Rough isometry between Gromov hyperbolic spaces and uniformization

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1 Introduction

Uniform domains play a special role in the study of planar quasiconformal mappings (see for example [MS] where the concept of uniform domains was first introduced, [Mar, GeO, BKR, H, KL]) and in potential theory (see for example [KP, KT, LLMS, A1, A2, HK, BSh]). The notion of uniform domains does not require the underlying space to be Euclidean or smooth, and so the notion of uniform domains has a natural extension to general metric spaces, see Definition 2.4 below. On the other hand, the notion of curvature, as defined in Riemannian geometry, is a second order calculus notion and so does not easily lend itself to the setting of more general metric spaces. Instead, in that non-smooth setting, the role of negative curvature is played by two possible alternatives, Alexandrov curvature and Gromov hyperbolicity, see the discussion in [BH, BuS, CDP]. Gromov hyperbolic spaces were first defined in [Gr] in the context of studying hyperbolic groups.

The work [BHK] demonstrates a strong connection between Gromov hyperbolic spaces and uniform domains. It was shown there that uniform domains in metric spaces, equipped with the quasihyperbolic metric k (see (1)) are necessarily a Gromov hyperbolic spaces. Conversely, given a geodesic Gromov hyperbolic space X, there is a positive number ε_0 such that whenever $0 < \varepsilon \leq \varepsilon_0$, the uniformization X_ε of X corresponding to the parameter ε is a uniform domain.

It is not difficult to see that if X and Y are two complete geodesic spaces with X a Gromov hyperbolic space, and if there is a rough isometry $\Phi : X \to Y$ as in Definition 2.6, then Y is also Gromov hyperbolic; that is, Gromov hyperbolicity is a large scale property and is not destroyed by small-scale perturbations. Therefore it is natural to ask whether the allowable range of uniformization parameters is preserved by rough isometries. This is

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the goal of this current note. In particular, we show that if X and Y are Gromov hyperbolic and $\Phi : Y \to X$ is a rough isometry, and if $\varepsilon > 0$ is such that X_ε is a uniform domain, then Y_ε is also a uniform domain, see Theorem 3.7. In [BBS] it was shown that if a Gromov hyperbolic space X is uniformized to a uniform domain X_ε (for sufficiently small $\varepsilon > 0$), and the subsequent boundary $Z := \partial X_\varepsilon$ has a hyperbolic filling Y with appropriate scaling parameters, then Y is roughly isometric to X. It follows from our results then that Y_ε is also a uniform domain (since we know that X_ε is). It is not difficult to see that ∂Y_ε is isometric to ∂X_ε, and hence our result ties the potential theoretic properties of ∂X_ε to those of Y_ε, even though X_ε itself could be ill-connected from the point of view of potential theory. It was shown in [BBS] that Y_ε has a suitable measure with respect to which Y_ε is doubling and supports a 1-Poincaré inequality if it is a uniform domain.

Observe that by the results of [BHK] we know that Y_ε is a uniform domain if ε is small enough, but here we do not require smallness of ε. The key reason in [BHK] for requiring ε be sufficiently small is that for small enough ε a Gehring-Hayman property holds for hyperbolic geodesics. Since we do not assume ε to be small, we cannot rely on this property; instead, our proof uses the technique of discretization of paths.

The next section is devoted to providing the relevant definitions. The first part of the third section develops the tools necessary to prove our main theorem, Theorem 3.7, and the proof of that theorem is given in the last part of that section. We adopt the convention that $Q_1 \gtrsim Q_2$ if there is a constant $C > 0$ such that $C\,Q_1 \geq Q_2$. We say that $Q_1 \lesssim Q_2$ if $Q_2 \gtrsim Q_1$, and we say that $Q_1 \simeq Q_2$ if $Q_1 \gtrsim Q_2$ and $Q_1 \lesssim Q_2$. We say that $Q_1 \simeq Q_2$ with comparison constant $C > 0$ if

$$\frac{1}{C}\,Q_1 \leq Q_2 \leq C\,Q_1.$$

2 Background

We provide the relevant definitions of the notions used in this note. In what follows, given a metric space (Z, d), $z \in Z$ and $r > 0$, we set $B(z, r) := \{x \in Z \,:\, d(x, z) < r\}$ and $\overline{B}(z, r) := \{x \in Z \,:\, d(x, z) \leq r\}$.

