completed only after the spectral gap for the operator $\mathcal{L}$ is established. Point (ii) has several consequences. The first is that the continuous flow provides a linear bound on the growth in complexity, which is exploited in [BD]. In the present work, this property is replaced by the complexity assumption (P1) introduced in Section 2.1; while the growth in complexity may be exponential for our class of maps, it is slow relative to the minimum hyperbolicity constant for the map (and therefore also relative to h_* by Lemma 3.6(d)). The second consequence of (ii) is that in [BD], there is a positive minimum distance between orbits that belong to different elements of $\mathcal{M}_0^n$. In the present context this may fail, so in Section 2.1 we define an adapted metric which we use to define the dynamical Bowen balls instrumental in the estimation of the entropy of μ_* in Section 5.

The structure of the paper is as follows. We begin by defining in Definition 2.5 the exponential rate of growth in complexity, h_*, which counts the number of domains of continuity $\mathcal{M}_0^n$ of T^n. This quantity dominates the measure-theoretic entropies of the invariant measures (Theorem 2.8). We then proceed to study the action of a weighted transfer operator, defined in Section 3.2. The Banach spaces we use are similar to those defined in [DL] (not [BD]) for this class of maps, yet the operator has significant differences from the transfer operator with respect to the SRB measure studied in [DL]. By proving a series of growth and fragmentation lemmas in Sections 3.5 and 3.6 that control the prevalence of short and long connected components of $T^{-n}W$ for local stable manifolds W, we are able to establish that the operator has a spectral gap in Section 4. Finally, in Section 5, we construct a measure μ_* out of the left and right eigenvectors of the transfer operator and show that it has exponential decay of correlations and that it is the unique invariant measure with entropy equal to h_*. The properties of the measure μ_* are summarized in Theorem 2.9. In Corollary 2.11, we derive our asymptotic bound on the growth rate of periodic orbits, applying results of [LiM] and [Bu]. Finally, as a byproduct of this approach, uniform growth rates are established for $\#\mathcal{M}_0^n$ and the length $|T^{-n}W|$ of stable manifolds W; these are stated in Proposition 2.12 and Corollary 2.13, respectively.

2. Setting, Definitions and Results

In this section, we introduce a set of formal assumptions on our class of piecewise hyperbolic maps and state the principal results of the paper.

2.1. **Piecewise Hyperbolic Maps.**

Let M be a compact two-dimensional Riemannian manifold, possibly with boundary and not necessarily connected, and let $T : M \circlearrowleft$ be a piecewise uniformly hyperbolic map in the sense described below. There exist a finite number of pairwise disjoint open, simply connected regions $\{M_i^+\}_{i=1}^d$ such that $\cup_i \overline{M_i^+} = M$ and ∂M_i^+ comprises finitely many $\mathcal{C}^1$ curves of finite length. We will refer to $\mathcal{S}^+ = M \setminus \cup_i M_i^+$ as the singularity set for T.

Define $M_i^- = T(M_i^+)$. We assume that $\cup_i \overline{M_i^-} = M$ and refer to the set $\mathcal{S}^- = M \setminus \cup_i M_i^-$ as the singularity set for T^{-1}. We require that $T \in \mathrm{Diff}^2(M \setminus \mathcal{S}^+, M \setminus \mathcal{S}^-)$ and that on each M_i^+, T has a $\mathcal{C}^2$ extension[2] to $\overline{M_i^+}$. Since the extension of T is defined on ∂M_i^+, we will write $T(\mathcal{S}^+)$ to denote the set of images of these boundary curves (on which the extension of T may be multi-valued). In this notation, $T(\mathcal{S}^+) = \mathcal{S}^-$ and $T^{-1}(\mathcal{S}^-) = \mathcal{S}^+$.

On each M_i, T is uniformly hyperbolic: i.e., there exist constants $\Lambda > 1$, $\kappa \in (0,1)$ and two DT-strictly-invariant families of cones C^u and C^s, continuous in each $\overline{M_i^+}$ which satisfy,

[2] This implies in particular that $\|DT\|$ is bounded on each M_i^+, so that this class of maps does not include dispersing billiards.

$DT(x)C^u(x) \subsetneq C^u(Tx)$, $DT^{-1}(x)C^s(x) \subsetneq C^s(T^{-1}x)$, and

$$\inf_{x \in M \setminus \mathcal{S}^+} \inf_{v \in C^u} \frac{\|DTv\|}{\|v\|} \geq \Lambda, \quad \inf_{x \in M \setminus \mathcal{S}^-} \inf_{v \in C^s} \frac{\|DT^{-1}v\|}{\|v\|} \geq \Lambda, \tag{2.1}$$
$$\text{and} \quad \kappa := \inf_{x \in M \setminus \mathcal{S}^+} \inf_{v \in C^s} \frac{\|DTv\|}{\|v\|}.$$

The strict invariance of the cone field together with the smoothness properties of the map implies that the stable and unstable directions are well-defined for each point whose trajectory does not meet a singularity line.

In Section 3.1, we define narrower cones with the same names and refer to them as the stable and unstable cones of T respectively. We assume the following uniform transversality properties: there is a uniform positive lower bound on the angle between vectors in $C^s(x)$ and $C^u(x)$ for all $x \in M \setminus \mathcal{S}^+$, the tangent vectors to the singularity curves in $\mathcal{S}^-$ are bounded away from C^s, and those of $\mathcal{S}^+$ are bounded away from C^u; lastly, curves in $\mathcal{S}^-$ either coincide with, or are uniformly transverse to, curves in $\mathcal{S}^+$. As mentioned in the introduction, this class of maps is similar to that studied in [P, L1, Y, DL, BG2]; see also [LW] for the symplectic case.

Convention 2.1. *(Doubling boundary points.) It will be convenient in what follows to have T defined pointwise on M, but a priori it is defined only on $\cup_i M_i^+$. Since T is C^2 up to the closure of each M_i^+, we may extend T to be defined on ∂M_i^+, making T multivalued where these boundaries overlap. Following [L1], we adopt the convention that the image of such a subset of M under T contains all such points, and continue to call this extended space M.*

We remark that although this convention is made for convenience, it follows from Theorem 2.9(a) that the measure μ_* is independent of how T is defined on ∂M_i^+.

Let $d(\cdot,\cdot)$ denote the Riemannian metric on M. The following related metric is better adapted to the dynamics. Define

$$\bar{d}(x,y) = d(x,y), \quad \text{whenever } x, y \text{ belong to the same component } \overline{M}_i^+, \tag{2.2}$$

and $\bar{d}(x,y) = 10\,\mathrm{diam}(M)$ otherwise. Since we have doubled boundary points in M according to Convention 2.1, the extended space M is compact in the metric $\bar{d}$.

Denote by $\mathcal{S}_n^+ = \cup_{i=0}^{n-1} T^{-i}\mathcal{S}^+$ the set of singularity curves for T^n and by $\mathcal{S}_n^- = \cup_{i=0}^{n-1} T^i\mathcal{S}^-$ the set of singularity curves for T^{-n}. Let $K(n)$ denote the maximum number of singularity curves in $\mathcal{S}_n^-$ or in $\mathcal{S}_n^+$ which intersect at a single point. We make the following assumption regarding the complexity of T.

(P1) There exist $\alpha_0 > 0$ and an integer $n_0 > 0$, such that $\Lambda\kappa^{\alpha_0} > 1$ and $(\Lambda\kappa^{\alpha_0})^{n_0} > K(n_0)$.

Condition (P1) can always be satisfied if $K(n)$ has polynomial growth (as is the case with a Sinai billiard on a torus); however, since (P1) is required only for some fixed n_0, it is not necessary to control $K(n)$ for all n in order to verify the condition.

Remark 2.2. *If property (P1) holds for α_0, then it holds for all $0 < \alpha < \alpha_0$ with the same n_0. Notice also that $K(kn_0) \leq K(n_0)^k$ which implies that the inequality in (P1) can be iterated to make $(\Lambda\kappa^{\alpha_0})^{-kn_0}K(kn_0)$ arbitrarily small once (P1) is satisfied for some n_0.*

In Section 3.1 we will define a set of admissible stable curves $\widehat{\mathcal{W}}^s$, with tangent vectors belonging to the stable cone, which we will use to define our norms. For $W \in \widehat{\mathcal{W}}^s$, let K_n denote the number of smooth connected components of $T^{-n}W$. For a fixed N, by shrinking the maximum length δ_0 of leaves in $\widehat{\mathcal{W}}^s$, we can require that $K_N \leq K(N) + 1$. This implies that choosing $N = kn_0$, we can make $(\Lambda\kappa^{\alpha_0})^{-N}K_N$ arbitrarily small.

Convention 2.3. *In what follows, we will assume that $n_0 = 1$. If this is not the case, we may always consider a higher iterate of T for which this is so by assumption (P1). We then choose δ_0 small enough that $K_1 \Lambda^{-1} \kappa^{-\alpha_0} =: \rho < 1$.*

We also assume the following.

(P2) T is topologically mixing and preserves a unique smooth invariant measure μ_{SRB}, i.e. there exists $f_{\mathrm{SRB}} \in \mathcal{C}^1(M_i^+)$ for each i such that $d\mu_{\mathrm{SRB}} = f_{\mathrm{SRB}} dm$, where m denotes the Riemannian volume on M.

Remark 2.4. *Property (P1) is standard for piecewise hyperbolic maps, and a variant of it has been used in* [P, L1, Y, DL, BG2]. *The most common form is only to require the complexity bound in one direction, for example on $\mathcal{S}_n^-$ in* [L1, DL]. *Here, we assume the symmetric version on both $\mathcal{S}_n^-$ and $\mathcal{S}_n^+$ in order to prove the super-multiplicativity property for $\#\mathcal{M}_0^n$, Proposition 3.12. In fact, the requirement for $\mathcal{S}_n^+$ is used only in the proof of Lemma 3.9.*

It follows from the piecewise hyperbolicity of T and (P1) that T admits an SRB measure [P, Theorem 1]. *The requirement that μ_{SRB} be smooth in Property (P2) is less essential to our argument. We use μ_{SRB} as our reference measure rather than the Riemannian volume m in order to simplify the estimates involving the transfer operator. Assuming that μ_{SRB} is smooth allows us to prove the embedding lemma, Lemma 3.3, connecting our Banach spaces to the standard spaces of distributions.*

Our assumptions on the hyperbolicity of T imply the following uniform expansion and bounded distortion properties along stable curves, which we record for future use. There exists $C_e > 0$ such that for any $W \in \widehat{\mathcal{W}}^s$ and $n \geq 0$,

$$|T^{-n}W| \geq C_e \Lambda^n |W| \,, \tag{2.3}$$

where $|W|$ denotes the arc length of W in the metric induced by the Riemannian metric on M.

Suppose $W \in \widehat{\mathcal{W}}^s$ is such that T^n is smooth on W and $T^iW \in \widehat{\mathcal{W}}^s$, for $i = 0, \ldots, n$. We denote by $J_W T^n$ the Jacobian of T^n along W with respect to arc length. There exists $C_d > 0$, independent of W, such that for all $x, y \in W$ and all $n \geq 0$,

$$\left| \frac{J_W T^n(x)}{J_W T^n(y)} - 1 \right| \leq C_d d_W(x, y) \,, \tag{2.4}$$

where $d_W(\cdot, \cdot)$ denotes arc length distance along W.

2.2. **A Definition of Topological Entropy.** Following [BD], for $k, n \geq 0$, let $\mathcal{M}_{-k}^n$ denote the set of maximal connected components of $M \setminus (\mathcal{S}_n^+ \cup \mathcal{S}_k^-)$, where we define $\mathcal{S}_0^{\pm} = \emptyset$. Note that by definition, elements of $\mathcal{M}_{-k}^n$ are open in M. With this notation, $\mathcal{M}_0^n$ denotes the set of maximal, open, simply connected components of M on which T^n is continuous, while $\mathcal{M}_{-n}^0$ has the analogous property for T^{-n}. We remark also that the requirement that each M_i^+ be open and simply connected prevents the partition $\mathcal{M}_0^1$ from being trivial, and implies in particular that the diameter of elements of $\mathcal{M}_{-n}^n$ tends to 0 as n gets large.[3] Similarly, the transversality assumptions coupled with the finiteness requirement on the number of smooth curves in $\mathcal{S}^+$ guarantee that $\#\mathcal{M}_{-k}^n$ is finite for each k and n.

Definition 2.5. *(Topological entropy of T.) Define* $h_*(T) = \limsup_{n \to \infty} \frac{1}{n} \log (\#\mathcal{M}_0^n)$.

[3] Thus, if one wants to apply the present results to a smooth map, for example a toral automorphism, one should first partition the torus into a finite number of simply connected 'rectangles' with boundaries transverse to C^u and C^s. Then Theorem 2.9 implies that the rate of growth in cardinality of dynamical refinements of this partition, h_*, will equal the topological entropy of the automorphism.

By definition, if $A \in \mathcal{M}_0^n$, then $T^n A \in \mathcal{M}_{-n}^0$, so that $\#\mathcal{M}_0^n = \#\mathcal{M}_{-n}^0$. Thus $h_*(T) = h_*(T^{-1})$, i.e. this definition is symmetric in time. Indeed, the limsup in the definition is in fact a limit, which follows from Proposition prop:MOn 2.12.

We begin by establishing that the quantity h_* is finite.

lem:finite **Lemma 2.6.** *For a piecewise hyperbolic map T as defined in Section sec:pw 2.1, but not necessarily satisfying conditions (P1) and (P2), the quantity $h_* < \infty$.*

Proof. The elements of $\mathcal{M}_0^1$ are simply the domains M_i^+. For any $n \geq 1$, elements of $\mathcal{M}_0^{n+1}$ are created by (the image under T^{-1} of) the connected components of the intersection of an element of $\mathcal{M}_0^n$ with one of the domains M_i^-. By assumption, $\mathcal{S}^+$ and $\mathcal{S}^-$ comprise finitely many $\mathcal{C}^1$ curves which either coincide or are uniformly transverse. Since T is $\mathcal{C}^2$ on the closure of each M_i^+, the same is true of the sets $\mathcal{S}_n^+$ and $\mathcal{S}^-$. Moreover, elements of $\mathcal{S}_n^+$ have a uniform bound (in n) on their derivative.

Consider the intersection $A \cap M_i^-$ for $A \in \mathcal{M}_0^n$. Connected components of this set are created by intersections of ∂A with elements of $\mathcal{S}^-$. Since $\partial A \subset \mathcal{S}_n^+$, by the compactness of M and uniform transversality, ∂A can intersect each smooth curve in $\mathcal{S}^-$ a finite number of times, with uniform upper bound $B > 0$ independent of n. Thus the number of connected components of $A \cap M_i^-$ is bounded by $B(\#\mathcal{S}^-)$. Since this bound holds for each $A \in \mathcal{M}_0^n$, we have

$$\#\mathcal{M}_0^{n+1} \leq (\#\mathcal{M}_0^n) B(\#\mathcal{S}^-) \leq d B^n (\#\mathcal{S}^-)^n ,$$

where d is the number of domains M_i^+. $\square$

In order to connect $h_* = h_*(T)$ to the dynamical refinements of a fixed partition, for each $k \in \mathbb{N}$, define $\mathcal{P}_k$ to be the maximal connected components of M on which T^k and T^{-k} are continuous. That is, $\mathcal{P}_k$ is the partition of M defined by $M \setminus (\mathcal{S}_k^+ \cup \mathcal{S}_k^-)$, together with the boundary curves associated to each element, according to Convention con:pointwise 2.1. If we let $\mathring{\mathcal{P}}_k$ denote the collection of interiors of elements of $\mathcal{P}_k$, then we have $\mathring{\mathcal{P}}_k = \mathcal{M}_{-k}^k$.

For $n \geq 1$, define $\mathcal{P}_k^n = \bigvee_{i=0}^n T^{-i}\mathcal{P}_k$. $\mathcal{P}_k^n$ is still a pointwise partition of M, yet its elements may not be open sets, and it may occur that $\mathcal{P}_k^n$ contains isolated points due to multiple boundary curves intersecting at one point. Furthermore, we do not assume that the elements of $\mathcal{P}_k^n$ are connected sets.[4] Thus, although the collection of interiors $\mathring{\mathcal{P}}_k^n$ is a partition of $M \setminus (\mathcal{S}_{k+n}^+ \cup \mathcal{S}_k^-)$, it may be that $\mathring{\mathcal{P}}_k^n \neq \mathcal{M}_{-k}^{k+n}$.

Our next lemma provides a rough upper bound on the number of isolated points that can be created by refinements of $\mathcal{P}_k$. Let $\#\mathcal{S}^\pm$ denote the number of smooth components of $\mathcal{S}^\pm$.

em:isolated **Lemma 2.7.** *For each $k, n \geq 1$, the number of isolated points in $\mathcal{P}_k^n$ is at most*

$$2(\#\mathcal{S}^- + \#\mathcal{S}^+) \sum_{j=1}^{k+n} \#\mathcal{M}_0^j.$$

Proof. By Convention con:pointwise 2.1, there are no isolated points in $\mathcal{P}_1$. Next, for each $n \geq 1$, at time n, isolated points in $\mathcal{P}_1^n$ can be produced by intersections of corner points in the boundary of $\mathcal{P}_1^{n-1}$ with elements of $\mathcal{S}^-$. Moreover, each pair of smooth curves $S \in \mathcal{S}_n^+$ and $S' \in \mathcal{S}^-$ intersect at most twice per element of $\mathcal{M}_0^n$. Thus the number of new isolated points created at time n is at most $2\#\mathcal{S}^- \#\mathcal{M}_0^n$. Applying this estimate inductively, we have

$$\text{number of isolated points in } \mathcal{P}_1^n \leq 2\#\mathcal{S}^- \sum_{j=1}^{n} \#\mathcal{M}_0^j .$$

[4]Contrast this with max [BD, Lemma 3.1], where the analogous construction yields connected elements due to the property of continuation of singularities enjoyed by dispersing billiards.

Next, for each k, applying a similar inductive argument to T^{-1}, we have

$$\begin{aligned}\text{number of isolated points in } \mathcal{P}_k &\le 2\#\mathcal{S}^+ \sum_{j=1}^{k} \#\mathcal{M}_{-j}^0 + 2\#\mathcal{S}^- \sum_{j=1}^{k} \#\mathcal{M}_0^j \\ &\le 2(\#\mathcal{S}^+ + \#\mathcal{S}^-) \sum_{j=1}^{k} \#\mathcal{M}_0^j \,,\end{aligned}$$

where we have used the fact that $\#\mathcal{M}_0^j = \#\mathcal{M}_{-j}^0$. Finally, refining $\mathcal{P}_k$, we create at most $2\#\mathcal{S}^-\#\mathcal{M}_0^{k+j}$ new isolated points in $\mathcal{P}_k^j$ at time j. Summing over $j \le n$, we complete the proof of the lemma. □

sec:main

2.3. **Statement of Main Results.** Our first result establishes a connection between the rates of growth of $\#\mathcal{P}_k^n$ and $\#\mathcal{M}_0^n$, and uses this to prove that h_* dominates the measure-theoretic entropies of the invariant measures of T.

thm:initial **Theorem 2.8.** *Let T be a piecewise hyperbolic map as defined in Section 2.1, but not necessarily satisfying conditions (P1) and (P2).*

a) *For each $k, n \ge 1$, $\#\mathring{\mathcal{P}}_k^n \le \#\mathcal{M}_{-k}^{k+n}$ and $\#\mathcal{P}_k^n \le C(k+n)\#\mathcal{M}_{-k}^{k+n}$, for some $C > 0$ depending only on T.*

b) *For all $k \ge 1$, $\limsup_{n\to\infty} \frac{1}{n} \log(\#\mathcal{M}_{-k}^n) = h_*$.*

c) $\sup_k \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}_k^n = \sup_k \lim_{n\to\infty} \frac{1}{n} \log \#\mathring{\mathcal{P}}_k^n \le h_*$.

d) $h_* \ge \sup\{h_\mu(T) : \mu$ *is an invariant probability measure for* $T\}$.

Proof. a) The first inequality is straightforward since by definition, both $\mathring{\mathcal{P}}_k^n$ and $\mathcal{M}_{-k}^{k+n}$ are partitions of $M \setminus (\mathcal{S}_{n+k}^+ \cup \mathcal{S}_k^-)$, yet $\mathring{\mathcal{P}}_k^n$ may have disconnected components. Thus $\mathcal{M}_{-k}^{k+n}$ is a refinement of $\mathring{\mathcal{P}}_k^n$. The second inequality follows by noting that $\#\mathcal{P}_k^n$ equals $\#\mathring{\mathcal{P}}_k^n$ plus isolated points, and then applying Lemma 2.7.

b) The value of the limsup is the same for each k since by definition, $A \in \mathcal{M}_{-k}^n$ if and only if $T^k A \in \mathcal{M}_0^{n+k}$. Thus $\#\mathcal{M}_{-k}^n = \#\mathcal{M}_0^{n+k}$.

c) We first remark that $\#\mathcal{P}_k^{n+m} \le \#\mathcal{P}_k^n \#\mathcal{P}_k^m$, and also $\#\mathring{\mathcal{P}}_k^{n+m} \le \#\mathring{\mathcal{P}}_k^n \#\mathring{\mathcal{P}}_k^m$ (which can be proved as in [BD, Lemma 3.3]), thus the two limits in part (c) exist by subadditivity. The fact that both limits are bounded by h_* follows from parts (a) and (b) of the theorem.

d) Let μ denote a T-invariant probability measure. The assumptions of uniform hyperbolicity imply that both T and T^{-1} are expansive with respect to the metric $\bar{d}$ defined in (2.2):

q:expansive (2.5) There exists $\varepsilon_0 > 0$ such that if $\bar{d}(T^j x, T^j y) < \varepsilon_0$ for all $j \in \mathbb{Z}$, then $x = y$.

