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ON ϵ-ESCAPING TRAJECTORIES IN HOMOGENEOUS SPACES

Federico Rodriguez Hertz and Zhiren Wang

Pennsylvania State University, State College
PA 16802, USA

Abstract. Let G/Γ be a finite volume homogeneous space of a semisimple Lie group G, and $\{\exp(tD)\}$ be a one-parameter Ad-diagonalizable subgroup inside a simple Lie subgroup G_0 of G. Denote by $Z_{\epsilon,D}$ the set of points $x \in G/\Gamma$ whose $\{\exp(tD)\}$-trajectory has an escape for at least an ϵ-portion of mass along some subsequence. We prove that the Hausdorff codimension of $Z_{\epsilon,D}$ is at least $c\epsilon$, where c depends only on G, G_0 and Γ.

1. Introduction.

1.1. Statements.

In this paper, we will work under the following setting: G is a connected semisimple Lie group, $\Gamma \subseteq G$ be a lattice in G. In addition, $G_0 \subseteq G$ is a connected simple Lie subgroup, and $A_0 \subseteq G_0$ is a Cartan subgroup of G_0. Let $\mathfrak{g}$, $\mathfrak{g}_0$ and $\mathfrak{a}_0$ denote respectively the Lie algebras of G, G_0 and A_0.

Fix a Cartan involution of the Lie algebra $\mathfrak{g}_0$, such that $\mathfrak{a}_0$ is the maximal abelian subalgebra of $\mathfrak{p}_0$ in the corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Then $\mathfrak{k}_0$ is the Lie algebra of a maximal compact subgroup $K_0 \subset G_0$. Equip $\mathfrak{a}_0$ with the inner product metric induced by the Killing form on $\mathfrak{g}_0$, and K_0 with the unique bi-invariant probability measure m_{K_0}.

The focus of this paper is the statistics of orbits of one-parameter subroups of A_0 on G/Γ that spends at least a prescribed amount of time out of a compact set.

Theorem 1.1. *Let G be a connected semisimple Lie group, $G_0 \subseteq G$ be a connected simple Lie subgroup and $\Gamma \subset G$ be a lattice. Then there exists a constant $c = c(G, G_0, \Gamma) > 0$ such that:*

For all $\epsilon \in (0, 1]$, Cartan subgroups $A_0 \subset G_0$, and compact subsets $\Omega_0 \subset G/\Gamma$, there exists a compact subset $\Omega = \Omega(G, G_0, \Gamma, A_0, \Omega_0, \epsilon) \subset G/\Gamma$, such that:

For all $x \in \Omega_0$, $T > 0$, and non-zero vectors $D \in \mathfrak{a}_0$,

$$\mathrm{m}_{K_0}\big(\big\{k \in K_0 : \big|\{t \in [0, T] :\ \exp(tD)k.x \notin \Omega\}\big| \geq \epsilon T\big\}\big) < e^{-c\epsilon T}. \tag{1.1}$$

The key feature of Theorem 1.1 is that c is independent of the choices of the Cartan subgroup $A_0 \subset G_0$ and the unit vector $D \in \mathfrak{a}_0$.

Our second result is a uniform upper bound on the Hausdorff dimension of the set of escaping trajectories.

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Definition 1.2. Given G/Γ, $D \in \mathfrak{g}$ and $\epsilon \in (0,1]$, we say a point $x \in G_0$ is **sequentially ϵ-escaping on average** with respect to the one parameter subgroup $\{\exp(tD)\}$, if there exists a sequence $T_k \to \infty$ and a weak-* limit μ of the sequence of probability measures

$$\mu_{T_k} := \frac{1}{T_k}\int_0^{T_k} \delta_{\exp(tD).x}\mathrm{d}t, \tag{1.2}$$

such that $\mu(G/\Gamma) \leq 1-\epsilon$.

The set of points that are sequentially ϵ-escaping on average with respect to $\{\exp(tD)\}$ is denoted by $Z_{\epsilon,D}$.

Theorem 1.3. *Let G be a connected semisimple Lie group, $G_0 \subseteq G$ be a connected simple Lie subgroup and $\Gamma \subset G$ be a lattice. Then there exists a positive constant $c = c(G, G_0, \Gamma)$, such that for all semisimple elements $D \in \mathfrak{g}_0$ and $\epsilon \in (0,1]$, the Hausdorff dimension of $Z_{\epsilon,D}$ satisfies*

$$\dim_H Z_{\epsilon,D} \leq \dim G - c\epsilon.$$

1.2. **Historical background.** In [2] Cheung proved the set of singular vectors in dimension 2 has Hausdorff dimension $\frac{4}{3}$ and thus Hausdorff codimension $\frac{2}{3}$. By a principle due to Dani [4], this set corresponds to the set of points of the form $\begin{pmatrix} 1 & & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix}\Gamma \in G/\Gamma$, where $G = \mathrm{SL}_3(\mathbb{R})$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, whose trajectories under the one parameter action by $\{\exp(tD)\}_{t\geq 0}$ is divergent, where $D = \mathrm{diag}\,(1,1,-2)$. It was also shown in [2] that the set of all points $x \in G/\Gamma$ with divergent trajectories under the same action also has Hausdorff codimension $\frac{2}{3}$ in G/Γ. These results were later extended by Cheung and Chevallier [3] (when $m=1$ below) and by Das, Fishman, Simmons and Urbanski [6, 7] to all $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ for divergent trajectories of one parameter flows

$$\{\exp(tD)\}_{t\geq 0},\ D = \mathrm{diag}\,(m,\cdots,m,-n,\cdots,-n) \tag{1.3}$$

where $m+n=d$. In this situation, the exact Hausdorff codimension is $\frac{mn}{m+n}$. As in the case of $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$, the codimension is found along orbits of the group $\begin{pmatrix} \mathrm{Id}_m & * \\ & \mathrm{Id}_n \end{pmatrix}$, which is the full unstable horospherical group of the flow (1.3).

A trajectory is divergent if it eventually leaves every compact set without returning. The works of Kadyrov [11] and Kadyrov-Kleinbock-Lindenstrauss-Margulis [12] considered, for the same flows as above, the set of points whose trajectories are escaping on average. This means the empirical probability measures along the trajectories converge to 0 in the weak-* topology. In other words, even though the trajectory is allowed to return to a given compact set, on average it only spends a zero portion of time inside the compact set. In [11] and [12], it was proved that, in the same settings from [2] and [3], the set of points whose trajectories escape on average has the same Hausdorff dimension as that of divergent trajectories. This yields the Hausdorff dimension of the so-called singular vectors on average.

Recently, Khalil [13] considered more general homogeneous spaces G/Γ. His result provided sharp upper bounds of the Hausdorff dimension of the set of escaping trajectories for certain one-parameter diagonal flows when either G is an almost product of factors of real rank 1, or the flow arises from a representation of $G_0 = \mathrm{SL}_2(\mathbb{R})$.

In the current paper, we work on homogeneous spaces in full generality. The ambient space G/Γ will be a general homogeneous space, and the direction D of the flow will be a general semisimple element instead of the special diagonal elements above. Instead of escaping on average, we will treat the set of trajectories which are sequentially ϵ-escaping on average. In this scenario, there exists a sequence of sampling times, up to which the trajectorie has spent no more than a $(1-\epsilon)$ portion of time inside any prescribed compact set. A similar but more restrictive notion of ϵ-escaping on average was studied in [12]*Theorem 1.5 for the diagonal flows (1.3).

Following the ideas from [2, 3, 11, 12], our main result Theorem 1.3 provides a positive lower bound of the form $c\epsilon$ to the Hausdorff codimension of the set of trajectories sequentially ϵ-escaping on average. A main ingredient of the proof is to control recurrence by a height function that is contracted on average over random tranjectories. This idea was first introduced by Eskin, Margulis and Mozes in [9], and also used in the works of Eskin-Margulis [8] and Benoist-Quint [1].

It should be noted that unlike in the works listed above, all of which focused on very particular diagonal flows, our bound is neither sharp nor explicit. Instead, the central motivation of the result is the uniform positiveness of the coefficient c when the flow direction varies over all semisimple elements inside a given simple Lie subgroup G_0. One obstacle when achieving such uniformity is that, in contrast to the flow in (1.3), a general flow of the form $\exp(tD)$ may expand part of its unstable horosphere arbitrarily slowly. This obstruction will be dealt with in Section 3, where the unstable horosphere is substituted with a strong unstable subgroup.

2. **Reduction to unstable horospheres.** Let $\mathfrak{g}_0 = \sum_{\chi\in\Sigma_0} \mathfrak{g}_0^\chi$ be the decomposition of $\mathfrak{g}_0$ into relative root spaces with respect to the Cartan subalgebra $\mathfrak{a}_0$. $\Sigma_0 \subset \mathfrak{a}_0^*$ is the collection of relative roots, and the relative root on $\mathfrak{g}_0^\chi$ is $\chi \in \Sigma_0$. Each hyperplane $\ker\chi$ where $\chi \in \Sigma\backslash\{0\}$ is a Weyl chamber wall in $\mathfrak{a}_0$.

In the remainder of the paper, $\mathfrak{g}_0$ will be intrinsically equipped with an inner product metric (for example, by reversing the sign on the negative part of the Killing form), such that the restriction on $\mathfrak{a}_0$ coincides with the one induced by Killing form. Denote by $B_r^{\mathfrak{g}_0}$ the ball of radius r centered at 0 in $\mathfrak{g}_0$ and let $B_r^{G_0} = \exp B_r^{\mathfrak{g}_0}$. Similar notions will be used, without further explanation, for other Lie algebras equipped with a metric and their corresponding Lie groups.

Define $\Sigma_0^+ := \{\chi \in \Sigma_0 : \chi(D) > 0\}$ and $\Sigma_0^{0-} := \{\chi \in \Sigma_0 : \chi(D) \le 0\}$. Let $\mathfrak{g}_0^+$, $\mathfrak{g}_0^{0-}$ be respectively the direct sums of relative root spaces $\mathfrak{g}_0^\chi$ with $\chi \in \Sigma_0^+$ and $\chi \in \{0\} \cup \Sigma_0^-$. It is easy to see that they are Lie subalgebras. Let G_0^+, G_0^{0-} denote the corresponding connected Lie subgroups of G_0.

Then the orbits of G_0^+ and G_0^{0-} are respectively the unstable, and central-stable leaves of the flow $x \to \exp(tD).x$ on G/Γ.

Moreover, as the Lie subalgebras $\mathfrak{g}_0^+$, $\mathfrak{g}_0^{0-}$ are transversal to each other and their sum is $\mathfrak{g}_0$, there exists $r_1 > 0$ such that the map $(a, b) \to ab$ from $B_{r_1}^{G_0^{0-}} \times B_{r_1}^{G_0^+}$ to G_0 is a C^∞ diffeomorphism to its image..

In particular, there exists a neighborhood $B_r^{G_0}$ of the identity in G_0 such that every g can be uniquely decomposed as $g_{G_0^{0-}} g_{G_0^+}$ with $g \in B_{r_1}^{G_0^+}$ and $g_{G_0^{0-}} \in B_{r_1}^{G_0^{0-}}$, moreover, the map $g \to (g_{G_0^{0-}}, g_{G_0^+})$ is a C^∞ diffeomorphism from $B_r^{G_0}$ to its image. Let $\pi_{G_0^+}$ denote the projection $g \to g_{G_0^+}$.

We consider the restriction of $\pi_{G_0^+}$ to $B_r^{K_0}$, and consider its derivative, which maps the Lie algebra $\mathfrak{k}_0$ of K_0 to $\mathfrak{g}_0^+$.

Lemma 2.1. *If r is sufficiently small, then the map $\pi_{G_0^+}: B_r^{K_0} \to G_0^+$ is regular, i.e. the derivative $D_k\pi_{G_0^+}: \mathfrak{k} \to \mathfrak{g}_0^+$ is surjective for all $k \in B_r^{K_0}$.*

Proof. As $\pi_{G_0^+}$ is smooth, $D_k\pi_{G_0^+}$ depends continuously on k and hence it being regular is an open condition in k. Therefore it suffices to show the regularity for the identity element $k = e$. As $D_e\pi_{G_0^+}$ is the quotient map from $\mathfrak{g}_0$ to $\mathfrak{g}_0^+ \cong \mathfrak{g}_0/\mathfrak{g}_0^{0-}$, this is equivalent to showing that $\mathfrak{k}_0 + \mathfrak{g}_0^{0-} = \mathfrak{g}_0$.

For all $D \in \mathfrak{a}_0$, the set $\{\chi \in \Sigma_0 : \chi(D) \leq 0\}$ of non-positive relative roots must contain the negative relative roots $\{\chi \in \Sigma_0 : \chi(D') < 0\}$ for some regular element $D' \in \mathfrak{a}_0$. By the Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$, where $\mathfrak{n}_0 = \bigoplus_{\substack{\chi\in\Sigma_0\setminus\\ \chi(D')<0}} \mathfrak{g}_0^\chi$. Furthermore, $\mathfrak{a}_0 \subseteq \mathfrak{g}_0^0$ and thus $\mathfrak{a}_0 \oplus \mathfrak{n} \subseteq \mathfrak{g}_0^0 + \mathfrak{n}_0 = \bigoplus_{\substack{\chi\in\Sigma_0\\ \chi(D)\leq 0}} \mathfrak{g}_0^\chi = \mathfrak{g}_0^{0-}$. It follows that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0 \subseteq \mathfrak{k}_0 + \mathfrak{g}_0^{0-}$, which is what we need. □

The Lie algebra $\mathfrak{g}_0^+$ is nilpotent, and thus G_0^+ is a connected nilpotent Lie group and is hence unimodular.

We also fix a neighborhood $B_{G_0^+}$ of identity in G_0^+, whose construction will be specified later. The Haar measure $\mathrm{m}_{G_0^+}$ on G_0^+ will be normalized so that

$$\mathrm{m}_{G_0^+}(B_{G_0^+}) = 1. \tag{2.1}$$

The choice of $B_{G_0^+}$ is determined by G_0^+, and is the same for different D's leading to the same G_0^+.

Remark 2.2. All Cartan subgroups are conjugate to each other and the norms induced by Killing forms on Cartan subalgebras are isometric to each other through these conjugacies. Therefore we can renormalize the metric and Haar measures on G_0, $G_0^\pm$ according to A_0 in the following way: first fix a Cartan subgroup $\widehat{A}_0$ as well as a non-zero vector $\widehat{D} \in \hat{\mathfrak{a}}_0$. This determines stable and unstable subgroups $G_0^\pm(\widehat{D})$ with respect to $\widehat{D}$.) Define metrics and Haar measures on G_0, $G_0^\pm(\widehat{D})$. (Note that $G_0^\pm(\widehat{D})$ depends only on the Weyl chamber containing $\widehat{D}$. However there are only finitely many such Weyl chambers.) Then for an arbitrary A_0 and $D \in \mathfrak{a}_0$, there exists $g \in G_0$ such that $A_0 = g\widehat{A}_0 g^{-1}$ and $D = \mathrm{Ad}_g \widehat{D}$ where $\widehat{D}$ is some unit vector with respect to the norm on $\mathfrak{a}_0$ induced by the Killing form of $\mathfrak{g}_0$. Redefine the metrics and Haar measures on G_0 and the stable and unstable subgroups $G_0^\pm$ with respect to D by pushforward the corresponding objects by the conjugacy by g.