Definition 2.1. A complete geodesic metric space (Z, d) is said to be *Gromov hyperbolic* if there exists $\delta \geq 0$ such that whenever $x, y, z \in Z$ and $[x, y], [y, z], [z, x]$ are geodesic paths in Z with end points x, y, end points y, z, and end points z, x respectively, then

$$[x, y] \subset \bigcup_{w \in [y,z] \cup [z,x]} \overline{B}(w, \delta).$$

Here, if $\delta = 0$, we interpret $\overline{B}(w, \delta)$ to be the set $\{w\}$.

Definition 2.2. We say that a Gromov hyperbolic space (Z, d) is *roughly starlike* if there exists $M \geq 0$ and $z_0 \in Z$ such that for all $z \in Z$ there is a geodesic ray $\gamma : [0, \infty) \to Z$ with $\gamma(0) = z_0$ and $t_0 \in [0, \infty)$ such that $d(z, \gamma(t_0)) \leq M$.

Trees are Gromov hyperbolic with $\delta = 0$ and are roughly starlike with $M = 0$. Uniform domains, equipped with the quaishyperbolic metric, are necessarily Gromov hyperbolic and rough starlike, see the discussion in [BHK, Chapter 3].

Following [BHK], for each $\varepsilon > 0$ we consider uniformization of Gromov hyperbolic spaces with parameter ε.

Definition 2.3. Let (Z, d) be a Gromov hyperbolic space, $z_0 \in Z$, and $\varepsilon > 0$. We consider the "density" function $\rho_\varepsilon^Z : Z \to (0, 1]$ given by

$$\rho_\varepsilon^Z(z) := e^{-\varepsilon d(z, z_0)}.$$

This density function induces a metric on Z, given by

$$d_\varepsilon(z_1, z_2) = \inf_\gamma \int_\gamma \rho_\varepsilon^Z \, ds,$$

for $z_1, z_2 \in Z$, where the infimum is over all rectifiable paths γ in Z with end points z_1 and z_2. We denote this induced metric space (Z, d_ε) by Z_ε.

The above construction of uniformization is from [BHK, Chapter 4]. As mentioned above, from [BHK] we know that if Z is Gromov hyperbolic and $\varepsilon \le \varepsilon_0 = \varepsilon_0(\delta)$, then Z_ε is a uniform domain, that is, it satisfies the following definition.

Definition 2.4. Let Z be a locally complete, non-complete metric space, and set $\partial Z := \overline{Z} \setminus Z$. We say that Z is a *uniform domain* (or a uniform space) if there is a constant $\lambda \ge 1$ such that for each pair of points $x, y \in Z$ there is a rectifiable curve γ in Z with end points x and y satisfying

1. $\ell(\gamma) \le \lambda \, d(x, y)$,
2. for each $z \in \gamma$,

$$\delta(z) := \operatorname{dist}(z, \partial Z) \ge \lambda^{-1} \min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\}.$$

Here $\gamma(x, z)$ is any of the subcurves of γ with end points x, z.

From [MS, GeO, BHK], there is a natural deformation of the metric on a uniform domain (Z, d), called the quasihyperbolic metric.

Definition 2.5. Given a locally compact, non-complete metric space (Z, d), the *quasihyperbolic metric* k on Z is given by

$$k(x, y) = \inf_\gamma \int_\gamma \frac{1}{\delta_Z(\gamma(t))} \, ds(t) \tag{1}$$

when $x, y \in Z$. Here the infimum is over all rectifiable curves γ in Z with end points x and y, and with $\partial Z := \overline{Z} \setminus Z$,

$$\delta_Z(w) := \operatorname{dist}(w, \partial Z)$$

whenever $w \in Z$.

We assume from now on that $(X,d),(Y,d)$ are geodesic Gromov hyperbolic spaces.

Definition 2.6. A map $\Phi\colon Y\to X$ is a *τ-rough isometry* if

$$d(x,y)-\tau\le d(\Phi(x),\Phi(y))\le d(x,y)+\tau$$

for all $x,y\in Y$ and $\Phi(Y)$ is *τ-dense* in X, that is, for each $x\in X$ there is some $z_x\in Y$ such that $d(x,\Phi(z_x))\le\tau$.

Note that we do not require Φ to be continuous nor do we require it to be injective or surjective.

