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Thurston norms of tunnel number-one manifolds

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ABSTRACT

The Thurston norm of a three-manifold measures the complexity of surfaces representing two-dimensional homology classes. We study the possible unit balls of Thurston norms of three-manifolds M with $b_1(M) = 2$, and whose fundamental groups admit presentations with two generators and one relator. We show that even among this special class, there are three-manifolds such that the unit ball of the Thurston norm has arbitrarily many faces.

Keywords: Thurston norm, three-manifold, one-relator groups.

Mathematics Subject Classification 2010: 57M27, 57M05, 20J05, 20J06, 57R19

1. Introduction

A *knot* is the image of a smooth embedding of the circle $\mathbb{S}^1$ into the three-sphere $\mathbb{S}^3$. Every knot $K \subset \mathbb{S}^3$ bounds an embedded orientable surface in $\mathbb{S}^3 - K$ called a *Seifert surface*. This leads to a useful knot invariant, called the *genus* of K, which is the minimal genus of such a Seifert surface. For example, the unknot is the only knot whose genus is zero. This invariant can be generalized to the *Thurston norm*, which is defined for all compact, connected, orientable three–manifolds.

Specifically, for each $\phi \in H^1(N, \mathbb{Z})$, let $PD(\phi)$ denote the homology class in $H_2(N, \partial N, \mathbb{Z})$ that is Poincaré dual to ϕ. This class can be represented by a properly embedded oriented surface Σ. Let $\chi_-(\Sigma) = \sum_{i=1}^{k} \max\{-\chi(\Sigma_i), 0\}$, where

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$\Sigma_1, \ldots, \Sigma_k$ are the connected components of Σ. The *Thurston norm* of ϕ is defined to be

$$x_N(\phi) := \min\{\chi_-(\Sigma) | [\Sigma] = PD(\phi) \in H_2(N, \partial N; \mathbb{Z})\}.$$

Thurston showed that x_N can be extended to a seminorm on $H^1(N, \mathbb{R})$. Since a Seifert surface of a knot K is dual to the generator of $H^1(\mathbb{S}^3 - K, \mathbb{Z})$ corresponding to a meridional loop, the Thurston norm of this generator is $2g(K)-3$, where $g(K)$ is the genus of K; hence, the Thurston norm generalizes the knot genus.

The Thurston norm is also useful in studying the possible ways that a three-manifold can fiber over a circle. Recall that a class $\phi \in H^1(N; \mathbb{R})$ is *fibered* if ϕ is represented by a non-degenerate closed 1-form; in particular, $\phi \in H_1(N, \mathbb{Z})$ is fibered if and only if it is induced by a fibration $N \to \mathbb{S}^1$.

The results of Thurston [9, Theorems 1–3] can be summarized in the following theorem. A *marked polytope* is a polytope with a distinguished subset of vertices, which we refer to as *marked vertices*.

Theorem 1.1 (Thurston). *Let N be a three-manifold. There exists a unique centrally symmetric marked polytope $\mathcal{M}_N$ in $H_1(N; \mathbb{R})$ such that for any $\phi \in H^1(N; \mathbb{R}) = \mathrm{Hom}(H_1(N, \mathbb{R}), \mathbb{R})$ we have*

$$x_N(\phi) = \max\{\phi(p) - \phi(q) : p, q \in \mathcal{M}_N\}.$$

Moreover, ϕ is fibered if and only if ϕ restricted to $\mathcal{M}_N$ attains its maximum on a marked vertex.

The polytope $\mathcal{M}_N$ is dual to the unit ball of the Thurston norm on $H^1(N, \mathbb{R})$, see Sec. 2. We let $\mathcal{P}_N$ denote $\mathcal{M}_N$ without the marked vertices. As a sample application, this immediately implies that if a three-manifold fibers over a circle and the first Betti number $b_1(N)$ is at least 2, then it fibers in infinitely many ways.

We are interested in the possible (marked) polytopes that arise as $\mathcal{M}_N$ for a three-manifold N. We first note some necessary conditions: If $\mathcal{M}$ is the marked polytope of some three-manifold, then

(1) $\mathcal{M}$ is centrally symmetric, bounded, and convex.
(2) $\mathcal{M}$ can be translated by a vector $v \in \frac{1}{2}H_1(N, \mathbb{Z})$ so that its vertices are in $H_1(N, \mathbb{Z})$/torsion (this follows from the fact that x_N assigns integral values to elements of $H_1(N, \mathbb{Z})$).