Corollary 2.3. *There exist $r > 0$ and $C = C(G_0) > 0$, such that for all non-zero vectors $D \in \mathfrak{a}_0$, $\pi_{G_0^+}(B_r^{K_0})$ is contained in $B_{G_0^+}$ and $(\pi_{G_0^+})_*(\mathrm{m}_{K_0}|_{B_r^{K_0}}) \leq C\mathrm{m}_{G_0^+}|_{B_{G_0^+}}$.*

Proof. As $\pi_{G_0^+}$ is smooth near identity, it is clear that the image of $B_r^{K_0}$ is in $B_{G_0}^+$ for sufficiently small r. In addition, because of the regularity from Lemma 2.1, when r is sufficiently small, the pushforward $(\pi_{G_0^+})_*(\mathrm{m}_{K_0}|_{B_r^{K_0}})$ is bounded by a multiple of $\mathrm{m}_{G_0^+}$. Finally, note that the pair (G_0^+, G_0^{0-}), which determines the map $\pi_{G_0^+}$, has only finitely many possibilities for all $D \in A$, as $\mathfrak{g}_0^+$ and $\mathfrak{g}_0^{0-}$ are direct sums of relative root spaces. Thus we can choose uniform values for r and C that are independent of D. In light of Remark 2.2, C can be made independent of the choice of A_0. □

Provided the radius r in Corollary 2.3, we fix from now on a finite covering of the compact group K_0 by sets of the form $B_r^{K_0} \cdot h_j$, where $h_j \in K_0$, $j = 1, \cdots, q$.

We now reduce Theorem 1.1 to the following

Theorem 2.4. *Let G be a connected semisimple Lie group, $G_0 \subseteq G$ be a connected simple Lie subgroup and $\Gamma \subset G$ be a lattice. Then there exists a constant $c = c(G, G_0, \Gamma) > 0$ such that:*

For all $\epsilon > 0$, Cartan subgroups $A_0 \subset G_0$, and compact subsets $\Omega_0 \subset G/\Gamma$, there exists a compact subset $\Omega = \Omega(G, G_0, \Gamma, A_0, \Omega_0, \epsilon) \subset G/\Gamma$, such that:

For all $x \in \Omega_0$, $T > 0$, and non-zero vectors $D \in \mathfrak{a}_0$,

$$\mathrm{m}_{G_0^+}\left(\left\{g \in B_{G_0^+} : \left|\{t \in [0,T] : \ \exp(tD)g.x \notin \Omega\}\right| \geq \epsilon T\right\}\right) < e^{-c\epsilon T}. \tag{2.2}$$

Proposition 2.5. *Theorem 2.4 implies Theorem 1.1.*

Proof. Apply Theorem 2.4 to $h_j x$, $1 \leq j \leq q$, then we have Ω such that for all $x \in \Omega_0$, $T > 0$ and non-zero vector $D \in \mathfrak{a}_0$,

$$\mathrm{m}_{G_0^+}\left(\left\{g \in B_{G_0^+} : \left|\{t \in [0,T] : \ \exp(tD)gh_j.x \notin \Omega\}\right| \geq \epsilon T\right\}\right) < e^{-c\epsilon T}. \tag{2.3}$$

Here the compact set $\bigcup_j h_j \Omega_0$ is used in lieu of Ω_0.

By Corollary 2.3, the inequality (2.3) implies

$$\mathrm{m}_{K_0}\left(\left\{k \in B_r^{K_0} : \left|\{t \in [0,T] : \ \exp(tD)k_{G_0^+}h_j.x \notin \Omega\}\right| \geq \epsilon T\right\}\right) < Ce^{-c\epsilon T}. \tag{2.4}$$

Remark that for $k \in B_r^{k_0}$ and $t \geq 0$,

$$\begin{aligned} \exp(tD)kh_j.x &= \exp(tD)k_{G_0^{0-}}k_{G_0^+}h_j.x \\ &= \exp(tD)k_{G_0^{0-}}\exp(-tD) \cdot \exp(tD)k_{G_0^+}h_j.x \\ &= \mathrm{Ad}_{\exp(tD)}(k_{G_0^{0-}}) \cdot \exp(tD)k_{G_0^+}h_j.x. \end{aligned} \tag{2.5}$$

Because tD is an element in the Cartan subalgebra $\mathfrak{a}_0$, $\mathrm{Ad}_{\exp(tD)}$ is semisimple in $\mathrm{GL}(\mathfrak{g}_0)$. As G_0^{0-} is the central-stable subgroup for the one-parameter subgroup $\{\exp(tD)\}$, this shows the distance from $\mathrm{Ad}_{\exp(tD)}(k_{G_0^{0-}})$ to the identity is bounded by a constant L determined by the right-invariant metric we choose on G and the compact subgroup K_0. Let Ω_1 be the set of points whose distance to Ω is at most L. Then it follows from (2.4) and (2.5) that for all $T > 0$,

$$\mathrm{m}_{K_0}\left(\left\{k \in B_r^{K_0} : \left|\{t \in [0,T] : \ \exp(tD)kh_j.x \notin \Omega_1\}\right| \geq \epsilon T\right\}\right) < Ce^{-c\epsilon T},$$

Or equivalently,

$$\mathrm{m}_{K_0}\left(\left\{k \in B_r^{K_0}h_j : \left|\{t \in [0,T] : \ \exp(tD)k.x \notin \Omega_1\}\right| \geq \epsilon T\right\}\right) < Ce^{-c\epsilon T}$$

for each $1 \leq j \leq q$. Since K_0 is covered by the union of the $B_r^{K_0}h_j$'s, we see that for all sufficiently large T,

$$\mathrm{m}_{K_0}\left(\left\{k \in K_0 : \left|\{t \in [0,T] : \ \exp(tD)k.x \notin \Omega_1\}\right| \geq \epsilon T\right\}\right) < qCe^{-c\epsilon T}. \tag{2.6}$$

To deduce Theorem 1.1 from (2.6), it suffices to rewrite $\frac{c}{2}$ as c and Ω_1 as Ω. □

3. **Reduction to strong unstable subgroups.** As the unit element $D \in \mathfrak{a}_0$ is chosen arbitrarily, a priori it may sit near a Weyl chamber wall and expand certain directions in its unstable horosphere very slowly. This subsection aims to overcome this obstacle.

Proposition 3.1. *Given a connected simple Lie group G_0, there are constants $\theta_1 = \theta_1(G_0) > 0$ and $M = \theta_1(G_0) > 0$ satisfying the following condition:*

Suppose A_0 is a Cartan subgroup and its Lie algebra $\mathfrak{a}_0$ is equipped with the norm induced by the Killing form on $\mathfrak{g}_0$. For all $1 \le i \le s$, unit vector $D \in \mathfrak{a}_0$ and $\theta \in (0, \theta_1)$, the unstable Lie subalgebra $\mathfrak{g}_0^+$ contains a Lie subalgebra $\mathfrak{u}$, and there exists $D' \in \mathfrak{a}_0$ such that:

1. *$|D'| = 1$ and $|D' - D| < M\theta$;*
2. *$\mathfrak{u} = \bigoplus_{\substack{\chi\in\Sigma_0 \\ \chi(D')>0}} \mathfrak{g}_0^\chi$;*
3. *$\chi(D) \ge \theta$ and $\chi(D') \ge \theta$ for each $\mathfrak{g}_0^\chi \subseteq \mathfrak{u}$.*

Here the constants θ_1 and M can be independent of A_0 and D for the reasons explained in Remark 2.2.

Lemma 3.2. *There exists a constant $\kappa_1 > 0$ that depends only on G_0, such that for all unit vector $D \in \mathfrak{a}_0$, the subset of roots $\{\chi \in \Sigma_0 : |\chi(D)| < \kappa_1\}$ is contained in a proper subspace of $\mathfrak{a}_0^*$.*

Proof. We view both $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ as $\mathbb{R}^{r_0}$ where $r_0 = \operatorname{rank}_{\mathbb{R}} G_0$, and equip them with Euclidean norms that are dual to each other. Let $L > 1$ be such that $|\chi| < L$ and all $\chi \in \Sigma_0 \subset \mathfrak{a}_0^*$. Since Σ_0 is a finite set which spans $\mathfrak{a}_0^*$, one can define $\kappa_0 \in (0,1)$ such that $\kappa_0 < |\chi_1 \wedge \cdots \wedge \chi_n|$ for all linearly independent elements $\chi_1, \cdots, \chi_n$ of Σ_0.

Let $\kappa_1 = \frac{1}{4} r_0^{-1} L^{-(r_0-1)} \kappa_0$. Note that r_0, κ_0 and L depends only on G_0 (see Remark 2.2). Therefore κ_1 depends only on G_0 as well. To see that κ_1 satisfies the conclusion of the lemma, it suffices to show that any r_0 elements $\chi_1, \cdots, \chi_{r_0} \in \Sigma_0$ with $|\chi_j(D)| < \kappa_1$ are linearly dependent. Indeed, let P be the $r_0 - 1$ dimensional annihilator of D: $P = \{\delta \in \mathfrak{a}_0^* : \delta(D) = 0\}$. For each χ_j, denote by $\chi_j = \chi_j^P + \chi_j^\perp$ the orthogonal decomposition in $P \oplus P^\perp$. Then $|\chi_j| \le L$, $|\chi_j^P| \le L$ and $|\chi_j^\perp| \le \kappa_1$, $\forall j = 1, \cdots, r_0$. Thus

$$\begin{aligned}
&|\chi_1 \wedge \cdots \wedge \chi_{r_0}| \\
=&\Big|\chi_1^P \wedge \cdots \wedge \chi_{r_0}^P + \sum_{i=1}^{r_0} \chi_1 \wedge \cdots \wedge \chi_{i-1} \wedge \chi_i^\perp \wedge \chi_{i+1}^P \wedge \cdots \wedge \chi_{r_0}^P\Big| \\
=&\Big|0 + \sum_{i=1}^{r_0} \chi_1 \wedge \cdots \wedge \chi_{i-1} \wedge \chi_i^\perp \wedge \chi_{i+1}^P \wedge \cdots \wedge \chi_{r_0}^P\Big| \\
\le&\sum_{i=1}^{r_0} |\chi_1 \wedge \cdots \wedge \chi_{i-1} \wedge \chi_i^\perp \wedge \chi_{i+1}^P \wedge \cdots \wedge \chi_{r_0}^P| \\
\le& r_0 \cdot L^{r_0-1}\kappa_1 < \kappa_0.
\end{aligned}$$

It follows from the choice of κ_0 that $\chi_1, \cdots, \chi_{r_0}$ are linearly dependent. The lemma is proved. □

Lemma 3.3. *There exist constants $M_0 > 1$, $\kappa_2 > 0$, both depending only on G_0, that satisfy the following condition:*

Suppose that for some $D, D' \in \mathfrak{a}_0$ with $|D| = 1$ and $\delta \in (0, \kappa_2)$,

$$\{\chi \in \Sigma_0 : \chi(D') > 0\} \subseteq \{\chi \in \Sigma_0 : \chi(D) > 0\}, \tag{3.1}$$

and

$$|D - D'| < \delta, \tag{3.2}$$

then for any $\delta' \in (0, \kappa_2)$*, one of the following holds:*

1. *The set*
$$\{\chi \in \Sigma_0 : \chi(D') > 0, \min(\chi(D), \chi(D')) < \delta'\} \tag{3.3}$$
is empty;
2. *There exists* $D'' \in \mathfrak{a}_0^*$ *with* $|D''| = 1$*, such that*
$$\{\chi \in \Sigma_0 : \chi(D'') > 0\} \subsetneq \{\chi \in \Sigma_0 : \chi(D') > 0\}, \tag{3.4}$$
and
$$|D - D''| < M_0(\delta + \delta'). \tag{3.5}$$

Proof. Let L, κ_0 and κ_1 be as in Lemma 3.2 and its proof. Take
$$M_0 = 4L^{r_0}\kappa_0^{-1} + 1,\ \ \kappa_2 = \min(\frac{1}{2M_0}, \frac{\kappa_1}{L+1}).$$

We first remark that as the set of roots Σ_0 is symmetric, (3.1) implies
$$\{\chi \in \Sigma_0 : \chi(D') < 0\} \subseteq \{\chi \in \Sigma_0 : \chi(D) < 0\}, \tag{3.6}$$
and thus also
$$\{\chi \in \Sigma_0 : \chi(D') = 0\} \supseteq \{\chi \in \Sigma_0 : \chi(D) = 0\}, \tag{3.7}$$

Assume that claim (1) does not hold. Let $\zeta \in \Sigma_0$ be an element of the set (3.3). Then either $|\zeta(D)| < \delta'$, which implies $|\zeta(D')| = |\zeta(D' - D) + \zeta(D)| \leq L|D - D'| + \delta' < L\delta + \delta' < (L+1)\kappa_2$; or $|\zeta(D')| < \delta' < \kappa_2$. As $(L+1)\kappa_2 \leq \kappa_1$, in both cases $\zeta(D') < (L+1)\kappa_2$.

As $(L+1)\kappa_2 \leq \kappa_1$, by Lemma 3.2, ζ and $\{\chi \in \Sigma_0 : \chi(D') = 0\}$ span a proper subspace P of $\mathfrak{a}_0^*$. P is strictly larger than the subspace spanned by $\{\chi \in \Sigma_0 : \chi(D') = 0\}$, as $\zeta(D') \neq 0$.

Fix a basis $\chi_1, \cdots, \chi_n$, chosen from $\{\zeta\} \cup \{\chi \in \Sigma_0 : \chi(D') = 0\}$, for P. We can assume $\chi_1 = \zeta$. Then for $2 \leq i \leq n$, $\chi_i(D') = 0$. On the other hand, for $i = 1$, as $\chi_1 = \zeta$, we have shown above $\chi_1(D') < (L+1)\kappa_2$.

Each element $\eta \in P$ with $|\eta| = 1$ can be written as
$$\sum_{i=1}^{r} \frac{\chi_1 \wedge \cdots \wedge \chi_{i-1} \wedge \eta \wedge \chi_{i+1} \wedge \cdots \wedge \chi_n}{\chi_1 \wedge \cdots \wedge \chi_n} \cdot \chi_i.$$
Here the notation $\frac{\chi_1 \wedge \cdots \wedge \chi_{i-1} \wedge \eta \wedge \chi_{i+1} \wedge \cdots \wedge \chi_n}{\chi_1 \wedge \cdots \wedge \chi_n}$ makes sense because both the numerator and the denominator lie in the one dimensional vector space $\wedge^n P$. Notice that the absolute value of each coefficient is less than $\frac{L^{n-1} \cdot 1}{\kappa_0}$, where κ_0 is as in the proof of Lemma 3.2. Thus
$$|\eta(D')| < L^{n-1}\kappa_0^{-1} \cdot |\chi_1(D')| \leq L^{r_0 - 1}\kappa_0^{-1}(L+1)\kappa_2. \tag{3.8}$$

Let $W \subseteq \mathfrak{a}_0$ be the annihilator of P and $W^\perp$ be the orthogonal complement of W. $W^\perp$ is non-trivial because so is P. The inequality (3.8) implies that the $W^\perp$ component $D'_\perp$ of D' in the decomposition $W \oplus W^\perp$ satisfies
$$|D'_\perp| < L^{r_0 - 1}\kappa_0^{-1}(L+1)\kappa_2 < \frac{1}{2}, \tag{3.9}$$
where the last inequality follows from the choice of κ_2.

Set
$$D'_t = \frac{D' - tD'_\perp}{|D' - tD'_\perp|}, \tag{3.10}$$
which is well-defined for $t \in [0, 1]$. Observe that $D'_1 \in W$ satisfies that $\chi(D'_1) = 0$ for each $\chi \in \Sigma_0$ with $\chi(D') = 0$ as well as for $\chi = \zeta$. In addition, $\chi(D'_t) = 0$ for

all D'_t and $\chi \in \Sigma_0$, because D'_t is proportional to a convex combination between D' and D'_1. Let $\mathcal{T} \subset [0, 1]$ be the set of values of t such that

$$\{\chi \in \Sigma_0 : \chi(D'_t) = 0\} \supsetneq \{\chi \in \Sigma_0 : \chi(D') = 0\}. \tag{3.11}$$

Then $\mathcal{T}$ is non-empty, as $1 \in \mathcal{T}$. Moreover, it is not difficult to verify that $\mathcal{T}$ is closed because $t \in \mathcal{T}$ is equivalent to that D'_t belongs to the union of finitely many predetermined subspaces of the form $\ker \chi$. Hence, $s = \min_{t\in\mathcal{T}} t$ is well defined. Let

$$D'' = D'_s. \tag{3.12}$$

We now verify (3.4) and (3.5).

