Polar Coded Repetition for Low-Capacity Channels

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***Abstract*—Constructing efficient low-rate error-correcting codes with low-complexity encoding and decoding have become increasingly important for applications involving ultra-low-power devices such as Internet-of-Things (IoT) networks. To this end, schemes based on concatenating the state-of-the-art codes at moderate rates with repetition codes have emerged as practical solutions deployed in various standards. In this paper, we propose a novel mechanism for concatenating outer polar codes with inner repetition codes which we refer to as *polar coded repetition*. More specifically, we propose to transmit a slightly modified polar codeword by deviating from Arıkan's standard 2×2 Kernel in a certain number of polarization recursions at each repetition block. We show how this modification can improve the asymptotic achievable rate of the polar-repetition scheme, while ensuring that the overall encoding and decoding complexity is kept almost the same. The achievable rate is analyzed for the binary erasure channels (BEC).**

I. Introduction

Recently, the Third Generation Partnership Project (3GPP) has introduced various features including Narrow-Band Internet of Things (NB-IoT) and enhanced Machine-Type Communications (eMTC) into the cellular standard in order to address the diverse requirements of massive IoT networks including low-power and wide-area (LPWA) cellular connectivity [4].

In general, devices in IoT networks have strict limitations on their total available power and are not equipped with advanced transceivers due to cost constraints. Consequently, they often need to operate at very low signal-to-noise ratio (SNR) necessitating ultra-low-rate error-correcting codes for reliable communications. For instance, the SNR of -13 dB is translated to capacity being 0.03 bits per transmission. The solution adopted in the 3GPP standard is to use the legacy turbo codes or convolutional codes at moderate rates, e.g., the turbo code of rate $1/3$, together with up to 2048 repetitions to support effective code rates as low as 1.6×10^{-4}. Although this repetition leads to efficient implementations with reduced computational complexity and latency, repeating a high-rate code to enable low-rate communication will result in rate loss and mediocre performance. As a result, studying efficient channel coding strategies for reliable communication in this low SNR regime, where channel coding is the only choice, is necessary [1].

The fundamental non-asymptotic laws for channel coding in the low-capacity regimes have been recently studied in [1]. Furthermore, the optimal number of repetitions with negligible rate loss, in terms of the code block length and the underlying channel capacity, is characterized in [1]. It is also shown in [1] that the state-of-the-art polar codes, proposed by Arıkan [2], naturally invoke this optimal number of repetitions when constructed for low-capacity channels. In another related work, low-rate codes for binary symmetric channels are constructed by concatenating high-rate i.e., rate close to 1, polar codes with repetitions [5].

In this paper, we propose an alternative mechanism called *coded repetition,* for the repetition concatenation scheme. A slightly modified codeword in each repetition block is transmitted instead of identical codewords in all repetition blocks. The goal is to reduce the rate loss due to the repetition while keeping the overall encoding and decoding complexity the same as in a standard repetition concatenation scheme. In particular, we consider polar codes as the outer code. In the proposed polar coded repetition scheme, a slightly modified polar codeword is transmitted in each repetition block by deviating from Arıkan's standard 2×2 Kernel in a certain number of polarization recursions at each repetition block. We show that our proposed scheme outperforms the straightforward polar-repetition scheme, in terms of the asymptotic achievable rate, for any given number of repetitions over the binary erasure channel (BEC). The proposed polar coded repetition has almost the same encoding and decoding complexity as the straightforward repetition scheme.

A. Background

Consider two copies of a binary discrete memoryless channel (B-DMC) $W : \mathcal{X} \rightarrow \mathcal{Y}$ with binary inputs $x_1,\ x_2 \in \mathcal{X}$ and outputs $y_1,\ y_2 \in \mathcal{Y}$. The transformation $G_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is applied on the inputs of these two channels and u_1 and u_2 are generated. Then, x_1 and x_2 are transmitted through the independent copies of W. At the decoder side, u_1 is decoded by using two observations y_1, y_2 and then u_2 is decoded by using the decoded sequence, $\hat{u}_1$, and the observations y_1, y_2. The transformation G_2 along with this successive decoding, referred to as *successive cancellation (SC)*, transforms the two copies of the channel W into two synthetic channels $W^0 : W \boxast W : \mathcal{X} \rightarrow \mathcal{Y}^2$ and $W^1 : W \circledast W : \mathcal{X} \rightarrow \mathcal{Y}^2 \times \mathcal{X}$ as follows:

$$
\begin{aligned}
W \boxast W(y_1, y_2|u_1) &= \sum_{u_2 \in \mathcal{X}} \frac{1}{2} W(y_1|u_1+u_2) W(y_2|u_2), \\
W \circledast W(y_1, y_2, u_1|u_2) &= \frac{1}{2} W(y_1|u_1+u_2) W(y_2|u_2).
\end{aligned} \tag{1}
$$