Thurston showed [9, Corollary to Theorem 6] that for any norm x on $\mathbb{R}^2$ that takes even integral values on $\mathbb{Z}^2$, there is a closed three-manifold N such that x_N is equal to x. We note that Thurston's proof, while explicit, gives no control over the complexity of the fundamental groups of the three-manifolds constructed.

Agol asked the second author whether any polygon in $\mathbb{R}^2$ satisfying (1) and (2) above is $\mathcal{M}_N$ for a very simple three-manifold N; in particular, a three-manifold whose fundamental group admits a $(2,1)$-presentation of the form $\pi_1(N) = \langle x, y \,|\, r\rangle$.

We cannot give a complete answer, but our main theorem states that the restriction to such simple manifolds does not restrict the complexity of the associated norms:

Main Theorem. *For any $n \in \mathbb{N}$, there exists a three-manifold N such that $\pi_1(N)$ admits a $(2,1)$-presentation, $\mathcal{M}_N$ is a polygon with $(2n)$-sides, every vertex of $\mathcal{M}_N$ is marked, and $\mathcal{M}_N$ has $\left\lfloor \frac{n}{2} \right\rfloor$ sides of different lengths.*

The manifolds in the Main Theorem are constructed by gluing a two-handle onto an orientable genus-two handlebody along a regular neighborhood of a homotopically non-trivial simple closed curve (these are usually called *tunnel number one manifolds*). Since we require $H_1(N, \mathbb{R}) \cong \mathbb{R}^2$, we will always choose the curve to be separating. By Theorem 1.1, every cohomology class in our examples which assumes its maximal value on a vertex of $\mathcal{M}$ is fibered.

To calculate the Thurston norm of these examples, we use an algorithm of Friedl, the second author, and Tillmann [5], which gives an easy way to read off $\mathcal{M}_N$ from the relator in a $(2,1)$-presentation of $\pi_1(N)$. To construct a wide variety of examples, we consider the orbit of a particular separating curve under the mapping class group of the marked boundary surface of the handlebody, which gives us many possible curves to glue two-handle onto. Dunfield and Thurston have used this same construction with a random walk in the mapping class group to show that a random tunnel number one manifold does not fiber over a circle [2, Theorem 2.4]. In this sense, our examples are non-generic.

In Sec. 2 we review the algorithm which computes the marked polytope $\mathcal{M}_N$. In Sec. 3, we recall standard generators of the mapping class group and how they behave as automorphisms of the fundamental group of a surface. In Sec. 4, we use these generators to derive a complicated relator r which produces the desired complicated Thurston norm unit ball. In the appendix, we do an explicit calculation of the Thurston norm of a three-manifold with fundamental group the Baumslag–Solitar group $BS(m,m)$.

2. Algorithm for the Thurston Norm

In [5], the following algorithm was given for computing the Thurston norm of three-manifold N with $\pi_1(N) = \langle x, y \,|\, r \rangle$ and $b_1(N) = 2$, see also [4]. The relator r determines a walk on $H_1(N;\mathbb{Z})$ in $H_1(N;\mathbb{R}) \cong \mathbb{R}^2$. We assume that r is reduced and cyclically reduced. The marked polytope $\mathcal{M}_M$ is constructed as follows (see Fig. 1 for an example):

(1) Start at the origin and walk on $\mathbb{Z}^2$ reading the word r from left to right.
(2) Take the convex hull $\mathcal{C}$ of the lattice points reached by the walk.
(3) Mark the vertices of $\mathcal{C}$ which the walk passes through exactly once.
(4) Consider the unit squares that are contained in $\mathcal{C}$ and touch a vertex of $\mathcal{C}$. Mark a midpoint of a square precisely when a vertex of $\mathcal{C}$ incident with the square is marked.

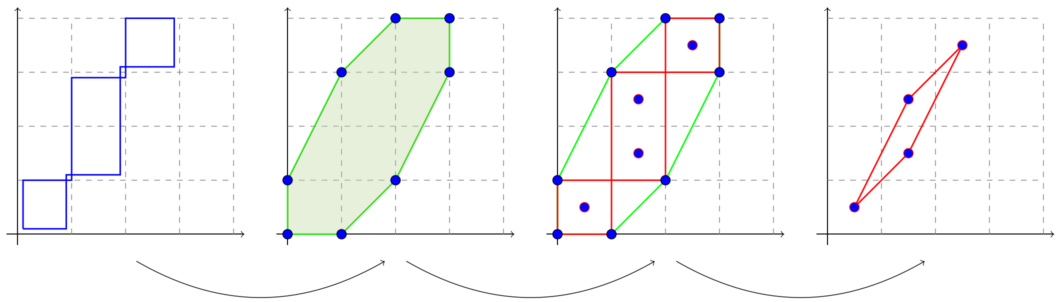

Fig. 1. The algorithm applied to the group: $\langle x, y \mid r = xyxyyxyXYXYYXY \rangle$, where capital letters denote inverses. This is the first case of our sequence of examples in Sec. 4.