By (2.1), the uniform transversality of stable and unstable cones, and the assumption that each M_i^+ is simply connected, the maximum diameter of elements of $\mathcal{M}_{-k}^k$ (and hence of $\mathcal{P}_k$) is bounded by $C\Lambda^{-k}$. Choosing k large enough that $C\Lambda^{-k} \le \varepsilon_0$, we conclude that $\mathcal{P}_k$ is a generator for T [W, Theorem 5.23]. Then by [W, Theorem 4.22],

$$h_\mu(T) = h_\mu(T, \mathcal{P}_k) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}_k^n) \le \lim_{n\to\infty} \frac{1}{n} \log(\#\mathcal{P}_k^n) \le h_* \,,$$

applying part (c) of the present theorem. Thus $h_\mu(T) \le h_*$. □

Next we state our main theorem, which requires the additional hypotheses (P1) and (P2).

Theorem 2.9. *Let T be a piecewise hyperbolic map as defined in Section 2.1, satisfying conditions (P1) and (P2).*

There exists a T-invariant probability measure μ_* with the following properties.

a) *The measure μ_* has no atoms, and there exists $C > 0$ such that for any $\varepsilon > 0$,*
$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}^\pm)) \le C\varepsilon^{1/p}\,,$$
where $p > 1$ is from (3.4) and $\mathcal{N}_\varepsilon(\cdot)$ denotes the ε-neighborhood of a set in the Riemannian metric on M. This implies in particular, that μ_*-a.e. $x \in M$ has a stable and unstable manifold of positive length, and that x approaches $\mathcal{S}^\pm$ at a subexponential rate.

b) *$\mu_*(O) > 0$ for any open set $O \subset M$.*

c) *(T^n, μ_*) is ergodic for all $n \in \mathbb{Z}^+$.*

d) *μ_* has exponential decay of correlations against Hölder continuous functions.*

e) *The measure μ_* is the unique T-invariant probability measure satisfying $h_{\mu_*}(T) = h_*$.*

Theorem 2.9 will be proved in Section 5. In particular, items (a)-(c) are proved in Section 5.1, item (d) is proved in Proposition 5.3, and item (e) is proved in Sections 5.2 and 5.3.

Corollary 2.10. *Let T be a piecewise hyperbolic map as defined in Section 2.1, satisfying conditions (P1) and (P2).*

T satisfies the following variational principle: For all $k \ge 0$,
$$\lim_{n\to\infty} \frac{1}{n} \log\left(\#\mathcal{M}^n_{-k}\right) = h_* = \sup\{h_\mu(T) : \mu \textit{ is an invariant probability measure for } T\}.$$

Proof. The fact that the limit defining h_* exists (rather than simply the lim sup from Definition 2.5) follows from Proposition 2.12, and the independence from k follows from Theorem 2.8(b). The second equality follows from Theorem 2.8(d) together with Theorem 2.9(e). □

Theorem 2.9(a) implies that $\int_M |\log d(x, \mathcal{S}^\pm)|\, d\mu_* < \infty$ (see Corollary 5.5(c)), so that μ_* is T-adapted in the language of [LiM]. This allows us to make the following connection to the growth of periodic orbits of T. Let $P_n(T) = \{x \in M : \#\{T^k x : k \in \mathbb{Z}\} = n\}$ denote the set of points of prime period n for T.

Corollary 2.11. *Under the assumptions of Theorem 2.9, $\liminf_{n\to\infty} \# P_n(T) e^{-nh_*} = 1$.*

Proof. The proof relies on the construction of a countable Markov partition for hyperbolic maps with singularities carried out in [LiM]. The class of maps in the present paper satisfy conditions (A1)-(A6) in [LiM], which are general enough to admit dispersing billiards. Since μ_* is T-adapted and hyperbolic (see Corollary 5.5), we may apply [LiM, Corollary 1.2] to conclude that there exist $p \ge 1$ and $C > 0$ such that the number of points of period np for T is at least Ce^{nph_*} for all n sufficiently large.

Next, applying [Bu, Main Theorem] as in [Bu, Theorem 1.5], we conclude that we may take $p = 1$ and asymptotically, $C = 1$ for large n. □

In the course of proving the growth lemmas in Section 3, we establish the following uniform bounds on the growth of $\#\mathcal{M}^n_0$, which may be of independent interest, and are needed for the proof of uniqueness in Section 5.3.

Proposition 2.12. *There exists a constant $C_\# > 0$ such that for all $n \ge 1$,*
$$C_\# e^{nh_*} \le \#\mathcal{M}^n_0 = \#\mathcal{M}^0_{-n} \le C_\#^{-1} e^{nh_*}\,.$$

Proof. The upper bound is Corollary 3.13, while the lower bound is Lemma 5.1. □

Corollary 2.13. *There exists $\bar{C} > 0$ such that for all stable curves $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$ and all $n \ge n_1$, where both $\delta_1 > 0$ and n_1 are from (3.9), we have*
$$\bar{C} e^{nh_*} \le |T^{-n}W| \le \bar{C}^{-1} e^{nh_*}\,.$$

Proof. Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$. We use the notation of Section 3.5 regarding the connected components $\mathcal{G}_n(W)$ of $T^{-n}W$. Lemma 3.6(b), Lemma 3.11 and Proposition 2.12 together yield,

$$c_0 C_\# e^{nh_*} \leq c_0 \# \mathcal{M}_0^n \leq \# \mathcal{G}_n(W) \leq C\delta_0^{-1} \# \mathcal{M}_0^n \leq C\delta_0^{-1} C_\#^{-1} e^{nh_*} .$$

Then on the one hand,

$$|T^{-n}W| = \sum_{W_i \in \mathcal{G}_n(W)} |W_i| \leq \delta_0 \# \mathcal{G}_n(W) ,$$

since each element of $\mathcal{G}_n(W)$ has length at most δ_0, completing the upper bound of the corollary. On the other hand, by (4.12),

$$|T^{-n}W| = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |W_i| \geq \tfrac{2\delta_1}{9} \# \mathcal{G}_n(W) ,$$

proving the lower bound. □

sec:banach

3. Banach Spaces and Growth Lemmas

In this section we define the Banach spaces we will use in the analysis of the transfer operator and prove several key lemmas controlling the growth in complexity of T^n.

:admissible

3.1. **Stable Curves.**

We begin with a definition of stable curves as graphs of functions in local charts, following [DL]. We will use the fact that the uniform hyperbolicity of T guarantees the existence of stable $E^s(x)$ and unstable $E^u(x)$ directions in the tangent space $\mathcal{T}_xM$ at Lebesgue-almost-every $x \in M$.

For τ sufficiently small, we define the stable cone at $x \in M$ by

$$\hat{C}^s(x) = \{u + v \in \mathcal{T}_xM : u \in E^s(x), v \perp E^s(x), \|v\| \leq \tau \|u\|\} .$$

Define $\hat{C}^u(x)$ analogously. These families of cones are strictly invariant, $DT^{-1}(x)\hat{C}^s(x) \subsetneq \hat{C}^s(T^{-1}x)$ and $DT(x)\hat{C}^u(x) \subsetneq C^u(Tx)$.

For each i, we choose a finite number of coordinate charts $\{\chi_j\}_{j=1}^L$, whose domains R_j are either $(-r_j, r_j)^2$ if χ_j maps only to the interior of M_i^+, or $(-r_j, r_j)$ restricted to one side of a piecewise $\mathcal{C}^1$ curve (the preimage of a piece of ∂M_i^+) which we place so that it passes through the origin. For each j, R_j has a centroid x_j, and χ_j satisfies,

(a) $D\chi_j(x_j)$ is an isometry;
(b) $D\chi_j(x_j) \cdot (\mathbb{R} \times 0) = E^s(\chi_j(x_j))$;
(c) The $\mathcal{C}^2$-norm of χ_j and its inverse are bounded by $1 + \tau$;
(d) There exists $c_j \in (\tau, 2\tau)$ such that the cone $C_j = \{u+v \in \mathbb{R}^2 : u \in \mathbb{R}\times\{0\}, v \in \{0\}\times\mathbb{R}, \|v\| \leq c_j\|u\|\}$ satisfies: For each $y \in R_j$ such that $\chi_j(y) \notin \mathcal{S}^-$, $D\chi_j(y)C_j \supset \hat{C}^s(\chi_j(y))$, and $DT^{-1}(D\chi_j(y)C_j) \subset \hat{C}^s(T^{-1}(\chi_j(y)))$;
(e) $M_i^+ \subset \cup_{j=1}^L \chi_j(R_j \cap (-\frac{r_j}{2}, \frac{r_j}{2})^2)$.

Choose $r_0 \leq \frac{1}{2}\min_j r_j$; r_0 may be further reduced later, depending on δ. Fix $B < \infty$ and consider the set of functions

$$\Xi := \{F \in \mathcal{C}^2([-r, r], \mathbb{R}) : r \in (0, r_0], F(0) = 0, |F|_{\mathcal{C}^1} \leq \tau, |F|_{\mathcal{C}^2} \leq B\} .$$

Define $I_r = (-r, r)$. For $x \in R_j \cap (-r_j/2, r_j/2)^2$ such that $x + (t, F(t)) \in R_j$ for $t \in I_r$, define $G(x, r, F)(t) := \chi_j(x + (t, F(t))$ for $t \in I_r$, i.e. $G(x, r, F)$ is the lift of the graph of F to M. To abbreviate notation, we will refer to $G(x, r, F)$ as G_F. It follows from the construction that $|G_F|_{\mathcal{C}^1} \leq (1 + \tau)^2$ and $G_F^{-1} \leq 1 + \tau$.

Our set of admissible stable curves is defined by,

$$\widehat{\mathcal{W}}^s := \{W = G(x, r, F)(I_r) : x \in R_j \cap (r_j/2, r_j/2)^2, r \leq r_0, F \in \Xi\} .$$

If necessary, we reduce r_0 so that $\sup_{W \in \widehat{\mathcal{W}}^s} |W| \le \delta_0$, where δ_0 is the length scale chosen in Convention 2.3. Due to the uniform hyperbolicity of T, if $T^{-n}\widehat{\mathcal{W}}^s$ represents the connected components of $T^{-n}W$ for $W \in \widehat{\mathcal{W}}^s$, then choosing B large enough, it follows that $T^{-n}\widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$, up to subdivision of long curves. With this choice of B, the set of real local stable manifolds of length at most δ_0, which we denote by $\mathcal{W}^s$, satisfies $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$.

Next, we define two notions of distance[5] which are used in the definition of our norms, namely the strong unstable norm. For two curves $W_1(\chi_{i_1}, x_1, r_1, F_1)$ and $W_2(\chi_{i_2}, x_2, r_2, F_2)$, we define the distance between them to be,

$$d_{\mathcal{W}^s}(W_1, W_2) = \eta(i_1, i_2) + |x_1 - x_2| + |r_1 - r_2| + |F_1 - F_2|_{\mathcal{C}^1(I_{r_1} \cap I_{r_2})},$$

where $\eta(i_1, i_2) = 0$ if $i_1 = i_2$ and $\eta(i_1, i_2) = \infty$ otherwise, i.e. we only compare curves in the same chart.

Given W_1, W_2 with $d_{\mathcal{W}^s}(W_1, W_2) < \infty$ and two functions $\psi_i \in \mathcal{C}^0(W_i)$, we define the distance between them to be

$$d_0(\psi_1, \psi_2) = |\psi_1 \circ G_{F_1} - \psi_2 \circ G_{F_2}|_{\mathcal{C}^0(I_{r_1} \cap I_{r_2})}\,.$$

3.2. **Transfer operator.** The main tool we will use to construct the measure of maximal entropy is a weighted transfer operator, $\mathcal{L}$. Because we do not have a conformal measure at our disposal a priori, we will define the transfer operator acting on distributions defined via local stable manifolds. Let $\widetilde{\mathcal{W}}^s$ denote the set of maximal connected local stable manifolds of T restricted to each M_i^+. Note that such manifolds have uniformly bounded length due to the the finite diameter of M and the assumption that M_i^+ is simply connected. Due to the uniform hyperbolicity of T, μ_{SRB}-almost every point in M has a stable manifold of positive length.

For any local stable manifold W, and $\alpha \in (0, 1]$, define the α-Hölder norm of a test function $\psi : M \to \mathbb{C}$ by

$$|\psi|_{\mathcal{C}^\alpha(W)} = |\psi|_{\mathcal{C}^0(W)} + H^\alpha_W(\psi) := \sup_W |\psi| + \sup_{x \ne y \in W} \frac{|\psi(x) - \psi(y)|}{d_W(x,y)^\alpha}\,, \tag{3.1}$$

where $d_W(\cdot,\cdot)$ denotes distance induced by the Riemannian metric restricted to W. Let $\tilde{\mathcal{C}}^\alpha(W)$ denote the set of functions in $\mathcal{C}^0(W)$ with finite $|\cdot|_{\mathcal{C}^\alpha(W)}$ norm. With this notation, $\tilde{\mathcal{C}}^1(W)$ denotes the set of Lipschitz functions on W.

Analogously, for each $n \ge 0$, define $H^\alpha_{\widetilde{\mathcal{W}}^s}(\psi) = \sup_{W \in \widetilde{\mathcal{W}}^s} H^\alpha_W(\psi)$, and

$$\tilde{\mathcal{C}}^\alpha(\widetilde{\mathcal{W}}^s) = \{\psi : M \to \mathbb{C} \mid |\psi|_\infty + H^\alpha_{\widetilde{\mathcal{W}}^s}(\psi) < \infty\}\,.$$

The set $\tilde{\mathcal{C}}^\alpha(\widetilde{\mathcal{W}}^s)$ together with the norm $|\psi|_{\mathcal{C}^\alpha(\widetilde{\mathcal{W}}^s)} := |\psi|_\infty + H^\alpha_{\widetilde{\mathcal{W}}^s}(\psi)$ is a Banach space.

Since stable manifolds cannot be cut under T^n, if $W \in \widetilde{\mathcal{W}}^s$, then $T^nW \subset V \in \widetilde{\mathcal{W}}^s$ for each $n \ge 0$. This together with the uniform hyperbolicity of T and (2.1) implies that if $\psi \in \mathcal{C}^\alpha(\widetilde{\mathcal{W}}^s)$, then $\psi \circ T \in \mathcal{C}^\alpha(\widetilde{\mathcal{W}}^s)$ (see also (4.4)).

Then if $f \in (\tilde{\mathcal{C}}^\alpha(\widetilde{\mathcal{W}}^s))^*$ belongs to the dual of $\mathcal{C}^\alpha(\widetilde{\mathcal{W}}^s)$, the operator $\mathcal{L} : (\tilde{\mathcal{C}}^\alpha(\widetilde{\mathcal{W}}^s))^* \to (\tilde{\mathcal{C}}^\alpha(\widetilde{\mathcal{W}}^s))^*$ is defined by,

$$\mathcal{L}f(\psi) = f\left(\frac{\psi \circ T}{J^sT}\right) \quad \forall \psi \in \mathcal{C}^\alpha(\widetilde{\mathcal{W}}^s)\,, \tag{3.2}$$

where J^sT denotes the stable Jacobian of T. By (2.4), it follows that[6] $J^sT^n \in \tilde{\mathcal{C}}^1(\widetilde{\mathcal{W}}^s)$ for each $n \ge 1$.

[5]Neither of these distances will satisfy the triangle inequality, but that is irrelevant for our purposes.

[6]For $x \in W \in \widetilde{\mathcal{W}}^s$, $J^sT(x) = J_WT(x)$.

If $f \in \mathcal{C}^0(M)$, then we identify f with a signed measure absolutely continuous with respect to μ_{SRB}. We denote this integration by,

$$f(\psi) = \int_M \psi \, f \, d\mu_{\mathrm{SRB}} \,,$$

for $\psi \in \mathcal{C}^0(M)$. With this identification, we consider $\mathcal{C}^0(M) \subset (\tilde{\mathcal{C}}^\alpha(\widetilde{\mathcal{W}}^s))^*$. Then also by (3.2), for any $n \geq 1$, $\mathcal{L}^n f$ is absolutely continuous with respect to μ_{SRB} with density,

$$\mathcal{L}^n f = \frac{f \circ T^{-n}}{J^s T^n \circ T^{-n}} \,. \tag{3.3}$$

3.3. **Definition of Norms.** Let $\mathcal{W}^s$ denote those local stable manifolds having length at most δ_0, where δ_0 is from Convention 2.3. Note that $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$, yet $\mathcal{W}^s \not\subset \widetilde{\mathcal{W}}^s$ since $\widetilde{\mathcal{W}}^s$ contains only maximal local stable manifolds (which are necessarily disjoint), while $\mathcal{W}^s$ contains stable manifolds of any length less than δ_0, many of which may overlap. We will define our norms by integrating on elements of $\mathcal{W}^s$ against Hölder continuous test functions.

For $W \in \mathcal{W}^s$ and $\alpha > 0$, let $\mathcal{C}^\alpha(W)$ denote the closure of $\tilde{\mathcal{C}}^1(W)$ in the $\mathcal{C}^\alpha$ norm, defined in (3.1).[7] In this notation, then $\mathcal{C}^1(W) = \tilde{\mathcal{C}}^1(W)$.

Now given a function $f \in \mathcal{C}^1(M)$, define the weak norm of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^1(W) \\ |\psi|_{\mathcal{C}^1(W)} \leq 1}} \int_W f \, \psi \, dm_W \,,$$

where m_W denotes arc length along W. Let $|W| = m_W(W)$.

Next, choose $\alpha, \beta < 1$ and $p > 1$ such that

$$0 < 2\beta \leq 1/p \leq 1 - \alpha \leq \alpha_0, \qquad \text{and} \quad 1/p < \alpha \,. \tag{3.4}$$

Define the strong stable norm of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \leq |W|^{-1/p}}} \int_W f \, \psi \, dm_W \,.$$

Recalling the notion of distance $d_{\mathcal{W}^s}(\cdot, \cdot)$ between curves $W \in \mathcal{W}^s$ and the distance $d_0(\cdot, \cdot)$ between test functions on nearby curves defined in Section 3.1 and fixing $\varepsilon_0 \leq r_0$, we define the strong unstable norm of f by,

$$\|f\|_u = \sup_{\varepsilon \leq \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \leq \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^1(W_i)} \leq 1 \\ d_0(\psi_1, \psi_2) = 0}} \varepsilon^{-\beta} \left| \int_{W_1} f \, \psi_1 \, dm_{W_1} - \int_{W_2} f \, \psi \, dm_{W_2} \right| \,.$$

Define the strong norm of f by $\|f\|_{\mathcal{B}} = \|f\|_s + c_u \|f\|_u$, where $c_u > 0$ is a constant to be chosen in the proof of Lemma 4.3.

Finally, our weak space $\mathcal{B}_w$ is defined to be the completion of $\mathcal{C}^1(M)$ in the weak norm, $|\cdot|_w$, while our strong space $\mathcal{B}$ is defined to be the completion of $\mathcal{C}^1(M)$ in the strong norm $\|\cdot\|_{\mathcal{B}}$.