Note $D'_0 = D'$. As the linear span of $\{\chi \in \Sigma_0 : \chi(D'_t) = 0\}$ remains constant for $t \in [0, s)$, the signs of $\chi(D'_t)$ does not change on $[0, s)$ for any $\chi \in \Sigma_0$ with $\chi(D') \neq 0$. It therefore follows from the fact that D'_s satisfies (3.11) that, the only change when t reaches s is that, for one or more $\chi \in \Sigma_0$, $\chi(D'_t)$ switches from positive to 0 (and at the same time, for the corresponding $-\chi \in \Sigma_0$, $-\chi(D'_t)$ switches from negative to 0). This shows (3.4).

Denote the right hand of (3.9) by $\tilde{\delta}$, which is in $(0, \frac{1}{2})$. Then $|D' - tD'_\perp| \in (1 - \tilde{\delta}, 1 + \tilde{\delta}) \subset (\frac{1}{2}, \frac{3}{2})$. It follows that $\left|1 - \frac{1}{|D' - tD'_\perp|}\right| < 2\tilde{\delta}$. Therefore,

$$\begin{aligned} |D' - D'_t| =& \left|(1 - \frac{1}{|D' - tD'_\perp|})D' + \frac{t}{|D' - tD'_\perp|}D'_\perp\right| \\ <& 2\tilde{\delta}|D'| + 2|D'_\perp| \leq 4\tilde{\delta}. \end{aligned}$$

In particular, $|D' - D''| < 4\tilde{\delta}$.

Adding (3.2), we obtain that

$$|D - D''| < 4\tilde{\delta} + \delta = (4L^{r_0}\kappa_0^{-1} + 1)\delta + 4L^{r_0-1}\kappa_0^{-1}\delta' \leq M_0(\delta + \delta'),$$

which is (3.5). Here we used that $L > 1$. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let M_0, κ_1 and κ_2 be as Lemmas 3.2 and 3.3. Define $f_0 = 0$ and inductively $f_{n+1} = M_0(f_n + 1)$. Choose $\theta_1 \in (0, \kappa_1)$ such that $f_{r_0-1}\theta_1 < \kappa_2$.

Fix a unit vector $D \in \mathfrak{a}_0$ and $\theta \in (0, \theta_1)$. Set $D_0 = D$. By applying Lemma 3.3 inductively (to D and D_{n-1} in each step, with $\delta' = \theta$ and $\delta = f_{n-1}\theta$), we can define $D_n \in \mathfrak{a}_0$ such that:

$$\{\chi \in \Sigma_0 : \chi(D_n) > 0\} \subsetneq \{\chi \in \Sigma_0 : \chi(D_{n-1}) > 0\}, \tag{3.13}$$

and

$$|D - D_n| < f_n\theta, \tag{3.14}$$

unless: either

$$\{\chi \in \Sigma_0 : \chi(D_{n-1}) > 0, \min(\chi(D), \chi(D_{n-1})) < \theta\} = \emptyset \tag{3.15}$$

or

$$f_{n-1}\theta \geq \kappa_2. \tag{3.16}$$

Notice that (3.13) implies

$$\{\chi \in \Sigma_0 : \chi(D_n) = 0\} \supsetneq \{\chi \in \Sigma_0 : \chi(D_{n-1}) = 0\},$$

and that the linear span P_n of $\{\chi \in \Sigma_0 : \chi(D_n) = 0\}$ strictly increases. Because Σ_0 spans $\mathfrak{a}_0$, whose dimension is r, $\dim P_n$ increases for at most $r_0 - 1$ steps before the inductive process stops. However, the choice of κ_1 guarantees that (3.16) does not happen for $n \le r$. Hence for some $n \le r$, D_{n-1} is well-defined and (3.15) holds. Since (3.13) holds for all $m \le n-1$,

$$\{\chi \in \Sigma_0 : \chi(D_{n-1}) > 0\} \subseteq \{\chi \in \Sigma_0 : \chi(D) > 0\}.$$

To conclude the proof of Proposition 3.1, it suffices to denote $D' = D_{n-1}$ and $M = f_{r_0-1}$. We remark that the subspace $\mathfrak{u}$ defined in the statement of the proposition form a Lie subalgebra, as for roots $\chi, \chi' \in \Sigma_0$ such that $\mathfrak{g}_0^{\chi}, \mathfrak{g}_0^{\chi'} \subseteq \mathfrak{u}$, either $\chi + \chi' \notin \Sigma_0$ or $\mathfrak{g}_0^{\chi+\chi'} \subseteq \mathfrak{u}$. □

The subgroup $U = \exp \mathfrak{u}$ is the strong horosphere for the one parameter subgroup $\exp(tD')$. For all $r > 0$, we define a bounded neighborhood B_r of the identity in U by

$$B_r = \exp B_r^{\mathfrak{u}} \subset B, \tag{3.17}$$

and we simply denote

$$B = B_1 = \exp B_1^{\mathfrak{u}}. \tag{3.18}$$

Throughout the rest of the paper, we will have the Haar measure m_U on U normalized so that $\mathrm{m}_U(B) = 1$.

The subalgebra $\mathfrak{g}_0^+$ splits as $\mathfrak{u} \oplus \mathfrak{u}^\perp$ where

$$\mathfrak{u}^\perp = \bigoplus_{\chi(D')=0, \chi(D)>0} \mathfrak{g}_0^{\chi}. \tag{3.19}$$

One can easily check that $\mathfrak{u}^\perp$ is also a subalgebra. Let $U^\perp = \exp \mathfrak{u}^\perp$. As G_0^+, U, $U^\perp$ are nilpotent groups, $G_0^+ = U \cdot U^\perp$ and the decomposition is a diffeomorphism between G_0^+ and $U^\perp \times U$ (see e.g. [17]). Moreover, as U and $U^\perp$ are both nilpotent and $U^\perp$ normalizes U, one can renormalize the volumes such that $\mathrm{m}_{G_0^+} = \mathrm{m}_U \times \mathrm{m}_{U^\perp}$. We will choose $B_{G_0^+}$ such that $B_{G_0^+} = B_{U^\perp} B_{\frac{1}{2}}$ for some bounded neighborhood $B_{U^\perp}$ of the identity in $U^\perp$, and $\mathrm{m}_{U^\perp}(B_{U^\perp}) = \frac{1}{\mathrm{m}_U(B_{\frac{1}{2}})}$.

Remark that there are only finitely many possible configurations for the triple $(G_0^+, U, U^\perp)$ for which we need to choose the neighborhoods $B_{G_0^+}$, B, $B_{U^\perp}$.

Remark 3.4. As in Remark 2.2, the objects U, $U^\perp$, B_r, $B_{U^\perp}$, as well as the metric and Haar measures on them, can be chosen according to the choice of A_0 in a way that is equivalent by conjugacy to the corresponding objects defined for a prescribed Cartan subgroup $\widehat{A}_0$.

Proposition 3.1 allows to further reduce Theorem 1.1 to the following:

Theorem 3.5. *Let G be a connected semisimple Lie group, $G_0 \subseteq G$ be a connected simple Lie subgroup and $\Gamma \subset G$ be a lattice. Then there exists a constant $c = c(G, G_0, \Gamma) > 0$, such that:*

For all $\epsilon > 0$, Cartan subgroups $A_0 \subset G_0$, and compact subsets $\Omega_0 \subset G/\Gamma$, there exists a compact subset $\Omega = \Omega(G, G_0, \Gamma, A_0, \Omega_0, \epsilon) \subset G/\Gamma$, such that:

For all $x \in \Omega_0$, $T > 0$, and unit vectors $D \in \mathfrak{a}_0$ (with respect to the norm induced by the Killing form on $\mathfrak{g}_0$),

$$\mathrm{m}_U\big(\{u \in B : \big|\{t \in [0,T] : \ \exp(tD)u.x \notin \Omega\}\big| \ge \epsilon T\}\big) < \mathrm{m}_U(B_{\frac{1}{2}})e^{-c\epsilon T}, \tag{3.20}$$

where U and B are constructed as above.

Proposition 3.6. *Theorem 3.5 implies Theorem 1.1.*

Proof. First observe that in the statement of Theorem 1.1, one may assume without loss of generality that D is a unit vector after replacing D with $\frac{D}{|D|}$ if necessary.

We assume Theorem 3.5 holds. By Proposition 2.5 it suffices to verify Theorem 2.4. Notice

$$\begin{aligned}
&\mathrm{m}_{G_0^+}\big(\{g\in B_{G_0^+}:\big|\{t\in[0,T]:\ \exp(tD)g.x\notin\Omega\}\big|\geq\epsilon T\}\big)\\
=&\int_{u^\perp\in B_{U^\perp}}\mathrm{m}_U\big(\{u\in B_{\frac{1}{2}}:\big|\{t\in[0,T]:\\
&\qquad\exp(tD)uu^\perp.x\notin\Omega\}\big|\geq\epsilon T\}\big)\mathrm{dm}_{U^\perp}
\end{aligned}\tag{3.21}$$

Given Ω_0 in the condition of Theorem 2.4, let Ω_0' be the union of the $\overline{B_{U^\perp}}\cdot\Omega_0$'s for all possible values of $U^\perp$. Then Ω_0' a compact subset of G/Γ, so we can apply Theorem 3.5 to it and get c, as well as Ω for all $\epsilon>0$. Then for $x\in\Omega_0$ and $T>0$, the integrand in (3.21) is less than $\mathrm{m}_U(B_{\frac{1}{2}})e^{-c\epsilon T}$ for all $v^\perp\in B_{U^\perp}$. (3.20) follows. ∎

4. **Reduction by Margulis arithmeticity theorem.** In order to prove Theorem 3.5, we begin by some elementary reductions.

Lemma 4.1. *Suppose for two connected semisimple Lie groups G and G^*, G^* is a factor group of G with compact or discrete kernel, and the projection of a lattice $\Gamma\subset G$ is commensurable to a lattice $\Gamma^*\subset G^*$. Then Theorem 3.5 is true for (G,Γ) if and only if it is true for (G^*,Γ^*).*

Proof. It suffices to consider two separate scenarios:

1. G^* is a factor of G with compact or discrete kernel, and Γ^* is the projection of Γ;
2. $G=G^*$ and Γ is a finite index sublattice in Γ^*.

In both cases, G^*/Γ^* is a factor of G/Γ with preimage fibers being compact or discrete. As both spaces have finite volume, the fibers are discrete if only if G/Γ is a finite cover of G^*/Γ^*.

The "if" direction: Suppose Theorem 3.5 holds on G^*/Γ^*. Let G_0, A_0, D and Ω_0 be defined in G or G/Γ as in Theorem 3.5. And let U and B be constructed correspondingly as in Section 3. We aim to show that Theorem 3.5 is true for these objects.

Let π denote indifferently the projections $G\to G^*$ and $G/\Gamma\to G^*/\Gamma^*$. We may naturally project G_0, A_0, D, Ω_0, U, m_U, B, $B_{\frac{1}{2}}$ under π. Denote the projected images respectively by G_0^*, A_0^*, D^*, Ω_0^*, U^*, m_{U^*}, B^* and $B^*_{\frac{1}{2}}$. Note that, G_0 can be assumed to be non-compact, otherwise A_0 is trivial and the statement of Theorem 3.5 is empty. As G_0 is a non-compact simple Lie group and $\ker\pi$ is compact or discrete, $G_0\cap\ker\pi$ must be discrete, in other words $\pi:G_0\to G_0^*$ is a covering map. In this case the Cartan subgroup A_0, the nilpotent subgroup $U\subseteq G_0$, as well as the set B, are bijectively projected. So A_0^*, U^* and B^* are isomorphic copies of A_0, U and B. Moreover, D^* and U^* still satisfy Proposition 3.1, which is a statement on the Lie algebra level. And $\mathrm{m}_{U^*}(B^*)=1$, $\mathrm{m}_{U^*}(B^*_{\frac{1}{2}})=\mathrm{m}_U(B_{\frac{1}{2}})$.

Apply Theorem 3.5 with respect to D^* and Ω_0^*. We obtain a constant $c=c(G^*,G_0^*,\Gamma^*)$ and a compact set $\Omega^*=\Omega^*(G^*,G_0^*,A_0^*,\Gamma^*,\Omega_0^*,\epsilon)$ that satisfy (3.20) with respect to G_0^*, A_0^*, D^* and Ω_0^*. Because the flow $\{\exp(tD)\}$ projects to $\{\exp(tD^*)\}$, (3.20) holds for c and the preimage Ω of Ω_*. Notice that since π is given, $c=c(G,G_0,\Gamma)$ and $\Omega=\Omega(G,G_0,A_0,\Gamma_0,\Omega_0,\epsilon)$.

The "only if" direction: The proof of this direction is similar. Suppose Theorem 3.5 holds on G/Γ. Let G_0^*, A_0^*, D^* and Ω_0^* be defined in G^* or G^*/Γ as in Theorem 3.5. And let U^* and B^* be constructed correspondingly as in Section 3.

As G^* is a factor of the semisimple Lie group G, $\mathfrak{g} = \mathfrak{g}^* \oplus \mathfrak{h}$ where $\mathfrak{h}$ is the Lie algebra of the compact kernel $\ker \pi$. Let $\mathfrak{g}_0$ be the image of the Lie algebra $\mathfrak{g}_0^* \subseteq \mathfrak{g}^*$ of G_0^* in $\mathfrak{g}$, and $G_0 \subseteq G$ be the connected simple Lie group corresponding to $\mathfrak{g}_0$. Similarly, define the Cartan subgroup A_0 and the nilpotent subgroup U inside G_0, and $D \in \mathfrak{g}_0$. Then $\mathfrak{g}_0^*$ is an isomorphic image of $\mathfrak{g}_0$ under $D\pi$, thus $G_0 \cap \ker \pi$ is again discrete. In this case, A_0^*, U^* are isomorphic images of A_0 and U under π. Define the neighborhoods B and $B_{\frac{1}{2}}$ as in Section 3, then they projects isomorphically to B^* and $B^*_{\frac{1}{2}}$. The Proposition 3.1 is satisfied by D and U.

Apply Theorem 3.5 with respect to D and the preimage $\Omega_0 = \pi^{-1}(\Omega_0^*)$. (Note that Ω_0 is compact because Ω_0^* is compact and the fibers of $\pi : G/\Gamma \to G^*/\Gamma^*$ is compact or finite.) We obtain a constant $c = c(G, G_0)$ and a compact set $\Omega = \Omega(G, G_0, A_0, \Omega_0, \epsilon)$ that satisfy (3.20) with respect to G_0, A_0, D and Ω_0. Because the flow $\{\exp(tD)\}$ projects to $\{\exp(tD^*)\}$, (3.20) holds for c and the projected image $\Omega^* = \pi(\Omega)$. Again, since π is given, $c = c(G^*, G_0^*, \Gamma^*)$ and $\Omega = \Omega(G^*, G_0^*, A_0^*, \Gamma_0^*, \Omega_0^*, \epsilon^*)$. $\square$

Lemma 4.2. *In order to prove Theorem 3.5, it suffices to consider the case when Γ is an irreducible lattice and G has trivial center and no compact factors.*

Recall that an irreducible lattice Γ in a semisimple Lie group G is one that projects densely into all non-trivial factors of G.

Proof. By Lemma 4.1, one may assume that G is centerless and has no compact almost simple factors (by quotienting them out if necessary). In this case G is the connected component $\mathbf{G}(\mathbb{R})^\circ$ of the real points of a linear algebraic group $\mathbf{G}$. Again by Lemma 4.1, after passing to a commensurable lattice if necessary, we may assume that $G = \prod_{i=1}^r G_i$ and $\Gamma = \prod_{i=1}^r \Gamma_i$, where each G_i is a connected semisimple Lie group without compact factors and Γ_i is an irreducible lattice in G_i.