Here, the channel W^0 is *weaker* (i.e., less reliable) compared to W, while the channel W^1 is *stronger* (i.e., more reliable) compared to the channel W. The quality of a channel is measured by a reliability metric such as the Bhattacharyya parameter defined as

$$Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}, \tag{2}$$

which is equal to the erasure probability for BECs, i.e., for BEC(ϵ), $Z(W) = \epsilon$. The Bathacharyya parameters of the synthetic channels follow the properties

$$\begin{aligned} Z(W^1) &= Z(W)^2, \\ Z(W^0) &\leq 2Z(W) - Z(W)^2, \end{aligned} \tag{3}$$

with equality in (3) iff W is a BEC.

If we continue applying the transformation G_2 recursively m times, we will obtain $n = 2^m$ synthetic channels $\{W_m^{(i)}\}_{i \in \{0,1,...,n-1\}}$. More specifically, if we let $\{i_1, i_2, ..., i_m\}$ be the binary expansion of $i = \{0, 1, ..., n-1\}$ over m bits, where i_1 is the most significant bit and i_m is the least significant one, then we define the synthetic channels $\{W_m^{(i)}\}_{i \in \{0,...,n-1\}}$ as

$$W_m^{(i)} = (((W^{i_1})^{i_2})^{\cdots})^{i_m}. \tag{4}$$

Arıkan in his seminal paper, [2], showed that as $m \to \infty$, these 2^m synthetic channels are either purely noiseless or purely noisy channels. Thus, on the encoder side, using k entries of the input vector u_0^{n-1} as the information bits and setting the remaining entries to zero (frozen bits) will provide almost error-free communication. Hence, an $(n = 2^m, k)$ polar code is a linear block code generated by k rows of $G_n = G_2^{\otimes m}$, which correspond to the best k synthetic channels. Here, $.^{\otimes m}$ is the m-times Kronecker product of a matrix with itself.

Repetition coding is a simple way of designing a practical low-rate code. Let r denote the number of the repetitions and N, the length of the code. For constructing the repetition code, first, one needs to design a smaller outer code (e.g. polar codes) of length $n = N/r$ for channel W^r and then repeat each of its code bits r times. Consequently, the length of the final code will be $n \times r = N$. This is equivalent to transmitting an input bit over the r-repetition channel W^r and outputs an r tuple. For example, if W is BEC(ϵ), then its corresponding r-repetition channel is $W^r = BEC(\epsilon^r)$. The main advantage of this concatenation scheme is that the decoding complexity and latency is essentially reduced to that of the outer code making it appealing to low-power applications. This comes at the expense of loss in the asymptotic achievable rate especially if the number of the repetitions is large. Suppose that $C(W)$ is the capacity of the channel W and $NC(W)$ is the capacity corresponding to N channel transmissions. With repetition coding, since we transmit n times over the channel W^r, the capacity will be reduced to $nC(W^r)$. Note that, in general, we have $nC(W^r) \leq NC(W)$ and the ratio vanishes with growing r. Let's consider BEC(ϵ) as an example with $r = 2$. If $\epsilon = 0.5$, then $\frac{1}{2}C(W^2) = 0.375$ whereas $C(W) = 0.5$. However, when ϵ is close to 1, $C(W^2) = 1 - \epsilon^2$ is very close to $2C(W) = 2(1 - \epsilon)$.

II. Proposed Scheme

In this section, the proposed polar coded repetition scheme is discussed. It is shown how to improve the performance of the straightforward repetition scheme in the low-rate regime, while keeping the computational complexity and latency almost the same as the original one.