Remark 3.1. *The definition of our spaces $\mathcal{B}$ and $\mathcal{B}_w$ is nearly the same as that in* [DL, Section 2.2]*, the key difference being that the norms in* [DL] *integrate along cone-stable curves $\widehat{\mathcal{W}}^s$, while our norms here integrate on local stable manifolds $\mathcal{W}^s$. This change is necessary since the potential for our weighted transfer operator, $1/J^sT$, is Hölder continuous along real stable manifolds, yet may only be measurable along arbitrary stable curves. By restricting our norms to this smaller set of curves, we are able to prove the essential Lasota-Yorke inequalities, Proposition 4.2.*

[7] This space is strictly smaller than the set of $\mathcal{C}^\alpha$ functions, yet contains $\mathcal{C}^{\alpha'}$ for each $\alpha' > \alpha$. We adopt this usage in order that the embedding of our strong space in our weak space is injective (Lemma 3.4).

sec:prelim

3.4. **Preliminary facts about the Banach spaces.**

lem:piece

Lemma 3.2. *Let $\mathcal{Q}$ be a (mod 0 w.r.t. μ_{SRB}) finite partition of M into open, simply connected sets such that there exist constants $\bar{K}, C_{\mathcal{Q}} > 0$ such that for each $Q \in \mathcal{Q}$, and $W \in \mathcal{W}^s$, $Q \cap W$ comprises at most $\bar{K}$ connected components and for any $\varepsilon > 0$, $m_W(\mathcal{N}_\varepsilon(\partial Q) \cap W) \le C_{\mathcal{Q}} \varepsilon^{1/2}$.*

a) *Let $\gamma > \beta/(1-\beta)$ and suppose φ is a function on M such that $\sup_{Q \in \mathcal{Q}} |\varphi|_{\mathcal{C}^\gamma(Q)} < \infty$. Then $\varphi \in \mathcal{B}$.*

b) *There exists $C > 0$ such that if φ is such that $\sup_{Q \in \mathcal{Q}} |\varphi|_{\mathcal{C}^1(Q)} < \infty$ and $f \in \mathcal{B}$, then $\varphi f \in \mathcal{B}$ and $\|\varphi f\|_{\mathcal{B}} \le C \|f\|_{\mathcal{B}} \sup_{Q \in \mathcal{Q}} |\varphi|_{\mathcal{C}^1(Q)}$.*

Proof. To prove (a), a function φ as in the statement of the lemma can be approximated by $\mathcal{C}^1$ functions using mollification precisely as in [DZ3, Lemma 3.5]. Part (b) follows along similar lines using [DZ3, Lemma 5.3]. Both proofs use the restrictions in (3.4) we have assumed for the parameters appearing in the norms. In particular. we need $\beta \le 1/(2p)$, rather than simply $\beta \le 1/p$, due to the weak transversality condition assumed on $\partial \mathcal{Q}$. □

lem:embed

Lemma 3.3. *Let $f \in \mathcal{C}^1(M)$ and $\psi \in \tilde{\mathcal{C}}^1(\widetilde{\mathcal{W}}^s)$. Then,*

$$|f(\psi)| = \left| \int_M f \, \psi \, d\mu_{SRB} \right| \le C |f|_w (|\psi|_\infty + H^1_{\widetilde{\mathcal{W}}^s}(\psi)) \, .$$

Proof. Let $f \in \mathcal{C}^1(M)$ and $\psi \in \tilde{\mathcal{C}}^1(\widetilde{\mathcal{W}}^s)$. We will estimate

$$f(\psi) = \int_M f \, \psi \, d\mu_{\mathrm{SRB}} \, .$$

To this end, we choose a foliation $\mathcal{F} = \{W_\xi\}_{\xi \in \Xi} \subset \mathcal{W}^s$ of maximal local stable manifolds subdivided according to the length scale δ_0. We then disintegrate the measure μ_{SRB} into conditional measures μ_{SRB}^ξ on $W_\xi \in \mathcal{F}$ and a factor measure $\hat{\mu}_{\mathrm{SRB}}(\xi)$ on the index set Ξ of stable manifolds. Since μ_{SRB} is smooth by assumption $(P2)$, it follows from [P, Proposition 6] (see also [CZ, eq. (3.7)]) that the conditional measures μ_{SRB}^ξ are absolutely continuous with respect to arc length, $d\mu_{\mathrm{SRB}}^\xi = |W_\xi|^{-1} g_\xi dm_{W_\xi}$, where g_ξ is given by[8]

$$\frac{g_\xi(x)}{g_\xi(y)} = \lim_{n \to \infty} \frac{J_{W_\xi} T^n(x)}{J_{W_\xi} T^n(y)} \quad \text{for all } x, y \in W_\xi \, .$$

This characterization, plus the normalization $\mu_{\mathrm{SRB}}^\xi(W_\xi) = 1$, uniquely determines g_ξ. It follows from a standard estimate[9] and (2.4) that g_ξ is uniformly log-Lipschitz continuous on W_ξ, i.e. there exists $C_g \ge 1$ such that

eq:log g

$$0 < C_g^{-1} \le \inf_{\xi \in \Xi} \inf_{W_\xi} g_\xi \le \sup_{\xi \in \Xi} |g_\xi|_{\mathcal{C}^1(W_\xi)} \le C_g < \infty \, . \tag{3.5}$$

[8]Both [P] and [CZ] give the analogous formula for the conditional measures of μ_{SRB} on unstable manifolds. Yet, due to our assumption (P2), μ_{SRB} is an SRB measure for T^{-1} as well, and so enjoys the analogous properties on stable manifolds of T.

[9]Note $J_{W_\xi} T^n(z) = \prod_{j=0}^{n-1} J_{T^j W_\xi} T(T^j z)$ and for brevity let $g_n = J_{W_\xi} T^n$. The limit of $g_n(x)/g_n(y)$ exists if the limit of $\log(g_n(x)/g_n(y))$ exists. Now for $n, k \ge 1$, we may estimate using (2.3) and (2.4),

$$\left| \log \frac{g_n(x)}{g_n(y)} - \log \frac{g_{n+k}(x)}{g_{n+k}(y)} \right| = \log \frac{J_{T^n W_\xi} T^k(T^n x)}{J_{T^n W_\xi} T^k(T^n y)} \le C_d d_{T^n W_\xi}(T^n x, T^n y) \le C_d C_e^{-1} \Lambda^{-n} d_{W_\xi}(x, y) \, ,$$

so that the sequence $\log(g_n(x)/g_n(y))$ is Cauchy and therefore converges. Thus the limit defining g_ξ exists. A similar estimate shows that g_n is log-Lipschitz with Lipschitz constant at most C_d, bounded independently of n, and so this bound carries over to g_ξ.

Using this disintegration, we write,

$$
\begin{aligned}
|f(\psi)| &= \left| \int_{\xi\in\Xi} \int_{W_\xi} f\, \psi\, g_\xi\, |W_\xi|^{-1} dm_{W_\xi} d\hat\mu_{\mathrm{SRB}}(\xi) \right| \\
&\le \int_{\xi\in\Xi} |f|_w |\psi|_{\mathcal{C}^1(W_\xi)} |g_\xi|_{\mathcal{C}^1(W_\xi)} |W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi) \qquad (3.6)\\
&\le C_g |f|_w (|\psi|_\infty + H^1_{\widehat{\mathcal{W}}^s}(\psi)) \int_{\xi\in\Xi} |W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi)\,.
\end{aligned}
$$

To bound this last integral, we will apply some results of [CZ], which studies hyperbolic maps with singularities in an axiomatic context (Assumptions (H.1)-(H.5) in that paper), which include the class of maps in the present paper, in addition to many dispersing and semi-dispersing billiards. Indeed, the final integral in (3.6) is precisely the $\mathcal{Z}$-function, $\mathcal{Z}_1(\mathcal{F})$, defined in [CZ, eq. (4.7)] which governs the average length of stable manifolds in the family $\mathcal{F}$. (See also [CM, Exercise 7.15 and Proposition 7.17] for a similar application of these ideas.) The parameters p and q in [CZ] are both equal to 1 in our context, due to our property (P1) and Convention 2.3, which imply that T satisfies the one-step expansion condition, [CZ, Condition (H.5)] with parameter $q = 1$,

$$
\sup_{W\in\mathcal{W}^s} \sum_{V_i \subset T^{-1}W} \left(\frac{|W|}{|V_i|}\right)^q \frac{|TV_i|}{|W|} \le K_1 \Lambda^{-1} \le \rho < 1\,, \qquad (3.7)
$$

where V_i are the maximal, connected components of $T^{-1}W$. The required bound on $\mathcal{Z}_1(\mathcal{F})$ follows from [CZ, Lemma 4] (again with $q = 1$) since μ_{SRB} is obtained as the limit of standard pairs with finite valued $\mathcal{Z}$-function. □

Lemma 3.4. *There is a sequence of continuous inclusions,*

$$\mathcal{C}^1(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^\alpha(\mathcal{W}^s))^*\,.$$

The first two inclusions are injective.

Proof. The continuity of the first inclusion follows from Lemma 3.2 and its injectivity is obvious. The continuity of the second inclusion follows from $|\cdot|_w \le \|\cdot\|_s$. Its injectivity is a result of the fact that we have defined $\|\cdot\|_s$ with respect to $\mathcal{C}^\alpha(W)$ rather than $\tilde{\mathcal{C}}^\alpha(W)$, and $\mathcal{C}^1(W)$ is dense in $\mathcal{C}^\alpha(W)$. Finally, the continuity of the third inclusion follows from Lemma 3.3. □

By adding an additional weight to the weak norm, one can make the third inclusion in Lemma 3.4 injective as well (see for example [DZ3, Lemma 3.8]), but we will not need this property here. Our final lemma in this section is essential for proving the quasi-compactness of $\mathcal{L}$ on $\mathcal{B}$.

Lemma 3.5. *The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.*

Proof. The lemma follows from [DL, Lemma 3.5]. The fact that [DL, Lemma 3.5] uses the family of admissible curves $\widehat{\mathcal{W}}^s$ while we use the smaller set $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ does not affect the argument since the family of functions defining $\mathcal{W}^s$ in each chart is still compact in the $\mathcal{C}^1$-metric. □

3.5. **Growth Lemmas.** In this section, we prove several growth lemmas which will be instrumental in establishing precise upper and lower bounds on the spectral radius of our transfer operator. Many of the results in this subsection and the next parallel those of [BD, Section 5].

Given a curve $W \in \widehat{\mathcal{W}}^s$, let $\mathcal{G}_1(W)$ denote the maximal connected components of $T^{-1}W$ on which T is smooth, with long pieces subdivided so that they have length between $\delta_0/2$ and δ_0. In particular, elements of $\mathcal{G}_1(W)$ must belong to a single element of $\mathcal{M}_0^1$, i.e. to a single component M_i^+ of M. Inductively, define $\mathcal{G}_n(W)$ to denote the collection of maximal connected components of $T^{-1}V$, where $V \in \mathcal{G}_{n-1}(W)$, again subdividing long pieces into curves of length between $\delta_0/2$ and δ_0. We call $\mathcal{G}_n(W)$, the nth generation of W.

For each n, let $L_n(W)$ denote those elements of $\mathcal{G}_n(W)$ having length at least $\delta_0/3$. Let $\mathcal{I}_n(W)$ denote those elements $W_i \in \mathcal{G}_n(W)$ such that for each $0 \le k \le n-1$, $T^k W_i \subset V \in \mathcal{G}_{n-k}(W)$ and $|V| < \delta_0/3$, i.e. $\mathcal{I}_n(W)$ represents those elements in $\mathcal{G}_n(W)$ that have always been contained in a short element of $\mathcal{G}_{n-k}(W)$ from time 1 to time n.

lem:growth **Lemma 3.6.** *There exists $C > 0$ such that for all $W \in \widehat{\mathcal{W}}^s$, and all $n \ge 0$,*

a) $\#\mathcal{I}_n(W) \le K_1^n \le \rho^n \kappa^{\alpha_0 n} \Lambda^n$;
b) $\#\mathcal{G}_n(W) \le C\delta_0^{-1} \#\mathcal{M}_0^n$;
c) $\displaystyle\sum_{W_i \in \mathcal{G}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} \le C\delta_0^{-1+1/p} \kappa^{-n/p} (\#\mathcal{M}_0^n)^{1-1/p}$;
d) $\#\mathcal{M}_0^n \ge C\delta_0 \Lambda^n$.

Proof. (a) This estimate follows from the fact that curves $W_i \in \mathcal{I}_n(W)$ have always been contained in a short element of $\mathcal{G}_{n-k}(W)$ for each k between 0 and $n-1$. Thus property (P1) (recalling also Convention convention: n_0=1 2.3) can be applied inductively in k to each element of $\mathcal{I}_{n-k}(W)$, yielding the claimed bound on the cardinality of these elements.

(b) The bound is trivial since each element of $\mathcal{G}_n(W)$ belongs by definition to one element of $\mathcal{M}_0^n$. Since the stable diameter of each component of $\mathcal{M}_0^n$ is uniformly bounded in n, the connected components of $T^{-n}W$ are subdivided into at most $C\delta_0^{-1}$ curves to form the elements of $\mathcal{G}_n(W)$, for some uniform $C > 0$.

(c) Note that for $W_i \in \mathcal{G}_n(W)$, using (eq:exp def 2.1),

$$|T^n W_i| = \int_{W_i} J_{W_i} T^n \, dm_{W_i} \ge |W_i| \kappa^n .$$

Thus,

$$\sum_{W_i \in \mathcal{G}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} \le \kappa^{-n/p} \sum_{W_i \in \mathcal{G}_n(W)} \frac{|T^n W_i|^{1/p}}{|W|^{1/p}} \le \kappa^{-n/p} \Bigg(\sum_{W_i \in \mathcal{G}_n(W)} 1 \Bigg)^{1-1/p}$$
$$\le C\kappa^{-n/p} \delta_0^{-1+1/p} (\#\mathcal{M}_0^n)^{1-1/p} ,$$

where we have used the Hölder inequality and part (b) of the lemma.

(d) Applying part (b) of the lemma, we have

$$|T^{-n}W| = \sum_{W_i \in \mathcal{G}_n(W)} |W_i| \le \delta_0 \#\mathcal{G}_n(W) \le C \#\mathcal{M}_0^n .$$

Then recalling (eq:one grow 2.3) and applying this to $W \in \mathcal{W}^s$ with $|W| = \delta_0$ completes the proof of the lemma. $\square$

Next we proceed to show that most elements of $\mathcal{G}_n(W)$ are long, if the length scale is chosen appropriately. For $\delta \in (0, \delta_0)$ and $W \in \widehat{\mathcal{W}}^s$, define $\mathcal{G}_n^\delta(W)$ to be the smooth components of $T^{-n}W$, with pieces longer than δ subdivided to have length between $\delta/2$ and δ, i.e. $\mathcal{G}_n^\delta(W)$ is defined precisely like $\mathcal{G}_n(W)$, but with δ_0 replaced by δ. Define $L_n^\delta(W)$ to be the set of curves in $\mathcal{G}_n^\delta(W)$ having length at least $\delta/3$, and let $S_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$. Similarly, let $\mathcal{I}_n^\delta(W)$ denote those elements of $S_n^\delta(W)$ that have no ancestors of length at least $\delta/3$.

m:most grow **Lemma 3.7.** *For all $\varepsilon > 0$, there exist $\delta \in (0, \delta_0)$ and $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$,*

$$\#L_n^\delta(W) \ge (1-\varepsilon) \#G_n^\delta(W) , \quad \text{for all } W \in \widehat{\mathcal{W}}^s \text{ with } |W| \ge \delta/3.$$

Proof. Fix $\varepsilon \in (0,1)$ and by Property (P1) and Remark 2.2, choose n_1 sufficiently large that $3C_e^{-1}(K(n_1+\ell)+1)\Lambda^{-n_1-\ell} < \varepsilon/2$ for all $0 \le \ell \le n_1 - 1$, where $C_e \le 1$ is from (2.3). Next, choose $\delta > 0$ sufficiently small that if $W \in \widehat{\mathcal{W}}^s$ with $|W| \le \delta$, then $T^{-n}W$ comprises at most $K(n)+1$ smooth components of length at most δ_0 for all $n \le 2n_1$.

Now let $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta/3$. We shall prove that for $n \ge n_1$,

$$\#S_n^\delta(W) \le \varepsilon \#\mathcal{G}_n^\delta(W)\,.$$

For $n \ge n_1$, write $n = kn_1 + \ell$ for some $0 \le \ell < n_1$. If $k = 1$, the above inequality follows immediately since there are at most $K(n_1+\ell)+1$ elements of $S_{n_1+\ell}^\delta(W)$ by choice of δ, while by (2.3), $|T^{-n_1-\ell}W| \ge C_e\Lambda^{n_1+\ell}|W| \ge C_e\Lambda^{n_1+\ell}\delta/3$. Thus $\mathcal{G}_n^\delta(W)$ must contain at least $C_e\Lambda^{n_1+\ell}/3$ curves since each has length at most δ. Thus,

$$\frac{\#S_{n_1+\ell}^\delta(W)}{\#\mathcal{G}_{n_1+\ell}^\delta(W)} \le 3C_e^{-1}\frac{K(n_1+\ell)+1}{\Lambda^{n_1+\ell}} < \frac{\varepsilon}{2}\,,$$

by assumption on n_1.

On the other hand, if $k > 1$ then we split n into $k-1$ blocks of length n_1 and one block of length $n_1+\ell$. We group elements $W_i \in S_{kn_1+\ell}^\delta(W)$ by most recent long ancestor $V_j \in L_{tn_1}^\delta(W)$: t is the greatest index $\le k-1$ such that $T^{(k-t)n_1+\ell}W_i \in V_j$ and $V_j \in L_{tn_1}^\delta(W)$. Note that we only consider ancestors occurring in blocks of length n_1. It is irrelevant for our estimate whether W_i has a long ancestor at an intermediate time.

Since each $|V_j| \ge \delta/3$, it follows that $\mathcal{G}_{(k-t)n_1+\ell}^\delta(V_j)$ must contain at least $C_e\Lambda^{(k-t)n_1}/3$ curves of length at most δ. Thus using Lemma 3.6(a), we have

$$\begin{aligned}
\frac{\#S_{kn_1+\ell}^\delta(W)}{\#\mathcal{G}_{kn_1+\ell}^\delta(W)} &= \frac{\#\mathcal{I}_{kn_1+\ell}^\delta(W)}{\#\mathcal{G}_{kn_1+\ell}^\delta(W)} + \frac{\sum_{t=1}^{k-1}\sum_{V_j\in L_{tn_1}^\delta(W)}\#\mathcal{I}_{(k-t)n_1+\ell}^\delta(V_j)}{\#\mathcal{G}_{kn_1+\ell}^\delta(W)} \\
&\le \frac{(K(n_1)+1)^k}{C_e\Lambda^{kn_1}/3} + \sum_{t=1}^{k-1}\frac{\sum_{V_j\in L_{tn_1}^\delta(W)}(K(n_1)+1)^{k-t}}{\sum_{V_j\in L_{tn_1}^\delta(W)}C_e\Lambda^{(k-t)n_1}/3} \qquad (3.8)\\
&\le 3C_e^{-1}\sum_{t=1}^{k}(K(n_1)+1)^t\Lambda^{-tn_1} \le \sum_{t=1}^{k}\left(\frac{\varepsilon}{2}\right)^t < \varepsilon\,.
\end{aligned}$$

□

The following corollary extends Lemma 3.7 to arbitrarily short curves, and is used in Lemma 4.5 to prove the positivity of our maximal eigenvector on all elements of $\mathcal{W}^s$.

Corollary 3.8. *There exists $C_2 > 0$ such that for any ε, δ and n_1 as in Lemma 3.7,*

$$\#L_n^\delta(W) \ge (1-2\varepsilon)\#\mathcal{G}_n^\delta(W)\,, \quad \forall W \in \widehat{\mathcal{W}}^s,\ \forall n \ge C_2 n_1\frac{|\log(|W|/\delta)|}{|\log\varepsilon|}\,.$$

Proof. Fix ε, δ and n_1 from Lemma 3.7. Suppose $W \in \widehat{\mathcal{W}}^s$ has $|W| < \delta/3$, and let $n > n_1$. We decompose $\mathcal{G}_n^\delta(W)$ as in Lemma 3.7, and estimate the second sum in (3.8) precisely as before.

The first term on the right hand side of (3.8), $\#\mathcal{I}_n^\delta(W)/\#\mathcal{G}_n^\delta(W)$, is handled differently. Let n_2 denote the least integer ℓ such that $\mathcal{G}_\ell^\delta(W)$ contains at least one element of length $\delta/3$. Since $|T^{-\ell}W| \ge C_e\Lambda^\ell|W|$ by (2.3), and $\mathcal{G}_\ell^\delta(W) \le K_1^\ell$ by (P1) and Convention 2.3, as long as $|T^{-\ell}W| \le \delta_0$, at least one element of $\mathcal{G}_\ell^\delta(W)$ must have length at least $\frac{C_e\Lambda^\ell|W|}{K_1^\ell} \ge C_e\rho^{-\ell}|W|$. Thus

$$n_2 \le \frac{|\log(3C_e|W|\delta^{-1})|}{|\log\rho|}\,.$$

Then calling V the element of $\mathcal{G}^\delta_{n_2}(W)$ having length at least $\delta/3$, we have

$$\#\mathcal{G}^\delta_n(W) \ge \#\mathcal{G}^\delta_{n-n_2}(V) \ge C_e \Lambda^{n-n_2}/3\,.$$

Thus

$$\frac{\#\mathcal{I}^\delta_n(W)}{\#\mathcal{G}^\delta_n(W)} \le \frac{3(K(n_1)+1)^{\lfloor n/n_1 \rfloor}}{C_e \Lambda^n} \Lambda^{n_2} \le \left(\frac{\varepsilon}{2}\right)^{\lfloor n/n_1 \rfloor} \Lambda^{n_2}\,.$$

Finally, since $n_2 = \mathcal{O}(|\log(|W|/\delta)|)$, we may choose C_2 sufficiently large, that if $n \ge C_2 n_1 \frac{|\log(|W|/\delta)|}{|\log \varepsilon|}$, then the quantity on the right is at most ε, completing the proof of the corollary. $\square$

Choosing $\varepsilon = 1/3$, we let $\delta_1 > 0$ and n_1 be the corresponding quantities from Lemma 3.7. Fixing this choice of δ_1 and n_1, we have

$$\#L^{\delta_1}_n(W) \ge \tfrac{2}{3}\#\mathcal{G}^{\delta_1}_n(W), \qquad \text{for all } W \in \widehat{\mathcal{W}}^s \text{ with } |W| \ge \delta_1/3 \text{ and all } n \ge n_1. \tag{3.9}$$

Our next lemma shows that a positive fraction of elements of $\mathcal{M}^n_0$ and $\mathcal{M}^0_{-n}$ have length at least δ_1 in some direction. This will be essential to establishing the lower bounds of Section 3.6. For $A \subset M$, let $\mathrm{diam}^s(A)$ denote the stable diameter of A, i.e. the length of the longest stable curve in A. Similarly, define the unstable diameter $\mathrm{diam}^u(A)$ to be the length of the longest unstable curve in A.

The boundary of the partition defined by $\mathcal{M}^0_{-n}$ is comprised of unstable curves belonging to $\mathcal{S}^-_n = \cup_{i=0}^{n-1} T^i(\mathcal{S}^-)$. Similarly, $\partial\mathcal{M}^n_0$ is comprised of the stable curves, $\mathcal{S}^+_n = \cup_{i=0}^{n-1} T^{-i}(\mathcal{S}^+)$. In what follows, we will find it convenient to invoke Convention 2.1 regarding the definition of $T^{\pm 1}$ on each smooth component of $\mathcal{S}^\pm$. Let $L_u(\mathcal{M}^0_{-n})$ denote those elements of $\mathcal{M}^0_{-n}$ whose unstable diameter is at least $\delta_1/3$, and let $L_s(\mathcal{M}^n_0)$ denote those elements of $\mathcal{M}^n_0$ whose stable diameter is at least $\delta_1/3$. The following lemma is the analogue of Lemma 3.7 for these dynamically defined partitions.