For each $i = 1, \cdots, n$, consider the simple subgroup $G_{0,i}$ and its Cartan subgroup $A_{0,i}$, which we define respectively as the projections of G_0 and A_0 in G_i. Notice that as G_0 is simple, $G_{0,i}$ and $A_{0,i}$ are either both trivial or respectively isomorphic to G_0 and A_0. Denote by D_i the i-th projection of D, this gives rises to nilpotent subgroups $U_i \subseteq G_{0,i}$ after applying Proposition 3.1. One can then define neighborhoods B_i and $(B_{\frac{1}{2}})_i$ accordingly. Then $B \subseteq \prod_{i=1}^r B_i$ and $B_{\frac{1}{2}} \subseteq \prod_{i=1}^r (B_{\frac{1}{2}})_i$. The Haar measure m_U is proportional to $\prod_{i=1}^r \mathrm{m}_{U_i}$.

Now suppose Theorem 3.5 is true for configuration $(G_i, G_{0,i}, A_{0,i}, \Omega_{0,i})$ for every i. On each G_i/Γ_i, we get a constant $c_i > 0$, that depends on G_i and $G_{0,i}$. If $G_{0,i}$ is non-trivial then there is a compact function $\Omega_i \subseteq G_i/\Gamma_i$ that is independent of D_i and satisifes (1.1). When $G_{0,i}$ is trivial, let $\Omega_i = \Omega_{0,i}$. Since $\Omega_0 \subseteq \prod_{i=1}^r \Omega_{0,i}$, for every $x \in \Omega_0$ and $\epsilon \in (0, 1]$, we have

$$\mathrm{m}_{U_i}\big(\{u \in B_i : \big|\{t \in [0, T] :\ \exp(tD)u.\pi_i(x) \notin \Omega_i\}\big| \geq \epsilon T\}\big) < e^{-c_i \epsilon T}.$$

Here π_i is the projection from G/Γ to G_i/Γ_i. Equivalently,

$$\mathrm{m}_U\big(\{u \in \prod_{i=1}^r B_i : \big|\{t \in [0, T] :\ \pi_i(\exp(tD)k.x) \notin \Omega_i\}\big| \geq \epsilon T\}\big)$$

$$<\mathrm{m}_U(\prod_{i=1}^r B_i)e^{-c_i\epsilon T}.$$

Let $\Omega^\# = \prod_{i=1}^r \Omega_i$, then for all $x \in \Omega_0$,

$$\begin{aligned} &\mathrm{m}_U\big(\{u \in B : \big|\{t \in [0,T] : \ \exp(tD)u.x \notin \Omega^\#\}\big| \geq \epsilon T\}\big) \\ \leq&\mathrm{m}_U\big(\{u \in \prod_{i=1}^r B_i : \big|\{t \in [0,T] : \ \exp(tD)u.x \notin \Omega^\#\}\big| \geq \epsilon T\}\big) \\ \leq&\sum_{i=1}^r \mathrm{m}_U\big(\{u \in \prod_{i=1}^r B_i : \big|\{t \in [0,T] : \ \pi_i(\exp(tD)u.x) \notin \Omega_i\}\big| \geq \frac{\epsilon}{n}T\}\big) \\ <&n \cdot \mathrm{m}_U(\prod_{i=1}^r B_i)e^{-\frac{c}{n}\epsilon T} \leq n_G \mathrm{m}_U(\prod_{i=1}^r B_i) \cdot e^{-\frac{c}{n_G}\epsilon T}. \end{aligned} \tag{4.1}$$

Here $c = \min_{i=1}^r c_i$ and n_G denotes the number of almost simple factors in G.

Recall that c_i is determined by G_i, $G_{i,0}$ and Γ_i, so c is determined by G , G_0 and Γ. Furthermore, given G_0 and A_0, the factors $G_{0,i}$ and $A_{0,i}$ are determined, and there are only finitely many possible choices for U and U_i, which in turn determine B and B_i (since the metric on G_0 is intrinsically defined using the Lie algebra structure of $\mathfrak{g}_0$), so the coefficient $\mathrm{m}_U(\prod_{i=1}^r B_i) = \dfrac{\mathrm{m}_U(\prod_{i=1}^r B_i)}{\mathrm{m}_U(B)}$ has only finitely many possible values once G_0 and A_0 are given. Indeed, these values are also independent of A_0 as all Cartan subgroups are conjugate. To summarize, $n_G\mathrm{m}_U(\prod_{i=1}^r B_i)$ admits only finitely many values determined by G and G_0.

Hence, there is $T^\# = T^\#(G, G_0, \epsilon)$, such that

$$n_G e^{-\frac{c}{n_G}\epsilon T} < e^{-\frac{c}{2n_G}\epsilon T}, \ \forall T > T^\#. \tag{4.2}$$

Write $\Omega = \Omega^\# \cup \exp(\overline{B_{\mathfrak{a}}^{T^\#}})B.\Omega_0$, then Ω is determined by G, G_0, A_0, Γ, Ω_0 and ϵ. Moreover, for all $x \in \Omega_0$ and $T \leq T^\#$,

$$\{u \in B : \big|\{t \in [0,T] : \ \exp(tD)u.x \notin \Omega^\#\}| \geq \epsilon T\}\big) = \emptyset, \forall T \in [0, T^\#]. \tag{4.3}$$

Combining (4.2) and (4.3), we know that for all $x \in \Omega_0$ and $T > 0$,

$$\mathrm{m}_U\big(\{u \in B : \big|\{t \in [0,T] : \ \exp(tD)u.x \notin \Omega\}\big| \geq \epsilon T\}\big) < e^{-\frac{c}{2n_G}\epsilon T}. \tag{4.4}$$

This is the content of Theorem 3.5, after renaming $\frac{c}{2n_G}$ by c. □

Corollary 4.3. *In order to prove Theorem 3.5, it suffices to consider the following special cases:*

1. *(Arithmetic lattices) $G = \mathbf{G}(\mathbb{R})^\circ$ is the connected component of the real points of a linear algebraic group $\mathbf{G} \subseteq \mathrm{SL}_d$ defined over $\mathbb{Q}$, $\mathbf{G}$ is $\mathbb{Q}$-almost simple and $\Gamma = \mathbf{G}(\mathbb{Z}) \cap G$, and G_0 is a connected simple Lie subgroup of G.*
2. *(Rank 1 homogeneous spaces) G is a connected simple Lie group of real rank 1, $G_0 \subseteq G$ is a connected simple Lie subgroup, and $\Gamma \subseteq G$ is a lattice.*

Proof. By Lemma 4.2, one may assume Γ is irreducible and G is a connected centerless semisimple Lie group has no compact factors. If G is trivial, then the statement is empty and it suffices set $c = 1$. When G is non-trivial, if $\mathrm{rank}_{\mathbb{R}} G = 1$, then G must be simple, which is Case (2). Otherwise, Margulis Arithmeticity Theorem [18]*Introduction, Theorem 1' states that there is a connected linear semisimple algebraic group $\mathbf{G} \subseteq \mathbf{SL}_d$ defined over $\mathbb{Q}$ and a surjective Lie group morphism

$\pi : \mathbf{G}(\mathbb{R})^\circ \to G$ with compact kernel, such that $\pi\big(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ\big)$ is commensurable to Γ. In order for Γ to be irreducible, $\mathbf{G}$ must be $\mathbb{Q}$-almost simple. By Lemma 4.1, one may work on the arithmetic homogeneous space $\mathbf{G}(\mathbb{R})^\circ/\big(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ\big)$, i.e. in Case (1) instead. □

5. **Expansion of vectors.** Following Eskin-Margulis-Mozes [9], Eskin-Margulis [8] and Benoist-Quint [1], we will prove in the next two sections that a random trajectory does return to a compact set in finite time with hight probability. This section will characterize the behaviour of the unipotent translates using the quantitative non-divergence property from Kleinbock-Margulis [16] and Kleinbock [14].

Let (ρ, V) be a non-trivial irreducible representation of G_0. We denote indifferently by ρ the derivative representation on V of the Lie algebra $\mathfrak{g}_0$. V decomposes as a direct sum $\bigoplus_{\xi\in\Xi} V^\xi$ of relative weight spaces.

Lemma 5.1. *For any connected simple Lie group G_0, there exists $\beta = \beta(G_0) > 0$, such that for all non-trivial irreducible representations (ρ, V) of G_0 and unit vector $D \in \mathfrak{a}_0$, $\max_{\xi\in\Xi} \xi(D) > \beta$.*

Proof. This follows from the following facts:

The convex hull of the set of weights of ρ is a polygon P in $\mathfrak{a}_0^*$. Furthermore, there is a small radius $r_0 > 0$ such that $B(0, r_0) \subseteq P$ for all non-trivial representations ρ of G_0. □

Recall that U is constructed in Proposition 3.1 together with a perturbation D' of D. Let $V_{\max} = \bigoplus_{\substack{\xi'\in\Xi \\ \xi'(D')=\max_{\xi\in\Xi}\xi(D')}} V^{\xi'}$, and $V_{\max}^\perp$ be the direct sum of the remaining relative weight spaces in V. Then $V = V_{\max} \oplus V_{\max}^\perp$. $V_{\max}$ is clearly a non-trivial subspace. Furthermore, $\max_{\xi\in\Xi} \xi(D') > \beta$ by Lemma 5.1, and similarly one can also prove $\min_{\xi\in\Xi} \xi(D') < -\beta$. This shows $V_{\max}$ is proper in V.

Lemma 5.2. *For all non-zero vectors $v \in V$, $\rho(U).v \not\subseteq V_{\max}^\perp$.*

Proof. It suffices to prove $\rho(\mathfrak{u}).v \not\subseteq V_{\max}^\perp$, where we indifferently denote by ρ the induced representation of $\mathfrak{g}_0$ on V.

Assume $\rho(\mathfrak{u}).v \subseteq V_{\max}^\perp$, then in particular $v \in V_{\max}^\perp$. However, for each ξ such that $\xi(D')$ doesn't achieve the maximal value $\max_{\xi\in\Xi}\xi(D')$, the sum of any non-positive root χ (with respect to D') with ξ remains non-maximal. Thus $\rho(\mathfrak{g}_0^\chi).v \subseteq V_{\max}^\perp$ as well. However, by Proposition 3.1, such $\mathfrak{g}_0^\chi$'s span $\mathfrak{g}_0$ together with $\mathfrak{u}$, it follows that $\rho(\mathfrak{g}_0).v \subseteq V_{\max}^\perp$. This makes the span of $\rho(\mathfrak{g}_0).v$ a proper subrepresentation of V. As V is irreducible, this must be a trivial subrepresentation, and thus $v = 0$. □

Corollary 5.3. *For any non-zero vector $v \in V$, $\rho(B_{\frac{1}{4}}).v \not\subseteq V_{\max}^\perp$.*

Proof. Suppose for the sake of contradiction that $\rho(B_{\frac{1}{4}}).v \subseteq V_{\max}^\perp$. Then by differentiating at the identity, $D\rho(\mathfrak{u}).v \subseteq V_{\max}^\perp$ and it follows that $\rho(U).v \subseteq V_{\max}^\perp$. This contradicts the lemma above. □

Write $\pi_{V_{\max}}$ for the projection from $V = V_{\max} \oplus V_{\max}^\perp$ to $V_{\max}$. By Corollary 5.3, $\sup_{u\in B} |\pi_{V_{\max}}(\rho(u).v)| > 0$ for all $v \in V\backslash\{0\}$. As $(u, v) \to \pi_{V_{\max}}(\rho(u).v)$ is continuous and the closure $\overline{B_{\frac{1}{4}}}$ of $B_{\frac{1}{4}}$ is compact, there exists $\eta \in (0, 1)$ such that

$$\sup_{u\in B_{\frac{1}{2}}} |\pi_{V_{\max}}(\rho(u).v)| \geq \sup_{u\in\overline{B_{\frac{1}{4}}}} |\pi_{V_{\max}}(\rho(u).v)| > \eta,\ \forall v \in V \text{ with } |v| = 1. \tag{5.1}$$

The constant $\eta = (G_0, A_0, \rho)$.

A function f on $\mathbb{R}^n$ is said to be (C,α)-*good* if for any open ball $B \subset \mathbb{R}^n$,

$$\mathrm{m}_{\mathbb{R}^n}\big(\{x \in B : |f(x)| < \epsilon\}\big) \leq C\Big(\frac{\epsilon}{\sup_{x\in B}|f(x)|}\Big)^{\alpha}\mathrm{m}_{\mathbb{R}^n}(B), \forall\epsilon > 0.$$

This definition was introduced in Kleinbock-Margulis [16] and can be traced to Dani-Margulis [5].

Lemma 5.4. *For $n, l \in \mathbb{N}$, there are constants $C = C(n,l) > 1, \alpha = \alpha(n,l) > 0$, such that all polynomials of degree at most l on $\mathbb{R}^n$ are (C,α)-good.*

The statement of Lemma 5.4 appeared in [14]*§1. Indeed, the $n = 1$ case was proved in [5]*Lemma 4.1 and [16]*Proposition 3.2. It is not difficult to deduce the general case by induction.

Given $t > 0$, $D \in \mathfrak{a}_0$ and $v \in V$ such that $|v| = 1$, consider the function

$$w(u) = w_{\rho,t,D,v}(u) = \rho(\exp(tD)u).v$$

on U. Then $w\circ\exp^{-1}$ is a polynomial map on $\mathfrak{u}$, where the degree of the polynomial is determined by ρ and the structure of $\mathfrak{u}$.

Let θ_1 and M be as in Proposition 3.1. Assume that in Proposition 3.1, $\theta < \zeta = \min(\theta_1, \frac{\beta}{2M\max_{\xi\in\Xi\setminus\{0\}}|\xi|})$, then

$$|\xi(D) - \xi(D')| < |\xi|\cdot M\theta < \frac{\beta}{2}.$$

So for each $V^\xi \subseteq V_{\max}$, $\xi(D) > \frac{\beta}{2}$. Note that $\zeta = \zeta(G_0, A_0, \rho)$.

In this case, for $v \in V_{\max}$, $|\rho(\exp(tD)).v| \geq e^{\frac{\beta t}{2}}|v|$. It follows from (5.1) that

$$\sup_{u\in B}|w(u)| \geq e^{\frac{\beta t}{2}}\sup_{u\in B}|\pi_{V_{\max}}(\rho(u).v)| > e^{\frac{\beta t}{2}}\eta. \tag{5.2}$$

In fact, for general representations (ρ, V) of G_0 without fixed vectors, as ρ is a direct sum $\oplus_{j=1}^q \rho_j$ of finitely many non-trivial irreducible representations (ρ_j, V_j), (5.2) remains true if $\theta < \zeta$ where ζ is a constant depending on G_0, A_0 and ρ. To see this, note that if $|v| = 1$, then for some V_j the component v_j of v in V_j is at least of modulus $\frac{1}{q}$. By (5.2) for irreducible representations, if $\theta < \zeta_j$ then

$$\sup_{u\in B}|w(u)| \geq e^{\frac{\beta t}{2}}\sup_{u\in B}|\pi_{V_{i,\max}}(\rho(u).v_j)| > e^{\frac{\beta t}{2}}\eta_j, \tag{5.3}$$

where ζ_j and η_j are constants depending only on G_0, A_0, and ρ_j. By taking $\zeta = \min_j \zeta_j$ and $\eta = \min_j \eta_j$, this verifies (5.2) for ρ.