Consider an outer polar code with $r = 2^t$ repetitions and let c denote a polar codeword of length $n = 2^m$ designed for transmission over a channel W, r times. Owing to the recursive structure of the polar codes, one can write the polarization transform matrix as $G_n = G'_{r'} \otimes G_2^{\otimes(m-t')}$, where $G'_{r'}$ is an $r' \times r'$ binary matrix with $r' = 2^{t'}$. In our proposed scheme, we consider a different $G'_{r'}$ in each repetition block, while keeping $G_2^{\otimes(m-t')}$ the same in all of them. In other words, the first t' recursions of Arıkan's polarization transform are modified in each repetition while the rest of $m - t'$ recursions are kept the same. Note that if one chooses $r' = n$, i.e., the transmission in each block being different, then the channel capacity $C(W)$ can be achieved. However, we choose $r' = r$ to have a comparable complexity with the straightforward polar-repetition scheme. The complexity of the simple polar-repetition and the proposed modified polar-repetition schemes will be provided at the end of this section.

We illustrate the idea through some examples with two and four repetitions and constructed with *regular* and *irregular* polar coding approaches. Then, we generalize the regular scheme to accommodate an arbitrary repetition r.

A. Examples for two and four repetitions

In this subsection, we provide three examples for two and four repetitions as follows.

Example 1 (Two repetitions): Consider an outer polar code with two repetitions. Hence, the polar codeword c needs to be designed for $W^2 = W \circledast W$. The recursive structure of polar codes implies that codeword $c = (c_1 \oplus c_2, c_2)$ is constructed from the generator matrix $G_n = G'_2 \otimes G_2^{\otimes(m-1)}$, where $G'_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and c_1 and c_2 are polar codewords of length $n/2$ generated from $G_2^{\otimes(m-1)}$.

Now, we consider an alternative scheme where in each repetition, we transmit different combinations of c_1 and c_2 by choosing different G'_2 in each of them. Let ${G'}_2^{(i)}$ be a lower triangular matrix[1] ${G'}_2^{(i)} = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$, where $e \in \mathcal{F}_2$ and $i = \{1, 2\}$ is the index of the transmission (see TABLE. I for two possible matrices). There are three possible cases for two transmissions as follows.

Table I: Two possible matrices for two repetitions

Pattern no.	${G'}_2^{(i)}$
$P_2^{(0)}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$P_2^{(1)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

[1] [6] showed that the column permutations and the one-directional row operations can always transform a non-singular kernel ${G'}_2^{(i)}$ to a lower triangular kernel G'' with the same polarization behavior.

1) $G'^{(1)}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $G'^{(2)}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$: In this case, $(c_1 \oplus c_2, c_2)$ and $(c_1 \oplus c_2, c_2)$ are transmitted in each repetition. By considering both transmissions, one concludes that codeword c_1 is implicitly designed for the effective channel that the sub-block of length $n/2$ observes, i.e., for $W^2 \boxast W^2$ and c_2 is designed for $W^2 \circledast W^2$. As a result, the capacity per channel use per transmission for this case and specifically for BEC will be

$$\begin{aligned} C^{(1)}_2 &= (C(W^2 \boxast W^2) + C(W^2 \circledast W^2))/4 \\ &= (1 - \epsilon^2)/2. \end{aligned}$$

2) $G'^{(1)}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $G'^{(2)}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: For this case, $(c_1 \oplus c_2, c_2)$ and (c_1, c_2) are transmitted in the first and second repetitions. Codeword c_1 is designed for the effective channel that the sub-block of length $n/2$ observes, i.e., for $(W \boxast W^2) \circledast W$, and c_2 is designed for $W^2 \circledast W$. As a result, the capacity per channel use per transmission for this case is

$$\begin{aligned} C^{(2)}_2 &= (C((W \boxast W^2) \circledast W) + C(W^2 \circledast W))/4 \\ &= (2 - \epsilon^2 - 2\epsilon^3 + \epsilon^4)/4. \end{aligned}$$

3) $G'^{(1)}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $G'^{(2)}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: In the first and second repetitions, (c_1, c_2) and (c_1, c_2) are transmitted. Both Codewords c_1 and c_2 are designed for the effective channel that the sub-block of length $n/2$ observes, i.e., for W^2. As a result, the capacity for this case will be

$$\begin{aligned} C^{(3)}_2 &= (C(W^2) + C(W^2))/4 \\ &= (1 - \epsilon^2)/2. \end{aligned}$$

It can be observed that for $0 < \epsilon < 1$, the capacity of case 2 is larger than the capacities of both cases 1 and 3, which are simple repetition schemes. In other words,

$$C((W \boxast W^2) \circledast W) + C(W^3) > 2C(W^2), \qquad (5)$$

where the right hand side of (5) is the capacity for the straightforward repetition scheme and the left hand side of (5) is the capacity of case 2.

