LARGE ANNIHILATOR CATEGORY $\mathcal{O}$ FOR $\mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$

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Abstract. We construct a new analogue of the BGG category $\mathcal{O}$ for the infinite-dimensional Lie algebras $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. A main difference with the categories studied in [N] and [CP] is that all objects of our category satisfy the large annihilator condition introduced in [DPS]. Despite the fact that the splitting Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ are not conjugate, one can eliminate the dependency on the choice of $\mathfrak{b}$ and introduce a universal highest weight category $\mathcal{OLA}$ of $\mathfrak{g}$-modules, the letters $\mathcal{LA}$ coming from "large annihilator". The subcategory of integrable objects in $\mathcal{OLA}$ is precisely the category $\mathbb{T}_{\mathfrak{g}}$ studied in [DPS]. We investigate the structure of $\mathcal{OLA}$, and in particular compute the multiplicities of simple objects in standard objects and the multiplicities of standard objects in indecomposable injectives. We also complete the annihilators in $U(\mathfrak{g})$ of simple objects of $\mathcal{OLA}$.

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1. Introduction

Let $\mathfrak{gl}(\infty)$ denote the Lie algebra of finitary infinite matrices over $\mathbb{C}$, and let $\mathfrak{sl}(\infty) \subset \mathfrak{gl}(\infty)$ be the Lie subalgebra of traceless matrices. One can consider the representation theory of $\mathfrak{sl}(\infty)$ as a way to study stabilization phenomena for representations of the Lie algebras $\mathfrak{sl}(n)$ when $n \to \infty$. In fact, the very language of representation theory suggests what kind of stabilization features it is natural to consider. In particular, the theory of tensor $\mathfrak{sl}(\infty)$-modules developed in [PStyr] shows that Weyl's semisimplicity theorem for $\mathfrak{sl}(n)$ does not stabilize when $n \to \infty$. This is because some morphisms of tensor modules over $\mathfrak{sl}(n)$ "persist at ∞" while others do not. For instance, the tautological morphism $\mathfrak{sl}(n) \to \mathfrak{gl}(n)$ persists at infinity and induces the tautological injective morphism $\mathfrak{sl}(\infty) \to \mathfrak{gl}(\infty)$. However the morphism of $\mathfrak{sl}(n)$-modules $\mathbb{C} \to \mathfrak{gl}(n)$ which induces the splitting $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{C}$ is lost "at ∞" as $\mathfrak{gl}(\infty)$ has no nonzero invariants as a module over $\mathfrak{sl}(\infty)$. Similarly, if one considers the Lie algebras $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$, and denotes their natural representations by V_{2n}, the respective morphisms $\mathbf{S}^2(V_{2n}) \to \mathbb{C}$ and $\mathbf{\Lambda}^2(V_{2n}) \to \mathbb{C}$ persist at ∞, while the (respective) morphisms $\mathbb{C} \to \mathbf{S}^2(V_{2n})$ and $\mathbb{C} \to \mathbf{\Lambda}^2(V_{2n})$ are lost at ∞.

An intrinsic viewpoint on these phenomena is presented in the paper [DPS] where a category of tensor modules $\mathbb{T}_{\mathfrak{g}}$ is introduced, and it is established that the tensor

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products of copies of the natural and conatural representations are injective objects of this category.

Let $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. The purpose of the present paper is to introduce and study an interesting category $\mathcal{OLA}$ of $\mathfrak{g}$-modules which is an analogue of Bernstein-Gelfand-Gelfand's category $\mathcal{O}$ [BGG], and contains the category of tensor modules $\mathbb{T}_{\mathfrak{g}}$ as a full subcategory. In the papers [N] and [CP], other "analogues at ∞" of the category $\mathcal{O}$ have been studied, however these categories are essentially different from the category $\mathcal{OLA}$. In particular, the integrable subcategories of the categories studied in [N] and [CP] are semisimple.

Recall that the category $\mathbb{T}_{\mathfrak{g}}$ consists of integrable $\mathfrak{g}$-modules (i.e., modules which decompose as sums of finite-dimensional modules over any finite-dimensional simple subalgebra of $\mathfrak{g}$) of finite length, satisfying the following three equivalent conditions:

(a) M is a weight module for any splitting Cartan subalgebra of $\mathfrak{g}$ (absolute weight module);
(b) M is $\left(\operatorname{Aut}\mathfrak{g}\right)^{\circ}$-invariant, where $\left(\operatorname{Aut}\mathfrak{g}\right)^{\circ}$ is the connected component of the group of automorphisms of $\mathfrak{g}$;
(c) the annihilator $\operatorname{Ann}_{\mathfrak{g}} m$ of every vector $m \in M$ contains the derived algebra of the centralizer of a finite-dimensional Lie subalgebra of $\mathfrak{g}$.

When one tries to extend $\mathbb{T}_{\mathfrak{g}}$ to an analogue of the BGG category $\mathcal{O}$, one notices that conditions (a) and (b) must be dropped as they no longer hold in the BGG category $\mathcal{O}$. On the other hand, condition (c) is empty for category $\mathcal{O}$, and therefore, it is the only condition among the three that can lead to an interesting "category $\mathcal{O}$ for $\mathfrak{g}$".

More precisely, we fix splitting Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} = \mathfrak{h} \supsetplus \mathfrak{n}$ and define the category $\mathcal{OLA}_{\mathfrak{b}}$ by the conditions that its objects are $\mathfrak{h}$-semisimple, satisfy condition c), and are locally finite under the action of any element of $\mathfrak{n}$. The first problem we address, is the dependence of $\mathcal{OLA}_{\mathfrak{b}}$ on $\mathfrak{b}$. The BGG category $\mathcal{O}$ is independent, up to equivalence, on the choice of a Borel subalgebra as all Borel subalgebras of a finite-dimensional reductive Lie algebra are conjugate. In our case the situation is more complicated and the main result of Section 3 is that there exist Borel subalgebras $\mathfrak{b}$, called perfect, such that for any other splitting Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ the category $\mathcal{OLA}_{\mathfrak{b}'}$ is naturally equivalent to $\mathcal{OLA}_{\mathfrak{b}}$ or to a proper full subcategory of $\mathcal{OLA}_{\mathfrak{b}}$.

In Sections 4-6 we fix a perfect Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and study the category $\mathcal{OLA} = \mathcal{OLA}_{\mathfrak{b}}$. We show that every simple object of $\mathcal{OLA}$ is a highest weight module and that $\mathcal{OLA}$ is a highest weight category. We also prove that every finitely generated object of $\mathcal{OLA}$ has finite length and that any object of $\mathcal{OLA}$ has an exhaustive socle filtration. Furthermore, we describe the blocks of $\mathcal{OLA}$ and prove that any finitely generated object of $\mathcal{OLA}$ has nonzero annihilator in $U(\mathfrak{g})$. These results manifest further differences with the categories studied in [N] and [CP].

Let us point out that, as a highest weight category, $\mathcal{OLA}$ admits only standard objects and no costandard objects. Costandard objects (analogues of Verma modules) are replaced by certain approximations which do not "converge" in $\mathcal{OLA}$, nevertheless provide stable Kazhdan-Lusztig multiplicities for a version of BGG-reciprocity which we establish. The indecomposable injectives in $\mathcal{OLA}$ admit finite filtrations whose successive quotients are standard objects, while the standard objects have infinite filtrations whose quotients are simple objects. It is essential that the multiplicities of simple objects in standard objects are finite. Interestingly, these latter multiplicities are a mixture of finite-dimensional Kazhdan-Lusztig numbers and Kostka numbers.

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2. The Set-Up

The base field is $\mathbb{C}$. The notations $\mathbf{S}(\cdot)$ and $\mathbf{\Lambda}(\cdot)$ stand respectively for symmetric and exterior algebra. The superscript * indicates dual space. Span over a monoid A is denoted by $\langle\cdot\rangle_A$. If μ is a partition, then $\mathbb{S}_\mu$ denotes the Schur functor associated with μ. In particular, $\mathbb{S}_{(k)}(\cdot) = \mathbf{S}^k(\cdot)$ and $\mathbb{S}_{\underbrace{(1,1,\dots,1)}_{\text{k times}}}(\cdot) = \mathbf{\Lambda}^k(\cdot)$. The sign $\Subset$ stands for semidirect sum of Lie algebras (the round part points to the respective ideal).

We fix a nondegenerate pairing of countable-dimensional vector spaces $\mathtt{p}\colon V \times V_* \to \mathbb{C}$, and define the Lie algebra $\mathfrak{gl}(\infty)$ as the Lie algebra arising from the associative algebra $V \otimes V_*$. Both spaces V and V_* carry obvious structures of $\mathfrak{gl}(\infty)$-modules. It is a well known fact (going back to G. Mackey [Mac]) that there exist dual bases $\{v_i\}_{i\in I}$ of V and $\{w_i\}_{i\in I}$ of V_* (i.e. a basis $\{v_i\}_{i\in I}$ of V and a basis $\{w_i\}_{i\in I}$ of V_* such that $\mathtt{p}(v_i, w_j) = \delta_{ij}$, where δ_{ij} is Kronecker's delta) where I is a fixed countable set. Then clearly $\mathfrak{gl}(\infty) = \langle v_i \otimes w_j | i, j \in I\rangle_{\mathbb{C}}$.

