

Bounded geometry and p -harmonic functions under uniformization and hyperbolization

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Dedicated to Pekka Koskela and the late Juha Heinonen on their 59th birthdays.

Abstract. The uniformization and hyperbolization transformations formulated by Bonk, Heinonen and Koskela in “*Uniformizing Gromov Hyperbolic Spaces*”, Astérisque **270** (2001), dealt with geometric properties of metric spaces. In this paper we consider metric measure spaces and construct a parallel transformation of measures under the uniformization and hyperbolization procedures. We show that if a locally compact roughly starlike Gromov hyperbolic space is equipped with a measure that is uniformly locally doubling and supports a uniformly local p -Poincaré inequality, then the transformed measure is globally doubling and supports a global p -Poincaré inequality on the corresponding uniformized space. In the opposite direction, we show that such global properties on bounded locally compact uniform spaces yield similar uniformly local properties for the transformed measures on the corresponding hyperbolized spaces.

We use the above results on uniformization of measures to characterize when a Gromov hyperbolic space, equipped with a uniformly locally doubling measure supporting a uniformly local p -Poincaré inequality, carries nonconstant globally defined p -harmonic functions with finite p -energy.

We also study some geometric properties of Gromov hyperbolic and uniform spaces. While the Cartesian product of two Gromov hyperbolic spaces need not be Gromov hyperbolic, we construct an indirect product of such spaces that does result in a Gromov hyperbolic space. This is done by first showing that the Cartesian product of two bounded uniform domains is a uniform domain.

Key words and phrases: bounded geometry, doubling measure, finite-energy Liouville theorem, Gromov hyperbolic space, hyperbolization, metric space, p -harmonic function, Poincaré inequality, quasihyperbolic metric, uniform space, uniformization.

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1. Introduction

Studies of metric space geometry usually consider two types of synthetic (i.e. axiomatic) negative curvature conditions: Alexandrov curvature (known as CAT(−1) spaces) and Gromov hyperbolicity. While the Alexandrov condition governs both small and large scale behavior of triangles, the Gromov hyperbolicity governs only the large scale behavior. As such, the Gromov hyperbolicity was eminently suited to the study of hyperbolic groups, see e.g. Gromov [24], Coornaert–Delzant–Papadopoulos [20] and Ghys–de la Harpe [23], while Bridson–Haefliger [17] gives an excellent overview of both notions of curvature.

Since the ground-breaking work of Gromov [24], the notion of Gromov hyperbolicity has found applications in other parts of metric space analysis as well. In [14, Theorem 1.1], Bonk, Heinonen and Koskela gave a link between quasiisometry classes of locally compact roughly starlike Gromov hyperbolic spaces and quasimilarity classes of locally compact bounded uniform spaces. In Buyalo–Schroeder [19] it was shown that every complete bounded doubling metric space is the visual boundary of a Gromov hyperbolic space, see also Bonk–Saksman [15].

While none of the above mentioned studies, involving Gromov hyperbolic spaces and uniform domains, considered how measures transform on such spaces (see also e.g. Buckley–Herron–Xie [18] and Herron–Shanmugalingam–Xie [31]), analytic studies on metric spaces require measures as well. Although [15] does consider function spaces on certain Gromov hyperbolic spaces, called hyperbolic fillings, these function spaces are associated with just the counting measure on the vertices of such hyperbolic fillings and so do not lend themselves to more general Gromov hyperbolic spaces. Similar studies were undertaken in Bonk–Saksman–Soto [16] and Björn–Björn–Gill–Shanmugalingam [8].

In this paper we seek to remedy this gap in the literature on analysis in Gromov hyperbolic spaces. Thus the primary focus of this paper is to construct transformations of measures under the uniformization and hyperbolization procedures, and to demonstrate how analytic properties of the measure are preserved by them. This does not seem to have been considered elsewhere. The analytic properties of interest here are the doubling property and the Poincaré inequality, assumed either globally on the uniform spaces, or uniformly locally (i.e. for balls up to some fixed radius) on the Gromov hyperbolic spaces. As trees are the quintessential models of Gromov hyperbolic spaces, the results in this paper are motivated in part by the results in [8].

The following is our main result, combining Theorems 4.9 and 6.2. Here, $z_0 \in X$ is a fixed uniformization center and $\varepsilon_0(\delta)$ is as in Bonk–Heinonen–Koskela [14], see later sections for relevant definitions.

Theorem 1.1. *Assume that (X, d) is a locally compact roughly starlike Gromov δ -hyperbolic space equipped with a measure μ which is doubling on X for balls of radii at most R_0 , with a doubling constant C_d . Let $X_\varepsilon = (X, d_\varepsilon)$ be the uniformization of X given for $0 < \varepsilon \leq \varepsilon_0(\delta)$ by*

$$d_\varepsilon(x, y) = \inf_{\gamma} \int_{\gamma} e^{-\varepsilon d(\cdot, z_0)} ds,$$

with the infimum taken over all rectifiable curves γ in X joining x to y . Also let

$$\beta > \frac{17 \log C_d}{3R_0} \quad \text{and} \quad d\mu_\beta = e^{-\beta d(\cdot, z_0)} d\mu.$$

Then the following are true:

- (a) μ_β is globally doubling both on X_ε and its completion \overline{X}_ε .

- (b) If μ supports a p -Poincaré inequality for balls of radii at most R_0 , then μ_β supports a global p -Poincaré inequality both on X_ε and \bar{X}_ε .

Along the way, we also show that if the assumptions hold with some value of R_0 then they hold for any value of R_0 at the cost of enlarging C_d , see Proposition 3.2 and Theorem 5.3.

We also obtain the following corresponding result for the hyperbolization procedure, see Propositions 7.3 and 7.4.

Theorem 1.2. *Let (Ω, d) be a locally compact bounded uniform space, equipped with a globally doubling measure μ . Let k be the quasihyperbolic metric on Ω , given by*

$$k(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds}{d_{\Omega}(\cdot)}, \quad (1.1)$$

where $d_{\Omega}(x) = \text{dist}(x, \partial\Omega)$ and the infimum is taken over all rectifiable curves γ in Ω connecting x to y . For $\alpha > 0$ we equip the corresponding Gromov hyperbolic space (Ω, k) with the measure μ^α given by $d\mu^\alpha = d_{\Omega}(\cdot)^{-\alpha} d\mu$. Let $R_0 > 0$.

Then the following are true:

- (a) μ^α is doubling on (Ω, k) for balls of radii at most R_0 .
- (b) If μ supports a global p -Poincaré inequality, then μ^α supports a p -Poincaré inequality for balls of radii at most R_0 .

We use Theorem 1.1 to study potential theory on locally compact roughly starlike Gromov hyperbolic spaces, equipped with a locally uniformly doubling measure supporting a uniformly local Poincaré inequality. In particular, we characterize when the finite-energy Liouville theorem holds on such spaces, i.e. when there exist no nonconstant globally defined p -harmonic functions with finite p -energy. The characterization is given in terms of the nonexistence of two disjoint compact sets of positive p -capacity in the boundary of the uniformized space, see Theorem 10.5. This characterization complements our results in [12].

As already mentioned, an in-depth study of locally compact roughly starlike Gromov hyperbolic spaces, as well as links between them and bounded locally compact uniform domains, was undertaken in the seminal work Bonk–Heinonen–Koskela [14]. They showed [14, the discussion before Proposition 4.5] that the operations of uniformization and hyperbolization are mutually opposite:

- A uniformization followed by a hyperbolization takes a given locally compact roughly starlike Gromov hyperbolic space X to a roughly starlike Gromov hyperbolic space which is biLipschitz equivalent to X , see [14, Proposition 4.37]. (Note that in [14] “quasiisometric” means biLipschitz.)
- A hyperbolization of a bounded locally compact uniform space Ω , followed by a uniformization, returns a bounded uniform space which is quasisimilar to Ω , see [14, Proposition 4.28].

Here, a homeomorphism $\Phi : X \rightarrow Y$ between two noncomplete metric spaces is *quasisimilar* if it is C_x -biLipschitz on every ball $B(x, c_0 \text{dist}(x, \partial X))$, for some $0 < c_0 < 1$ independent of x , and there exists a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that for each distinct triple of points $x, y, z \in X$,

$$\frac{d_Y(\Phi(x), \Phi(y))}{d_Y(\Phi(x), \Phi(z))} \leq \eta \left(\frac{d_X(x, y)}{d_X(x, z)} \right).$$

It was also shown in [14, Theorem 4.36] that two roughly starlike Gromov hyperbolic spaces are biLipschitz equivalent if and only if any two of their uniformizations are quasisimilar.

We continue the study of Gromov hyperbolic spaces in this spirit by considering pairs of Gromov hyperbolic spaces in Section 8. Note that the Cartesian product of two Gromov hyperbolic spaces need not be Gromov hyperbolic, as demonstrated by $\mathbf{R} \times \mathbf{R}$, which is *not* a Gromov hyperbolic space even though \mathbf{R} is. On the other hand, in Section 8 we obtain the following result, see Proposition 8.3 for a more precise result.

Proposition 1.3. *Let (Ω, d) and (Ω', d') be two bounded uniform spaces. Then $\Omega \times \Omega'$ is a bounded uniform space with respect to the metric*

$$\tilde{d}((x, x'), (y, y')) = d(x, y) + d'(x', y').$$

We use this, together with the results of [14], to construct an indirect product $X \times_{\varepsilon} Y$ of two Gromov hyperbolic spaces which is also Gromov hyperbolic, see Section 8. In this section we also study properties of such product hyperbolic spaces. For a fixed Gromov δ -hyperbolic space X , there is a whole family of uniformizations X_{ε} , one for each $0 < \varepsilon \leq \varepsilon_0(\delta)$. As mentioned above, X_{ε} is quasisimilar to $X_{\varepsilon'}$ when $0 < \varepsilon, \varepsilon' \leq \varepsilon_0(\delta)$.

On the other hand, we show in Proposition 8.5 that the canonical identity mapping between $X \times_{\varepsilon} Y$ and $X \times_{\varepsilon'} Y$ is never biLipschitz if $\varepsilon \neq \varepsilon'$, and it is even possible that the two indirect products are not even quasiisometric. Here, a map $\Phi : Z \rightarrow W$ is a *quasiisometry* (also called, perhaps more accurately, rough quasiisometry as in [14] and [8]) if there are $C > 0$ and $L \geq 1$ such that the C -neighborhood of $\Phi(Z)$ contains W and for all $z, z' \in Z$,

$$\frac{d(z, z')}{L} - C \leq d(\Phi(z), \Phi(z')) \leq Ld(z, z') + C.$$

It is not difficult to show that visual boundaries of quasiisometric locally compact roughly starlike Gromov hyperbolic spaces are quasisymmetric, see e.g. Bridson–Haefliger [17, Theorem 3.22]. We take advantage of this to show the quasiisometric nonequivalence of two indirect products of the hyperbolic disk and \mathbf{R} , see Example 8.7.

The broad organization of the paper is as follows. Background definitions and preliminary results are given in Sections 2 and 3, while the definition of Poincaré inequalities is given in Section 5. The main aims in Sections 4 and 6 are to deduce parts (a) and (b), respectively, of Theorem 1.1. The dual transformation of hyperbolization, via the quasihyperbolic metric (1.1), is discussed in Section 7, where also Theorem 1.2 is shown. The above sections fulfill the main goal of this paper, and form a basis for comparing the potential theories on Gromov hyperbolic spaces and on uniform spaces.

The remaining sections are devoted to applications of the results obtained in the preceding sections. In Section 8 we construct and study the indirect product, providing a family of new Gromov hyperbolic spaces from a pair of Gromov hyperbolic spaces. The subsequent sections are devoted to the impact of uniformization and hyperbolization procedures on nonlinear potential theory. In Section 9 we discuss Newton–Sobolev spaces and p -harmonic functions, and then in Section 10 we show that under certain natural conditions, the class of p -harmonic functions is preserved under the uniformization and hyperbolization procedures. In this final section, we also characterize which Gromov hyperbolic spaces with bounded geometry support the finite-energy Liouville theorem for p -harmonic functions.

In the beginning of each section, we list the standing assumptions for that section in italicized text; in Sections 2 and 4 these assumptions are given a little later.

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2. Gromov hyperbolic spaces

A *curve* is a continuous mapping from an interval. Unless stated otherwise, we will only consider curves which are defined on compact intervals. We denote the length of a curve γ by $l_\gamma = l(\gamma)$, and a curve is *rectifiable* if it has finite length. Rectifiable curves can be parametrized by arc length ds .

A metric space $X = (X, d)$ is *L -quasiconvex* if for each $x, y \in X$ there is a curve γ with end points x and y and length $l_\gamma \leq Ld(x, y)$. X is a *geodesic space* if it is 1-quasiconvex, and γ is then a *geodesic* from x to y . We will consider a related metric, called the *inner metric*, given by

$$d_{\text{in}}(x, y) := \inf_{\gamma} l_{\gamma} \quad \text{for all } x, y \in X, \quad (2.1)$$

where the infimum is taken over all curves γ from x to y . If (X, d) is quasiconvex, then d and d_{in} are biLipschitz equivalent metrics on X . The space X is a *length space* if $d_{\text{in}}(x, y) = d(x, y)$ for all $x, y \in X$. By Lemma 4.43 in [5], arc length is the same with respect to d and d_{in} , and thus (X, d_{in}) is a length space. A metric space is *proper* if all closed bounded sets are compact. A proper length space is necessarily a geodesic space, by Ascoli's theorem or the Hopf–Rinow theorem below. To avoid pathological situations, all metric spaces in this paper are assumed to contain at least two points.

We denote balls in X by $B(x, r) = \{y \in X : d(y, x) < r\}$ and the scaled concentric ball by $\lambda B(x, r) = B(x, \lambda r)$. In metric spaces it can happen that balls with different centers and/or radii denote the same set. We will however adopt the convention that a ball comes with a predetermined center and radius. Similarly, when we say that $x \in \gamma$ we mean that $x = \gamma(t)$ for some t . If γ is noninjective, this t may not be unique, but we are always implicitly referring to a specific such t .

Theorem 2.1. (Hopf–Rinow theorem) *If X is a complete locally compact length space, then it is proper and geodesic.*

This version is a generalization of the original theorem, see e.g. Gromov [25, p. 9] for a proof.

Definition 2.2. A complete unbounded geodesic metric space X is *Gromov hyperbolic* if there is a *hyperbolicity constant* $\delta \geq 0$ such that whenever $[x, y]$, $[y, z]$ and $[z, x]$ are geodesics in X , every point $w \in [x, y]$ lies within a distance δ of $[y, z] \cup [z, x]$.

The ideal Gromov hyperbolic space is a metric tree, which is Gromov hyperbolic with $\delta = 0$. A *metric tree* is a tree where each edge is considered to be a geodesic of unit length.

Definition 2.3. An unbounded metric space X is *roughly starlike* if there are some $z_0 \in X$ and $M > 0$ such that whenever $x \in X$ there is a geodesic ray γ in X , starting from z_0 , such that $\text{dist}(x, \gamma) \leq M$. A *geodesic ray* is a curve $\gamma : [0, \infty) \rightarrow X$ with infinite length such that $\gamma|_{[0, t]}$ is a geodesic for each $t > 0$.

If X is a roughly starlike Gromov hyperbolic space, then the roughly starlike condition holds for every choice of z_0 , although M may change.

Definition 2.4. A nonempty open set $\Omega \subsetneq X$ in a metric space X is an A -uniform domain, with $A \geq 1$, if for every pair $x, y \in \Omega$ there is a rectifiable arc length parametrized curve $\gamma : [0, l_\gamma] \rightarrow \Omega$ with $\gamma(0) = x$ and $\gamma(l_\gamma) = y$ such that $l_\gamma \leq Ad(x, y)$ and

$$d_\Omega(\gamma(t)) \geq \frac{1}{A} \min\{t, l_\gamma - t\} \quad \text{for } 0 \leq t \leq l_\gamma,$$

where

$$d_\Omega(z) = \text{dist}(z, X \setminus \Omega), \quad z \in \Omega.$$

The curve γ is said to be an A -uniform curve. A noncomplete metric space (Ω, d) is A -uniform if it is an A -uniform domain in its completion.

A ball $B(x, r)$ in a uniform space Ω is a *subWhitney ball* if $r \leq c_0 d_\Omega(x)$, where $0 < c_0 < 1$ is a predetermined constant. We will primarily use $c_0 = \frac{1}{2}$.

The completion of a locally compact uniform space is always proper, by Proposition 2.20 in Bonk–Heinonen–Koskela [14]. Unlike the definition used in [14], we do not require uniform spaces to be locally compact.

It follows directly from the definition that an A -uniform space is A -quasiconvex. One might ask if the uniformity assumption in Proposition 2.20 in [14] can be replaced by a quasiconvexity assumption, i.e. if the completion of a locally compact quasiconvex space is always proper, however the following example shows that this can fail even if the original space is geodesic. Thus the uniformity assumption in Proposition 2.20 in [14] is really crucial.

Example 2.5. Let

$$X = \left\{ \{x_j\}_{j=1}^\infty : \sum_{j=1}^\infty |x_j| \leq 1, 0 < x_1 \leq 1, \text{ and } x_n = 0 \text{ if } x_1 > \frac{1}{n}, n = 2, 3, \dots \right\},$$

equipped with the ℓ^1 -metric. Then X is a bounded locally compact geodesic space which is not totally bounded, and thus has a nonproper completion.

We assume from now on that X is a locally compact roughly starlike Gromov δ -hyperbolic space. We also fix a point $z_0 \in X$ and let M be the constant in the roughly starlike condition with respect to z_0 .

By the Hopf–Rinow Theorem 2.1, X is proper. The point z_0 will serve as a center for the uniformization X_ε of X . Following Bonk–Heinonen–Koskela [14], we define

$$(x|y)_{z_0} := \frac{1}{2}[d(x, z_0) + d(y, z_0) - d(x, y)], \quad x, y \in X,$$

and, for a fixed $\varepsilon > 0$, the *uniformized metric* d_ε on X as

$$d_\varepsilon(x, y) = \inf_\gamma \int_\gamma \rho_\varepsilon ds, \quad \text{where } \rho_\varepsilon(x) = e^{-\varepsilon d(x, z_0)}$$

and the infimum is taken over all rectifiable curves γ in X joining x to y . Note that if γ is a compact curve in X , then ρ_ε is bounded from above and away from 0 on γ , and in particular γ is rectifiable with respect to d_ε if and only if it is rectifiable with respect to d .

The set X , equipped with the metric d_ε , is denoted by X_ε . We let \overline{X}_ε be the completion of X_ε , and let $\partial_\varepsilon X = \overline{X}_\varepsilon \setminus X_\varepsilon$. When writing e.g. B_ε , diam_ε and dist_ε the ε indicates that these notions are taken with respect to $(\overline{X}_\varepsilon, d_\varepsilon)$. The length of

the curve γ with respect to d_ε is denoted by $l_\varepsilon(\gamma)$, and arc length ds_ε with respect to d_ε satisfies

$$ds_\varepsilon = \rho_\varepsilon ds.$$

It follows that X_ε is a length space, and thus also \bar{X}_ε is a length space. By a direct calculation (or [14, (4.3)]), $\text{diam}_\varepsilon \bar{X}_\varepsilon = \text{diam}_\varepsilon X_\varepsilon \leq 2/\varepsilon$. Note that as a set, $\partial_\varepsilon X$ is independent of ε and depends only on the Gromov hyperbolic structure of X , see e.g. [14, Section 3]. The notation adopted in [14] is $\partial_G X$.