Lemma 2.7. *Given a τ-rough isometry $\Phi : Y\to X$, there exists a 3τ-rough isometry $\Phi^{-1}: X\to Y$ such that for all $y\in Y$ and $x\in X$ we have*

$$d(y,\Phi^{-1}(\Phi(y)))\le 2\tau,\qquad d(x,\Phi(\Phi^{-1}(x)))\le\tau.$$

This seems to be well-known (see for example [BS]), but as we were not able to find a published proof of this fact, we provide the proof here for the convenience of the reader.

Proof. We first construct $\Phi^{-1}: X\to Y$ as follows. Given $x\in X$, by the fact that every point in X is within a distance τ of $\Phi(Y)$, we can find a point $y_x\in Y$ such that $d(\Phi(y_x),x)\le\tau$. We choose one such y_x and set $\Phi^{-1}(x)=y_x$. Note that

$$d(\Phi(\Phi^{-1}(x)),x)=d(\Phi(y_x),x)\le\tau.$$

Moreover, for $y\in Y$, we see that with the choice of $x=\Phi(y)$, we have the point $y_{\Phi(y)}$ as a point in Y that Φ^{-1} maps x to; then $d(\Phi(y_{\Phi(y)}),x)\le\tau$, and so

$$d(\Phi^{-1}(\Phi(y)),y)=d(y_{\Phi(y)},y)\le\tau+d(\Phi(y_{\Phi(y)}),\Phi(y))\le 2\tau.$$

For $x,x'\in X$, we have

$$\begin{aligned}d(\Phi^{-1}(x),\Phi^{-1}(x'))=d(y_x,y_{x'})&\le\tau+d(\Phi(y_x),\Phi(y_{x'}))\\&\le\tau+d(\Phi(y_x),x)+d(x,x')+d(x',\Phi(y_{x'}))\\&\le 3\tau+d(x,x').\end{aligned}$$

Furthermore,

$$\begin{aligned}d(\Phi^{-1}(x),\Phi^{-1}(x'))=d(y_x,y_{x'})&\ge d(\Phi(y_x),\Phi(y_{x'}))-\tau\\&\ge -d(\Phi(y_x),x)+d(x,x')-d(x',\Phi(y_{x'}))-\tau\\&\ge d(x,x')-3\tau.\end{aligned}$$

Finally, given $y\in Y$, we set $x=\Phi(y)$ and note from the first part of the argument that

$$d(y,\Phi^{-1}(x))=d(y,\Phi^{-1}(\Phi(y)))\le 2\tau.$$

This concludes the proof. □

Remark 2.8. Note that if Φ is a τ-rough isometry, then it is also a 3τ-rough isometry. Hence, by replacing τ with 3τ if necessary, we will assume in the rest of the paper that both Φ and Φ^{-1} are τ-rough isometries with

$$d(y, \Phi^{-1}(\Phi(y))) \leq \tau, \qquad d(x, \Phi(\Phi^{-1}(x))) \leq \tau.$$

The density ρ_ε^Z as considered in Definition 2.3 is an example of a large class of densities, called *conformal densities*, used to deform metrics on a given metric space, see for example [KL, BKR]. A non-negative continuous function ρ on a metric space Z is a *conformal density* or *Harnack weight* if there is a constant $A \geq 1$ such that whenever $x, y \in X$ with $d(x, y) \leq 1$, we have

$$\frac{1}{A} \leq \frac{\rho(x)}{\rho(y)} \leq A. \tag{2}$$

The nomenclature is justified by the fact that if ρ is a conformal density on (Z, d) and the metric on Z is modified to a new metric d_ρ according to the scheme given in Definition 2.3 with ρ playing the role of ρ_ε^Z, then the natural identity map $\mathrm{Id} : (Z, d) \to (Z, d_\rho)$ is a (metrically) 1-quasiconformal map.

We are concerned with two densities,

$$\rho_\varepsilon^X(x) = e^{-\varepsilon d(x_0, x)}, \quad \rho_\varepsilon^Y(y) = e^{-\varepsilon d(y_0, y)}.$$

We denote by X_ε and Y_ε the ε-uniformizations of X and Y. We also assume that X is *roughly starlike* with respect to x_0, with constant $M \geq 0$.