By $\mathfrak{sl}(\infty)$ we denote the Lie algebra $\ker\mathtt{p}$; this is a codimension-1 Lie subalgebra of $\mathfrak{gl}(\infty)$. Moreover, we fix the abelian subalgebra

$$\mathfrak{h} := \langle h_i := v_i \otimes w | i \in I\rangle_{\mathbb{C}} \cap \mathfrak{sl}(\infty) \subset \mathfrak{sl}(\infty).$$

Next, assume that V is endowed with non-degenerate symmetric or antisymmetric form $\mathtt{b} : V \otimes V \to \mathbb{C}$. If $\mathtt{b}$ is symmetric, we define the Lie algebra $\mathfrak{o}(\infty)$ as the vector space $\mathbf{\Lambda}^2(V)$ with commutator satisfying

$$[u \wedge v, w \wedge z] = -\mathtt{b}(u, w)v \wedge z + \mathtt{b}(u, z)v \wedge w + \mathtt{b}(v, w)u \wedge z - \mathtt{b}(v, z)u \wedge w.$$

According to [Mac] there exist a basis $\{u, v_i, w_i\}_{i\in I}$ of V such that

$$\mathtt{b}(u, v_i) = \mathtt{b}(u, w_j) = \mathtt{b}(v_i, v_j) = \mathtt{b}(w_i, w_j) = 0, \quad \mathtt{b}(u, u) = 1, \quad \mathtt{b}(v_i, w_j) = \delta_{ij}, \tag{2.1}$$

and a basis $\{v_i, w_i\}_{i\in I}$ of V such that

$$\mathtt{b}(v_i, v_j) = \mathtt{b}(w_i, w_j) = 0, \quad \mathtt{b}(v_i, w_j) = \delta_{ij}. \tag{2.2}$$

In both cases, we set

$$\mathfrak{h} := \langle h_i := v_i \wedge w_i | i \in I \rangle_{\mathbb{C}} \subset \mathfrak{o}(\infty).$$

If $\mathtt{b}$ is antisymmetric, we define the Lie algebra $\mathfrak{sp}(\infty)$ as the space $\mathbf{S}^2(V)$ with commutator satisfying

$$[uv, wz] = \mathtt{b}(u, w)vz + \mathtt{b}(u, z)vw + \mathtt{b}(v, w)uz + \mathtt{b}(v, z)uw.$$

Furthermore, there exists a basis $\{v_i, w_i\}_{i\in I}$ of V satisfying (2.2). We set

$$\mathfrak{h} := \langle h_i := v_i w_i | i \in I \rangle_{\mathbb{C}} \subset \mathfrak{sp}(\infty).$$

We denote by $\mathfrak{g}$ one of the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. In all four cases above, $\mathfrak{h}$ is a *splitting Cartan subalgebra of* $\mathfrak{g}$ according to [DPS]. Furthermore, $\mathfrak{g}$ has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where Δ is the *root system of* $\mathfrak{g}$. We define $\varepsilon_i \in \mathfrak{h}^*$ by setting

$$\varepsilon_i(h_j) := \delta_{ij}.$$

Then the root system of $\mathfrak{sl}(\infty)$ is

$$\Delta = A_\infty = \{\varepsilon_i - \varepsilon_j \,|\, i, j \in I, i \neq j\},$$

and the root system of $\mathfrak{sp}(\infty)$ is

$$\Delta = C_\infty = \{\varepsilon_i - \varepsilon_j \,|\, i, j \in I, i \neq j\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \,|\, i, j \in I\}.$$

The Lie algebra $\mathfrak{o}(\infty)$ has two root systems depending on whether $\mathfrak{h}$ is of type B or type D:

$$\Delta = B_\infty = \{\varepsilon_i - \varepsilon_j \,|\, i, j \in I, i \neq j\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \,|\, i, j \in I, i \neq j\} \cup \{\pm\varepsilon_i \,|\, i \in I\}$$

if (2.1) holds, and

$$\Delta = D_\infty = \{\varepsilon_i - \varepsilon_j \,|\, i, j \in I, i \neq j\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \,|\, i, j \in I, i \neq j\}$$

if (2.2) holds.

For a $\mathfrak{g}$-module M which is semisimple as an $\mathfrak{h}$-module, we put

$$\operatorname{supp} M := \{\lambda \in \mathfrak{h}^* \,| M_\lambda := \{m \in M | hm = \lambda(h)m \;\forall h \in \mathfrak{h}\} \neq 0\}.$$

Next, set

$$\tilde{I} := \begin{cases} \{\varepsilon_i \,|\, i \in I\}, & \text{for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \{\pm\varepsilon_i \,|\, i \in I\}, & \text{for } \mathfrak{g} = \mathfrak{o}(\infty),\, \mathfrak{sp}(\infty). \end{cases}$$

Note that V, as well as V_* for $\mathfrak{g} = \mathfrak{sl}(\infty)$ is a $\mathfrak{g}$-module which is semisimple as an $\mathfrak{h}$-module. We refer to V (respectively, V_*) as the *natural* (respectively, *conatural*) $\mathfrak{g}$-*module*. In all cases except $\Delta = B_\infty$, we have $\operatorname{supp} V = \tilde{I}$. If $\Delta = B_\infty$ then $\operatorname{supp} V = \tilde{I} \sqcup 0$. Finally, $\operatorname{supp} V_* = -\tilde{I}$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$ (note that the pairing $\mathtt{p}$ makes V_* a $\mathfrak{g}$-submodule of $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$).

For $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ we call a subset J of $\tilde{I}$ *symmetric* if $J = -J$. For any subset $J \subset \tilde{I}$, which we assume symmetric if $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, put

$$\Delta_J := \Delta \cap \langle J \rangle_{\mathbb{Z}},$$

and let $\mathfrak{g}_J$ be the root subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ for $\alpha \in \Delta_J$. By $\mathfrak{g}_J^c$ we denote the centralizer of $\mathfrak{g}_{\tilde{I} \setminus J}$ in $\mathfrak{g}$. For the root systems C_∞ and D_∞ we have $\mathfrak{g}_J^c = \mathfrak{g}_J$. This holds also for A_∞ under the assumption that J is not cofinite in $\tilde{I}$, otherwise $\mathfrak{g}_J = [\mathfrak{g}_J^c, \mathfrak{g}_J^c]$. For the root system B_∞, we have $\mathfrak{g}_J^c \subset \mathfrak{g}_J$: if $\mathfrak{g}_J$ has root system $B_{|J|/2}$, then $\mathfrak{g}_J^c$ has root system $D_{|J|/2}$ where $|J| = \operatorname{card} J$ (if $|J| < \infty$, the root systems $B_{|J|/2}$ and $D_{|J|/2}$ are the classical finite root systems of respective types B or D).

A *splitting Borel subalgebra* $\mathfrak{b}$ containing $\mathfrak{h}$ [DP], has the form

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

for an arbitrary decomposition $\Delta = \Delta^+ \sqcup \Delta^-$ such that $\Delta^- = -\Delta^+$ and $\alpha + \beta \in \Delta^+$ whenever $\alpha, \beta \in \Delta^+$, $\alpha + \beta \in \Delta$.

All splitting Borel subalgebras containing $\mathfrak{h}$ are in a natural bijection with the set of total orders $\prec$ on $\tilde{I}$, subject to the condition that $a \prec b$ implies $-b \prec -a$ in the case $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. In what follows, we call such orders *symmetric* or $\mathbb{Z}_2$-*linear*. Indeed, given a (symmetric) total order $\prec$ on $\tilde{I}$, we set

$$\Delta^+ := \begin{cases} \{\varepsilon_i - \varepsilon_j \,|\, \varepsilon_i < \varepsilon_j\} & \text{if } \Delta = A_\infty, \\ \{\alpha \,|\, \alpha \prec -\alpha\} \sqcup \{\alpha + \beta \,|\, \alpha \prec -\alpha, \beta \prec -\beta\} \sqcup & \\ \sqcup \{\alpha - \beta \,|\, \alpha \prec \beta\} \text{ for } \alpha, \beta \in \tilde{I} & \text{if } \Delta = B_\infty, \\ \{2\alpha \,|\, \alpha \prec -\alpha\} \sqcup \{\alpha + \beta \,|\, \alpha \prec -\alpha, \beta \prec -\beta\} \sqcup & \\ \sqcup \{\alpha - \beta \,|\, \alpha \prec \beta\} \text{ for } \alpha, \beta \in \tilde{I} & \text{if } \Delta = C_\infty, \\ \{\alpha + \beta \,|\, \alpha \prec -\alpha, \beta \prec -\beta\} \sqcup & \\ \sqcup \{\alpha - \beta \,|\, \alpha \prec \beta\} \text{ for } \alpha, \beta \in \tilde{I} & \text{if } \Delta = D_\infty. \end{cases}$$

In the remainder of the paper we assume that all total orders $\prec$ on $\tilde{I}$ considered are symmetric for $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$.