The following important theorem is due to Bonk–Heinonen–Koskela [14].

Theorem 2.6. *There is a constant $\varepsilon_0 = \varepsilon_0(\delta) > 0$ only depending on δ such that if $0 < \varepsilon \leq \varepsilon_0(\delta)$, then X_ε is an A -uniform space for some A depending only on δ , and \bar{X}_ε is a compact geodesic space.*

If $\delta = 0$, then $\varepsilon_0(0)$ can be chosen arbitrarily large.

In the proof below we recall the relevant references from [14] and specify the dependence on δ .

Proof. By Proposition 4.5 in [14] there is $\varepsilon_0(\delta) > 0$ such that if $0 < \varepsilon \leq \varepsilon_0(\delta)$, then X_ε is an A -uniform space for some A depending only on δ . As X_ε is bounded, it follows from Proposition 2.20 in [14] that \bar{X}_ε is a compact length space, which by Ascoli's theorem or the Hopf–Rinow Theorem 2.1 is geodesic.

The bound $\varepsilon_0(\delta)$ in Proposition 4.5 in [14] is only needed for the Gehring–Hayman lemma to be true, see [14, Theorem 5.1]. If $\delta = 0$, then any curve from x to y contains the unique geodesic $[x, y]$ as a subcurve. From this the Gehring–Hayman lemma follows directly without any bound on ε . Note that in this case it also follows that a curve in $X = X_\varepsilon$ is simultaneously a geodesic with respect to d and d_ε . \square

We recall, for further reference, the following key estimates from [14].

Lemma 2.7. ([14, Lemma 4.10]) *There is a constant $C(\delta) \geq 1$ such that for all $0 < \varepsilon \leq \varepsilon_0 = \varepsilon_0(\delta)$ and all $x, y \in X$,*

$$\frac{1}{C(\delta)} d_\varepsilon(x, y) \leq \frac{\exp(-\varepsilon(x|y)_{z_0})}{\varepsilon} \min\{1, \varepsilon d(x, y)\} \leq C(\delta) d_\varepsilon(x, y). \quad (2.2)$$

Lemma 2.8. ([14, Lemma 4.16]) *Let $\varepsilon > 0$. If $x \in X$, then*

$$\frac{e^{-\varepsilon d(x, z_0)}}{e\varepsilon} \leq \text{dist}_\varepsilon(x, \partial_\varepsilon X) =: d_\varepsilon(x) \leq C_0 \frac{e^{-\varepsilon d(x, z_0)}}{\varepsilon}, \quad (2.3)$$

where $C_0 = 2e^{\varepsilon M} - 1$. In particular, $\varepsilon d_\varepsilon(x) \simeq \rho_\varepsilon(x)$, and $x \rightarrow \partial_\varepsilon X$ with respect to d_ε if and only if $d(x, z_0) \rightarrow \infty$.

Note that one may choose $C_0 = 2e^{\varepsilon_0 M} - 1$ for it to be independent of ε , provided that $0 < \varepsilon \leq \varepsilon_0$.

Corollary 2.9. *Assume that $0 < \varepsilon \leq \varepsilon_0(\delta)$. Let $x, y \in X$. If $\varepsilon d(x, y) \geq 1$ then*

$$\exp(\varepsilon d(x, y)) \simeq \frac{d_\varepsilon(x, y)^2}{d_\varepsilon(x) d_\varepsilon(y)},$$

where the comparison constants depend only on δ , M and ε_0 .

Proof. Since $\varepsilon d(x, y) \geq 1$, (2.2) can be written as

$$\exp(-2\varepsilon(x|y)_{z_0}) \simeq (\varepsilon d_\varepsilon(x, y))^2, \quad (2.4)$$

where the comparison constants depend only on δ . Moreover, (2.3) gives

$$\exp(-\varepsilon d(x, z_0)) \simeq \varepsilon d_\varepsilon(x) \quad \text{and} \quad \exp(-\varepsilon d(y, z_0)) \simeq \varepsilon d_\varepsilon(y)$$

with comparison constants depending only on M and ε_0 . Dividing (2.4) by the last two formulas, and using the definition of $(x|y)_{z_0}$ concludes the proof. \square

We now wish to show that subWhitney balls in the uniformization X_ε are contained in balls of a fixed radius with respect to the Gromov hyperbolic metric d of X .

Theorem 2.10. *For all $0 < \varepsilon \leq \varepsilon_0(\delta)$, $x \in X$ and $0 < r \leq \frac{1}{2}d_\varepsilon(x)$, we have*

$$B\left(x, \frac{C_1 r}{\rho_\varepsilon(x)}\right) \subset B_\varepsilon(x, r) \subset B\left(x, \frac{C_2 r}{\rho_\varepsilon(x)}\right),$$

where $C_1 = e^{-(1+\varepsilon M)}$ and $C_2 = 2e(2e^{\varepsilon M} - 1)$. If $d_\varepsilon(x, y) < C_1 d_\varepsilon(x)/2C_2$, then

$$\frac{\rho_\varepsilon(x)}{C_2} d(x, y) < d_\varepsilon(x, y) \leq e^{1/e} \rho_\varepsilon(x) d(x, y).$$

Remark 2.11. As in Lemma 2.8, the constants C_1 and C_2 obtained for ε_0 will do for $\varepsilon < \varepsilon_0$ as well. The proof also shows that the condition $0 < r \leq \frac{1}{2}d_\varepsilon(x)$ can be replaced by $0 < r \leq c_0 d_\varepsilon(x)$ for any fixed $0 < c_0 < 1$, but then C_1 and C_2 also depend on c_0 and get progressively worse as c_0 approaches 1.

Proof. Assume that $y \in B(x, C_1 r/\rho_\varepsilon(x))$ and let γ be a d -geodesic from x to y . The assumption $r \leq \frac{1}{2}d_\varepsilon(x)$ and (2.3) then imply that for all $z \in \gamma$,

$$d(x, z) \leq d(x, y) < \frac{C_1 r}{\rho_\varepsilon(x)} \leq \frac{C_1 d_\varepsilon(x)}{2\rho_\varepsilon(x)} \leq \frac{C_1(2e^{\varepsilon M} - 1)}{2\varepsilon} < \frac{C_1 e^{\varepsilon M}}{\varepsilon} = \frac{1}{\varepsilon e}. \quad (2.5)$$

The triangle inequality then yields $d(z, z_0) \geq d(x, z_0) - d(x, z) \geq d(x, z_0) - 1/\varepsilon e$ and hence

$$\rho_\varepsilon(z) = e^{-\varepsilon d(z, z_0)} \leq e^{1/e} \rho_\varepsilon(x).$$

From this and (2.5) it readily follows that

$$d_\varepsilon(x, y) \leq \int_\gamma \rho_\varepsilon ds \leq e^{1/e} \rho_\varepsilon(x) d(x, y) < C_1 e^{1/e} r < r. \quad (2.6)$$

To see the other inclusion, assume that $d_\varepsilon(x, y) < r \leq \frac{1}{2}d_\varepsilon(x)$ and let γ_ε be a geodesic curve in \bar{X}_ε connecting x to y . Then for all $z \in \gamma_\varepsilon$, we have by the triangle inequality that

$$d_\varepsilon(z) \geq d_\varepsilon(x) - d_\varepsilon(x, z) \geq d_\varepsilon(x) - d_\varepsilon(x, y) > \frac{1}{2}d_\varepsilon(x),$$

in particular $\gamma_\varepsilon \subset X_\varepsilon$. It now follows from (2.3) that

$$\rho_\varepsilon(z) \geq \frac{\varepsilon d_\varepsilon(z)}{2e^{\varepsilon M} - 1} > \frac{\varepsilon d_\varepsilon(x)}{2(2e^{\varepsilon M} - 1)} \geq \frac{\rho_\varepsilon(x)}{C_2},$$

where C_2 is as in the statement of the theorem. This implies that

$$r > d_\varepsilon(x, y) = \int_{\gamma_\varepsilon} \rho_\varepsilon ds > \frac{\rho_\varepsilon(x)}{C_2} d(x, y), \quad (2.7)$$

and hence $B_\varepsilon(x, r) \subset B(x, C_2 r/\rho_\varepsilon(x))$.

Finally, if $d_\varepsilon(x, y) < C_1 d_\varepsilon(x)/2C_2$, then from the last inclusion we see that $y \in B(x, C_1 s/\rho_\varepsilon(x))$ with $s = \frac{1}{2}d_\varepsilon(x)$. Therefore we can apply (2.6) and (2.7) to obtain the last claim of the lemma. \square

In this paper, the letter C will denote various positive constants whose values may change even within a line. We write $Y \lesssim Z$ if there is an implicit constant $C > 0$ such that $Y \leq CZ$, and analogously $Y \gtrsim Z$ if $Z \lesssim Y$. We also use the notation $Y \simeq Z$ to mean $Y \lesssim Z \lesssim Y$. We will point out how the comparison constants depend on various other constants related to the metric measure spaces under study.

3. Doubling property

In the rest of this paper, we will continue to assume that X is a locally compact roughly starlike Gromov hyperbolic space. For general definitions and some results, we will assume that Y is a metric space equipped with a Borel measure ν .

Just as for X , we will denote the metric on Y by d , and balls in Y by $B(x, r)$, but it should always be clear from the context in which space these concepts are taken.

Definition 3.1. A Borel measure ν , defined on a metric space Y , is *globally doubling* if

$$0 < \nu(B(x, 2r)) \leq C_d \nu(B(x, r)) < \infty$$

whenever $x \in Y$ and $r > 0$. If this holds only for balls of radii $\leq R_0$, then we say that ν is *doubling for balls of radii at most R_0* , and also that ν is *uniformly locally doubling*.

The following result shows that the last condition is independent of R_0 , provided that Y is quasiconvex. Without assuming quasiconvexity this is not true as shown by Example 3.3 below.

Proposition 3.2. *Assume that Y is L -quasiconvex and that ν is doubling on Y for balls of radii at most R_0 , with a doubling constant C_d . Let $R_1 > 0$. Then ν is doubling on Y for balls of radii at most R_1 with a doubling constant depending only on R_1/R_0 , L and C_d .*

Example 3.3. Let $X = ([0, \infty) \times \{0, 1\}) \cup (\{0\} \times [0, 1])$ equipped with the Euclidean distance and the measure $d\mu = w d\mathcal{L}^1$, where \mathcal{L}^1 is the Lebesgue measure and

$$w(x, y) = \begin{cases} 1, & \text{if } y < 1, \\ e^x, & \text{if } y = 1. \end{cases}$$

Then X is a connected nonquasiconvex space and μ is doubling for balls of radii at most R_0 if and only if $R_0 \leq \frac{1}{2}$. This shows that the quasiconvexity assumption in Proposition 3.2 cannot be dropped.

Before proving Proposition 3.2 we deduce the following lemmas. In particular, Lemma 3.5 covers Proposition 3.2 under the extra assumption that Y is a length space, but with better control of the doubling constant than what is possible in general quasiconvex spaces.

Lemma 3.4. *Assume that ν is doubling on Y for balls of radii at most R_0 , with a doubling constant C_d . Then every ball B of radius $r \leq \frac{7}{4}R_0$ can be covered by at most C_d^7 balls with centers in B and radius $\frac{1}{4}r$.*

Proof. Find a maximal pairwise disjoint collection of balls B_j with centers in B and radii $\frac{1}{4}r$. Note that for each j ,

$$B_j \subset \frac{15}{14}B \quad \text{and} \quad \frac{15}{112}B \subset \frac{127}{112} \cdot 14B_j \subset 16B_j.$$

The doubling property then implies that

$$\nu(\frac{15}{14}B) \leq C_d^3 \nu(\frac{15}{112}B) \leq C_d^7 \nu(B_j).$$

From this and the pairwise disjointness of all B_j we thus obtain

$$\nu(\frac{15}{14}B) \geq \sum_j \nu(B_j) \geq \frac{1}{C_d^7} \nu(\frac{15}{14}B) \sum_j 1,$$

i.e. there are at most C_d^7 such balls. As the balls $2B_j$ cover B , we are done. \square

Lemma 3.5. *Assume that Y is a length space and that ν is doubling on Y for balls of radii at most R_0 , with a doubling constant C_d . Let n be a positive integer. Then the following are true:*

(a) *If $x, x' \in Y$, $0 < r \leq R_0$ and $d(x, x') < nr$, then*

$$\nu(B(x', r)) \leq C_d^n \nu(B(x, r)).$$

(b) *Every ball B of radius nr , with $r \leq \frac{1}{4}R_0$, can be covered by at most $C_d^{7(n+4)/6}$ balls of radius r , $n = 1, 2, \dots$.*

In particular, for any $R_1 > 0$, ν is doubling on Y for balls of radii at most R_1 with a doubling constant depending only on R_1/R_0 and C_d .

Proof. (a) Connect x and x' by a curve of length $l_\gamma < nr$. Along this curve, we can find balls B_j of radius r , $j = 0, 1, \dots, n$, such that $B_0 = B(x, r)$, $B_n = B(x', r)$ and $B_j \subset 2B_{j-1}$. An iteration of the doubling property gives the desired estimate.

(b) Suppose that $\varphi(n)$ is the smallest number such that each ball $B(x, nr)$ is covered by $\varphi(n)$ balls B_j of radius r . Since Y is a length space, the balls $7B_j$ cover $B(x, (n+6)r)$. Using Lemma 3.4, each $7B_j$ can in turn be covered by at most C_d^7 balls of radius r , which implies that $\varphi(n+6) \leq C_d^7 \varphi(n)$. As, $\varphi(1) = 1$ and φ is nondecreasing, the statement follows by induction. \square

Proof of Proposition 3.2. We will use the inner metric d_{in} , defined in (2.1), and denote balls with respect to d_{in} by B_{in} . It follows from the inclusions

$$B_{\text{in}}(x, r) \subset B(x, r) \subset B_{\text{in}}(x, Lr), \quad (3.1)$$

together with a repeated use of the doubling property for metric balls, that ν is doubling for inner balls of radii at most R_0 . As (X, d_{in}) is a length space, it thus follows from Lemma 3.5 that ν is doubling for inner balls of radii at most LR_1 . Hence, using the inclusions (3.1) again, ν is doubling for metric balls of radii at most R_1 . \square

4. The measure μ_β is globally doubling on X_ε

Standing assumptions for this section will be given after Example 4.3.

Given a uniformly locally doubling measure μ on the Gromov hyperbolic space X , we wish to obtain a globally doubling measure on its uniformization X_ε . We do so as follows.

Definition 4.1. Assume that X is a locally compact roughly starlike Gromov hyperbolic space equipped with a Borel measure μ , and that $z_0 \in X$.

Fix $\beta > 0$, and set μ_β to be the measure on $X = X_\varepsilon$ given by

$$d\mu_\beta = \rho_\beta d\mu, \quad \text{where } \rho_\beta(x) = e^{-\beta d(x, z_0)}.$$

We also extend this measure to \bar{X}_ε by letting $\mu_\beta(\bar{X}_\varepsilon \setminus X_\varepsilon) = 0$.

Our aim in this section is to show that μ_β is a globally doubling measure on \overline{X}_ε , under suitable assumptions (see Theorem 4.9).

Bonk–Heinonen–Koskela [14, Theorem 1.1] showed that there is a kind of duality between Gromov hyperbolic spaces and bounded uniform domains, see the introduction for further details. Here we also equip these spaces with measures. The following examples illustrate what happens in a simple case.

Example 4.2. The Euclidean real line $X = \mathbf{R}$ is Gromov hyperbolic, because it is a metric tree. Since $\delta = 0$, any $\varepsilon > 0$ is allowed in the uniformization process, by Theorem 2.6. Setting $z_0 = 0$, we now determine what X_ε is. For $x, y \in \mathbf{R}$, the uniformized metric is given by

$$d_\varepsilon(x, y) = \left| \int_x^y e^{-\varepsilon|t|} dt \right| = \begin{cases} \frac{1}{\varepsilon} |e^{-\varepsilon|x|} - e^{-\varepsilon|y|}|, & \text{if } xy \geq 0, \\ \frac{1}{\varepsilon} (2 - (e^{-\varepsilon|x|} + e^{-\varepsilon|y|})), & \text{if } xy \leq 0. \end{cases}$$

With $y = 0$ we get $d_\varepsilon(x, 0) = (1 - e^{-\varepsilon|x|})/\varepsilon$. Hence the map $\Phi : X_\varepsilon \rightarrow (-1/\varepsilon, 1/\varepsilon)$ given by

$$\Phi(x) = \frac{1}{\varepsilon} (1 - e^{-\varepsilon|x|}) \operatorname{sign} x$$

is an isometry, identifying X_ε with the open interval $(-1/\varepsilon, 1/\varepsilon)$.

However, when X is equipped with the Lebesgue measure \mathcal{L}^1 , the measure μ_β is not the Lebesgue measure on $(-1/\varepsilon, 1/\varepsilon)$. To determine μ_β , note that it is absolutely continuous with respect to the Lebesgue measure \mathcal{L}^1 on the interval $(-1/\varepsilon, 1/\varepsilon)$. So we compute the Radon–Nikodym derivative of μ_β with respect to \mathcal{L}^1 . By symmetry, it suffices to consider $x > 0$. Then

$$d\mu_\beta(\Phi(x)) = e^{-\beta x} J_{\Phi^{-1}(\Phi(x))} d\mathcal{L}^1(\Phi(x)) = e^{(\varepsilon-\beta)x} d\mathcal{L}^1(\Phi(x)).$$

Substituting $\Phi(x) = z$ in the above, we get

$$d\mu_\beta(z) = (1 - \varepsilon z)^{-1+\beta/\varepsilon} d\mathcal{L}^1(z) = (\varepsilon d_\varepsilon(z))^{-1+\beta/\varepsilon} d\mathcal{L}^1(z), \quad (4.1)$$

where $d_\varepsilon(z) = 1/\varepsilon - z$ is the distance from $\Phi(x) = z \geq 0$ to the boundary $\{\pm 1/\varepsilon\}$ of $\Phi(X_\varepsilon)$.

Similarly, if $X = \mathbf{R}$ is equipped with a weighted measure

$$d\mu(x) = w(x) d\mathcal{L}^1(x),$$

then as in (4.1),

$$d\mu_\beta(z) = (\varepsilon d_\varepsilon(z))^{-1+\beta/\varepsilon} w(\Phi^{-1}(z)) d\mathcal{L}^1(z). \quad (4.2)$$

The following example reverses the procedure in Example 4.2.