Recall that the exponential map identifies $\mathrm{m}_{\mathfrak{u}}$ with m_U, and that $\mathrm{m}_U(B) = 1$. By Lemma 5.4, there exists $C > 1$ and $\alpha > 0$ determined by G_0 and ρ (because the degree l of w and dimension n of V are bounded when (ρ, V) is given), such that

$$\mathrm{m}_U\big(\{x \in B : |w(u)| < \epsilon\}\big) \leq C(e^{-\frac{\beta t}{2}}\eta^{-1}\epsilon)^{\alpha}, \forall\epsilon > 0. \tag{5.4}$$

Therefore,

$$\begin{aligned}
&\int_B |w(u)|^{-\theta} \mathrm{dm}_U(u) \\
=&\int_0^\infty \mathrm{m}_U\big(\{x \in B : |w(u)|^{-\theta} > y\}\big)\mathrm{d}y \\
=&\int_0^\infty \mathrm{m}_U\big(\{x \in B : |w(u)| < y^{-\frac{1}{\theta}}\}\big)\mathrm{d}y \\
\leq&\int_0^\infty \min\big(1, Ce^{-\frac{\alpha\beta t}{2}}\eta^{-\alpha}y^{-\frac{\alpha}{\theta}}\big)\mathrm{d}y
\end{aligned} \tag{5.5}$$

Since for all $R > 0$ and $\gamma > 1$,

$$\begin{aligned}
\int_0^\infty \min\big(1, Ry^{-\gamma}\big)\mathrm{d}y =& \int_0^{R^{\frac{1}{\gamma}}} 1\mathrm{d}y + \int_{R^{\frac{1}{\gamma}}}^\infty Ry^{-\gamma}\mathrm{d}y \\
=& R^{\frac{1}{\gamma}} + \frac{R}{\gamma - 1} \cdot R^{\frac{1-\gamma}{\gamma}} = \frac{\gamma}{\gamma - 1} R^{\frac{1}{\gamma}},
\end{aligned} \tag{5.6}$$

we have, for all $\delta \leq \frac{\alpha}{2}$,

$$\int_B |w(u)|^{-\delta} \mathrm{dm}_U(u) = \frac{\alpha}{\alpha - \delta} C^{\frac{\delta}{\alpha}} e^{-\frac{\beta\delta t}{2}} \eta^{-\delta} \leq 2C^{\frac{1}{2}} e^{-\frac{\beta\delta t}{2}} \eta^{-\frac{\alpha}{2}}. \tag{5.7}$$

Recall that $\beta = \beta(G_0)$ and the constants C, α, ζ, and η are determined by G_0, A_0, and ρ. After rewrite $\frac{\alpha}{2}$ as α, we have proved the following:

Proposition 5.5. *Suppose A_0 is a Cartan subgroup in G_0. For all representations $\rho : G_0 \to \mathrm{SL}(V)$ without non-zero fixed vectors, there exist positive constants $\zeta = \zeta(G_0, A_0, \rho)$ and $\alpha = \alpha(G_0, A_0, \rho)$, such that:*

If $\theta \in (0, \zeta)$ in Proposition 3.1, then for the strong unstable subgroup $U \subseteq G_0^+$ and neighborhood $B \subseteq U$, for all $\delta \in (0, \alpha]$ and $a > 0$, there exists $s = s(G_0, A_0, \rho, a, \delta) > 0$, such that for all unit vector $D \in \mathfrak{a}_0$ with respect to the norm induced by Killing form on $\mathfrak{g}_0$, and $t \geq s$, then

$$\int_B |\rho(\exp(tD)u).v|^{-\delta} \mathrm{dm}_U(u) \leq a|v|^{-\delta}.$$

The proposition above is analogous to [8]*Lemma 4.2 and [1]*Lemma 4.4.

6. **Contraction of height functions.** Proposition 5.5 leads to the following important contraction property:

Proposition 6.1. *In both special cases described in Corollary 4.3, there exist positive constants t_0, θ_0, determined by G, G_0, A_0 and Γ, such that for all compact set $\Omega_0 \subseteq G/\Gamma$, there is a proper lower semi-continuous function $f : G/\Gamma \to [0, \infty]$ and $b > 0$, determined by G, G_0, A_0, Γ and Ω_0, such that:*

1. *For all unit vectors $D \in \mathfrak{a}_0$ with respect to the norm induced by the Killing form on $\mathfrak{g}_0$, $t \geq t_0$, $x \in G/\Gamma$, $\theta \in (0, \theta_0)$, for the subgroup U defined in Proposition 3.1 with parameter θ, and the neighborhood B in (3.18),*

$$\int_B f(\exp(tD)u.x)\mathrm{dm}_U(u) \leq e^{-\frac{1}{2}} f(x) + b. \tag{6.1}$$

2. *f is bounded on Ω_0.*

We also claim a uniform Lipschitz property of $\log f$.

Corollary 6.2. *In the setting of Proposition 6.1, there exists an $C_1 > 0$ that depends only on G, G_0, A_0, and Γ, such that for all $x \in G/\Gamma$ and $Y \in \mathfrak{g}_0$,*

$$e^{-C_1|Y|}f(x) \leq f((\exp Y).x) \leq e^{C_1|Y|}f(x). \tag{6.2}$$

Following [1]*§5-6, the proof of Proposition 6.1 and Corollary 6.2 is divided into the arithmetic case and the rank 1 case based on Margulis Arithmeticity Theorem.

6.1. **Case I: Arithmetic lattices.** We first assume the special case (1) from Corollary 4.3. Recall that in this case $G = \mathbf{G}(\mathbb{R})^\circ$ is the connected component of the real points of a linear algebraic group $\mathbf{G} \subseteq \mathbf{SL}_d$ defined over $\mathbb{Q}$, and $\Gamma = \mathrm{SL}_d(\mathbb{Z}) \cap G$. Then G/Γ is naturally embedded in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$.

We set

$$\theta_0 = \min(\theta_1, \min_\rho \zeta(G_0, A_0, \rho)), \delta_0 = \min_\rho \alpha(G_0, A_0, \rho) \tag{6.3}$$

where $\theta_1 = \theta_1(G_0)$ comes from Proposition 3.1 and the minimum is taken over all subrepresentations ρ without fixed vectors in all representations of the form $\wedge^k \rho_0$ where ρ_0 is the standard representation of $\mathrm{SL}_d(\mathbb{R})$ on $\mathbb{R}^d$, restricted to G_0. Here we recall $G_0 \subseteq G \subseteq \mathrm{SL}_d(\mathbb{R})$.

The constants θ_0, δ_0 depend only on G, G_0, and the rational embedding of G in $\mathrm{SL}_d(\mathbb{R})$, as the collection of subrepresentations ρ is finite. It should be emphasized that the rational embedding of G is not intrinsic to the Lie group structure of G, but instead determined by Γ via Margulis Arithmeticity Theorem (see Corollary 4.3). So we have $\theta_0 = \theta(G, G_0, A_0, \Gamma)$ and $\delta_0 = \delta_0(G, G_0, A_0, \Gamma)$.

Proof of Proposition 6.1 for arithmetic latices. The proof of this case is the same as that of Proposition 5.3 in [1] on recurrence properties of semisimple random walks.

To be precise, in [1]*Lemma 4.4, it was proved that for some semisimple subgroup $H \subseteq \mathrm{SL}(d, \mathbb{R})$ and a family of probability measures μ^{*n} of finite exponential moments on H, for all representations (ρ, V) of H without fixed vectors, there exists $\delta_0' > 0$ such that for all $\delta \in (0, \delta_0')$ and $a > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, then

$$\int |\rho(g).v|^{-\delta} \mathrm{d}\mu^{*n}(g) \leq a|v|^{-\delta}.$$

With the semisimple subgroup $G_0 \subseteq G \subseteq \mathrm{SL}_d(\mathbb{R})$ in place of H and the family of compactly supported probability measures $\{(u \mapsto \exp(tD)u)_* \mathrm{m}_U|_B\}_{t \geq 0}$ in place of $\{\mu^{*n}\}_{n \in \mathbb{N}}$, Proposition 5.5 replaces [1]*Lemma 4.4.

A family of proper lower semi-continuous functions

$$f_\sigma : \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}) \to [0, \infty]$$

are defined in [1]*equation (5.1). The proof of [1]*Proposition 5.3, with the subsitutions above, shows that:

For some $\sigma_0 = \sigma_0(G, G_0, A_0, \Gamma) > 0$, for all $a \in (0, 1)$, there exists $t_0 = t_0(G, G_0, A_0, \Gamma, a) > 0$ satisfying:

For all $\sigma \in (0, \sigma_0)$ and $\theta \in (0, \theta_0)$, $\delta \in (0, \delta_0)$, and $t > t_0$, there is $b > 0$ such that for all $x \in G/\Gamma$

$$\int_B f_\sigma(\exp(tD)u.x)^\delta \mathrm{dm}_U(u) \leq \frac{1 + ad}{2} f_\sigma(x)^\delta + b.$$

Moreover, the family $\{f_\sigma\}$ is such that for every compact subset $Z \subseteq \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d$ $(\mathbb{Z})$, f_σ is bounded on Z for sufficiently small $\sigma > 0$ (by [1]*Remark 5.2 and Mahler's compactness criterion.) We choose such an σ such that f_σ is bounded on the compact subset $\Omega_0 \subset G/\Gamma$, which is embedded in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. The choice of σ depends on G, Γ and Ω_0.

To conclude the proposition, it suffices to let $a = \frac{e^{-\frac{1}{2}-\frac{1}{2}}}{d}$ and $f = f_\sigma^{\frac{\delta_0}{2}}|_{G/\Gamma}$. This omits the dependence of t_0 on a. Finally, we note that the function f_σ and the constant b are determined by G, G_0, A_0, Γ and σ. So as $\sigma = \sigma(G, \Gamma, \Omega_0)$, and $\delta_0 = \delta_0(G, G_0, A_0, \Gamma)$, f and b depend only on G, G_0, A_0, Γ and Ω_0. $\square$

Next, we recall the construction of the family $\{f_\sigma\}$ in [1] as they will become useful later.

Fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and a Weyl chamber $\mathcal{C} \subset \mathfrak{a}$. Let $P^+ \subset \mathfrak{a}^*$ be the set of highest weights in all representations of G with respect to $\mathcal{C}$. Fix an element $E \in \mathfrak{a}$ which lies in the interior of $\mathcal{C}$. For $\lambda \in P^+$, let $\Delta_\lambda = \lambda(E)$. For all vector v in a representative (ρ, V), $q_\lambda(v)$ denotes the ρ-equivariant projection to the direct sum of all irreducible components of highest weight λ.

For every x in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, which is the moduli space of unimodular lattices in $\mathbb{R}^d$, write $\mathcal{L}_x$ for the lattice that x represents. For $0 \le p \le d$, let $\Delta_p = p(d-p)$.

According to [1]*equations (4.3) and (5.1),

$$f_\sigma(x) = \sup_{\substack{0<p<d \\ v \in (\bigwedge^p \mathcal{L}_x)\setminus\{0\}}} \phi_\sigma(v), \tag{6.4}$$

where ϕ_σ is defined as follows: for all vectors v in $\bigwedge^p \mathbb{R}^d$, on which the natural exterior product representation acts,

$$\phi_\sigma(v) = \mathbf{1}_{\{|q_0(v)|<\sigma^{\Delta_p}\}}(v) \cdot \min_{\lambda \in P^+\setminus\{0\}} \sigma^{\frac{\Delta_p}{\Delta_\lambda}} |q_\lambda(v)|^{-\frac{1}{\Delta_\lambda}}. \tag{6.5}$$

To obtain Corollary 6.2 in this case we will need is the following observation:

Lemma 6.3. *There exists $C > 0$ that depends only on G, G_0, and the embedding of G in $\mathrm{SL}_d(\mathbb{R})$, such that for all $\sigma > 0$, $x \in G/\Gamma$ and $Y \in \mathfrak{g}_0$,*

$$e^{-C|Y|} f_\sigma(x) \le f_\sigma((\exp Y).x) \le e^{C|Y|} f_\sigma(x).$$

Proof. Let σ be fixed. Let $C' = C'(d) = \max_{p=1}^{d-1} \| \wedge^p \rho_0\|$. Here $\wedge^p\rho_0 : \mathfrak{sl}_d(\mathbb{R}) \to \mathrm{End}(\bigwedge^p \mathbb{R}^d)$ denotes the standard Lie algebra representation of $\mathfrak{sl}_d(\mathbb{R})$ on $\mathbb{R}^d$, and $\| \wedge^p \rho_0\| := \sup_{Y\in\mathfrak{sl}_d(\mathbb{R}), |Y|=1} \| \wedge^p \rho(Y)\|_{\mathrm{End}(\bigwedge^p \mathbb{R}^d)}$ denotes its norm, while the vector space $\mathrm{End}(\bigwedge^p \mathbb{R}^d)$ is equipped with the operator norm.

We first note that $f_\sigma(x) = 0$ if and only if for all $0 < p < d$ and all $v \in (\bigwedge^p \mathcal{L}_x) \setminus \{0\}$ for one of the p's, such that $q_0(v) \ge \sigma^{\Delta_p}$. As $q_0(v)$ is the equivariant projection to the subspace of fixed vectors, $q_0((\exp Y).v) = q_0(v)$. It follows that $f_\sigma(x) = 0$ if and only if $f_\sigma((\exp Y).x) = 0$. Therefore we may assume $f_\sigma(x) \ne 0$.

Suppose $v \in (\bigwedge^p \mathcal{L}_x) \setminus \{0\}$. We have $|q_0(v)| < \sigma^{\Delta_p}$ and $|q_\lambda(v)| = \phi_\sigma(v)^{-\Delta_\lambda}\sigma^{\Delta_p}$ for some $\lambda \in P^+ \setminus \{0\}$. Thus

$$|q_0((\exp Y).v)| = |(\exp Y).q_0(v)| = |q_0(v)| < \sigma^{\Delta_p},$$

and

$$|q_\lambda((\exp Y).v)| = |(\exp Y).q_\lambda(v)| \ge e^{-C'|Y|}|q_\lambda(v)|.$$

Thus, $|q_\lambda((\exp Y).v)| \ge e^{-C'|Y|}\phi_\sigma(v)^{-\Delta_\lambda}\sigma^{\Delta_p}$ and

$$\begin{aligned}\phi_\sigma((\exp Y).v) \le& \sigma^{\frac{\Delta_p}{\Delta_\lambda}} |q_\lambda((\exp Y).v)|^{-\frac{1}{\Delta_\lambda}} \le e^{\frac{C'}{\Delta_\lambda}|Y|}\phi_\sigma(v) \\ \le& e^{\frac{C'}{\Delta_\lambda}|Y|} f_\sigma(x).\end{aligned} \tag{6.6}$$

Let $C = \max_\lambda \frac{C'}{\Delta_\lambda}$, where the maximum is taken over all $0 < p < d$, and all highest weights $\lambda \in \mathfrak{a}^*$ appearing in components of $\bigwedge^p \mathbb{R}^d$. As there are only finitely many possible λ's to consider for a given group G, C depends only on G and d.

Then (6.6) implies

$$f_\sigma((\exp Y).x) = \sup_{\substack{0<p<d \\ v\in(\bigwedge^p \mathcal{L}_x)\setminus\{0\}}} \phi_\sigma((\exp Y).v) \leq e^{C|Y|} f_\sigma(x).$$

Similarly we can show $f_\sigma(x) \leq e^{C|Y|} f_\sigma((\exp Y).x)$. □

Proof of Corollary 6.2 for arithmetic lattices. To deduce this from Lemma 6.3, it suffices to remember that $f = f_\sigma^{\frac{\delta_0}{2}}$, where $\delta_0 = \delta_0(G, G_0, A_0, \Gamma)$, in the proof of Proposition 6.1. □

6.2. **Case II: Rank 1 homogeneous spaces.** We now assume the special case (2) from Corollary 4.3, which is that the semisimple Lie groups G has real rank 1.

Proof of Prop. 6.1 when $\operatorname{rank}_\mathbb{R} G = 1$. In this case, a continuous and proper height function $f_0 : G/\Gamma \to [0, \infty)$ was given in [1]*(6.3) following [8]. The function has the form

$$f_0(g\Gamma) = \max_{1\leq i\leq r} \max_{\gamma\in\Gamma} |\rho(g\gamma)v_i|^{-1}, \tag{6.7}$$

where the v_i's are non-trivial vectors from a fixed faithful irreducible representation (ρ, V) of G. Moreover, v_i is invariant under N_i, where $\{N_i\}_{i=1}^r$ is a maximal set of maximal unipotent subgroups of G which intersect Γ in a lattice and are not conjugate to each other by elements from Γ. (This set is known to be finite by Garland-Raghunathan[10].)