Given a total order $\prec$ on the set $\tilde{I}$, we define subsets S_{max} and S_{min} of $\tilde{I}$ as follows: S_{min} (respectively, S_{max}) is the set of all $\alpha \in \tilde{I}$ such that there exists a cofinite subset $A \subset \tilde{I}$ in which α is minimal (respectively, maximal). Note that for $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, we have $S_{min} = -S_{max}$. A total order $\prec$ on $\tilde{I}$ is *ideal* if both S_{min} and S_{max} are infinite; a total order $\prec$ on $\tilde{I}$ is *perfect* if it is ideal and $\tilde{I} = S_{min} \cup S_{max}$. The corresponding Borel subalgebras are also called *ideal* or *perfect*. Note that all perfect total orders on $\tilde{I}$ are isomorphic, which implies that all perfect Borel subalgebras are conjugate under $\operatorname{Aut}\mathfrak{g}$.

A root $\alpha \in \Delta^+$ is *simple* if α cannot be decomposed as a sum $\beta + \gamma$ for $\beta, \gamma \in \Delta^+$. If a root can be written as a linear combination of simple roots we call it a $\mathfrak{b}$-*finite* root. All other roots are *infinite* by definition. For instance, if $\mathfrak{b}$ is perfect with positive roots $\varepsilon_i - \varepsilon_j$ for $\varepsilon_i \prec \varepsilon_j$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, then the $\mathfrak{b}$-finite roots are of the form $\varepsilon_i - \varepsilon_j$ for $\varepsilon_i, \varepsilon_j \in S_{min}$ or $\varepsilon_i, \varepsilon_j \in S_{max}$.

If M is a $\mathfrak{g}$-module for $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, or for a finite-dimensional Lie algebra $\mathfrak{g}$, the Fernando-Kac subalgebra $\mathfrak{g}[M]$ of $\mathfrak{g}$ consists of all vectors $g \in \mathfrak{g}$ which act locally finitely on M, i.e. such that $\dim(\langle m, gm, g^2m, \dots \rangle_{\mathbb{C}}) < \infty$ for any $m \in M$. The fact that $\mathfrak{g}[M]$ is indeed a Lie subalgebra has been proved independently in [K] and [Fe].

We say that a $\mathfrak{g}$-module M satisfies the *large annihilator condition* if, for any $m \in M$, the annihilator in $\mathfrak{g}$ of m contains the commutator subalgebra of the centralizer of a finite-dimensional Lie subalgebra of $\mathfrak{g}$ (i.e. if M satisfies condition (c) from the Introduction).

Finally, recall that the *socle* of a $\mathfrak{g}$-module M, $\mathrm{soc}M$, is the sum of all simple submodules of M. It is a standard fact that $\mathrm{soc}M$ is the largest semisimple submodule of M. The *socle fltration* of M is

$$0 \subset \mathrm{soc}M = \mathrm{soc}^0 M \subset \mathrm{soc}^1 M \subset \mathrm{soc}^2 M \subset \dots$$

where $\mathrm{soc}^i M := \pi_i^{-1}(\mathrm{soc}(M/\mathrm{soc}^{i-1}M))$ and $\pi_i : M \to M/\mathrm{soc}^{i-1}M$ is the canonical homomorphism. We say that the socle filtration of M is *exhaustive* if $M = \bigcup_{i \geq 0} \mathrm{soc}^i M$.

3. The category $\mathcal{OLA}_{\mathfrak{b}}$

Let $\mathfrak{h}$ be the fixed splitting Cartan subalgebra of $\mathfrak{g}$, see Section 2, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a fixed splitting Borel subalgebra containing $\mathfrak{h}$ and corresponding to a total order $\prec$ on $\tilde{I}$. We define $\mathcal{OLA}_{\mathfrak{b}}$ as the full subcategory of the category of all $\mathfrak{g}$-modules, consisting of $\mathfrak{g}$-modules M satisfying the following conditions:

(i) M satisfies the large annihilator condition;
(ii) M is $\mathfrak{h}$-semisimple;
(iii) every $x \in \mathfrak{n}$ acts locally nilpotently on M.

The first problem we address, is to what extent $\mathcal{OLA}_{\mathfrak{b}}$ depends on the choice of $\mathfrak{b}$.

Set $S := S_{min} \cup S_{max} \subset \tilde{I}$. For a, $b \in \mathbb{Z}_{\geq 0}$, define $S_{min}(a) \subset S_{min}$ and $S_{max}(b) \subset S_{max}$ to be respectively the first a elements of S_{min} and the last b elements of S_{max}. Here we assume $S_{min}(0) = S_{max}(0) = \emptyset$. Put $\mathfrak{g}_{a,b} := \mathfrak{g}_{\tilde{I} \setminus (S_{min}(a) \cup S_{max}(b))}$, where for $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$ we suppose that $a = b$ and that all subsets of $\tilde{I}$ we consider are symmetric.

The large annihilator condition can be rewritten in the form

$$\text{for every } m \in M \text{ there exists a cofinite set } J \subset \tilde{I} \text{ such that } \mathfrak{g}^c_J m = 0. \tag{3.1}$$

Lemma 3.1. *Let $M \in \mathcal{OLA}_{\mathfrak{b}}$.*

(a) If M is finitely generated, then there exist $a, b \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{g}_{a,b} \subset \mathfrak{g}[M]$.
(b) For an arbitrary M, we have $\mathfrak{g}_{\tilde{I} \setminus S} \subset \mathfrak{g}[M]$.

Proof. It suffices to prove (a) for a cyclic module. Let M be generated by a vector $m \in M$. By (3.1) there exists a cofinite set $J \subset \tilde{I}$ such that $\mathfrak{g}^c_J m = 0$. Since the action of $\operatorname{ad} x$ on $U(\mathfrak{g})$ is locally finite for all $x \in \mathfrak{g}$, and $M = U(\mathfrak{g})m$, we conclude that $\mathfrak{g}^c_J \subset \mathfrak{g}[M]$. On the other hand, by (iii) we have $\mathfrak{n} \subset \mathfrak{g}[M]$. It is easy to check that the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}$ and $\mathfrak{g}^c_J$ equals $\mathfrak{g}_{J'}$ where J' is the minimal interval containing J. By the cofiniteness of J' we get $\mathfrak{g}_{J'} = \mathfrak{g}_{a,b}$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Hence $\mathfrak{g}_{a,b} \subset \mathfrak{g}[M]$.

(b) is a consequence of (a) since $\mathfrak{g}_{\tilde{I}\setminus S}$ equals the intersection $\bigcap_{a,b} \mathfrak{g}_{a,b}$. □

Theorem 3.2. *Assume that $\mathfrak{b}$ is ideal, and let J and K be infinite (symmetric) subsets of $\tilde{I}$ such that $\tilde{I} = J \sqcup K$. Suppose further that $S \subset K$, and set $\mathfrak{b}_K := \mathfrak{g}_K \cap \mathfrak{b}$. Let $\mathcal{OLA}_{\mathfrak{b}_K}$ be the category of $\mathfrak{g}_K$-modules satisfying the conditions (i)–(iii) with respect to $\mathfrak{b}_K$.*

(a)The categories $\mathcal{OLA}_{\mathfrak{b}_K}$ and $\mathcal{OLA}_{\mathfrak{b}}$ are equivalent.

(b) If the root system of $\mathfrak{g}$ is B_∞, there is also an equivalence of the categories $\mathcal{OLA}_{\mathfrak{b}^c_K}$ and $\mathcal{OLA}_{\mathfrak{b}}$ where $\mathfrak{b}^c_K := \mathfrak{b} \cap \mathfrak{g}^c_K$.

Proof. (a) Consider the functor

$$\Phi_K : \mathcal{OLA}_{\mathfrak{b}} \to \mathcal{OLA}_{\mathfrak{b}_K},\ \Phi_K(M) := M^{\mathfrak{g}^c_J},$$

where the superscript $(\cdot)^{\mathfrak{g}^c_J}$ indicates taking invariants. We shall prove that Φ_K is an equivalence of categories.