Example 4.3. The interval $X = (-1, 1)$ is a uniform domain and so, by Theorem 3.6 in Bonk–Heinonen–Koskela [14], it becomes a Gromov hyperbolic space when equipped with the quasihyperbolic metric k . The *quasihyperbolic metric* is for $0 \leq y < z < 1$ given by

$$k(y, z) = \int_y^z \frac{1}{1-t} dt = \log\left(\frac{1-y}{1-z}\right),$$

cf. Section 7. With $z_0 = 0$, by symmetry, we have $k(z, z_0) = \log(1/(1 - |z|))$ for $z \in X$. Hence we consider the map $\Psi : (-1, 1) \rightarrow \mathbf{R}$ given by

$$\Psi(z) = (\operatorname{sign} z) \log \frac{1}{1 - |z|},$$

and see that Ψ is an isometry between the Gromov hyperbolic space (X, k) and the Euclidean line \mathbf{R} . By Example 4.2 with $\varepsilon = 1$, the uniformization of \mathbf{R} gives back the Euclidean interval $(-1, 1)$.

We wish to find a measure μ on $(X, k) = \mathbf{R}$ such that the weighted measure μ_β given by Definition 4.1 becomes the Lebesgue measure on $(-1, 1)$. In view of (4.2) with $\varepsilon = 1$ and $\Phi = \Psi^{-1}$, μ is given by $d\mu(x) = w(x) d\mathcal{L}^1(x)$, where

$$w(x) = d_1(\Phi(x))^{1-\beta} = (1 - |\Phi(x)|)^{1-\beta} = e^{(\beta-1)|x|}.$$

In the rest of this section, we assume that X is a locally compact roughly starlike Gromov δ -hyperbolic space equipped with a measure μ which is doubling on X for balls of radii at most R_0 , with a doubling constant C_d . We also fix a point $z_0 \in X$, let M be the constant in the roughly starlike condition with respect to z_0 , and assume that

$$0 < \varepsilon \leq \varepsilon_0(\delta) \quad \text{and} \quad \beta > \beta_0 := \frac{17 \log C_d}{3R_0}. \quad (4.3)$$

Finally, we let X_ε be the uniformization of X with uniformization center z_0 .

In specific cases one may want to consider how to optimally choose R_0 , and the corresponding C_d , in the formula for β_0 . The factor $\frac{17}{3}$ comes from various estimates leading up to the proof of Proposition 4.7, and is not likely to be optimal. The following example shows however that it is not too far from optimal and that it cannot be replaced by any constant < 1 .

Example 4.4. Let X be the infinite regular K -ary metric tree, equipped with the Lebesgue measure μ , as considered in Björn–Björn–Gill–Shanmugalingam [8, Section 3]. As it is a tree, any $\varepsilon > 0$ is allowed for uniformization.

If $C_d(R)$ is the optimal doubling constant for radii $\leq R$, then a straightforward calculation shows that

$$\lim_{R \rightarrow \infty} \frac{C_d(R)}{K^R} = 1,$$

and thus we are allowed, in this paper, to use any

$$\beta > \frac{17 \log K^R}{3R} = \frac{17}{3} \log K.$$

In this specific case, it was shown in [8, Corollary 3.9] that μ_β is globally doubling and supports a global 1-Poincaré inequality on X_ε whenever $\beta > \log K$. For $\beta \leq \log K$, $\mu_\beta(X_\varepsilon) = \infty$ and μ_β cannot possibly be globally doubling on the bounded space X_ε .

The following lemma gives us an estimate of $\mu_\beta(B)$ for subWhitney balls B .

Lemma 4.5. *Let $x \in X$ and $0 < r \leq \frac{1}{2}d_\varepsilon(x)$. Then*

$$\mu_\beta(B_\varepsilon(x, r)) \simeq \rho_\beta(x) \mu \left(B \left(x, \frac{r}{\rho_\varepsilon(x)} \right) \right)$$

with comparison constants depending only on M , ε , C_d , R_0 and β .

Proof. By Lemma 2.8, we have for all $y \in B_\varepsilon(x, r)$,

$$\rho_\beta(y) = \rho_\varepsilon(y)^{\beta/\varepsilon} \simeq (\varepsilon d_\varepsilon(y))^{\beta/\varepsilon} \simeq (\varepsilon d_\varepsilon(x))^{\beta/\varepsilon} \simeq \rho_\beta(x). \quad (4.4)$$

Moreover, Theorem 2.10 implies that

$$B \left(x, \frac{C_1 r}{\rho_\varepsilon(x)} \right) \subset B_\varepsilon(x, r) \subset B \left(x, \frac{C_2 r}{\rho_\varepsilon(x)} \right). \quad (4.5)$$

This yields

$$\mu_\beta(B_\varepsilon(x, r)) \simeq \rho_\beta(x)\mu(B_\varepsilon(x, r)) \lesssim \rho_\beta(x)\mu\left(B\left(x, \frac{C_2r}{\rho_\varepsilon(x)}\right)\right)$$

and similarly, $\mu_\beta(B_\varepsilon(x, r)) \gtrsim \rho_\beta(x)\mu(B(x, C_1r/\rho_\varepsilon(x)))$. Finally, Lemma 3.5 shows that the last two balls in X have measure comparable to $\mu(B(x, r/\rho_\varepsilon(x)))$, which concludes the proof. \square

Remark 4.6. Lemma 4.5 implies that if μ_β is globally doubling on X_ε then μ is uniformly locally doubling on X , i.e. the converse of Theorem 1.1 (a) holds. Indeed, if $0 < r \leq \frac{1}{4e\varepsilon}$ and $x \in X$, then $2r\rho_\varepsilon(x) \leq \frac{1}{2}d_\varepsilon(x)$, by (2.3). Lemma 4.5, with r replaced by $2r\rho_\varepsilon(x)$ and $r\rho_\varepsilon(x)$, respectively, then gives

$$\mu(B(x, 2r)) \simeq \frac{\mu_\beta(B_\varepsilon(x, 2r\rho_\varepsilon(x)))}{\rho_\beta(x)} \simeq \frac{\mu_\beta(B_\varepsilon(x, r\rho_\varepsilon(x)))}{\rho_\beta(x)} \simeq \mu(B(x, r)).$$

Similar arguments, combined with the arguments in the proof of Lemma 6.1, show that if μ_β also supports a global p -Poincaré inequality on X_ε or \bar{X}_ε then μ supports a uniformly local p -Poincaré inequality on X , i.e. the converse of Theorem 1.1 (b) holds.

We shall now estimate $\mu_\beta(B)$ for balls B centered at $\partial_\varepsilon X$ in terms of the (essentially) largest Whitney ball contained in B . The existence of such balls is given by Lemma 4.8 below.

Proposition 4.7. *Let $\xi \in \partial_\varepsilon X$ and $0 < r \leq 2 \operatorname{diam}_\varepsilon X_\varepsilon$. Assume that $a_0 > 0$ and $z \in X$ are such that $B_\varepsilon(z, a_0r) \subset B_\varepsilon(\xi, r)$ and $d_\varepsilon(z) \geq 2a_0r$. Then,*

$$\mu_\beta(B_\varepsilon(\xi, r)) \simeq \rho_\beta(z)\mu(B(z, R_0)) \simeq \rho_\beta(z)\mu\left(B\left(z, \frac{a_0r}{\rho_\varepsilon(z)}\right)\right) \quad \text{and} \quad \rho_\beta(z) \simeq (\varepsilon r)^{\beta/\varepsilon},$$

where the comparison constants depend only on $\delta, M, \varepsilon, C_d, R_0, \beta$ and a_0 .

Proof. For $n = 1, 2, \dots$, define the boundary layers

$$A_n = \{x \in B_\varepsilon(\xi, r) : e^{-n}r \leq d_\varepsilon(x) \leq e^{1-n}r\}.$$

Corollary 2.9 implies that for every $x \in A_n$, either $\varepsilon d(x, z) < 1$ or

$$\exp(\varepsilon d(x, z)) \simeq \frac{d_\varepsilon(x, z)^2}{d_\varepsilon(x)d_\varepsilon(z)} \leq \frac{(d_\varepsilon(x, \xi) + d_\varepsilon(\xi, z))^2}{2a_0e^{-n}r^2} \leq \frac{2e^n}{a_0},$$

and hence $\varepsilon d(x, z) < n + C$, where C depends only on δ, M, ε and a_0 .

Using Lemma 3.5 (b), we can thus cover each layer $A_n \subset B(z, (n + C)/\varepsilon)$ by $N_n \lesssim C_d^{14n/3\varepsilon R_0}$ balls $B_{n,j}$ with centers in $B(z, (n + C)/\varepsilon)$ and radius R_0 . Since X_ε is geodesic, Lemma 3.5 (a) implies that each of these balls satisfies

$$\mu(B_{n,j}) \lesssim C_d^{n/\varepsilon R_0} \mu(B(z, R_0)),$$

Moreover, as in (4.4) we see that $\rho_\beta(z) = \rho_\varepsilon(z)^{\beta/\varepsilon} \simeq (\varepsilon d_\varepsilon(z))^{\beta/\varepsilon} \simeq (\varepsilon r)^{\beta/\varepsilon}$ and

$$\rho_\beta(x) = \rho_\varepsilon(x)^{\beta/\varepsilon} \simeq (\varepsilon d_\varepsilon(x))^{\beta/\varepsilon} \simeq (e^{-n}\varepsilon r)^{\beta/\varepsilon} \quad \text{for all } x \in A_n.$$

It thus follows that

$$\mu_\beta(A_n \cap B_{n,j}) \lesssim (e^{-n}\varepsilon r)^{\beta/\varepsilon} \mu(B_{n,j}) \lesssim C_d^{n/\varepsilon R_0} e^{-n\beta/\varepsilon} \rho_\beta(z) \mu(B(z, R_0))$$

and hence for $\beta > \beta_0 = 17(\log C_d)/3R_0$,

$$\begin{aligned} \mu_\beta(B_\varepsilon(\xi, r)) &\leq \sum_{n=1}^{\infty} \sum_{j=1}^{N_n} \mu_\beta(A_n \cap B_{n,j}) \\ &\lesssim \rho_\beta(z) \mu(B(z, R_0)) \sum_{n=1}^{\infty} (C_d^{17/3R_0})^{n/\varepsilon} e^{-n\beta/\varepsilon} \simeq \rho_\beta(z) \mu(B(z, R_0)). \end{aligned}$$

Since $a_0 r \leq \frac{1}{2} d_\varepsilon(z)$, Lemma 4.5 implies that

$$\mu_\beta(B_\varepsilon(\xi, r)) \geq \mu_\beta(B_\varepsilon(z, a_0 r)) \simeq \rho_\beta(z) \mu\left(B\left(z, \frac{a_0 r}{\rho_\varepsilon(z)}\right)\right).$$

By (2.3) we see that

$$r > d_\varepsilon(z) \geq \frac{\rho_\varepsilon(z)}{e\varepsilon},$$

and hence, by the doubling property for μ on X ,

$$\mu\left(B\left(z, \frac{a_0 r}{\rho_\varepsilon(z)}\right)\right) \geq \mu\left(B\left(z, \frac{a_0}{e\varepsilon}\right)\right) \simeq \mu(B(z, R_0)). \quad \square$$

The following lemma shows how to pick z and a_0 in Proposition 4.7.

Lemma 4.8. *Let $0 < a_0 < a := \min\{\frac{1}{8}, \frac{1}{6A}\}$, where $A = A(\delta)$ is as in Theorem 2.6. Then for every $x \in \bar{X}_\varepsilon$ and every $0 < r \leq 2 \operatorname{diam}_\varepsilon X_\varepsilon$ we can find a ball $B_\varepsilon(z, a_0 r) \subset B_\varepsilon(x, r)$ such that $d_\varepsilon(z) \geq 2a_0 r$.*

Proof. First, assume that $x \in X_\varepsilon$. By Theorem 2.6, there is an A -uniform curve γ from x to z_0 , parametrized by arc length ds_ε . If $l_\varepsilon(\gamma) \geq \frac{2}{3}r$ then for $z = \gamma(\frac{1}{3}r)$ we have

$$d_\varepsilon(z) \geq \frac{r}{3A} \quad \text{and} \quad B_\varepsilon\left(z, \frac{r}{6A}\right) \subset B_\varepsilon\left(x, \frac{r}{3} + \frac{r}{6A}\right) \subset B_\varepsilon(x, r).$$

Thus, any $a_0 \leq \frac{1}{6A}$ will do in this case. If $l_\varepsilon(\gamma) < \frac{2}{3}r$, then letting $z = z_0$ yields

$$B_\varepsilon(z, \frac{1}{3}r) \subset B_\varepsilon(x, l_\varepsilon(\gamma) + \frac{1}{3}r) \subset B_\varepsilon(x, r),$$

and for $a_0 \leq \frac{1}{8}$,

$$d_\varepsilon(z) = d_\varepsilon(z_0) \geq 4a_0 \operatorname{diam}_\varepsilon X_\varepsilon \geq 2a_0 r.$$

This proves the lemma for $x \in X_\varepsilon$. For $x \in \partial_\varepsilon X$ and any $0 < a_0 < a$, choose $r' = a_0 r / a$ and $x' \in X_\varepsilon$ sufficiently close to x so that, with the corresponding z ,

$$B_\varepsilon(z, a_0 r) = B_\varepsilon(z, ar') \subset B_\varepsilon(x', r') \subset B(x, r) \quad \text{and} \quad d_\varepsilon(z) \geq 2ar' = 2a_0 r. \quad \square$$

Lemma 4.5 and Proposition 4.7 can be summarized in the following result, which roughly says that in $(X_\varepsilon, \mu_\beta)$, the measure of every ball is comparable to the measure of the (essentially) largest Whitney ball contained in it.

Theorem 4.9. *The measure μ_β is globally doubling on \bar{X}_ε .*

Moreover, with a_0 and z provided by Lemma 4.8, we have for every $x \in \bar{X}_\varepsilon$ and $0 < r \leq 2 \operatorname{diam}_\varepsilon X_\varepsilon$,

$$\mu_\beta(B_\varepsilon(x, r)) \simeq \mu_\beta(B_\varepsilon(z, a_0 r)), \quad (4.6)$$

where the comparison constants depend only on δ , M , ε , C_d , R_0 , β and a_0 .

It follows directly that μ_β is globally doubling also on X_ε . The optimal doubling constants are the same, by Proposition 3.3 in Björn–Björn [7].

Proof. We start by proving the measure estimate (4.6). Since $a_0 r \leq \frac{1}{2}d_\varepsilon(z)$, Lemma 4.5 applied to $B_\varepsilon(z, a_0 r)$ implies that

$$\mu_\beta(B_\varepsilon(z, a_0 r)) \simeq \rho_\beta(z) \mu\left(B\left(z, \frac{a_0 r}{\rho_\varepsilon(z)}\right)\right). \quad (4.7)$$

If $0 < r \leq \frac{1}{2}d_\varepsilon(x)$ then by (4.4), (4.5) and Lemma 2.8,

$$\rho_\beta(z) \simeq \rho_\beta(x) \quad \text{and} \quad d(x, z) \leq \frac{C_2 r}{\rho_\varepsilon(x)} \lesssim \frac{1}{2}.$$

Lemma 3.5 then implies that

$$\mu\left(B\left(z, \frac{a_0 r}{\rho_\varepsilon(z)}\right)\right) \simeq \mu\left(B\left(z, \frac{r}{\rho_\varepsilon(z)}\right)\right) \simeq \mu\left(B\left(x, \frac{r}{\rho_\varepsilon(x)}\right)\right),$$

and another application of Lemma 4.5, this time to $B_\varepsilon(x, r)$, proves (4.6) in this case.

If $r \geq \frac{1}{2}d_\varepsilon(x)$ then $B_\varepsilon(x, r) \subset B_\varepsilon(\xi, 3r)$ for some $\xi \in \partial_\varepsilon X$. Proposition 4.7 (with a_0 replaced by $\frac{1}{3}a_0$) then implies

$$\mu_\beta(B_\varepsilon(\xi, 3r)) \simeq \rho_\beta(z) \mu\left(B\left(z, \frac{a_0 r}{\rho_\varepsilon(z)}\right)\right),$$

which together with (4.7) proves (4.6) also in this case.

To conclude the doubling property, use the Whitney ball $B_\varepsilon(z, a_0 r)$ for both $B_\varepsilon(x, r)$ and $B_\varepsilon(x, 2r)$, with constants a_0 and $a'_0 = \frac{1}{2}a_0$, respectively. Since $d_\varepsilon(z) \geq 2a_0 r = 2a'_0 \cdot 2r$, we have by (4.6), first used with a'_0 and then with a_0 ,

$$\mu_\beta(B_\varepsilon(x, 2r)) \simeq \mu_\beta(B_\varepsilon(z, 2a'_0 r)) = \mu_\beta(B_\varepsilon(z, a_0 r)) \simeq \mu_\beta(B_\varepsilon(x, r)). \quad \square$$

We conclude this section with an estimate of the lower and upper dimensions for the measure μ_β at $\partial_\varepsilon X$.

Lemma 4.10. *Let*

$$s_\pm = \frac{\beta}{\varepsilon} \pm \frac{\log C_d}{\varepsilon R_0}.$$

Then for all $\xi \in \partial_\varepsilon X$ and all $0 < r \leq r' \leq 2 \text{diam}_\varepsilon X_\varepsilon$,

$$\left(\frac{r}{r'}\right)^{s_+} \lesssim \frac{\mu_\beta(B_\varepsilon(\xi, r))}{\mu_\beta(B_\varepsilon(\xi, r'))} \lesssim \left(\frac{r}{r'}\right)^{s_-},$$

with comparison constants depending only on $\delta, M, \varepsilon, C_d, R_0, \beta$ and the constant a_0 from Theorem 2.6.

Note that $0 < s_- \leq s_+$, because $\beta > \beta_0$, where β_0 is as in (4.3).

Proof. Proposition 4.7 and Lemma 4.8 imply that there are $z, z' \in X$ such that

$$\mu_\beta(B_\varepsilon(\xi, r)) \simeq (\varepsilon r)^{\beta/\varepsilon} \mu(B(z, R_0)) \quad \text{and} \quad \mu_\beta(B_\varepsilon(\xi, r')) \simeq (\varepsilon r')^{\beta/\varepsilon} \mu(B(z', R_0)), \quad (4.8)$$

where

$$\begin{aligned} B_\varepsilon(z, a_0 r) &\subset B_\varepsilon(\xi, r), & d_\varepsilon(z) &\geq 2a_0 r, \\ B_\varepsilon(z', a_0 r') &\subset B_\varepsilon(\xi, r'), & d_\varepsilon(z') &\geq 2a_0 r'. \end{aligned}$$

From Corollary 2.9 we conclude that if $\varepsilon d(z, z') \geq 1$ then

$$\exp(\varepsilon d(z, z')) \simeq \frac{d_\varepsilon(z, z')^2}{d_\varepsilon(z)d_\varepsilon(z')} \leq \frac{(2r')^2}{(2a_0 r)(2a_0 r')} = \frac{r'}{a_0^2 r},$$

and hence $d(z, z') \leq \frac{1}{\varepsilon}(C + \log(r'/r))$ holds regardless of the value of $d(z, z')$. Lemma 3.5(a) with $n = \lceil d(z, z')/R_0 \rceil$ (the smallest integer $\geq d(z, z')/R_0$) then implies that

$$\frac{\mu(B(z, R_0))}{\mu(B(z', R_0))} \geq C_d^{-n} \gtrsim \left(\frac{r}{r'}\right)^{(\log C_d)/\varepsilon R_0}, \quad (4.9)$$

which together with (4.8) proves the first inequality in the lemma. The second inequality follows similarly by interchanging z and z' in (4.9). \square

5. Upper gradients and Poincaré inequalities

We assume in this section that $1 \leq p < \infty$ and that $Y = (Y, d, \nu)$ is a metric space equipped with a complete Borel measure ν such that $0 < \nu(B) < \infty$ for all balls $B \subset Y$.