(5) The set of vertices of $\mathcal{M}_N$ is the set of midpoints of these squares, and a vertex of $\mathcal{M}_N$ is marked precisely when it is a marked midpoint of a square.

To obtain the unit ball of the Thurston norm x_N, we consider the dual polytope

$$\mathcal{M}_N^* := \{\phi \in H^1(N, \mathbb{R}) : \phi(v) - \phi(w) \leq 1 \text{ for all } v, w \in \mathcal{M}_N\}.$$

3. Mapping Class Groups

For background and details on mapping class groups, we refer the reader to [3]. For our purposes, we focus on a single surface: the closed genus two surface Σ_2. As we will be working with fundamental groups, we need to keep track of a basepoint. Fix a point p in Σ_2. Let $\mathrm{Homeo}^+(\Sigma_2, p)$ denote the group of all orientation-preserving self-homeomorphisms of Σ_2 fixing the point p, and let $\mathrm{Homeo}_0(\Sigma_2, p)$ be the subgroup of $\mathrm{Homeo}^+(\Sigma_2, p)$ consisting of homeomorphisms isotopic to the identity through isotopies that fix the point p at every stage.

Definition 3.1. The *mapping class group* of the marked surface (Σ_2, p) is the group

$$\mathrm{Mod}(\Sigma_2, p) = \mathrm{Homeo}^+(\Sigma_2, p) / \mathrm{Homeo}_0(\Sigma_2, p).$$

By keeping track of the point p, we obtain a natural action of $\mathrm{Mod}(\Sigma_2, p)$ on $\pi_1(\Sigma_2, p)$. In fact, this action induces an isomorphism from $\mathrm{Mod}(\Sigma_2, p)$ to an index-2 subgroup of $\mathrm{Aut}(\pi_1(\Sigma_2, p))$ (see [3, Chap. 8]).

Given a simple closed curve c on Σ_2, we let $T_c \in \mathrm{Mod}(\Sigma_2, p)$ denote the isotopy class of a right Dehn twist about c. The five Dehn twists T_a, T_b, T_c, T_d, T_e about the curves a, b, c, d, e as shown in Fig. 2 generate $\mathrm{Mod}(\Sigma_2, p)$ (see [3, Theorem 4.13]). In the proof of Theorem 4.3, we will need explicit computations of the action of mapping classes on elements of the fundamental group; the action of the generators is given in the lemma below, whose proof we leave as an exercise for the reader.

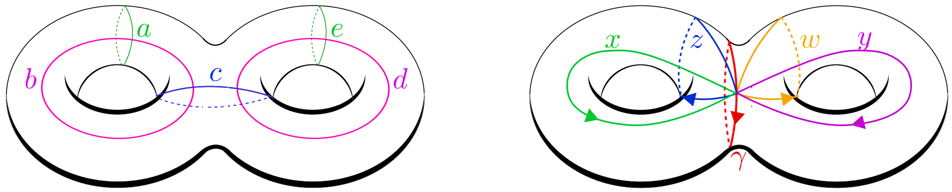

Fig. 2. The right Dehn twists about the curves a, b, c, d, and e generate $\mathrm{Mod}(\Sigma_2)$. The loops w, x, y, and z are standard generators for $\pi_1(\Sigma_2)$, which we will use throughout the paper. The curve γ in $\pi_1(\Sigma_2)$ is the commutator $[x, z]$.

Lemma 3.2. *Let w, x, y and z be the standard generators of $\pi_1(\Sigma_2, p)$ shown in Fig. 2. If T_a, T_b, T_c, T_d and T_e are the Dehn twists about the curves a, b, c, d and e as in Fig. 2, then*

- $T_a : x \mapsto z^{-1}x$
- $T_b : z \mapsto xz$
- $T_c : x \mapsto xwz^{-1}, \quad y \mapsto ywz^{-1}$
- $T_d : w \mapsto y^{-1}w$
- $T_e : y \mapsto wy$
- $T_a^{-1} : x \mapsto zx$
- $T_b^{-1} : z \mapsto x^{-1}z$
- $T_c^{-1} : x \mapsto xzw^{-1}, \quad y \mapsto yzw^{-1}$
- $T_d^{-1} : w \mapsto yw$
- $T_e^{-1} : y \mapsto w^{-1}y,$

where an automorphism fixes a generator of $\pi_1(\Sigma_2, p)$ if it is not listed.