Lemma 3.9. *There exist $C_{n_1} > 0$ and $n_3 \ge n_1$ such that for all $n \ge n_3$,*

$$\#L_s(\mathcal{M}^n_0) \ge C_{n_1}\delta_1 \#\mathcal{M}^n_0 \quad \textit{and} \quad \#L_u(\mathcal{M}^0_{-n}) \ge C_{n_1}\delta_1 \#\mathcal{M}^0_{-n}\,.$$

Proof. We prove the bound for $L_s(\mathcal{M}^n_0)$. In order to prove the lemma, we will use the fact that the boundary of $\mathcal{M}^n_0$ is the set $\cup_{j=0}^{n-1} T^{-j}\mathcal{S}^+$.

Let $S_s(\mathcal{M}^n_0)$ denote the elements of $\mathcal{M}^n_0$ whose stable diameter is less than $\delta_1/3$. We have $\mathcal{M}^n_0 = L_s(\mathcal{M}^n_0) \cup S_s(\mathcal{M}^n_0)$. Similarly, let $S_s(T^{-j}\mathcal{S}^+)$ denote the set of stable curves in $T^{-j}\mathcal{S}^+$ whose length is less than $\delta_1/3$.

The following sublemma will prove useful for establishing key claim in the proof.

Sublemma 3.10. *If a smooth stable curve $V_i \in T^{-i}\mathcal{S}^+$ intersects a smooth curve $V_j \subset T^{-j}\mathcal{S}^+$ for $i < j$, then V_j must terminate on V_i.*

Proof of Sublemma 3.10. Suppose such an intersection occurs for $j > i$. Then $T^{i+1}(V_i) \subset \mathcal{S}^-$ is an unstable curve, while $T^{i+1}(V_j) \subset \mathcal{S}^+_{j-i-1}$ is a stable curve. Thus $T^{i+1}(V_j)$ must cross $\mathcal{S}^-$ transversally, and so $T^i(V_j)$ will be split into at least two smooth components since $\mathcal{S}^-$ is the singularity set for T^{-1}. This implies that V_j cannot be a single smooth curve. $\square$

Using the sublemma, we establish the following claim:

$$\#S_s(\mathcal{M}^n_0) \le 2\sum_{j=0}^{n-1} \#S_s(T^{-j}\mathcal{S}^+) + B_1 n\,, \tag{3.10}$$

for some $B_1 > 0$. According to the sublemma, if $A \in S_s(\mathcal{M}^n_0)$, then either ∂A contains a short curve in $T^{-j}\mathcal{S}^+$ or ∂A contains an intersection point of two curves in $T^{-j}\mathcal{S}^+$, for some $0 \le j \le n-1$. But intersections of curves within $T^{-j}\mathcal{S}^+$ are images of intersections of curves within $\mathcal{S}^+$, and the cardinality of cells created by such intersections is bounded by some uniform constant $B_1 > 0$

depending only on $\mathcal{S}^+$. Since each short curve in $T^{-j}\mathcal{S}^+$ belongs to the boundary of at most two elements of $S_s(\mathcal{M}_0^n)$, the claim follows.

Now, we subdivide $\mathcal{S}^+$ into ℓ_0 smooth curves V_i of length between $\delta_1/3$ and δ_1. For $j \ge n_1$, recalling the notation $S_j^{\delta_1}(V_i)$ for the short elements of the jth generation $\mathcal{G}_j^{\delta_1}(V_i)$ of subcurves in $T^{-j}V_i$, we have by (3.9),

$$\# S_s(T^{-j}\mathcal{S}^+) = \sum_{i=1}^{\ell_0} \# S_j^{\delta_1}(V_i) \le \tfrac{1}{3} \sum_{i=1}^{\ell_0} \# L_j^{\delta_1}(V_i) \, . \tag{3.11}$$

Next, using (3.11), we estimate the sum over j in (3.10) by splitting it over two parts,

$$\# S_s(\mathcal{M}_0^n) \le B_1 n + 2 \sum_{j=0}^{n_1-1} \# S_s(T^{-j}\mathcal{S}^+) + \tfrac{2}{3} \sum_{j=n_1}^{n-1} \sum_{i=0}^{\ell_0} \# L_j^{\delta_1}(V_i) \, . \tag{3.12}$$

The cardinality of the first sum up to $n_1 - 1$ is bounded by some constant $\bar{C}_{n_1}$ depending only on the map T and n_1, but independent of n.

Next, we wish to relate $\# L_j^{\delta_1}(V_i)$ to $\# L_s(\mathcal{M}_0^n)$ for $j \ge n_1$. Note that if $V' \in L_j^{\delta_1}(V_i)$, then $|T^{n-j}V'| \ge C\Lambda^{n-j}\delta_1/3$, so that $\#\mathcal{G}_{n-j}^{\delta_1}(V') \ge C\Lambda^{n-j}/3$.

Now for each j such that $n_1 \le j \le n-1-n_1$, and $V' \in L_j^{\delta_1}(V_i)$, we may apply (3.9), so that

$$\# L_{n-1}^{\delta_1}(V_i) \ge \sum_{V' \in L_j^{\delta_1}(V_i)} \# L_{n-1-j}^{\delta_1}(V') \ge C'\Lambda^{n-1-j} \# L_j^{\delta_1}(V_i) \, . \tag{3.13}$$

For $j > n - n_1$, we compare $L_j^{\delta_1}(V_i)$ with $L_{n-1}^{\delta_1}(V_i)$. Since $K_1 < \Lambda$, there is at least one element of $L_{j+1}^{\delta_1}(V_i)$ for each element of $L_j^{\delta_1}(V_i)$. Applying this inductively to j, we conclude,

$$\# L_{n-1}^{\delta_1}(V_i) \ge \# L_j^{\delta_1}(V_i) \, .$$

Putting together this estimate with (3.13) in (3.12), we estimate,

$$\begin{aligned} \# S_s(\mathcal{M}_0^n) &\le B_1 n + \bar{C}_{n_1} + \sum_{j=n_1}^{n-1-n_1} C\Lambda^{j+1-n} \# L_s(T^{-n+1}\mathcal{S}^+) + \sum_{j=n-n_1}^{n-1} \# L_s(T^{-n+1}\mathcal{S}^+) \\ &\le B_1 n + \bar{C}_{n_1} + C\delta_1^{-1} \# L_s(\mathcal{M}_0^n) + n_1 C \delta_1^{-1} \# L_s(\mathcal{M}_0^n) \, , \end{aligned} \tag{3.14}$$

where in the second line we have used the fact that $\# L_s(T^{-n+1}\mathcal{S}^+) \le C\delta_1^{-1} \# L_s(\mathcal{M}_0^n)$, which follows from Sublemma 3.10.

Finally, since $\#\mathcal{M}_0^n = \# L_s(\mathcal{M}_0^n) + \# S_s(\mathcal{M}_0^n)$, we estimate,

$$\# L_s(\mathcal{M}_0^n) \ge \frac{\#\mathcal{M}_0^n - \bar{C}_{n_1} - B_1 n}{1 + C\delta_1^{-1}(1 + n_1)} \, .$$

Since $\#\mathcal{M}_0^n \ge C\delta_0\Lambda^n$ by Lemma 3.6(d) and n_1 is fixed, we may choose $n_2 \in \mathbb{N}$ such that $\#\mathcal{M}_0^n - \bar{C}_{n_1} - B_1 n \ge \frac{1}{2}\#\mathcal{M}_0^n$, for all $n \ge n_2$. We conclude that there exists $C_{n_1} > 0$ such that for $n \ge n_2$, $\# L_s(\mathcal{M}_0^n) \ge C_{n_1}\delta_1 \#\mathcal{M}_0^n$, completing the proof of the lemma for $L_s(\mathcal{M}_0^n)$.

The lower bound for $\# L_u(\mathcal{M}_{-n}^0)$ follows similarly, using the fact that (P1) also allows us to control the evolution of unstable curves under T^n by controlling the complexity of $\mathcal{S}_n^+$. Note that the analogue of Lemma 3.7 holds for forward iterates of unstable curves using precisely the same proof. The constant κ does not appear in this argument, i.e. the fact that the rate of expansion has a maximum is not needed for the proof. $\square$

sec:lower

3.6. **Lower bounds on growth.** The prevalence of long pieces established in Lemmas 3.7 and 3.9 have the following important consequences.

lem:lower

Lemma 3.11. *Let δ_1 be the length scale from (3.9). There exists $c_0 > 0$, depending on δ_1, such that for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$ and $n \geq 1$, we have $\#\mathcal{G}_n(W) \geq c_0 \#\mathcal{M}_0^n$.*

This lemma, in turn, implies the supermultiplicativity property for $\#\mathcal{M}_0^n$.

prop:super

Proposition 3.12. *There exists $c_1 > 0$ such that for all $j, n \in \mathbb{N}$ with $j \leq n$, it holds,*

$$\#\mathcal{M}_0^n \geq c_1 \#\mathcal{M}_0^{n-j} \#\mathcal{M}_0^j \,.$$

In order to establish Lemma 3.11, we recall the construction of Cantor rectangles. For $x \in M$, let $W^s(x)$ and $W^u(x)$ denote the maximal smooth components of the local stable and unstable manifolds of x (which, by definition, belong to a single domain M_i^+).

We begin by defining a solid rectangle $D \subset M$ to be a closed region whose boundary comprises exactly two stable manifolds and two unstable manifolds of positive length. Given such a region D, define the *locally maximal Cantor rectangle R in D* to be the union of all points in D whose local stable and unstable manifolds completely cross D. Locally maximal Cantor rectangles are endowed with a natural product structure: for any $x, y \in R$, $W^u(x) \cap W^s(y)$ belongs to R. Such rectangles are closed, so their boundary coincides with the boundary of D. In this case, we write $D = D(R)$ to denote the fact that D is the smallest solid rectangle containing R.

Following [L1], for a Cantor rectangle R, we call the *core* of R to be $R \cap D_{1/4}$, where $D_{1/4}$ is an approximately concentric rectangle in $D(R)$ with side lengths 1/4 the side lengths of D.

For a locally maximal Cantor rectangle R, we say that a stable (respectively unstable) curve W *properly crosses* R if W intersects the rectangle $D_{1/4}(R)$, but does not terminate in $D(R)$, and W does not cross either of the stable (resp. unstable) boundaries of both $D(R)$ and $D_{1/4}(R)$.

Proof of Lemma 3.11. Applying [L1, Theorem 4.10], we may choose locally maximal Cantor rectangles $\mathcal{R}_{\delta_1} = \{R_1, \cdots, R_k\}$, with $\mu_{\mathrm{SRB}}(R_i) > 0$, whose stable and unstable boundaries have length at most $\frac{1}{10}\delta_1$ such that any stable or unstable curve of length at least $\delta_1/3$ properly crosses at least one of them.[10] Furthermore, we may choose the rectangles sufficiently small that both R_i and $R_i \cap D_{1/4}(R_i)$ have positive μ_{SRB}-measure for each i. The number of rectangles k depends on δ_1.

For brevity, denote by $R_i^* = R_i \cap D_{1/4}(R_i)$, the core of R_i. Due to the mixing property of (T, μ_{SRB}), there exist $\varepsilon > 0$ and $n_4 \in \mathbb{N}$ such that for all $n \geq n_4$, and all $1 \leq i, j, \leq k$, $\mu_{\mathrm{SRB}}(R_i^* \cap T^{-n} R_j) \geq \varepsilon$.

We claim that for each n, at least one Cantor rectangle $R_i \in \mathcal{R}_{\delta_1}$ is fully crossed in the unstable direction by at least $\frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$ elements of of $\mathcal{M}_{-n}^0$. This is because if $A \in \mathcal{M}_{-n}^0$, then ∂A is comprised of unstable curves belonging to $\mathcal{S}_n^-$. Since unstable manifolds cannot be cut under iteration by T^{-n}, $\mathcal{S}_n^-$ cannot intersect the unstable boundaries of R_i. Thus if $A \cap R_i \neq \emptyset$, then either ∂A terminates inside R_i or A fully crosses R_i. This implies that elements of $L_u(\mathcal{M}_{-n}^0)$ fully cross at least one R_i, and so at least one R_i must be fully crossed by at least $\frac{1}{k}$ such elements.

With the claim established, for each n, let R_{i_n} denote a Cantor rectangle that is fully crossed by at least $\frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$ elements of $\mathcal{M}_{-n}^0$.

Now take $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$. By construction, there exists $R_j \in \mathcal{R}_{\delta_1}$ such that W properly crosses R_j in the stable direction. For each $n \in \mathbb{N}$, using mixing, we have $\mu_{\mathrm{SRB}}(R_{i_n}^* \cap T^{-n_4} R_j) \geq \varepsilon$. By [L1, Lemma 4.13], there is a curve $V \in \mathcal{G}_{n_4}^{\delta_1}(W)$ that properly crosses R_{i_n} in the stable direction. By choice of R_{i_n}, this implies that $\#\mathcal{G}_n(V) \geq \frac{1}{k}\#L_u(\mathcal{M}_{-n}^0)$. Thus,

$$\#\mathcal{G}_{n+n_4}(W) \geq \tfrac{1}{k}\#L_u(\mathcal{M}_{-n}^0) \implies \#\mathcal{G}_n(W) \geq \tfrac{C'}{k}\#L_u(\mathcal{M}_{-n}^0)\,,$$

[10] Once a Cantor rectangle of some size is constructed around μ_{SRB}-almost-every $x \in M$, the existence of such a finite family for any fixed length scale δ_1 follows from the compactness of the set of stable (and also unstable) curves of length $\geq \delta_1/3$ in the Hausdorff metric, as in [CM, Lemma 7.87].

where $(C')^{-1} = C\delta_0^{-1}\#\mathcal{M}_0^{n_4}$ since $\#\mathcal{G}_{n+n_4}(W) \le C\delta_0^{-1}\#\mathcal{M}_0^{n_4}\#\mathcal{G}_n(W)$ by Lemma 3.6(b).

Finally, by Lemma 3.9, $\#L_u(\mathcal{M}_{-n}^0) \ge C_{n_1}\delta_1\#\mathcal{M}_{-n}^0$, which proves the lemma for $n \ge \max\{n_3, n_4\}$ since $\#\mathcal{M}_{-n}^0 = \#\mathcal{M}_0^n$. The lemma extends to all $n \in \mathbb{N}$ by possibly reducing the constant c_0 since there are only finitely many values to correct for. □

Proof of Proposition 3.12. Recall that since $T^{-j}(\mathcal{S}_j^- \cup \mathcal{S}_{n-j}^+) = \mathcal{S}_n^+$, there is a one-to-one correspondence between elements of $\mathcal{M}_{-j}^{n-j}$ and $\mathcal{M}_0^n$ for each $j < n$. Thus $\#\mathcal{M}_0^n = \#\mathcal{M}_{-j}^{n-j}$, and this latter partition is obtained by taking the maximal connected components of $\mathcal{M}_{-j}^0 \bigvee \mathcal{M}_0^{n-j}$.

To prove the lemma, we will show that a positive fraction, independent of j and n, of elements of $\mathcal{M}_0^{n-j}$ intersect a positive fraction of elements of $\mathcal{M}_{-j}^0$. Recall that $L_u(\mathcal{M}_{-j}^0)$ denotes those elements of $\mathcal{M}_{-j}^0$ with unstable diameter of length at least $\delta_1/3$ while $L_s(\mathcal{M}_0^{n-j})$ denotes those elements of $\mathcal{M}_0^{n-j}$ with stable diameter of length at least $\delta_1/3$.

If $A \in L_s(\mathcal{M}_0^{n-j})$ and $V \subset A$ is a stable curve with $|V| \ge \delta_1/3$, then $\#\mathcal{G}_j(V) \ge c_0\#\mathcal{M}_0^j$ by Lemma 3.11. Remark that up to subdivision of long pieces, each component of $\mathcal{G}_j(V)$ corresponds to one component of $V \setminus \mathcal{S}_j^-$. Thus V intersects at least $c_0\#\mathcal{M}_0^j = c_0\#\mathcal{M}_{-j}^0$ elements of $\mathcal{M}_{-j}^0$. Applying this estimate to each $A \in L_s(\mathcal{M}_0^{n-j})$, we obtain

$$\#\mathcal{M}_0^n \ge \#L_s(\mathcal{M}_0^{n-j}) \cdot c_0\#\mathcal{M}_0^j \ge C_{n_1}\delta_1 c_0\#\mathcal{M}_0^{n-j}\#\mathcal{M}_0^j\,,$$

where we have applied Lemma 3.9 in the second inequality. This proves the lemma when $n - j \ge n_3$. For $n - j \le n_3$, since $\#\mathcal{M}_0^{n-j} \le \#\mathcal{M}_0^{n_3}$, we obtain the lemma by possibly decreasing the value of c_1 since there are only finitely many values to correct for. □

Corollary 3.13. *For all $n \in \mathbb{N}$, $\#\mathcal{M}_0^n \le 2c_1^{-1}e^{nh_*}$, where $c_1 > 0$ is from Proposition 3.12.*

Proof. The proof follows using Proposition 3.12, precisely as in [BD, Proposition 4.6]. □

4. Spectral Properties of $\mathcal{L}$

In this section, we prove the following theorem.

Theorem 4.1. *The operator $\mathcal{L}$ acting on $\mathcal{B}$ is quasi-compact, with spectral radius equal to e^{h_*} and essential spectral radius bounded by $\max\{\Lambda^{-\beta}, \rho\}e^{h_*}$.*

Since T is topologically mixing, $\mathcal{L}$ has a spectral gap: e^{h_*} is a simple eigenvalue (multiplicity 1 and no Jordan blocks) and the rest of the spectrum of $\mathcal{L}$ is contained in a disk of radius strictly smaller than e^{h_*}.

Let $\nu_0 \in \mathcal{B}$ be an eigenfunction for eigenvalue e^{h_*} defined by

$$\nu_0 := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}\mathcal{L}^k 1\,.$$

Then $\nu_0 \neq 0$ is a non-negative Radon measure on M.

The quasi-compactness of $\mathcal{L}$ is proved in Lemma 4.3, following the Lasota-Yorke inequalities of Proposition 4.2. The fact that $\mathcal{L}$ has a spectral gap is proved in Lemma 4.6, while the characterization of ν_0 is proved in Lemma 4.4.

4.1. Lasota-Yorke Inequalities.

The following proposition is the key component in establishing the quasi-compactness of $\mathcal{L}$.

Proposition 4.2. *There exists $C > 0$ such that for all $n \ge 0$ and $f \in \mathcal{B}$,*

$$|\mathcal{L}^n f|_w \le C\delta_0^{-1}(\#\mathcal{M}_0^n)|f|_w\,, \tag{4.1}$$

$$\|\mathcal{L}^n f\|_s \le C\delta_0^{-2}(\#\mathcal{M}_0^n)\big((\Lambda^{-\alpha n} + \rho^n)\|f\|_s + \kappa^{-n/p}|f|_w\big)\,, \tag{4.2}$$

$$\|\mathcal{L}^n f\|_u \le C\delta_0^{-1}(\#\mathcal{M}_0^n)(\Lambda^{-\beta n}\|f\|_u + \kappa^{-n/p}\|f\|_s)\,. \tag{4.3}$$

By density, it suffices to prove the proposition for $f \in \mathcal{C}^1(M)$.

4.1.1. *Weak norm bound.* Take $f \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^1(W)$, $|\psi|_{\mathcal{C}^1(W)} \le 1$. Recalling that $\mathcal{G}_n(W)$ denotes the decomposition of $T^{-n}W$ into elements of $\mathcal{W}^s$, we estimate for $n \ge 1$,

$$\int_W \mathcal{L}^n f\, \psi\, dm_W = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f\, \psi \circ T^n\, dm_{W_i} \le |f|_w \sum_{W_i \in \mathcal{G}_n(W)} |\psi \circ T^n|_{C^1(W_i)}\,,$$

where we have applied the weak norm of f to the integral on each W_i. Next, using the uniform contraction of T along stable curves, we have

$$\frac{|\psi \circ T^n(x) - \psi \circ T^n(y)|}{d_{W_i}(x,y)} = \frac{|\psi \circ T^n(x) - \psi \circ T^n(y)|}{d_W(T^n x, T^n y)} \frac{d_W(T^n x, T^n y)}{d_{W_i}(x,y)} \le C |J^s T^n|_{\mathcal{C}^0(W_i)} H^1_W(\psi)\,, \tag{4.4}$$

for some uniform constant $C > 0$, using (2.1). Then since $|\psi \circ T^n|_{\mathcal{C}^0(W_i)} \le |\psi|_{\mathcal{C}^0(W)}$, we have $|\psi \circ T^n|_{\mathcal{C}^1(W_i)} \le C |\psi|_{\mathcal{C}^1(W)} \le C$. Finally, applying Lemma 3.6(b) to the sum over $\mathcal{G}_n(W)$ and taking the supremum over $\psi \in \mathcal{C}^1(W)$ and $W \in \mathcal{W}^s$ completes the proof of (4.1).