Let $\mathcal{OLA}^{a,b}_{\mathfrak{b}}$ denote the full subcategory of $\mathcal{OLA}_{\mathfrak{b}}$ consisting of modules such that $\mathfrak{g}_{a,b} \subset \mathfrak{g}[M]$. By Lemma 3.1(a),

$$\mathcal{OLA}_{\mathfrak{b}} = \varinjlim \mathcal{OLA}^{a,b}_{\mathfrak{b}}.$$

Similarly, we define the category $\mathcal{OLA}^{a,b}_{\mathfrak{b}_K}$ as the subcategory of modules M satisfying $\mathfrak{g}_{a,b,K} := \mathfrak{g}_K \cap \mathfrak{g}_{a,b} \subset \mathfrak{g}[M]$. Then

$$\mathcal{OLA}_{\mathfrak{b}_K} = \varinjlim \mathcal{OLA}^{a,b}_{\mathfrak{b}_K}.$$

Clearly, Φ_K induces well-defined functors

$$\Phi^{a,b}_K : \mathcal{OLA}^{a,b}_{\mathfrak{b}} \to \mathcal{OLA}^{a,b}_{\mathfrak{b}_K},$$

and it suffices to prove that $\Phi^{a,b}_K$ are equivalences of categories for all $a, b \in \mathbb{Z}_{\geq 0}$. Denote by $\tilde{\mathbb{T}}_{\mathfrak{g}_{a,b}}$ the inductive completion of the category $\mathbb{T}_{\mathfrak{g}_{a,b}}$. Then, for any fixed a, $b \in \mathbb{Z}_{\geq 0}$, we have the following commutative diagram of functors

$$\begin{tikzcd} \mathcal{OLA}^{a,b}_{\mathfrak{b}} \arrow[r, "\Phi^{a,b}_K"] \arrow[d, "\operatorname{Res}_{\mathfrak{g}_{a,b}}"'] & \mathcal{OLA}^{a,b}_{\mathfrak{b}_K} \arrow[d, "\operatorname{Res}_{\mathfrak{g}_{a,b,K}}"] \\ \tilde{\mathbb{T}}_{\mathfrak{g}_{a,b}} \arrow[r, "\Phi^{a,b}_K"] & \tilde{\mathbb{T}}_{\mathfrak{g}_{a,b,K}}. \end{tikzcd}$$

We claim that $\Phi_K^{a,b}$ is an equivalence of symmetric monoidal categories downstairs. This follows directly from Lemma 5.13 and 5.14 in [PS] which prove that $\Phi_K^{a,b}$ establishes an equivalence between $\mathbb{T}_{\mathfrak{g}_{a,b}}$ and $\mathbb{T}_{\mathfrak{g}_{a,b,K}}$. The passage to the respective inductive completions $\tilde{\mathbb{T}}_{\mathfrak{g}_{a,b}}$ and $\tilde{\mathbb{T}}_{\mathfrak{g}_{a,b,K}}$ is automatic because $\Phi_K^{a,b}$ commutes with direct limits.

Next we show that $\Phi_K^{a,b}$ remains an equivalence upstairs. Indeed, consider the decomposition (of vector spaces) $\mathfrak{g} = \mathfrak{g}_{a,b} \oplus \mathfrak{r}$, where $\mathfrak{r}$ is a $\mathfrak{g}_{a,b}$-stable subspace. The objects of $\mathcal{OLA}_{\mathfrak{b}}^{a,b}$ are pairs (M, φ) where $M \in \tilde{\mathbb{T}}_{\mathfrak{g}_{a,b}}$ and $\varphi : M \otimes \mathfrak{r} \to M$ is a morphism satisfying a certain set of tensor identities. Note that $\mathfrak{g}_K = \Phi_K(\mathfrak{g})$, and set $\mathfrak{r}_K := \Phi_K(\mathfrak{r})$. We have $\mathfrak{g}_K = \mathfrak{g}_{a,b,K} \oplus \mathfrak{r}_K$. The objects of $\mathcal{OLA}_{\mathfrak{b}_K}^{a,b}$ are pairs (N, ψ) where $N \in \tilde{\mathbb{T}}_{\mathfrak{g}_{a,b,K}}$ and $\psi : N \otimes \mathfrak{r}_K \to N$ is a morphism satisfying the same set of tensor identities. Obviously, $\Phi_K^{a,b}(\mathfrak{r}) = \mathfrak{r}_K$ and $\Phi_K^{a,b}(\varphi) = \psi$. This completes the proof of (a).

To prove (b), define the functor

$$\Phi'_K : \mathcal{OLA}_{\mathfrak{b}} \to \mathcal{OLA}_{\mathfrak{b}_K^c},\ \Phi'_K(M) = M^{\mathfrak{g}_J},$$

The proof that Φ'_K is an equivalence of categories is similar to the proof of (a). $\square$

Corollary 3.3. *If $\mathfrak{b} \subset \mathfrak{g}$ is an ideal subalgebra, the category $\mathcal{OLA}_{\mathfrak{b}}$ is equivalent to the category $\mathcal{OLA}_{\mathfrak{b}'}$ for a perfect subalgebra $\mathfrak{b}' \subset \mathfrak{g}$.*

Proof. First we prove that $\mathcal{OLA}_{\mathfrak{b}}$ is equivalent to $\mathcal{OLA}_{\mathfrak{b}_S}$. If S is coinfinite in $\tilde{I}$, this is established in Theorem 3.2(a). Therefore, assume that S is cofinite in $\tilde{I}$. Extend $\tilde{I}$ to a totally ordered set $\tilde{P}$ by replacing the interval $\tilde{I} \setminus S$ by an infinite interval (symmetric in the case $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$). Then $\mathfrak{g}$ and $\mathfrak{g}_S$ are embedded into an isomorphic copy $\mathfrak{g}_{\tilde{P}}$ of $\mathfrak{g}$ in which the role of $\tilde{I}$ is played by $\tilde{P}$. Let $\tilde{\mathfrak{b}}$ be the Borel subalgebra of $\mathfrak{g}_{\tilde{P}}$ defined by the ordered set $\tilde{P}$. Now Theorem 3.2 implies that both categories $\mathcal{OLA}_{\mathfrak{b}_S}$ and $\mathcal{OLA}_{\mathfrak{b}}$ are equivalent to $\mathcal{OLA}_{\tilde{\mathfrak{b}}}$. Hence, $\mathcal{OLA}_{\mathfrak{b}_S}$ and $\mathcal{OLA}_{\mathfrak{b}}$ are equivalent.

Furthermore, $\mathfrak{b}_S$ is a perfect Borel subalgebra of $\mathfrak{g}_s$ and $\mathfrak{g}_s \simeq \mathfrak{g}$. Consider an isomorphism $\varphi : \mathfrak{g}_S \to \mathfrak{g}$ and set $\mathfrak{b}' := \varphi(\mathfrak{b}_s)$. This isomorphism extends to an equivalence between $\mathcal{OLA}_{\mathfrak{b}_S}$ and $\mathcal{OLA}_{\mathfrak{b}'}$. The statement follows. $\square$

Corollary 3.4. *Assume that the root system of $\mathfrak{g}$ is B_∞ and the root system of $\mathfrak{g}'$ is D_∞. Then, for any ideal Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ there exists a perfect subalgebra $\mathfrak{b}' \subset \mathfrak{g}'$ such that the category $\mathcal{OLA}_{\mathfrak{b}}$ is equivalent to the category $\mathcal{OLA}_{\mathfrak{b}'}$.*

Proof. The proof is similar to the proof of Corollary 3.3 via application of Theorem 3.2(b). $\square$

In the rest of the section, $\mathfrak{b}$ is an arbitrary splitting Borel subalgebra containing $\mathfrak{h}$.

Proposition 3.5. *Assume that S is finite. Then there exists a perfect Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ such that $\mathcal{OLA}_{\mathfrak{b}} = \mathcal{OLA}_{\mathfrak{b}'}^{a,b}$ for some $a, b \geq 0$. In particular, if $S = \emptyset$ then $\mathcal{OLA}_{\mathfrak{b}} = \tilde{\mathbb{T}}_{\mathfrak{g}}$.*

Proof. Set $a = |S_{min}|$, $b = |S_{max}|$. Define a perfect order on $\tilde{I}$ such that $S_{min} \subset \tilde{I}$ (respectively, $S_{max} \subset \tilde{I}$) are the first (respectively, the last) elements of $\tilde{I}$. Denote by $\mathfrak{b}'$ the Borel subalgebra corresponding to this order. Then $\mathcal{OLA}_{\mathfrak{b}}^{a,b} = \mathcal{OLA}_{\mathfrak{b}'}^{a,b}$, and by Lemma 3.1(b) $\mathcal{OLA}_{\mathfrak{b}} = \mathcal{OLA}_{\mathfrak{b}}^{a,b}$. The assertion follows. □

Proposition 3.6. *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Suppose that exactly one of S_{min} and S_{max} is finite.*

(a) The categories $\mathcal{OLA}_{\mathfrak{b}_S}$ and $\mathcal{OLA}_{\mathfrak{b}}$ are equivalent.

(b) Set

$$\mathcal{OLA}_{\mathfrak{b}}^{a,\infty} := \varinjlim \mathcal{OLA}_{\mathfrak{b}}^{a,b} \text{ for } b \to \infty,$$

$$\mathcal{OLA}_{\mathfrak{b}}^{\infty,b} := \varinjlim \mathcal{OLA}_{\mathfrak{b}}^{a,b} \text{ for } a \to \infty.$$

Then there exists a perfect Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ such that

(1) if $|S_{min}| = a$ and S_{max} is infinite, then $\mathcal{OLA}_{\mathfrak{b}}$ is equivalent to $\mathcal{OLA}_{\mathfrak{b}'}^{a,\infty}$;

(2) if S_{min} is infinite and $|S_{max}| = b$, then $\mathcal{OLA}_{\mathfrak{b}}$ is equivalent to $\mathcal{OLA}_{\mathfrak{b}'}^{\infty,b}$.