We follow Heinonen and Koskela [29] in introducing upper gradients as follows (in [29] they are referred to as very weak gradients).

Definition 5.1. A Borel function $g : Y \rightarrow [0, \infty]$ is an *upper gradient* of an extended real-valued function u on Y if for all arc length parametrized curves $\gamma : [0, l_\gamma] \rightarrow Y$,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \quad (5.1)$$

where we follow the convention that the left-hand side is considered to be ∞ whenever at least one of the terms therein is $\pm\infty$. If g is a nonnegative measurable function on Y and if (5.1) holds for p -almost every curve (see below), then g is a *p -weak upper gradient* of u .

We say that a property holds for *p -almost every curve* if it fails only for a curve family Γ with *zero p -modulus*, i.e. there is a Borel function $0 \leq \rho \in L^p(Y)$ such that $\int_\gamma \rho \, ds = \infty$ for every curve $\gamma \in \Gamma$. The p -weak upper gradients were introduced in Koskela–MacManus [34]. It was also shown therein that if $g \in L^p_{\text{loc}}(Y)$ is a p -weak upper gradient of u , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of u such that $\|g_j - g\|_{L^p(Y)} \rightarrow 0$.

If u has an upper gradient in $L^p_{\text{loc}}(Y)$, then it has a *minimal p -weak upper gradient* $g_u \in L^p_{\text{loc}}(Y)$ in the sense that for every p -weak upper gradient $g \in L^p_{\text{loc}}(Y)$ of u we have $g_u \leq g$ a.e., see Shanmugalingam [37] (or [5] or [30]). The minimal p -weak upper gradient is well defined up to a set of measure zero.

Definition 5.2. Y (or ν) supports a *global p -Poincaré inequality* if there exist constants $\lambda \geq 1$ (called *dilation*) and $C_{\text{PI}} > 0$ such that for all balls $B \subset Y$, all bounded measurable functions u on Y and all upper gradients g of u ,

$$\int_B |u - u_B| \, d\nu \leq C_{\text{PI}} \text{diam}(B) \left(\int_{\lambda B} g^p \, d\nu \right)^{1/p}, \quad (5.2)$$

where $u_B := u_{B, \nu} := \int_B u \, d\nu := \int_B u \, d\nu / \nu(B)$.

If this holds only for balls B of radii $\leq R_0$, then we say that ν supports a *p -Poincaré inequality for balls of radii at most R_0* , and also that Y (or ν) supports a *uniformly local p -Poincaré inequality*.

Multiplying bounded measurable functions by suitable cut-off functions and truncating integrable functions shows that one may replace “bounded measurable” by “integrable” in the definition. On the other hand, the proofs of [30, Lemma 8.1.5 and Theorem 8.1.53] show that (5.2) can equivalently be required for all (not necessarily bounded) measurable functions u on λB and all upper (or p -weak upper)

gradients g of u . See also [5, Proposition 4.13], [30, Theorem 8.1.49], Hajlasz–Koskela [26, Theorem 3.2], Heinonen–Koskela [29, Lemma 5.15] and Keith [32, Theorem 2] for further equivalent versions.

Theorem 5.3. *Assume that ν is doubling and supports a p -Poincaré inequality, both properties holding for balls of radii at most R_0 . Also assume that Y is L -quasiconvex and that $R_1 > 0$.*

Then ν supports a p -Poincaré inequality, with dilation L , for balls of radii at most R_1 .

The proof below can be easily adapted to show that the same is true for so-called (q, p) -Poincaré inequalities. The following examples show that the quasiconvexity assumption cannot be dropped even if one assumes that ν is globally doubling, and that one cannot replace L in the conclusion by the dilation constant in the assumed p -Poincaré inequality, nor any fixed multiple of it.

Example 5.4. Let $X = ([0, \infty) \times \{0, 1\}) \cup (\{0\} \times [0, 1])$ equipped with the Euclidean distance and the Lebesgue measure \mathcal{L}^1 . Then X is a connected nonquasiconvex space and \mathcal{L}^1 is globally doubling on X . However, \mathcal{L}^1 supports a p -Poincaré inequality on X , $p \geq 1$, for balls of radii at most R_0 if and only if $R_0 \leq 1$. In this case one can choose the dilation constant $\lambda = 1$. This shows that the quasiconvexity assumption in Theorem 5.3 cannot be dropped.

Example 5.5. For $a \geq 1$, let $X = ([0, a] \times \{0, 1\}) \cup (\{0\} \times [0, 1])$, equipped with the Euclidean distance and the Lebesgue measure \mathcal{L}^1 . Then X is a connected $(2a + 1)$ -quasiconvex space and \mathcal{L}^1 is globally doubling on X . In this case, \mathcal{L}^1 supports a p -Poincaré inequality on X , $p \geq 1$, for balls of radii at most R_0 for any $R_0 > 0$, with the optimal dilation

$$\begin{cases} 1, & \text{if } R_0 \leq 1, \\ \sqrt{1 + a^2}, & \text{if } R_0 > 1. \end{cases}$$

This shows that the dilation constant L in the conclusion of Theorem 5.3 cannot in general be replaced by the dilation constant in the p -Poincaré inequality assumed for balls $\leq R_0$, nor any fixed multiple of it.

Proof of Theorem 5.3. The arguments are similar to the proof of Theorem 4.4 in Björn–Björn [6]. Let C_d , C_{PI} and λ be the constants in the doubling property and the p -Poincaré inequality for balls of radii $\leq R_0$. Let B be a ball with radius $r_B \leq \frac{5}{2}LR_1 =: R_2$. We can assume that $r_B > R_0$.

First, note that the conclusions in the first paragraph of the proof in [6] with $B_0 = B$, $\sigma = L$, $r' = R_0/\lambda$ and μ replaced by ν , follow directly from our assumptions, without appealing to Lemma 4.7 nor Proposition 4.8 in [6]. This and the use of Lemma 3.4 explains why there is no need to assume properness here.

By Lemma 3.5, ν is doubling for balls of radii $\leq 7LR_2$, with doubling constant C'_d , depending only on C_d and LR_2/R_0 . Hence, using Lemma 3.4, we can cover B by at most $(C'_d)^{\lceil \log_7(R_2/r') \rceil}$ balls B'_j with radius r' . Their centers can then be connected by L -quasiconvex curves. As in the proof of [6, Theorem 4.4], we then construct along these curves a chain $\{B_j\}_{j=1}^N$ of balls of radius r' , covering B and with a uniform bound on N . It follows that the constant C''' in the proof of [6, Theorem 4.4] only depends on C_d , C_{PI} , λ , L and R_2/R_0 . Thus we conclude from the last but one displayed formula in the proof of [6, Theorem 4.4] (with $B_0 = B$) that ν supports a p -Poincaré inequality for balls of radii at most R_2 , with dilation $2L$.

That we can replace $2L$ by L now follows from [6, Theorem 5.1], provided that we decrease the bound on the radii to R_1 . \square

6. Poincaré inequality on X_ε

In this section, we assume that X is a locally compact roughly starlike Gromov δ -hyperbolic space equipped with a Borel measure μ . We also fix a point $z_0 \in X$, let M be the constant in the roughly starlike condition with respect to z_0 , and assume that

$$0 < \varepsilon \leq \varepsilon_0(\delta) \quad \text{and} \quad 1 \leq p < \infty.$$

Finally, we let X_ε be the uniformization of X with uniformization center z_0 .

The following lemma shows that the p -Poincaré inequality holds for μ_β on sufficiently small subWhitney balls in X_ε . Recall that $\beta_0 = (17 \log C_d)/3R_0$ as in (4.3).

Lemma 6.1. *Assume that μ is doubling, with constant C_d , and supports a p -Poincaré inequality, with constants C_{PI} and λ , both properties holding for balls of radii at most R_0 . Let $\beta > \beta_0$.*

Then there exists $c_0 > 0$, depending only on δ , M , ε , R_0 and λ , such that for all $x \in X_\varepsilon$ and all $0 < r \leq c_0 d_\varepsilon(x)$, the p -Poincaré inequality for μ_β holds on $B_\varepsilon = B_\varepsilon(x, r)$, i.e. for all bounded measurable functions u and upper gradients g_ε of u on X_ε we have

$$\int_{B_\varepsilon} |u - u_{B_\varepsilon, \mu_\beta}| d\mu_\beta \leq Cr \left(\int_{\tau B_\varepsilon} g_\varepsilon^p d\mu_\beta \right)^{1/p},$$

where $\tau = C_2\lambda/C_1$, with C_1 and C_2 from Theorem 2.10, and C depends only on δ , M , ε , C_d , R_0 , β , λ and C_{PI} .

Proof. Theorem 2.10 shows that if $c_0 \leq C_1/2C_2\lambda$ then

$$B_\varepsilon \subset B\left(x, \frac{C_2 r}{\rho_\varepsilon(x)}\right) =: B \subset \lambda B \subset B_\varepsilon\left(x, \frac{C_2 \lambda r}{C_1}\right) = \tau B_\varepsilon. \quad (6.1)$$

Moreover, as in (4.4) we have for all $y \in \tau B_\varepsilon$,

$$\rho_\beta(y) = \rho_\varepsilon(y)^{\beta/\varepsilon} \simeq \rho_\varepsilon(x)^{\beta/\varepsilon} = \rho_\beta(x). \quad (6.2)$$

Hence, by Theorem 4.9, all the balls in (6.1) have comparable μ_β -measures, as well as comparable μ -measures.

Let u be a bounded measurable function on X_ε , or equivalently on X , and let g_ε be an upper gradient of u on X_ε . Since the arc length parametrization ds_ε with respect to d_ε satisfies $ds_\varepsilon = \rho_\varepsilon ds$, we conclude that for all compact rectifiable curves γ in X_ε ,

$$\int_\gamma g_\varepsilon ds_\varepsilon = \int_\gamma g_\varepsilon \rho_\varepsilon ds, \quad (6.3)$$

and thus $g := g_\varepsilon \rho_\varepsilon$ is an upper gradient of u on X . (Note that a compact curve in X is rectifiable with respect to d if and only if it is rectifiable with respect to d_ε .) If $c_0 \leq R_0\varepsilon/C_2(2e^{\varepsilon_0 M} - 1)$, then by Lemma 2.8,

$$\frac{C_2 r}{\rho_\varepsilon(x)} \leq \frac{C_2 c_0 d_\varepsilon(x)}{\rho_\varepsilon(x)} \leq \frac{C_2 c_0 (2e^{\varepsilon_0 M} - 1)}{\varepsilon} \leq R_0,$$

and thus the p -Poincaré inequality holds on B . Using (6.2) we then obtain

$$\begin{aligned} \int_{B_\varepsilon} |u - u_{B_\varepsilon, \mu}| d\mu_\beta &\lesssim \int_B |u - u_{B, \mu}| d\mu \leq \frac{C_{\text{PI}} C_2 r}{\rho_\varepsilon(x)} \left(\int_{\lambda B} g^p d\mu \right)^{1/p} \\ &\simeq \frac{r}{\rho_\varepsilon(x)} \left(\int_{\lambda B} (g_\varepsilon \rho_\varepsilon)^p d\mu_\beta \right)^{1/p} \lesssim r \left(\int_{\tau B_\varepsilon} g_\varepsilon^p d\mu_\beta \right)^{1/p}. \end{aligned}$$

Finally, a standard argument [5, Lemma 4.17] makes it possible to replace $u_{B, \mu}$ on the left-hand side by $u_{B_\varepsilon, \mu_\beta}$. \square

Bonk, Heinonen and Koskela [14, Section 6] proved that if Ω is a locally compact uniform space equipped with a measure μ such that (Ω, μ) is uniformly Q -Loewner in subWhitney balls, then Ω is globally Q -Loewner, where $Q > 1$. If μ is locally doubling with $\mu(B(x, r)) \gtrsim r^Q$ whenever $B(x, r)$ is a subWhitney ball, then the local Q -Loewner condition is equivalent to an analogous local Q -Poincaré inequality, see [29, Theorems 5.7 and 5.9].

We have shown above that the measure μ_β on the uniformized space X_ε is globally doubling and supports a p -Poincaré inequality for subWhitney balls. Following the philosophy of [14, Theorem 6.4], the next theorem demonstrates that the p -Poincaré inequality is actually global on X_ε .

Theorem 6.2. *Assume that μ is doubling and supports a p -Poincaré inequality on X , both properties holding for balls of radii at most R_0 . Let $\beta > \beta_0$ and $\lambda > 1$.*

Then μ_β is globally doubling and supports a global p -Poincaré inequality on \overline{X}_ε with dilation 1, and on X_ε with dilation λ .

If X_ε happens to be geodesic, then it follows from the proof below that we can choose the dilation constant $\lambda = 1$ also on X_ε .

Proof. The global doubling property follows from Theorem 4.9, both on X_ε and \overline{X}_ε . Since X_ε is a length space and Lemma 6.1 shows that the p -Poincaré inequality on X_ε holds for subWhitney balls, the global p -Poincaré inequality on X_ε , with dilation $\lambda > 1$, follows from the following proposition. Moreover, as \overline{X}_ε is geodesic, the global p -Poincaré inequality on \overline{X}_ε , with dilation 1, also follows from the following proposition. \square

Proposition 6.3. *Let (Ω, d) be a bounded A -uniform space equipped with a globally doubling measure ν , which supports a p -Poincaré inequality for all subWhitney balls corresponding to some fixed $0 < c_0 < 1$. Assume that Ω is L -quasiconvex. Then ν supports a global p -Poincaré inequality on Ω with dilation L .*

If moreover the completion $\overline{\Omega}$ of Ω is L' -quasiconvex, then ν , extended by $\nu(\partial\Omega) = 0$, supports a global p -Poincaré inequality on $\overline{\Omega}$ with dilation L' .

Recall that Ω is always A -quasiconvex by the A -uniformity condition, but that L may be smaller than A . Also, $\overline{\Omega}$ is always L -quasiconvex, but it is possible to have $L' < L$.

Proof. Let $x_0 \in \Omega$, $0 < r \leq 2 \operatorname{diam} \Omega$ and $B_0 = B(x_0, r)$ be fixed. The balls in this proof are with respect to Ω . It is well known, and easily shown using the arguments in the proof of Lemma 4.8, that uniform spaces satisfy the corkscrew condition, i.e. there exists a_0 (independent of x_0 and r) and z such that $d_\Omega(z) \geq 2a_0r$ and $B(z, a_0r) \subset B_0$, cf. Björn–Shanmugalingam [13, Lemma 4.2]. With c_0 as in the assumptions of the proposition, let

$$r_0 = \frac{a_0 c_0 r}{8A} \leq \frac{c_0 d_\Omega(z)}{16A} \quad \text{and} \quad r_i = 2^{-i} r_0, \quad i = 1, 2, \dots$$

Since Ω is A -uniform, [13, Lemma 4.3] with $\rho_0 = r_0$ and $\sigma = 1/c_0$ provides us for every $x \in B_0$ with a chain

$$\mathcal{B}_x = \{B_{i,j} = B_\varepsilon(x_{i,j}, r_i) : i = 0, 1, \dots \text{ and } j = 0, 1, \dots, m_i\}$$

of balls connecting the ball $B_{0,0} := B(z, r_0)$ to x as follows:

- (a) For all i and j we have $m_i \leq Ar/r_0 = 8A^2/a_0c_0$,

$$4r_i \leq c_0 d_\Omega(x_{i,j}) \quad \text{and} \quad d(x_{i,j}, x) \leq 2^{-i} Ad(x, z) < 2^{-i} Ar.$$

- (b) For large i , we have $m_i = 0$ and the balls $B_{i,0}$ are centered at x .
(c) The balls are ordered lexicographically, i.e. $B_{i,j}$ comes before $B_{i',j'}$ if and only if $i < i'$, or $i = i'$ and $j < j'$. If B^* denotes the immediate successor of $B \in \mathcal{B}_x$ then $B \cap B^*$ is nonempty.