In the proposed modified approach, which we refer to as coded repetition scheme, we consider case 2. This modified scheme has the same encoding/decoding complexity as well as latency compared to a simple repetition scheme.

Example 2 (Four repetitions with regular polar codes): Consider an outer polar codes with four repetitions. Since we intend to keep the complexity of the proposed scheme the same as the complexity of the simple repetition one, let's consider all possible Kronecker products of the patterns $P^{(0)}_2$ and $P^{(1)}_2$ for $G'^{(i)}_{4R}$, $i = \{1, 2, 3, 4\}$ as the ones depicted in Table II. We call these patterns *regular* polar codes. Then, for four transmissions, we try all 35 multi-subsets of size 4 from the set $\{P^{(0)}_{4R}, P^{(1)}_{4R}, P^{(2)}_{4R}, P^{(3)}_{4R}\}$ to find the best one in terms of the capacity. The channel that each codeword c_i observes follows the recursive structure shown in Fig. 1. With a simple search among these 35 multi-subsets, it is found that the pattern $(P^{(0)}_{4R}, P^{(3)}_{4R}, P^{(3)}_{4R}, P^{(3)}_{4R})$ has the largest capacity.

Table II: All possible cases for four repetitions

Pattern no.	$G'^{(i)}_{4R}$
$P^{(0)}_{4R}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$P^{(1)}_{4R}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$P^{(2)}_{4R}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$P^{(3)}_{4R}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In this modified repetition scheme, $(c_1 \oplus c_2 \oplus c_3 \oplus c_4, c_2 \oplus c_4, c_3 \oplus c_4, c_4), (c_1, c_2, c_3, c_4), (c_1, c_2, c_3, c_4)$ and (c_1, c_2, c_3, c_4) are transmitted in the first, second, third and fourth transmissions, respectively. Codword c_1 is constructed for the effective channel that the first sub-block of length $n/4$ observes, i.e., for $W_1 = ((W \boxast W^2) \boxast (W \boxast W^2)^2) \circledast W^3$, c_2 for $W_2 = (W \boxast W^2)^2 \circledast W^3$, c_3 for $W_3 = (W^2 \boxast W^4) \circledast W^3$ and c_4 for $W_4 = W^4 \circledast W^3$. For BEC W, the capacity of the modified scheme is larger than that of the repetition scheme for $0 < \epsilon < 1$:

$$C_{4\mathrm{R}} = C(W_1) + C(W_2) + C(W_3) + C(W_4) > 4C(W^4). \quad (6)$$

Example 3 (Four repetitions with irregular polar codes[2]): We consider an alternative type of patterns for 4 repetitions, referred to as *irregular* polar codes, which have the same computational complexity as the simple repetition scheme. These 8 irregular patterns are constructed with $G'^{(i)}_{4I} = \begin{pmatrix} P^{(j)}_2 & 0 \\ P^{(j)}_2 & P^{(j)}_2 \end{pmatrix}$ and $G'^{(i)}_{4I} = \begin{pmatrix} P^{(j)}_2 & 0 \\ 0 & P^{(j)}_2 \end{pmatrix}$, where $j = \{0, 1\}$ and $i = \{1, 2, \ldots, 8\}$ (see Fig. 2).

With a simple search among all 330 multi-subsets of size 4 from the set $\{P^{(k)}_{4I}\}_{k=0}^{7}$, it is found that the pattern $(P^{(2)}_{4I}, P^{(5)}_{4I}, P^{(7)}_{4I}, P^{(7)}_{4I})$ has the largest capacity. The channel that each codeword c_i observes follows the recursive structure shown in Fig. 2. In this scheme, $(c_1 \oplus c_3 \oplus c_4, c_2 \oplus c_4, c_3 \oplus c_4, c_4)$, $(c_1 \oplus c_2, c_2, c_3, c_4)$, (c_1, c_2, c_3, c_4) and (c_1, c_2, c_3, c_4) are transmitted in the first, second, third and fourth transmissions, respectively. Codeword c_1 is constructed for the effective channel $W_1 = (W \boxast W^2) \circledast (W \boxast W^2) \circledast W \circledast W$, c_2 for $W_2 = (W \boxast W^2) \circledast W^2 \circledast W \circledast W$, c_3 for $W_3 = (W^2 \boxast W^4) \circledast W \circledast W \circledast W$ and c_4 for $W_4 = W^4 \circledast W \circledast W \circledast W$. For BEC W, the capacity of the modified scheme with irregular polar codes is larger than the one with regular polar codes for $0 < \epsilon < 1$. In other words,