Proof. (a) can be proven in the same way as Corollary 3.3, and we leave the proof to the reader.

Let us prove (b) in the case (1). Case (2) is similar. By (a) we may assume that $\tilde{I} = S$. We include S_{min} into an ordered set L isomorphic to $\mathbb{Z}_{\geq 0}$ such that S_{min} is identified with the first a elements of L. Set $\tilde{P} := L \sqcup S_{max}$, $L \prec S_{max}$ and consider the corresponding Lie algebra $\mathfrak{g}_{\tilde{P}}$ with Borel subalgebra $\tilde{\mathfrak{b}}$. Define the functor

$$\Phi_S : \mathcal{OLA}_{\tilde{\mathfrak{b}}} \to \mathcal{OLA}_{\mathfrak{b}}, \quad \Phi_S(M) := M^{\mathfrak{g}_{L \setminus S_{min}}}.$$

As in the proof of Theorem 3.2, one can show that the restriction of Φ_S to $\mathcal{OLA}_{\tilde{\mathfrak{b}}}^{a,\infty}$ is an equivalence between the categories $\mathcal{OLA}_{\tilde{\mathfrak{b}}}^{a,\infty}$ and $\mathcal{OLA}_{\mathfrak{b}}$. Since $\mathfrak{g}_{\tilde{P}}$ is isomorphic to $\mathfrak{g}$, the Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ can be chosen as the image of $\tilde{\mathfrak{b}}$ under an isomorphism $\mathfrak{g}_{\tilde{P}} \backsimeq \mathfrak{g}$, and the statement follows. □

Corollary 3.7. *If $\mathfrak{b}$ is an arbitrary splitting Borel subalgebra of $\mathfrak{g}$, there exists a perfect Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ such that the category $\mathcal{OLA}_{\mathfrak{b}}$ is equivalent to a full subcategory of $\mathcal{OLA}_{\mathfrak{b}'}$.*

Proof. Follows from Theorem 3.2, Proposition 3.5 and Proposition 3.6. □

4. $\mathcal{OLA}$: SIMPLE AND PARABOLICALLY INDUCED MODULES

Corollary 3.7 suggests that it makes sense to restrict our study of the category $\mathcal{OLA}_{\mathfrak{b}}$ to the case when $\mathfrak{b}$ is a fixed perfect Borel subalgebra. In the rest of the paper we do this and write $\mathcal{OLA}$, omitting the subscript $\mathfrak{b}$. Furthermore, Corollary 3.4 allows us to disregard the case $\Delta = B_\infty$ and assume that $\Delta = D_\infty$ for $\mathfrak{g} = \mathfrak{o}(\infty)$.

By $\bar{\mathfrak{b}} = \mathfrak{h} \ni \bar{\mathfrak{n}}$ we denote the opposite Borel subalgebra, $\mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{h}$. In addition, for $\mathfrak{g} = \mathfrak{sl}(\infty)$ we identify the ordered set $\tilde{I}$ with $\mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}$ where $i < -j$ for $i, j \in \mathbb{Z}_{>0}$, so that $S_{min} = \{\varepsilon_i | i \in \mathbb{Z}_{>0}\}$, $S_{max} = \{\varepsilon_i | i \in \mathbb{Z}_{<0}\}$. For $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ we identify S_{min} with $\mathbb{Z}_{>0}$, and write $S_{min} = \{\varepsilon_i | i \in \mathbb{Z}_{>0}\}$; then $S_{max} = -S_{min} = \{-\varepsilon_i | i \in \mathbb{Z}_{>0}\}$.

Let $\mathfrak{k}_n$ be the centralizer of $\mathfrak{g}_{n,n}$ in $\mathfrak{g}$. Note that $\mathfrak{g}_{n,n} \simeq \mathfrak{g}$ and

$$\mathfrak{k}_n \simeq \begin{cases} \mathfrak{sl}(2n) \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \mathfrak{o}(2n) \text{ for } \mathfrak{g} = \mathfrak{o}(\infty) \\ \mathfrak{sp}(2n) \text{ for } \mathfrak{g} = \mathfrak{sp}(\infty). \end{cases}$$

Next, fix compatible nodegenerate invariant forms on $\mathfrak{k}_n$ which define a nondegenerate invariant form $(\cdot, \cdot)$ on $\mathfrak{g}$. We will use the same notation when considering $(\cdot, \cdot)$ as a form on $\mathfrak{h}^*$.

In what follows we will use the family of parabolic subalgebras of $\mathfrak{g}$

$$\mathfrak{p}_n := \mathfrak{b} + \mathfrak{g}_{n,n}$$

with reductive parts $\mathfrak{l}_n = \mathfrak{h} + \mathfrak{g}_{n,n}$. By $\bar{\mathfrak{p}}_n$ we denote the parabolic subalgebra opposite to $\mathfrak{p}_n$, $\mathfrak{p}_n \cap \bar{\mathfrak{p}}_n = \mathfrak{l}_n$. Furthermore, we define $\mathfrak{m}_n$ as the nilpotent ideal of $\mathfrak{p}_n$ such that $\mathfrak{p}_n = \mathfrak{l}_n \ni \mathfrak{m}_n$. The space of $\mathfrak{g}_{n,n}$-invariants $\mathfrak{m}_n^{\mathfrak{g}_{n,n}}$ is finite dimensional, and the decomposition of $\mathfrak{g}_{n,n}$-modules $\mathfrak{m}_n = \mathfrak{r}_n \oplus \mathfrak{m}_n^{\mathfrak{g}_{n,n}}$ defines $\mathfrak{r}_n \subset \mathfrak{m}_n$.

In addition, we introduce the subalgebras $\mathfrak{s} \subset \mathfrak{g}$ and $\mathfrak{s}_n \subset \mathfrak{k}_n$ by setting

$$\mathfrak{s} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{fin}} \mathfrak{g}_\alpha, \quad \mathfrak{s}_n := \mathfrak{s} \cap \mathfrak{k}_n,$$

where Δ_{fin} stands for the $\mathfrak{b}$-finite roots. We have

$$\mathfrak{s} \simeq \begin{cases} \mathfrak{sl}(\infty) \oplus \mathfrak{sl}(\infty) \oplus \mathbb{C} \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \mathfrak{gl}(\infty) \text{ for } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty) \end{cases}.$$

and

$$\mathfrak{s}_n \simeq \begin{cases} \mathfrak{sl}(n) \oplus \mathfrak{sl}(n) \oplus \mathbb{C} \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \mathfrak{gl}(n) \text{ for } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty) \end{cases}.$$

Note that $\mathfrak{h}_n := \mathfrak{h} \cap \mathfrak{k}_n$ is a Cartan subalgebra of $\mathfrak{k}_n$ as well as of $\mathfrak{s}_n$.

For $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ we denote by V_n the natural $\mathfrak{gl}_n = \mathfrak{s}_n$-module. For $\mathfrak{g} = \mathfrak{sl}(\infty)$ we set $V_n = V_n^L \oplus V_n^R$, where $\operatorname{supp} V_n^L = \{\varepsilon_i | 1 \leq i \leq n\}$ and $\operatorname{supp} V_n^R = \{\varepsilon_i | -n \leq i \leq -1\}$. Then there is canonical decomposition

$$V = V_n \oplus \bar{V}_n$$

where $\bar{V}_n$ is the natural $\mathfrak{g}_{n,n}$-module (the notion of natural module makes sense for $\mathfrak{g}_{n,n}$ as $\mathfrak{g}_{n,n}$ is isomorphic to $\mathfrak{g}$). Moreover, we have the following isomorphism of

$\mathfrak{g}_{n,n}$-modules:

$$\mathfrak{r}_n \simeq \begin{cases} \bar{V}_n^{\oplus n} \oplus (\bar{V}_n)_*^{\oplus n} \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \bar{V}_n^{\oplus n} \text{ for } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty) \end{cases} .$$

4.1. **Simple modules.** We start with the following lemma.

Lemma 4.1. *There exists a finite-dimensional* $\mathrm{ad}(\mathfrak{m}_n^{\mathfrak{g}_{n,n}})$*-stable subspace* $\mathfrak{u} \subset \mathfrak{r}_n$ *such that* $\mathbf{S}(\mathfrak{u})$ *generates* $\mathbf{S}(\mathfrak{r}_n)$ *as a module over* $\mathfrak{g}_{n,n}$.

