

3 + 1 formulation of the standard model extension gravity sectorKellie O’Neal-Ault^{*} and Quentin G. Bailey[†]*Embry-Riddle Aeronautical University, 3700 Willow Creek Road, Prescott, Arizona 86301, USA*Nils A. Nilsson[‡]*National Centre for Nuclear Research, ul. Pasteura 7, 00-293, Warsaw, Poland*

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We present a $3 + 1$ formulation of the effective field theory framework called the standard model extension in the gravitational sector. The explicit local Lorentz and diffeomorphism symmetry breaking assumption is adopted, and we perform a Dirac-Hamiltonian analysis. We show that the structure of the dynamics presents significant differences from general relativity and other modified gravity models. We explore Hamilton’s equations for some special choices of the coefficients. Our main application is cosmology, and we present the modified Friedmann equations for this case. The results show some intriguing modifications to standard cosmology. In addition, we compare our results to existing frameworks and models, and we comment on the potential impact to other areas of gravitational theory and phenomenology.

DOI: [10.1103/PhysRevD.103.044010](https://doi.org/10.1103/PhysRevD.103.044010)**I. INTRODUCTION**

It is generally expected that general relativity (GR) and the standard model (SM) of particle physics are not the ultimate descriptions of nature, but rather low-energy effective field theories which accurately describe physics at energy scales available to us. This point of view is motivated by the expectation that there exists a single unified theory encompassing all the known fundamental interactions. This implies the existence of a renormalizable quantum theory of gravity which has GR as its low-energy limit. GR, being an effective field theory, is then expected to hold up to some ultraviolet (UV) cutoff scale, normally taken to be the Planck energy, $E_{\text{Pl}} \approx 10^{19}$ GeV. Any theory attempting to bridge GR and SM should, on dimensional grounds, contain all the characteristic constants of the constituent theories. As E_{Pl} represents the UV cutoff scale of GR, new physics should appear close to this energy, and a promising avenue to find new physics is to search for deviations from fundamental principles of GR.

Local Lorentz invariance is one of the fundamental symmetries of relativity as well as particle physics, stating that any local experiment is independent of both orientation and velocity of both the experiment and observer, and it is a key ingredient of GR. As such, precision tests of local Lorentz symmetry are an excellent way to test for new physics [1,2].

The standard model extension (SME) is a general effective field theory framework for testing Lorentz and *CPT* symmetries [3,4]. It has become a standard framework for constraining Lorentz violation in a systematic way (for a list of all current measurements, see Ref. [5]). The SME contains GR and the standard model of particle physics, as well as generic Lorentz-violating terms up to arbitrary order. The terms are constructed by contracting operators built from known fields with coefficients for Lorentz violation, the latter of which control the degree of symmetry breaking and can be constrained by experiments.

In principle the SME contains an arbitrary number of terms, but is frequently truncated at low order in mass dimension of the field operators used. A much-studied truncation is called the *minimal* SME and contains operators of mass dimension three or four.

Whereas many limits have to date been set in the matter sector of the SME, gravitational-sector coefficients have also been constrained. These include analysis with short-range gravity tests [6], gravimeters [7], solar-system tests [8,9], pulsars [10], gravitational waves [11], and others [12].

Much of the theoretical phenomenology that experiments and observations have used is based on weak-field gravity analysis [13–17]. So-called “exact” results beyond this regime in the SME gravity sector have just begun to be explored [18–23]. The aim of this work is in part to extend results to situations where weak gravitational fields cannot be assumed, for example in cosmology. Furthermore, we begin a study of the $3 + 1$ formulation of this framework, which allows for a Dirac-Hamiltonian analysis [24]. Note that this type of analysis has been performed for vector and

^{*}aultk@my.erau.edu[†]baileyq@erau.edu[‡]albin.nilsson@ncbj.gov.pl

tensor models of spontaneous Lorentz violation [25–28] and other related models [29], but as of yet has not been attempted for the SME, and we seek to fill this gap in this work. Primarily we shall adopt the explicit symmetry breaking scenario, though some of our results can be extended to spontaneous symmetry breaking. Ultimately, we aim to push the application of the SME framework in a new direction in order to explore more broadly the consequences of Lorentz violation in gravity.

The SME as a framework for testing Lorentz symmetry naturally contains specific models of Lorentz violation as subsets. Much work in the literature has involved the study of such models, particularly in the gravity context [30–36]. The connection between the coefficients for Lorentz violation in the SME and proposed models in the literature has been established for some quantum gravity approaches [37,38], massive gravity models [39], noncommutative geometry [40], as well as vector and tensor models of spontaneous Lorentz symmetry breaking. In this paper, we use our results to match to yet another model which involved explicit Lorentz breaking.

The paper is organized as follows: in Sec. II we give an overview of the key features of the SME. The details of the $3 + 1$ decomposition are presented in Sec. III, starting with a geometric overview followed by the discussion of the SME action terms. In Sec. IV we perform a Hamiltonian analysis, starting with general features, and we then focus on two special cases. As an application of the results, we study cosmological solutions in Sec. V. We connect our results to existing frameworks and models in Sec. VI. Finally we discuss our results and conclusions in Sec. VII, along with remarks on future work.

Notational conventions in this paper match prior work as much as possible [4,13]. Greek letters are used for spacetime indices and latin letters $i, j, k, \dots$, for spatial indices. For local Lorentz frame (vierbein) indices we use the latin letters $a, b, c, \dots$, when needed. The metric $g_{\mu\nu}$ signature matches the standard GR choice $(-+++)$, and we use units where $\hbar = c = 1$. One important notational difference in this work is that we use here ∇_μ for the spacetime covariant derivative, reserving $\mathcal{D}_\mu$ for covariant derivatives defined on constant-time spatial hypersurfaces.

II. BASIC FRAMEWORK

In Riemann spacetimes, the lowest order terms in the GR + SME gravitational Lagrange density can be written as

$$\mathcal{L}_{\text{SME}} = \frac{\sqrt{-g}}{2\kappa} (R - 2\Lambda + (k_R)_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}) + \mathcal{L}'. \quad (1)$$

In this expression, $R^{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor, Λ is the cosmological constant, $(k_R)_{\alpha\beta\gamma\delta}$ are the SME coefficient fields [4], and $\kappa = 8\pi G_N$, where G_N is the gravitational constant. A generic Lagrangian $\mathcal{L}'$ appears in the

case when the coefficients arise dynamically, as in spontaneous Lorentz-symmetry breaking. The coefficients $(k_R)_{\alpha\beta\gamma\delta}$ can be written as a scalar u , two tensor $s_{\mu\nu}$, and four tensor $t_{\alpha\beta\gamma\delta}$ through a Ricci decomposition. So we can write

$$(k_R)_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = -uR + s_{\mu\nu} (R^{(T)})^{\mu\nu} + t_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta}, \quad (2)$$

where $(R^{(T)})^{\mu\nu}$ is the trace-free Ricci tensor, and $W^{\alpha\beta\gamma\delta}$ is the Weyl curvature tensor. The coefficients u and $s_{\mu\nu}$ can be moved to the matter sector by field redefinitions at first order in these coefficients [18], so it is important to consider also the matter sector when calculating physically measurable results. In this work we choose our conventions such that these coefficients reside in the gravity sector.

The action obtained from integrating (1) over $\int d^4x$ has several features that play a key role in the analysis to follow. The relevant symmetries in spacetime are the four-dimensional diffeomorphism symmetry group and local Lorentz transformations. First, the action is invariant under general coordinate transformations; i.e., it is *observer* diffeomorphism invariant. This means specifically that the curvature tensors and the coefficients $(k_R)_{\alpha\beta\gamma\delta}$ transform under observer transformations (change of coordinates) as tensors; however, for *particle* diffeomorphisms the curvature, metric, and other fields transform as tensors but the coefficients $(k_R)_{\alpha\beta\gamma\delta}$ remain fixed. Therefore, the action associated with (1) breaks particle diffeomorphisms.

In the standard vierbein formalism where the metric is reduced to Minkowski form η_{ab} at each point with a set of four vectors e^μ_a via $\eta_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$, similar considerations apply for local Lorentz transformations Λ^a_b . Therefore, we distinguish *observer* local Lorentz transformations, under which local tensors R_{abcd} transform as well as the coefficients $(k_R)_{abcd}$. In contrast, under *particle* local Lorentz transformations, the coefficients remain fixed, and therefore the action also breaks local Lorentz transformations. Note that the details of these transformations can involve the notion of a background vierbein, and so care is required, as discussed in Ref. [39].

The action formed with (1) can be interpreted as explicit symmetry breaking, where the coefficients are nondynamical, or through spontaneous Lorentz symmetry breaking. In the latter case, the underlying model retains the *particle* local Lorentz and diffeomorphism symmetries because the coefficients are dynamical fields. There must then be a dynamical mechanism, such as a potential function of the fields, that triggers a vacuum expectation value $\langle (k_R)_{abcd} \rangle$ of the fields [30]. Upon specifying the vacuum one can still obtain an effective model of the form (1). Indeed, this has been demonstrated for a variety of models with vector and tensor couplings to curvature. These results have been discussed extensively elsewhere in the literature, particularly for spontaneous symmetry breaking [33,34].

For either Lorentz and diffeomorphism symmetry breaking scenario, there are conservation equations which hold based on the action formed from (1). The field equations for the metric $g_{\mu\nu}$ obtained from the action take the form

$$G^{\mu\nu} = (T_{\text{ust}})^{\mu\nu} + \kappa(T_M)^{\mu\nu}, \quad (3)$$

where the explicit form of $(T_{\text{ust}})^{\mu\nu}$ can be found in Ref. [4] and $(T_M)^{\mu\nu}$ is the energy-momentum tensor obtained from the matter sector. As a consequence of the traced Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$, four conservation laws which must hold are given by

$$\nabla_\mu (T_{\text{ust}})^{\mu\nu} = -\kappa \nabla_\mu (T_M)^{\mu\nu}. \quad (4)$$

That these conservation laws hold will be a key point in this work. There are also six conservation laws associated with local Lorentz symmetry breaking, which we do not display here for brevity. In generality, the recent works of Bluhm and collaborators clarify the differences between explicit and spontaneous local Lorentz and diffeomorphism symmetry breaking [39,41], and the intricacies of the conservation laws. Note that alternatives to Riemann geometry exist, such as Riemann-Finsler geometry which has recently garnered attention as an additional avenue of pursuit in Lorentz violation theory and phenomenology [42].

III. 3 + 1 VARIABLES AND DECOMPOSITION

A. 3 + 1 basics

We start with a four-dimensional manifold $\mathcal{M}$ with associated metric $g_{\mu\nu}$. Following standard methods [43–46], we decompose $\mathcal{M}$ into constant-time spatial hypersurfaces Σ_t with associated timelike normal vector n^μ (normalized to $n_\mu n^\mu = -1$). In a commonly used coordinate representation the components are $n_\mu = (-\alpha, 0, 0, 0)$, where α is the Arnowitt-Deser-Misner (ADM) lapse function. Referring to Fig. 1, $t^\mu dt$ is a vector connecting points on neighboring hypersurfaces with fixed spatial coordinates x^i (cf. Fig. 2.4 in [47]). The shift vector is β^i , which arises from the dif-

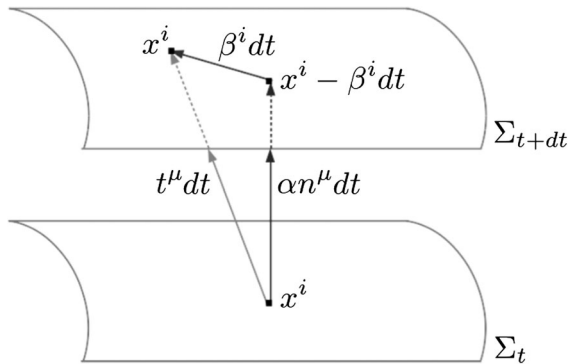


FIG. 1. The ADM variables connecting spatial hypersurfaces Σ at time t and $t + dt$.

ference between t^μ and n^μ , while the inverse spatial metric $\gamma^{\mu\nu}$ is given by

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu. \quad (5)$$

Using n^μ and $\gamma^{\mu\nu}$, the four-dimensional curvature $R_{\alpha\beta\gamma\delta}$ is decomposed into a three-dimensional spatial curvature $\mathcal{R}_{\alpha\beta\gamma\delta}$ and extrinsic curvature $K_{\mu\nu}$. The extrinsic curvature is defined in terms of the Lie derivative along n^μ as

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}. \quad (6)$$

A spatial covariant derivative $\mathcal{D}_\mu$ is obtained from γ -projection of the covariant derivative of a tensor. For a tensor with mixed indices $T^\mu{}_\nu$, for example, it is given by

$$\mathcal{D}_\mu T^\alpha{}_\beta = \gamma^\delta{}_\mu \gamma^\alpha{}_\epsilon \gamma^\zeta{}_\beta \nabla_\delta T^\epsilon{}_\zeta. \quad (7)$$

It will be useful also to use the “acceleration” $a_\mu = n^\nu \nabla_\nu n_\mu$, which is orthogonal to n^μ . The three-dimensional curvature $\mathcal{R}^\alpha{}_{\beta\gamma\delta}$ is defined by the commutator of the spatial covariant derivatives as

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] v_\delta = -\mathcal{R}^\epsilon{}_{\delta\alpha\beta} v_\epsilon \quad (8)$$

and satisfies $\mathcal{R}^\epsilon{}_{\delta\alpha\beta} n_\epsilon = 0$, where v_ϵ is assumed spatial ($n^\epsilon v_\epsilon = 0$). Some more key results in the 3 + 1 decomposition are included in Appendix A.