Remark 2.9. Given a conformal density ρ on Z as in (2), and Z a geodesic space, we see that whenever $K \in \mathbb{N}$ and $x, y \in X$ such that $d(x, y) \leq K$, then

$$\frac{1}{A^K} \leq \frac{\rho(x)}{\rho(y)} \leq A^K.$$

Note that by the triangle inequality,

$$\frac{\rho_\varepsilon^X(x)}{\rho_\varepsilon^X(y)} = e^{-\varepsilon[d(x,x_0) - d(y,x_0)]} \geq e^{-\varepsilon d(x,y)} \geq e^{-\varepsilon}$$

when $d(x, y) \leq 1$. Similarly, we get

$$\frac{\rho_\varepsilon^X(x)}{\rho_\varepsilon^X(y)} = e^{-\varepsilon[d(x,x_0) - d(y,x_0)]} \leq e^{\varepsilon d(x,y)} \leq e^{\varepsilon}.$$

Thus both ρ_ε^X and ρ_ε^Y satisfy (2) with $A = e^\varepsilon$.

As described above, a given roughly starlike Gromov hyperbolic space can be uniformized and then the resulting space can be equipped with its quasihyperbolic metric (see (1) above for the definition of quasihyperbolic metric). The outcome may not be isometric to the original Gromov hyperbolic space, but as the next lemma shows, it is biLipschitz equivalent.

Lemma 2.10. *Let (X,d) be a roughly starlike Gromov hyperbolic space and $\varepsilon > 0$. Then (X_ε, k) is biLipschitz equivalent to (X,d).*

In the above lemma, k is the quasihyperbolic metric with respect to the uniformized space that is X_ε. Note that we do not assume any condition on ε apart from that it is positive. The above lemma was proved in [BHK, Proposition 4.37] for the setting where $\varepsilon \leq \varepsilon_0$. For the convenience of the reader, we provide this short proof here.

Proof. Note that the quasihyperbolic distance k is given by

$$k(x,y) = \inf_\gamma \int_\gamma \frac{1}{\delta_\varepsilon(\gamma(t))}\, ds_\varepsilon(t),$$

where we took γ to be arc-length parametrized with respect to the metric d on X with end points x and y, and ds_ε is the arc-length metric with respect to the uniformized metric d_ε. By the construction of uniformization, we have that $ds_\varepsilon(z) = e^{-\varepsilon d(z,x_0)}\, ds$. On the other hand, from Lemma 3.4 we know that $\delta_\varepsilon(z) \simeq e^{-\varepsilon d(z,x_0)}$. It follows that

$$k(x,y) \simeq \inf_\gamma \ell_d(\gamma) = d(x,y).$$

$\square$

What about starting from a uniform space, quasihyperbolize it, and then try to uniformize it; is there a choice of ε for which we get (biLipschitzly) the original uniform domain back?

3 Results

In what follows, all curves are assumed to be parametrized by (hyperbolic) arclength unless otherwise specified.

Lemma 3.1. *Suppose that $\rho\colon Y \to (0,\infty)$ satisfies the Harnack condition (2) with constant A. Let $L > 1$ and $\gamma\colon [0,L] \to Y$ be a curve with $\ell(\gamma) = L$, $\gamma(0) = a$ and $\gamma(L) = b$. Choose $N \in \mathbb{N}$ such that $N \leq L < N+1$. Then*

$$\int_\gamma \rho ds \simeq \sum_{i=0}^{N-1} \rho(a_i), \tag{3}$$

where $a_i = \gamma(iq)$ with $q := \frac{L}{N}$. The comparison constant in (3) can be taken to be $2A^2$.

If $L \leq 1$ we instead have $\int_\gamma \rho ds \simeq L \cdot \rho(\gamma(0))$ with comparison constant A.

Proof. Note that $1 \le q < 2$. For $0 \le i \le N-1$, let $\gamma_i \colon [0,q] \to Y$ be the curve given by $\gamma_i(t) = \gamma(iq+t)$. Note that γ_i is parametrized by arclength because γ is. Hence the length $\ell(\gamma_i)$ of γ_i satisfies $1 \le \ell(\gamma_i) < 2$. By condition (2), it follows that

$$\frac{1}{A^2}\rho(a_i) \le \int_{\gamma_i} \rho ds \le 2A^2\rho(a_i).$$

Hence

$$\sum_{i=0}^{N-1} \rho(a_i) \simeq \sum_{i=0}^{N-1} \int_{\gamma_i} \rho ds = \int_{\gamma} \rho ds$$

with comparison constant $2A^2$. □

Remark 3.2. Lemma 3.1 holds in X as well.