By Proposition 5.5, for all $\delta \in (0, \alpha]$ where $\alpha = \alpha(G_0, A_0, \rho) > 0$, there exists $s = s(G_0, A_0, \rho, \delta) > 0$ such that for all $t \geq s$,

$$\int_B f_0(\exp(tD)u.x)^\delta \mathrm{d}m_U(u) \leq e^{-\frac{1}{2}} f_0(x)^\delta. \tag{6.8}$$

This shows Proposition 6.1 in this case by letting $f = f_0^\alpha$. □

Lemma 6.4. *There exists $C > 0$ that depends only on G and Γ, such that for all $x \in G/\Gamma$ and $Y \in \mathfrak{g}_0$,*

$$e^{-C|Y|} f_0(x) \leq f_0((\exp Y).x) \leq e^{C|Y|} f_0(x).$$

The proof of the lemma is the same as that of Lemma 6.3, while using a different representation ρ.

Proof of Cor. 6.2 when $\operatorname{rank}_\mathbb{R} G = 1$. This follows directly from Lemma 6.4, as $f = f_0^\alpha$ and α depends on G, G_0, and ρ, and ρ is an arbitrarily fixed faithful irreducible representation of G. □

6.3. **Independence of parameters on A_0.** We can eliminate the dependence of t_0 and θ_0 on A_0.

Lemma 6.5. *In Proposition 6.1, the parameters t_0, θ_0 can be made to be dependent only on G, G_0 and Γ, and b can be made to be dependent only on G, G_0, Γ and Ω_0. Corollary 6.2 is not affected by these changes.*

Proof. Recall that all Cartan subgroups of G_0 are conjugate to each other. Fix a Cartan subgroup $\widehat{A}_0$, and choose $g \in G$ such that $A_0 = g\widehat{A}_0 g^{-1}$. Choose t_0 and θ_0 in Proposition 6.1 with respect to $\widehat{A}_0$. Then for all compact set $\Omega_0 \subseteq G/\Gamma$, there is a proper lower semicountinuous function $\hat{f} : G/\Gamma \to [0, \infty]$ and b such that:

1. For all unit vectors $\widehat{D} \in \hat{\mathfrak{a}}_0$ with respect to the norm induced by the Killing form on $\mathfrak{g}_0$, $t \geq t_0$, $x \in G/\Gamma$, $\theta \in (0, \theta_0)$, for the subgroup $\widehat{U}$ defined in Proposition 3.1 with parameter θ with respect to $\widehat{A}_0$ and $\widehat{D}$, and the neighborhood $\widehat{B} \subset \widehat{U}$ given by (3.18),
$$\int_{\widehat{B}} \hat{f}(\exp(t\widehat{D})u.x)\mathrm{dm}_{\widehat{U}}(u) \leq e^{-\frac{1}{2}}\hat{f}(x) + b. \tag{6.9}$$
2. $\hat{f}$ is bounded on $g^{-1}\Omega_0$.

For a unit vector $D \in \mathfrak{a}_0$ with respect to the norm on $\mathfrak{a}_0$ induced by the Killing form on $\mathfrak{g}_0$, $\widehat{D} := \mathrm{Ad}_{g^{-1}} D$ is a unit vector with respect to the norm on $\hat{\mathfrak{a}}_0$ induced by the Killing form on $\mathfrak{g}_0$. The group $U = g\widehat{U}g^{-1}$ satisfies Proposition 3.1 with respect to $D \in \mathfrak{a}_0$ and θ. We normalize the metric on G_0 and U so that the neighborhood B in (3.18) satisfies $B = g\widehat{B}g^{-1}$. (Note that doing so would not affect the reductions in §2 and §3, see Remarks 2.2 and 3.4.) Then m_U is equivalent to $\mathrm{m}_{\widehat{U}}$ via the conjugacy $u \to gug^{-1}$.

Define $f(x) = \hat{f}(g^{-1}x)$, which is bounded on Ω_0. Because
$$f(\exp(tD)ux) = \hat{f}(g^{-1}\exp(tD)ux) = \hat{f}(\exp(t\widehat{D})g^{-1}ug.g^{-1}x),$$
(6.1) follows from (6.9) with the same value b for all $t \geq t_0$ and $\theta \in (0, \theta_0)$. This eliminates the dependence on A_0 from t_0, θ_0 and b.

By Corollary 6.2, $\log \hat{f}$ is Lipschitz with respect to left translations with a Lipschtiz constant $\widehat{C}_1$ depending on G, G_0, Γ and the choice of $\widehat{A}_0$, since $f((\exp Y).x) = \hat{f}(g^{-1}(\exp Y).x) = \hat{f}(\exp(\mathrm{Ad}_g Y).g^{-1}x)$, $\log f$ is also Lipschitz continuous with a Lipschitz constant C_1 depending on $\widehat{C}_1$ and g. As $\widehat{A}_0$ is fixed and g depends on A_0 and $\widehat{A}_0$, $C_1 = C_1(G, G_0, A_0, \Gamma)$. Thus Corollary 6.2 remains valid after the substitution above. □

7. **Non-escape of mass for random walks.** In this section, we assume G, G_0, and Γ are as in at least one of the conditions from Corollary 4.3. Fix $\Omega_0 \subset G/\Gamma$. Let θ_0, t_0, b and f be as in Proposition 6.1 (and Lemma 6.5). Fix a flow time $\tau \geq t_0$, the choice of which will depend only on G, G_0, Γ, and be specified later. We also fix $\theta = \frac{1}{2}\theta_0$, a unit vector D in one of the $\mathfrak{a}_0$'s, and let U be as in Proposition 3.1. The next goal is to show escape of mass is exponentially rare for the random walk generated by $(u \to \exp(\tau D)u)_*\mathrm{m}_U|B$ on G/Γ.

In the remainder of this part, we roughly follow the approach of Kadyrov, Kleinbock, Lindenstrauss and Margulis [12]*§5, but work on a nilpotent scheme instead.

For all $t \geq 0$, define a probability measure
$$\nu_t = (\mathrm{Ad}_{\exp(-tD)})_*\mathrm{m}_U|_B. \tag{7.1}$$
Remark that $\nu_0 = \mathrm{m}_U|_B$.

The convolution between two probability measures μ and ν on U is defined by
$$\int_U \phi(u)\mathrm{d}(\mu * \nu)(u) = \int_U \int_U \phi(uv)\mathrm{d}\mu(u)\mathrm{d}\nu(v). \tag{7.2}$$

Lemma 7.1. *There exists $t_1 = t_1(G, G_0, \Gamma) > 0$ such that for all $\tau \geq t_1$ and $N \geq 0$, the probability measures $\mathrm{m}_U|_B$ and $\nu_{N\tau} * \nu_{(N-1)\tau} * \cdots * \nu_\tau * \nu_0$ coincide on $B_{\frac{1}{2}}$, i.e.*

$$\mathrm{m}_U(Q) = (\nu_{N\tau} * \nu_{(N-1)\tau} * \cdots * \nu_\tau * \nu_0)(Q)$$

for all subsets $Q \subseteq B_{\frac{1}{2}}$.

Proof. By the Baker-Campbell-Hausdorff formula on the nilpotent Lie algebra $\mathfrak{u}$, for $X, Y \in \mathfrak{u}$ such that $|X|, |Y| \leq 2$,

$$\exp^{-1}\big(\exp Y \exp X\big) = X + Y + O_U(XY).$$

In other words, there is a constant $C_2 > 1$ such that for all $X, Y \in \mathfrak{u}$ with $|X|, |Y| \leq 2$,

$$\big|\exp^{-1}\big(\exp Y \exp X\big) - X\big| \leq C_2|Y|.$$

The constant C_2 is determined by the metric and Lie structure of U, hence it can be chosen to be a constant that depends only on G_0 in light of Remark 3.4.

In consequence, if $X_0, X_1, \cdots, X_N \in \mathfrak{u}$ satisfy $|X_0| \leq 1$ and $\sum_{j=1}^N |X_j| \leq \frac{1}{C_2}$, then $\exp X_N \cdots \exp X_1 \exp X_0 = \exp X$ where

$$|X| \leq |X_0| + C_2 \sum_{j=1}^N |X_j|. \tag{7.3}$$

Choose $t_1 > 0$ such that

$$\frac{e^{-t_1\theta}}{1 - e^{-t_1\theta}} < \frac{1}{2C_2}. \tag{7.4}$$

t_1 is determined by G, G_0 and Γ as $C_2 = C_2(G_0)$ and $\theta = \frac{1}{2}\theta_0$ with $\theta_0 = \theta_0(G, , G_0, \Gamma)$ (see Lemma 6.5).

We now start with the equality

$$\begin{aligned} &(\nu_{N\tau} * \nu_{(N-1)\tau} * \cdots * \nu_\tau * \nu_0)(Q) \\ =& \int_{\substack{(u_N, \cdots, u_0) \in U^{N+1} \\ u_N u_{N-1} \cdots u_0 \in Q}} \prod_{j=0}^N \mathrm{d}\nu_{j\tau}(u_j) \\ =& \int_{(u_N, \cdots, u_1) \in U^N} \Big(\int_{\substack{u_0 \in U \\ u_N u_{N-1} \cdots u_0 \in Q}} \mathrm{d}\mathrm{m}_U|_B(u_0)\Big) \prod_{j=0}^N \mathrm{d}\nu_{j\tau}(u_j). \end{aligned} \tag{7.5}$$

By Proposition 3.1, for $t \geq 0$, $\chi(-tD) \leq -t\theta$ for all $\mathfrak{g}_0^\chi \subseteq \mathfrak{u}$. Moreover, the adjoint action of G_0 on $\mathfrak{u} \subseteq \mathfrak{g}_0$ is semisimple. Thus

$$\mathrm{Ad}_{\exp(-tD)}(B) = \exp(\mathrm{Ad}_{\exp(-tD)}(B_1^{\mathfrak{u}})) \subseteq \exp B_{e^{-t\theta}}^{\mathfrak{u}}. \tag{7.6}$$

If $(u_1, \cdots, u_N)$ is in the support of $\prod_{j=0}^N \mathrm{d}\nu_{j\tau}$, then as $\nu_{j\tau}$ is supported on $\mathrm{Ad}_{\exp(-j\tau D)}(B)$, $|\exp^{-1} u_j| \leq e^{-j\tau\theta}$. By (7.4), $\sum_{j=1}^N |\exp^{-1} u_j| < \frac{1}{2C_2}$.

For $Q \subseteq B_{\frac{1}{2}}$, by (7.3), every element u of the set

$$\exp(-u_1)\exp(-u_2)\cdots\exp(-u_N)Q$$

verifies $|\exp^{-1} u| \le \frac{1}{2} + C_2 \cdot \frac{1}{2C_2} = 1$ and thus belongs to B. Thus, by (7.5),

$$\begin{aligned}&(\nu_{N\tau} * \nu_{(N-1)\tau} * \cdots * \nu_\tau * \nu_0)(Q)\\ =&\int_{(u_N,\cdots,u_1)\in U^N} \mathrm{m}_U\Big(\exp(-u_1)\exp(-u_2)\cdots\exp(-u_N)Q\Big)\prod_{j=0}^N \mathrm{d}\nu_{j\tau}(u_j)\\ =&\int_{(u_N,\cdots,u_1)\in U^N} \mathrm{m}_U(Q)\prod_{j=0}^N \mathrm{d}\nu_{j\tau}(u_j)\\ =&\mathrm{m}_U(Q).\end{aligned}$$

The proof is completed. □

Hereafter, we fix $\tau = \tau(G, G_0, \Gamma)$ by making

$$\tau = \max(t_0, t_1). \tag{7.7}$$

Define, for $N \ge n \ge 0$ and $\bar{u} = (u_0, \cdots, u_{N-1}) \in B^N$,

$$\psi_n(\bar{u}) = (\mathrm{Ad}_{\exp(-(n-1)\tau D)} u_{n-1}) \cdots (\mathrm{Ad}_{\exp(-\tau D)} u_1) u_0, \tag{7.8}$$

with the convention that $\psi_0(\bar{u}) = e$. The construction (7.1) guarantees that

$$(\psi_n)_*(\mathrm{m}_U|_B)^N = \nu_{nq\tau} * \cdots * \nu_\tau * \nu_0. \tag{7.9}$$

Lemma 7.2. *For all $n \ge 0$ and $\bar{u} \in B^N$,*

$$\exp(n\tau D)\psi_n(\bar{u}) = \exp(\tau D)u_{n-1}\exp(\tau D)u_{n-2}\cdots\exp(\tau D)u_1\exp(\tau D)u_0.$$

Proof. This follows from direct computation. □

Lemma 7.3. *For all $N \ge n \ge 0$ and $\bar{u} \in B^N$,*

1. *$\psi_n(\bar{u}) \in B_{\frac{3}{2}}$;*
2. *For all $x \in G/\Gamma$,*
$$e^{-C_1(n\tau+\frac{3}{2})} f(x) \le f\big(\exp(n\tau D)\psi_n(\bar{u}).x\big) \le e^{C_1(n\tau+\frac{3}{2})} f(x),$$
3. *For all $x \in G/\Gamma$,*
$$\begin{aligned}e^{-\frac{3C_1}{2}} f\big(\exp(n\tau D)\psi_N(\bar{u}).x\big) &\le f\big(\exp(n\tau D)\psi_n(\bar{u}).x\big)\\ &\le e^{\frac{3C_1}{2}} f\big(\exp(n\tau D)\psi_N(\bar{u}).x\big).\end{aligned}$$

Here C_1 is as in Corollary 6.2.

Proof. (1) By (7.6), $\mathrm{Ad}_{\exp(-k\tau D)} u_k \in B_{e^{-k\tau}} = \exp B^{\mathfrak{u}}_{e^{-k\tau}}$. Then by (7.4), (7.3) and (7.7), for all n and $\bar{u}$,

$$|\exp^{-1}\psi_n(\bar{u})| \le 1 + C_2\sum_{k=1}^n e^{-k\tau\theta} \le 1 + C_2 \cdot \frac{1}{2C_2} = \frac{3}{2}.$$

(2) Part (2) is a direct consequence of Corollary 6.2 and part (1).
(3) By Lemma 7.2,

$$\begin{aligned}&\exp(n\tau D)\psi_N(\bar{u})\\ =&\exp(n\tau D)\big(\mathrm{Ad}_{\exp(-n\tau D)}\psi_N(u_n,\cdots,u_{N-1})\big)\psi_n(\bar{u})\\ =&\psi_N(u_n,\cdots,u_{N-1})\exp(n\tau D)\psi_n(\bar{u}),\end{aligned}$$

Part (3) is proved by applying Corollary 6.2 and part (1). □

Corollary 7.4. *For $x \in G/\Gamma$ and $q > 0$, if there exists $\bar{u}^* \in B^q$ such that $f\big(\exp(q\tau D)\psi_q(\bar{u}^*).x\big) \geq \frac{1}{e^{-\frac{1}{3}}-e^{-\frac{1}{2}}}e^{C_1(2q\tau+3)}b$, then*

$$\int_{B^q} f\big(\exp(q\tau D)\psi_q(\bar{u})\big)\mathrm{dm}_U^q(\bar{u}) \leq e^{-\frac{1}{3}q}f(x).$$

Proof. For $0 \leq k \leq q-1$ and $\bar{u} \in B^k$, by Lemma 7.3.(2),

$$\begin{aligned}&f\big(\exp(k\tau D)\psi_k(\bar{u}).x\big)\\ \geq& e^{C_1(k\tau+\frac{3}{2})}f(x) \geq e^{-C_1(k\tau+\frac{3}{2})}e^{-C_1(q\tau+\frac{3}{2})}f\big(\exp(q\tau)\psi_q(\bar{u}^*).x\big)\\ \geq& e^{-C_1(2q\tau+3)}f\big(\exp(q\tau)\psi_q(\bar{u}^*).x\big) \geq \frac{1}{e^{-\frac{1}{3}}-e^{-\frac{1}{2}}}b.\end{aligned}$$

Hence, by Lemma 7.2, (7.7) and Proposition 6.1,

$$\begin{aligned}&\int_{B^{k+1}} f\big(\exp((k+1)\tau D)\psi_k(\bar{u}).x\big)\mathrm{dm}_U^{k+1}(\bar{u})\\ =&\int_{B^k}\int_B f\big(\exp(\tau D)u_k\exp(k\tau D)\psi_k(\bar{u}).x\big)\mathrm{dm}_U(u_k)\mathrm{dm}_U^k(\bar{u})\\ \leq&\int_{B^k}\Big(e^{-\frac{1}{2}}f\big(\exp(k\tau D)\psi_k(\bar{u}).x\big)+b\Big)\mathrm{dm}_U^k(\bar{u})\\ \leq&\int_{B^k} e^{-\frac{1}{3}}f\big(\exp(k\tau D)\psi_k(\bar{u}).x\big)\mathrm{dm}_U^k(\bar{u})\\ =&e^{-\frac{1}{3}}\int_{B^k} f\big(\exp(k\tau D)\psi_k(\bar{u}).x\big)\mathrm{dm}_U^k(\bar{u}).\end{aligned}$$

The corollary is established by using this inequality q times. □

For $N \in \mathbb{N}$, write $[N] = \{1, \cdots, N\}$. For $x \in G/\Gamma$, $M > 0$, $N, q \in \mathbb{N}$, $\bar{u} \in B^{Nq}$, define

$$J_x(M,N,q,\tau,\bar{u}) = \{n \in [N] : f\big(\exp(nq\tau D)\psi_{nq}(\bar{u}).x\big) > M\}. \tag{7.10}$$

Given a subset $J \subseteq [N]$, we denote

$$Z_x(M,N,q,\tau,J) = \{\bar{u} \in B^{Nq} : J_x(M,N,q,\tau,\bar{u}) = J\}, \tag{7.11}$$

and try to estimate its size.