4. Main Result

A *handlebody* is a compact, orientable, irreducible three-manifolds with a non-empty connected boundary whose inclusion is π_1-surjective. The *genus* of a handlebody H is defined to be the genus of ∂H. We will focus on the genus two handlebody, which one can visualize as the compact three-manifolds bounded by a standard embedded copy of a genus two surface in $\mathbb{R}^3$ (as drawn in Fig. 2).

Given a simple closed curve γ in the boundary of a genus two handlebody H, we define the three-manifolds M_γ to be

$$M_\gamma = H \bigcup_{\nu(\gamma)} (D^2 \times [0, 1]),$$

where D^2 denotes the 2-disk, $\nu(\gamma)$ is a regular neighborhood of γ in ∂H homeomorphic to $\gamma \times [0, 1]$, and $\nu(\gamma)$ is identified with $\partial D^2 \times [0, 1]$. Colloquially, M_γ is the manifold obtained by gluing a two-handle onto γ.

Lemma 4.1. *Let γ be a simple closed curve on the boundary of a genus two handlebody H. If γ separates ∂H, then $b_1(M_\gamma) = 2$, where b_1 denotes the first Betti number.*

Proof. We can deformation retract M_γ to a complex with 2 one-cells and 1 2-cell. Therefore, $b_1 = 1$ or 2. Let x and y denote the generators of $\pi_1(H)$, and x, y, z, w

the generators of $\pi_1(\partial H)$. The attaching map of the 2-cell comes from removing the letters z and w from the word that γ represents in $\pi_1(\partial H)$.

If γ is separating, then γ bounds a subsurface of ∂H, and hence $[\gamma] \in [\pi_1(\partial H), \pi_1(\partial H)]$. Therefore, the total sum of the exponents of x and y in γ is zero, so the boundary map of the 2-cell is zero on homology. □

We now give the construction we will be using in the proof of the Main Theorem. For the remainder of the section, we fix H to be a genus two handlebody, p to be a point in ∂H, w, x, y, z to be the standard generators of $\pi_1(\partial H, p)$ shown in Fig. 2, and a, b, c, d, e to be the simple closed curves on ∂H as shown in Fig. 2. In addition, fix a simple, separating closed curve γ on ∂H representing the commutator $[x, z] \in \pi_1(\partial H)$.

Definition 4.2. Given an element $g \in \mathrm{Mod}(\partial H, p)$, define M_g to be the manifold

$$M_g = M_{g(\gamma)}.$$

Theorem 4.3. *For each $n \in \mathbb{N}$ let $g_n = T_b^{-1}(T_d^{-1}T_c)^{n+1}$. The polygon $\mathcal{M}_{M_{g_n}}$ has $4n$ marked vertices.*

Remark 4.4. Here are the first two relators that this algorithm constructs (capital letters denote inverses).

(1) xyxyyxyXYXYYXY
(2) xyxyyxyxyyxyyxyxyyxyXYXYYXYXYYXYYXYXYYXY.

Proof (Proof of Theorem 4.3). Let $g = T_b^{-1}T_d^{-1}T_cT_b$, so that $g_{n+1} = g \cdot g_n$. We will inductively define two sequences of words in the generators of $\pi_1(\partial H, p)$:

$$A_1 = ywz^{-1}x \qquad A_{n+1} = B_nA_n$$

$$B_1 = y^2wz^{-1}x \qquad B_{n+1} = B_nB_nA_n = B_nA_{n+1}.$$

We claim that

$$A_{n+1} = gA_n \quad \text{and} \quad B_{n+1} = gB_n. \tag{4.1}$$

We proceed by induction. The base case is established by a straightforward computation:

$$gA_1 = B_1A_1 = A_2 \quad \text{and} \quad gB_1 = B_1B_1A_1 = B_2.$$

Now let $n \in \mathbb{N}$ and assume $gA_{n-1} = A_n$ and $gB_{n-1} = B_n$. We then have that