Lemma 3.3. *Suppose $x, y \in Y$ with $d(x,y) > 1$. Let $L > 1$ and $\gamma \colon [0,L] \to Y$ be a curve with $\gamma(0) = x$ and $\gamma(L) = y$. Fix $N \in \mathbb{N}$ such that $N \le L < N+1$. Then,*

$$\int_{\gamma} \rho_{\varepsilon}^{Y} ds \simeq \sum_{i=0}^{N-1} \rho_{\varepsilon}^{X}(\Phi(a_i)) \simeq \left(\sum_{i=0}^{N-2} \rho_{\varepsilon}^{X}(\Phi(a_i))\right) + \rho_{\varepsilon}^{X}(\Phi(y))$$

where $q = \frac{L}{N}$ and $a_i = \gamma(iq)$ for $0 \le i \le N$. In the above, we adopt the convention that $\sum_{i=0}^{N-2} \rho_{\varepsilon}^{X}(\Phi(a_0)) = 0$ if $N = 1$. The comparison constants depend solely on ε and τ.

Proof. Note that $a_0 = x$ and $a_N = y$. For $0 \le i \le N$, let $b_i = \Phi(a_i)$. By Lemma 3.1, we have

$$\int_{\gamma} \rho_{\varepsilon}^{Y} ds \simeq \sum_{i=0}^{N-1} \rho_{\varepsilon}^{Y}(a_i)$$

with comparison constant $e^{2\varepsilon}$. Now, $\rho_{\varepsilon}^{Y}(a_i) = e^{-\varepsilon d(y_0, a_i)}$ and, as Φ is a τ-rough isometry, we have

$$d(y_0, a_i) - \tau \le d(x_0, b_i) \le d(y_0, a_i) + \tau.$$

In particular,

$$e^{-\tau\varepsilon} \le \frac{\rho_{\varepsilon}^{Y}(a_i)}{\rho_{\varepsilon}^{X}(b_i)} \le e^{\tau\varepsilon}$$

for all i. Hence we have

$$\sum_{i=0}^{N-1} \rho_{\varepsilon}^{Y}(a_i) \simeq \sum_{i=0}^{N-1} \rho_{\varepsilon}^{X}(\Phi(a_i)),$$

with comparison constant $e^{\tau\varepsilon}$. Hence

$$\int_{\gamma} \rho_{\varepsilon}^{Y} ds \simeq \sum_{i=0}^{N-1} \rho_{\varepsilon}^{X}(\Phi(a_i))$$

with comparison constant $2e^{2\varepsilon+\tau\varepsilon}$.

The second comparability follows as $d(a_{N-1}, y) \le 2$, and so $\rho^Y_\varepsilon(a_{N-1}) \simeq \rho^Y_\varepsilon(y)$ with comparison constant $e^{2\varepsilon}$, see Remark 2.9. $\square$

Lemma 3.4. *Let Y be a roughly starlike Gromov hyperbolic space and $\varepsilon > 0$. Then for each $x \in Y$ we have*

$$\delta_\varepsilon(x) := \operatorname{dist}(x, \partial Y_\varepsilon) := \operatorname{dist}(x, \overline{Y_\varepsilon} \setminus Y_\varepsilon) \simeq e^{-\varepsilon d(x,y_0)}, \tag{4}$$

with comparison constant $[M + \varepsilon^{-1}]e^{\varepsilon M}$.

Proof. We set $\delta_\varepsilon(x) := \inf_{\zeta\in\partial Y_\varepsilon} d_\varepsilon(x, \zeta)$ for $x \in Y$. Recall that Y is roughly starlike with starlikeness constant M. Let $x \in Y$ and $\gamma : [0, \infty) \to Y$ be a geodesic ray from y_0 so that there is some $t_0 \in [0, \infty)$ for which we have $d(x, \gamma(t_0)) \le M$. Let β be a geodesic with end points x and $\gamma(t_0)$; then the concatenation γ_* of $\gamma|_{[t_0,\infty)}$ and β gives us that

$$\delta_\varepsilon(x) \le \int_{\gamma_*} e^{-\varepsilon d(\gamma_*(t),y_0)}\, dt.$$

Note that for points $w \in \beta$, $d(x, y_0) - M \le d(w, y_0) \le d(x, y_0) + M$, and so

$$\delta_\varepsilon(x) \le Me^{\varepsilon M} e^{-\varepsilon d(x,y_0)} + \int_{t_0}^\infty e^{-\varepsilon t}\, dt \le Me^{\varepsilon M} e^{-\varepsilon d(x,y_0)} + \varepsilon^{-1} e^{-\varepsilon t_0}.$$