One can write $[N]$ as a disjoint union of non-empty segments $\bigsqcup_{l=1}^{L} I_l$, where $I_1, \cdots, I_L$ are listed in increasing order, such that either $J = \bigsqcup_{l\text{ odd}} I_l$ or $J = \bigsqcup_{l\text{ even}} I_l$. Then each I_l is contained either in J or in $[N] \setminus J$. Write $I_k = \{N_{k-1}+1, \cdots, N_k\}$.

Denote

$$E_n = \int_{Z_x(M,n,q,\tau,J\cap[n])} f\big(\exp(nq\tau D)\psi_{nq}(\bar{u}).x\big)\mathrm{dm}_U^{nq}(\bar{u}), \tag{7.12}$$

with the convention that

$$E_0 = f(x). \tag{7.13}$$

We now prove the following key claim:

Lemma 7.5. *If $M \geq \frac{1}{e^{-\frac{1}{3}}-e^{-\frac{1}{2}}}e^{C_1(2q\tau+3)}b$, then for $k \geq 1$,*

1. *If $I_k \subseteq J$, then $E_{N_k} \leq e^{-\frac{1}{3}(N_k-N_{k-1})q}E_{N_{k-1}}$;*
2. *If $I_k \subseteq [N] \setminus J$, then $E_{N_k} \leq M$. Suppose in addition that $k \geq 2$, then $E_{N_k} \leq E_{N_{k-1}}$.*

Proof. (1) We inductively bound E_n for $N_{k-1}+1 \le n \le N_k+1$.

For $N_{k-1}+1 \le n \le N_k$, $n \in I_k \subset J$. Remark that $Z_x(M,n,q,\tau,J\cap[n]) \subseteq Z_x(M,n-1,q,\tau,J\cap[n-1]) \times B^q$. Write

$$\begin{aligned}Y_x(M,n-1,q,\tau,J) =&\{\bar u \in Z_x(M,n-1,q,\tau,J\cap[n-1]) \\ &\text{s.t. } \exists \bar w^* \in B^q \text{ with } (\bar u,\bar w^*) \in Z_x(M,n,q,\tau,J\cap[n])\}.\end{aligned}$$

Then by Corollary 7.4,

$$\begin{aligned}
&E_n \\
\le& \int_{\substack{(\bar u,u_{nq-q},\cdots,u_{nq-1})\\ \in Z_x(M,n,q,\tau,J\cap[n])}} f\big(\exp(q\tau D)\psi_q(u_{nq-q},\cdots,u_{nq-1}) \\
&\qquad \exp(nq\tau D)\psi_{nq}(\bar u).x\big)\mathrm{dm}_U^q(u_{nq-q},\cdots,u_{nq-1})\mathrm{dm}_U^{(n-1)q}(\bar u) \\
\le& \int_{Y_x(M,n-1,q,\tau,J)}\int_{\bar w\in B^q} f\big(\exp(q\tau D)\psi_q(\bar w) \qquad (7.14)\\
&\qquad \exp((n-1)q\tau D)\psi_{(n-1)q}(\bar u).x\big)\mathrm{dm}_U^q(\bar w)\mathrm{dm}_U^{(n-1)q}(\bar u) \\
\le& \int_{Y_x(M,n-1,q,\tau,J)} e^{-\frac13 q} f\big(\exp((n-1)q\tau D)\psi_{(n-1)q}(\bar u).x\big)\mathrm{dm}_U^{(n-1)q}(\bar u) \\
\le& e^{-\frac13 q}E_{n-1}.
\end{aligned}$$

Here Corollary 7.4 applies because, as $(\bar u,\bar w^*) \in Z_x(M,n,q,\tau,J\cap[n])$ and $n\in J$,

$$\begin{aligned}
&f\big(\exp(q\tau D)\psi_q(\bar w^*)\exp((n-1)q\tau D)\psi_{(n-1)q}(\bar u).x\big) \\
=&f\big(\exp(nq\tau D)\psi_{nq}(\bar u,\bar w^*).x\big) > M \\
\ge& \frac{1}{e^{-\frac13}-e^{-\frac12}}e^{C_1(2q\tau+3)}b.
\end{aligned}$$

The inequality in part (1) follows by applying (7.14) repeatedly.

(2) Remark that for all $\bar u \in Z_x(M,N_k,q,\tau,J\cap[N_k])$,

$$f\big(\exp(N_kq\tau D)\psi_{N_kq}(\bar u).x\big) \le M.$$

This guarantees $E_{N_K} \le M$. And, if $k\ge 2$, then

$$f\big(\exp(N_{k-1}q\tau D)\psi_{N_{k-1}q}(\bar u).x\big) > M.$$

Furthermore,

$$Z_x(M,N_k,q,\tau,J\cap[N_k]) \subseteq Z_x(M,N_{k-1},q,\tau,J\cap[N_{k-1}]) \times B^{(N_k-N_{k-1})q}.$$

Therefore, as $\mathrm{m}_U(B)=1$,

$$\begin{aligned}
E_{N_k} =& \int_{Z_x(M,N_k,q,\tau,J\cap[N_k])} f\big(\exp(N_kq\tau D)\psi_{N_kq}(\bar u).x\big)\mathrm{dm}_U^{N_kq}(\bar u) \\
\le& \int_{Z_x(M,N_{k-1},q,\tau,J\cap[N_{k-1}])} f\big(\exp(N_kq\tau D)\psi_{N_kq}(\bar u).x\big)\mathrm{dm}_U^{N_{k-1}q}(\bar u) \\
\le& \int_{Z_x(M,N_{k-1},q,\tau,J\cap[N_{k-1}])} f\big(\exp(N_{k-1}q\tau D)\psi_{N_{k-1}q}(\bar u).x\big)\mathrm{dm}_U^{N_{k-1}q}(\bar u) \\
=& E_{N_{k-1}},
\end{aligned}$$

which proves part (2). □

Lemma 7.5 leads to:

Proposition 7.6. *Suppose G, G_0 and Γ are as in one of the conditions from Corollary 4.3. Then there exists $M_1 > 0$, determined by G, G_0, Γ, Ω_0 and q, such that for all $N \in \mathbb{N}$, $M \geq M_1$, $x \in G/\Gamma$ satisfying $f(x) \leq M$, and $J \subseteq [N]$*

$$\mathrm{m}_U^{Nq}\big(Z_x(M, N, q, \tau, J)\big) \leq e^{-\frac{1}{3}q|J|}.$$

Proof. Set $M_1 = \frac{1}{e^{-\frac{1}{3}} - e^{-\frac{1}{2}}} e^{C_1(2q\tau+3)} b$. Recall that b, C_1 and τ are all determined by G, G_0, Γ and Ω_0 (see Proposition 6.1, Corollary 6.2 and Lemma 6.5). Thus M_1 depends only on G, G_0, Γ and q. Assume $M \geq M_1$.

Suppose $I_1 \subseteq J$, then $E_1 \leq e^{-\frac{1}{3}q|I_1|} E_0 = e^{-\frac{1}{3}q|I_1|} f(x) \leq e^{-\frac{1}{3}q|I_1|} M$ by Lemma 7.5.(1). Suppose $I_1 \subset [N] \setminus J$, then by Lemma 7.5.(2), $E_1 \leq M$. In both cases, inductively applying Lemma 7.5 shows

$$E_{N_k} \leq e^{-\frac{1}{3}q \sum_{h \leq k, I_h \subseteq J} |I_h|} M. \tag{7.15}$$

Let $k = K$ or $K - 1$, depending on which one makes $I_k \subset J$. Then (7.15) writes

$$\int_{Z_x(M, N_k, q, \tau, J)} f\big(\exp(N_k q \tau D)\psi_{N_k q}(\bar{u}).x\big) \mathrm{d}\mathrm{m}_U^{N_k q}(\bar{u}) \leq e^{-\frac{1}{3}q|J|} M.$$

Since $N_k \in J$, $f\big(\exp(N_k q\tau D)\psi_{N_k q}(\bar{u}).x\big) \geq M$ for all $\bar{u} \in Z_x(M, N_k, q, \tau, J)$. Therefore,

$$\mathrm{m}_U^{N_k q}\big(Z_x(M, N_k, q, \tau, J)\big) \leq e^{-\frac{1}{3}q|J|}.$$

Finally, as $Z_x(M, N, q, \tau, J) \subseteq Z_x(M, N_k, q, \tau, J \cap [N_k]) \times B^{(N-N_k)q}$ and $\mathrm{m}_U(B) = 1$, the corollary follows. ☐

For a real number $\epsilon \in (0, 1)$, denote

$$Z_x(M, N, q, \tau, \epsilon) = \{\bar{u} \in B^N : |J_x(M, N, q, \tau, \bar{u})| \geq \epsilon N\}. \tag{7.16}$$

Corollary 7.7. *Suppose G, G_0 and Γ are as in one of the conditions from Corollary 4.3. For all $\epsilon \in (0, 1)$, there exist $M_1 = M_1(G, G_0, \Gamma, \Omega_0, \epsilon) > 0$ and $q = q(\epsilon) \in \mathbb{N}$, such that for all $N \in \mathbb{N}$, $M \geq M_1$, $x \in G/\Gamma$ satisfying $f(x) \leq M$, and $J \subseteq [N]$*

$$\mathrm{m}_U^{Nq}\big(Z_x(M, N, q, \tau, \epsilon)\big) \leq e^{-\frac{1}{6}q\epsilon N}.$$

Proof. $Z_x(M, N, q, \tau, \epsilon) = \bigcup_{\substack{J \subseteq [N] \\ |J| \geq \epsilon N}} Z_x(M, N, q, \tau, J)$. So by the proposition above, its measure is at most $2^N e^{-\frac{1}{3}q\epsilon N}$ where 2^N is the number of subsets in $[N]$. For $q \geq 6\epsilon^{-1} \log 2$, $2e^{-\frac{1}{3}q\epsilon} \leq e^{-\frac{1}{6}q\epsilon}$. The corollary follows. ☐

8. **Non-escape of mass for diagonal flows.** We produce the proof of Theorem 3.5, which in turn implies Theorem 1.1, in this section. For now we continue to assume that at least one of the conditions from Corollary 4.3 holds.

In addition to (7.10), (7.16), set

$$J'_x(M, N, q, \tau, \bar{u}) = \{n \in [N] : f\big(\exp(nq\tau D)\psi_{Nq}(\bar{u}).x\big) > M\}, \tag{8.1}$$

and

$$Z'_x(M, N, q, \tau, \epsilon) = \{\bar{u} \in B^N : |J'_x(M, N, q, \tau, \bar{u})| \geq \epsilon N\}. \tag{8.2}$$

The difference is that $\psi_{nq}(\bar{u})$ is replaced with $\psi_{Nq}(\bar{u})$ in this new definition.

We deduce from Lemma 7.3.(3) that

$$J'_x(M, N, q, \tau, \bar{u}) \subseteq J_x(e^{-\frac{3C_1}{2}} M, N, q, \tau, \bar{u}).$$

Hence

$$Z'_x(M, N, q, \tau, \bar{u}) \subseteq Z_x(e^{-\frac{3C_1}{2}} M, N, q, \tau, \bar{u}).$$

With $M_2 = e^{\frac{3C_1}{2}} M_1$, Corollary 7.7 implies

Corollary 8.1. *For all $\epsilon \in (0,1)$, there exist $M_2 = M_2(G, G_0, \Gamma, \Omega_0, \epsilon) > 0$ and $q = q(\epsilon) \in \mathbb{N}$, determined by G, G_0, A_0 and ϵ, such that for all $N \in \mathbb{N}$, $M \geq M_2$, $x \in G/\Gamma$ satisfying $x \leq M$, and $J \subseteq [N]$*

$$\mathrm{m}_U^{Nq}\big(Z'_x(M,N,q,\tau,\epsilon)\big) \leq e^{-\frac{1}{6}q\epsilon N}.$$

By switching from the discretized setting in Corollary 8.1 to a continuous flow, we are now able to prove Theorem 3.5.

Proof of Theorem 3.5. By Corollary 4.3, one may assume that either Γ is an arithmetic lattice or G has real rank 1, so that all the earlier discussions can be applied.

Similarly to (8.1) and (8.2), for $u \in U$, define

$$J^*_x(M,N,q,\tau,u) = \{n \in [N] : f\big(\exp(nq\tau D)u.x\big) > M\}, \tag{8.3}$$

and

$$Z^*_x(M,N,q,\tau,\epsilon) = \{u \in B_{\frac{1}{2}} : |J^*_x(M,N,q,\tau,u)| \geq \epsilon N\}. \tag{8.4}$$

Then $\psi_{Nq}^{-1} Z^*_x(M,N,q,\tau,\epsilon) \subseteq Z'_x(M,N,q,\tau,\epsilon)$.

Since $Z^*_x(M,N,q,\tau,\epsilon) \subseteq B_{\frac{1}{2}}$, by Lemma 7.1, Lemma 7.2, equality (7.9) and Corollary 8.1,

$$\begin{aligned}
&\mathrm{m}_U\big(Z^*_x(M,N,q,\tau,\epsilon)\big)\\
=&(\nu_{(Nq-1)\tau} * \cdots * \nu_\tau * \nu_0)\big(Z^*_x(M,N,q,\tau,\epsilon)\big)\\
=&\mathrm{m}_U^{Nq}\big(\psi_{Nq}^{-1} Z^*_x(M,N,q,\tau,\epsilon)\big)\\
\leq&\mathrm{m}_U^{Nq}\big(Z'_x(M,N,q,\tau,\epsilon)\big)\\
\leq&e^{-\frac{1}{6}q\epsilon N}
\end{aligned} \tag{8.5}$$

assuming $M \geq M_2$ and $f(x) \leq M$.