Let u be a bounded measurable function on Ω and g be an upper gradient of u in Ω . If $x \in B_0$ is a Lebesgue point of u then

$$|u(x) - u_{B_{0,0}}| = \lim_{i \rightarrow \infty} |u_{B_{i,0}} - u_{B_{0,0}}| \leq \sum_{B \in \mathcal{B}_x} |u_{B^*} - u_B|, \quad (6.4)$$

where $u_B = u_{B,\nu}$ and similarly for other balls. Moreover, $B^* \subset 3B$ and

$$|u_{B^*} - u_B| \leq |u_{B^*} - u_{3B}| + |u_{3B} - u_B|.$$

As $3r_i \leq c_0 d_\Omega(x_{i,j})$ and the radii of B and B^* differ by at most a factor 2, an application of the p -Poincaré inequality on $3B$ shows that

$$|u_{B^*} - u_{3B}| \lesssim \int_{3B} |u - u_{3B}| d\nu \leq Cr(B) \left(\int_{3\lambda B} g^p d\nu \right)^{1/p},$$

where $r(B)$ is the radius of B and λ is the dilation constant in the assumed p -Poincaré inequality for subWhitney balls. The difference $|u_{3B} - u_B|$ is estimated in the same way. Hence, inserting these estimates into (6.4),

$$|u(x) - u_{B_{0,0}}| \lesssim \sum_{B \in \mathcal{B}_x} r(B) \left(\int_{3\lambda B} g^p d\nu \right)^{1/p}.$$

We now wish to estimate the measure of level sets of the function $x \mapsto |u(x) - u_{B_{0,0}}|$ in B_0 . Assume that $|u(x) - u_{B_{0,0}}| \geq t$ and write $t = C_\alpha N t \sum_{i=0}^{\infty} 2^{-i\alpha}$, where $\alpha \in (0, 1)$ will be chosen later, and $N \leq 1 + Ar/R_0 = 1 + 8A^2/a_0 c_0$ is the maximal number of balls in \mathcal{B}_x with the same radius. Then

$$C_\alpha N t \sum_{i=0}^{\infty} 2^{-i\alpha} = t \lesssim \sum_{B \in \mathcal{B}_x} r(B) \left(\int_{3\lambda B} g^p d\nu \right)^{1/p}.$$

Hence, there exists $B_x = B(x_{i,j}, r_i) \in \mathcal{B}_x$ such that

$$C_\alpha 2^{-i\alpha} t \lesssim r_i \left(\int_{3\lambda B_x} g^p d\nu \right)^{1/p}.$$

We have $2^{-i} = r_i/r_0 = 8Ar_i/a_0 c_0 r$, and inserting this into the last inequality yields

$$t \lesssim r \left(\frac{r_i}{r} \right)^{1-\alpha} \left(\int_{3\lambda B_x} g^p d\nu \right)^{1/p}.$$

As ν is globally doubling, there exists $s > 0$ independent of B_x such that

$$\frac{r_i}{r} \lesssim \left(\frac{\nu(3\lambda B_x)}{\nu(B_0)} \right)^{1/s},$$

see e.g. [5, Lemma 3.3] or [30, (3.4.9)]. Hence

$$t \lesssim r \left(\frac{\nu(3\lambda B_x)}{\nu(B_0)} \right)^{(1-\alpha)/s} \left(\int_{3\lambda B_x} g^p d\nu \right)^{1/p},$$

and choosing $\alpha \in (0, 1)$ so that $\theta := 1 - (1 - \alpha)p/s \in (0, 1)$, we obtain

$$\nu(3\lambda B_x)^\theta \lesssim \frac{r^p}{t^p \nu(B_0)^{1-\theta}} \int_{3\lambda B_x} g^p d\nu. \quad (6.5)$$

Let $E_t = \{x \in B_0 : |u(x) - u_{B_{0,0}}| \geq t\}$ and F_t be the set of all points in E_t which are Lebesgue points of u . The global doubling property of ν guarantees that a.e. x is a Lebesgue point of u , see Heinonen [27, Theorem 1.8]. By the above, for every $x \in F_t$ there exists $B_x \in \mathcal{B}_x$ satisfying (6.5). Note also that by construction of the chain, we have $x \in B'_x := 8(a_0 c_0)^{-1} A^2 B_x$. The balls $\{B'_x\}_{x \in F_t}$, therefore cover F_t . The 5-covering lemma (Theorem 1.2 in Heinonen [27]) provides us with a pairwise disjoint collection $\{\lambda B'_{x_i}\}_{i=1}^\infty$ such that the union of all balls $5\lambda B'_{x_i}$ covers F_t . Then the balls $3\lambda B_{x_i} \subset \lambda B'_{x_i}$ are also pairwise disjoint and the global doubling property of ν , together with (6.5), yields

$$\begin{aligned} \nu(E_t) = \nu(F_t) &\leq \sum_{i=1}^\infty \nu(5\lambda B'_{x_i}) \lesssim \sum_{i=1}^\infty \nu(3\lambda B_{x_i}) \\ &\lesssim \frac{r^{p/\theta}}{t^{p/\theta} \nu(B_0)^{1/\theta-1}} \sum_{i=1}^\infty \left(\int_{3\lambda B_{x_i}} g^p d\nu \right)^{1/\theta} \\ &\leq \frac{r^{p/\theta}}{t^{p/\theta} \nu(B_0)^{1/\theta-1}} \left(\int_{\Lambda B_0} g^p d\nu \right)^{1/\theta}, \end{aligned}$$

where Λ depends only on A, λ, a_0 and c_0 . Lemma 4.22 in Heinonen [27], which can be proved using the Cavalieri principle, now implies that

$$\int_{B_0} |u - u_{B_{0,0}}| d\nu \lesssim r \left(\int_{\Lambda B_0} g^p d\nu \right)^{1/p}$$

and a standard argument [5, Lemma 4.17] allows us to replace $u_{B_{0,0}}$ by u_{B_0} .

Since Ω is L -quasiconvex, it follows from [5, Theorem 4.39] that the dilation Λ in the obtained global p -Poincaré inequality can be replaced by L .

Finally, by Proposition 7.1 in Aikawa–Shanmugalingam [1] (or the proof above applied within $\bar{\Omega}$ and with $x_0 \in \bar{\Omega}$), ν supports a global p -Poincaré inequality, where, again using [5, Theorem 4.39], the dilation constant can be chosen to be L' . \square

7. Hyperbolization

We assume in this section that (Ω, d) is a noncomplete L -quasiconvex space which is open in its completion $\bar{\Omega}$, and let $\partial\Omega$ be its boundary within $\bar{\Omega}$.

We define the *quasihyperbolic metric* on Ω by

$$k(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds}{d_{\Omega}(\gamma(s))}, \quad \text{where } d_{\Omega}(x) = \text{dist}(x, \partial\Omega),$$

ds is the arc length parametrization of γ , and the infimum is taken over all rectifiable curves in Ω connecting x to y . It follows that (Ω, k) is a length space. Balls with respect to the quasihyperbolic metric k will be denoted by B_k .

Even though our main interest is in hyperbolizing uniform spaces, the quasihyperbolic metric makes sense in greater generality. In fact, the results in this section hold also if we let $\Omega \subsetneq Y$ be an L -quasiconvex open subset of a (not necessarily complete) metric space Y and the quasihyperbolic metric k is defined using $d_{\Omega}(x) = \text{dist}(x, Y \setminus \Omega)$.

If Ω is a locally compact uniform space, then Theorem 3.6 in Bonk–Heinonen–Koskela [14] shows that the space (Ω, k) is a proper geodesic Gromov hyperbolic space. Moreover, if Ω is bounded, then (Ω, k) is roughly starlike.

As described in the introduction, the operations of uniformization and hyperbolization are mutually opposite, by Bonk–Heinonen–Koskela [14, the discussion before Proposition 4.5].

Lemma 7.1. *Let $x, y \in \Omega$. Then the following are true:*

$$\begin{aligned} k(x, y) &\geq \frac{d(x, y)}{2d_\Omega(x)}, & \text{if } d(x, y) \leq d_\Omega(x), \\ k(x, y) &\geq \frac{1}{2}, & \text{if } d(x, y) \geq d_\Omega(x), \\ \frac{d(x, y)}{2d_\Omega(x)} \leq k(x, y) &\leq \frac{2Ld(x, y)}{d_\Omega(x)}, & \text{if } d(x, y) \leq \frac{d_\Omega(x)}{2L}. \end{aligned} \quad (7.1)$$

Moreover,

$$\begin{aligned} B\left(x, \frac{rd_\Omega(x)}{2L}\right) &\subset B_k(x, r) \subset B(x, 2rd_\Omega(x)), & \text{if } r \leq \frac{1}{2}, \\ B_k\left(x, \frac{r}{2d_\Omega(x)}\right) &\subset B(x, r) \subset B_k\left(x, \frac{2Lr}{d_\Omega(x)}\right), & \text{if } r \leq \frac{d_\Omega(x)}{2L}. \end{aligned}$$

If Ω is A -uniform it is possible to get an upper bound similar to the one in (7.1) also when $d(x, y) \leq \frac{1}{2}d_\Omega(x)$, albeit with a little more complicated expression for the constant. As we will not need such an estimate, we leave it to the interested reader to deduce such a bound.

Proof. Without loss of generality we assume that $x \neq y$.

Assume first that $d(x, y) \leq d_\Omega(x)$. Let $\gamma : [0, l_\gamma] \rightarrow \Omega$ be a curve from x to y . All curves in this proof will be arc length parametrized rectifiable curves in Ω . Then $l_\gamma \geq d(x, y)$ and

$$\int_\gamma \frac{ds}{d_\Omega(\gamma(s))} \geq \int_0^{d(x, y)} \frac{dt}{d_\Omega(x) + t} > \int_0^{d(x, y)} \frac{dt}{2d_\Omega(x)} = \frac{d(x, y)}{2d_\Omega(x)}.$$

Taking infimum over all such γ shows that $k(x, y) \geq d(x, y)/2d_\Omega(x)$.

Suppose next that $d(x, y) \geq d_\Omega(x)$. Let $\gamma : [0, l_\gamma] \rightarrow \Omega$ be a curve from x to y . Then $l_\gamma \geq d(x, y) \geq d_\Omega(x)$ and

$$\int_\gamma \frac{ds}{d_\Omega(\gamma(s))} \geq \int_0^{d_\Omega(x)} \frac{dt}{d_\Omega(x) + t} > \int_0^{d_\Omega(x)} \frac{dt}{2d_\Omega(x)} = \frac{1}{2}.$$

Taking infimum over all such γ shows that $k(x, y) \geq \frac{1}{2}$.

Assume finally that $d(x, y) \leq d_\Omega(x)/2L$. As Ω is L -quasiconvex, there is a curve $\gamma : [0, l_\gamma] \rightarrow \Omega$ from x to y with length $l_\gamma \leq Ld(x, y) \leq \frac{1}{2}d_\Omega(x)$. Then

$$k(x, y) \leq \int_\gamma \frac{ds}{d_\Omega(\gamma(s))} \leq l_\gamma \frac{2}{d_\Omega(x)} \leq \frac{2Ld(x, y)}{d_\Omega(x)}.$$

The ball inclusions now follow directly from this. \square

We shall now equip (Ω, k) with a measure determined by the original measure μ on Ω . As before, for the results in this section it will be enough to assume that Ω is quasiconvex.

Definition 7.2. Let Ω be equipped with a Borel measure μ . For measurable $A \subset \Omega$ and $\alpha > 0$, let

$$\mu^\alpha(A) = \int_A \frac{d\mu(x)}{d_\Omega(x)^\alpha}.$$

Proposition 7.3. Assume that μ is globally doubling in (Ω, d) with doubling constant C_μ . Then μ^α is doubling for B_k -balls of radii at most $R_0 = \frac{1}{8}$, with doubling constant $C_d = 4^\alpha C_\mu^m$, where $m = \lceil \log_2 8L \rceil$.

Moreover, if $R_1 > 0$, then μ^α is doubling for B_k -balls of radii at most R_1 .

Proof. Let $x \in \Omega$, $r \leq \frac{1}{8}$, $B_k = B_k(x, r)$ and $B = B(x, rd_\Omega(x))$. By Lemma 7.1,

$$\mu^\alpha(B_k) \geq \mu^\alpha\left(\frac{1}{2L}B\right) \geq \left(\frac{1}{2d_\Omega(x)}\right)^\alpha \mu\left(\frac{1}{2L}B\right)$$

and hence, again using Lemma 7.1,

$$\begin{aligned} \mu^\alpha(2B_k) &\leq \mu^\alpha(4B) \leq \left(\frac{2}{d_\Omega(x)}\right)^\alpha \mu(4B) \leq \left(\frac{2}{d_\Omega(x)}\right)^\alpha C_\mu^m \mu\left(\frac{1}{2L}B\right) \\ &\leq \left(\frac{2}{d_\Omega(x)}\right)^\alpha (2d_\Omega(x))^\alpha C_\mu^m \mu^\alpha(B_k) = C_d \mu^\alpha(B_k). \end{aligned}$$

As (Ω, k) is a length space, Lemma 3.5 shows that μ^α is doubling for B_k -balls of radii at most R_1 for any $R_1 > 0$. \square

Proposition 7.4. Assume that (Ω, d) is equipped with a globally doubling measure μ supporting a global p -Poincaré inequality with dilation λ and $p \geq 1$. Let $\alpha > 0$ and $R_1 > 0$. Then (Ω, k) , equipped with the measure μ^α , supports a p -Poincaré inequality for balls of radii at most R_1 with dilation L and the other Poincaré constant depending only on L , R_1 and the global doubling and Poincaré constants.

Proof. Let u be a bounded measurable function on Ω and \hat{g} be an upper gradient of u with respect to k . Since the arc length parametrization ds_k with respect to k satisfies

$$ds_k = \frac{ds}{d_\Omega(\cdot)},$$

we conclude that

$$\int_\gamma \hat{g} ds_k = \int_\gamma \frac{\hat{g}}{d_\Omega(\cdot)} ds$$

and thus $g(z) := \hat{g}(z)/d_\Omega(z)$ is an upper gradient of u with respect to d , see the proof of Lemma 6.1 for further details.

Next, let $x \in \Omega$, $0 < r \leq R_0 := 1/8\lambda L$, $B_k = B_k(x, r)$ and $B = B(x, 2rd_\Omega(x))$. We see, by Lemma 7.1, that

$$\frac{1}{4L}B \subset B_k \subset B \quad \text{and} \quad \lambda B \subset 4\lambda L B_k,$$

where all the above balls have comparable μ -measures, as well as comparable μ^α -measures. Note that $d_\Omega(z) \simeq d_\Omega(x)$ for all $z \in 4\lambda L B_k$. Thus,

$$\begin{aligned} \int_{B_k} |u - u_{B, \mu}| d\mu^\alpha &\lesssim \int_B |u - u_{B, \mu}| d\mu \lesssim rd_\Omega(x) \left(\int_{\lambda B} g^p d\mu \right)^{1/p} \\ &\simeq r \left(\int_{\lambda B} \hat{g}^p d\mu \right)^{1/p} \lesssim r \left(\int_{4\lambda L B_k} \hat{g}^p d\mu^\alpha \right)^{1/p}. \end{aligned}$$

A standard argument as in [5, Lemma 4.17] makes it possible to replace $u_{B, \mu}$ on the left-hand side by u_{B_k, μ^α} , and thus Y supports a p -Poincaré inequality for balls of radii $\leq R_0$, with dilation $4\lambda L$. The conclusion now follows from Theorem 5.3. \square

Remark 7.5. Let X be a Gromov hyperbolic space, equipped with a measure μ , and consider its uniformization X_ε , together with the measure μ_β , $\beta > 0$, as in Definition 4.1. With $\alpha = \beta/\varepsilon$, it is then easily verified that the pull-back to X of the measure $(\mu_\beta)^\alpha$, defined on the hyperbolization (X_ε, k) of X_ε , is comparable to the original measure μ .

8. An indirect product of Gromov hyperbolic spaces

We assume in this section that X and Y are two locally compact roughly starlike Gromov δ -hyperbolic spaces. We fix two points $z_X \in X$ and $z_Y \in Y$, and let M be a common constant for the roughly starlike conditions with respect to z_X and z_Y . We also assume that $0 < \varepsilon \leq \varepsilon_0(\delta)$ and that z_X and z_Y serve as centers for the uniformizations X_ε and Y_ε .

In general, the Cartesian product $X \times Y$ of two Gromov hyperbolic spaces X and Y need not be Gromov hyperbolic; for example, $\mathbf{R} \times \mathbf{R}$ is not Gromov hyperbolic. In this section, we shall construct an indirect product metric on $X \times Y$ that does give us a Gromov hyperbolic space, namely we set $X \times_\varepsilon Y$ to be the Gromov hyperbolic space $(X_\varepsilon \times Y_\varepsilon, k)$. To do so, we first need to show that the Cartesian product of two uniform spaces, equipped with the sum of their metrics, is a uniform domain. This can be proved using Theorems 1 and 2 in Gehring–Osgood [22] together with Proposition 2.14 in Bonk–Heinonen–Koskela [14], but this would result in a highly nonoptimal uniformity constant. We instead give a more self-contained proof that also yields a better estimate of the uniformity constant for the Cartesian product.

Example 8.1. Recall that the uniformization \mathbf{R}_ε of the hyperbolic 1-dimensional space \mathbf{R} is isometric to $(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$, see Example 4.2. Hence, for all $\varepsilon > 0$, $\mathbf{R}_\varepsilon \times \mathbf{R}_\varepsilon$ is a planar square region, which is biLipschitz equivalent to the planar disk. Thus also its hyperbolization $\mathbf{R} \times_\varepsilon \mathbf{R}$ is biLipschitz equivalent to the hyperbolic disk, which is the model 2-dimensional hyperbolic space.

Lemma 8.2. *Let (Ω, d) be a bounded A -uniform space. Then for every pair of points $x, y \in \Omega$ and for every L with $d(x, y) \leq L \leq \text{diam } \Omega$, there exists a curve $\gamma \subset \Omega$ of length*

$$\frac{L}{5A} \leq l(\gamma) \leq (A+1)L, \quad (8.1)$$

connecting x to y and such that for all $z \in \gamma$,

$$d_\Omega(z) \geq \frac{1}{16A^2} \min\{l(\gamma_{x,z}), l(\gamma_{z,y})\},$$

where $\gamma_{x,z}$ and $\gamma_{z,y}$ are the subcurves of γ from x to z and from z to y , respectively.

Proof. Choose $x_0 \in \Omega$ such that $d_\Omega(x_0) \geq \frac{4}{5} \sup_{z \in \Omega} d_\Omega(z)$. Then for all $z \in \Omega$, with γ_{z,x_0} being an A -uniform curve from z to x_0 , and z' its midpoint,

$$d(z, x_0) \leq l(\gamma_{z,x_0}) \leq 2Ad_\Omega(z') \leq \frac{5}{2}Ad_\Omega(x_0).$$

Hence $\text{diam } \Omega \leq 5Ad_\Omega(x_0)$. Now, let $x, y \in \Omega$ and L be as in the statement of the lemma. Let γ_{x,x_0} be an A -uniform curve from x to x_0 . We shall distinguish two cases:

1. If $L \leq 5Al(\gamma_{x,x_0})$ then let $\hat{\gamma}_x$ be the restriction of γ to $[0, L/10A]$ and $\hat{x} = \gamma(L/10A)$ be its new endpoint.
2. If $L \geq 5Al(\gamma_{x,x_0})$ then let γ_x be the restriction of γ to $[0, \frac{1}{2}l(\gamma_{x,x_0})]$ and $\hat{x} = \gamma(\frac{1}{2}l(\gamma_{x,x_0}))$ be its new endpoint. Note that

$$d_\Omega(\hat{x}) \geq d_\Omega(x_0) - \frac{l(\gamma_{x,x_0})}{2} \geq \frac{\text{diam } \Omega}{5A} - \frac{L}{10A} \geq \frac{L}{10A}.$$

Choose a curve γ' of length $L/10A$, which starts and ends at \hat{x} . Then for all $z \in \gamma'$,

$$d_\Omega(z) \geq d_\Omega(\hat{x}) - \frac{L}{20A} \geq \frac{L}{20A}.$$

Thus, concatenating γ' to γ_x we obtain a curve $\hat{\gamma}_x$ from x to \hat{x} of length

$$\frac{L}{10A} \leq l(\hat{\gamma}_x) \leq \frac{L}{5A} \quad (8.2)$$

and such that for all $z \in \hat{\gamma}_x$,

$$d_\Omega(z) \geq \frac{1}{\max\{4, A\}} l(\hat{\gamma}_{x,z}) \geq \frac{1}{4A} l(\hat{\gamma}_{x,z}), \quad (8.3)$$

where $\hat{\gamma}_{x,z}$ is the part of $\hat{\gamma}_x$ from x to z . The curve $\hat{\gamma}_x$, obtained in case 1, clearly satisfies (8.2) and (8.3) as well.

A similar construction, using an A -uniform curve from y to x_0 , provides us with a curve $\hat{\gamma}_y$ from y to \hat{y} , satisfying (8.2) and (8.3) with x replaced by y .

Now, let $\tilde{\gamma}$ be a uniform curve from \hat{x} to \hat{y} and let γ be the concatenation of $\hat{\gamma}_x$ with $\tilde{\gamma}$ and $\hat{\gamma}_y$ (reversed). Since $d(\hat{x}, \hat{y}) \leq d(x, y) + 2L/5A \leq (1 + 2/5A)L$, we see that

$$l(\gamma) \leq A\left(1 + \frac{2}{5A}\right)L + \frac{2L}{5A} \leq (A+1)L$$

and the right-hand side inequality in (8.1) holds, while the left-hand side follows from (8.2).