When necessary we will refer specifically to the space-time metric in standard ADM form expressed as a line element,

$$ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_j dt dx^j + \gamma_{ij} dx^i dx^j. \quad (9)$$

This form makes plain that the principle variables for gravity are the 10 degrees of freedom α , β^j , and γ_{ij} .

B. GR and the SME action

1. GR action

Of principle importance in what follows is that in the SME action, spacetime covariant derivatives occur which act on the coefficients for Lorentz violation, and do not generally vanish. To see this, we decompose the Lagrangian (1) using the 3 + 1 curvature projections in Appendix A. For reference, we examine first the GR Lagrange density which is

$$\mathcal{L}_{\text{GR}} = \frac{\sqrt{-g}}{2\kappa} [\mathcal{R} + K^{\alpha\beta} K_{\alpha\beta} - K^2 - 2\nabla_\alpha (n^\alpha K + a^\alpha)]. \quad (10)$$

Note here that the last terms form a three-dimensional surface term in the action that normally does not affect the

dynamical field equations, and thus they are usually dropped.¹ What is left contains the extrinsic curvature $K_{\alpha\beta}$, which can be seen from (6) to have time derivatives of $\gamma_{\mu\nu}$ via the Lie derivative along n^μ . Specifically, if one evaluates the Lie derivative in (6), one obtains the standard result

$$K_{ij} = -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - \mathcal{D}_i \beta_j - \mathcal{D}_j \beta_i), \quad (11)$$

and the other components $K_{0\mu}$ contain no new time derivatives other than those in (11). The spatial curvature term in (10) contains no such time derivatives, depending only on the curvature in each spatial hypersurface.

The presence of the time derivatives determines the dynamical variables used for a Hamiltonian formulation; in GR, only time derivatives of γ_{ij} occur, and thus these six components are the only dynamical degrees of freedom in the Hamiltonian formulation.² The other metric degrees of freedom α and β^j are nondynamical, corresponding to the 4 gauge degrees of freedom in diffeomorphism symmetry. This leads to the four primary constraints in a Hamiltonian analysis of GR. Note also that, while it does not occur in (11), the acceleration a_μ has only spatial derivatives, as it can be shown that $a_j = \partial_j \ln \alpha$ and $a_0 = \beta^j a_j$.

2. SME action and global background coefficients

We next examine the contribution of the $(k_R)_{\alpha\beta\gamma\delta}$ coefficients in the SME action. Using the general curvature expression in Appendix A this term can be manipulated into the form

$$\begin{aligned} \mathcal{L}_{k_R} = \frac{\sqrt{-g}}{2\kappa} \{ & (k_R)_{\alpha\beta\gamma\delta} [\mathcal{R}^{\alpha\beta\gamma\delta} - 6K^{\alpha\gamma} K^{\beta\delta} \\ & + 4n^\alpha n^\gamma (K^{\beta\epsilon} K^\delta_\epsilon - K^{\beta\delta} K^\epsilon_\epsilon) + 4a^\beta n^\gamma K^{\alpha\delta}] \\ & - 4(n^\alpha n^\gamma (n^\epsilon K^{\beta\delta} + \gamma^{\delta\epsilon} a^\beta) \\ & - 2n^\alpha \gamma^{\gamma\epsilon} K^{\beta\delta}) \nabla_\epsilon (k_R)_{\alpha\beta\gamma\delta} \}. \end{aligned} \quad (12)$$

We can see that a term with the covariant derivative of the coefficients occurs while the remaining terms are expressible in terms of the extrinsic curvature K_{ij} or the acceleration a^μ . In general spacetimes this term $\nabla_\epsilon (k_R)_{\alpha\beta\gamma\delta}$ cannot be made to vanish [4]. Since we are interested in the dynamical content of the framework we can use the 3 + 1 decomposition to interpret such terms.

Consider first the simpler case of the covariant derivative of a covariant vector b_μ . Using projection and the definition

(7), as well as properties of the Lie derivative along n^μ , we can write this in terms of the spatial covariant derivative, the Lie derivative, and the extrinsic curvature, as

$$\begin{aligned} \nabla_\mu b_\nu = & \mathcal{D}_\mu b_\nu - n_\nu \mathcal{D}_\mu (n^\lambda b_\lambda) - 2n_{(\mu} K_{\nu)}^\lambda b_\lambda \\ & - n_\mu n_\nu (a^\lambda b_\lambda) + n_\mu n_\nu \mathcal{L}_n (n^\lambda b_\lambda) - n_\mu \gamma^\beta_\nu \mathcal{L}_n b_\beta. \end{aligned} \quad (13)$$

It can be checked using (A5) that the spatial covariant derivative of b_ν will only contain spatial partial derivatives ∂_j of components of b_ν , the functions α and β^j , the extrinsic curvature K_{ij} , or the three-dimensional connection coefficients ${}^{(3)}\Gamma^i_{jk}$, the latter of which contain only spatial derivatives of γ_{ij} . Thus $\mathcal{D}_\mu$ acting in (13) cannot introduce any time derivatives of the metric functions α and β^j . From a geometrical perspective, this is because the $\mathcal{D}_\mu$ derivative describes changes in the three-dimensional hypersurface Σ_t .

Since the acceleration a_μ depends on spatial derivatives of α , we are left with the final two terms in (13) as places where time derivatives of α and β_j might reside. The projection of $n^\lambda b_\lambda$ can be written as

$$n^\lambda b_\lambda = \frac{1}{\alpha} (b_0 - \beta^j b_j), \quad (14)$$

while its Lie derivative is

$$\mathcal{L}_n (n^\lambda b_\lambda) = -\frac{\dot{\alpha} (n^\lambda b_\lambda) + b_i \dot{\beta}^i}{\alpha^2} + \frac{1}{\alpha} n^\mu \dot{b}_\mu - \frac{1}{\alpha} \beta^j \mathcal{D}_j (n^\lambda b_\lambda). \quad (15)$$

Note the appearance of $\dot{\alpha} = \partial_t \alpha$ and $\dot{\beta}^j = \partial_t \beta^j$ for the lapse and shift functions. This implies that in the Hamiltonian analysis we will generally not obtain the usual four primary momentum constraints as in GR. The final Lie derivative term in (13) is proportional to $\gamma^i_\nu \mathcal{L}_n b_i$, which can be shown not to contain time derivatives of the gravitational variables α , β^j , and γ_{ij} .

One might immediately suspect that the appearance of $\dot{\alpha}$ and $\dot{\beta}^j$ is merely a coordinate artifact and can be removed by general coordinate transformations. Indeed, the SME maintains general coordinate invariance of the action. Under a general coordinate transformation, b_μ transforms as a covariant vector,

$$b'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} b_\nu, \quad (16)$$

with other quantities transforming as usual. However, the quantity $n^\mu b_\mu$ occurring in (15) is a scalar and is the projection of the hypersurface normal n^μ along the fixed, *a priori* unknown, background b_μ . The appearance of α and β^j in (14), in a certain sense, describes the unknown

¹See Ref. [46] for details on surface terms.

²Note that we adopt the first order derivative form for the action as much as possible, particularly for time derivatives. Otherwise, the Hamiltonian formalism gets modified to accommodate higher derivatives [48]. This would also occur for models with higher than second derivatives present.

orientation of the background and the orientation of the hypersurface geometry, the latter being tied to the source of the gravitational fields. If an alternative coordinate system $x^{\mu'}$ is chosen so that $n^{\mu'}b_{\mu'} = b_0$ and we then suppose that in the new coordinate system $b_{\mu'}$ is now *the* fixed background that is independent of the gravitational fields, we have effectively made a different choice of background, and because of the explicit breaking of diffeomorphism symmetry, we have chosen a different model.³ We return to this point later when we consider alternative ways of specifying the background fields.

The vector example can be extended to general tensors; since our focus is on the SME gravity action, in the minimal case, it is possible to manipulate the Lagrange density into a form where time derivative dependencies are more transparent just as in (13). In fact, we can write (12) as

$$\begin{aligned} \mathcal{L}_{k_R} = \frac{\sqrt{-g}}{2\kappa} \{ & (k_R)_{\alpha\beta\gamma\delta} [\mathcal{R}^{\alpha\beta\gamma\delta} + 2K^{\alpha\gamma}K^{\beta\delta} \\ & - 12n^\alpha n^\gamma K^{\beta\epsilon}K^\epsilon_\delta + 4n^\alpha n^\gamma K^{\beta\delta}K^\epsilon_\epsilon + 8K^{\alpha\gamma}n^\beta n^\delta] \\ & + 8K^{\epsilon\zeta}\mathcal{D}_\lambda((k_R)_{\alpha\beta\gamma\delta}\gamma^\alpha_\epsilon\gamma^{\beta\lambda}_\zeta\gamma^\gamma_\epsilon n^\delta) \\ & - 4a^\epsilon\mathcal{D}_\zeta((k_R)_{\alpha\beta\gamma\delta}\gamma^{\alpha\zeta}_\epsilon\gamma^\gamma_\epsilon n^\beta n^\delta) \\ & - 4K^{\epsilon\zeta}\mathcal{L}_n((k_R)_{\alpha\beta\gamma\delta}\gamma^\alpha_\epsilon\gamma^\gamma_\zeta n^\beta n^\delta)\}. \end{aligned} \quad (17)$$

Any nonstandard time derivative terms will be contained in the last Lie derivative term. The appearance of $\dot{\alpha}$ and $\dot{\beta}^j$ terms can be verified by working out the Lie derivative term explicitly. We find the relevant piece to be

$$\mathcal{L}_{k_R} \supset \frac{4\sqrt{-g}}{\kappa a^2} K^{ij}n^\delta((k_R)_{i\beta j\delta}n^\beta\dot{\alpha} + (k_R)_{i\ell j\delta}\dot{\beta}^\ell). \quad (18)$$

As in Eq. (15) we obtain a combination of $\dot{\alpha}$ and $\dot{\beta}^j$ terms, and we thus expect this to hold for general tensors in the SME [15].

This rather interesting result, the occurrence of $\dot{\alpha}$ and $\dot{\beta}^j$, is in contrast with GR and many modified models of gravity. It is somewhat unsurprising in that we are considering the SME framework interpreted in the context of explicit diffeomorphism symmetry breaking, which breaks the gauge symmetry of GR. Other models, such as massive gravity, which also have explicit diffeomorphism breaking, modify mass-type terms with no derivatives in them. They generally do not modify the kinetic structure of the theory and thus do not introduce such terms. As another example, for models with curvature contractions in the Lagrangian such as $R_{\alpha\beta}R^{\alpha\beta}$, even though they have higher than second derivatives of the metric, the lapse and shift functions remain gauge [49]. More varied results exist for other models with higher than second derivatives [50].

³This choice corresponds to Gaussian normal coordinates [45].

In the case of spontaneous symmetry breaking, where for example $b_\mu \rightarrow B_\mu$ is dynamical, there are separate field equations for B_μ which must also be considered. The net effect in this case, since the underlying diffeomorphism symmetry remains, is that $\dot{\alpha}$ and $\dot{\beta}^j$ can be eliminated by a particle diffeomorphism. Or alternatively, one can see that the dynamics of α and β^j become linked to the field B_μ , as the time derivatives always occur in the combination in (15), and they thus do not represent independent degrees of freedom. This point about the spontaneous-breaking case parallels the reasoning behind the observation that any diffeomorphism Nambu-Goldstone modes Ξ^μ vanish from terms in the action as $\nabla_\mu B_\nu$ [33,34].

3. Local background coefficients

In the explicit breaking case considered above, the coefficients $k_{\alpha\beta\gamma\delta}$ with spacetime indices are considered as the fixed background fields independent of the gravitational variables. There are alternative choices that could change the results. For instance, using the vierbein formalism it may be more natural to treat the coefficients in a different way. From the basic definition of the vierbein, we can find its 3 + 1 components from the metric (9). The components e_μ^a are given by

$$\begin{aligned} e_t^{\bar{0}} &= \alpha, \\ e_j^{\bar{0}} &= 0, \\ e_t^{\bar{j}} &= e_t^{\bar{j}}\beta^j, \\ \gamma_{ij} &= e_t^{\bar{j}}e_{j\bar{j}}, \end{aligned} \quad (19)$$

where we use t and $i, j, \dots$, for time and space indices while for the local frame we use a bar over the index. The last equation merely defines the spatial piece of the vierbein $e_t^{\bar{j}}$, since we have not specified the spatial metric. The explicit decomposition can be performed once a spatial metric is chosen. The vierbein in (19) is not unique; one may apply a local Lorentz transformation $\Lambda^a_b(x)$ and generally mix components.