$$gA_n = g(B_{n-1})g(A_{n-1}) = B_nA_n = A_{n+1},$$

and

$$gB_n = g(B_{n-1})g(B_{n-1})g(A_{n-1}) = B_nB_nA_n = B_{n+1}.$$

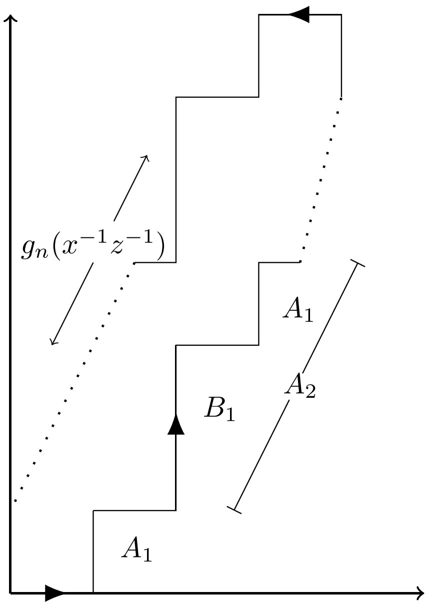

Fig. 3. A general picture of $g_n([x, z])$, after removing the letters z and w. The relator starts at the origin and climbs up the bottom "staircase" which is $g_n(xz)$. Then $g_n(x^{-1}z^{-1})$ travels back down.

Given any word A in $\{w, x, y, z\}$, let $\bar{A}$ denote the word obtained by eliminating the letters w and z, and then reducing and cyclically reducing. The images of x and y generate $\pi_1(H, p)$, and the word $\bar{A}$ determines a walk in $H_1(M_{g_n}, \mathbb{Z}) \cong \mathbb{Z}^2$ as in Sec. 2.

Let $w(\bar{A}_n)$ and $w(\bar{B}_n)$ denote the width of these walks and $h(\bar{A}_n)$ and $h(\bar{B}_n)$ their height. We have

$$w(\bar{A}_1) = 1, \quad h(\bar{A}_1) = 1, \quad w(\bar{B}_1) = 1, \quad h(\bar{B}_1) = 2,$$

and it follows from the observation that A_n and B_n contain only positive powers of x and y that

$$\begin{aligned} w(\bar{A}_{n+1}) &= w(\bar{B}_n) + w(\bar{A}_n) \\ h(\bar{A}_{n+1}) &= h(\bar{B}_n) + h(\bar{A}_n) \\ w(\bar{B}_{n+1}) &= w(\bar{B}_n) + w(\bar{A}_{n+1}) \\ h(\bar{B}_{n+1}) &= h(\bar{B}_n) + h(\bar{A}_{n+1}). \end{aligned}$$

Let f_n denote the nth term of the Fibonacci sequence starting with $f_0 = f_1 = 1$. Another short induction argument yields

$$w(\bar{A}_n) = f_{2n-2}, \quad h(\bar{A}_n) = f_{2n-1}, \quad w(\bar{B}_n) = f_{2n-1}, \quad \text{and} \quad h(\bar{B}_n) = f_{2n}. \tag{4.2}$$

We claim that

$$g_n([x, z]) = xA_1A_2 \ldots A_nB_nB_{n-1} \ldots B_1yw(g_n(x^{-1}z^{-1})). \tag{4.3}$$

This can easily be checked for $n = 1$ and follows inductively from (4.1) and the fact that

$$g(x) = xA_1 \quad \text{and} \quad g(yw) = B_1 yw.$$

Fix $n \in \mathbb{N}$ and define the list of points P_n in $\mathbb{Z}^2$ as follows:

$$\begin{aligned} P_n &= \{(1,0), (w(\bar{A}_1), h(\bar{A}_1)), \dots, (w(\bar{A}_n), h(\bar{A}_n)), \\ &\qquad (w(\bar{B}_n), h(\bar{B}_n)), \dots, (w(\bar{B}_1), h(\bar{B}_1)), (0,1)\} \\ &= \{(1,0), (f_0, f_1), (f_2, f_3), \dots, (f_{2n-2}, f_{2n-1}), \\ &\qquad (f_{2n-1}, f_{2n}), (f_{2n-3}, f_{2n-1}), \dots, (f_1, f_2), (0,1)\}. \end{aligned}$$

Note that $|P_n| = 2n + 2$. For $k \in \{0, 1, \dots, |P_n|\}$, define the point p_k in $\mathbb{Z}^2$ by setting $p_0 = (0,0)$ and

$$p_k = p_{k-1} + P_n(k),$$

for $0 < k \leq |P_n|$, where $P_n(k)$ denotes the kth element in the list P_n. Additionally, we define ℓ_k to be the line segment connecting consecutive points, that is, for $k \in \{1, \dots, |P_n|\}$ we set

$$\ell_k = \overline{p_{k-1} p_k}.$$

It follows from the construction that for each $k \in \{0, \dots, |P_n|\}$ the point p_k lies on the walk determined by

$$g_n(xz) = xA_1 A_2 \dots A_n B_n B_{n-1} \dots B_1 yw.$$

Moreover, for $k \in \{1, \dots, n+1\}$, the point p_{k+1} is the endpoint of the walk determined by $xA_1 \dots A_k$. Similarly, for $k \in \{n+2, \dots, 2n+1\}$, the point p_{k+1} is the end point of the walk determined by $xA_1 \dots A_n B_n \dots B_{2n-k+2}$. We now want to prove that the walk determined by $g_n(xz)$ is bounded on the right by $\bigcup_k \ell_k$.