4.1.2. *Strong stable norm bound.* Let $f \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^\alpha(W)$ with $|\psi|_{\mathcal{C}^\alpha(W)} \le |W|^{-1/p}$. Let $n \ge 1$. For each $W_i \in \mathcal{G}_n(W)$, define $\overline{\psi}_i = |W_i|^{-1} \int_{W_i} \psi \circ T^n\, dm_{W_i}$. Proceeding as before, we estimate

$$\int_W \mathcal{L}^n f\, \psi\, dm_W = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f\, (\psi \circ T^n - \overline{\psi}_i)\, dm_{W_i} + \sum_{W_i \in \mathcal{G}_n(W)} \overline{\psi}_i \int_{W_i} f\, dm_{W_i}\,. \tag{4.5}$$

To each term in the first sum on the right hand side, we apply the strong stable norm,

$$\int_{W_i} f\, (\psi \circ T^n - \overline{\psi}_i) \le \|f\|_s |W_i|^{1/p} |\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^\alpha(W_i)} \le C \|f\|_s \frac{|W_i|^{1/p}}{|W|^{1/p}} |J^s T^n|^\alpha_{\mathcal{C}^0(W_i)}\,,$$

where we have applied the analogous estimate to (4.4) to the difference $\psi \circ T^n - \overline{\psi}_i$ with the exponent α. Since $\alpha > 1/p$, using bounded distortion (2.4), we estimate

$$|W_i|^{1/p} |J^s T^n|^\alpha_{\mathcal{C}^0(W_i)} \le C |T^n W_i|^{1/p} \Lambda^{-n(\alpha - 1/p)}\,.$$

Finally, summing over W_i, we obtain,

$$\begin{aligned}
\sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f\, (\psi \circ T^n - \overline{\psi}_i) &\le C \|f\|_s \Lambda^{-n(\alpha - 1/p)} \sum_{W_i \in \mathcal{G}_n(W)} \frac{|T^n W_i|^{1/p}}{|W|^{1/p}} \\
&\le C \|f\|_s \Lambda^{-n(\alpha - 1/p)} \left(\sum_i \frac{|T^n W_i|}{|W|} \right)^{1/p} (\# \mathcal{G}_n(W_i))^{1 - 1/p} \qquad (4.6) \\
&\le C \delta_0^{-1+1/p} \|f\|_s \Lambda^{-n(\alpha - 1/p)} (\# \mathcal{M}_0^n)^{1-1/p} \le C \delta_0^{-1} \Lambda^{-\alpha n} \|f\|_s \# \mathcal{M}_0^n\,,
\end{aligned}$$

where in the second line we have used the Hölder inequality and in the third we have used Lemma 3.6(b) and (d).

Next, we estimate the second sum in (4.5). For this estimate, we group $W_i \in \mathcal{G}_n(W)$ by most recent long ancestor as follows. Recall that $L_k(W)$ denotes those elements of $\mathcal{G}_k(W)$ whose length is at least $\delta_0/3$. If $V_j \in L_k(W)$ is such that $T^{n-k}(W_i) \subset V_j$ and $k \le n$ is the largest such index with this property, then we say that V_j is the most recent long ancestor of W_i. Let $\mathcal{I}_{n-k}(V_j)$ denote those elements of $\mathcal{G}_n(W)$ whose most recent long ancestor is V_j. If no such ancestor exists, then $W_i \in \mathcal{I}_n(W)$. Thus,

$$\sum_{W_i \in \mathcal{G}_n(W)} \overline{\psi}_i \int_{W_i} f\, dm_{W_i} = \sum_{k=1}^n \sum_{V_j \in L_k(W)} \sum_{W_i \in \mathcal{I}_{n-k}(V_j)} \overline{\psi}_i \int_{W_i} f\, dm_{W_i} + \sum_{W_i \in \mathcal{I}_n(W)} \overline{\psi}_i \int_{W_i} f\, dm_{W_i}\,.$$

We use the strong stable norm to estimate the terms in $\mathcal{I}_n(W)$,

$$\sum_{W_i \in \mathcal{I}_n(W)} \overline{\psi}_i \int_{W_i} f\, dm_{W_i} \le \|f\|_s \sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} \le \|f\|_s \kappa^{-n/p} \sum_{W_i \in \mathcal{I}_n(W)} \frac{|T^n W_i|^{1/p}}{|W|^{1/p}} \tag{4.7}$$
$$\le \|f\|_s \kappa^{-n/p} K_1^{n(1-1/p)} \;\le\; \|f\|_s \kappa^{-n/p} \rho^n \kappa^{\alpha_0 n} \Lambda^n \;\le\; \|f\|_s \rho^n C \delta_0^{-1} \# \mathcal{M}_0^n \,,$$

where we have used (2.1) for the second inequality, the Hölder inequality and Lemma 3.6(a) for the third and fourth inequalities, and the fact that $\alpha_0 \ge 1/p$ (from (3.4)) and Lemma 3.6(d) for the last inequality.

For the remainder of the terms, we use the weak norm of f, and sum using Lemma 3.6(a) from time k to time n,

$$\sum_{k=1}^{n} \sum_{V_j \in L_k(W)} \sum_{W_i \in \mathcal{I}_{n-k}(V_j)} \overline{\psi}_i \int_{W_i} f\, dm_{W_i} \le \sum_{k=1}^{n} \sum_{V_j \in L_k(W)} \sum_{W_i \in \mathcal{I}_{n-k}(V_j)} |W|^{-1/p} |f|_w$$
$$\le \sum_{k=1}^{n} \sum_{V_j \in L_k(W)} 3\delta_0^{-1/p} K_1^{n-k} \frac{|V_j|^{1/p}}{|W|^{1/p}} |f|_w \le \sum_{k=1}^{n} C\delta_0^{-1} K_1^{n-k} \kappa^{-k/p} (\#\mathcal{M}_0^k)^{1-1/p} |f|_w$$
$$\le C\delta_0^{-1} |f|_w \kappa^{-n/p} \sum_{k=1}^{n} \rho^{n-k} \kappa^{\alpha_0 (n-k)} \Lambda^{n-k} \# \mathcal{M}_0^k \le C \delta_0^{-2} c_1^{-1} \kappa^{-n/p} \# \mathcal{M}_0^n |f|_w \,,$$

where we have used Lemma 3.6(c) to sum over $V_j \in L_k(W)$, as well as the fact that

$$\Lambda^{n-k} \# \mathcal{M}_0^k \le C\delta_0^{-1} \# \mathcal{M}_0^{n-k} \# \mathcal{M}_0^k \le C \delta_0^{-1} c_1^{-1} \# \mathcal{M}_0^n \,,$$

by Proposition 3.12. Putting this estimate together with (4.7) and (4.6) in (4.5) yields,

$$\int_W \mathcal{L}^n f\, \psi\, dm_W \le C\delta_0^{-2} \big((\Lambda^{-\alpha n} + \rho^n) \|f\|_s + \kappa^{-n/p} |f|_w \big) \# \mathcal{M}_0^n \,,$$

and taking the appropriate suprema over W and ψ completes the proof of (4.2).

4.1.3. *Strong unstable norm bound.* Let $f \in \mathcal{C}^1(M)$ and $\varepsilon \in (0, \varepsilon_0)$. Take $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$, and $\psi_k \in \mathcal{C}^1(W^k)$ such that $|\psi_k|_{\mathcal{C}^1(W^k)} \le 1$ and $d_0(\psi_1, \psi_2) = 0$. For $n \ge 1$, we subdivide $\mathcal{G}_n(W^k)$ into matched and unmatched pieces as follows.

To each $W_i^1 \in \mathcal{G}_n(W^1)$, we associate a family of vertical (in the chart) segments $\{\gamma_x\}_{x \in W_i^1}$ of length at most $C\Lambda^{-n}\varepsilon$ such that if γ_x is not cut by an element of $\mathcal{S}_n^+$, its image $T^n \gamma_x$ will have length $C\varepsilon$ and will intersect W^2. Due to the uniform transversality of stable and unstable cones, such a segment $T^i \gamma_x$ will belong to the unstable cone for each $i = 0, \dots, n$, and so undergo the uniform expansion due to (2.1).

In this way, we obtain a partition of W^1 into intervals for which $T^n \gamma_x$ is not cut and intersects W^2 and subintervals for which this is not the case. This defines an analogous partition of $T^{-n} W^1$ and $T^{-n} W^2$. We call two curves $U_j^1 \subset T^{-n} W^1$ and $U_j^2 \subset T^{-n} W^2$ *matched* if they are connected by the foliation γ_x and their images under T^n are connected by $T^n \gamma_x$. We call the remaining components of $T^{-n} W^k$ *unmatched* and denote them by V_i^k. With this decomposition, there is at most one matched piece and two unmatched pieces for each $W_i^k \in \mathcal{G}_n(W^k)$, and we may write $T^{-n} W^k = (\cup_j U_j^k) \cup (\cup_i V_i^k)$.

We proceed to estimate,

$$\left| \int_{W^1} \mathcal{L}^n f\, \psi_1 - \int_{W^2} \mathcal{L}^n f\, \psi_2 \right| \le \sum_j \left| \int_{U_j^1} f\, \psi_1 \circ T^n - \int_{U_j^2} f\, \psi_2 \circ T^n \right| + \sum_{k,i} \left| \int_{V_i^k} f\, \psi_k \circ T^n \right| . \tag{4.8}$$

We begin by estimating the contribution from unmatched pieces. We say a curve V_i^1 is created at time j, $1 \le j \le n$, if j is the first time that $T^{n-j} V_i^1$ is not part of a matched curve in $T^{-j} W^1$.

Define,

$$\mathcal{V}_{j,\ell} = \{ i : V_i^1 \text{ is created at time } j \text{ and } T^{n-j} V_i^1 \subset W_\ell^1 \in \mathcal{G}_j(W^1) \} .$$

Note that $\cup_{i \in \mathcal{V}_{j,\ell}} V_i^1 = W_\ell^1$. Due to the expansion of T in the unstable cone and the uniform transversality of $\mathcal{S}_j^-$ with the stable cone, it follows that $|W_\ell^1| \le C \Lambda^{-j} \varepsilon$. Now applying the strong stable norm to each such curve at the time it is created,

(4.9)

$$\begin{aligned}
\sum_i \int_{V_i^1} f \, \psi_1 \circ T^n &= \sum_{j=1}^n \sum_{W_\ell^1 \in \mathcal{G}_j(W)} \int_{W_\ell^1} \mathcal{L}^{n-j} f \, \psi_i \circ T^{n-j} \\
&\le \sum_{j=1}^n \sum_{W_\ell^1 \in \mathcal{G}_j(W)} |W_\ell^1|^{1/p} \| \mathcal{L}^{n-j} f \|_s |\psi \circ T^{n-j}|_{\mathcal{C}^\alpha(W_\ell^1)} \\
&\le \sum_{j=1}^n \sum_{W_\ell^1 \in \mathcal{G}_j(W)} C \Lambda^{-j/p} \varepsilon^{1/p} \delta_0^{-1} \kappa^{-(n-j)/p} (\# \mathcal{M}_0^{n-j}) \|f\|_s \\
&\le C \delta_0^{-1} \varepsilon^{1/p} \|f\|_s \kappa^{-n/p} \sum_{j=1}^n \Lambda^{-j/p} \# \mathcal{M}_0^j \# \mathcal{M}_0^{n-j} \le C \delta_0^{-1} \varepsilon^{1/p} \|f\|_s \kappa^{-n/p} \# \mathcal{M}_0^n ,
\end{aligned}$$

where we have applied (4.2) in the second inequality (actually, a simpler version suffices with no need to subtract the average of the test function on each W_i), and Proposition 3.12 in the fourth. A similar estimate holds over the curves V_i^2.

Next, we estimate the matched pieces. Recall that according to our notation in Section 3.1 the curve U_j^1 is associated with the quadruple (i_j, x_j, r_j, F_j^1) so that F_j^1 is defined in the chart χ_{i_j} and $U_j^1 = G(x_j, r_j, F_j^1)(I_{r_j})$. By definition of our matching process, it follows that $U_j^2 = G(x_j, r_j, F_j^2)(I_{r_j})$ for some function F_j^2 defined in the same chart, so that the point $x_j + (t, F_j^1(t))$ is associated with the point $x_j + (t, F_j^2(t))$ by the vertical line $(0, s)_{s \in \mathbb{R}}$ in the chart.

Recall that $G_{F_j^k} = \chi_{i,j}(x_j + (t, F_j^k(t))$, for $t \in I_{r_j}$. Define

$$\widetilde{\psi}_j = \psi_1 \circ T^n \circ G_{F_j^1} \circ G_{F_j^2}^{-1} .$$

The function $\widetilde{\psi}_j$ is well-defined on U_j^2 and $d_0(\widetilde{\psi}_j, \psi_1 \circ T^n) = 0$. We can then estimate,

(4.10)

$$\sum_j \left| \int_{U_j^1} f \, \psi_1 \circ T^n - \int_{U_j^2} f \, \psi_2 \circ T^n \right| \le \sum_j \left| \int_{U_j^1} f \, \psi_1 \circ T^n - \int_{U_j^2} f \, \widetilde{\psi}_j \right| + \left| \int_{U_j^2} f \, (\widetilde{\psi}_j - \psi_2 \circ T^n) \right| .$$

We estimate the first term on the right side of (4.10) using the strong unstable norm. It follows from the uniform hyperbolicity of T and the usual graph transform arguments (see [DL, Section 4.3]), that

$$d_{\mathcal{W}^s}(U_j^1, U_j^2) \le C \Lambda^{-n} \varepsilon .$$

Moreover, by definition $G_{F_j^1}, G_{F_j^2}^{-1} \in \mathcal{C}^1$ so that by (4.4), $|\widetilde{\psi}_j|_{\mathcal{C}^1(U_j^2)} \le C |\psi_1|_{\mathcal{C}^1(W^1)}$ for some uniform constant C. Thus,

$$\sum_j \left| \int_{U_j^1} f \, \psi_1 \circ T^n - \int_{U_j^2} f \, \widetilde{\psi}_j \right| \le C \varepsilon^\beta \Lambda^{-\beta n} \|f\|_u \delta_0^{-1} \# \mathcal{M}_0^n , \tag{4.11}$$

where we have used Lemma 3.6(b) to sum over the matched pieces since there is at most one matched piece per element of $\mathcal{G}_n(W^1)$.

We estimate the second term on the right side of (4.10) using the strong stable norm,

$$\sum_j \left| \int_{U_j^2} f\,(\widetilde{\psi}_j - \psi_2 \circ T^n) \right| \le \sum_j \|f\|_s |U_j^2|^{1/p} |\widetilde{\psi}_j - \psi_2 \circ T^n|_{\mathcal{C}^\alpha(U_j^2)} \,.$$

It follows from [DL, Lemma 4.2 and eq. (4.20)] that,

$$|\widetilde{\psi}_j - \psi_2 \circ T^n|_{\mathcal{C}^\alpha(U_j^2)} \le C\varepsilon^{1-\alpha} \,.$$

Putting this together with the above estimate and summing over j yields,

$$\sum_j \left| \int_{U_j^2} f\,(\widetilde{\psi}_j - \psi_2 \circ T^n) \right| \le C\varepsilon^{1-\alpha} \|f\|_s \delta_0^{-1} \# \mathcal{M}_0^n \,.$$

Finally, collecting the above estimate with (4.11) in (4.10) and adding the estimate over unmatched pieces from (4.9), yields by (4.8),

$$\left| \int_{W^1} \mathcal{L}^n f\, \psi_1 - \int_{W^2} \mathcal{L}^n f \psi_2 \right| \le C\delta_0^{-1} (\varepsilon^\beta \Lambda^{-\beta n} \|f\|_u + \varepsilon^{1-\alpha}\|f\|_s + \varepsilon^{1/p} \kappa^{-n/p} \|f\|_s) \# \mathcal{M}_0^n \,.$$

Then, since $\beta \le \min\{1-\alpha, 1/p\}$ according to (3.4), we may divide through by ε^β, and take the appropriate suprema to complete the proof of (4.3).

4.2. **A spectral gap for $\mathcal{L}$.** We prove that $\mathcal{L}$ has a spectral gap in a series of lemmas, first establishing its quasi-compactness, Lemma 4.3, then characterizing elements of its peripheral spectrum, Lemmas 4.4 and 4.5, and finally concluding the existence of a spectral gap, Lemma 4.6. These are all the items of Theorem 4.1.

Lemma 4.3. *The spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is e^{h_*}, while its essential spectral radius is at most σe^{h_*} for any $\sigma > \max\{\Lambda^{-\beta}, \rho\}$. Thus $\mathcal{L}$ is quasi-compact on $\mathcal{L}$. Moreover, the peripheral spectrum of $\mathcal{L}$ contains no Jordan blocks.*

Proof. First we establish the upper bound on the spectral radius of $\mathcal{L}$ using Proposition 4.2 and Corollary 3.13. Fix $\sigma < 1$ such that $\sigma > \max\{\Lambda^{-\beta}, \rho\}$. Next, choose $N > 0$ such that $C\delta_0^{-2} 2c_1^{-1} \max\{\Lambda^{-\beta N}, \rho^N\} \le \frac{1}{2}\sigma^N$. Finally, choose $c_u > 0$ such that $c_u C \delta_0^{-2} 2 c_1^{-1} \kappa^{-N/p} \le \frac{1}{2}\sigma^N$. Then,

$$\begin{aligned} \|\mathcal{L}^N f\|_{\mathcal{B}} &= \|\mathcal{L}^N f\|_s + c_u \|\mathcal{L}^N f\|_u \\ &\le \left(\tfrac{1}{2}\sigma^N \|f\|_s + C\delta_0^{-2} 2c_1^{-1} \kappa^{-N/p} |f|_w + c_u \tfrac{1}{2}\sigma^N \|f\|_u + c_u C \delta_0^{-1} 2 c_1^{-1} \kappa^{-N/p} \|f\|_s \right) e^{N h_*} \\ &\le \left(\sigma^N \|f\|_{\mathcal{B}} + C' \delta_0^{-2} \kappa^{-N/p} |f|_w \right) e^{N h_*} \,. \end{aligned}$$

This is the standard Lasota-Yorke inequality for $\mathcal{L}$, which, coupled with the compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ (Lemma 3.5), is sufficient to conclude [H] that the essential spectral radius of $\mathcal{L}$ is at most σe^{h_*}, and its spectral radius is at most e^{h_*}.

To prove the lower bound on the spectral radius, we estimate using (3.9) and Lemma 3.11. Take $W \in \mathcal{W}^s$ with $|W| \ge \delta_1/3$. Then for $n \ge n_1$ we have,

$$\begin{aligned} \|\mathcal{L}^n 1\|_{\mathcal{B}} &\ge \int_W \mathcal{L}^n 1\, dm_W = \sum_{W_i \in \mathcal{G}_n^{\delta_1}} |W_i| \ge \sum_{W_i \in L_n^{\delta_1}(W)} \delta_1/3 \\ &\ge \frac{2\delta_1}{9} \# \mathcal{G}_n(W) \ge \frac{2\delta_1}{9} c_0 \# \mathcal{M}_0^n \,. \end{aligned} \tag{4.12}$$

Then taking the limit as $n \to \infty$ and using the definition of h_*,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^n\|_{\mathcal{B}} \ge \limsup_{n\to\infty} \frac{1}{n} \log \left(\|\mathcal{L}^n 1\|_{\mathcal{B}} / \|1\|_{\mathcal{B}} \right) \ge \limsup_{n\to\infty} \frac{1}{n} \log \left(\# \mathcal{M}_0^n \right) = h_* \,,$$

which proves that the spectral radius of $\mathcal{L}$ is at least e^{h_*}. We conclude that the spectral radius of $\mathcal{L}$ is in fact e^{h_*} and so $\mathcal{L}$ is quasi-compact since its essential spectral radius is bounded by σe^{h_*}.

Finally, the lack of Jordan blocks stems from Corollary 3.13 and Proposition 4.2, which together imply $\|\mathcal{L}^n\|_{\mathcal{B}} \le Ce^{nh_*}$ for all $n \ge 0$. $\square$

Let $\mathbb{V}_\theta$ denote the eigenspace associated to the eigenvalue $e^{h_*+2\pi i\theta}$. Due to the quasi-compactness of $\mathcal{L}$ and the absence of Jordan blocks, the spectral projector $\Pi_\theta : \mathcal{B} \to \mathbb{V}_\theta$ is well-defined in the uniform topology of $L(\mathcal{B},\mathcal{B})$ and can be realized as,

$$\Pi_\theta = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} e^{-2\pi i\theta k} \mathcal{L}^k \,. \tag{4.13}$$

Let $\mathbb{V} = \oplus_\theta \mathbb{V}_\theta$, where the sum is taken over θ corresponding to eigenvalues of $\mathcal{L}$. Note that $\mathbb{V}$ is finite dimensional by the quasi-compactness of $\mathcal{L}$. Analogously, and as in the statement of Theorem 4.1, define

$$\nu_0 = \Pi_0 1 := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_*} \mathcal{L}^k 1 \,. \tag{4.14}$$

Since we have proved uniform bounds of the form $\|\mathcal{L}^k\|_{\mathcal{B}} \le Ce^{kh_*}$, the limit above exists and satisfies $\mathcal{L}\nu_0 = e^{h_*}\nu_0$. A priori, however, ν_0 may be 0 (if e^{h_*} is not in the spectrum of $\mathcal{L}$). The following lemma shows this is not the case, and provides an important characterization of the peripheral spectrum of $\mathcal{L}$.