Returning to the vector example above, when using the vierbein it is natural to consider the local covariant vector field b_a as the fundamental background object which breaks the spacetime symmetries [4,51]. For instance, using the vierbein and the vector n^μ the projection which occurs in Eq. (15) can be written

$$b_\mu n^\mu = b_{\bar{0}}. \quad (20)$$

In this case the Lie derivative term in (15) yields

$$\mathcal{L}_n(n^\lambda b_\lambda) = \frac{\partial_t b_{\bar{0}}}{\alpha} - \frac{1}{\alpha}\beta^j\partial_j(b_{\bar{0}}). \quad (21)$$

Now we can see that no time derivatives of α and β^j occur, provided that b_a is *the* independent background.⁴ It should be noted that this choice does not make use of a background vierbein, as discussed in Ref. [39] and may result in more severe constraints on explicit breaking models via the conservation laws.

How does all of this play into the dynamical and propagation structure that is known from weak-field studies of the SME and models of spacetime symmetry breaking? To answer this we also perform a comparison with what is known about the weak-field quadratic limit [17], including generic gauge-violating terms in Sec. VI A. Ultimately in this work, we look at cases of explicit breaking with both choices of the background coefficients corresponding to the “global” background in (14) and the “local” background (20).

IV. HAMILTONIAN ANALYSIS

A. Generalities

Working with the Lagrange density in the form of Eq. (17), we carry out a Riemann decomposition of $(k_R)_{\alpha\beta\gamma\delta}$ into u , $s_{\mu\nu}$, and $t_{\kappa\lambda\mu\nu}$. The Lagrange density for $t_{\kappa\lambda\mu\nu}$ will take the same form as (17) with the replacement of $(k_R)_{\alpha\beta\gamma\delta} \rightarrow t_{\alpha\beta\gamma\delta}$. For u and $s_{\mu\nu}$ we obtain

$$\begin{aligned}\mathcal{L}_u &= \frac{\sqrt{-g}}{2\kappa} [u(\mathcal{R} + K^{\alpha\beta}K_{\alpha\beta} - K^2) \\ &\quad + 2(K\mathcal{L}_n u + a^\mu \mathcal{D}_\mu u)], \\ \mathcal{L}_s &= \frac{\sqrt{-g}}{2\kappa} [s_{\mu\nu} \mathcal{R}^{\mu\nu} - n^\alpha n^\beta s_{\alpha\beta} (K^{\mu\nu} K_{\mu\nu} - K^2) \\ &\quad + 2s_{\alpha\beta} K^{\alpha\delta} K^\beta_\delta + K^{\mu\nu} \mathcal{L}_n s_{\mu\nu} - K\mathcal{L}_n (n^\mu n^\nu s_{\mu\nu}) \\ &\quad + 2K(s_{\mu\nu} n^\mu a^\nu + \mathcal{D}_\lambda (s_{\mu\nu} n^\mu \gamma^{\nu\lambda})) \\ &\quad - 2K^\lambda_\kappa \mathcal{D}_\lambda (s_{\mu\nu} n^\mu \gamma^{\nu\kappa}) \\ &\quad + a_\kappa \mathcal{D}_\lambda (s_{\mu\nu} \gamma^{\mu\lambda} \gamma^{\nu\kappa}) - a^\lambda \mathcal{D}_\lambda (s_{\mu\nu} n^\mu n^\nu)].\end{aligned}\quad (22)$$

Note that one can also consider other possibilities such as substituting $u \rightarrow g^{\mu\nu} s_{\mu\nu}$.

In the Hamiltonian analysis, one first finds the canonical momentum densities using $L = \int d^3x \mathcal{L}$ via the standard variational definition

$$\Pi_n = \frac{\delta L}{\delta \dot{\phi}_n}. \quad (23)$$

In the present case the ϕ_n correspond to α , β^i , and γ_{ij} . To describe the results for the SME actions we show the canonical momenta for the u , $s_{\mu\nu}$, and $t_{\alpha\beta\gamma\delta}$ terms, assuming global background coefficients as in Sec. III B 2.

⁴An alternative way to arrive at Eq. (20) is to define $\mathbf{n} \cdot \mathbf{b} = b_\mu n^\mu = b_\perp$ as *the* time component [52].

For $\Pi_\alpha = \delta L / \delta \dot{\alpha}$ and $(\Pi_\beta)_i = \delta L / \delta \dot{\beta}^i$ we obtain

$$\Pi_\alpha = \frac{\sqrt{\gamma}}{\kappa\alpha} n^\mu n^\nu (K s_{\mu\nu} + 4K^{ij} t_{i\mu j\nu}), \quad (24)$$

$$\Pi_{\beta,i} = \frac{\sqrt{\gamma}}{\kappa\alpha} n^\mu (K s_{\mu i} + 4K^{jk} t_{jik\mu}). \quad (25)$$

Note that no nonzero terms appear here for the case of u ; however, if one chooses u to be composite, such as $u = s^\mu_\mu$, a different result ensues. For the $s_{\mu\nu}$ and $t_{\kappa\lambda\mu\nu}$ coefficients the expressions for $\Pi_\gamma^{ij} = \delta L / \delta \dot{\gamma}_{ij}$ are lengthy and omitted here. These expressions contain terms which generally mix the components of s_{ij} and $\dot{\gamma}_{ij}$ in an anisotropic manner; for instance,

$$\Pi_\gamma^{ij} \supset \frac{\sqrt{\gamma}}{2\kappa} (K\gamma^{ij} - K^{ij} - s_{lm} \gamma^{li} K^{jm} - s_{lm} \gamma^{lj} K^{im} + \dots),$$

where the first two terms are the GR result and the displayed remaining terms show a mixing of the components of s_{ij} and $K_{ij} \sim \dot{\gamma}_{ij} - \mathcal{D}_i \beta_j - \mathcal{D}_j \beta_i$.

To construct the Hamiltonian density $\mathcal{H} = \Pi^n \dot{\phi}_n - \mathcal{L}$, one needs to express $\dot{\phi}_n$ in terms of the momenta Π_n . Since obtaining the general expression for $\dot{\gamma}_{ij}$ involves a lengthy process of inversion due to the anisotropic components of the coefficients, we endeavor in this work to begin an investigation by studying special limiting cases of the underlying action.

B. Case study 1

We consider a special case with one nonzero component s_{00} of $s_{\mu\nu}$ in the chosen coordinate system.⁵ In this case, using the specific components of the metric (9) the Lagrange density simplifies to

$$\begin{aligned}\mathcal{L}_1 &= \frac{\alpha\sqrt{\gamma}}{2\kappa} \left[\mathcal{R} + \frac{\alpha^2 - s_{00}}{\alpha^2} (K^{ij} K_{ij} - K^2) \right. \\ &\quad + K \left(\frac{2}{\alpha^4} s_{00} (\dot{\alpha} - \alpha \beta^i a_i) - \frac{1}{\alpha^3} (\dot{s}_{00} - \beta^i \partial_i s_{00}) \right) \\ &\quad \left. + \frac{2}{\alpha^2} s_{00} a^i a_i - \frac{1}{\alpha^2} a^i \partial_i s_{00} \right] + \mathcal{L}_M.\end{aligned}\quad (26)$$

When constructing the Hamiltonian, the variables of the system are the α , β^μ , and $\gamma_{\mu\nu}$ fields along with their conjugate momentum densities:

⁵Note that alternative choices exist such as considering the contravariant coefficients as the fixed background; for example, $n_\mu n_\nu s^{\mu\nu} = \alpha^2 s^{00}$ for arbitrary $s^{\mu\nu}$.

$$\Pi_{\gamma}^{ij} = \frac{\sqrt{\gamma}}{2\kappa} \left[\frac{\alpha^2 - s_{00}}{\alpha^2} (K\gamma^{ij} - K^{ij}) + \frac{1}{2\alpha} \gamma^{ij} (\partial_t - \beta^k \partial_k) \left(\frac{s_{00}}{\alpha^2} \right) \right], \quad (27)$$

$$\Pi_{\beta,i} = 0, \quad (28)$$

$$\Pi_{\alpha} = \frac{\sqrt{\gamma} s_{00}}{\kappa \alpha^3} K. \quad (29)$$

From now on we drop the γ label on Π^{ij} and abbreviate its trace as $\Pi = \Pi^{ij} \gamma_{ij}$, and we drop the β label so that $\Pi_i = \Pi_{\beta,i}$.

Examining the momenta, we see that Eq. (28) gives three primary constraints. Equations (27) and (29), along with Eq. (11), can be inverted to solve for $\dot{\gamma}_{ij}$ and $\dot{\alpha}$. Following standard procedure [52], we first find the *base* Hamiltonian density of the system through a Legendre transformation on the Lagrange density $\mathcal{H}_0 = \pi^{ij} \dot{\gamma}_{ij} + \pi_{\alpha} \dot{\alpha} - \mathcal{L}$. For the base Hamiltonian density we find

$$\begin{aligned} \mathcal{H}_0 = & \frac{2\kappa\alpha^3}{\sqrt{\gamma}(\alpha^2 - s_{00})} \left(\Pi_{ij} \Pi^{ij} - \frac{1}{3} \Pi^2 \right) \\ & + \frac{\kappa\alpha^5(\alpha^2 - s_{00})}{3\sqrt{\gamma}s_{00}^2} \Pi_{\alpha}^2 - \frac{2\kappa\alpha^4}{3\sqrt{\gamma}s_{00}} \Pi_{\alpha} \Pi + \frac{\alpha\dot{s}_{00}}{2s_{00}} \Pi_{\alpha} \\ & - \frac{\sqrt{\gamma}}{\kappa\alpha} \left(s_{00} a^i a_i - \frac{1}{2} a^i \partial_i s_{00} \right) - \frac{\alpha\sqrt{\gamma}}{2\kappa} \mathcal{R} \\ & + \beta^i \left(\Pi_{\alpha} \left[\alpha a_i - \frac{\alpha}{2s_{00}} \partial_i s_{00} \right] - 2\mathcal{D}_j \Pi^j_i \right). \end{aligned} \quad (30)$$

Note that by comparison, in GR, the momentum for α is absent; thus Eq. (29) is replaced with $\Pi_{\alpha} = 0$. The GR Hamiltonian is

$$\mathcal{H}_{\text{GR}} = \frac{2\kappa\alpha}{\sqrt{\gamma}} \left(\Pi_{ij} \Pi^{ij} - \frac{1}{2} \Pi^2 \right) - \frac{\alpha\sqrt{\gamma}}{2\kappa} \mathcal{R} - 2\beta^i \mathcal{D}_j \Pi^j_i. \quad (31)$$

Examining (31) and (30) it can be seen that taking the limit of $s_{00} \rightarrow 0$ does not smoothly connect the Hamiltonians. This is an artifact of the Hamiltonian method; in particular it results from solving Eq. (29) for $\gamma^{ij} \dot{\gamma}_{ij}$, which requires $s_{00} \neq 0$. We return to this point later in the process.

To Eq. (30) we add a term involving the primary constraint contracted with a Lagrange multiplier to obtain the *augmented* Hamiltonian $\mathcal{H}_A = \mathcal{H}_0 + \xi^i \Pi_i$. We then check the consistency condition, or evolution, for this primary constraint by taking its Poisson bracket with the augmented Hamiltonian $\dot{\Pi}_i = \{\Pi_i, H_A\}$. This yields a secondary constraint

$$\dot{\Pi}_i = 2\gamma_{ij} \mathcal{D}_k \Pi^{jk} - \Pi_{\alpha} \left(\alpha a_i - \frac{\alpha}{2s_{00}} \partial_i s_{00} \right) \approx 0. \quad (32)$$

Note that the $\approx$ symbol here refers to an expression that is “weakly” equal to zero; i.e., when the constraints are imposed, it vanishes [24]. This secondary constraint can also be observed in the last line of Eq. (30) multiplying the β^i .

We continue to check consistency conditions with the secondary constraint $\Phi_i = \dot{\Pi}_i$. The full expression for the evolution of Φ_i is needed, including the explicit time dependence since there may be additional time dependence in s_{00} . A lengthy calculation reveals

$$\begin{aligned} \frac{d\Phi_i}{dt} &= \{\Phi_i, H_A\} + \frac{\partial\Phi_i}{\partial t} \\ &= \mathcal{D}_j (\beta^j \Phi_i) + \Phi_j \mathcal{D}_i \beta^j + \Psi \partial_i s_{00}, \end{aligned} \quad (33)$$

where Ψ is a function of the coordinates and momenta equal to

$$\begin{aligned} \Psi = & -\frac{\kappa\alpha^3}{\sqrt{\gamma}(\alpha^2 - s_{00})s_{00}} \left(\Pi_{ij} \Pi^{ij} - \frac{1}{3} \Pi^2 \right) \\ & - \frac{\kappa\alpha^5(\alpha^2 - s_{00})}{6\sqrt{\gamma}s_{00}^3} \Pi_{\alpha}^2 + \frac{\kappa\alpha^4}{3\sqrt{\gamma}s_{00}^2} \Pi_{\alpha} \Pi + \frac{\alpha\sqrt{\gamma}}{4\kappa s_{00}} \mathcal{R} \\ & + \frac{\sqrt{\gamma}}{4\kappa s_{00} \alpha} \mathcal{D}^2 s_{00} - \frac{\sqrt{\gamma}}{2\kappa \alpha^2} \mathcal{D}^2 \alpha \\ & - \frac{\sqrt{\gamma}}{\kappa s_{00} \alpha} a^i \partial_j s_{00} + \frac{3\sqrt{\gamma}}{2\kappa \alpha} a^j a_j. \end{aligned} \quad (34)$$

The implications of (33) are as follows. Examining this expression the first two terms are linear in the original constraint Φ_i , and so are weakly equal to zero—providing no new constraints. The last terms would appear to give new constraints, but this depends on the properties of the background coefficients $s_{\mu\nu}$. If we insisted that the coefficients and their derivatives remain arbitrary we would have to take the last terms in (33) as new constraints and again check the consistency using Poisson brackets with the Hamiltonian. On the other hand, if we merely insist that in the chosen coordinate system $\partial_i s_{00} = 0$, then (33) will be weakly equal to zero and no new constraints are needed. That a constraint on s_{00} has arisen directly from this analysis can be traced to the fact that (32) is a modification of the usual momentum constraint of GR. The term $\partial_i s_{00}$ represents an additional kind of “shift” in the momentum conservation law.