To prove the second property, in view of (8.3), it suffices to consider $z \in \tilde{\gamma}$. Without loss of generality, assume that the part $\tilde{\gamma}_{\hat{x},z}$ of $\tilde{\gamma}$ from \hat{x} to z has length at most $\frac{1}{2}l(\tilde{\gamma})$. Note that (8.3), applied to the choice $z = \hat{x}$, gives

$$d_\Omega(\hat{x}) \geq \frac{1}{4A} l(\hat{\gamma}_x). \quad (8.4)$$

Again, we distinguish two cases.

1. If $\frac{1}{2}d_\Omega(\hat{x}) \geq l(\tilde{\gamma}_{\hat{x},z})$ then by (8.4),

$$d_\Omega(z) \geq d_\Omega(\hat{x}) - l(\tilde{\gamma}_{\hat{x},z}) \geq \frac{1}{2}d_\Omega(\hat{x}) \geq \max\left\{l(\tilde{\gamma}_{\hat{x},z}), \frac{1}{8A}l(\hat{\gamma}_x)\right\},$$

and hence we obtain that

$$d_\Omega(z) \geq \frac{1}{2}d_\Omega(\hat{x}) \geq \frac{1}{16A}(l(\tilde{\gamma}_{\hat{x},z}) + l(\hat{\gamma}_x)) = \frac{1}{16A}l(\gamma_{x,z}),$$

where $\gamma_{x,z}$ is the part of γ from x to z .

2. On the other hand, if $\frac{1}{2}d_\Omega(\hat{x}) \leq l(\tilde{\gamma}_{\hat{x},z})$ then by (8.4) again,

$$d_\Omega(z) \geq \frac{1}{A}l(\tilde{\gamma}_{\hat{x},z}) \geq \frac{1}{2A}d_\Omega(\hat{x}) \geq \frac{1}{8A^2}l(\hat{\gamma}_x),$$

We conclude that

$$d_\Omega(z) \geq \frac{1}{16A^2}(l(\tilde{\gamma}_{\hat{x},z}) + l(\hat{\gamma}_x)) = \frac{1}{16A^2}l(\gamma_{x,z}). \quad \square$$

Proposition 8.3. *Let (Ω, d) and (Ω', d') be two bounded uniform spaces, with diameters and uniformity constants D, D', A and A' , respectively. Then $\tilde{\Omega} = \Omega \times \Omega'$ is also a bounded uniform space with respect to the metric*

$$\tilde{d}((x, x'), (y, y')) = d(x, y) + d'(x', y'), \quad (8.5)$$

with uniformity constant

$$\tilde{A} = \frac{80[(A+1)D + (A'+1)D']}{\min\{D/A^3, D'/(A')^3\}}.$$

Proof. The boundedness is clear. Let $\tilde{x} = (x, x')$ and $\tilde{y} = (y, y')$ be two distinct points in $\tilde{\Omega}$, and let

$$\Lambda = \max\left\{\frac{d(x, y)}{D}, \frac{d'(x', y')}{D'}\right\} \leq 1, \quad L = \Lambda D \geq d(x, y) \quad \text{and} \quad L' = \Lambda D' \geq d'(x', y').$$

Note that

$$\Lambda(D + D') \geq \tilde{d}(\tilde{x}, \tilde{y}) \geq \Lambda \min\{D, D'\} \geq \Lambda \min\left\{\frac{D}{A^3}, \frac{D'}{(A')^3}\right\}. \quad (8.6)$$

We use Lemma 8.2 to find curves $\gamma \subset \Omega$ and $\gamma' \subset \Omega'$, connecting x to y and x' to y' , respectively, of lengths

$$\frac{L}{5A} \leq l(\gamma) \leq (A + 1)L \quad \text{and} \quad \frac{L'}{5A'} \leq l(\gamma') \leq (A' + 1)L', \quad (8.7)$$

and such that for all $z \in \gamma$,

$$d_{\Omega}(z) \geq \frac{1}{16A^2} \min\{l(\gamma_{x,z}), l(\gamma_{z,y})\}, \quad (8.8)$$

where $\gamma_{x,z}$ and $\gamma_{z,y}$ are the parts of γ from x to z and from z to y , respectively; similar statements holding true for $z' \in \gamma'$ and A' . Note that $\Lambda > 0$ since $\tilde{x} \neq \tilde{y}$. Hence $L, L' > 0$ and, by (8.7), the curves γ and γ' are nonconstant.

Next, assuming that γ and γ' are arc length parametrized, we show that the curve

$$\tilde{\gamma}(t) = \left(\gamma\left(\frac{t}{l(\gamma)}\right), \gamma'\left(\frac{t}{l(\gamma')}\right)\right), \quad t \in [0, 1],$$

is an \tilde{A} -uniform curve in $\tilde{\Omega}$ connecting \tilde{x} to \tilde{y} . To see this, note that we have by the definition (8.5) of \tilde{d} that for all $0 \leq s \leq t \leq 1$, using (8.7) and then (8.6),

$$\begin{aligned} l(\tilde{\gamma}|_{[s,t]}) &= (t-s)(l(\gamma) + l(\gamma')) \leq (t-s)[(A+1)D + (A'+1)D']\Lambda \\ &\leq (t-s)\tilde{A}\tilde{d}(\tilde{x}, \tilde{y}). \end{aligned}$$

In particular, $\tilde{\gamma}$ has the correct length. Since

$$\partial\tilde{\Omega} = (\partial\Omega \times \Omega') \cup (\Omega \times \partial\Omega') \cup (\partial\Omega \times \partial\Omega'),$$

we see that for all $\tilde{\gamma}(t) = (z, z')$ with $0 \leq t \leq \frac{1}{2}$, using (8.8) and then (8.7),

$$d_{\Omega}(z) \geq \frac{l(\gamma)t}{16A^2} \geq \frac{Lt}{80A^3} = \frac{\Lambda Dt}{80A^3},$$

and similarly $d_{\Omega'}(z') \geq \Lambda D't/80(A')^3$. Thus, using (8.7) for the last inequality,

$$\begin{aligned} d_{\tilde{\Omega}}(\tilde{\gamma}(t)) &= \min\{d_{\Omega}(z), d_{\Omega'}(z')\} \geq \frac{\Lambda t}{80} \min\left\{\frac{D}{A^3}, \frac{D'}{(A')^3}\right\} \\ &= \frac{t}{\tilde{A}}[(A+1)L + (A'+1)L'] \geq \frac{t}{\tilde{A}}[l(\gamma) + l(\gamma')] = \frac{l(\tilde{\gamma}_{\tilde{x}, \tilde{\gamma}(t)})}{\tilde{A}}. \end{aligned}$$

As a similar estimate holds for $\frac{1}{2} \leq t \leq 1$, we see that $\tilde{\gamma}$ is indeed an \tilde{A} -uniform curve. \square

We next see that the projection map $\pi : X \times_{\varepsilon} Y \rightarrow X$ given by $\pi((x, y)) = x$ is Lipschitz continuous.

Proposition 8.4. *The above-defined projection map $\pi : X \times_\varepsilon Y \rightarrow X$ is (C/ε) -Lipschitz continuous, with C depending only on ε_0 and M .*

Proof. Since $X \times_\varepsilon Y$ is geodesic, it suffices to show that π is locally (C/ε) -Lipschitz with C independent of the locality. With $C_1 = e^{-(1+\varepsilon M)}$ and $C_2 = 2\varepsilon(2e^{\varepsilon M} - 1)$ as in Theorem 2.10, for $(x, y) \in X \times Y$ let

$$r = \frac{C_1 \min\{d_\varepsilon(x), d_\varepsilon(y)\}}{2C_2}, \quad x' \in B_\varepsilon(x, r) \quad \text{and} \quad y' \in B_\varepsilon(y, r).$$

The last part of Theorem 2.10 together with Lemma 2.8 then gives

$$d(\pi(x, y), \pi(x', y')) = d(x, x') \simeq \frac{d_\varepsilon(x, x')}{\rho_\varepsilon(x)} \simeq \frac{d_\varepsilon(x, x')}{\varepsilon d_\varepsilon(x)},$$

with comparison constants depending only on ε_0 and M .

Let k_ε denote the quasihyperbolic metric on $\Omega := X_\varepsilon \times Y_\varepsilon$. Note that since both X_ε and Y_ε are length spaces, so is Ω . As $C_2/C_1 > 2e$, we see that

$$d_\Omega((x, y)) = \min\{d_\varepsilon(x), d_\varepsilon(y)\} > 2e(d_\varepsilon(x, x') + d_\varepsilon(y, y'))$$

and thus (7.1) in Lemma 7.1 with $L = e$ yields

$$k_\varepsilon((x, y), (x', y')) \simeq \frac{d_\varepsilon(x, x') + d_\varepsilon(y, y')}{\min\{d_\varepsilon(x), d_\varepsilon(y)\}}. \quad (8.9)$$

It follows that

$$d(\pi(x, y), \pi(x', y')) \lesssim \frac{1}{\varepsilon} k_\varepsilon((x, y), (x', y')). \quad \square$$

Next, we shall see how $X \times_\varepsilon Y$ compares to $X \times_{\varepsilon'} Y$.

Proposition 8.5. *Let $0 < \varepsilon' < \varepsilon \leq \varepsilon_0(\delta)$. The canonical identity maps*

$$\Phi : X \times_\varepsilon Y \rightarrow X \times_{\varepsilon'} Y \quad \text{and} \quad \Psi : X_{\varepsilon'} \times Y_{\varepsilon'} \rightarrow X_\varepsilon \times Y_\varepsilon$$

are Lipschitz continuous. More precisely, there is a constant C' , depending only on ε_0 and M , such that Φ is $(C'\varepsilon'/\varepsilon)$ -Lipschitz while Ψ is C' -Lipschitz.

Moreover, neither Φ^{-1} nor Ψ^{-1} is Lipschitz continuous.

Proof. We first consider Φ . Since $X \times_\varepsilon Y$ is geodesic, it suffices to show that Φ is locally $(C'\varepsilon'/\varepsilon)$ -Lipschitz with C' independent of the locality. As in the proof of Proposition 8.4, for $(x, y) \in X \times Y$ and C_1, C_2 from Theorem 2.10, let

$$r = \frac{C_1 \min\{d_\varepsilon(x), d_{\varepsilon'}(x), d_\varepsilon(y), d_{\varepsilon'}(y)\}}{2C_2}, \quad x' \in B_\varepsilon(x, r) \quad \text{and} \quad y' \in B_\varepsilon(y, r).$$

Theorem 2.10 then gives

$$d_\varepsilon(x, x') \simeq \rho_\varepsilon(x) d(x, x') \quad \text{and} \quad d_\varepsilon(y, y') \simeq \rho_\varepsilon(y) d(y, y'). \quad (8.10)$$

Let $\tilde{d}_\varepsilon, \tilde{d}_{\varepsilon'}, k_\varepsilon$ and $k_{\varepsilon'}$ denote the product metrics as in (8.5) and the quasihyperbolic metrics on $X_\varepsilon \times Y_\varepsilon$ and $X_{\varepsilon'} \times Y_{\varepsilon'}$ respectively. As in (8.9), we conclude that

$$k_\varepsilon((x, y), (x', y')) \simeq \frac{d_\varepsilon(x, x') + d_\varepsilon(y, y')}{\min\{d_\varepsilon(x), d_\varepsilon(y)\}}.$$

Without loss of generality we assume that $\rho_\varepsilon(x) \leq \rho_\varepsilon(y)$, and then using Lemma 2.8,

$$d_\varepsilon(x) \simeq \frac{\rho_\varepsilon(x)}{\varepsilon} \leq \frac{\rho_\varepsilon(y)}{\varepsilon} \simeq d_\varepsilon(y),$$

in which case we also have that

$$d_{\varepsilon'}(x) \simeq \frac{\rho_{\varepsilon'}(x)}{\varepsilon'} \leq \frac{\rho_{\varepsilon'}(y)}{\varepsilon'} \simeq d_{\varepsilon'}(y).$$

Therefore, using (8.10),

$$k_{\varepsilon}((x, y), (x', y')) \simeq \frac{d_{\varepsilon}(x, x') + d_{\varepsilon}(y, y')}{d_{\varepsilon}(x)} \simeq \varepsilon \left(d(x, x') + \frac{\rho_{\varepsilon}(y)}{\rho_{\varepsilon}(x)} d(y, y') \right), \quad (8.11)$$

with a similar statement holding true for ε' . Since

$$\frac{\rho_{\varepsilon'}(y)}{\rho_{\varepsilon'}(x)} = \left(\frac{\rho_{\varepsilon}(y)}{\rho_{\varepsilon}(x)} \right)^{\varepsilon'/\varepsilon} \leq \frac{\rho_{\varepsilon}(y)}{\rho_{\varepsilon}(x)},$$

we conclude from (8.11) that

$$k_{\varepsilon'}((x, y), (x', y')) \lesssim \frac{\varepsilon'}{\varepsilon} k_{\varepsilon}((x, y), (x', y')),$$

which proves the Lipschitz continuity of Φ .

We now compare the product uniform domains $X_{\varepsilon} \times Y_{\varepsilon}$ and $X_{\varepsilon'} \times Y_{\varepsilon'}$. With $(x, y), (x', y') \in X \times Y$ as in the first part of the proof, we have by (8.10) and the assumption $0 < \varepsilon' < \varepsilon$ that

$$\begin{aligned} \tilde{d}_{\varepsilon}((x, y), (x', y')) &= d_{\varepsilon}(x, x') + d_{\varepsilon}(y, y') \\ &\simeq \rho_{\varepsilon}(x) d(x, x') + \rho_{\varepsilon}(y) d(y, y') \\ &\leq \rho_{\varepsilon'}(x) d(x, x') + \rho_{\varepsilon'}(y) d(y, y') \\ &\simeq \tilde{d}_{\varepsilon'}((x, y), (x', y')), \end{aligned}$$

which proves the Lipschitz continuity of Ψ . On the other hand, choosing $y = y' = z_Y$, with $\rho_{\varepsilon}(z_Y) = 1$, gives

$$\frac{\tilde{d}_{\varepsilon'}((x, z_Y), (x', z_Y))}{\tilde{d}_{\varepsilon}((x, z_Y), (x', z_Y))} \simeq \frac{\rho_{\varepsilon'}(x)}{\rho_{\varepsilon}(x)} = \rho_{\varepsilon}(x)^{-1+\varepsilon'/\varepsilon}.$$

Since $\varepsilon' < \varepsilon$, letting $d(x, z_X) \rightarrow \infty$ and so $\rho_{\varepsilon}(x) \rightarrow 0$ shows that Ψ^{-1} is *not* Lipschitz.

To show that Φ^{-1} is *not* Lipschitz, let $x_j \in X$ be such that $\rho_{\varepsilon}(x_j) \rightarrow 0$ (and equivalently, $\rho_{\varepsilon'}(x_j) \rightarrow 0$) as $j \rightarrow \infty$. With $C(\delta)$ as in (2.2) and C_1, C_2 as in Theorem 2.10, for $j = 1, 2, \dots$ we choose $y_j \in Y$ such that

$$d(z_Y, y_j) = \frac{C_1 d_{\varepsilon}(x_j)}{4C_2 C(\delta)}.$$

This is possible since Y is geodesic. Then, for sufficiently large j , we have $\varepsilon d(z_Y, y_j) \leq 1$ and hence by (2.2),

$$d_{\varepsilon}(z_Y, y_j) \leq C(\delta) d(z_Y, y_j) = \frac{C_1 d_{\varepsilon}(x_j)}{4C_2}.$$

Since also $\rho_{\varepsilon}(x_j) \leq 1 = \rho_{\varepsilon}(z_Y)$, we thus conclude from (8.11), with the choice $x = x' = x_j$, $y = z_Y$ and $y' = y_j$, that

$$k_{\varepsilon}((x_j, z_Y), (x_j, y_j)) \simeq \frac{\varepsilon d(z_Y, y_j)}{\rho_{\varepsilon}(x_j)},$$

with a similar statement holding also for ε' . This shows that

$$\frac{k_{\varepsilon}((x_j, z_Y), (x_j, y_j))}{k_{\varepsilon'}((x_j, z_Y), (x_j, y_j))} \simeq \frac{\varepsilon \rho_{\varepsilon'}(x_j)}{\varepsilon' \rho_{\varepsilon}(x_j)} = \frac{\varepsilon}{\varepsilon'} \rho_{\varepsilon}(x_j)^{-1+\varepsilon'/\varepsilon} \rightarrow \infty, \quad \text{as } j \rightarrow \infty.$$

i.e. Φ^{-1} is not Lipschitz. \square

Remark 8.6. If $X = Y = \mathbf{R}$ then, according to Example 8.1, all the indirect products $\mathbf{R} \times_\varepsilon \mathbf{R}$ are mutually biLipschitz equivalent. However, Proposition 8.5 shows that this equivalence cannot be achieved by the canonical identity map Φ .

By Theorem 1.1 in Bonk–Heinonen–Koskela [14], Φ is biLipschitz if and only if Ψ is a quasimilarity. Note that X_ε and $X_{\varepsilon'}$ are quasisymmetrically equivalent by [14], and so are Y_ε and $Y_{\varepsilon'}$. On the other hand, products of quasisymmetric maps need not be quasisymmetric, as exhibited by the Rickman’s rug $([0, 1], d_{\text{Euc}}) \times ([0, 1], d_{\text{Euc}}^\alpha)$ for $0 < \alpha < 1$, see Bishop–Tyson [2, Remark 1, Section 5] and DiMarco [21, Section 1]. This seems to happen whenever one of the component spaces has dimension 1 and the other has dimension larger than 1.

Example 8.7. Let X be the unit disk in \mathbf{R}^2 , equipped with the Poincaré metric k , making it a Gromov hyperbolic space. Let $Y = (-1, 1)$ be equipped with the quasihyperbolic metric (and so it is isometric to \mathbf{R} , see Examples 4.2 and 4.3). For both X and Y we can choose $\varepsilon = 1$, resulting in X_1 being the Euclidean unit disk and Y_1 being the Euclidean interval $(-1, 1)$. Thus $X_1 \times Y_1$ is a solid 3-dimensional Euclidean cylinder, with boundary made up of $\mathbf{S}^1 \times [-1, 1]$ together with two copies of the disk.

Choosing $0 < \varepsilon < 1$, we instead obtain X_ε and Y_ε , with Y_ε isometric to the Euclidean interval $(-1/\varepsilon, 1/\varepsilon)$, see Example 4.2. The boundary of $X_\varepsilon \times Y_\varepsilon$ is made up of two copies of X_ε together with $Z \times [-1/\varepsilon, 1/\varepsilon]$, where Z is the ε -snowflaking of \mathbf{S}^1 , which results in Z being biLipschitz equivalent to a generalized von Koch snowflake loop.

If $X \times_\varepsilon Y$ were biLipschitz equivalent to $X \times_1 Y$, then $Z \times [-1/\varepsilon, 1/\varepsilon]$ would be quasisymmetrically equivalent to a 2-dimensional region in $\partial(X_1 \times Y_1)$, which is impossible as pointed out before this example.

9. Newtonian spaces and p -harmonic functions

We assume in this section that $1 \leq p < \infty$ and that $Y = (Y, d, \nu)$ is a metric space equipped with a complete Borel measure ν such that $0 < \nu(B) < \infty$ for all balls $B \subset Y$.

For proofs of the facts stated in this section we refer the reader to Björn–Björn [5] and Heinonen–Koskela–Shanmugalingam–Tyson [30].