Lemma 4.4. *(Peripheral spectrum of $\mathcal{L}$)*

a) *The distribution $\nu_0 = \Pi_0 1 \neq 0$ is a non-negative Radon measure and e^{h_*} is in the spectrum of $\mathcal{L}$.*
b) *All elements of $\mathbb{V}$ are signed measures, absolutely continuous with respect to ν_0.*
c) *The spectrum of $e^{-h_*}\mathcal{L}$ consists of a finite number of cyclic groups; in particular, each θ is rational.*

Proof. (a) By density of $\mathcal{C}^1(M)$ in $\mathcal{B}$, since $\mathbb{V}_\theta$ is finite-dimensional, it follows that $\Pi_\theta \mathcal{C}^1(M) = \mathbb{V}_\theta$. Thus for each $\nu \in \mathbb{V}$, $\nu \neq 0$, there exists $f \in \mathcal{C}^1(M)$ such that $\Pi_\theta f = \nu$. Moreover, for every $\psi \in \mathcal{C}^1(M)$, we have

$$|\nu(\psi)| = |\Pi_\theta f(\psi)| \le \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-h_* k} |\mathcal{L}^k f(\psi)| \le |f|_\infty \Pi_0 1(|\psi|) \,, \tag{4.15}$$

so that $\Pi_0 1 \neq 0$ since $\nu \neq 0$. In particular, e^{h_*} is an eigenvalue of $\mathcal{L}$. Moreover, since $\Pi_0 1$ is positive as an element of $(\mathcal{C}^1(M))^*$, it follows from [Sch, Sect. I.4] that $\nu_0 = \Pi_0 1$ is a non-negative Radon measure on M.

(b) Applying (4.15) again to $\nu \in \mathbb{V}_\theta$, we conclude that every element of $\mathbb{V}_\theta$ is a signed measure, absolutely continuous with respect to ν_0. Moreover, setting $f_\nu = \frac{d\nu}{d(\nu_0)}$, it follows that $f_\nu \in L^\infty(M, \nu_0)$.

(c) Suppose $\nu \in \mathbb{V}_\theta$. Then using part (b), for any $\psi \in \mathcal{C}^1(M)$,

$$\begin{aligned} \int_M \psi\, f_\nu \, d\nu_0 &= \nu(\psi) = e^{-h_*} e^{-2\pi i\theta} \mathcal{L}\nu(\psi) = e^{-h_*} e^{-2\pi i\theta} \nu\Big(\frac{\psi \circ T}{J^sT}\Big) \\ &= e^{-h_*} e^{-2\pi i\theta} \nu_0\Big(f_\nu \frac{\psi \circ T}{J^sT}\Big) = e^{-h_*} e^{-2\pi i\theta} \mathcal{L}\nu_0(\psi f_\nu \circ T^{-1}) \\ &= e^{-2\pi i\theta} \int_M \psi\, f_\nu \circ T^{-1} \, d\nu_0 \,. \end{aligned} \tag{4.16}$$

Thus $f_\nu \circ T^{-1} = e^{2\pi i\theta} f_\nu$, ν_0-a.e. Define $f_{\nu,k} = (f_\nu)^k \in L^\infty(\nu_0)$. It follows as in [DL, Lemma 5.5], that $d\nu_k := f_{\nu,k} d\nu_0 \in \mathcal{B}$ for each $k \in \mathbb{N}$. Then since $\mathcal{L}\nu_k = e^{2\pi ik\theta}\nu_k$, it follows that $e^{2\pi ik\theta}$ is in the peripheral spectrum of $\mathcal{L}$ for each k. By the quasi-compactness of $\mathcal{L}$, this set must be finite, and so θ must be rational. □

We remark that elements of $\mathcal{B}_w$ can be viewed as both distributions on M, as well as families of *leafwise distributions* on stable manifolds as follows (cf. [BD, Definition 7.5]). For $f \in \mathcal{C}^1(M)$, the map defined by

$$\mathcal{K}_{(W,f)}(\psi) = \int_W f\psi \, dm_W, \qquad \psi \in \mathcal{C}^1(W)\,,$$

can be viewed as a distribution of order 1 on W. Since $\mathcal{K}_{(W,f)}(\psi) \le |f|_w |\psi|_{\mathcal{C}^1(W)}$, $\mathcal{K}_{(W,\,\cdot\,)}$ can be extended to $f \in \mathcal{B}_w$. We denote this extension by $\int_W \psi f$, and we call the associated family of distributions the *leafwise distribution* $(f,W)_{W\in\mathcal{W}^s}$ corresponding to f. If, in addition, $f \in \mathcal{B}_w$ satisfies $\int_W \psi f \ge 0$ for all $\psi \ge 0$, then by [Sch, Section I.4], the leafwise distribution is in fact a leafwise measure.

Recall the disintegration of μ_{SRB} used in the proof of Lemma 3.3 into conditional measures μ^ξ_{SRB} on the family of stable manifolds $\mathcal{F} = \{W_\xi\}_{\xi\in\Xi}$, and a factor measure $\hat\mu_{\mathrm{SRB}}$ on the index set Ξ. We have $d\mu^\xi_{\mathrm{SRB}} = |W_\xi|^{-1} g_\xi dm_{W_\xi}$, where g_ξ is uniformly log-Hölder continuous by (3.5).

Lemma 4.5. *Let ν_0^ξ and $\hat\nu_0$ denote the conditional measures on W_ξ and factor measure on Ξ, respectively, obtained by disintegrating ν_0 on the family of stable manifolds $\mathcal{F}$. For all $\psi \in \mathcal{C}^1(M)$,*

$$\int_{W_\xi} \psi \, d\nu_0^\xi = \frac{\int_{W_\xi} \psi \, g_\xi \, \nu_0}{\int_{W_\xi} g_\xi \, \nu_0} \quad \text{for all } \xi \in \Xi, \text{ and} \quad d\hat\nu_0(\xi) = |W_\xi|^{-1}\Big(\int_{W_\xi} g_\xi \, \nu_0\Big)\, d\hat\mu_{SRB}(\xi)\,.$$

Moreover, viewed as a leafwise measure, $\nu_0(W) > 0$ for all $W \in \mathcal{W}^s$.

Proof. We prove the last claim first. For $W \in \mathcal{W}^s$, let $n_2 \le \bar C_2 |\log(|W|/\delta_1)|$ be the constant from the proof of Corollary 3.8 applied in the case $\varepsilon = 1/3$ and δ_1 as chosen in (3.9). Let $V \in \mathcal{G}^{\delta_1}_{n_2}(W)$ have $|V| \ge \delta_1/3$. Then using (3.9) and Lemma 3.11,

$$\begin{aligned}\int_W \nu_0 &= \lim_{n\to\infty} \frac1n \sum_{k=0}^{n-1} e^{-kh_*} \int_W \mathcal{L}^k 1 \, dm_W \ge \lim_{n\to\infty} \frac1n \sum_{k=n_1+n_2}^{n-1} e^{-kh_*} \sum_{W_i \in \mathcal{G}_{k-n_2}(V)} |W_i| \\ &\ge \lim_{n\to\infty} \frac1n \sum_{k=n_1+n_2}^{n-1} e^{-kh_*} \tfrac{2\delta_1}{9} c_0 \# \mathcal{M}_0^{k-n_2} = \tfrac{2c_0\delta_1}{9} e^{-(n_1+n_2)h_*} \lim_{n\to\infty} \sum_{k=0}^{\infty} e^{-kh_*} \#\mathcal{M}_0^k\,.\end{aligned}$$

We claim that the last limit cannot be 0. For suppose it were 0. Then for any $W \in \mathcal{W}^s$, $\psi \in \mathcal{C}^1(W)$, we would have by Lemma 3.6(b),

$$\begin{aligned}\int_W \psi \, \nu_0 &= \lim_{n\to\infty} \frac1n \sum_{k=0}^{n-1} e^{-kh_*} \int_W \psi \, \mathcal{L}^k 1 \, dm_W \le \lim_{n\to\infty} \frac1n \sum_{k=0}^{n-1} e^{-kh_*} \sum_{W_i \in \mathcal{G}_k(W)} |\psi|_\infty |W_i| \\ &\le \lim_{n\to\infty} \frac1n \sum_{k=0}^{n-1} e^{-kh_*} C \#\mathcal{M}_0^k = 0\,,\end{aligned}$$

which would imply $\nu_0 = 0$, a contradiction. This proves the claim, and recalling the definition of n_2, we conclude that

$$\nu_0(W) \ge C' |W|^{h_* \bar C_2} \qquad \text{for all } W \in \mathcal{W}^s. \tag{4.17}$$

With (4.17) established, the remainder of the proof follows from the definition of convergence in the weak norm, precisely as in [BD, Lemma 7.7]. □

We are finally ready to prove the final point of our characterization of the peripheral spectrum of $\mathcal{L}$.

Lemma 4.6. *$\mathcal{L}$ has a spectral gap on $\mathcal{B}$.*

Proof. Recalling Lemma 4.4(c), suppose $\nu_q \in \mathbb{V}_{p/q}$. Then $\mathcal{L}^q \nu_q = e^{qh_*}\nu_q$ and $\mathcal{L}^q \nu_0 = e^{qh_*}\nu_0$. Since T^q is also mixing and the spectral radius of $\mathcal{L}^q$ is e^{qh_*}, it suffices to prove that mixing implies the eigenspace corresponding to e^{h_*} is simple in order to conclude that $\mathcal{L}$ can have no other eigenvalues of modulus e^{h_*}, i.e. $\mathcal{L}$ has a spectral gap. We proceed to prove this claim.

Suppose $\nu_1 \in \mathbb{V}_0$. We will show that $\nu_1 = c\nu_0$ for some constant $c > 0$. By (4.16), there exists $f_1 \in L^\infty(\nu_0)$ such that $f_1 \nu_1 = \nu_0$ and $f_1 \circ T = f_1$, ν_0-a.e. Letting

$$S_n f_1(x) = \sum_{k=0}^{n-1} f_1 \circ T^k(x) \, ,$$

it follows that the ergodic average $\frac{1}{n} S_n f_1 = f_1$ for all $n \geq 0$. This implies that f_1 is constant on stable manifolds. In addition, since by Lemma 4.5 and (4.17), the factor measure $\hat\nu_0$ is equivalent to $\hat\mu_{\mathrm{SRB}}$ on the index set Ξ, we have that $f_1 = f_1 \circ T$ on $\hat\mu_{\mathrm{SRB}}$ a.e. $W_\xi \in \mathcal{F}$, i.e. $f_1 = f_1 \circ T$, μ_{SRB}-a.e. By the ergodicity of μ_{SRB}, $f_1 =$ constant μ_{SRB}-a.e. But since this constant value holds on each stable manifold $W_\xi \in \mathcal{F}$, using again the equivalence of $\hat\nu_0$ and $\hat\mu_{\mathrm{SRB}}$, we conclude that f_1 is constant ν_0-a.e. □

5. Construction and Properties of the Measure of Maximal Entropy

Since $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ has a spectral gap, we may decompose $\mathcal{L}$ as

$$\mathcal{L}^n f = e^{nh_*}\Pi_0 f + R^n f \ \text{ for any } n \geq 1, \ f \in \mathcal{B}, \tag{5.1}$$

where $\Pi_0^2 = \Pi_0$, $\Pi_0 R = R\Pi_0 = 0$ and there exists $\bar\sigma < 1$ and $C > 0$ such that $\|e^{-nh_*}R^n\|_{\mathcal{B}} \leq C\bar\sigma^n$. Indeed, we may recharacterize the definition of the spectral projector Π_0 in (4.13) as,

$$\Pi_0 f = \lim_{n\to\infty} e^{-nh_*}\mathcal{L}^n f \, ,$$

where convergence is in the $\mathcal{B}$ norm. Indeed, letting $W \in \mathcal{W}^s$ with $|W| \geq \delta_1/3$, we have by Lemma 3.6(b) and (4.17),

$$\begin{aligned} 0 < \nu_0(W) &= \lim_{n\to\infty} e^{-nh_*}\int_W \mathcal{L}^n 1 \, dm_W = \lim_{n\to\infty} e^{-nh_*} \sum_{W_i \in \mathcal{G}_n(W)} |W_i| \\ &\leq \liminf_{n\to\infty} C e^{-nh_*} \# \mathcal{M}_0^n \, . \end{aligned}$$

This implies the final limit cannot be 0. We have proved the following.

Lemma 5.1. *There exists $\bar c_1 > 0$ such that $\#\mathcal{M}_0^n \geq \bar c_1 e^{nh_*}$ for all $n \geq 1$.*

Next, consider the dual operator, $\mathcal{L}^* : \mathcal{B}^* \to \mathcal{B}^*$, which also has a spectral gap. Recalling our identification of $f \in \mathcal{C}^1(M)$ with the measure $f d\mu_{\mathrm{SRB}}$ from Section 3.2, define

$$\tilde\nu_0 := \lim_{n\to\infty} e^{-nh_*}(\mathcal{L}^*)^n d\mu_{\mathrm{SRB}} \, , \tag{5.2}$$

where convergence is in the dual norm, $\|\cdot\|_{\mathcal{B}^*}$. Clearly, $\tilde\nu_0 \in \mathcal{B}^*$, and $\mathcal{L}^*\tilde\nu_0 = e^{h_*}\tilde\nu_0$. By the positivity of the operator $\mathcal{L}^*$, we have $\tilde\nu_0(f) \geq 0$ for each $f \in \mathcal{C}^1(M)$ with $f \geq 0$ (recalling $\mathcal{C}^1(M) \subset \mathcal{B}$). Thus again applying [Sch, Section I.4], we conclude that $\tilde\nu_0$ is a Radon measure on M.

Next, defining $f_n = e^{-nh_*}\mathcal{L}^n 1 \in \mathcal{B}$ for $n \geq 1$, we have,

$$\tilde\nu_0(f_n) = \lim_{k\to\infty} e^{-kh_*}\langle f_n, (\mathcal{L}^*)^k d\mu_{\mathrm{SRB}}\rangle = \lim_{k\to\infty} e^{-kh_*}\langle \mathcal{L}^k f_n, d\mu_{\mathrm{SRB}}\rangle \, ,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between an element of $\mathcal{B}$ and an element of $\mathcal{B}^*$. Then, decomposing μ_{SRB} into its conditional measures μ^{ξ}_{SRB} and factor measure $\hat\mu_{\mathrm{SRB}}$ on W_ξ, $\xi \in \Xi$, as in the proof of Lemma 3.3, and letting $\Xi^{\delta_1} \subset \Xi$ denote the set of indices such that $|W_\xi| \ge \delta_1/3$, we estimate

$$\begin{aligned}
\tilde\nu_0(f_n) &= \lim_{k\to\infty} \int_M f_{n+k}\, d\mu_{\mathrm{SRB}} = \lim_{k\to\infty} \int_\Xi d\hat\mu_{\mathrm{SRB}}(\xi)\, e^{-(n+k)h_*} \int_{W_\xi} \mathcal{L}^{n+k} 1\, g_\xi \, dm_{W_\xi} |W_\xi|^{-1} \\
&\ge \lim_{k\to\infty} \int_{\Xi^{\delta_1}} d\hat\mu_{\mathrm{SRB}}(\xi)\, e^{-(n+k)h_*} \sum_{W_{\xi,i} \in L^{\delta_1}_{n+k}(W_\xi)} \inf_{W_\xi} g_\xi \, |W_{\xi,i}| |W_\xi|^{-1} \\
&\ge \lim_{k\to\infty} \int_{\Xi^{\delta_1}} d\hat\mu_{\mathrm{SRB}}(\xi)\, e^{-(n+k)h_*} C_g^{-1} \tfrac{2c_0}{9} \# \mathcal{M}_0^{n+k} \ge \hat\mu_{\mathrm{SRB}}(\Xi^{\delta_1}) C_g^{-1} \tfrac{2c_0}{9} \bar c_1 \,,
\end{aligned}$$

for all $n \ge 1$, where we have used (3.9) and Lemma 3.11 for the second inequality, and Lemma 5.1 for the third. Since this lower bound is independent of n, we have $\tilde\nu_0(\nu_0) > 0$.

We can at last formulate the following definition, which is our candidate for the measure of maximal entropy.

Definition 5.2. *For $\psi \in \mathcal{C}^1(M)$, define,*

$$\mu_*(\psi) := \frac{\langle \psi \nu_0, \tilde\nu_0 \rangle}{\langle \nu_0, \tilde\nu_0 \rangle} \,.$$

The measure μ_* is a probability measure on M due to the positivity of ν_0 and $\tilde\nu_0$, and since $\langle \nu_0, \tilde\nu_0 \rangle \neq 0$. Moreover, $\mu_*(\psi \circ T) = \mu_*(\psi)$ so that μ_* is an invariant measure for T.

We may also characterize the spectral projector Π_0 in terms of this pairing: for any $f \in \mathcal{B}$, it follows from (5.1) and (5.2) that,

$$\Pi_0 f = \frac{\langle f, \tilde\nu_0 \rangle}{\langle \nu_0, \tilde\nu_0 \rangle} \nu_0 \,. \tag{5.3}$$

It follows immediately from the spectral gap of $\mathcal{L}$ that μ_* has exponential decay of correlations.

Proposition 5.3. *For all $q > 0$, there exist constants $C = C(q)$ and $\gamma = \gamma(q) > 0$ such that for all $\varphi, \psi \in \mathcal{C}^q(M)$,*

$$\left| \int_M \varphi\, \psi \circ T^n \, d\mu_* - \int_M \varphi \, d\mu_* \int_M \psi \, d\mu_* \right| \le C |\varphi|_{\mathcal{C}^q(M)} |\psi|_{\mathcal{C}^q(M)} e^{-\gamma n} \qquad \textit{for all } n \ge 0.$$

Proof. We prove the proposition for $\varphi, \psi \in \mathcal{C}^1(M)$. The result for $q \in (0,1)$ then follows by a standard approximation argument.

First we verify that $\psi \circ T^n \tilde\nu_0$ is an element of $\mathcal{B}^*$ for $\psi \in \mathcal{C}^1(M)$ and $n \ge 1$. We do this by noting that for any $\psi \in \mathcal{C}^1(M)$, $\psi \tilde\nu_0 \in \mathcal{B}^*$ by simply defining,

$$\langle f, \psi \tilde\nu_0 \rangle := \langle \psi f, \tilde\nu_0 \rangle \qquad \text{for any } f \in \mathcal{B},$$

and the expression on the right is bounded by $|\psi|_{\mathcal{C}^1} \|f\|_{\mathcal{B}} \|\nu_0\|_{\mathcal{B}^*}$ by Lemma 3.2(b), and so the pairing defines a bounded, linear functional on $\mathcal{B}$, with norm at most $|\psi|_{\mathcal{C}^1} \|\tilde\nu_0\|_{\mathcal{B}^*}$. Next, define for $n \ge 1$,

$$\langle f, \psi \circ T^n \tilde\nu_0 \rangle := \langle e^{-nh_*} \mathcal{L}^n f, \psi \nu_0 \rangle = \langle \psi\, e^{-nh_*} \mathcal{L}^n f, \nu_0 \rangle \,. \tag{5.4}$$

The expression on the right is bounded by

$$\|\psi e^{-nh_*} \mathcal{L}^n f\|_{\mathcal{B}} \|\tilde\nu_0\|_{\mathcal{B}^*} \le |\psi|_{\mathcal{C}^1(M)} e^{-nh_*} \|\mathcal{L}^n f\|_{\mathcal{B}} \|\tilde\nu_0\|_{\mathcal{B}^*} \le C |\psi|_{\mathcal{C}^1(M)} \|f\|_{\mathcal{B}} \|\tilde\nu_0\|_{\mathcal{B}^*} \,,$$

where we have used Lemma 3.2(b) for the first inequality and (5.1) for the second, since in particular, $e^{-nh_*} \|\mathcal{L}^n f\|_{\mathcal{B}} \le C$. Thus (5.4) defines a bounded, linear functional on $\mathcal{B}$, so $\psi \circ T^n \tilde\nu_0 \in \mathcal{B}^*$.