Further insight can be gained by examining the traced Bianchi identities (4) for the choice of coefficients we have made. From Ref. [4] we have

$$\nabla_{\mu} (T_s)^{\mu}_{\nu} = \frac{1}{2} R^{\mu\lambda} \nabla_{\nu} s_{\mu\lambda} - \nabla_{\mu} (R^{\mu\lambda} s_{\lambda\nu}). \quad (35)$$

We next reexpress this equation in 3 + 1 form by using Appendix Eq. (A2), the decomposition of the covariant derivative as in Eq. (13), and we use Eqs. (27)–(29) to write the expression in terms of the Hamiltonian variables α , β^j , γ_{ij} , Π_{α} , and Π^{ij} . Examining the case of s_{00} only then yields

$$\nabla_\mu (T_s)^\mu_j = \frac{\kappa}{\sqrt{\gamma}\alpha} \Psi \partial_j s_{00}, \quad (36)$$

which contains the same terms as in (33). Therefore, we see that the Hamiltonian evolution has produced a constraint that we expect from the field equations.

We will proceed with the assumption that the coefficients s_{00} are independent of spatial coordinates:

$$\partial_i s_{00} = 0. \quad (37)$$

Note that this is a coordinate-dependent statement which may be more properly understood as saying that s_{00} does not change within the spatial hypersurface at fixed t .

Hamilton's equations of motion can now be obtained in the standard way through the Poisson bracket $\dot{p}_n = \{p_n, H\}$, where H is the *final* Hamiltonian with the primary constraint added. In principle, one adds the secondary constraint to the Hamiltonian with an additional three Lagrange multipliers. Since the secondary constraint can be seen to be already contained in the β^j term in the Hamiltonian (30), it is not strictly necessary to add this term. This reflects the remaining gauge freedom in this limit of the framework.

We then find Hamilton's equations of motion for the momentum variables to be

$$\begin{aligned} \dot{\Pi}_\alpha = & -\frac{2\kappa\alpha^2(\alpha^2 - 3s_{00})}{\sqrt{\gamma}(\alpha^2 - s_{00})^2} \left(\Pi^{ij}\Pi_{ij} - \frac{1}{3}\Pi^2 \right) + \frac{8\kappa\alpha^3}{3\sqrt{\gamma}s_{00}} \Pi_\alpha \Pi - \frac{\kappa\alpha^4}{3\sqrt{\gamma}s_{00}^2} (7\alpha^2 - 5s_{00})\Pi_\alpha^2 + \mathcal{D}_k(\beta^k \Pi_\alpha) \\ & - \frac{1}{2s_{00}} \Pi_\alpha \dot{s}_{00} + \frac{s_{00}\sqrt{\gamma}}{\kappa\alpha^2} (a^i a_i - 2\mathcal{D}_i a^i) + \frac{\sqrt{\gamma}}{2\kappa} \mathcal{R}, \end{aligned} \quad (38)$$

$$\dot{\Pi}_i = 2\gamma_{ij}\mathcal{D}_k \Pi^{jk} - \Pi_\alpha \alpha a_i, \quad (39)$$

$$\begin{aligned} \dot{\Pi}^{ij} = & -\frac{4\kappa\alpha^3}{\sqrt{\gamma}(\alpha^2 - s_{00})} \left(\Pi^i_k \Pi^{jk} - \frac{1}{3}\Pi \Pi^{ij} \right) + \frac{\kappa\alpha^3}{\sqrt{\gamma}(\alpha^2 - s_{00})} \gamma^{ij} \left(\Pi^{kl}\Pi_{kl} - \frac{1}{3}\Pi^2 \right) + \frac{\kappa\alpha^5(\alpha^2 - s_{00})}{6\sqrt{\gamma}s_{00}^2} \gamma^{ij}\Pi_\alpha^2 \\ & - \frac{\kappa\alpha^4}{3\sqrt{\gamma}s_{00}} \Pi_\alpha (\Pi \gamma^{ij} - 2\Pi^{ij}) - 2\Pi^{k(i} \mathcal{D}_k \beta^{j)} + \mathcal{D}_k (\Pi^{ij} \beta^k) - \frac{\sqrt{\gamma}}{2\kappa} \left(\alpha \mathcal{R}^{ij} - \frac{1}{2} \gamma^{ij} \alpha \mathcal{R} - \mathcal{D}^i \mathcal{D}^j \alpha + \gamma^{ij} \mathcal{D}^2 \alpha \right) \\ & + \frac{\sqrt{\gamma}s_{00}}{\kappa\alpha} \left(\frac{1}{2} \gamma^{ij} a^k a_k - a^i a^j \right), \end{aligned} \quad (40)$$

and

$$\begin{aligned} \dot{\alpha} = & -\frac{2\kappa\alpha^4}{3s_{00}\sqrt{\gamma}} \left[\Pi - \frac{\alpha(\alpha^2 - s_{00})}{s_{00}} \Pi_\alpha \right] + \alpha \beta^k a_k \\ & + \frac{\alpha}{2s_{00}} \dot{s}_{00}, \end{aligned} \quad (41)$$

$$\dot{\beta}^i = \xi^i, \quad (42)$$

$$\begin{aligned} \dot{\gamma}_{ij} = & \frac{4\kappa\alpha^3}{\sqrt{\gamma}(\alpha^2 - s_{00})} \left(\Pi_{ij} - \frac{1}{3}\Pi \gamma_{ij} \right) \\ & - \frac{2\kappa\alpha^4}{3\sqrt{\gamma}s_{00}} \Pi_\alpha \gamma_{ij} + \mathcal{D}_i \beta_j + \mathcal{D}_j \beta_i. \end{aligned} \quad (43)$$

Note that we have implemented the condition (37).

At this point it is useful to remark upon the degrees of freedom in this special case of the SME.⁶ We began with

⁶Degrees of freedom represent pairs of coordinate and momenta variables freely specifiable on a hypersurface of fixed t [27].

up to 10 degrees of freedom in the variables α , β^j , and γ_{ij} . With our choice of s_{00} we have three primary constraints and three secondary constraints, along with six undetermined Lagrange multipliers. According to the standard recipe [see, for example, Eq. (B11) in Appendix B of Ref. [52]] one can use the equation

$$N_{\text{dof}} = N_{\text{dof,initial}} - \frac{1}{2} (\# \text{ constraints}) - \frac{1}{2} (\# \text{ undetermined Lagrange multipliers}) \quad (44)$$

to determine the number of degrees of freedom. In the case above, we have $10 - (1/2)(6) - (1/2)(6) = 4$ degrees of freedom in our model. In GR, by contrast, there are four primary constraints, four secondary constraints, and in principle eight undetermined Lagrange multipliers which leaves $10 - (1/2)8 - (1/2)8 = 2$ degrees of freedom.

Note the appearance of the inverse of s_{00} in the expressions above. This does not represent a phase space singularity, but rather a parameter singularity. Its appearance is tied to the Hamiltonian method, where one inverts,

for example, Eq. (29), while in contrast results are generally linear in the parameter s_{00} in the standard Euler-Lagrange equations. There is also a denominator in (30) that appears to have a singular point when $\alpha = \pm\sqrt{s_{00}}$. It is not clear if anything keeps the field configurations away from this region of phase space, and this remains to be investigated.

The Hamiltonian and Hamilton's equations in this special example, or a related one with different choices of $s_{\mu\nu}$, can form the basis for future work in a variety of areas such as studying the initial value formulation of the system of equations. This could lead to modeling the effects of SME coefficients in strong field gravity systems, for example using numerical techniques of integration [47]. In this paper, we content ourselves with a cosmology application in Sec. V.

C. Case study 2

In contrast to the case considered above, we can make an alternative choice for the background coefficients. In this example we choose s_{ab} to be diagonal and isotropic in the local Lorentz frame

$$s_{ab} = \begin{pmatrix} s_{\bar{0}\bar{0}} & 0 & 0 & 0 \\ 0 & \frac{1}{3}s & 0 & 0 \\ 0 & 0 & \frac{1}{3}s & 0 \\ 0 & 0 & 0 & \frac{1}{3}s \end{pmatrix}, \quad (45)$$

where the nonzero components $s_{\bar{0}\bar{0}}$ and s are left as arbitrary functions of the spacetime. Using the vierbein in (19) we can find the components $s_{\mu\nu} = e_\mu^a e_\nu^b s_{ab}$ in the spacetime coordinates of the metric (9). Simplifying the action for $s_{\mu\nu}$ in (22) we obtain an alternative explicit breaking Lagrangian:

$$\mathcal{L}_2 = \frac{\alpha\sqrt{\gamma}}{2\kappa} \left[\mathcal{R} \left(1 + \frac{1}{3}s \right) + (K^{ij}K_{ij} - K^2)(1 - s_{\bar{0}\bar{0}}) + K\mathcal{L}_n\Omega + a^i\partial_i\Omega \right], \quad (46)$$

where we use the abbreviation $\Omega = s/3 - s_{\bar{0}\bar{0}}$.

In this case the terms involving the time derivatives of α and β^j are absent, and except for the time and space dependence of the coefficients which we take as arbitrary for the moment, the Lagrange density resembles that of GR with scalings of the extrinsic curvature and spatial curvature terms. The canonical momenta are calculated to be

$$\Pi^{ij} = \frac{\sqrt{\gamma}}{2\kappa} \left[(K\gamma^{ij} - K^{ij})(1 - s_{\bar{0}\bar{0}}) - \frac{1}{2}\gamma^{ij}\mathcal{L}_n\Omega \right], \quad (47)$$

$$\Pi_{\beta,i} = 0, \quad (48)$$

$$\Pi_\alpha = 0. \quad (49)$$

Note the appearance of the Lie derivative of the coefficients directly in the momentum and that we get four primary constraints for α and β^j , as in GR. This is in contrast to the case of global coefficients in the general analysis of Sec. IV A, and so (49) is not a limit of Eq. (25). The base Hamiltonian density for this case is given by

$$\begin{aligned} \mathcal{H}_0 = & \frac{2\kappa\alpha}{\sqrt{\gamma}(1 - s_{\bar{0}\bar{0}})} \left(\Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi^2 \right) + 2\Pi^{ij}\mathcal{D}_i\beta_j \\ & - \frac{\alpha\sqrt{\gamma}}{2\kappa} \left(1 + \frac{1}{3}s \right) \mathcal{R} - \frac{1}{2(1 - s_{\bar{0}\bar{0}})} \Pi\Omega' \\ & - \frac{3\sqrt{\gamma}}{16\kappa\alpha(1 - s_{\bar{0}\bar{0}})} (\Omega')^2 - \frac{\alpha\sqrt{\gamma}}{2\kappa} a^i\partial_i\Omega, \end{aligned} \quad (50)$$

where for convenience we define $\Omega' = (\partial_0 - \beta^i\partial_i)\Omega$.

The evolution of the primary constraints with respect to the augmented Hamiltonian

$$H_A = \int d^3x (\mathcal{H}_0 + v\Pi_\alpha + \xi^i\Pi_i), \quad (51)$$

where v and ξ^j are Lagrange multipliers, yields the following secondary constraints:

$$\begin{aligned} \{\Pi_\alpha, H_A\} = & -\frac{2\kappa}{\sqrt{\gamma}(1 - s_{\bar{0}\bar{0}})} \left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2 \right) \\ & + \frac{\sqrt{\gamma}}{2\kappa} \left(1 + \frac{1}{3}s \right) \mathcal{R} - \frac{3\sqrt{\gamma}}{16\kappa(1 - s_{\bar{0}\bar{0}})\alpha^2} (\Omega')^2 \\ & - \frac{\sqrt{\gamma}}{2\kappa} \mathcal{D}^2\Omega, \end{aligned} \quad (52)$$

$$\begin{aligned} \{\Pi_i, H_A\} = & 2\mathcal{D}_j\Pi_i^j - \frac{\Pi}{2(1 - s_{\bar{0}\bar{0}})} \partial_i\Omega \\ & - \frac{3\sqrt{\gamma}}{8\kappa(1 - s_{\bar{0}\bar{0}})\alpha} \Omega' \partial_i\Omega. \end{aligned} \quad (53)$$

These secondary constraints contain the GR secondary constraints but they differ in the extra terms involving time and space derivatives of the coefficients in Ω . The standard procedure is to check the consistency of these secondary constraints. Due to the presence of the spatial derivatives in Ω , we expect a result similar to that for the case 1 model, whereupon we obtain a lengthy function of the canonical variables multiplied by terms proportional to $\partial_i\Omega$. This is indeed confirmed by calculation, and so we proceed with the simplifying assumption that $\partial_i\Omega = 0$. This assumption has the immediate effect of reducing the secondary constraints in (53) for Π_i to that of the standard ones for GR, $\dot{\Pi}_i = 2\mathcal{D}_j\Pi_i^j \approx 0$.