$$C_{4\mathrm{I}} = C(W_1) + C(W_2) + C(W_3) + C(W_4) > C_{4\mathrm{R}}. \quad (7)$$

B. General case for regular polar codes

For the general case of $r = 2^t$ repetitions with regular polar codes, we consider all r possible t times Kronecker products of the patterns $P^{(0)}_2$ and $P^{(1)}_2$, as $P^{(i)}_r$, $i = 0, 1, \ldots, r-1$. In the proposed scheme, we use $P^{(0)}_r = (P^{(0)}_2)^{\otimes t}$ for the first transmission and $P^{(r-1)}_r = (P^{(1)}_2)^{\otimes t}$ for the rest $r - 1$ ones. For BEC W with an erasure probability ϵ, let's define $Z_{P^{(i)}_r}(W^{(k)}_r) \triangleq Z_{(i_1,\ldots,i_t)}(W^{(k)}_r)$ as the erasure probabilities of the channels that each codeword c_k, $k = \{1, 2, \ldots, r\}$ for pattern $P^{(i)}_r$ observes and $\{i_1, i_2, \ldots i_t\}$ as the t-bit binary

[2]Note that regular scheme is a special case of the irregular scheme.

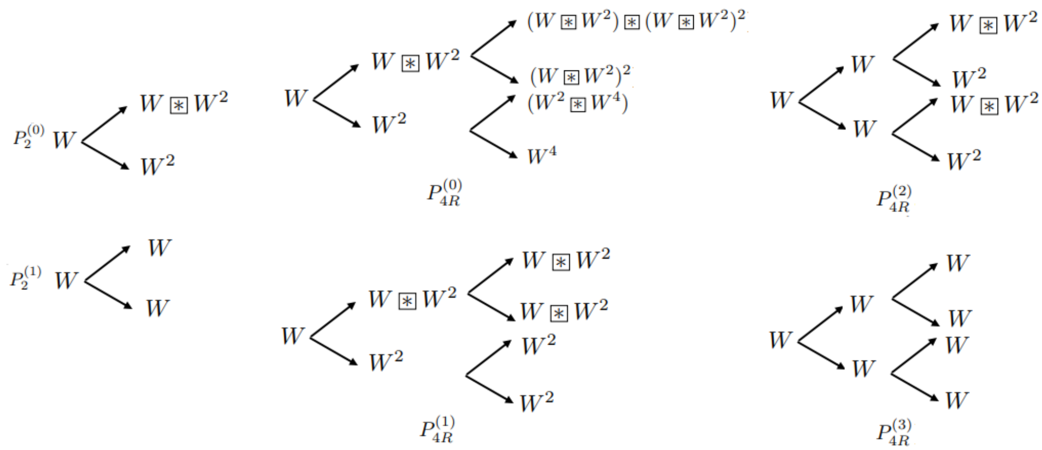

Figure 1: The recursive structure of the channels that each codeword c_i observes for two and four transmissions.