Proof. Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then

$$\mathbf{S}(\mathfrak{r}_n) = \mathbf{S}((\bar{V}_n)_*^{\oplus n} \oplus \bar{V}_n^{\oplus n}) = \bigoplus_{\lambda,\mu} (\mathbb{S}_\lambda((\bar{V}_n)_*) \otimes \mathbb{S}_\mu(\bar{V}_n))^{\oplus c(\lambda,\mu)}$$

for some $c(\lambda,\mu) \in \mathbb{Z}_{\geq 0}$, where the summation is taken over all partitions λ, μ with at most n parts. Recall that, by Lemma 4.1(a) in [DPS], the $\mathfrak{g}$-module $\mathbb{S}_\lambda((\bar{V}_n)_*) \otimes \mathbb{S}_\mu(\bar{V}_n)$ is generated by $\mathbb{S}_\lambda(Z'_n) \otimes \mathbb{S}_\mu(Z_n)$ for some n-dimensional subspaces $Z'_n \subset (\bar{V}_n)_*$ and $Z_n \subset \bar{V}_n$. Therefore, $\mathbf{S}(\mathfrak{r}_n)$ is also generated by $\mathbf{S}(\mathfrak{u})$ for some finite-dimensional space $\mathfrak{u} \subset \mathfrak{r}_n$. As $\mathfrak{m}_n^{\mathfrak{g}_{n,n}}$ is finite dimensional and its elements act locally finitely on $\mathfrak{r}_n$, the subspace $\mathfrak{u}$ can clearly be chosen $\mathrm{ad}(\mathfrak{m}_n^{\mathfrak{g}_{n,n}})$-stable.

In the orthogonal and symplectic case we have the decomposition

$$\mathbf{S}(\mathfrak{r}_n) = \mathbf{S}(\bar{V}_n^{\oplus n}) = \bigoplus_{\lambda} (\mathbb{S}_\lambda(\bar{V}_n))^{\oplus c(\lambda)},$$

for some $c(\lambda) \in \mathbb{Z}_{\geq 0}$, where λ runs over all partitions with at most n parts. Here, application of Lemma 4.1(b) from [DPS] leads to the result. $\square$

By $U(\cdot)$ we denote as usual the enveloping algebra of a Lie algebra, and $U^k(\cdot)$ stands for the k-th term of the PBW filtration on $U(\cdot)$.

Proposition 4.2. *Let* $M \in \mathcal{OLA}$ *and* $0 \neq v \in M$ *satisfy* $\mathfrak{g}_{n,n} v = 0$.
(a) There exists $m \in \mathbb{Z}_{>0}$ *such that* $\mathfrak{m}_n^m v = 0$.
(b) $(U(\mathfrak{m}_n)v)^{\mathfrak{m}_n} \neq 0$.

Proof. Let us prove (a). Since every element of $\mathfrak{m}_n$ acts locally nilpotently on M, it suffices to check that $U^k(\mathfrak{m}_n)v = U(\mathfrak{m}_n)v$ for sufficiently large k. Then m can be chosen as $k+1$. Note that $\mathfrak{m}_n$ is $\mathrm{ad}(\mathfrak{g}_{n,n})$-stable, therefore $U^k(\mathfrak{m}_n)v$ is also $\mathrm{ad}(\mathfrak{g}_{n,n})$-stable. Choose $\mathfrak{u} \subset \mathfrak{r}_n$ as in Lemma 4.1 and set $\mathfrak{a} := \mathfrak{u} \niplus \mathfrak{m}_n^{\mathfrak{g}_{n,n}}$. Since $\mathfrak{a}$ is a nilpotent finite-dimensional Lie algebra we have $U^k(\mathfrak{a})v = U(\mathfrak{a})v$ for sufficiently large k. On the other hand, $U^k(\mathfrak{a})$ (respectively, $U(\mathfrak{a})$) generates $U^k(\mathfrak{m}_n)$ (respectively, $U(\mathfrak{m}_n)$) as an adjoint $\mathfrak{g}_{n,n}$-module. This implies that the $\mathfrak{g}_{n,n}$-submodules of $U(\mathfrak{g})v$ generated respectively by $U^k(a)v$ and $U(a)v$ coincide. As these modules equal respectively $U^k(\mathfrak{m}_n)v$ and $U(\mathfrak{m}_n)v$, we obtain $U^k(\mathfrak{m}_n)v = U(\mathfrak{m}_n)v$.

(b) follows immediately from (a). $\square$

Theorem 4.3. *Let $L \in \mathcal{OLA}$ be a simple object. Then there exist $n \in \mathbb{Z}_{\geq 0}$ and a weight $\lambda \in \mathfrak{h}^*$, such that $\lambda|_{\mathfrak{h}\cap\mathfrak{g}_{n,n}} = 0$ and L is isomorphic to the unique simple quotient of the induced module $\mathrm{Ind}_{\mathfrak{p}_n}^{\mathfrak{g}} \mathbb{C}_\lambda$. In particular, L is a highest weight module with highest weight λ, and we denote it by $L(\lambda)$.*

Proof. The large annihilator condition ensures that for any $v \in L$ we have $\mathfrak{g}_{k,k}v = 0$ for some $k \in \mathbb{Z}_{\geq 0}$. Therefore Proposition 4.2(b) implies $L^{\mathfrak{m}_k} \neq 0$ for some k. Since L is simple, $L^{\mathfrak{m}_k}$ is a simple $\mathfrak{l}_k$-module. Moreover, as a $\mathfrak{g}_{k,k}$-module, $L^{\mathfrak{m}_k}$ is integrable and satisfies the large annihilator condition. Hence, by Theorem 4.2 in [DPS], $L^{\mathfrak{m}_k}$ has a highest weight vector u with respect to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}_{k,k}$. Since $\mathfrak{n} = \mathfrak{m}_k \subsetplus (\mathfrak{n} \cap \mathfrak{g}_{k,k})$, we obtain $\mathfrak{n}u = 0$. Denote by λ the weight of u. By the large annihilator condition, there exists $n \geq k$ such that $\mathfrak{g}_{n,n}u = 0$. This implies that $\mathbb{C}u$ is a one-dimensional $\mathfrak{p}_n$-module isomorphic to $\mathbb{C}_\lambda$. Then by Frobenius reciprocity L is isomorphic a quotient of $\mathrm{Ind}_{\mathfrak{p}_n}^{\mathfrak{g}} \mathbb{C}_\lambda$. □

In what follows, we call a weight $\lambda \in \mathfrak{h}^*$ *eligible* if $\lambda|_{\mathfrak{h}\cap\mathfrak{g}_{n,n}} = 0$ for some $n \in \mathbb{Z}_{\geq 0}$. The set of eligible weights coincides with the subspace $\langle \tilde{I} \rangle_{\mathbb{C}} \subset \mathfrak{h}^*$. Note that for $\mathfrak{g} = \mathfrak{sl}(\infty)$ an eligible weight λ has the form $\lambda^L + \lambda^R$ for uniquely determined eligible weights $\lambda^L := \sum_{i\in\mathbb{Z}_{>0}} \lambda_i\varepsilon_i$ and $\lambda^R := \sum_{i\in\mathbb{Z}_{<0}} \lambda_i\varepsilon_i$ (recall that in this case $S_{min} = \{\varepsilon_i | i \in \mathbb{Z}_{>0}\}$, $S_{max} = \{\varepsilon_i | i \in \mathbb{Z}_{<0}\}$). Furthermore, Theorem 4.3 claims that any simple object of $\mathcal{OLA}$ is a $\mathfrak{b}$-highest weight module with an eligible highest weight.

A weight λ is $\mathfrak{b}$-*dominant* if $2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta^+$. We observe that for $\mathfrak{g} = \mathfrak{sl}(\infty)$ an eligible weight λ is $\mathfrak{b}$-dominant iff $\lambda^1 := (\lambda_1^L \geq \dots \geq \lambda_l^L \geq \dots)$ and $\lambda^2 := (-\lambda_{-1}^R \geq \dots \geq -\lambda_{-r}^R \geq \dots)$ are partitions where $\lambda_i \in \mathbb{Z}_{\geq 0}$ for $i \in \mathbb{Z}_{>0}$, $\lambda_i \in \mathbb{Z}_{\leq 0}$ for $i \in \mathbb{Z}_{<0}$. For $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ an eligible weight λ is $\mathfrak{b}$-dominant iff $\lambda = (\lambda_1, \dots, \lambda_k, \dots)$ is itself a partition. In [DPS] the simple modules of the category $\mathbb{T}_{\mathfrak{g}}$ are parametrized as $V_{(\lambda^1,\lambda^2)}$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, and as V_λ for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, where $\lambda^1, \lambda^2, \lambda$ are partitions. As we pointed out in the introduction, $\mathbb{T}_{\mathfrak{g}}$ is a full subcategory of $\mathcal{OLA}$, and the simple modules $V_{(\lambda^1,\lambda^2)}$ are denoted in the present paper as $L(\lambda)$ where $\lambda = (\lambda^1, \lambda^2)$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, and where λ is considered both as an eligible weight and as a partition for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$.

4.2. **Parabolically induced modules** $\mathrm{Ind}_{\mathfrak{p}_n}^{\mathfrak{g}} \mathbb{C}_\lambda$**.** For an eligible weight λ, we set

$$M_n(\lambda) := \mathrm{Ind}_{\mathfrak{p}_n}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where we always assume that n is large enough to ensure that $\mathbb{C}_\lambda$ is a trivial $\mathfrak{g}_{n,n}$-module.