Still allowing for arbitrary time dependence of the coefficients $s_{\bar{0}\bar{0}}$ and s , we proceed with the calculation of the secondary constraint evolution. Denoting $\Phi_\alpha = \{\Pi_\alpha, H_A\}$ and $\Phi_i = \{\Pi_i, H_A\}$, we obtain the following results for their evolution:

$$\begin{aligned} \{\Phi_\alpha, H_A\} + \frac{\partial \Phi_\alpha}{\partial t} = & \mathcal{D}_i(\beta^i \Phi_\alpha) + \frac{2(1+\frac{1}{3}s)}{(1-s_{\bar{0}\bar{0}})} \Phi_i \mathcal{D}^i \alpha + \frac{(1+\frac{1}{3}s)\alpha}{(1-s_{\bar{0}\bar{0}})} \mathcal{D}^i \Phi_i + v \frac{3\sqrt{\gamma}}{8\kappa\alpha^3(1-s_{\bar{0}\bar{0}})} \dot{\Omega}^2 + \frac{9\sqrt{\gamma}}{64\kappa\alpha^2(1-s_{\bar{0}\bar{0}})^2} \dot{\Omega}^3 \\ & + \frac{3}{8\alpha(1-s_{\bar{0}\bar{0}})^2} \dot{\Omega}^2 \Pi - \frac{\kappa(\dot{s} + \dot{s}_{\bar{0}\bar{0}})}{2\sqrt{\gamma}(1-s_{\bar{0}\bar{0}})^2} \left(\Pi_{ij} \Pi^{ij} - \frac{1}{2} \Pi^2 \right) - \frac{3\sqrt{\gamma}}{8\kappa\alpha^3(1-s_{\bar{0}\bar{0}})} \dot{\Omega}^2 \beta^i \mathcal{D}_i \alpha \\ & - \frac{\sqrt{\gamma}}{8\kappa(1-s_{\bar{0}\bar{0}})} \left(\left(1 + \frac{1}{3}s \right) \dot{\Omega} - \frac{4}{3} \dot{s}(1-s_{\bar{0}\bar{0}}) \right) \mathcal{R} - \frac{3\sqrt{\gamma}}{8\kappa\alpha^2(1-s_{\bar{0}\bar{0}})} \dot{\Omega} \ddot{\Omega} - \frac{3\sqrt{\gamma}}{16\kappa\alpha^2(1-s_{\bar{0}\bar{0}})^2} \dot{\Omega}^2 \dot{s}_{\bar{0}\bar{0}}, \quad (54) \end{aligned}$$

$$\{\Phi_j, H_A\} = \Phi_i \mathcal{D}_j \beta^i + \mathcal{D}_i(\beta^i \Phi_j) + \Phi_\alpha \mathcal{D}_j \alpha. \quad (55)$$

Examining these expressions reveals two things. First, from (55), we see that the secondary constraint $\mathcal{D}_i \Pi^i_j$ is preserved since its evolution is linear in the secondary constraints, which weakly vanish. Note in particular in (55) that terms linear in Φ_i and Φ_α appear, in contrast to case study 1 in Sec. IV B [see Eq. (33)], where there was no primary constraint Π_α nor a secondary constraint Φ_α . Second, the Lagrange multiplier v appears in the evolution equation for the Φ_α constraint (54). This latter result also differs from case study 1, where v did not even occur because there was no Φ_α constraint, and in GR, v remains an undetermined Lagrange multiplier.

In this case, the standard procedure is to solve for v from (54) by demanding that the equation weakly vanish. The first three terms vanish weakly by the prior secondary constraints, so this amounts to the v term canceling all remaining terms. When we solve for v in this manner, we obtain

$$\begin{aligned} v = & -\frac{3\alpha\dot{\Omega}}{8(1-s_{\bar{0}\bar{0}})} - \frac{\kappa\alpha^2\Pi}{(1-s_{\bar{0}\bar{0}})\sqrt{\gamma}} + \beta^i \mathcal{D}_i \alpha \\ & - \frac{4\kappa^2\alpha^3(\dot{s} + \dot{s}_{\bar{0}\bar{0}})}{3\gamma(1-s_{\bar{0}\bar{0}})\dot{\Omega}^2} \left(\Pi_{ij} \Pi^{ij} - \frac{1}{2} \Pi^2 \right) \\ & + \left(\left(1 + \frac{1}{3}s \right) \dot{\Omega} - \frac{4}{3} \dot{s}(1-s_{\bar{0}\bar{0}}) \right) \frac{\alpha^3 \mathcal{R}}{3\dot{\Omega}^2} \\ & + \frac{\alpha\ddot{\Omega}}{\dot{\Omega}} + \frac{\alpha\dot{s}_{\bar{0}\bar{0}}}{2(1-s_{\bar{0}\bar{0}})}. \quad (56) \end{aligned}$$

As can be seen from this equation, there is a problematic denominator in some of the terms. One would thus demand that the solution only include cases where $\dot{\Omega} \neq 0$.

Denoting the solution of (56) with capital V , the expression would then be inserted back into the Hamiltonian and the final form would be

$$\begin{aligned} H_F = & \int d^3x (\mathcal{H}_0 + V(\alpha, \gamma_{ij}, \beta^i, \Pi^{ij}, \dots) \Pi_\alpha + \xi^i \Pi_i \\ & + \zeta^j \Phi_j), \quad (57) \end{aligned}$$

where $\mathcal{H}_0$ (50) is evaluated with $\partial_i \Omega = 0$ and we have indicated that V is now a function of the canonical variables

and the coefficients. We have added three additional Lagrange multipliers ζ^j for the secondary constraints $\Phi_j \approx 0$. Note that, upon doing this, we end up with one of Hamilton's equations specifying $\dot{\alpha} = V(\alpha, \gamma_{ij}, \beta, \Pi^{ij}, \dots)$, again in contrast to GR where α is pure gauge. The full Hamilton's equations for this case are lengthy and omitted here, but it would be of interest in future work to study these types of cases in more detail.

In the result, Eq. (57), we have four primary constraints (48) and (49), four secondary constraints (52) and (53), and a total of six Lagrange multipliers ξ^i and ζ^j . Note that the Lagrange multiplier v was solved for, and so does not count as an undetermined Lagrange multiplier. Using the counting scheme in Eq. (44), for this case we obtain $10 - (1/2)(8) - (1/2)6 = 3$ degrees of freedom, one more than GR.

Another choice is to set s and $s_{\bar{0}\bar{0}}$ to be constants. This choice reduces the Hamiltonian to one where there are scalings of GR terms, obtainable from (50) by setting the Ω' , $\partial_i \Omega$ terms to zero. Indeed, it is this choice that forms the starting point for the match of explicit breaking models to the SME, as we discuss in Sec. VI B. For this latter choice, the number of degrees of freedom reduces to the GR result of 2.

D. Addition of matter

To apply the results above to physically relevant situations, we address the addition of the matter sector to the Hamiltonian analysis. We assume here that the matter sector does not couple to any coefficients for Lorentz violation and is minimally coupled to gravity. Depending on the area of study, the description of matter could be as basic as a perfect fluid or a set of scalar fields, or more sophisticated with gauge fields and/or spinors. For this work we shall leave this specification generic and comment on how the matter sector feeds into the analysis above.

First note that when performing variations of the matter action with respect to the spacetime metric $g_{\mu\nu}$, we have

$$(T_M)^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}}. \quad (58)$$

Upon constructing the Hamiltonian for the matter sector, we can use (58) and the 3 + 1 decomposition to show that the following hold in space and time components:

$$\begin{aligned}
\frac{\delta H_M}{\delta \alpha} &= \alpha^2 \sqrt{\gamma} (T_M)^{00}, \\
\frac{\delta H_M}{\delta \beta^i} &= -\alpha \sqrt{\gamma} [(T_M)^{00} \beta_i + (T_M)^{0j} \gamma_{ij}], \\
\frac{\delta H_M}{\delta \gamma_{ij}} &= -\frac{1}{2} \alpha \sqrt{\gamma} [(T_M)^{ij} + \beta^i \beta^j (T_M)^{00} + 2(T_M)^{0(i} \beta^{j)}].
\end{aligned} \tag{59}$$

In the Dirac-Hamiltonian analysis, one checks the consistency or evolution of the secondary constraints. If you add the matter sector, minimally coupled to gravity, certain combinations of the terms in (59) are involved in these calculations. For example, in the secondary constraints in (53), an extra term for the matter sector $-\delta H_M / \delta \beta^i$ is added, and its evolution is governed by the expression

$$\begin{aligned}
\left\{ \frac{\delta H_M}{\delta \beta^i}, H_A \right\} &= \frac{\delta H_M}{\delta \alpha} \mathcal{D}_i \alpha + \mathcal{D}_j \left(\beta^j \frac{\delta H_M}{\delta \beta^i} \right) \\
&\quad + \frac{\delta H_M}{\delta \beta^j} (\mathcal{D}_i \beta^j) - 2\gamma_{ki} \mathcal{D}_j \frac{\delta H_M}{\delta \gamma_{jk}}, \tag{60}
\end{aligned}$$

where H_A is the augmented Hamiltonian including the matter sector. Similar results hold for $\delta H / \delta \alpha$. Finally, we note that, while we do not address it here, the addition of even minimally coupled tensor fields can significantly affect the constraint structure of the model, as shown in Ref. [52].

V. COSMOLOGICAL SOLUTIONS

In this section we apply Hamilton's equations for the case study 1 subset of the SME discussed in Sec. IV B to search for solutions in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime [45]. We use the general FLRW metric in spherical coordinates

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{61}$$

where $a(t)$ is the scale factor and $k = \{+1, 0, -1\}$ represents a closed, flat, and open universe, respectively. For this metric, the lapse and shift can be seen by comparison with (9) to be $\alpha = 1$, $\beta^i = 0$ and the acceleration vanishes $a_i = 0$.

We proceed with evaluating Hamilton's equations for this case. Using the result $\dot{\gamma}_{ij} = 2\dot{a}\gamma_{ij}/a$, Eqs. (27) and (29), we can find the canonical momenta to be

$$\begin{aligned}
\Pi^{ij} &= -\frac{\sqrt{\gamma}}{\kappa} (1 - s_{00}) \frac{\dot{a}}{a} \gamma^{ij} + \frac{\sqrt{\gamma}}{4\kappa} \dot{s}_{00} \gamma^{ij}, \\
\Pi_\alpha &= -\frac{3s_{00}}{\kappa} \sqrt{\gamma} \frac{\dot{a}}{a},
\end{aligned} \tag{62}$$

where $\sqrt{\gamma} = a^3 r^2 \sin \theta / \sqrt{1 - kr^2}$. These results allow us to establish that since Π^{ij} is proportional to γ^{ij} , quantities such

as $\Pi^{ij} \Pi_{ij} - \frac{1}{3} \Pi^2$ will vanish. Indeed, evaluation of the other set of Hamilton's equations (38)–(40) for this case yields

$$\begin{aligned}
\dot{\Pi}^{ij} &= \frac{\kappa(1 - s_{00})}{6\sqrt{\gamma}s_{00}^2} \gamma^{ij} \Pi_\alpha^2 - \frac{\kappa}{3\sqrt{\gamma}s_{00}} \Pi_\alpha (\Pi \gamma^{ij} - 2\Pi^{ij}) \\
&\quad - \frac{\sqrt{\gamma}}{2\kappa} \mathcal{G}^{ij} + \frac{\sqrt{\gamma}}{2} (T_M)^{ij}, \\
\dot{\Pi}_\alpha &= \frac{\sqrt{\gamma}}{2\kappa} \mathcal{R} + \frac{8\kappa}{3\sqrt{\gamma}s_{00}} \Pi_\alpha \Pi - \frac{\kappa}{3\sqrt{\gamma}s_{00}^2} (7 - 5s_{00}) \Pi_\alpha^2 \\
&\quad - \frac{1}{2s_{00}} \Pi_\alpha \dot{s}_{00} - \sqrt{\gamma} (T_M)^{00},
\end{aligned} \tag{63}$$

where we have used the matter couplings in (59) and $\mathcal{G}^{ij}$ is the three-dimensional Einstein tensor. With the choice of metric and $\beta^i = 0$ the constraint equation (39) is satisfied, as can be checked directly. Also, note the fact that α is dynamical in our model, and even though it is fixed to unity, it still plays a role through the momenta Π_α .

For matter we use the usual perfect fluid model for a homogeneous and isotropic universe, $(T_M)^\mu{}_\nu = \text{diag}(-\rho, p, p, p)$, where ρ and p are the energy density and pressure, respectively. They are related through the equation of state $p = w\rho$, where w is the barotropic index.