Following Shanmugalingam [36], we define a version of Sobolev spaces on Y .

Definition 9.1. For a measurable function $u : Y \rightarrow [-\infty, \infty]$, let

$$\|u\|_{N^{1,p}(Y)} = \left(\int_Y |u|^p d\nu + \inf_g \int_Y g^p d\nu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of u . The *Newtonian space* on Y is

$$N^{1,p}(Y) = \{u : \|u\|_{N^{1,p}(Y)} < \infty\}.$$

In this paper we assume that functions in $N^{1,p}(Y)$ are defined everywhere (with values in $[-\infty, \infty]$), not just up to an equivalence class in the corresponding function space. This is important in Definition 5.1, to make sense of g being an upper gradient of u . The space $N^{1,p}(Y)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(Y)} = 0$, is a Banach space and a lattice. For a measurable set $E \subset Y$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \nu|_E)$ as a metric space in its own right. We say that $f \in N_{\text{loc}}^{1,p}(\Omega)$, where Ω is an open subset of X , if for every $x \in \Omega$ there exists $r_x > 0$ such that $B(x, r_x) \subset \Omega$ and $f \in N^{1,p}(B(x, r_x))$. The space $L_{\text{loc}}^p(\Omega)$ is defined similarly.

Definition 9.2. The (Sobolev) *capacity* of a set $E \subset Y$ is the number

$$C_p(E) := C_p^Y(E) := \inf_u \|u\|_{N^{1,p}(Y)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(Y)$ such that $u = 1$ on E .

A property is said to hold *quasi*everywhere (q.e.) if the set of all points at which the property fails has C_p -capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(Y)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in N_{\text{loc}}^{1,p}(Y)$ and $u = v$ a.e., then $u = v$ q.e.

We will also need the variational capacity.

Definition 9.3. Let $\Omega \subset Y$ be open. Then

$$N_0^{1,p}(\Omega) := \{u|_\Omega : u \in N^{1,p}(Y) \text{ and } u = 0 \text{ on } Y \setminus \Omega\}.$$

The *variational capacity* of $E \subset \Omega$ with respect to Ω is

$$\text{cap}_p(E, \Omega) := \text{cap}_p^Y(E, \Omega) := \inf_u \int_\Omega g_u^p d\nu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u = 1$ on E .

The following lemma provides us with a sufficient condition for when a set has positive capacity, in terms of Hausdorff measures. It is similar to Proposition 4.3 in Lehrbäck [35], but the dimension condition for s is weaker here and is only required for $x \in K$. For the reader's convenience, we provide a complete proof. We will use Lemma 9.4 to deduce Proposition 10.10.

Lemma 9.4. *Let (Y, d, ν) be a complete metric space equipped with a globally doubling measure ν supporting a global p -Poincaré inequality. Let $E \subset Y$ be a Borel set of positive κ -dimensional Hausdorff measure and assume that for some $C, s, r_0 > 0$,*

$$\nu(B(x, r)) \geq Cr^s \quad \text{for all } x \in E \text{ and all } 0 < r \leq r_0, \quad (9.1)$$

Then $C_p^Y(E) > 0$ whenever $p > s - \kappa$.

Note that if (9.1) holds for some r_0 , then it holds with $r_0 = 1$, although C may change.

Proof. By the regularity of the Hausdorff measure, there is a compact set $K \subset E$ with positive κ -dimensional Hausdorff measure. Assume that $C_p^Y(K) = 0$. Then also the variational capacity $\text{cap}_p^Y(K, B) = 0$ for every ball $B \supset K$. By splitting K into finitely many pieces if necessary, and shrinking B , we can assume that $\nu(2B \setminus B) > 0$.

As $\text{cap}_p^Y(K, B) = 0$, it follows from [5, Theorem 6.19] that there are

$$u_k \in \text{Lip}_0(B) := \{\varphi \in \text{Lip}(Y) : \varphi = 0 \text{ in } Y \setminus B\}$$

with upper gradients g_k such that $u_k = 1$ on K , $0 \leq u_k \leq 1$ on Y and

$$\int_Y g_k^p d\nu \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We can assume that $r_0 \leq \text{dist}(K, Y \setminus B)$ and set $r_j = 2^{-j}r_0$, $j = 0, 1, \dots$. For a fixed $x \in K$, consider the balls $B_j = B(x, r_j)$. A standard telescoping argument,

using the doubling property of ν together with the p -Poincaré inequality, then shows that for a fixed k and $u := u_k$,

$$\begin{aligned} |u(x) - u_{B_0}| &\leq \sum_{j=0}^{\infty} |u_{B_{j+1}} - u_{B_j}| \lesssim \sum_{j=0}^{\infty} \int_{B_j} |u - u_{B_j}| d\nu \\ &\lesssim \sum_{j=0}^{\infty} \frac{r_j}{\nu(B_j)^{1/p}} \left(\int_{\lambda B_j} g_k^p d\nu \right)^{1/p}. \end{aligned} \quad (9.2)$$

Because u vanishes outside B and $\nu(2B \setminus B) > 0$, we see that

$$u_{2B} := \int_{2B} u d\nu \leq \frac{\nu(B)}{\nu(2B)} =: 1 - 2\theta < 1.$$

Moreover,

$$|u_{2B} - u_{B_0}| \lesssim \frac{r_B}{\nu(2\lambda B)^{1/p}} \left(\int_{2\lambda B} g_k^p d\nu \right)^{1/p} \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

where r_B stands for the radius of B . Since $u(x) = 1$, we conclude that for sufficiently large k , independently of $x \in K$,

$$|u(x) - u_{B_0}| \geq |u(x) - u_{2B}| - |u_{2B} - u_{B_0}| \geq 2\theta - |u_{2B} - u_{B_0}| \geq \theta \simeq \sum_{j=0}^{\infty} r_j^\tau,$$

where $\tau = 1 - (s - \kappa)/p > 0$. Inserting this into (9.2) and comparing the sums, we see that for each $x \in K$ there exists a ball $B_x = B_{j(x)}$ centered at x and with radius $r_x = r_{j(x)}$ such that

$$\int_{\lambda B_x} g_k^p d\nu \gtrsim \frac{\nu(B_x)}{r_x^{p(1-\tau)}} \gtrsim r_x^\kappa, \quad (9.3)$$

because of the assumption (9.1). Using the 5-covering lemma, we can out of the balls λB_x choose a countable pairwise disjoint subcollection $\lambda \widehat{B}_j$, $j = 1, 2, \dots$, with radii \widehat{r}_j , so that $K \subset \bigcup_{j=1}^{\infty} 5\lambda \widehat{B}_j$. Hence using (9.3) we obtain

$$\sum_{j=1}^{\infty} \widehat{r}_j^\kappa \lesssim \sum_{j=1}^{\infty} \int_{\lambda \widehat{B}_j} g_k^p d\nu \lesssim \int_B g_k^p d\nu \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

showing that the κ -dimensional Hausdorff content (and thus also the corresponding measure) is zero. This causes a contradiction, which concludes the proof. \square

Definition 9.5. Let $\Omega \subset Y$ be open. Then $u \in N_{\text{loc}}^{1,p}(\Omega)$ is p -harmonic in Ω if it is continuous and

$$\int_{\varphi \neq 0} g_u^p d\nu \leq \int_{\varphi \neq 0} g_{u+\varphi}^p d\nu \quad \text{for all } \varphi \in N_0^{1,p}(\Omega). \quad (9.4)$$

This is one of several equivalent definitions in the literature, see Björn [3, Proposition 3.2 and Remark 3.3] (or [5, Proposition 7.9 and Remark 7.10]). In particular, multiplying φ by suitable cut-off functions shows that the inequality in (9.4) can equivalently be required for all $\varphi \in N_0^{1,p}(\Omega)$ with bounded support.

If ν is locally doubling and supports a local p -Poincaré inequality then every $u \in N_{\text{loc}}^{1,p}(\Omega)$ satisfying (9.4) can be modified on a set of zero capacity to become continuous, and thus p -harmonic, see Kinnunen–Shanmugalingam [33, Theorem 5.2]. Moreover, it follows from [33, Corollary 6.4] that p -harmonic functions obey the strong maximum principle, i.e. if Ω is connected, then they cannot attain their maximum in Ω without being constant.

Definition 9.6. A metric space Y is *locally annularly quasiconvex* around a point x_0 if there exist $\Lambda \geq 2$ and $r_0 > 0$ such that for every $0 < r \leq r_0$, each pair of points $x, y \in B(x_0, 2r) \setminus B(x_0, r)$ can be connected within $B(x_0, \Lambda r) \setminus B(x_0, r/\Lambda)$ by a curve of length at most $\Lambda d(x, y)$.

Lemma 9.7. *Let Y be a complete metric space equipped with a globally doubling measure ν supporting a global p -Poincaré inequality. Assume that a connected open set $\Omega \subset Y$ is locally annularly quasiconvex around $x_0 \in \Omega$, with parameters Λ and $r_0 < \text{dist}(x_0, Y \setminus \Omega)/2\Lambda$. Let u be a p -harmonic function in $\Omega \setminus \{x_0\}$. Then for every $0 < r \leq r_0$,*

$$\text{osc}_{B(x_0, 2r) \setminus \{x_0\}} u \leq C \left(\sum_{k=0}^{\infty} \left(\frac{(2^{-k}r)^p}{\nu(B(x_0, 2^{-k}r))} \right)^{1/(p-1)} \right)^{1-1/p} \left(\int_{B(x_0, 2\Lambda r)} g_u^p d\nu \right)^{1/p}, \quad (9.5)$$

where C depends only on Λ and the global doubling and Poincaré constants.

Here $\text{dist}(x_0, \emptyset)$ is considered to be ∞ .

Remark 9.8. Under the assumptions of Lemma 9.7 and the additional assumption $r < \frac{1}{4} \text{diam } Y$, the sum in (9.5) is, by e.g. [5, Proposition 6.16], comparable to

$$\sum_{k=0}^{\infty} \text{cap}_p^Y(B(x_0, 2^{-k-1}r), B(x_0, 2^{-k}r))^{1/(1-p)} \leq \text{cap}_p^Y(\{x_0\}, B(x_0, r))^{1/(1-p)},$$

where the last inequality follows from Lemma 2.6 in Heinonen–Kilpeläinen–Martio [28] whose proof applies verbatim also in the metric space setting. Thus if $\text{cap}_p^Y(\{x_0\}, B(x_0, r))$ is positive, then the above sum is finite.

Proof of Lemma 9.7. We can assume that $r_0 < \frac{1}{2} \text{diam } \Omega$. For $0 < \rho \leq r_0$, find $x, y \in B(x_0, 2\rho) \setminus B(x_0, \rho)$ so that

$$|u(x) - u(y)| \geq \frac{1}{2} \text{osc}_{B(x_0, 2\rho) \setminus B(x_0, \rho)} u. \quad (9.6)$$

Let γ be a curve in the annulus $B(x_0, \Lambda\rho) \setminus B(x_0, \rho/\Lambda)$ provided by the annular quasiconvexity. Along this curve, we can find a chain of balls $\{B_j\}_{j=1}^N$ of radius $\rho/4\Lambda$, such that N is bounded by a constant depending only on Λ and the dilation λ from the p -Poincaré inequality, and

$$\begin{aligned} 2\lambda B_j &\subset B(x_0, 2\Lambda\rho) \setminus B(x_0, \rho/2\Lambda), \quad \text{for } j = 1, \dots, N, \\ B_j \cap B_{j+1} &\neq \emptyset \quad \text{for } j = 1, \dots, N-1. \end{aligned}$$

Using Lemma 4.1 in Björn–Björn–Shanmugalingam [12], we thus get that

$$|u(x) - u(y)| \leq \sum_{j=1}^N \text{osc}_{B_j} u \lesssim \frac{\rho}{4\lambda\Lambda} \sum_{j=1}^N \frac{1}{\nu(B_j)^{1/p}} \left(\int_{2\lambda B_j} g_u^p d\nu \right)^{1/p}.$$

Since ν is globally doubling, we have $\nu(B_j) \simeq \nu(B(x_0, \rho))$ and so by (9.6) and the uniform bound on N ,

$$\text{osc}_{B(x_0, 2\rho) \setminus B(x_0, \rho)} u \lesssim \frac{\rho}{\nu(B(x_0, \rho))^{1/p}} \left(\int_{B(x_0, 2\Lambda\rho) \setminus B(x_0, \rho/2\Lambda)} g_u^p d\nu \right)^{1/p}.$$

Hölder's inequality, together with the last estimate applied to $\rho = r_k = 2^{-k}r$ then yields

$$\begin{aligned} \text{osc}_{B(x_0, 2r) \setminus \{x_0\}} u &\leq \sum_{k=0}^{\infty} \text{osc}_{B(x_0, 2r_k) \setminus B(x_0, r_k)} u \\ &\lesssim \left(\sum_{k=0}^{\infty} \left(\frac{r_k}{\nu(B(x_0, r_k))^{1/p}} \right)^{p/(p-1)} \right)^{1-1/p} \left(\sum_{k=0}^{\infty} \int_{A_k} g_u^p d\nu \right)^{1/p}, \end{aligned}$$

where $A_k = B(x_0, 2\Lambda r_k) \setminus B(x_0, r_k/2\Lambda)$. These annuli have clearly bounded overlap depending only on Λ , and so (9.5) follows. \square

The following lemma will be used when proving Theorem 10.5.

Lemma 9.9. *Let (Ω, d) be an A -uniform space and $a \in \partial\Omega$. Then $\overline{\Omega} \setminus \{a\}$ is locally annularly quasiconvex around a with $\Lambda = 4A$.*

Proof. Let $r > 0$ and assume that $x, y \in \Omega \cap B(a, 2r) \setminus B(a, \frac{1}{2}r)$. Let γ be an arc length parametrized A -uniform curve joining x to y . Since $l_\gamma \leq Ad(x, y) \leq 4Ar$, we have $\gamma \subset B(a, (2 + 2A)r)$. Now, if $\frac{1}{4}r \leq t \leq l_\gamma - \frac{1}{4}r$, then

$$d(a, \gamma(t)) \geq d_\Omega(\gamma(t)) \geq \frac{1}{A} \min\{t, l_\gamma - t\} \geq \frac{r}{4A}.$$

Similarly, if $d(x, \gamma(t)) < \frac{1}{4}r$ or $d(y, \gamma(t)) < \frac{1}{4}r$ then $d(a, \gamma(t)) > \frac{1}{4}r$. In both cases it follows that $\gamma \cap B(a, \frac{1}{4}r) = \emptyset$, and the lemma is proved under the assumption that $x, y \in \Omega$.

Finally, if $x, y \in \overline{\Omega} \cap B(a, 2r) \setminus B(a, r)$ with $x \neq y$, then we find

$$x', y' \in \Omega \cap (B(a, 2r) \setminus B(a, \frac{1}{2}r)) \quad \text{with} \quad d(x, x') \leq \frac{d(x, y)}{8A} \quad \text{and} \quad d(y, y') \leq \frac{d(x, y)}{8A}.$$

By the definition of uniform space, Ω is A -quasiconvex, and hence so is $\overline{\Omega}$. Join x' to y' by a curve γ as in the first part of the proof. Concatenating γ with the A -quasiconvex curves, joining x to x' and y' to y , gives a suitable curve $\tilde{\gamma}$ with length

$$l_{\tilde{\gamma}} \leq Ad(x', y') + 2 \cdot \frac{1}{8}d(x, y) \leq (A + 1)d(x, y),$$

which concludes the proof. \square

10. p -harmonic functions on X and X_ε

In this section, we assume that X is a locally compact roughly starlike Gromov δ -hyperbolic space equipped with a complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We also fix a point $z_0 \in X$, let M be the constant in the roughly starlike condition with respect to z_0 , and assume that

$$0 < \varepsilon \leq \varepsilon_0(\delta), \quad \beta > 0 \quad \text{and} \quad 1 \leq p < \infty.$$

Finally, we let X_ε be the uniformization of X with uniformization center z_0 . When discussing the uniformization X_ε , and in particular $C_p^{\overline{X}_\varepsilon}$, it will always be assumed to be equipped with μ_β for the β given above.

In this section we shall see that with suitable choices of p , ε and β satisfying $\beta = p\varepsilon$, each p -harmonic function on the unbounded Gromov hyperbolic space (X, d, μ) transforms into a p -harmonic function on the bounded space $(X_\varepsilon, d_\varepsilon, \mu_\beta)$. This fact will make it possible to characterize, under uniformly local assumptions, when there are no nonconstant p -harmonic functions with finite p -energy on X , i.e. when the *finite-energy Liouville theorem* holds. A function u has *finite p -energy* with respect to (X, d, μ) if $\int_X g_u^p d\mu < \infty$.

In the setting of complete metric spaces, equipped with a globally doubling measure supporting a global p -Poincaré inequality, it was shown in Björn–Björn–Shanmugalingam [12, Theorem 1.1] that the finite-energy Liouville theorem holds on X whenever X is either annularly quasiconvex around a point or

$$\limsup_{r \rightarrow \infty} \frac{\mu(B(x_0, r))}{r^p} > 0 \quad \text{for some fixed point } x_0.$$

The focus of this section will be to consider the finite-energy Liouville theorem for Gromov hyperbolic spaces under uniformly local assumptions.

Proposition 10.1. *Let $\Omega \subset X$ be open and $u : \Omega \rightarrow [-\infty, \infty]$ be measurable. Then the following are true:*

- (a) *With g_u and $g_{u,\varepsilon}$ denoting the minimal p -weak upper gradients of u with respect to (d, μ) and $(d_\varepsilon, \mu_\beta)$, respectively, we have*

$$g_{u,\varepsilon}(x) = g_u(x)e^{\varepsilon d(x,z_0)} \quad (10.1)$$

and

$$\int_{\Omega} g_u(x)^p d\mu(x) = \int_{\Omega} g_{u,\varepsilon}(x)^p e^{(\beta-p\varepsilon)d(x,z_0)} d\mu_\beta(x). \quad (10.2)$$

- (b) $N_{\text{loc}}^{1,p}(\Omega, d, \mu) = N_{\text{loc}}^{1,p}(\Omega, d_\varepsilon, \mu_\beta)$.

- (c) *If Ω is bounded, then $N^{1,p}(\Omega, d, \mu) = N^{1,p}(\Omega, d_\varepsilon, \mu_\beta)$, as sets and with comparable norms (depending only on ε, β, p and Ω).*

Remark 10.2. At first glance it would seem that the minimal p -weak upper gradient $g_{u,\varepsilon}$ of u would also depend on the ambient measure μ_β , but because of the local nature of minimal weak upper gradients and by the fact that the weight $x \mapsto e^{-\beta d(x,z_0)}$ is locally bounded away from both 0 and ∞ , it follows that $g_{u,\varepsilon}$ indeed does not depend on the choice of β , see the proof below.

Proof of Proposition 10.1. Clearly, (b) follows directly from (c). To prove (a) and (c), we conclude from (6.3) that $g_\varepsilon(x) := g(x)e^{\varepsilon d(x,z_0)}$ is an upper gradient of u with respect to d_ε if and only if g is an upper gradient of u with respect to d . Since p -weak upper gradients can be approximated by upper gradients, both in the L^p -norm and also pointwise almost everywhere with respect to μ and (equivalently) μ_β , this identity holds also for p -weak upper gradients. In particular, (10.1) and (10.2) hold, which proves part (a).