Moreover, $t_0 = d(\gamma(t_0), y_0) \ge d(y_0, x) - M$. Therefore

$$\delta_\varepsilon(x) \le [M + \varepsilon^{-1}]e^{\varepsilon M}\, e^{-\varepsilon d(y_0,x)}.$$

On the other hand, if γ is any path from x that leaves every compact subset of Y, then we have

$$\int_\gamma e^{-\varepsilon d(\gamma(t),y_0)}\, dt \ge \int_0^\infty e^{-\varepsilon[d(y_0,x)+t]}\, dt = \frac{e^{-\varepsilon d(y_0,x)}}{\varepsilon}.$$

It follows that

$$\delta_\varepsilon(z) \simeq e^{-\varepsilon d(z,y_0)}, \tag{5}$$

with comparison constant $[M + \varepsilon^{-1}]e^{\varepsilon M}$. $\square$

Lemma 3.5. *Let $x, y \in Y$ such that $d(x, y) \le 4 + \tau$, and let γ be a Gromov hyperbolic geodesic in Y with end points x, y. Then*

$$\ell_\varepsilon(\gamma) \simeq d_\varepsilon(x, y) \simeq e^{-\varepsilon d(x,y_0)} d(x, y) \tag{6}$$

and γ is a uniform curve with respect to the metric d_ε on Y_ε, with uniformity constant depending only on ε, M, and τ.

Proof. If $x, y \in Y$ with $d(x,y) \le 4+\tau$, then set γ to be a Gromov hyperbolic geodesic curve with end points x, y. Then the length $\ell_\varepsilon(\gamma)$ of γ in the uniformized metric d_ε is given by

$$\ell_\varepsilon(\gamma) = \int_\gamma e^{-\varepsilon d(\gamma(t), y_0)}\, dt,$$

and as

$$d(x,y_0) - 4 - \tau \le d(x,y_0) - d(x,z) \le d(y_0,z) \le d(x,y_0) + d(x,z) \le d(x,y_0) + 4 + \tau.$$

for each z in the trajectory of γ, we see that

$$\ell(\gamma) e^{-\varepsilon d(x,y_0)} e^{-\varepsilon(4+\tau)} \le \ell_\varepsilon(\gamma) \le \ell(\gamma) e^{-\varepsilon d(x,y_0)} e^{\varepsilon(4+\tau)}.$$

Observe that $d(x,y) = \ell(\gamma)$. On the other hand, with β any rectifiable non-geodesic curve in Y with end points x and y, we must have $\ell(\beta) > d(x,y)$, and so with $t_0 \in [0, \ell(\beta)]$ the smallest number for which $d(x, \beta(t_0)) = d(x,y)$, we get

$$\int_\beta \rho_\varepsilon^Y\, ds \ge \int_0^{t_0} \rho_\varepsilon^Y \circ \beta(t)\, dt \ge d(x,y) e^{-\varepsilon d(x,y_0)} e^{-\varepsilon(4+\tau)}.$$

Therefore

$$d(x,y) e^{-\varepsilon d(x,y_0)} e^{\varepsilon(4+\tau)} \ge \ell_\varepsilon(\gamma) \ge d_\varepsilon(x,y) = \inf_\beta \int_\beta \rho_\varepsilon^Y\, ds \ge d(x,y) e^{-\varepsilon d(x,y_0)} e^{-\varepsilon(4+\tau)}.$$

Hence γ is a quasigeodesic in Y_ε, with constant depending only on ε and τ. Moreover, from Lemma 3.4 and the fact that $d(x,y) \le 4+\tau$ we know that for $z \in \gamma$,

$$\delta_\varepsilon(z) \gtrsim e^{-\varepsilon d(z,y_0)} \gtrsim e^{-\varepsilon d(x,y_0)} \gtrsim \ell_\varepsilon(\gamma),$$

that is, γ is a uniform curve, with uniformity constants that depend only on M, ε, τ. $\square$

From the above lemma, to show that Y_ε is a uniform domain it suffices to show that $x, y \in Y$ can be connected by a uniform curve when $d(x,y) \ge 4+\tau$. This is the focus of the remaining discussion.

Lemma 3.6. *Let $x, y \in Y$ be such that $d(x,y) \ge 2+\tau$. Then*

$$d_\varepsilon(x,y) \simeq d_\varepsilon(\Phi(x), \Phi(y)).$$

In the proof of this lemma we use Φ^{-1} together with Φ, see Lemma 2.7 regarding the construction of Φ^{-1}.