For $M, T, \epsilon > 0$, $u \in U$, denote

$$J^*_x(M,T,u) = \{t \in T : f\big(\exp(tD)u.x\big) > M\}, \tag{8.6}$$

and

$$Z^*_x(M,T,\epsilon) = \{u \in B_{\frac{1}{2}} : |J^*_x(M,T,u)| \geq \epsilon T\}. \tag{8.7}$$

By Lemma 6.2, we have

$$[t - q\tau, t + q\tau] \subseteq J^*_x(e^{-C_1 q\tau} M, T, u), \forall t \in J^*_x(M,T,u). \tag{8.8}$$

Given $T > 0$, let $N = \lfloor \frac{T}{q\tau} \rfloor$. Then for all $u \in Z^*_x(M,T,\epsilon)$, the number of the intervals among $\big((n-1)q\tau, nq\tau\big]$ who intersect $J^*_x(M,T,u)$ is at least ϵN. For these values of n, $n \in J^*_x(e^{-C_1 q\tau} M, N, q, \tau, u)$ by (8.8). This demonstrates that

$$Z^*_x(M,T,\epsilon) \subseteq Z^*_x(e^{-C_1 q\tau} M, N, q, \tau, \epsilon). \tag{8.9}$$

It now follows from (8.4) that, assuming $T \geq 2q\tau - \frac{12\tau}{\epsilon} \log \mathrm{m}_U(B_{\frac{1}{2}})$, $M \geq e^{C_1 q\tau} M_2$ and $f(x) \leq M$, then

$$\mathrm{m}_U\big(Z^*_x(M,T,\epsilon)\big) \leq e^{-\frac{1}{6}q\epsilon N} \leq e^{-\frac{1}{6}q\epsilon(\frac{T}{q\tau}-1)} \leq \mathrm{m}_U(B_{\frac{1}{2}}) e^{-\frac{1}{12\tau}\epsilon T}. \tag{8.10}$$

Under the hypothesis in Theorem 3.5, take

$$T^\# = 2q\tau - \frac{12\tau}{\epsilon} \log \mathrm{m}_U(B_{\frac{1}{2}}),$$

$$c = \frac{1}{12\tau},$$

$$\Omega^{\#} = f^{-1}\big([0, \max(e^{C_1 q}M_2, \sup_{x\in\Omega_0} f(x))]\big).$$

Recall that τ is determined by G, G_0 and Γ; q depends only on ϵ; M_2 is determined by G, G_0, Γ, Ω_0 and ϵ; and f is determined by G, G_0, A_0 and Γ. Also there are only finitely many possible values for $\mathrm{m}_U(B_{\frac{1}{2}})$ when D varies as a unit vector in $\mathfrak{a}_0$. Thus, $c = c(G, G_0, \Gamma)$, $T^{\#} = T^{\#}(G, G_0, \Gamma, \epsilon)$ and $\Omega^{\#} = \Omega^{\#}(G, G_0, A_0, \Gamma, \Omega_0, \epsilon)$.

Since the function f, defined by Proposition 6.1, is bounded on Ω_0, the threshold value $M^{\#} := \max(e^{C_1 q}M_2, \sup_{x\in\Omega_0} f(x))$ is finite. Moreover, as f is also proper, $\Omega^{\#}$ is compact.

Assume $x \in \Omega_0$, then $f(x) \leq M^{\#}$. As $M^{\#} \geq e^{C_1 q}M_2$, $\mathrm{m}_U\big(Z^*_x(M^{\#}, T, \epsilon)\big) \leq e^{-c\epsilon T}$ for all $T \geq T^{\#}$ by (8.10). Equivalently, for all $x \in \Omega_0$ and $T \geq T^{\#}$,

$$\begin{aligned} &\mathrm{m}_U\big(\{u \in B : \big|\{t \in [0,T] :\ \exp(tD)u.x \notin \Omega^{\#}\}\big| \geq \epsilon T\}\big) \\ <&\mathrm{m}_U(B_{\frac{1}{2}})e^{-c\epsilon T}. \end{aligned} \tag{8.11}$$

Now let $\Omega = \Omega^{\#} \cup \exp(\overline{B^{T^{\#}}_{\mathfrak{a}}})B.\Omega_0$. Then (3.20) holds for all $x \in \Omega_0$ and $T > 0$, where $\overline{B^{T^{\#}}_{\mathfrak{a}}}$ is the closed ball of radius $T^{\#}$ in $\mathfrak{a}_0$. Indeed, this follows from (8.11) when $T \geq T^{\#}$; and is automatically true when $T < T^{\#}$, as in this case $\exp(tD)u.x \in \exp(\overline{B^{T^{\#}}_{\mathfrak{a}}})B.\Omega_0$.

Finally, remark that B, $\Omega^{\#}$ and $T^{\#}$ are all determined by G, G_0, A_0, Ω_0, Γ and ϵ, and thus so is Ω. □

Therefore, due to Proposition 3.6, the proof of Theorem 1.1 is completed.

9. **Hausdorff dimension estimate.**

We now prove Theorem 1.3. To do so we will work first under the additional assumption that D belongs to a Cartan subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}_0$, and extend to the general case later.

Proof of Theorem 1.3 assuming $D \in \mathfrak{a}_0$. First of all, by passing to a commensurable lattice one may again assume $\Gamma = \mathbf{G}(\mathbb{Z})$.

The vector D can be assumed to be non-trivial, as otherwise $Z_{\epsilon,D}$ is empty and its Hausdorff dimension is 0. Furthermore, one may fix a norm on $\mathfrak{g}$ and assume without loss of generality that D is a unit vector with respect to it. This is because replacing D with the unit vector in its direction would not affect the definition of $Z_{\epsilon,D}$.

Let U be as in Proposition 3.1 and define the identity neighborhood B inside U as in (3.18). We may fix a transversal manifold $B^* \subset G$ of dimension $\dim G - \dim U$ such that:

1. $e \in B^*$;
2. the multiplication $(g,h) \to gh$ is a diffeomorphism from $B \times B^*$ to its image in G;
3. $\mathrm{m}_G(BB^*) = 1$.

As m_G is bi-invariant and $\mathrm{m}_U(B) = 1$, there is a probability measure μ on B^* such that $\mathrm{dm}_G(gh) = \mathrm{dm}_U(g)\mathrm{d}\mu(h)$. As there are only finitely many choices of U once G, G_0 and A_0 are given, the same can be made true for the choices of B, B^* and μ. In parcticular, BB^* is uniformly bounded given G, G_0 and A_0.

Fix a coordinate system in $\mathfrak{g}$, and let $\widehat{B}^r_{\mathfrak{g}}$ be the closed cube of diameter r centered at 0 in these coordinates. Because of the uniform boundedness of BB^*, the following claim is evident:

There exists $\kappa = \kappa(G, G_0, A_0) > 0$, such that for all $r \in (0,1)$ and $v \in \exp^{-1}(BB^*)$,

$$\exp(\widehat{B}_{\mathfrak{g}}^{\kappa r} + v) \subseteq B_G^r . \exp(v). \tag{9.1}$$

As G/Γ is covered by countably many precompact sets of the form $BB^*.x$, it suffices to prove that

$$\dim_H(BB^*.x_0 \cap Z_{\epsilon,D}) \leq \dim G - c\epsilon,\ \forall x_0 \in G/\Gamma. \tag{9.2}$$

Denote

$$Z_{x_0}(\Omega, D, T, \epsilon) := \{g \in BB^* : \left|\{t \in [0,T] :\ \exp(tD)g.x_0 \notin \Omega\}\right| \geq \epsilon T\}.$$

Fix x_0 and $\epsilon' \in (0, \epsilon)$, and let $\Omega_0 = \overline{BB^*.x_0}$. Note that D is not necessarily a unit vector in terms of the norm $|\cdot|$ induced by the Killing form of $\mathfrak{a}_0$. However, we can still apply Theorem 3.5 to $\frac{D}{|D|}$, and find a compact set $\Omega_1 = \Omega_1(G, G_0, \Gamma, A_0, \Omega_0, \epsilon')$ such that for all $x \in B^*x_0 \subseteq \Omega$ and $T > 0$,

$$\mathrm{m}_U\big(\{u \in B : \left|\{t \in [0,T] :\ \exp(\frac{tD}{|D|_{\mathfrak{a}_0}})u.x \notin \Omega_1\}\right| \geq \epsilon' T\}\big) < \mathrm{m}_U(B_{\frac{1}{2}})e^{-c\epsilon' T}.$$

This is equivalent to that

$$\mathrm{m}_U\big(\{u \in B : \left|\{t \in [0,T] :\ \exp(tD)u.x \notin \Omega_1\}\right| \geq \epsilon' T\}\big) < \mathrm{m}_U(B_{\frac{1}{2}})e^{-c\epsilon' T},$$

for all $x \in B^*x_0$ and $T > 0$. After integrating with respect to μ, we obtain

$$\mathrm{m}_G(Z_{x_0}(\Omega_1, D, T, \epsilon')) < \mathrm{m}_U(B_{\frac{1}{2}})e^{-c\epsilon' T}, \forall T > 0. \tag{9.3}$$

To make use of (9.3), we claim that for the set $\Omega = B_G^1.\Omega_1$ and some contant $C = C(G) > 0$,

$$B_G^{e^{-CT}}.Z_{x_0}(\Omega, D, T, \epsilon')) \subseteq Z_{x_0}(\Omega_1, D, T, \epsilon'), \forall T > 0. \tag{9.4}$$

Indeed, if $g \in Z_{x_0}(\Omega, D, T, \epsilon')$ and $h \in B_G^{e^{-CT}}.g$, then

$$\exp(tD)h \in \mathrm{Ad}_{\exp(tD)}(B_G^{e^{-CT}}). \exp(tD)h.$$

When $C = C(G, G_0, A_0)$ is chosen by $C := \max_{X \in \mathfrak{a}_0, |X|=1} \| \mathrm{ad}_X \|_{\mathrm{GL}(\mathfrak{g})}$,

$$\mathrm{Ad}_{\exp(tD)}(B_G^{e^{-CT}}) \subseteq B_G^1, \forall t \in [0,T].$$

So $\exp(tD)h \in B_G^1. \exp(tD)g$, and thus

$$\{t \in [0,T] :\ \exp(tD)h.x_0 \notin \Omega\} \subseteq \{t \in [0,T] :\ \exp(tD)g.x_0 \notin \Omega_1\},$$

which in turn implies (9.4).

We also remark that, if $T_k \to \infty$ and $|T_k' - T_k| \leq 1$, then μ_{T_k} and $\mu_{T_k'}$ have the same weak-* limit in Definition 1.2. So in that definition, one can assume without loss of generality that $T_k \in \mathbb{N}$ for all k. By Definition 1.2 and the remark earlier, for all $g \in BB^*$ such that $g.x_0 \in Z_{\epsilon,D}$, there are infinitely many $T_k \in \mathbb{N}$ such that $g \in Z_{x_0}(\Omega, D, T_k, \epsilon')$. In other words,

$$\{g \in BB^* : g.x_0 \in Z_{\epsilon,D}\} \subseteq \bigcap_{n\geq 1} \bigcup_{m \geq n} Z_{x_0}(\Omega, D, m, \epsilon'). \tag{9.5}$$

We now bound the Hausdorff dimension of (9.5). For every $r > 0$, choose n to be the smallest positive integer such that $\kappa e^{-Cn} < r$. For all $m \geq n$, cover $\exp^{-1}(BB^*) \subseteq \mathfrak{g}$ by translates $\{Y_{m,i}\}_i$ of $\widehat{B}_{\mathfrak{g}}^{\kappa e^{-Cm}}$ which overlap only along their boundaries. If $Y_{m,i}$ intersects $\exp^{-1}(Z_{x_0}(\Omega, D, m, \epsilon'))$, then by (9.1) and (9.4), it is completely contained in

$$\exp^{-1}(B_G^{e^{-Cm}}.Z_{x_0}(\Omega, D, m, \epsilon')) \subseteq \exp^{-1}(Z_{x_0}(\Omega_1, D, m, \epsilon')).$$

By (9.3), the number of such indices i is at most $O_{G,G_0,A_0}(e^{-c\epsilon' m} \cdot (e^{Cm})^{\dim G})$. So $\exp^{-1}(Z_{x_0}(\Omega, D, m, \epsilon'))$ can be covered by a subcollection $\mathcal{I}_m$ of $Y_{m,i}$'s, that satisfies

$$\sum_{i\in\mathcal{I}_m} |Y_{m,i}|^s \leq O_{G,G_0,A_0}(e^{c\epsilon' m} \cdot (e^{Cm})^{\dim G} \cdot (e^{-Cm})^s). \tag{9.6}$$

For all $s > \dim G - \frac{c\epsilon'}{C}$,

$$\sum_{m\geq n} \sum_{i\in\mathcal{I}_m} |Y_{m,i}|^s \leq O_{G,G_0,A_0}\big(e^{(C(\dim G - s) - c\epsilon')n}\big), \tag{9.7}$$

which tends to 0 as $\delta \to 0$, or equivalently, as $n \to \infty$. Because $\{Y_{m,i}\}_{\substack{m\geq n,\\ i\in\mathcal{I}_m}}$ is a covering of the set

$$\exp^{-1}\left(\{g \in BB^* : g.x_0 \in Z_{\epsilon,D}\}\right)$$

by sets of diameter at most $\kappa e^{-cn} < r$, the Hausdorff dimension of (9.5) satisfies

$$\dim_H\left(\{g \in BB^* : g.x_0 \in Z_{\epsilon,D}\}\right) \leq \dim G - \frac{c\epsilon'}{C}. \tag{9.8}$$

Since $\epsilon' \in (0, \epsilon)$ is arbitary, we deduce that

$$\dim_H(BB^*.x_0 \cap Z_{\epsilon,D}) \leq \dim G - \frac{c\epsilon}{C}. \tag{9.9}$$

To complete the proof of Theorem 3.5 in this special case, it now suffices to rewrite $\frac{c}{C}$ as c, after which c depends only on G, G_0, A_0 and Γ.

Finally, like in Lemma 6.5, the dependence of c on A_0 can be removed. In fact, if $D = g\widehat{D}g^{-1}$, then $\exp(tD).x = g^{-1}\exp(t\widehat{D}).gx$. Thus $x \in Z_{\epsilon,D}$ if and only if $gx \in Z_{\epsilon,\widehat{D}}$, and therefore $\dim_H Z_{\epsilon,D} = \dim_H Z_{\epsilon,\widehat{D}}$. Since one can conjugate A_0 to a prescribed Cartan subgroup $\widehat{A}_0$, and D to some $\widehat{D} \in \hat{\mathfrak{a}}_0$ by some element g, the choice of constant c can be made independent of A_0. □

Proof of Theorem 1.3, general case. Now let $D \in \mathfrak{g}_0$ be a semisimple element, which does not necessarily belong to a Cartan subalgebra of $\mathfrak{a}_0$.

Similar to Lemma 4.1, after taking the quotient of G by its center (so that a point $x \in G/\Gamma$ is sequentially ϵ-escaping on average if and only if its projection is), we may assume that $G = \mathbf{G}(\mathbb{R})^\circ \subseteq \mathrm{SL}_d(\mathbb{R})$ where $\mathbf{G}$ is a semisimple linear algebraic group.

In this case, it is known that G_0, being a simple Lie subgroup, must be the connected component $\mathbf{G}_0(\mathbb{R})^\circ$ of a simple linear algebraic group $\mathbf{G}_0 \subseteq \mathbf{SL}_d$ as well. The semisimple element D decomposes as $D_s + D_a$, where $D_s, D_a \in \mathfrak{g}_0(\mathbb{R})$ and commute with each other, $\exp D_s$ belongs to an $\mathbb{R}$-split torus $\mathbf{T}_s \subseteq \mathbf{G}_0$ and $\exp D_a$ belongs to an $\mathbb{R}$-anisotropic torus $\mathbf{T}_a \subseteq \mathbf{G}_0$. Then $\exp(tD) = \exp(tD_a)\exp(tD_s)$ for all t. Moreover, $\exp(tD_a)$ lies in the compact group $\mathbf{T}_a(\mathbb{R})$ for all t. It follows that the sets $Z_{\epsilon,D}$ and Z_{ϵ,D_s} are equal. So we may assume without loss of generality that D sits in an $\mathbb{R}$-split torus $\mathbf{T}_s \subseteq \mathbf{G}_0$. Under this assumption, $\mathbf{T}_s$ is contained in a maximal $\mathbb{R}$-split torus $\mathbf{T}$ in $\mathbf{G}_0$, so $\exp(D)$ belongs to $\mathbf{T}(\mathbb{R})$, which is a Cartan subgroup of G_0. This reduces to the previous case. □

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E-mail address: `hertz@math.psu.edu`
E-mail address: `zhirenw@psu.edu`