We again proceed by induction on both sequences: As our base case, we observe that the vertices in the walk determined by $\bar{A}_1$ are $(0,0), (0,1)$ and $(1,1)$; hence the walk is bounded to the right by the line connecting the endpoints of the walk. Similarly one checks this for $\bar{B}_1$. Now the walk determined by $\bar{A}_k$ is the concatenation of the walk determined by $\bar{B}_{k-1}$ followed by that of $\bar{A}_{k-1}$. Now it follows from (4.2) that the slope of the line connecting the endpoints of the walk corresponding to $\bar{B}_{k-1}$ is $\frac{f_{2k-2}}{f_{2k-3}}$ and the slope connecting the endpoints of the walk given by $\bar{A}_k$ is $\frac{f_{2k-1}}{f_{2k-2}}$. Standard properties of the Fibonacci sequence guarantee the latter is always smaller. If we now assume that all the vertices in the walks $\bar{B}_{k-1}$ and $\bar{A}_{k-1}$ are bounded to the right by the line segments connecting their respective endpoints, then the same is true for the walk given by $\bar{A}_k$. A similar argument can be made for $\bar{B}_k$. As the walk given by $g_n(xz)$ is a concatenation of the walks given by the $\bar{A}_k$ and $\bar{B}_k$'s, we see that $\bigcup_k \ell_k$ bounds this walk on the right.

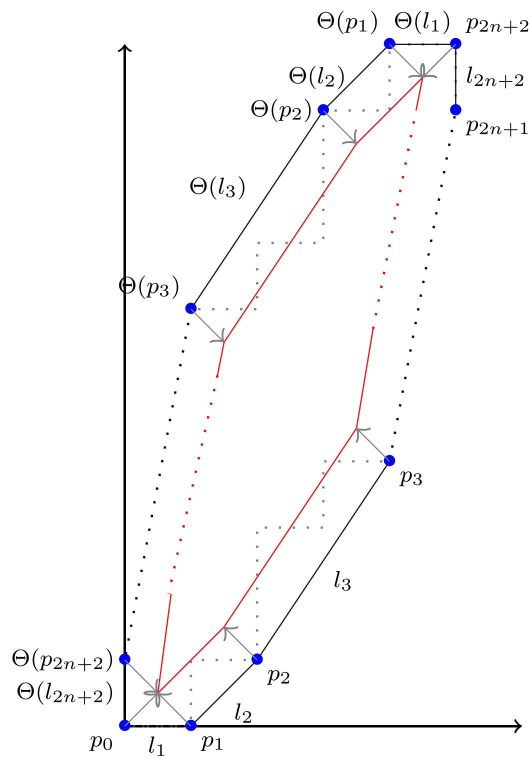

Fig. 4. The marked polytope $\mathcal{M}_{M_{g_n}}$ with $4n$ vertices.

Let $\Theta_n : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\Theta_n(v) = -v + p_{2n+2}$. We now claim that the convex hull $\mathcal{C}_n$ of the walk given by $g_n([x, z])$ has vertex set

$$V_n = \{p_k : 0 \leq k \leq 2n + 2\} \cup \{\Theta_n(p_k) : 0 < k < 2n + 2\},$$

and edge set

$$E_n = \{\ell_k : 1 \leq k \leq 2n + 2\} \cup \{\Theta_n(\ell_k) : 1 \leq k \leq 2n + 2\}.$$

Let $\hat{\ell}_k$ be the complete line containing the segment ℓ_k. Again using basic properties of the Fibonacci sequence, the slope of $\hat{\ell}_k$ is strictly greater than that of $\hat{\ell}_{k+1}$ implying that each $\hat{\ell}_k$ bounds $\mathcal{C}_n$ from the right. Now combining this with the fact that ℓ_k is contained in $\mathcal{C}_n$, we must have that ℓ_k is an edge of $\mathcal{C}_n$. As the walk is invariant under the transformation Θ_n, we see that $\Theta_n(\ell_k)$ is also an edge of $\mathcal{C}_n$. Finally, the pointwise union of the elements of E_n bounds a convex polygon and hence E_n contains all the edges of $\mathcal{C}_n$.