Proof. Let $\gamma\colon [0,L] \to Y$ be any curve with $\gamma(0) = x$, $\ell(\gamma) = L$, and $\gamma(L) = y$. Note that $L \geq 2+2\tau \geq 2$. We fix $N \in \mathbb{N}$ such that $N \leq L < N+1$. Let $q = \frac{L}{N}$ and, for $0 \leq i \leq N$, let $a_i = \gamma(iq)$ with $b_i = \Phi(a_i)$. Then $d(b_i, b_{i+1}) \leq d(a_i, a_{i+1}) + \tau \leq 4+\tau$, and so by Lemma 3.5 we have

$$d_\varepsilon(b_i, b_{i+1}) \lesssim e^{-\varepsilon d(b_i, x_0)} = \rho_\varepsilon^X(b_i)$$

with comparability constant depending only on ε, τ, and M. It follows that

$$d_\varepsilon(\Phi(x), \Phi(y)) \leq \sum_{i=0}^{N-1} d_\varepsilon(b_i, b_{i+1}) \lesssim \sum_{i=0}^{N-1} \rho_\varepsilon^X(b_i).$$

By Lemma 3.3, we have $\sum_{i=0}^{N-1} \rho_\varepsilon^X(b_i) \simeq \int_\gamma \rho_\varepsilon^Y ds$. Infimizing over all paths γ connecting x to y yields

$$d_\varepsilon(\Phi(x), \Phi(y)) \lesssim \inf_\gamma \int_\gamma \rho_\varepsilon^Y ds = d_\varepsilon(x,y).$$

Next, note that $d(\Phi(x), \Phi(y)) \geq d(x,y) - \tau \geq 2$. Hence, for $\Phi^{-1}(\Phi(x)) = x'$ and $\Phi^{-1}(\Phi(y)) = y'$ we can apply the same argument above to conclude that

$$d_\varepsilon(x', y') \lesssim d_\varepsilon(\Phi(x), \Phi(y)).$$

It remains to relate $d_\varepsilon(x', y')$ with $d_\varepsilon(x,y)$. As $d(\Phi^{-1} \circ \Phi(z), z) \leq \tau$ for each $z \in Y$, it follows from Lemma 3.5 that $d_\varepsilon(x', x) \lesssim e^{-\varepsilon d(x, y_0)}$ and $d_\varepsilon(y', y) \lesssim e^{-\varepsilon d(y, y_0)}$. Moreover, $d(x, y_0) \geq d(\Phi(x), x_0) - \tau$ and $d(y, y_0) \geq d(\Phi(y), x_0) - \tau$. Hence

$$d_\varepsilon(x,y) \leq d_\varepsilon(x,x') + d_\varepsilon(x',y') + d_\varepsilon(y',y) \lesssim d_\varepsilon(\Phi(x), \Phi(y)) + e^{-\varepsilon d(\Phi(x), x_0)} + e^{-\varepsilon d(\Phi(y), x_0)}.$$

Since $d_\varepsilon(\Phi(x), \Phi(y)) \geq d_\varepsilon(x,y) - \tau \geq 2$, we can apply Lemma 3.3 together with Lemma 3.1 to see that

$$d_\varepsilon(\Phi(x), \Phi(y)) \gtrsim e^{-\varepsilon d(\Phi(x), x_0)} + e^{-\varepsilon d(\Phi(y), x_0)},$$

from which we obtain the desired conclusion

$$d_\varepsilon(x,y) \lesssim d_\varepsilon(\Phi(x), \Phi(y)).$$

□

Theorem 3.7. *Let (X,d) and (Y,d) be two complete Gromov hyperbolic geodesic spaces, and suppose that there exists a τ-rough isometry $\Phi : Y \to X$. Let $y_0 \in Y$ and set $x_0 = \Phi(y_0)$. If X is roughly starlike with constant $M > 0$ with respect to x_0, and $\varepsilon > 0$ such that $(X_\varepsilon, d_\varepsilon)$ is a uniform domain, then $(Y_\varepsilon, d_\varepsilon)$ is also a uniform domain.*