The three-dimensional Ricci scalar and the three-dimensional Einstein tensor are $\mathcal{R} = 6k/a^2$ and $\mathcal{G}^{ij} = -k\gamma^{ij}/a^2$. Combining with (62) and (63), we obtain two equations:

$$\left(\frac{\dot{a}}{a} \right)^2 (1 - s_{00}) = \frac{\kappa\rho}{3} - \frac{k}{a^2} - s_{00} \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \frac{\dot{s}_{00}}{2}, \tag{64}$$

$$\left[\frac{\ddot{a}}{a} + \frac{1}{2} \left(\frac{\dot{a}}{a} \right)^2 \right] (1 - s_{00}) = -\frac{\kappa p}{2} - \frac{k}{2a^2} + \frac{\dot{a}}{a} \dot{s}_{00} + \frac{1}{4} \ddot{s}_{00}, \tag{65}$$

which have been written to match the standard FLRW equations of GR as closely as possible. Indeed, one recovers GR in the limit that $s_{00} \rightarrow 0$. The modifications include terms with first and second time derivatives of s_{00} , scalings by $1 - s_{00}$, and an extra $\ddot{a}$ term in the first equation. In principle, one could decouple the equations to obtain one with only the acceleration $\ddot{a}$ and one with only the Hubble factor $\dot{a}$, in the standard Friedmann equation form. However, s_{00} is an as yet unspecified function associated with explicit breaking of the underlying symmetries in the action (1).

In order to understand the role of s_{00} in this context more plainly, we examine the remaining conservation laws implied by the underlying action. These were given in Eq. (4) and for this particular subset of the SME in Eq. (35), Eq. (4) must be satisfied for consistency:

$$\nabla_\mu (T_s)^{\mu\nu} = -\kappa \nabla_\mu (T_M)^{\mu\nu}. \tag{66}$$

The $\nu = j$ component has already been satisfied by the assumption in (37), and thus we can assume correspondingly that part of the usual matter conservation law is satisfied, $\nabla_\mu (T_M)^\mu_j = 0$. The Hamiltonian method did not directly involve the $\nu = 0$ component, so we must ensure that it holds as well in the cosmological solutions here.

After some computation, we obtain for the left-hand side of (66),

$$\nabla_\mu (T_s)^\mu_0 = \frac{\ddot{a}}{a} \left(\frac{3}{2} \dot{s}_{00} + 6s_{00} \frac{\dot{a}}{a} \right) + 3s_{00} \frac{\ddot{a}}{a}. \quad (67)$$

For the matter part, we obtain

$$\nabla_\mu (T_M)^\mu_0 = -\dot{\rho} - 3 \frac{\dot{a}}{a} (\rho + p). \quad (68)$$

Consistency of these results, i.e., left-hand side equals right-hand side, can be verified with Eqs. (64) and (65) by solving for ρ and p and inserting the expressions into (68), to recover (67).

We are now in a position to examine the consequences of different choices for s_{00} . Among the myriad of possible functional forms for s_{00} , we study two cases here. First, we look at the case where s_{00} is determined by demanding that the matter stress-energy tensor by itself is completely conserved and thus Eq. (68) vanishes. Second, we look at a case when Eq. (68) does not vanish, yet the total conservation law (66) holds.

A. FLRW example 1

For the first case, we enforce the matter energy-momentum conservation law. Note that if the matter equations of motion are satisfied (“on shell”) this condition would necessarily hold. This condition implies that $\nabla_\mu (T_s)^\mu_0 = 0$ and thus the following expression must be solved for s_{00} :

$$-\frac{\ddot{a}}{a} \left(\frac{3}{2} \dot{s}_{00} + 6s_{00} \frac{\dot{a}}{a} \right) = 3s_{00} \frac{\ddot{a}}{a}. \quad (69)$$

It turns out that an analytical solution for s_{00} given by

$$s_{00} = \frac{\zeta}{a^4 \ddot{a}^2} \quad (70)$$

solves this equation, where ζ is an arbitrary constant. This solution has intriguing and yet pathological features. Obviously, if the acceleration $\ddot{a} = 0$, as it does in the past for standard cosmological solutions, it diverges. Also, if we could assume constant behavior for $\ddot{a}$, then the result shows that the coefficient s_{00} would naturally decrease with the expanding universe.

The next step to pursue the case just outlined would be to insert the solution (70) back into the modified equations (64) and (65) and attempt to solve the resulting

system of equations for $a(t)$ for different choices of sources ρ and p . However, one finds that the resulting equations have up to fourth order time derivatives in them, if they are solved with no approximations made. Furthermore, it is challenging to approach the equations from a perturbative point of view, where the dimensionless s_{00} is “small” compared to unity, as can be seen from plugging a GR solution into (70): again when $\ddot{a}$ approaches zero, this grows large and conflicts with perturbation theory. In this work, we do not pursue this solution further and leave it as an open problem to explore.

B. FLRW example 2

We now turn to the case where we do not impose the vanishing of $\nabla_\mu (T_s)^\mu_0$. The coefficient s_{00} remains arbitrary, and so for this work we examine the simplest case of a constant coefficient, $\dot{s}_{00} = 0$. As a consequence of this choice, the matter conservation law gets modified and matter exhibits a modified cosmological evolution in the presence of $s_{00} \neq 0$. First, we write the Friedmann equations for this case as

$$\begin{aligned} \left(\frac{\dot{a}}{a} \right)^2 &= \frac{\kappa \rho}{3(1 - \frac{3}{2}s_{00})} - \frac{k}{a^2(1 - s_{00})} \\ &\quad + \frac{\kappa p s_{00}}{(2 - 3s_{00})(1 - s_{00})}, \\ \frac{\ddot{a}}{a} &= -\frac{\kappa(\rho + 3p)}{6(1 - \frac{3}{2}s_{00})}. \end{aligned} \quad (71)$$

These results contain various scalings of the usual GR terms but also a nonstandard appearance of the pressure in the first equation.

Using (71) and (68) we obtain the modified conservation law, or continuity equation, as

$$\dot{\rho} + 3 \frac{\dot{a}}{a} f(w, s_{00}) \rho = 0, \quad (72)$$

where we have introduced the auxiliary equation $f(w, s_{00})$ as

$$f(w, s_{00}) = \frac{2(1 + w - s_{00})}{2 + s_{00}(3w - 2)}, \quad (73)$$

which reduces to the proper GR limit, where $f \rightarrow 1 + w$ as $s_{00} \rightarrow 0$. We integrate the modified continuity equation to find that

$$\rho = \rho_0 \left(\frac{a}{a_0} \right)^{-3f(w, s_{00})}, \quad (74)$$

where a_0 is the present value of the scale factor. For matter as a dust $w = 0$ and $f = 1$ so there is no modification to the cosmological evolution $\rho \sim a^{-3}$. However, for radiation

$w = 1/3$ and the cosmological constant $w = -1$, the evolution equation is modified, as occurs in other modifications to GR [53,54].

This leads to interesting type-dependent evolution of the different cosmological fluids. We can put together the Friedmann equations, paralleling the usual methods, by using dimensionless density parameters Ω_X , where $X = \{m, r, \Lambda, k\}$ represents the energy density of matter, radiation, cosmological constant, and curvature, respectively. We divide the first of Eqs. (71) by the square of the present value of the Hubble constant $H_0^2 = \dot{a}_0^2/a_0^2$ and use the evolution equation (74). The result can be written

$$\frac{H^2}{H_0^2} = \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4\eta_r} + \Omega_{\Lambda0}a^{\eta_\Lambda} + \Omega_{k0}a^{-2}, \quad (75)$$

where $H = \dot{a}/a$, $\eta_r = (1 - \frac{3}{4}s_{00})/(1 - \frac{1}{2}s_{00})$, and $\eta_\Lambda = 3s_{00}/(1 - \frac{5}{2}s_{00})$. Note that matter (m) and curvature (k) behave normally while radiation (r) and the cosmological constant (Λ) differ from GR, and that the density parameters in (75) are evaluated at the present epoch t_0 , as indicated by the subscript 0. The density parameters here can be found for each universe constituent from

$$\Omega_X = \frac{\kappa\rho}{3H^2(1 - \frac{3}{2}s_{00})} \frac{2 + (3w - 2)s_{00}}{2(1 - s_{00})}, \quad (76)$$

and for curvature $\Omega_k = -k/[H^2(1 - s_{00})]$. Note that scalings by s_{00} have been absorbed into the definitions of the Ω_X values.

Next we examine the acceleration equation for this case. Using the same density parameters, the second of equations (71) can be written

$$\begin{aligned} \frac{\ddot{a}}{aH_0^2} = & -\frac{1}{2}\Omega_{m0}a^{-3} - \Omega_{r0}\frac{2(1-s_{00})}{2-s_{00}}a^{-4\eta_r} \\ & + \Omega_{\Lambda0}\frac{2(1-s_{00})}{2-5s_{00}}a^{\eta_\Lambda}. \end{aligned} \quad (77)$$

Here we can see that the scalings appearing cannot be completely removed by redefining constants.

Equation (77) gives us the deceleration parameter, $q \equiv -(\ddot{a}/a)H^{-2}$, which we can attempt to use to find a crude constraint on s_{00} . Since the value of q at the present epoch ($t = t_0$) needs to be negative in order to match the observed accelerated expansion we can conservatively write the inequality

$$-\frac{1}{2}\Omega_{m0} - \Omega_{r0}\frac{2(1-s_{00})}{2-s_{00}} + \Omega_{\Lambda0}\frac{2(1-s_{00})}{2-5s_{00}} > 0. \quad (78)$$

Other than showing that s_{00} is less than order unity, this result is not particularly useful for placing constraints, since it is challenging to disentangle the density parameters from

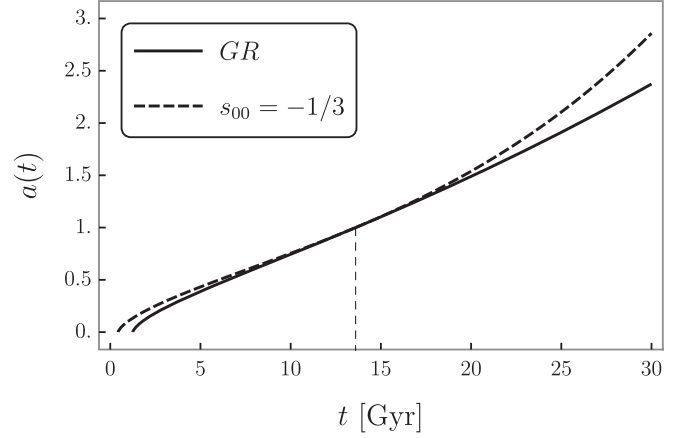


FIG. 2. Evolution of the scale factor for the constant s_{00} case of the flat FLRW solutions compared to GR, assuming $\Omega_{r0} = 0$, $\Omega_{m0} = 0.31$, and $\Omega_{\Lambda0} = 0.69$. The dashed vertical line represents the present day.

the s_{00} coefficient. Thus a complete analysis using cosmological data [55] should be attempted in the future. To display what the effects of the modified evolution would look like, we solve the first Friedmann equation in (75) and plot in Fig. 2.

VI. CONNECTION TO MODELS AND FRAMEWORKS

The SME is a test framework, and as such, any action-based model that describes coordinate-independent Lorentz violation should in principle be contained in some subset of terms. In practice, this can be challenging when certain assumptions are made in the SME to afford tractable phenomenological analyses [13,31], while these assumptions can differ from those made in specific models. We show here first how the results in this paper match to prior work in linearized gravity, and then we find a match to models formulated in the 3 + 1 formalism.

A. Quadratic SME gravity sector

In Refs. [13,14,16,17], results in the linearized gravity limit have been developed. In particular, a classification of all possible Lorentz-breaking Lagrangian terms at quadratic order in the metric fluctuations $h_{\mu\nu}$ around a flat background has been performed. Such terms take the form $\mathcal{L} \sim h_{\mu\nu}\hat{\mathcal{K}}^{\mu\nu\rho\sigma}h_{\rho\sigma}$ and much phenomenological analysis already exists, including results at leading order in the coefficients in propagation studies. In linearized gravity, diffeomorphism invariance can be described using the gauge transformation of the metric fluctuations $h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$. The analysis of the quadratic action terms includes both gauge symmetry breaking and gauge symmetric terms, though scant phenomenological attention has been put on the former.

We seek here to match the explicit breaking limit of the SME that we have used in this work to a subset of these terms in the weak-field limit. This will illuminate how results in this work may match to those previously obtained. Curiously, the SME Lagrangian with u , $s_{\mu\nu}$, and $t_{\alpha\beta\gamma\delta}$ terms can be shown to have both gauge-breaking and gauge-symmetric terms in the quadratic action limit when taken in the explicit breaking limit. Furthermore, one can trace the occurrence of dynamical pieces of the metric fluctuations $h_{\mu\nu}$ that are nondynamical in GR and in gauge-symmetric models. To see this we first examine the contributions of the $s_{\mu\nu}$ -type term only,

$$\begin{aligned}\mathcal{L}_s &= \frac{\sqrt{-g}}{2\kappa} s_{\mu\nu} R^{\mu\nu} \\ &= \frac{\sqrt{-g}}{2\kappa} \left(s_{\mu\nu} G^{\mu\nu} + \frac{1}{2} s^\lambda{}_\lambda R \right),\end{aligned}\quad (79)$$

where no linear approximations have yet been made.