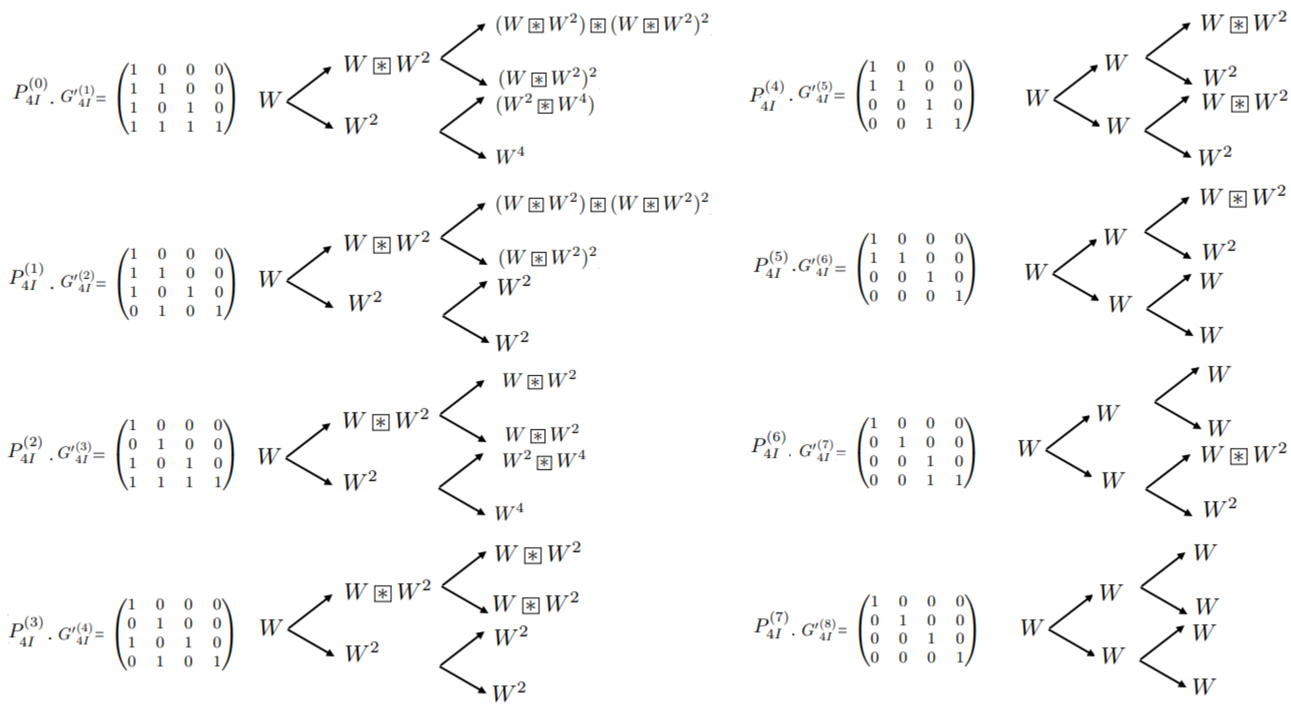

Figure 2: All 8 possible irregular kernels $G'^{(i)}_{4I}$ for 4 transmissions and the corresponding recursive structure of the channels that each codeword c_i observes.

expansion of i. Then, the recursive formula for computing $Z_{P_r^{(i)}}(W_r^{(k)})$ can be written as

$$\begin{aligned} Z_{(i_1,\ldots,i_t)}(W_r^{(2j-1)}) &= Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)})\times \\ &\quad [1+Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)})-Z^2_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)})]^{(1-i_t)}, \\ Z_{(i_1,\ldots,i_t)}(W_r^{(2j)}) &= Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)})\times \\ &\quad [Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)})]^{(1-i_t)}, \end{aligned} \tag{8}$$

where $Z(W_1^{(1)})=\epsilon$ and $j=1,2,\ldots,\frac{r}{2}$. Hence, the capacity for the proposed scheme will be

$$C_{rR} = \frac{r-\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) \times (Z_{P_r^{(r-1)}}(W_r^{(k)}))^{r-1}}{r^2}. \tag{9}$$

Since $Z_{P_r^{(r-1)}}(W_r^{(k)})=\epsilon$, for all $k=1,2,\ldots,r$, we will have

$$C_{rR} = \frac{r-\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) \times \epsilon^{r-1}}{r^2}. \tag{10}$$

Next, we show that $C_{rR} > \frac{C(W^r)}{r}$ for any r repetitions and

$0 < \epsilon < 1$. In other words,

$$\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) < r\epsilon. \quad (11)$$

To this end, we first prove that $\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) - r\epsilon$ has zeros at $\epsilon = 0$ and $\epsilon = 1$.

Theorem 1. *$Z_{P_r^{(0)}}(W_r^{(k)}) = 0$ at $\epsilon = 0$ and $Z_{P_r^{(0)}}(W_r^{(k)}) = 1$ at $\epsilon = 1$ for all $k = \{1, 2, \ldots, r\}$.*

Proof. Let us write the recursive formula for erasure probability as $Z_{P_r^{(0)}}(W_r^{(k)}) = f_{k_1}(f_{k_2}(\ldots f_{k_t}(\epsilon)))$, where $k_i = \{0, 1\}$, $i = \{1, 2, \ldots, t\}$ and $f_0(a) = a + a^2 - a^3$, $f_1(a) = a^2$, $\forall k = \{1, 2, \ldots, r\}$.