Proof. Let $x, y \in Y$. If $d(x,y) \le 4+\tau$, then by Lemma 3.5 we know that the hyperbolic geodesic connecting x to y is a uniform curve in $(Y_\varepsilon, d_\varepsilon)$. Therefore to verify that Y_ε is a uniform domain, it suffices to consider only points $x, y \in Y$ with $d(x,y) > 4+\tau$. For such x, y we have that $d(\Phi(x), \Phi(y)) \ge 4$. Let γ be a uniform curve in X_ε with end points $\Phi(x), \Phi(y)$. Then $\ell(\gamma) \ge 4$, and so we can apply Lemma 3.1 to γ. With $a_i = \gamma(iq)$, $q = L/N$, we see that

$$d_\varepsilon(x,y) \simeq d_\varepsilon(\Phi(x), \Phi(y)) \simeq \int_\gamma \rho_\varepsilon^X \, ds.$$

Here we have also used Lemma 3.6. Now applying Lemma 3.3 with $\Phi^{-1} : X \to Y$ playing the role of Φ there, we obtain

$$d_\varepsilon(x,y) \simeq \sum_{i=0}^{N-2} \rho_\varepsilon^Y(\Phi^{-1}(a_i)) + \rho_\varepsilon^Y(\Phi^{-1}(\Phi(y))).$$

As $d(y, \Phi^{-1} \circ \Phi(y)) \le \tau$ and $d(x, \Phi^{-1} \circ \Phi(x)) \le \tau$, we have that

$$d_\varepsilon(x,y) \simeq \rho_\varepsilon^Y(x) + \rho_\varepsilon^Y(y) + \sum_{i=1}^{N-2} \rho_\varepsilon^Y(\Phi^{-1}(a_i)).$$

Note that $d(a_i, a_{i+1}) \le 2$, and so $d(\Phi^{-1}(a_i), \Phi^{-1}(a_{i+1})) \le 2+\tau$. Similarly, $d(x, \Phi^{-1}(a_2)) \le 2+2\tau$, $d(y, \Phi^{-1}(a_{N-2})) \le 2+2\tau$. We set β_1 to be the hyperbolic geodesic with end points $\Phi^{-1}(a_2)$ and x, and set β_{N-1} to be the hyperbolic geodesic with end points y and $\Phi^{-1}(a_{N-2})$. For $i = 2, \cdots, N-2$ let β_i be the hyperbolic geodesic in Y with end points $\Phi^{-1}(a_i)$ and $\Phi^{-1}(a_{i+1})$. By Lemma 3.5 we have that

$$\ell_\varepsilon(\beta_i) \simeq \rho_\varepsilon^Y(\Phi^{-1}(a_i)) d(\Phi^{-1}(a_i), \Phi^{-1}(a_{i+1})) \lesssim \rho_\varepsilon^Y(\Phi^{-1}(a_i)),$$

and so

$$d_\varepsilon(x,y) \gtrsim \sum_{i=0}^{N-1} \ell_\varepsilon(\beta_i) = \ell_\varepsilon(\beta),$$

where β is the concatenation of the finitely many curves β_i, $i = 1, \cdots, N-1$. Thus β is a quasiconvex curve connecting x to y in Y. We now show that this curve is a uniform curve, that is, it satisfies Condition 2 of Definition 2.4.

Let $z \in \beta$. If $z \in \beta_1 \cup \beta_{N-1}$, then the result follows from Lemma 3.5. Thus we may assume that $z \in \beta_i$ for some $i \in \{2, \cdots, N-2\}$. Then by Lemma 3.4,

$$\delta_\varepsilon(z) \simeq \rho_\varepsilon^Y(z) \simeq \rho_\varepsilon^Y(\Phi^{-1}(a_i)).$$

On the other hand, as $d(a_i, x_0) - \tau \le d(\Phi^{-1}(a_i), y_0) \le d(a_i, x_0) + \tau$, we have that

$$\delta_\varepsilon(z) \simeq \rho_\varepsilon^X(a_i) \simeq \delta_\varepsilon(a_i) \ge \frac{1}{\lambda} \ell_\varepsilon(\gamma[\Phi(x), a_i]).$$

A repeat of the arguments above also tell us that

$$\ell_\varepsilon(\gamma[\Phi(x),a_i]) \simeq \sum_{j=0}^{i} \rho_\varepsilon^X(a_j) \simeq \sum_{j=0}^{i} \rho_\varepsilon^Y(\Phi^{-1}(a_j)) \gtrsim \ell_\varepsilon(\beta[x,z]).$$

Combining the above estimates, we obtain

$$\delta_\varepsilon(z) \gtrsim \ell_\varepsilon(\beta[x,z]).$$

$\square$

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