Finally, using Definition 5.2 and (5.4), noting that $\varphi\nu_0 \in \mathcal{B}$ by Lemma 3.2(b), and recalling again (5.1), we write

$$\int_M \varphi\,\psi\circ T^n\,d\mu_* = \frac{\langle \varphi\,\nu_0, \psi\circ T^n\tilde\nu_0\rangle}{\langle \nu_0,\tilde\nu_0\rangle} = \frac{\langle e^{-nh_*}\mathcal{L}^n(\varphi\,\nu_0), \psi\tilde\nu_0\rangle}{\langle \nu_0,\tilde\nu_0\rangle}$$
$$= \frac{\langle \Pi_0(\varphi\nu_0) + e^{-nh_*}R^n(\varphi\nu_0), \psi\tilde\nu_0\rangle}{\langle \nu_0,\tilde\nu_0\rangle} = \frac{\langle \varphi\nu_0,\tilde\nu_0\rangle}{\langle \nu_0,\tilde\nu_0\rangle}\frac{\langle \nu_0,\psi\tilde\nu_0\rangle}{\langle \nu_0,\tilde\nu_0\rangle} + \frac{\langle e^{-nh_*}R^n(\varphi\nu_0), \psi\tilde\nu_0\rangle}{\langle \nu_0,\tilde\nu_0\rangle}\,,$$

where we have used (5.3). The first term on the right is simply $\int_M \varphi\,d\mu_* \int_M \psi\,d\mu_*$. The second term is bounded by,

$$Ce^{-nh_*}\|R^n(\varphi\nu_0)\|_{\mathcal{B}}\|\psi\tilde\nu_0\|_{\mathcal{B}^*} \le C'\bar\sigma^n\|\varphi\nu_0\|_{\mathcal{B}}|\psi|_{\mathcal{C}^1}\|\tilde\nu_0\|_{\mathcal{B}^*} \le C''\bar\sigma^n|\varphi|_{\mathcal{C}^1}|\psi|_{C^1}\,,$$

where we have used Lemma 3.2(b) and C'' depends on $\|\nu_0\|_{\mathcal{B}}$, $\|\tilde\nu_0\|_{\mathcal{B}^*}$, and $\langle\nu_0,\tilde\nu_0\rangle$. □

sec:hyper

5.1. **Hyperbolicity and Ergodicity of μ_*.** We begin by showing that μ_* gives small measure to ε-neighborhoods of the singularity sets $\mathcal{S}_n^\pm$.

lem:control **Lemma 5.4.** *For any $k\in\mathbb{N}$, there exists $C_k>0$ such that*

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k^\pm)) \le C_k\varepsilon^{1/p}\,.$$

In particular, for any $\gamma>p$ and $k\in\mathbb{N}$, for μ_*-a.e. $x\in M$, there exists $C>0$ such that

eq:approach
$$d(T^nx,\mathcal{S}_k^\pm) \ge Cn^{-\gamma}\,, \quad \textit{for all } n\ge 0. \tag{5.5}$$

Proof. First we prove the claimed bounds with respect to ν_0 for each $\mathcal{S}_k^-$, $k\ge1$. Let $1_{k,\varepsilon}$ denote the indicator function of the set $\mathcal{N}_\varepsilon(\mathcal{S}_k^-)$. Since $\mathcal{S}_k^-$ comprises finitely many smooth curves, all uniformly transverse to the stable cone, by Lemma 3.2(b), $1_{k,\varepsilon}\nu_0\in\mathcal{B}$, and as a consequence, $1_{k,\varepsilon}\nu_0\in\mathcal{B}_w$. We claim that,

eq:nu bound
$$\nu_0(\mathcal{N}_\varepsilon(\mathcal{S}_k^-)) \le C|1_{k,\varepsilon}\nu_0|_w \le C_k\varepsilon^{1/p}\,. \tag{5.6}$$

Indeed, the first inequality follows from Lemma 3.3. To prove the second inequality, let $W\in\mathcal{W}^s$ and $\psi\in\mathcal{C}^1(W)$ with $|\psi|_{\mathcal{C}^1(W)}\le1$. Due to the transversality of $\mathcal{S}_k^-$ with the stable cone, $W\cap\mathcal{N}_\varepsilon(\mathcal{S}_k^-)$ comprises at most a finite number N_k of curves, depending only on $\mathcal{S}_k^-$ and δ_0, and not on W, each having length at most $C\varepsilon$. Thus,

$$\int_W 1_{k,\varepsilon}\,\psi\,\nu_0 = \sum_i\int_{W_i}\psi\,\nu_0 \le \sum_i \|\nu_0\|_{\mathcal{B}}|W_i|^{1/p}|\psi|_{\mathcal{C}^\alpha(W_i)} \le CN_k\varepsilon^{1/p}\,,$$

and taking the supremum over ψ and W proves the second inequality in (5.6).

Next, it follows from (5.2) and Lemma 3.3 that

q:weak dual
$$|\tilde\nu_0(f)| \le C|f|_w, \qquad \text{for all } f\in\mathcal{B}_w, \tag{5.7}$$

so that in fact $\tilde\nu_0\in\mathcal{B}_w^*\subset\mathcal{B}^*$. Thus for each $k\ge1$, by (5.6),

$$\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k^-)) = \frac{\tilde\nu_0(1_{k,\varepsilon}\nu_0)}{\tilde\nu_0(\nu_0)} \le C|1_{k,\varepsilon}\nu_0|_w \le CC_k\varepsilon^{1/p}\,.$$

To prove the bound for $\mathcal{S}_k^+$, we use the invariance of μ_* together with the fact that $T^{-k}\mathcal{S}_k^- = \mathcal{S}_k^+$. Moreover, we have $T^k(\mathcal{N}_\varepsilon(\mathcal{S}_k^+))\subset\mathcal{N}_{C\kappa_+^k\varepsilon}(\mathcal{S}_k^-)$, where κ_+ is the maximum rate of expansion in the unstable cone.

Finally, to prove (5.5), we fix $\gamma>p$ and estimate for each $k\in\mathbb{N}$,

$$\sum_{n\ge1}\mu_*(\mathcal{N}_{n^{-\gamma}}(\mathcal{S}_k^\pm)) \le C_k\sum_{n\ge1}n^{-\gamma/p} < \infty\,.$$

Thus by the Borel-Cantelli Lemma, μ_*-a.e. $x\in M$ visits $\mathcal{N}_{n^{-\gamma}}(\mathcal{S}_k^\pm)$ only finitely many times along its orbit, completing the proof of the lemma. □

Lemma 5.4 immediately implies the following corollary.

Corollary 5.5. *The following items establish the hyperbolicity of the measure μ_*.*

a) *For any $\mathcal{C}^1$ curve V uniformly transverse to the stable cone, there exists $C > 0$ such that $\nu_0(\mathcal{N}_\varepsilon(V)) \le C\varepsilon$ for all $\varepsilon > 0$.*
b) *The measures ν_0 and μ_* have no atoms, and $\mu_*(W) = 0$ for all local stable and unstable manifolds, W.*
c) $\int_M |\log d(x, \mathcal{S}_1^{\pm})|\, d\mu_* < \infty$.
d) *μ_*-a.e. $x \in M$ has a stable and an unstable manifold of positive length.*

Proof. The proof follows directly from the control established on the measures of the neighborhoods of the singularity sets in Lemma 5.4. The argument follows exactly as in [BD, Corollary 7.4]. □

With the control established in Lemma 5.4, we may follow the same arguments as in [BD, Section 7.3] to establish the ergodicity of the measure μ_*. Indeed, our control is stronger than the bounds $\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k^{\pm})) \le C_k |\log \varepsilon|^\gamma$ for some $\gamma > 1$ available in [BD], and the Hölder continuity of our strong norm $\|\cdot\|_u$ is stronger than the logarithmic modulus of continuity available in [BD]. The key result is establishing the absolute continuity of the unstable foliation with respect to μ_*. Given a locally maximal Cantor rectangle R, let $\mathcal{W}^{s/u}(R)$ be the set of stable/unstable manifolds that cross $D(R)$ completely (see Section 3.6).

Proposition 5.6. *Let R be a locally maximal Cantor rectangle with $\mu_*(R) > 0$. Fix $W^0 \in \mathcal{W}^s(R)$, and for $W \in \mathcal{W}^s(R)$, let $\Theta_W : W^0 \cap R \to W \cap R$ denote the holonomy map sliding along unstable manifolds in $\mathcal{W}^u(R)$. Then Θ_W is absolutely continuous with respect to μ_*.*

Proof. This is [BD, Corollary 7.9]. Its proof relies on the analogous property of absolute continuity for ν_0, which in turn follows from the control established by the strong norm and Lemma 5.4. The final step in the proof is to show that on each $W \in \mathcal{W}^s(R)$, the conditional measure μ_*^W of μ_* is equivalent to the leafwise measure ν_0 restricted to W, i.e. there exists $C_W > 0$ such that

$$C_W \mu_*^W \le \nu_0|_W \le C_W^{-1} \mu_*^W \,. \tag{5.8}$$

This equivalence of the measures follows from the representation of ν_0 as a family of leafwise measures given by Lemma 4.5 as well as the characterization of μ_* via the limit,

$$\mu_*(\psi) = \tilde\nu_0(\nu)^{-1} \tilde\nu_0(\psi\nu) = \tilde\nu_0(\nu_0)^{-1} \lim_{n\to\infty} e^{-nh_*} (\mathcal{L}^*)^n d\mu_{\mathrm{SRB}}(\psi\nu)\,,$$

from (5.2). □

Corollary 5.7. *The absolute continuity of the unstable holonomy with respect to μ_* implies the following additional properties.*

a) *(T^n, μ_*) is ergodic for all $n \ge 1$.*
b) *For any open set $O \subset M$, we have $\mu_*(O) > 0$.*

Proof. a) Using absolute continuity, one establishes that each Cantor rectangle belongs to a single ergodic component following the usual Hopf argument [BD, Lemma 7.15]. Then the ergodicity of T^n follows from the assumption that T is topologically mixing [BD, Proposition 7.16].

b) The proof is identical to the proof of [BD, Proposition 7.11]. □

5.2. **Entropy of μ_*.** In this section, we prove that the measure-theoretic entropy of μ_* is h_*, by estimating the measure of dynamically defined Bowen balls for T^{-1}. Recall the metric $\bar d$ defined in (2.2). For $n \ge 0$ and $\varepsilon > 0$ and $x \in M$, define

$$B_n(x, \varepsilon) = \{y \in M : \bar d(T^{-j}y, T^{-j}x) \le \varepsilon, \forall\, 0 \le j \le n\}\,.$$

lem:bowen **Lemma 5.8.** *There exists $C > 0$ such that for all $\varepsilon > 0$ sufficiently small and all $n \geq 0$, we have*[11]

$$\mu_*(B_n(x, \varepsilon)) \leq Cne^{-nh_*} .$$

Proof. Fix $x \in M$, $\varepsilon > 0$ and $n \geq 0$, and let $1^B_{n,\varepsilon}$ denote the indicator function of the Bowen ball $B_n(x, \varepsilon)$. We shall prove

eq:goal
$$\mu_*(B_n(x, \varepsilon))) = \frac{\tilde{\nu}_0(1^B_{n,\varepsilon} \nu_0)}{\tilde{\nu}_0(\nu_0)} \leq C|1^B_{n,\varepsilon}\nu_0|_w \leq Cne^{-nh_*} , \tag{5.9}$$

where $C > 0$ can be chosen independent of ε. The first inequality follows from (5.7), once we show that $1^B_{n,\varepsilon}\nu_0 \in \mathcal{B}_w$. To see this, write

$$1^B_{n,\varepsilon} = \prod_{j=0}^{n} 1_{\mathcal{N}_\varepsilon(T^{-j}x)} \circ T^{-j} = \prod_{j=0}^{n} \mathcal{L}^j_{\mathrm{SRB}}(1_{\mathcal{N}_\varepsilon(T^{-j}x)}) ,$$

where $\mathcal{L}_{\mathrm{SRB}}$ denotes the transfer operator with respect to μ_{SRB}. Since $\mathcal{L}_{\mathrm{SRB}}$ preserves $\mathcal{B}_w$ (and also $\mathcal{B}$) by [DL], the claim follows since $1_{\mathcal{N}_\varepsilon(T^{-j}x)}$ satisfies the assumptions of Lemma 3.2: $\partial \mathcal{N}_\varepsilon(T^{-j}x)$ consists of a single circular arc, together with possibly part of ∂M, both of which satisfy the weak transversality condition of that lemma for ε sufficiently small. Applying Lemma 3.2(b) inductively in j completes the proof of the claim, and of the first inequality in (5.9).

Next, since ν_0 is a non-negative leafwise measure by Lemma 4.5, we have $\int_W \psi \, \nu_0 \geq 0$ for all $W \in \mathcal{W}^s$ and $\psi \geq 0$. Then since $| \int_W \psi \, \nu_0| \leq \int_W |\psi| \, \nu_0$, we can achieve the supremum in the weak norm of ν_0 by restricting to test functions $\psi \geq 0$.

Now take $W \in \mathcal{W}^s$, $\psi \in \mathcal{C}^1(W)$ with $\psi \geq 0$ and $|\psi|_{\mathcal{C}^1(W)} \leq 1$, and suppose that $W \cap B_n(x, \varepsilon) \neq \emptyset$. Then using that ν_0 is an eigenfunction of $\mathcal{L}$,

$$\int_W \psi \, 1^B_{n,\varepsilon} \, \nu_0 = \int_W \psi \, 1^B_{n,\varepsilon} \, e^{-nh_*} \mathcal{L}^n \nu_0 = e^{-nh_*} \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} \psi \circ T^n \, 1^B_{n,\varepsilon} \circ T^n \, \nu_0 \, .$$

Observe that $1^B_{n,\varepsilon} \circ T^n = 1_{T^{-n}(B_n(x,\varepsilon))}$, and that

$$T^{-n}(B_n(x, \varepsilon)) = \{y \in M : \bar{d}(T^{j-n}x, T^j y) \leq \varepsilon, \forall \, 0 \leq j \leq n\} .$$

Thus on each $W_i \in \mathcal{G}_n(W)$ such that $W_i \cap T^{-n}(B_n(x, \varepsilon)) \neq \emptyset$, the positivity of ν_0 implies,

$$\int_{W_i} \psi \circ T^n \, 1^B_{n,\varepsilon} \circ T^n \, \nu_0 \leq \nu_0(W_i) \leq |\nu_0|_w \, .$$

It remains to estimate the cardinality of such W_i. Recalling (2.2), if $\varepsilon < 10 \, \mathrm{diam}(M)$, and $\bar{d}(T^{j-n}x, T^j y) \leq \varepsilon$, then $T^{j-n}x$ and $T^j y$ belong to the same set $\overline{M}^+_{i_j}$ for each j. We would like to conclude that then $T^{-n}(B_n(x, \varepsilon))$ belongs to a single element of $\mathcal{M}^n_0$, yet this may fail since both the dynamical refinements of $\mathcal{M}^1_0$ and the local components of $T^{-j}W \subset \overline{M}^+_{i_j}$ may not be connected. Figure ?? shows an example of how these multiple components may arise due to intersections of $\mathcal{S}^+_j$ with $\mathcal{S}^-$.

Yet suppose $V \subset V' \in \mathcal{G}_j(W)$, $|V| < \varepsilon$. Since $\mathcal{S}^-$ is fixed and uniformly transverse to the stable cone, for ε sufficiently small there can be at most two connected components of V that lie in the same $M^-_{i_j}$; these will be mapped to the same $M^+_{i_j}$ under T^{-1}. Since this subdivision of a set of radius ε can occur at most once per iterate, we have at most n elements $W_i \in \mathcal{G}_n(W)$ such that $W_i \cap T^{-n}(B_n(x, \varepsilon)) \neq 0$ for ε sufficiently small. Putting these estimates together yields,

$$\int_W \psi \, 1^B_{n,\varepsilon} \, \nu_0 \leq e^{-nh_*} n |\nu_0|_w \, ,$$

and taking the supremum over ψ and W yields the final inequality in (5.9). $\square$

[11] The extra factor of n in this estimate is due to the fact that we do not assume the dynamical refinements of $\mathcal{M}^1_0$ are simply connected. Such an assumption would allow us to eliminate this factor, as in [BD, Proposition 7.12].

Proposition 5.9. *For μ_* defined by Definition 5.2, we have $h_{\mu_*}(T) = h_*$.*

Proof. Recall that $\int_M |\log d(x, \mathcal{S}_1^{\pm})| \, d\mu_* < \infty$ by Corollary 5.5(c), and that μ_* is ergodic by Corollary 5.7. Thus applying [DWY, Proposition 3.1],[12] we conclude that for μ_*-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_*(B_n(x, \varepsilon)) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_*(B_n(x, \varepsilon)) = h_{\mu_*}(T^{-1}) = h_{\mu_*}(T) \, .$$

On the other hand, Lemma 5.8 implies that for all $\varepsilon > 0$ sufficiently small,

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_*(B_n(x, \varepsilon)) \geq h_* \, .$$

Thus $h_{\mu_*}(T) \geq h_*$. But $h_{\mu_*}(T) \leq h_*$ by Theorem 2.8(d), so equality follows. □

5.3. **Uniqueness of μ_*.** In this section we prove that μ_* is the unique invariant probability measure with $h_{\mu_*}(T) = h_*$.

The proof of uniqueness follows very closely the proof of uniqueness in [BD, Section 7.7]. We include the proof to point out several differences in the initial estimates on elements of $\mathcal{M}_{-n}^0$, and for completeness. The idea of the proof is to adapt Bowen's proof of the uniqueness of equilibrium states to the setting of maps with discontinuities. The key estimates will be to show that while not all elements of $\mathcal{M}_{-n}^0$ satisfy good lower bounds on their measure, most elements (in the sense of Lemma 5.10) have satisfied good lower bounds at some point in the recent past (in the sense of Lemma 5.11). Recall that $\mathcal{M}_0^n$ denotes the set of maximal, open connected components on which T^n is smooth, while $\mathcal{M}_{-n}^0$ denotes the analogous set for T^{-n}.

Choose $\delta_2 > 0$ sufficiently small that for all $n, k \in \mathbb{N}$, if $A \in \mathcal{M}_{-k}^n$ is such that $\mathrm{diam}^u(A) \leq \delta_2$ and $\mathrm{diam}^s(A) \leq \delta_2$, then $A \setminus \mathcal{S}^{\pm}$ consists of no more that K_1 connected components. Such a choice of δ_2 is possible by property (P1) and Convention 2.3.

For $n \geq 1$, define

$$B_{-2n}^0 = \{A \in \mathcal{M}_{-2n}^0 : \forall j, \, 0 \leq j \leq n/2, \, T^{-j} A \subset E \in \mathcal{M}_{-n+j}^0 \text{ such that } \mathrm{diam}^u(E) < \delta_2\} \, .$$

Define $B_0^{2n} \subset \mathcal{M}_0^{2n}$ analogously with $\mathrm{diam}^u(E)$ replaced by $\mathrm{diam}^s(E)$. Next, let

$$B_{2n} := \{A \in \mathcal{M}_{-2n}^0 : \text{ either } A \in B_{-2n}^0 \text{ or } T^{-2n} A \in B_0^{2n} \ \} \, , \tag{5.10}$$

and $G_{2n} = \mathcal{M}_{-2n}^0 \setminus B_{2n}$. We think of B_{2n} as the set of 'bad' elements and G_{2n} as the set of 'good' elements.

Note that for any $n \geq 1$, each $A \in \mathcal{M}_{-n}^0$ satisfies $\mathrm{diam}^s(A) \leq C\Lambda^{-n}$. We choose $\bar{n} \in \mathbb{N}$ such that $C\Lambda^{-\bar{n}} \leq \delta_2$. Our first lemma shows that the cardinality of B_{2n} is small relative to e^{2nh_*} for large n.

Lemma 5.10. *There exists $C > 0$ such that for all $n \geq \bar{n}$,*

$$\# B_{2n} \leq C e^{3nh_*/2} K_1^{n/2} \leq C \rho^{n/2} e^{2nh_*} \, .$$

Proof. For $n \geq \bar{n}$, suppose $A \in B_{-2n}^0 \subset \mathcal{M}_{-2n}^0$. For simplicity assume n is even; otherwise, we may use $\lfloor n/2 \rfloor$ in place of $n/2$. For $0 \leq j \leq n/2$, let A_j denote the element of $\mathcal{M}_{-3n/2-j}^0$ containing $T^{-(n/2-j)} A \in \mathcal{M}_{-3n/2-j}^{n/2-j}$.

Since $A \in B_{-2n}^0$ and by choice of $\bar{n}$, it follows that $\max\{\mathrm{diam}^s(A_j), \mathrm{diam}^u(A_j)\} \leq \delta_2$ for each $0 \leq j \leq n/2$. By choice of δ_2, the number of connected components of $\mathcal{M}_{-3n/2-j}^1$ in each A_j is at most K_1. Fixing $A_0 \in \mathcal{M}_{-3n/2}^0$ and applying this estimate inductively in j, we conclude that $\#\{A' \in B_{-2n}^0 : T^{-n/2} A' \subset A_0\} \leq K_1^{n/2}$. Summing over the possible $A_0 \in \mathcal{M}_{-3n/2}^0$ yields,

$$\# B_{-2n}^0 \leq \# \mathcal{M}_{-3n/2}^0 K_1^{n/2} \leq C e^{3nh_*/2} \rho^{n/2} \Lambda^{n/2} \leq C \rho^{n/2} e^{2nh_*} \, ,$$

[12] Which is a slight modification of the Brin-Katok local entropy theorem [BK], applying [M, Lemma 2]. See also [BD, Corollary 7.17].

where we have used Proposition 2.12 and Convention 2.3 for the second inequality, and Lemma 3.6(d) for the third.

Next, if $A \in \mathcal{M}_0^n$, then $\mathrm{diam}^u(A) \le C\Lambda^{-n}$ as well, so the same choice of $\bar{n}$ permits the analogous estimate to hold for $\#B_0^{2n}$ for $n \ge \bar{n}$. Finally, since there is a one-to-one correspondence between elements of $\mathcal{M}_0^n$ and $\mathcal{M}_{-n}^0$, we have $\#B_{2n} \le \#B_{-2n}^0 + \#B_0^{2n}$, completing the proof of the lemma. □

Our next lemma shows that long elements of $\mathcal{M}_{-j}^0$ enjoy good lower bounds on their μ_*-measure. These lower bounds will eventually be linked to elements of G_{2n}.

Lemma 5.11. *There exists a constant $C_{\delta_2} > 0$ such that for all $j \ge 1$ and $A \in \mathcal{M}_{-j}^0$ such that $\min\{diam^u(A), diam^s(T^{-j}A)\} \ge \delta_2$, it follows that,*

$$\mu_*(A) \ge C_{\delta_2} e^{-jh_*} .$$

Proof. As in the proof of Lemma 3.11, we choose a finite set $\mathcal{R}_{\delta_2} = \{R_1, \dots, R_\ell\}$ of locally maximal Cantor rectangles with $\mu_*(R_i) > 0$, such that every stable curve of length δ_2 properly crosses at least one R_i in the stable direction, and every unstable curve of length δ_2 properly crosses at least one R_i in the unstable direction.

Now let $j \ge 1$ and $A \in \mathcal{M}_{-j}^0$ be as in the statement of the lemma. By choice of $\mathcal{R}_{\delta_2}$, an unstable curve in A properly crosses at least one $R_i \in \mathcal{R}_{\delta_2}$. Since $\partial A \subset \mathcal{S}_n^-$, ∂A cannot intersect any unstable manifolds in R_i since unstable manifolds cannot be cut under T^{-n}. Thus A must fully cross R_i in the unstable direction. Similarly, $T^{-j}A \in \mathcal{M}_0^j$ must fully cross at least one rectangle $R_k \in \mathcal{R}_{\delta_2}$ in the stable direction.