To finish, we note that $\mathcal{C}_n$ has $4n + 4$ vertices and applying the remaining steps of the algorithm presented in Sec. 2 results in the polygon $\mathcal{M}_{M_{g_n}}$ with $4n$ vertices. In particular, the two collections of vertices $\{\Theta_n(p_{2n+1}), p_0, p_1\}$ and $\{p_{2n+1}, p_{2n+2}, \Theta_n(p_1)\}$ each collapse to a single vertex in the process of obtaining $\mathcal{M}_{M_{g_n}}$ from $\mathcal{C}_n$ (see Fig. 4). It is easy to see that two vertices v and v' of the convex hull collapse to a single vertex in $\mathcal{M}_\pi$ only if they lie on the same square, so no other vertices of the convex hull collapse. $\square$

Similar arguments give the following theorem; we only sketch the proof.

Theorem 4.5. *For each $n \in \mathbb{N}$ let $h_n = T_b^{-2}(T_d^{-1}T_c)^{n+1}$. The polygon $\mathcal{M}_{M_{h_n}}$ has $4n+2$ marked vertices.*

Proof. With g_n, A_k and B_k as in the proof of Theorem 4.3, the composition of T_b^{-1} with g_n simply increases the width of A_1, B_1 by 1. Therefore, the Fibonacci sequences start at $f_0 = f_1 = 2$ and thus doubles the width of A_k and B_k. Exactly the same argument as above applies to obtain a convex hull of $4n+4$ vertices, which yields a polygon of $4n+2$ vertices. $\square$

Remark 4.6. We do not know if the three-manifolds that we construct have any alternative descriptions. For example, it would be interesting if these manifolds were all link complements that had a uniform construction. In [7], Licata used Heegaard Floer homology to compute $\mathcal{M}_\pi$ for all two-components, four-strands pretzel links, and the unit balls are shaped similarly to the polytopes we construct. On the other hand, all of Licata's examples have eight vertices, so cannot match our examples.

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Appendix A. Baumslag–Solitar Fundamental Groups

We study in detail a specific collection of three-manifolds that were helpful for us in understanding the Thurston norm of tunnel number one manifolds. In particular, we look at three-manifolds whose fundamental group is isomorphic to a Baumslag–Solitar group of the form $\mathrm{BS(m,m)}$ for some positive integer m, that is,

$$\pi_1 = \langle x, y \mid xy^m x^{-1} y^{-m} \rangle.$$

For such a manifold, the Thurston norm assigns 0 to the cohomology class dual to $x \in H_1(M)$, and $m-1$ to the class dual to y. This can be seen using the algorithm presented in Sec. 2.

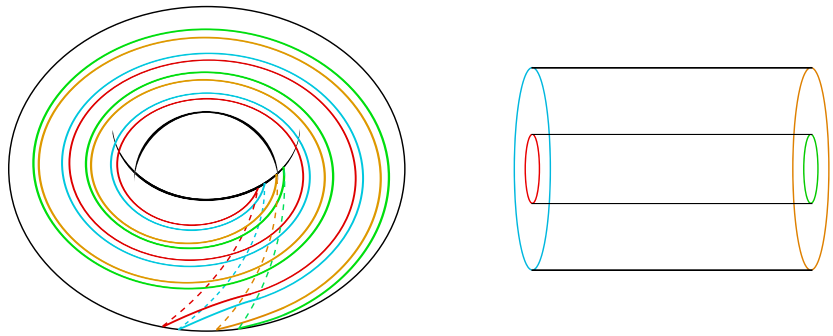

Fig. A.1. Shown here is the manifold $M_{f_m} \cong N = M_1 \cup M_2$ with $m = 4$. The manifold M_1 is drawn on the left and M_2 on the right. Curves of the same color are identified in M_{f_m}.

In the notation of Sec. 4, if for a positive integer m we define $f_m \in \mathrm{Mod}(\partial H)$ to be

$$f_m = T_d^m T_c^{-1} T_d T_c,$$

then $\pi_1(M_{f_m}) = \mathrm{BS}(m, m)$.

For the purpose of this appendix, we give another useful construction of the manifold M_{f_m}: Let $M_1 = \mathbb{S}^1 \times \mathbb{D}$ be a solid torus, $(p, q) \in \partial M_1$, $\lambda = \mathbb{S}^1 \times \{q\}$, and $\mu = \{p\} \times \partial\mathbb{D}$. Let γ be a curve in ∂M_1 representing the homology class $m[\lambda] + [\mu]$ in $H_1(\partial M_1)$ and let γ_0 and γ_1 be distinct parallel copies of γ, that is, γ, γ_0, and γ_1 are pairwise disjoint and homologous. Now, let $M_2 = \mathbb{S}^1 \times [0, 1] \times [0, 1]$.