Next we assume a weak-field expansion around a flat background for both the metric and $s_{\mu\nu}$:

$$\begin{aligned}g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta}, \\ s_{\alpha\beta} &= \bar{s}_{\alpha\beta} + \tilde{s}_{\alpha\beta}.\end{aligned}\quad (80)$$

We keep fluctuations for $s_{\alpha\beta}$ for generality at this point, and we will assume that the partial derivatives of $\bar{s}_{\mu\nu}$ vanish. The Lagrange density (79) is then expanded in the quadratic action limit (keeping terms of order h^2 , $h\tilde{s}$, $\tilde{s}^2$ and discarding total derivatives). It can then be written as

$$\begin{aligned}\mathcal{L}_s &\approx \frac{1}{2\kappa} \left[\left(1 + \frac{1}{2} h \right) \bar{s}_{\alpha\beta} G^{\alpha\beta} + \bar{s}_{\alpha\beta} (G_L)^{\alpha\beta} \right. \\ &\quad \left. + \frac{1}{2} \left(1 + \frac{1}{2} h \right) \eta^{\mu\nu} \tilde{s}_{\mu\nu} R + \frac{1}{2} (\eta^{\mu\nu} \tilde{s}_{\mu\nu} - h^{\mu\nu} \tilde{s}_{\mu\nu}) R_L \right],\end{aligned}\quad (81)$$

where curvature terms with the subscript L are linearized and those without are taken to quadratic order. It turns out that the first term on the first line of (81) by itself reproduces the gauge invariant contribution to the SME quadratic action expansion for the $\bar{s}_{\mu\nu}$ term,

$$\frac{1}{2\kappa} \left(1 + \frac{1}{2} h \right) \bar{s}_{\alpha\beta} G^{\alpha\beta} = \frac{1}{4\kappa} \bar{s}_{\alpha\beta} h_{\gamma\delta} G^{\alpha\gamma\beta\delta}, \quad (82)$$

where $G^{\alpha\gamma\beta\delta}$ is the linearized double dual curvature tensor [56]. Thus, if we take the explicit-breaking limit by discarding the fluctuations $\tilde{s}_{\mu\nu}$ entirely, we end up with the sum of the gauge invariant quadratic action terms and gauge-violating terms.

To summarize so far: in the quadratic action limit

$$\mathcal{L}_{s,\text{explicit}} = \frac{1}{4\kappa} (\bar{s}_{\alpha\beta} h_{\gamma\delta} G^{\alpha\gamma\beta\delta} - h^{\mu\nu} \bar{s}_{\mu\nu} R_L), \quad (83)$$

where the second term is explicitly gauge violating and can be matched to the general expansion of Ref. [17], and we have discarded the trace $\eta^{\mu\nu} s_{\mu\nu}$ term that merely scales GR. Among the gauge-violating terms in [17], at mass dimension four, there are two types of terms which are relevant for the second term in (83). They are contained in the general expansion in Table 1 of [17]:

$$h_{\mu\nu} \hat{\mathcal{K}}^{\mu\nu\rho\sigma} h_{\rho\sigma} \supset h_{\mu\nu} (s^{(4,1)\mu\rho\nu\sigma\alpha\beta} + k^{(4,1)\mu\rho\nu\sigma\alpha\beta}) \partial_\alpha \partial_\beta h_{\rho\sigma}, \quad (84)$$

where the $\mu\rho\nu\sigma$ indices are totally symmetric in the $s^{4,1}$ coefficients and of Riemann tensor symmetry $[\mu\rho][\nu\sigma]$ for the $k^{4,1}$ coefficients. The match to these terms for the present case can be obtained using the form

$$h^{\mu\nu} \bar{s}_{\mu\nu} R_L = \frac{1}{2} h_{\mu\nu} (\bar{s}^{\mu\nu} \hat{K}^{\rho\sigma} + \bar{s}^{\rho\sigma} \hat{K}^{\mu\nu}) h_{\rho\sigma}, \quad (85)$$

where $\hat{K}^{\mu\nu} = \partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^\lambda \partial_\lambda$. To complete the match one has to take the appropriate symmetric and antisymmetric combinations of the quantity in parentheses in (85).

Finally, we note that the fact that the terms studied in this paper correspond to the gauge-violating limit of the SME quadratic expansion explains, in part, why additional degrees of freedom beyond GR are found, as in Sec. IV B. For instance, because of the symmetries of the operator $\hat{\mathcal{K}}^{\mu\nu\rho\sigma}$ for gauge-symmetric terms, it can be shown that no time derivatives of h_{00} appear when the Lagrange density is written in the first-order derivative form $\sim \partial h \mathcal{K} \partial h$. Any such terms would correspond to time derivatives of the lapse function α via $\alpha \approx 1 + h_{00}/2$ in the weak-field limit. For gauge-violating terms, as in Eq. (85), such terms can appear because the symmetries of the operators $\hat{\mathcal{K}}^{\mu\nu\rho\sigma}$ allow for them, as they are less restrictive. In the case of $\bar{s}_{00}$ being the only nonzero coefficient we have

$$\mathcal{L}_s \supset \frac{1}{4\kappa} \bar{s}_{00} \left[\partial_0 h_{00} \left(\partial_0 h_{jj} - \frac{1}{2} \partial_j h_{j0} \right) + \dots \right]. \quad (86)$$

Despite this interesting feature there are likely severe constraints on any such models via the traced Bianchi identities, even in the linearized gravity limit. For example, for the case (83), the field equations from the first term are gauge invariant and automatically satisfy the traced Bianchi identities. The second term, however, would yield a constraint in the presence of matter given by

$$\frac{1}{2} \partial_\mu (\bar{s}^{\mu\nu} R_L) = \kappa \partial_\mu (T_M)^{\mu\nu}. \quad (87)$$

Thus one has a Ricci flat restriction which is challenging to reconcile in the presence of matter, or one has a modified

conservation law for matter, or one must reject such cases (“no-go”). We showed in Sec. V that modified behavior of matter may be an acceptable solution in some cases, such as cosmology.

B. Match to 3 + 1 models

Matches of specific models of Lorentz violation to the SME has been accomplished in the gravitational sector for a variety of models including those with dynamical vectors and tensors, noncommutative geometry, and massive gravity models [39]. Among the proposals for renormalizable quantum gravity is the approach known as Hořava-Lifshitz (HL) gravity [57]. Since this model is based on a 3 + 1 formalism, we should be able to match it to the SME in the present work. We shall focus on a simpler version of this model where the action is written in the 3 + 1 form:

$$\mathcal{L}_H = \alpha\sqrt{\gamma}(K_{ij}K^{ij} - \lambda K^2 + \xi\mathcal{R} + \eta a^i a_i + \dots), \quad (88)$$

where the ellipses include possible higher order spatial derivative terms and the matter sector [58,59]. (For simplicity in the remainder of this section we set the coupling $2\kappa = 1$.)

Note that the insertion of a parameter in front of the terms that occur in GR is akin to early kinematic approaches of tests of special relativity and dressed-metric based approaches for tests of GR [60]. That approach seems somewhat *ad hoc* from the SME point of view, since the SME is based on observer covariant terms added to the action with coefficients with indices. Nonetheless, we can possibly accommodate these terms with certain components of the SME coefficients in a particular coordinate system, as has been done for other models [61]. Equation (88) takes a rotationally isotropic form. If we proceed with the $s_{\mu\nu}R^{\mu\nu}$ coupling in the isotropic limit presented in Sec. IV C, assuming the coefficients are constant in time and space, we obtain

$$\mathcal{L}_2 = \alpha\sqrt{\gamma}\left[\mathcal{R}\left(1 + \frac{1}{3}s\right) + (K^{ij}K_{ij} - K^2)(1 - s_{\bar{0}\bar{0}})\right]. \quad (89)$$

Note that in the isotropic limit the combination $K^{ij}K_{ij} - K^2$ cannot be broken apart with an $s_{\mu\nu}$ -type term alone; however, in the conception of the SME as a limit of spontaneous symmetry breaking we have the freedom to add dynamical terms to the action. For example, for the $s_{\mu\nu}$ coefficients we can add general dynamical terms [22], which are included in Appendix C, to match (88).

We take first the case where $s_{\bar{0}\bar{0}} = 0$ in (89) and add the terms labeled 5 and 12 in the Appendix with a distinct set of coefficients that we denote with a capital $S_{\mu\nu}$. This yields

$$\begin{aligned} \mathcal{L}_{\text{SME,Match}} = & \alpha\sqrt{\gamma}\left[\mathcal{R}\left(1 + \frac{1}{3}s\right) + K^{ij}K_{ij} - K^2 \right. \\ & + a_5\frac{1}{2}(\nabla_\mu S^{\mu\lambda})(\nabla_\nu S^\nu{}_\lambda) \\ & \left. + a_{12}(S^{\mu\nu}\nabla_\mu S_{\nu\lambda})(S^{\kappa\rho}\nabla_\kappa S_{\rho}{}^\lambda)\right]. \quad (90) \end{aligned}$$

All terms in this Lagrange density are now treated as nondynamical. We next assume for the last two terms that the only nonzero coefficient in the local frame is $S_{\bar{0}\bar{0}} = 1$ —note the precise value of the coefficient needed. Using the vierbein (19) one can show that this is equivalent to $S_{\mu\nu} = n_\mu n_\nu$. This kind of choice has been used to match HL gravity to vector models [62]. With these assumptions we arrive at

$$\begin{aligned} \mathcal{L}_{\text{SME,Match}} = & \alpha\sqrt{\gamma}\left[\mathcal{R}\left(1 + \frac{1}{3}s\right) + K^{ij}K_{ij} \right. \\ & \left. - K^2\left(1 + \frac{1}{2}a_5\right) + \left(a_{12} + \frac{1}{2}a_5\right)a^i a_i\right]. \quad (91) \end{aligned}$$

It is now clear that if we make the following choice, $\lambda = 1 + a_5/2$, $\xi = 1 + s/3$, and $\eta = a_{12} + a_5/2$, then HL gravity in the form (88) can be matched to this limit of the SME. Note that the extra terms added to the SME are of second order and fourth order in $S_{\mu\nu}$, and two distinct sets of coefficients were used in this match. Finally, while we do not discuss it here, matter couplings proposed in the literature have also been matched to the matter sector of the SME in Ref. [39].

VII. DISCUSSION AND CONCLUSION

In this work, we have taken initial steps toward exploring the SME effective field theory framework description of local Lorentz and diffeomorphism breaking in the areas of the 3 + 1 formalism, Dirac-Hamilton analysis of the dynamics, and cosmology. We have examined consequences of adopting the explicit symmetry breaking paradigm, which is complementary to existing work assuming spontaneous symmetry breaking. Furthermore, we have established results without using the weak-field gravity approximation.

The key results of this work include a 3 + 1 decomposition of the SME gravity sector actions in Sec. III B, including a general analysis of the time derivative terms that occur, relevant for Hamiltonian analysis. We studied two example subsets of the SME using the Dirac-Hamiltonian analysis in Sec. IV. The results of one of these cases, Hamilton’s equations in (38)–(43), were studied for FLRW cosmological solutions in Sec. V, where some novel cosmological evolution was found. Further analysis for other strong-field gravity solutions can be the

subject of future work, for instance black hole spacetimes or other exotic solutions [63]. We also established a link between the explicit breaking terms in this work and existing SME studies in linearized gravity, and we further elucidated the match to Hořava-Lifshitz gravity in Sec. VI.

A set of Hamilton's equations such as those found in Sec. IV for a subset of the SME can be used to study the initial value formulation and develop numerical techniques to simulate Lorentz-breaking effects on strong-field gravitational systems [64]. Results in this paper can also be applied to a 3 + 1 and Dirac-Hamiltonian analysis of spontaneous symmetry breaking scenarios, for example by using the second order $s_{\mu\nu}$ terms in (C1).

One of the notable results of this work is the identification of subsets of the SME, whereupon in the explicit breaking limit, extra degrees of freedom, normally gauge in GR, occur in the Hamiltonian analysis. In light of this, it would be of interest to investigate approaches to quantum gravity [44] and the role of the "problem of time" in the SME framework [65]. Also, we explored the severe constraints that exist on such a model, due to constraints imposed by the Bianchi identities coming from the underlying geometrical framework [23]. The analysis in this paper represents a first step toward studying this phenomena, and it remains an open problem to fully understand the nature of these extra degrees of freedom.

As a preview of future work, we note that the cosmological solutions in Sec. V can be obtained from an effective classical Hamiltonian for homogeneous spacetimes with the variables $a(t)$ and $\alpha(t)$, their conjugate momenta $p_a = \partial L / \partial \dot{a}$ and $p_\alpha = \partial L / \partial \dot{\alpha}$, and matter variables, where L is the classical Lagrangian. This "mini-superspace" Hamiltonian takes the form, for vanishing spatial curvature and up to scalings,

$$H = \frac{\kappa \alpha^5 (\alpha^2 - s_{00}) p_a^2}{3a^3 s_{00}^2} - \frac{\kappa \alpha^4 p_a p_\alpha}{3a^2 s_{00}} + H_M, \quad (92)$$

with matter Hamiltonian H_M . This would modify the widely studied Wheeler-deWitt equation [66], for which α is nondynamical and a p_a^2 term is present instead. Indeed, since the usual Hamiltonian constraint is absent in this model, the wave function $\Psi = \Psi(a, \alpha, \dots, t)$ would depend on time t and evolve according to the Schrödinger equation $i\partial_t \Psi = H\Psi$. We expect this could offer a new area of exploration in quantum cosmology and will be studied in the future.