Let Ξ_i denote the index set of the family of stable manifolds comprising R_i. If $\xi \in \Xi$, set $W_{\xi,A} = W_\xi \cap A$. Since T^{-j} is smooth on A and $T^{-j}A$ fully crosses R_k in the stable direction, it must be that $T^{-j}(W_{\xi,A})$ is a single curve that properly crosses R_k, and so contains a stable manifold in the family corresponding to R_k.

Let $s > 0$ denote the length of the shortest stable manifold in the rectangles belonging to $\mathcal{R}_{\delta_2}$. Applying (4.17), we estimate for $\xi \in \Xi_i$,

$$\int_{W_{\xi,A}} \nu_0 = e^{-jh_*} \int_{W_{\xi,A}} \mathcal{L}^j \nu_0 = e^{-jh_*} \int_{T^{-j}(W_{\xi,A})} \nu_0 \ge e^{-jh_*} C' s^{h_* \bar{C}_2} .$$

Next, we let $D(R_i)$ denote the smallest solid rectangle containing R_i, and disintegrate μ_* on $\{W_\xi\}_{\xi \in \Xi_i}$ into conditional measures μ_*^ξ and a factor measure $\hat{\mu}_*$ on Ξ_i. Then using the equivalence of the conditional measure μ_*^ξ with ν_0 on μ_*-a.e. $\xi \in \Xi_i$ from (5.8), we have

$$\begin{aligned} \mu_*(A) &\ge \mu_*(A \cap D(R_i)) \ge \int_{\Xi_i} \mu_*^\xi(A)\, d\hat{\mu}_*(\xi) \\ &\ge \int_{\Xi_i} C_\xi^{-1} \nu_0(W_{\xi,A})\, d\hat{\mu}_*(\xi) \ge C' s^{h_* \bar{C}} e^{-jh_*} \int_{\Xi_i} C_\xi^{-1}\, d\hat{\mu}_*(\xi) , \end{aligned}$$

which completes the proof of the lemma due to the finiteness of $\mathcal{R}_{\delta_2}$. □

Our main proposition of the section is the following.

Proposition 5.12. *The measure μ_* is the unique measure of maximal entropy.*

Proof. Since μ_* is ergodic, it suffices to prove that if μ is an invariant probability measure that is singular with respect to μ_*, then $h_\mu(T) < h_{\mu_*}(T)$.

Recall from (2.5) that with respect to the metric $\bar{d}$ defined in (2.2), T and T^{-1} are expansive: there exists $\varepsilon_0 > 0$ such that if $\bar{d}(T^j x, T^j y) < \varepsilon_0$ for all $j \in \mathbb{Z}$, then $x = y$.

For $n \ge 1$, define $\mathcal{Q}_n$ to be the partition of maximal, connected components of M (with boundary points doubled according to Convention 2.1) on which T^{-n} is continuous. By the discussion of Section 2.2, $\mathcal{Q}_n$ consists of elements with non-empty interior which correspond to elements of $\mathcal{M}_{-n}^0$, plus isolated points. Since the entropy of an atomic measure is 0, we may assume that μ gives 0 mass to the isolated points, and it follows from Lemma 5.4 that μ_* does as well. Thus the only elements of $\mathcal{Q}_n$ with positive measure correspond to elements of $\mathcal{M}_{-n}^0 = B_n \cup G_n$. Accordingly, we throw out the atoms in $\mathcal{Q}_n$ and continue to call this collection of sets by the same name.

Since μ is singular with respect to μ_*, there exists a Borel set $F \subset M$ with $T^{-1}F = F$, $\mu_*(F) = 0$, and $\mu(F) = 1$. Our first step is to approximate F by elements of $\mathcal{Q}_n$.

Sublemma 5.13. *For each $n \geq \bar{n}$, there exists a finite union $\mathcal{C}_n$ of elements of $\mathcal{Q}_n$ such that*

$$\lim_{n\to\infty} (\mu + \mu_*)((T^{-n/2}\mathcal{C}_n) \triangle F) = 0\,.$$

This is [BD, Sublemma 7.24], and its proof relies on the fact that the diameters of elements of $T^{-n/2}(\mathcal{Q}_n)$ tend to 0 as n increases due to the uniform hyperbolicity of T. The invariance of F implies in addition that

$$\lim_{n\to\infty} (\mu + \mu_*)(\mathcal{C}_n \triangle F) = \lim_{n\to\infty} (\mu + \mu_*)((T^{n/2}\mathcal{C}_n) \triangle F) = 0\,.$$

By the proof of [BD, Sublemma 7.24], for each n, there exists a compact set $\mathcal{K}(n)$ that defines the approximating collection $\tilde{\mathcal{C}}_n = T^{-n/2}\mathcal{C}_n \subset \mathcal{M}_{-n/2}^{n/2}$, and satisfying $\mathcal{K}(n) \nearrow F$ as $n \to \infty$. To exploit this approximation, we group elements $Q \in \mathcal{Q}_{2n}$ according to whether $T^{-n}Q \subset \cup\tilde{\mathcal{C}}_n$ or $T^{-n}Q \cap (\cup\tilde{\mathcal{C}}_n) = \emptyset$, where $\cup\tilde{\mathcal{C}}_n$ denotes the union of elements of $\tilde{\mathcal{C}}_n$ in M. Since we have eliminated isolated points, if $T^{-n}Q \cap (\cup\tilde{\mathcal{C}}_n) \neq \emptyset$, then $T^{-n}Q \in \mathcal{M}_{-n}^n$ is contained in an element of $\mathcal{M}_{-n/2}^{n/2}$ that intersects $\mathcal{K}(n)$. Thus $Q \subset \cup T^n\tilde{\mathcal{C}}_n = \cup T^{n/2}\mathcal{C}_n$.

As noted above, the diameters of $T^{-n}\mathcal{Q}_{2n}$ tend to 0 as $n \to \infty$, so by the expansive property of T, since the image under T^{2n} of each element of $\mathcal{Q}_{2n}$ is connected, $\mathcal{Q}_{2n}$ is a generating partition for T^{2n} for n large enough. Thus,

$$h_\mu(T^{2n}) = h_\mu(T^{2n}, \mathcal{Q}_{2n}) \leq H_\mu(\mathcal{Q}_{2n}) = -\sum_{Q\in\mathcal{Q}_{2n}} \mu(Q)\log\mu(Q)\,.$$

And so,

$$\begin{aligned}
2nh_\mu(T) = h_\mu(T^{2n}) &\leq -\sum_{Q\in\mathcal{Q}_{2n}} \mu(Q)\log\mu(Q) \\
&\leq -\sum_{Q\subset\cup T^n\tilde{\mathcal{C}}^n} \mu(Q)\log\mu(Q) - \sum_{Q\cap(\cup T^n\tilde{\mathcal{C}}^n)=\emptyset} \mu(Q)\log\mu(Q) \\
&\leq \frac{2}{e} + \mu(\cup T^n\tilde{\mathcal{C}}^n)\log\#(\mathcal{Q}_{2n}\cap T^n\tilde{\mathcal{C}}_n) + \mu(M\setminus(\cup T^n\tilde{\mathcal{C}}^n))\log\#(\mathcal{Q}_{2n}\setminus(T^n\tilde{\mathcal{C}}_n))\,,
\end{aligned}$$

where in the last line we have used that for $p_j > 0$, $\sum_{j=1}^N p_j \leq 1$, it holds that

$$-\sum_{j=1}^N p_j\log p_j \leq \frac{1}{e} + (\log N)\sum_{j=1}^N p_j\,;$$

see for example [KH, eq. (20.3.5)]. We have applied this fact with $p_j = \mu(Q)$ to both sums separately. Next, since $-h_{\mu_*}(T) = \Big(\mu(\cup T^n \tilde{\mathcal{C}}_n) + \mu(M \setminus (\cup T^n \tilde{\mathcal{C}}_n))\Big) \log e^{-h_*}$, we estimate for $n \ge \bar{n}$,

$$\begin{aligned}
&2n(h_\mu(T) - h_{\mu_*}(T)) - \frac{2}{e} \\
&\quad \le \mu(\cup T^n \tilde{\mathcal{C}}_n) \log \sum_{Q \subset \cup T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} + \mu(M \setminus (\cup T^n \tilde{\mathcal{C}}_n)) \log \sum_{Q \in \mathcal{Q}_{2n} \setminus (T^n \tilde{\mathcal{C}}_n)} e^{-2nh_*} \\
&\quad \le \mu(\cup \mathcal{C}_n) \log \left(\sum_{Q \in G_{2n} \cap T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} + \sum_{Q \in B_{2n} \cap T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} \right) \\
&\qquad + \mu(M \setminus (\cup \mathcal{C}_n)) \log \left(\sum_{Q \in G_{2n} \setminus T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} + \sum_{Q \in B_{2n} \setminus T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} \right),
\end{aligned} \tag{5.11}$$

where for the last inequality, we have used the invariance of μ. By Lemma 5.10, the sums over the two subsets of B_{2n} are bounded by $C\rho^{n/2}$. We focus on estimating the sums over the two subsets of G_{2n}.

The following is proved in [BD, Section 7.7]: For each $Q \in G_{2n} \subset \mathcal{M}^0_{-2n}$, there exists $j, k \in \mathbb{N}$, $0 \le j, k \le n/2$ and $\bar{E} \in \mathcal{M}^0_{-2n+j+k}$ such that $T^{-j}Q \subset \bar{E}$ and $\min\{\text{diam}^u(\bar{E}), \text{diam}^s(T^{-2n+j+k})\} \ge \delta_2$. We call such a triple $(\bar{E}, j, k)$ an *admissible triple* for $Q \in G_{2n}$, and note that by Lemma 5.11,

$$\mu_*(\bar{E}) \ge C_{\delta_2} e^{(-2n+j+k)h_*} . \tag{5.12}$$

There may be many admissible triples for a fixed $Q \in G_{2n}$. Define the unique *maximal triple* for Q by taking first the maximum j, then the maximum k over all admissible triples for Q.

Denote by $\mathcal{E}_{2n}$ the set of maximal triples corresponding to elements of G_{2n}, and for $(\bar{E}, j, k) \in \mathcal{E}_{2n}$, set

$$\mathcal{A}_M(\bar{E}, j, k) = \{Q \in G_{2n} : (\bar{E}, j, k) \text{ is the maximal triple for } Q\} .$$

Since $\bar{E} \in \mathcal{M}^0_{-2n+j+k}$ and $G_{2n} \subset \mathcal{M}^0_{-2n}$, it follows from Proposition 2.12 that $\#\mathcal{A}_M(\bar{E}, j, k) \le Ce^{(j+k)h_*}$ for some C independent of $(\bar{E}, j, k)$ and n.

The following sublemma is [BD, Sublemma 7.25], which implies that if we organize our counting according to maximal triples, we avoid unwanted redundancies.

Sublemma 5.14. *If $(\bar{E}_1, j_1, k_1)$ and $(\bar{E}_2, j_2, k_2)$ are distinct elements of $\mathcal{E}_{2n}$ with $j_2 \ge j_1$, then $T^{-(j_2-j_1)}\bar{E}_1 \cap \bar{E}_2 = \emptyset$.*

If $Q \in T^n \tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k)$, then by definition of maximal triple, $T^{-n+j}\bar{E} \in \mathcal{M}^{n-j}_{-n+k}$ contains $T^{-n}Q$. Since $j, k \le n/2$, $T^{-n+j}\bar{E}$ is contained in an element of $\mathcal{M}^{n/2}_{-n/2}$ that also contains $T^{-n}Q$ and intersects $\mathcal{K}(n)$. Thus $T^{-n+j}\bar{E} \subset \cup \tilde{\mathcal{C}}_n$ whenever $T^n \tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) \ne \emptyset$, and so $\mathcal{A}_m(\bar{E}, j, k) \subset T^n \tilde{\mathcal{C}}_n$ whenever $T^n \tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) \ne \emptyset$.

Using these observations together with (5.12), we estimate

$$\begin{aligned}
&\sum_{Q \in G_{2n} \cap T^n \tilde{\mathcal{C}}_n} e^{-2nh_*} \le \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} \; \sum_{Q \in \mathcal{A}_M(\bar{E},j,k)} e^{-2nh_*} \\
&\le \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} Ce^{(-2n+j+k)h_*} \le \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C' \mu_*(\bar{E}) \\
&\le \sum_{(\bar{E},j,k) \in \mathcal{E}_{2n} : \bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n} C' \mu_*(T^{-n+j}\bar{E}) \le C'\mu_*(\cup \tilde{\mathcal{C}}_n) = C'\mu_*(\cup \mathcal{C}_n) ,
\end{aligned}$$

where we have used the invariance of μ_* and the constant C' is independent of n. In the last line we have used Sublemma 5.14 in order to sum over the elements of $\mathcal{E}_{2n}$ without double counting.

Similarly, since $T^{-n+j}\bar{E} \subset M \setminus \tilde{\mathcal{C}}_n$ whenever $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) = \emptyset$, the sum over $Q \in G_{2n} \setminus T^n\tilde{\mathcal{C}}_n$ in (5.11) is bounded by $C'\mu_*(M \setminus (\cup\mathcal{C}_n))$.

Putting these estimates together with (5.11) allows us to conclude the argument,

$$2n(h_\mu(T) - h_{\mu_*}(T)) - \frac{2}{e} \le \mu(\cup\mathcal{C}_n)\log\left(C'\mu_*(\cup\mathcal{C}_n) + C\rho^{n/2}\right) + \mu(M \setminus (\cup\mathcal{C}_n))\log\left(C'\mu_*(M \setminus (\cup\mathcal{C}_n)) + C\rho^{n/2}\right).$$

Then since $\mu(\cup\mathcal{C}_n) \to 1$ and $\mu_*(\cup\mathcal{C}_n) \to 0$ as $n \to \infty$, the quantity on the right side of the inequality tends to $-\infty$. This forces $h_\mu(T) < h_{\mu_*}(T)$ to permit the left side to tend to $-\infty$ as well. □

References

[BG1] V. Baladi and S. Gouëzel, *Good Banach spaces for piecewise hyperbolic maps via interpolation,* Annales de l'Institut Henri Poinaré, Analyse nonlinéaire **26** (2009), 1453–1481.

[BG2] V. Baladi and S. Gouëzel, *Banach spaces for piecewise cone hyperbolic maps,* J. Modern Dynam. **4** (2010), 91–137.

[BDL] V. Baladi, M.F. Demers, and C. Liverani, *Exponential decay of correlations for finite horizon Sinai billiard flows,* Invent. Math. **211** (2018), 39–177.

[BD] V. Baladi and M.F. Demers, *On the measure of maximal entropy for finite horizon Sinai billiard maps*, Journal Amer. Math. Soc. **33**:2 (2020), 381–449.

[Bo1] R. Bowen, *Periodic points and measures for Axiom A diffeomorphisms,* Trans. Amer. Math. Soc. **154** (1971) 377–397

[Bo2] R. Bowen, *Topological entropy for non-compact sets,* Trans. Amer. Math. Soc. **49** (1973) 125–136

[Bo3] R. Bowen, *Maximizing entropy for a hyperbolic flow,* Math. Systems Theory **7** (1974) 300–303

[Bo4] R. Bowen, *Some systems with unique equilibrium states,* Math. Systems Theory **8** (1974/75) 193–202

[BR] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Inventiones Math. **29** (1975), 181–202.

[BK] M. Brin and A. Katok, *On local entropy,* Geometric Dynamics (Rio de Janeiro, 1981) Lecture Notes in Mathematics **1007,** Springer: Berlin (1983) 30–38

[BCFT] K. Burns, V. Climenhaga, T. Fisher, and D.J. Thompson, *Unique equilibrium states for geodesic flows in nonpositive curvature,* Geom. Funct. Anal. **28** (2018) 1209–1259

[Bu] J. Buzzi, *The degree of Bowen factors and injective codings of diffeomorphisms,* arXiv:1807.04017, v3 (December 2019).

[BS] J. Buzzi, S. Crovisier and O. Sarig, *Measures of maximal entropy for surface diffeomorphisms*, arXiv:1811.02240, v2 (January 2019).

[CM] N.I. Chernov and R. Markarian, *Chaotic Billiards,* Math. Surveys and Monographs **127**, Amer. Math. Soc. (2006)

[CZ] N.I. Chernov and H.-K. Zhang, *On statistical properties of hyperbolic systems with singularities*, J. Stat. Phys. **136** (2009), 615–642.

[CFT] V. Climenhaga, T. Fisher, and D.J. Thompson, *Unique equilibrium states for Bonatti-Viana diffeomorphisms*, Nonlinearity **31**:6 (2018), 2532–2577.

[CKW] V. Climenhaga, G. Knieper, and K. War, *Uniqueness of the measure of maximal entropy for geodesic flows on certain manifolds without conjugate points,* arXiv:1903.09831, v1 (March 2019).

[CPZ] V. Climenhaga, Ya. Pesin and A. Zelerowicz, *Equilibrium measures for some partially hyperbolic systems*, arXiv:1810.08663, v3 (July 2019).

[DL] M.F. Demers and C. Liverani, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps*, Trans. Amer. Math. Soc. **360**:9 (2008), 4777-4814.

[DWY] M.F. Demers, P. Wright, and L.-S. Young, *Entropy, Lyapunov exponents and escape rates in open systems,* Ergod. Th. Dynam. Sys. **32** (2012) 1270–1301

[DZ1] M.F. Demers and H.-K. Zhang, *Spectral analysis for the transfer operator for the Lorentz gas,* J. Mod. Dyn. **5** (2011) 665–709

[DZ2] M.F. Demers and H.-K. Zhang, *A functional analytic approach to perturbations of the Lorentz gas*, Comm. Math. Phys. **324** (2013) 767–830

[DZ3] M.F. Demers and H.-K. Zhang, *Spectral analysis of hyperbolic systems with singularities,* Nonlinearity **27** (2014) 379–433

[Do] D. Dolgopyat, *On decay of correlations in Anosov flows,* Ann. of Math. **147** (1998), 357–390.

[GL] S. Gouëzel and C. Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties,* J. Diff. Geom. **79** (2008), 433–477.

hennion [H] H. Hennion, *Sur un théorème spectral et son application aux noyaux Lipchitziens,* Proc. Amer. Math. Soc. **118**:2 (1993), 627–634.

KH [KH] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems,* Cambridge University Press (1995).

lima [LiM] Y. Lima and C. Matheus, *Symbolic dynamics for non-uniformly hyperbolic surface maps with discontinuities*, Ann. Sci. Éc. Norm. Supér. **51**:1 (2018), 1–38.

Li1 [L1] C. Liverani, *Decay of correlations,* Ann. of Math. **142** (1995), 239–301.

Li2 [L2] C. Liverani, *On contact Anosov flows*, Ann. of Math. **159**:3 (2004), 1275–1312.

LiWo [LW] C. Liverani and M.P. Wojtkowski, *Ergodicity in Hamiltonian systems*, Dynamics Reported **4** (1995), 130–202.

mane [M] R. Mañé, *A proof of Pesin's formula,* Ergodic Th. Dynam. Sys. **1** (1981), 95–102.

Mar0 [Ma1] G.A. Margulis, *Certain applications of ergodic theory to the investigation of manifolds of negative curvature* (Russian) Funkcional. Anal. i Pril. **3** (1969) 89–90

Mar [Ma2] G.A. Margulis, *On some Aspects of the Theory of Anosov systems,* with a survey by R. Sharp: *Periodic orbits of hyperbolic flows,* Springer: Berlin (2004)

PaPo83 [PaP] W. Parry and M. Pollicott, *An analogue of the prime number theorem for closed orbits of Axiom A flows,* Ann. of Math. **118** (1983), 573–591.

Pesin1 [P] Ya.B. Pesin, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergod. Th. and Dynam. Sys. **12**:1 (1992), 123–151.

PS [PS] M. Pollicott, and R. Sharp, *Exponential error terms for growth functions on negatively curved surfaces,* Amer. J. Math. **120** (1998) 1019–1042

ruelle1 [Ru1] D. Ruelle, *Thermodynamic Formalism: The Mathematical Structures of Classical Equilibrium Statistical Mechanics*, Addison-Wesley, 1978, 183 pp.

ruelle2 [Ru2] D. Ruelle, *Locating resonances for Axiom A dynamical systems*, J. Stat. Phys. **44** (1986), 281–292.

Sar1 [Sa1] O. Sarig, *Bernoulli equilibrium states for surface diffeomorphisms,* J. Mod. Dyn. **5** (2011) 593–608

Sar2 [Sa2] O. Sarig, *Symbolic dynamics for surface diffeomorphisms with positive entropy,* J. Amer. Math. Soc. **26** (2013) 341–426

Sch [Sch] L. Schwartz, *Théorie des distributions,* Publications de l'Institut de Mathématique de l'Université de Strasbourg, Hermann: Paris (1966)

Sin72 [S] Ya. Sinai, *Gibbs measures in ergodic theory*, Russian Math. Surveys **27**:4 (1972), 21–69.

walters [W] P. Walters, *An Introduction to Ergodic Theory,* Graduate Texts in Math. **79** Springer: New York (1982).

Y98 [Y] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity,* Ann. of Math. **147** (1998), 585–650.

Department of Mathematics, Fairfield University, Fairfield CT 06824, USA
Email address: `mdemers@fairfield.edu`