Since $f_{k_i}(a)|_{a=1} = 1$ and $f_{k_i}(a)|_{a=0} = 0$, by using recursion, we conclude $Z_{P_r^{(0)}}(W_r^{(k)}) = 1$ at $\epsilon = 1$ and $Z_{P_r^{(0)}}(W_r^{(k)}) = 0$ at $\epsilon = 0$ $\forall k = \{1, 2, \ldots r\}$. ■

Then, one can use Sturm algorithm[3] [7] to show that $\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) - r\epsilon$ does not have any root in $\epsilon = (0, 1)$. Finally, one can choose an ϵ in the interval $(0, 1)$ and compare the values of $\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)})$ and $r\epsilon$ at that point to see that the capacity of proposed modified scheme is greater than the repetition one for r number of repetitions. Fig. (3) shows the left and the right sides of eq. (11) for $r = 4$.

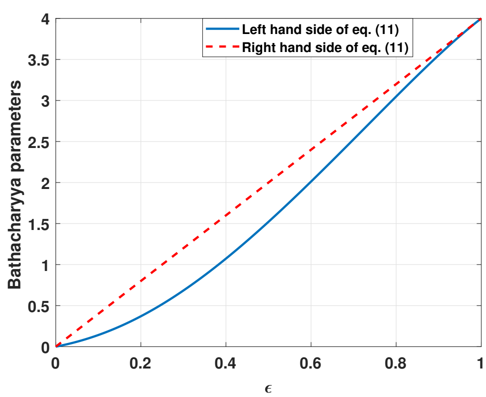

Figure 3: Comparison between the left and the right hand sides of eq. (11).

Note that the decoding complexity of the simple polar-repetition scheme and the proposed modified polar-repetition are $O(nr + n\log n)$ and $O(nr + n\log n + n(r-1)\log r)$, respectively, [9].

III. Numerical Results

In this section, we provide numerical results for the capacity of the proposed polar coded repetition scheme for different numbers of repetitions over BEC and compare them with the capacity of the simple repetition scheme and the Shannon bound. Fig. (4) illustrates the capacities of the proposed schemes for 2, 4 and 8 repetitions. It can be observed that the proposed scheme outperforms the simple repetition scheme for all of these repetitions. The irregular scheme also slightly outperforms the regular one for 4 repetitions. On the other hand, as the number of repetitions increases, the gap to the Shannon bound increases as expected.

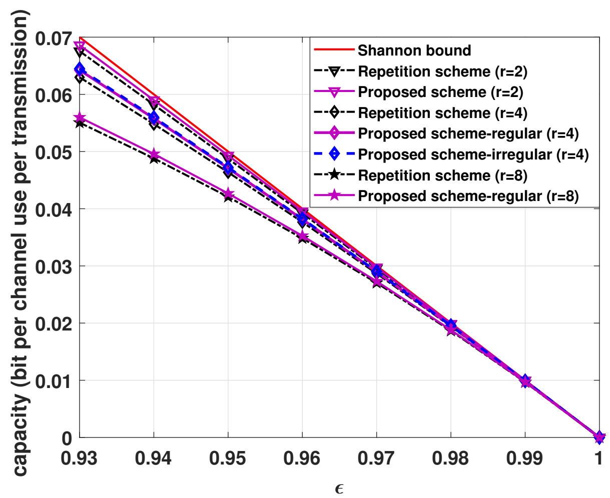

Figure 4: Capacity of the proposed scheme compared with the capacity of the repetition scheme for $r = 2, 4, 8$.

IV. Conclusion

In this paper, we proposed a modified approach for the repetition scheme. In this scheme, we used polar codes as the outer code and proposed to transmit slightly modified codeword in each repetition. We showed that the proposed scheme outperforms the simple repetition scheme, in terms of the asymptotic achievable rate, over BEC while it keeps the decoding complexity almost the same as the repetition scheme.

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[3] Although Sturm's theorem is a complete solution for finding the number of the real roots of the polynomials, when the degree of the polynomial increases, it isn't efficient in terms of implementation. The algorithm proposed in [8] is more efficient for higher degrees.