Lemma 4.4. *A nonzero integrable quotient of $M_n(\lambda)$ is simple.*

Proof. Since $\mathfrak{b} \subset \mathfrak{p}_n$, any quotient of $M_n(\lambda)$ is a $\mathfrak{b}$-highest weight module. An integrable quotient of $M_n(\lambda)$ is an object $\mathbb{T}_{\mathfrak{g}}$, and is hence isomorphic to a submodule of a finite direct sum $\bigoplus_i V^{\otimes n_i} \otimes (V_*)^{\otimes m_i}$ for some $m, n \in \mathbb{Z}_{\geq 0}$. (In the case

$\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ we assume $V = V_*$.) However, the explicit form of the socle filtration of $\bigoplus_i V^{\otimes n_i} \otimes (V_*)^{\otimes m_i}$, see [PStyr] or [DPS], implies that a $\mathfrak{b}$-highest weight submodule of $\bigoplus_i V^{\otimes n_i} \otimes (V_*)^{\otimes m_i}$ is necessarily simple. $\square$

Lemma 4.5. *The module $M_n(\lambda)$, considered as an $\mathfrak{l}_n$-module, has a decomposition $\bigoplus M_i$ such that each M_i is a finite-length $\mathfrak{l}_n$-module. Moreover, the Jordan-Hölder multiplicity of every simple $\mathfrak{l}_n$-module in $M_n(\lambda)$ is finite.*

Proof. We have an isomorphism of $\mathfrak{l}_n$-modules

$$M_n(\lambda) \simeq \mathbf{S}(\bar{\mathfrak{m}}_n) \otimes \mathbb{C}_\lambda$$

where $\bar{\mathfrak{m}}_n$ is the nilpotent ideal such that $\bar{\mathfrak{p}}_n = \mathfrak{l}_n \ni \bar{\mathfrak{m}}_n$. Let $z \in \mathfrak{l}_n$ be a central element which defines a finite $\mathbb{Z}_{<0}$-grading on $\bar{\mathfrak{m}}_n$. Consider the decomposition $M_n(\lambda) = \bigoplus_i M_i$ into ad$z$-eigenspaces. Then every M_i is isomorphic to a submodule in $(\oplus_{i-k<j<i+k}\mathbf{S}^j(\bar{\mathfrak{m}}_n)) \otimes \mathbb{C}_\lambda$ for sufficiently large k. Thus M_i is a finite-length $\mathfrak{l}_n$-module, and the statement follows. $\square$

Corollary 4.6. *There is a descending filtration*

$$M_n(\lambda) = (M_n(\lambda))_0 \supset (M_n(\lambda))_1 \supset \cdots \supset (M_n(\lambda))_i \supset \dots$$

such that $\bigcap_i (M_n(\lambda))_i = 0$ and $(M_n(\lambda))_i/(M_n(\lambda))_{i+1}$ is simple for all $i \geq 0$. Furthermore, the subquotient multiplicity $[M_n(\lambda) : L(\mu)]$ of any simple module $L(\mu)$ defined by such a filtration is finite and does not depend on the choice of a filtration.

Proof. Lemma 4.5 implies the statement if we consider $M_n(\lambda)$ as a module over $\mathfrak{l}$. Hence, the statement holds also for $\mathfrak{g} \supset \mathfrak{l}$. $\square$

4.3. **Jordan–Hoelder multiplicities for parabolically induced modules.** Consider the functor

$$\Phi_n : \mathcal{OLA} \to \tilde{\mathcal{O}}_{\mathfrak{k}_n},\ \Phi_n(M) := M^{\mathfrak{g}_{n,n}},$$

$\tilde{\mathcal{O}}_{\mathfrak{k}_n}$ being the inductive completion of the BGG category $\mathcal{O}$ for the finite-dimensional Lie algebra $\mathfrak{k}_n$. The large annihilator condition ensures that for any $M \in \mathcal{OLA}$

$$M = \varinjlim \Phi_n(M).$$

Lemma 4.7. *For $m \geq n$ we have an isomorphism of $\mathfrak{k}_m$-modules*

$$\Phi_m(M_n(\lambda)) \simeq \mathrm{Ind}^{\mathfrak{k}_m}_{\mathfrak{p}_n \cap \mathfrak{k}_m} \mathbb{C}_\lambda.$$

Proof. Note that the result of application of Φ_m depends only on the restriction to $\mathfrak{g}_{m,m}$. Therefore, the statement follows from the isomorphism of $\mathfrak{g}_{m,m}$-modules

$$M_n(\lambda) \simeq \mathbf{S}(\bar{\mathfrak{m}}_n) \otimes \mathbb{C}_\lambda$$

and the fact that

$$\mathbf{S}(\bar{\mathfrak{m}}_n)^{\mathfrak{g}_{m,m}} = \mathbf{S}(\bar{\mathfrak{m}}_n \cap \mathfrak{k}_m).$$

$\square$

Lemma 4.8. *Let $M, N \in \mathcal{OLA}$ and $U(\mathfrak{g})\Phi_n(M) = M$. Then the natural map $\mathrm{Hom}_{\mathfrak{g}}(M, N) \to \mathrm{Hom}_{\mathfrak{k}_n}(\Phi_n(M), \Phi_n(N))$ is injective.*

Proof. Straightforward. $\square$

Corollary 4.9. *Let $m \geq n$. The natural map*

$$\mathrm{Hom}_{\mathfrak{g}}(M_n(\lambda), N) \to \mathrm{Hom}_{\mathfrak{k}_m}(\mathrm{Ind}_{\mathfrak{p}_n \cap \mathfrak{k}_m}^{\mathfrak{k}_m} \mathbb{C}_\lambda, \Phi_m(N))$$

is injective.

Lemma 4.10. *If $\mathrm{Hom}_{\mathfrak{g}}(M_n(\lambda), M_m(\mu)) \neq 0$ for $\lambda \neq \mu$, then $\lambda - \mu$ is a sum of positive finite roots.*

Proof. By Corollary 4.9, $\mathrm{Hom}_{\mathfrak{g}}(M_n(\lambda), M_m(\mu)) \neq 0$ implies

$$\mathrm{Hom}_{\mathfrak{k}_p}(\mathrm{Ind}_{\mathfrak{p}_n \cap \mathfrak{k}_p}^{\mathfrak{k}_p} \mathbb{C}_\lambda, \mathrm{Ind}_{\mathfrak{p}_m \cap \mathfrak{k}_p}^{\mathfrak{k}_p} \mathbb{C}_\mu) \neq 0$$

for all $p \geq m, n$. Consequently $\mu|_{\mathfrak{h}_p} + \rho_p$ and $\lambda|_{\mathfrak{h}_p} + \rho_p$ lie in one orbit of the Weyl group W_p of $\mathfrak{k}_p$, where ρ_p denotes the half-sum of positive roots of $\mathfrak{k}_p$. In other words,

$$\lambda|_{\mathfrak{h}_p} - w_p(\mu|_{\mathfrak{h}_p}) = w_p(\rho_p) - \rho_p$$

for some $w_p \in W_p$. When $p \to \infty$ the quantity $|(\lambda|_{\mathfrak{h}_p} - w_p(\mu|_{\mathfrak{h}_p}), \alpha)|$ remains bounded for any fixed $\alpha \in \Delta$ and any $w_p \in W_p$, while the quantity $|(w_p(\rho_p) - \rho_p, \alpha)|$ remains bounded if and only if w_p is a product of reflections corresponding to simple roots of $\mathfrak{k}_p$ which are finite as roots of $\mathfrak{g}$. Therefore w_p must have the latter property, and this implies the statement. $\square$

Lemma 4.11. *Let $L_{\mathfrak{k}_m}(\mu|_{\mathfrak{h}_m})$ denote a simple $\mathfrak{k}_m$-module with a $\mathfrak{b} \cap \mathfrak{k}_m$-highest weight vector of weight $\mu|_{\mathfrak{h}_m}$. If $[M_n(\lambda) : L(\mu)] \neq 0$, then $[\Phi_m(M_n(\lambda)) : L_{\mathfrak{k}_m}(\mu|_{\mathfrak{h}_m})] \neq 0$ for sufficiently large m.*

Proof. If $[M_n(\lambda) : L(\mu)] \neq 0$, then there exists a nonzero vector $u \in M_n(\lambda)$ of weight μ and a submodule $X \subset M_n(\lambda)$ such that $\mathfrak{n}u \in X$ and $u \notin X$. For all sufficiently large m, we have $u \in \Phi_m(M_n(\lambda))$. Then $(\mathfrak{n} \cap \mathfrak{k}_m)u \in \Phi_m(X)$. Therefore $[\Phi_m(M_n(\lambda)) : L_{\mathfrak{k}_m}(\mu|_{\mathfrak{h}_m})] \neq 0$. $\square$

Lemma 4.12. *If $[M_n(\lambda) : L(\mu)] \neq 0$ for $\lambda \neq \mu$, then $\lambda - \mu$ is a sum of positive finite roots.*

Proof. The previous lemma implies $[\Phi_m(M_n(\lambda)) : L_{\mathfrak{k}_m}(\mu|_{\mathfrak{h}_m})] \neq 0$ for all sufficiently large m. Therefore we can use the same argument as in the proof of Lemma 4.10. $\square$

Let $\mathcal{W}$ be the group generated by all reflections with respect to the simple roots of our fixed Borel subalgebra $\mathfrak{b}$. Then $\mathcal{W} \simeq \mathcal{S}_\infty \times \mathcal{S}_\infty$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$ and $\mathcal{W} \simeq \mathcal{S}_\infty$ for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$; here $\mathcal{S}_\infty$ denotes the infinite symmetric group. We fix $\rho \in \mathfrak{h}^*$ such that $2\frac{(\rho,\alpha)}{(\alpha,\alpha)} = 1$ for any simple root α.