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APPENDIX A: 3 + 1 FORMALISM

In the 3 + 1 formalism we can express projections of the curvature tensors in terms of the timelike normal to the spatial hypersurfaces n^μ , the inverse spatial metric $\gamma^{\mu\nu}$, the extrinsic curvature $K_{\mu\nu}$, the spatial covariant derivative $\mathcal{D}_\mu$, the Lie derivative along the normal vector $\mathcal{L}_n$, the acceleration a_μ , and the three-dimensional curvature tensor $\mathcal{R}_{\alpha\beta\gamma\delta}$. This decomposition is standard in the literature [43,45], but for completeness we record here some useful results that can be derived from existing published ones. First, the basic relations for the 3 + 1 projections of the four-dimensional curvature tensor are given by

$$\begin{aligned} \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\kappa \gamma^\delta_\lambda R_{\alpha\beta\gamma\delta} &= \mathcal{R}_{\mu\nu\kappa\lambda} + K_{\mu\kappa} K_{\nu\lambda} - K_{\mu\lambda} K_{\nu\kappa}, \\ \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\kappa n^\delta R_{\alpha\beta\gamma\delta} &= \mathcal{D}_\nu K_{\mu\kappa} - \mathcal{D}_\mu K_{\nu\kappa}, \\ \gamma^\beta_\mu \gamma^\delta_\nu n^\alpha n^\gamma R_{\alpha\beta\gamma\delta} &= \mathcal{L}_n K_{\mu\nu} + \frac{1}{\alpha} \mathcal{D}_\mu \mathcal{D}_\nu \alpha + K^\beta_\mu K_{\nu\beta}. \end{aligned} \quad (A1)$$

From these, by taking contractions, we have the following decomposition of the four-dimensional curvature Ricci tensor:

$$\begin{aligned} R^{\mu\nu} &= \mathcal{R}^{\mu\nu} + n^\mu K^{\nu\alpha} a_\alpha + n^\nu K^{\mu\alpha} a_\alpha + K K^{\mu\nu} - \mathcal{L}_n K^{\mu\nu} \\ &\quad + 2K^{\alpha\mu} K^\nu_\alpha - a^\mu a^\nu - \mathcal{D}^\mu a^\nu - n^\mu \mathcal{D}^\nu K - n^\nu \mathcal{D}^\mu K \\ &\quad + n^\mu \mathcal{D}_\alpha K^{\alpha\nu} + n^\nu \mathcal{D}_\alpha K^{\alpha\mu} \\ &\quad + n^\mu n^\nu (\mathcal{L}_n K - K^{\alpha\beta} K_{\alpha\beta} + a^2 + \mathcal{D}_\alpha a^\alpha). \end{aligned} \quad (A2)$$

It is also useful to have a form for the curvature tensors which includes total spacetime covariant derivatives rather than Lie derivatives and spatial covariant derivatives. Using the definitions and properties of spatial covariant derivatives and Lie derivatives, Eqs. (A1) can be manipulated to the following forms:

$$\begin{aligned} R &= \mathcal{R} + K^{\alpha\beta} K_{\alpha\beta} - K^2 - 2\nabla_\alpha (n^\alpha K + a^\alpha), \\ R^{\alpha\beta} &= \mathcal{R}^{\alpha\beta} - 2K^{\alpha\beta} K + 2K^\alpha_\delta K^{\delta\beta} - n^\alpha a^\beta K \\ &\quad + n^\alpha K^\beta_\delta a^\delta - n^\alpha n^\beta (K^2 - K^{\alpha\beta} K_{\alpha\beta}) \\ &\quad + \nabla_\delta [n^\alpha n^\beta (n^\delta K + a^\delta) - n^\delta K^{\alpha\beta} - \gamma^{\delta\beta} a^\alpha] \\ &\quad - (n^\alpha \gamma^{\beta\delta} + n^\beta \gamma^{\alpha\delta}) K + n^\alpha K^{\beta\delta} + n^\beta K^{\alpha\delta}, \\ R^{\alpha\beta\gamma\delta} &= \mathcal{R}^{\alpha\beta\gamma\delta} - 3(K^{\alpha\gamma} K^{\beta\delta} - K^{\beta\gamma} K^{\alpha\delta}) \\ &\quad + (K^{\alpha\epsilon} K^\gamma_\epsilon n^\beta n^\delta + \text{sym}) - (K^{\alpha\gamma} n^\beta n^\delta K + \text{sym}) \\ &\quad - (K^{\alpha\gamma} n^{(\beta} a^{\delta)} + \text{sym}) \\ &\quad + \nabla_\epsilon [n^\epsilon (K^{\alpha\gamma} n^\beta n^\delta + \text{sym}) \\ &\quad + (\gamma^{\epsilon(\alpha} a^{\gamma)} n^\beta n^\delta + \text{sym}) \\ &\quad - 2(K^{\alpha\gamma} n^{(\beta} \gamma^{\delta)\epsilon} + \text{sym})], \end{aligned} \quad (A3)$$

where in the last equation, "sym" refers to the Riemann symmetric combination of terms involving the indices α, β, γ ,

δ . For instance, for two symmetric tensors $A^{\alpha\gamma}B^{\beta\delta} + \text{sym} = A^{\alpha\gamma}B^{\beta\delta} - A^{\beta\gamma}B^{\alpha\delta} - A^{\alpha\delta}B^{\beta\gamma} + A^{\beta\delta}B^{\alpha\gamma}$.

Results using the explicit form for the metric (9) are used throughout this paper, and some key expressions are collected here. The three-dimensional connection coefficients are given explicitly in terms of the metric γ_{ij} :

$${}^{(3)}\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}), \quad (\text{A4})$$

where γ^{il} is the inverse of the three metric and satisfies $\gamma^{il}\gamma_{lk} = \delta^i_k$. The components of the spatial covariant derivative acting on an arbitrary covariant vector v_μ are given by

$$\begin{aligned} \mathcal{D}_0 v_0 &= \beta^i \beta^j (\partial_i v_j - {}^{(3)}\Gamma^k_{ij} v_k + n^\mu v_\mu K_{ij}), \\ \mathcal{D}_0 v_i &= \beta^j (\partial_j v_i - {}^{(3)}\Gamma^k_{ij} v_k + n^\mu v_\mu K_{ij}), \\ \mathcal{D}_i v_0 &= \beta^j (\partial_i v_j - {}^{(3)}\Gamma^k_{ij} v_k + n^\mu v_\mu K_{ij}), \\ \mathcal{D}_i v_j &= \partial_i v_j - {}^{(3)}\Gamma^k_{ij} v_k + n^\mu v_\mu K_{ij}, \end{aligned} \quad (\text{A5})$$

where $n^\mu v_\mu = (1/\alpha)(v_0 - \beta^i v_i)$.

APPENDIX B: POISSON BRACKET ANALYSIS

In this subsection we collect some key results on Poisson brackets in field theory for the Dirac-Hamiltonian analysis that we use in the paper. Some results can be found in various places in the literature [27,46], but some subtleties arise in the calculations and it is useful to record them explicitly here. First, for fields $q_n(t, \vec{r})$, momenta $p^n(t, \vec{r})$, and functions of the fields and momenta $f(q, p)$ and $g(q, p)$, the Poisson bracket definition is formally

$$\{f, g\} = \int d^3z \left(\frac{\delta f}{\delta q_n(t, \vec{z})} \frac{\delta g}{\delta p^n(t, \vec{z})} - \frac{\delta f}{\delta p^n(t, \vec{z})} \frac{\delta g}{\delta q_n(t, \vec{z})} \right), \quad (\text{B1})$$

where f and g may depend on different spatial points via their dependence on the fields and momenta. Note also the equal times for all the fields. As an example, if we examine a single scalar field and let $q_1 = \phi(t, \vec{r})$ and the conjugate momenta $p_1 = \Pi = \Pi(t, \vec{r}')$, then we obtain

$$\{\phi(t, \vec{r}), \Pi(t, \vec{r}')\} = \delta^3(\vec{r} - \vec{r}'). \quad (\text{B2})$$

In classical mechanics, the functions f and g are algebraic functions of the coordinates and momenta. In field theory, however, one often encounters spatial derivatives in the calculations of Hamilton evolution via Poisson brackets. Generically, for a partial spatial derivative ∂_i of a function f of the canonical variables, its Poisson bracket with another function g can be shown to obey

$$\{\partial_i f, g\} = \partial_i \{f, g\}, \quad (\text{B3})$$

where the derivative acts on the space dependence x^j of the result of the bracket of f and g . This result can be extended to covariant spatial derivatives. For example, for the quantity which occurs in GR and the SME for the momentum constraint $\mathcal{D}_i \Pi^i_k$, and its Poisson bracket with the Hamiltonian H , using (B1) and (B3) we find

$$\begin{aligned} \{\gamma_{kl} \mathcal{D}_i \Pi^{il}, H\} &= \{\gamma_{kl}, H\} \mathcal{D}_i \Pi^{il} + \gamma_{kl} \mathcal{D}_i \{\Pi^{il}, H\} \\ &+ \Pi^{ij} \mathcal{D}_i \{\gamma_{jk}, H\} - \frac{1}{2} \Pi^{il} \mathcal{D}_k \{\gamma_{jl}, H\}. \end{aligned} \quad (\text{B4})$$

It is important to note that we used the fact that Π^{ij} is a three-dimensional tensor density of weight -1 and that the spatial covariant derivative has a dependence on the spatial metric γ_{ij} , resulting in the last two terms.

APPENDIX C: HIGHER-ORDER TERMS

The following terms generalize gravitational couplings to curvature for the SME for the $s_{\mu\nu}$ term with scalar dimensionless coupling parameters a_n :

$$\begin{aligned} \mathcal{L}_s &= \frac{\sqrt{-g}}{2\kappa} \left[a_1 s^\lambda_\lambda R + a_2 s_{\mu\nu} R^{\mu\nu} \right. \\ &+ a_3 \frac{1}{2} (\nabla_\mu s_{\nu\lambda}) (\nabla^\mu s^{\nu\lambda}) + a_4 \frac{1}{2} (\nabla_\mu s^{\mu\lambda}) (\nabla_\lambda s^\beta_\beta) \\ &+ a_5 \frac{1}{2} (\nabla_\mu s^{\mu\lambda}) (\nabla_\nu s^\nu_\lambda) + a_6 \frac{1}{2} (\nabla_\mu s^\nu_\nu) (\nabla^\mu s^\lambda_\lambda) \\ &+ a_7 s_{\mu\nu} s_{\kappa\lambda} R^{\mu\kappa\nu\lambda} + a_8 s_{\mu\nu} s^\mu_\lambda R^{\nu\lambda} \\ &\left. + a_9 s^\lambda_\lambda s_{\mu\nu} R^{\mu\nu} + a_{10} s^{\mu\nu} s_{\mu\nu} R + a_{11} s^\lambda_\lambda s^\mu_\mu R \right]. \end{aligned} \quad (\text{C1})$$

The first two terms are just the originally proposed SME couplings, linear in the coefficients $s_{\mu\nu}$. The remaining terms are second order in the coefficients $s_{\mu\nu}$ [22]. Since $s_{\mu\nu}$ are dimensionless and normally assumed small compared to unity, these terms represent a step beyond the minimal SME, which assumes first-order terms in the coefficients, and they are a special case of the terms outlined in Table V and Table VII in Ref. [23].

Many of these terms for a symmetric two-tensor have been proposed in modified gravity models in the literature in different contexts [60]. Also, other possible terms are omitted due to equivalence via integration by parts. For example,

$$0 = \int d^4x \sqrt{-g} (\nabla_\gamma s_{\alpha\beta} \nabla^\beta s^{\alpha\gamma} - \nabla_\beta s_\alpha^\beta \nabla_\gamma s^{\alpha\gamma} - s_{\alpha\beta} s_{\gamma\delta} R^{\alpha\delta\beta\gamma} + s_{\alpha\delta} s_\beta^\alpha R^{\delta\beta}). \quad (\text{C2})$$

Note also that one can add general potential terms for a symmetric two-tensor of the form $V(s_\mu^\mu, s_{\mu\nu} s^{\mu\nu}, \dots)$ for the case of spontaneous symmetry breaking, as detailed elsewhere [35]. In the particular case of the match to $3 + 1$

models in Sec. VI B, the possibility exists of using a term quartic in the coefficients $s_{\mu\nu}$:

$$\Delta\mathcal{L}_s = a_{12} \frac{\sqrt{-g}}{2\kappa} (s^{\mu\nu} \nabla_\mu s_{\nu\lambda}) (s^{\kappa\rho} \nabla_\kappa s_{\rho}^\lambda). \quad (\text{C3})$$

An analysis of these and other possible terms in the SME is forthcoming.

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