Let $\nu(\gamma_0)$ and $\nu(\gamma_1)$ be disjoint regular neighborhoods of γ_0 and γ_1, respectively. Let γ_i^+ and γ_i^- denote the boundary components of $\nu(\gamma_i)$ labeled such that γ_0 and γ_1 are contained in distinct components of $\partial M_1 \smallsetminus (\gamma_0^+ \cup \gamma_1^+)$. Let N be the manifold obtained by identifying $\nu(\gamma_i)$ in ∂M_1 with the annulus $\mathbb{S}^1 \times [0, 1] \times \{i\} \subset \partial M_2$ such that

- γ_i^+ is identified with $\mathbb{S}^1 \times \{1 - i\} \times \{i\}$ and
- γ_i^- is identified with $\mathbb{S}^1 \times \{i\} \times \{i\}$.

Up to homotopy, we may assume that γ is contained in the annulus A bounded by γ_0^+ and γ_1^- in ∂M_1. Pick a point $t \in \mathbb{S}^1$ and let δ be the loop obtained by closing up the segment $\{t\} \times \{1\} \times [0, 1]$ in ∂M_2 with an arc contained in A and intersecting γ once. We can then see that γ and δ contained a torus component of ∂N and hence commute as elements of the fundamental group. Furthermore, an application of Van Kampen's theorem shows that the curves λ and δ generate $\pi_1(N)$ and yields the presentation $\pi_1(N) = \langle \delta, \lambda \,|\, \delta\lambda^m\delta^{-1}\lambda^{-m} \rangle$. This construction is shown in Fig. A.1.

Now, it is easy to see that $\mathrm{BS}(m, m)$ has an infinite and normal cyclic subgroup (e.g. take the group generated by λ^m in $\pi_1(N)$). The Seifert fibered space theorem (see [1, 6]) implies that both M_{f_m} and N are Seifert fibered. In [8, Theorem 3.1],

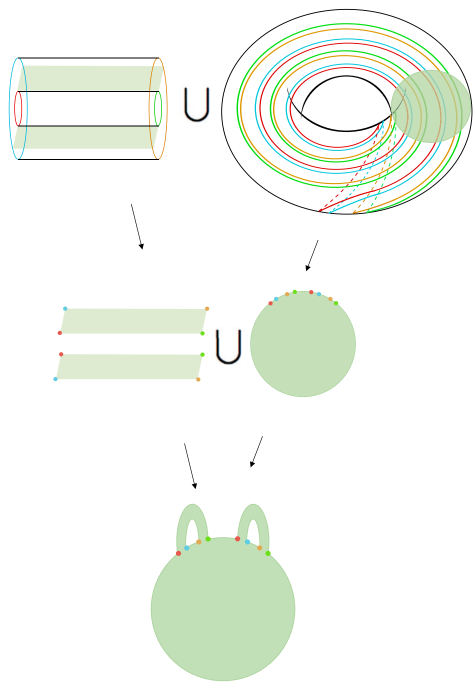

Fig. A.2. The dual disk to λ in the solid torus, and its extension to a dual surface in N.

Scott showed that if two Seifert fibered spaces have infinite and isomorphic fundamental groups, then they are in fact homeomorphic; hence, N and M_{f_m} are homeomorphic.

Using this new description of M_{f_m}, we can find the minimal dual surfaces to λ and δ. By applying the algorithm in Sec. 2, we know there exists a surface of Euler characteristic 0 dual to δ, e.g. the surface $\mathbb{S}^1 \times [0,1] \times \{\frac{1}{2}\} \subset M_2$.

Let us now focus on λ. In M_2, λ is dual to the disk D bounded by μ; however, μ does not live in the boundary of M; we will fix this with a surgery. For $i \in \{0,1\}$, the intersection $\nu(\gamma_i) \cap \mu$ can be written as a disjoint union of m intervals, call them $I_1^i, \dots, I_m^i$ (recall that $|\gamma_0 \cap \mu| = m$ by construction). The intervals I_k^0 and I_k^1 bound a rectangle R_k in M_2 of the form $I_k^0 \times [0,1]$ for each $k \in \{1, \dots, m\}$. Now the surface

$$\Sigma = D \cup R_1 \cup \cdots \cup R_m,$$

is obtained from a disk by attaching m rectangular strips, so it is a genus-0 surface with m boundary components. In particular, $\chi(\Sigma) = 1 - m$. Furthermore, Σ is dual to λ. The algorithm guarantees this dual surface is minimal.

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