If Ω is bounded, we also have that μ and μ_β are comparable on Ω , which implies that

$$\int_{\Omega} |u|^p d\mu \simeq \int_{\Omega} |u|^p d\mu_\beta$$

with comparison constants depending on β and Ω . Together with (10.2), this implies that $u \in N^{1,p}(\Omega, d, \mu)$ if and only if $u \in N^{1,p}(\Omega, d_\varepsilon, \mu_\beta)$, with comparable norms. \square

Remark 10.3. The proof of Proposition 10.1 also shows that even if Ω is not bounded then for $\beta \geq p\varepsilon$,

$$\|u\|_{N^{1,p}(\Omega, d_\varepsilon, \mu_\beta)} \leq \|u\|_{N^{1,p}(\Omega, d, \mu)}$$

and thus $N^{1,p}(\Omega, d, \mu) \subset N^{1,p}(\Omega, d_\varepsilon, \mu_\beta)$.

Proposition 10.4. *Let $\Omega \subset X$ be open. If $p = \beta/\varepsilon > 1$, then a function $u : \Omega \rightarrow \mathbf{R}$ is p -harmonic in Ω with respect to (d, μ) if and only if it is p -harmonic in Ω with respect to $(d_\varepsilon, \mu_\beta)$. Moreover, its p -energy is the same in both cases, i.e.*

$$\int_{\Omega} g_u^p d\mu = \int_{\Omega} g_{u,\varepsilon}^p d\mu_\beta. \quad (10.3)$$

Proof. By Proposition 10.1 (b), $u \in N_{\text{loc}}^{1,p}(\Omega, d, \mu)$ if and only if $u \in N_{\text{loc}}^{1,p}(\Omega, d_\varepsilon, \mu_\beta)$. Let φ be a function with bounded support in X . Then $\varphi \in N_0^{1,p}(\Omega, d, \mu)$ if and only if $\varphi \in N_0^{1,p}(\Omega, d_\varepsilon, \mu_\beta)$. Thus (10.2), together with a similar identity for the minimal p -weak upper gradients of $u + \varphi$, shows that

$$\int_{\varphi \neq 0} g_u^p d\mu \leq \int_{\varphi \neq 0} g_{u+\varphi}^p d\mu \quad \text{if and only if} \quad \int_{\varphi \neq 0} g_{u,\varepsilon}^p d\mu_\beta \leq \int_{\varphi \neq 0} g_{u+\varphi,\varepsilon}^p d\mu_\beta.$$

It then follows from the discussion after Definition 9.5 that u is p -harmonic with respect to (d, μ) if and only if it is p -harmonic with respect to $(d_\varepsilon, \mu_\beta)$. Moreover, (10.3) follows directly from (10.2). \square

Theorem 10.5. *Assume that μ is doubling and supports a p -Poincaré inequality, both properties holding for balls of radii at most R_0 . Let $\beta > \beta_0$ and assume that $p = \beta/\varepsilon > 1$. Then the following are equivalent:*

- (a) *There exists a nonconstant p -harmonic function on (X, d, μ) with finite p -energy, i.e. the finite-energy Liouville theorem fails for X .*
- (b) *There are two disjoint compact sets $K_1, K_2 \subset \partial_\varepsilon X$ with positive $C_p^{\bar{X}_\varepsilon}$ capacity.*

After proving the theorem we will give some illustrating examples. But first, before proving the theorem, we will provide several useful characterizations of the second condition (b). The characterization (e), applied to the restriction of $C_p^{\bar{X}_\varepsilon}$ to the boundary $\partial_\varepsilon X$, will be used in the proof of Theorem 10.5.

Lemma 10.6. *Let Z be a separable metric space, and $\text{Cap}(\cdot)$ be a monotone, countably subadditive set-function with values in $[0, \infty)$, defined for all subsets of Z . Assume that for each Borel set $E \subset Z$,*

$$\text{Cap}(E) = \sup_K \text{Cap}(K), \quad (10.4)$$

where the supremum is taken over all compact subsets $K \subset E$.

Define the support of Cap as

$$\text{supp Cap} = \{x \in Z : \text{Cap}(B(x, r)) > 0 \text{ for all } r > 0\}.$$

Then the following are equivalent:

- (a) *There are two disjoint compact sets $K_1, K_2 \subset Z$ such that $\text{Cap}(K_1) > 0$ and $\text{Cap}(K_2) > 0$.*
- (b) *There is a Borel set $E \subset Z$ such that $\text{Cap}(E) > 0$ and $\text{Cap}(Z \setminus E) > 0$.*
- (c) *There is an open set $G \subset Z$ such that $\text{Cap}(G) > 0$ and $\text{Cap}(Z \setminus G) > 0$.*
- (d) *The support supp Cap contains at least two points.*
- (e) *Cap is not concentrated to one point, i.e. $\text{Cap}(Z \setminus \{a\}) > 0$ for each $a \in Z$.*

If Y is equipped with a globally doubling measure ν supporting a global p -Poincaré inequality and $p > 1$, then C_p^Y is a Choquet capacity, by [5, Theorem 6.11], and thus satisfies the assumptions above. Hence the assumptions are also satisfied for any restriction of C_p^Y to any closed subset of Y as well. Example 6.6 in [5] shows that (10.4) can fail if $p = 1$.

The assumption (10.4) is only needed to establish the equivalence of (a) and (b). On the other hand, separability is only used to deduce the identity (10.5) below, which in turn is used to show the equivalence of (b)–(e).

Proof. We start by showing that

$$\text{Cap}(Z \setminus \text{supp Cap}) = 0. \quad (10.5)$$

To this end, for each $x \in Z \setminus \text{supp Cap}$ there is $r_x > 0$ so that $\text{Cap}(B(x, r_x)) = 0$. As Z is Lindelöf (which for metric spaces is equivalent to separability, see e.g. [5, Proposition 1.5]), we can write $Z \setminus \text{supp Cap}$ as a countable union of such balls, each of which has zero capacity. Hence the countable subadditivity shows that (10.5) holds.

Now we are ready to prove the equivalences of (a)–(e).

(b) \Rightarrow (a) By (10.4) there are $K_1 \subset E$ and $K_2 \subset Z \setminus E$ such that $\text{Cap}(K_1) > 0$ and $\text{Cap}(K_2) > 0$.

(a) \Rightarrow (c) \Rightarrow (b) These implications are trivial.

(b) \Rightarrow (e) Let $a \in Z$. If $a \in E$, then $\text{Cap}(Z \setminus \{a\}) \geq \text{Cap}(Z \setminus E) > 0$. Similarly, if $a \notin E$, then $\text{Cap}(Z \setminus \{a\}) \geq \text{Cap}(E) > 0$.

(e) \Rightarrow (d) As $\text{Cap}(Z \setminus \{a\}) > 0$ for each $a \in Z$, it follows from (10.5) that supp Cap is nonempty. Let $a \in \text{supp Cap}$. As again $\text{Cap}(Z \setminus \{a\}) > 0$, and (10.5) holds, there is $b \in \text{supp Cap} \setminus \{a\}$.

(d) \Rightarrow (c) Let $a, b \in \text{supp Cap}$, $a \neq b$, and then let $G = B(a, \frac{1}{2}d(a, b))$. Thus $\text{Cap}(G) > 0$ as $a \in \text{supp Cap}$, while $\text{Cap}(Z \setminus G) \geq \text{Cap}(B(b, \frac{1}{2}d(a, b))) > 0$ since $b \in \text{supp Cap}$. \square

Proof of Theorem 10.5. By Theorem 6.2, the uniformized space $(X_\varepsilon, d_\varepsilon, \mu_\beta)$, as well as its closure \bar{X}_ε , supports a global p -Poincaré inequality and μ_β is globally doubling. Moreover, \bar{X}_ε is complete. It thus follows from [5, Theorem 6.11], that $C_p^{\bar{X}_\varepsilon}$ is a Choquet capacity, and in particular satisfies the assumptions in Lemma 10.6, and so does its restriction to $\partial_\varepsilon X$.

(b) \Rightarrow (a) Let $f(x) := \text{dist}_\varepsilon(x, K_1)$. Since \bar{X}_ε is bounded, we have $f \in N^{1,p}(\bar{X}_\varepsilon)$ and hence there exists a p -harmonic function u in X_ε such that $u - f \in N_0^{1,p}(X_\varepsilon)$, see Shanmugalingam [37, Theorem 5.6] (or [5, Theorem 8.28 and Definition 8.31]). The function u is denoted $H_p f$ in Björn–Björn–Shanmugalingam [10] (and Hf in [5]). By the Kellogg property ([10, Theorem 3.9] or [5, Theorem 10.5]), we have $\lim_{X_\varepsilon \ni y \rightarrow x} u(y) = f(x)$ on $\partial_\varepsilon X$, except possibly for a set of zero $C_p^{\bar{X}_\varepsilon}$ -capacity. Consequently, as $f = 0$ on K_1 , $f > 0$ on K_2 and both K_1 and K_2 have positive $C_p^{\bar{X}_\varepsilon}$ -capacity, u must be nonconstant on X_ε .

Proposition 10.4 implies that $u \in N_{\text{loc}}^{1,p}(X, d, \mu)$ is p -harmonic in X with respect to (d, μ) as well, and from (10.2) with $\beta = p\varepsilon$ it follows that

$$\int_X g_u^p d\mu = \int_{X_\varepsilon} g_{u,\varepsilon}^p d\mu_\beta \leq \|u\|_{N^{1,p}(X_\varepsilon, d_\varepsilon, \mu_\beta)}^p < \infty.$$

\neg (b) \Rightarrow \neg (a) By Lemma 10.6, there is $a \in \partial_\varepsilon X$ such that $C_p^{\bar{X}_\varepsilon}(\partial_\varepsilon X \setminus \{a\}) = 0$. The capacity of $\{a\}$ can be zero or positive.

Let u be a p -harmonic function in (X, d, μ) with finite p -energy. Then u is also p -harmonic on $(X_\varepsilon, d_\varepsilon, \mu_\beta)$ with finite p -energy, by Proposition 10.4. Applying the global p -Poincaré inequality to the ball $B_\varepsilon(x_0, 2 \text{diam}_\varepsilon X) \cap X_\varepsilon = X_\varepsilon$, with an arbitrary $x_0 \in X$, shows that $u \in N^{1,p}(X_\varepsilon)$, cf. [5, Proposition 4.13 (d)].

If $\partial_\varepsilon X$ has zero $C_p^{\bar{X}_\varepsilon}$ -capacity then it is removable for p -harmonic functions in $N^{1,p}(X_\varepsilon)$, by Theorem 6.2 in Björn [4] (or [5, Theorem 12.2]). Hence, an extension of u is p -harmonic on the compact connected set \bar{X}_ε and is thus constant by the strong maximum principle.

Finally, assume that $C_p^{\bar{X}_\varepsilon}(\{a\}) > 0$. Then $E := \partial_\varepsilon X \setminus \{a\}$ has zero capacity and is thus removable for p -harmonic functions in $N^{1,p}(X_\varepsilon)$, by [4, Theorem 6.2] (or [5, Theorem 12.2]). Since $u \in N^{1,p}(X_\varepsilon)$, it follows that an extension of u is p -harmonic in the open set $\bar{X}_\varepsilon \setminus \{a\} = X_\varepsilon \cup E$. By Lemma 9.9, we know that $X_\varepsilon \cup E$ is annularly quasiconvex at a . Since $C_p^{\bar{X}_\varepsilon}(\{a\}) > 0$, it is also true that $\text{cap}_p^{\bar{X}_\varepsilon}(\{a\}, B_\varepsilon(a, \rho)) > 0$ if $\rho < \frac{1}{4} \text{diam}_\varepsilon X_\varepsilon$, by e.g. [5, Proposition 6.16]. Moreover, \bar{X}_ε is connected. Thus Lemma 9.7, together with the remark after it, implies that for sufficiently small $r > 0$,

$$\text{osc}_{B_\varepsilon(a, 2r) \setminus \{a\}} u \lesssim \left(\int_{B_\varepsilon(a, 2\Lambda r)} g_{u,\varepsilon}^p d\mu_\beta \right)^{1/p}.$$

Since $g_{u,\varepsilon} \in L^p(\bar{X}_\varepsilon, \mu_\beta)$, the last integral tends to 0 as $r \rightarrow 0$ and we conclude that $\lim_{\bar{X}_\varepsilon \setminus \{a\} \ni y \rightarrow a} u(y)$ exists. In particular, u is bounded on the compact set \bar{X}_ε . Finally, the strong maximum principle for p -harmonic functions on $\bar{X}_\varepsilon \setminus \{a\}$ shows that u must be constant on \bar{X}_ε . \square

Example 10.7. (Continuation of Example 4.2.) We have $C_d = 2$, and all choices of R_0 are acceptable. Hence any $\varepsilon, \beta > 0$ are allowed. Fixing $\varepsilon > 0$ and $1 < p < \infty$ and choosing $\beta = p\varepsilon$, we see that the weight in (4.1) becomes

$$w(z) = \varepsilon^{-1+\beta/\varepsilon} (1/\varepsilon - |z|)^{-1+\beta/\varepsilon} = \varepsilon^{p-1} (1/\varepsilon - |z|)^{p-1}.$$

By considering the functions

$$u_j(z) = \begin{cases} \min\left\{1, \frac{1}{j} \log \frac{1}{1-\varepsilon|z|}\right\}, & \text{if } |z| < \frac{1}{\varepsilon}, \\ 1, & \text{if } |z| = \frac{1}{\varepsilon}, \end{cases}$$

for which $\|u_j\|_{N^{1,p}(X_\varepsilon, \mu_\beta)} \rightarrow 0$, as $j \rightarrow \infty$, we see that $C_p^{\bar{X}_\varepsilon}(\{\pm 1/\varepsilon\}) = 0$.

Note that \mathbf{R} does not admit any nonconstant p -harmonic function with finite p -energy.

Example 10.8. Consider $X = \mathbf{R} \times [-1, 1]$, which is a Gromov hyperbolic space when equipped with the Euclidean metric. We equip X with a weighted measure

$$d\mu(x, y) = w(x, y) d\mathcal{L}^2(x, y)$$

such that (X, μ) is uniformly locally doubling and supports a uniformly local p -Poincaré inequality. Fixing $z_0 = (0, 0)$, the uniformization with $\varepsilon = 1$ gives a uniform domain X_1 such that $\partial_1 X$ consists of two points.

To understand the potential theory and geometry of X_1 near these two points, consider $z = (x, y) \in X$ such that $x \gg 1$. Then, with d_1 denoting the uniformized metric on X_1 , we have

$$d_1(z, z_0) \approx \int_0^x e^{-t} dt = 1 - e^{-x} \quad \text{and} \quad d_1((x, -1), (x, 1)) \approx e^{-x}.$$

Here by $d_1(z, z_0) \approx 1 - e^{-x}$ we mean that $d_1(z, z_0)/(1 - e^{-x}) \rightarrow 1$ as $x \rightarrow \infty$. Thus, near the two boundary points, X_1 is (biLipschitz equivalent to) the diamond region in \mathbf{R}^2 with corners $(\pm 1, 0)$ and $(0, \pm 1)$. The two boundary points of X_1 are $z_\pm := (\pm 1, 0)$.

Let $\beta = p > 1$ and let μ_β be the weighted measure on X_1 , given by Definition 4.1. By Theorem 10.5, X supports a nonconstant p -harmonic function with finite p -energy if and only if both boundary points z_\pm have positive $C_p^{\bar{X}_1}$ -capacity. By Björn–Björn–Lehrbäck [9, Proposition 5.3], $C_p^{\bar{X}_1}(\{z_+\}) > 0$ if and only if

$$\int_0^{r_0} \left(\frac{r^p}{\mu_\beta(B_1(z_+, r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty$$

for some (all) sufficiently small r_0 , where the balls $B_1(z_+, r)$ are with respect to the metric d_1 . By the global doubling property of μ_β we see that

$$\mu_\beta(B_1(z_+, r)) \simeq \mu_\beta(B_1(z_+, r) \setminus B_1(z_+, \frac{1}{2}r)).$$

In view of (2.3), each of these annuli is (roughly) the image of a rectangular region with fixed size and at distance approximately $\log(1/r)$ from the base point z_0 . Letting $Q(t) = [t-1, t+1] \times [-1, 1]$, we therefore have

$$\mu_\beta(B_1(z_+, r)) \simeq \int_{Q(\log(1/r))} e^{-\beta x} w(x, y) d\mathcal{L}^2(x, y).$$

Since $e^{-\beta x} \simeq r^\beta$ on $Q(\log(1/r))$, we therefore conclude that $C_p^{\overline{X}^1}(\{z_+\}) > 0$ if and only if

$$\int_0^{r_0} \left(\int_{Q(\log(1/r))} w d\mathcal{L}^2 \right)^{1/(1-p)} \frac{dr}{r} < \infty,$$

or equivalently,

$$\int_0^\infty \left(\int_{Q(t)} w d\mathcal{L}^2 \right)^{1/(1-p)} dt < \infty. \quad (10.6)$$

An analogous condition holds for z_- .

Note that when $w \equiv 1$, both (10.6) and its analogue for z_- fail, showing that the unweighted strip $\mathbf{R} \times [-1, 1]$ satisfies the finite-energy Liouville theorem. This special case was obtained in Björn–Björn–Shanmugalingam [12] by a more direct method, without the use of uniformization.

Remark 10.9. The weighted Euclidean real line (\mathbf{R}, μ) , where $d\mu = w dx$ is uniformly locally doubling and supports a uniformly local p -Poincaré inequality, can be treated similarly and we obtain that (\mathbf{R}, μ) supports nonconstant p -harmonic functions with finite energy if and only if a condition similar to (10.6) holds on it:

$$\int_0^\infty \left(\int_{t-1}^{t+1} w(x) dx \right)^{1/(1-p)} dt < \infty. \quad (10.7)$$

In [12], this question was studied by different methods and under local assumptions on w . It follows from the results in Björn–Björn–Shanmugalingam [11] on local A_p weights, that the condition

$$\int_0^\infty w(x)^{1/(1-p)} dx < \infty,$$

obtained in [12], is equivalent to (10.7) under the local assumptions on w .

We end the paper with the following result which is a direct consequence of Theorem 6.2 together with Lemmas 4.10 and 9.4. In combination with Theorem 10.5, it provides a sufficient condition for the existence of nonconstant p -harmonic functions on (X, d, μ) with finite p -energy. Note that the Hausdorff dimension depends only on ε , C_d and R_0 , but not on β or p .

Proposition 10.10. *Assume that μ is doubling and supports a p -Poincaré inequality, both properties holding for balls of radii at most R_0 . Let $\beta > \beta_0$. Assume that the Borel set $E \subset \partial_\varepsilon X$ has positive κ -dimensional Hausdorff measure for some $\kappa > (\log C_d)/\varepsilon R_0$. If $p = \beta/\varepsilon \geq 1$, then $C_p^{\overline{X}^\varepsilon}(E) > 0$.*

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