We define a partial order $\leq_{fin}$ on the set of eligible weights by setting $\mu \leq_{fin} \lambda$ if $\mu = \lambda$ or $\lambda - \mu$ is a sum of positive simple roots and $(\lambda + \rho) = w(\mu + \rho)$ for some

$w \in \mathcal{W}$. This order is interval-finite. In fact, the following stronger property holds: for any eligible weight μ, the set

$$\mu^+_{fin} := \{\lambda \,|\, \mu \leq_{fin} \lambda\}$$

is finite.

Lemmas 4.11 and 4.12 imply the following.

Corollary 4.13. *If $[M_n(\lambda) : L(\mu)] \neq 0$, then $\mu \leq_{fin} \lambda$.*

Lemma 4.14. *Given two eligible weights λ and μ, there exists $N \in \mathbb{Z}_{\geq 0}$ such that the multiplicity $[M_n(\lambda) : L(\mu)]$ is constant for $n > N$. We denote this constant multiplicity by $m(\lambda, \mu)$.*

Proof. Choose N such that $\lambda - \mu$ is a sum of roots of $\mathfrak{k}_{N,N}$. For $n > N$, consider the canonical surjection homomorphism $\varphi : M_n(\lambda) \to M_N(\lambda)$. We have $\mu \notin \operatorname{supp} \ker \varphi$. Hence $[\ker \varphi : L(\mu)] = 0$. This implies $[M_n(\lambda) : L(\mu)] = [M_N(\lambda) : L(\mu)]$. □

Lemma 4.15. *The $\mathfrak{g}$-module $M_n(\lambda)$ has finite length.*

Proof. We claim that there are finitely many weights μ for which $[M_n(\lambda) : L(\mu)] \neq 0$. Indeed, $[M_n(\lambda) : L(\mu)] \neq 0$ implies that $\mathfrak{g}[L(\mu)] \subset \mathfrak{g}_{n,n}$ and hence the restriction of μ to $\mathfrak{h} \cap \mathfrak{g}_{n,n}$ is $\mathfrak{b} \cap \mathfrak{g}_{n,n}$-dominant. On the other hand, by Corollary 4.13 $\mu = w(\lambda+\rho)-\rho$ for some $w \in \mathcal{W}$. This is possible only for finitely many w, and hence for finitely many μ. □

The following lemma shows that the multiplicities $m(\lambda, \mu)$ can be expressed in terms of Kazhdan-Lusztig multiplicities for the BGG category $\mathcal{O}_{\mathfrak{s}_n}$ of the reductive Lie algebra $\mathfrak{s}_n$ for sufficiently large n.

Lemma 4.16. *Let λ, μ be eligible weights such that $\mu \leq_{fin} \lambda$, and let $\lambda|_{\mathfrak{h}\cap\mathfrak{g}_{n,n}} = \mu|_{\mathfrak{h}\cap\mathfrak{g}_{n,n}} = 0$ for some n. Then*

$$m(\lambda, \mu) = [M_{\mathfrak{s}_n}(\lambda|_{\mathfrak{h}_n}) : L_{\mathfrak{s}_n}(\mu|_{\mathfrak{h}_n})]$$

where $M_{\mathfrak{s}_n}(\lambda|_{\mathfrak{h}_n})$ and $L_{\mathfrak{s}_n}(\mu|_{\mathfrak{h}_n})$ denote the respective Verma and simple module over $\mathfrak{s}_n$.

Proof. Consider the parabolic subalgebra $\mathfrak{q}_n = \mathfrak{s}_n + \mathfrak{p}_n$. Then $M_n(\lambda) \simeq \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{q}_n} M_{\mathfrak{s}_n}(\lambda|_{\mathfrak{h}_n})$. Since $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{q}_n}$ is an exact functor, we have $m(\lambda, \mu) \geq [M_{\mathfrak{s}_n}(\lambda|_{\mathfrak{h}_n}) : L_{\mathfrak{s}_n}(\mu|_{\mathfrak{h}_n})]$. Choose $h \in \mathfrak{h}$ such that $[h, \mathfrak{s}_n] = [h, \mathfrak{g}_{n,n}] = 0$ and $\alpha(h) = 1$ for the simple roots α which are not roots of $\mathfrak{s}_n \oplus \mathfrak{g}_{n,n}$. Then $L(\mu)^{h-\lambda(h)} \simeq L_{\mathfrak{s}_n}(\mu|_{\mathfrak{h}_n})$ and $M_n(\lambda)^{h-\lambda(h)} \simeq M_{\mathfrak{s}_n}(\lambda|_{\mathfrak{h}_n})$, the superscript indicating taking invariants. Hence $m(\lambda, \mu) \leq [M_{\mathfrak{s}_n}(\lambda|_{\mathfrak{h}_n}) : L_{\mathfrak{s}_n}(\mu|_{\mathfrak{h}_n})]$. The statement follows. □

Proposition 4.17. *Any finitely generated module in $\mathcal{OLA}$ has finite length.*

Proof. It suffices to check the statement for a cyclic module. Assume that M is generated by some weight vector v annihilated by $\mathfrak{g}_{n,n}$. Then $\mathfrak{m}^m_n v = 0$ for some m

by Proposition 4.2 (a). Therefore $\dim U(\mathfrak{m}_n)v < \infty$, and there is a finite filtration $\{(U(\mathfrak{m}_n)v)_i\}$ of $U(\mathfrak{m}_n)v$ such that every quotient $(U(\mathfrak{m}_n)v)_i/(U(\mathfrak{m}_n)v)_{i-1}$ is annihilated by $\mathfrak{m}_n$. Moreover, $(U(\mathfrak{m}_n)v)_i/(U(\mathfrak{m}_n)v)_{i-1}$ is an object of the category $\mathbb{T}_{\mathfrak{g}_{n,n}}$. Hence one can refine this filtration of $U(\mathfrak{m}_n)v$ and obtain a finite filtration

$$0 \subset F_1 \subset \cdots \subset U(\mathfrak{m}_n)v,$$

such that F_i/F_{i-1} is a simple integrable $\mathfrak{g}_{n,n}$-module annihilated by $\mathfrak{m}_n$.

Consider the induced filtration of M:

$$0 \subset U(\mathfrak{g})F_1 \subset \cdots \subset U(\mathfrak{g})v = M.$$

Then $U(\mathfrak{g})F_i/U(\mathfrak{g})F_{i-1}$ is isomorphic to a quotient of the induced module $\mathrm{Ind}_{\mathfrak{p}_n}^{\mathfrak{g}}(F_i/F_{i-1})$, and the latter module is isomorphic to a quotient of $M_t(\lambda)$ for some $t > n$ and some λ. Since $M_t(\lambda)$ has finite length, the same is true for $U(\mathfrak{g})F_i/U(\mathfrak{g})F_{i-1}$, and thus for M. □

Corollary 4.18. *(a) Any $M \in \mathcal{OLA}$ is the union of finite-length submodules.*

(b) Any $M \in \mathcal{OLA}$ has an exhausting socle filtration.

Proof. Any module is the union of its finitely generated submodules. Therefore (a) follows from Proposition 4.17. The same proposition implies (b) as any module of finite length has a finite exhausting socle filtration.

□

4.4. **Canonical filtration on $\mathcal{OLA}$.** For an eligible weight $\lambda = \sum_{\varepsilon_i \in I} \lambda_i \varepsilon_i$ we set

$$d(\lambda) = \begin{cases} \frac{1}{2}(\sum_{i \in \mathbb{Z}_{>0}} \lambda_i - \sum_{j \in \mathbb{Z}_{<0}} \lambda_j) \text{ if } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \frac{1}{2}\sum_{i \in \mathbb{Z}_{>0}} \lambda_i \text{ if } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty). \end{cases}$$

Note that if $\lambda - \mu \in \langle \Delta^+ \rangle_{\mathbb{Z}_{\geq 0}}$, then $d(\lambda) - d(\mu) \in \mathbb{Z}_{\geq 0}$.

Lemma 4.19. *Assume* $\mathrm{Ext}^1_{\mathcal{OLA}}(L(\lambda), L(\mu)) \neq 0$*. Then* $d(\lambda) - d(\mu) \in \mathbb{Z}_{\leq 0}$*.*

Proof. Recall that if M is a $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$, then M_λ is the weight space of weight λ,

$$M_\lambda := \{m \in M | hm = \lambda(h)m \ \forall h \in \mathfrak{h}\}.$$

Consider a nonsplit exact sequence in $\mathcal{OLA}$

$$0 \to L(\mu) \to M \to L(\lambda) \to 0.$$

Since M is a $\mathfrak{h}$-semisimple, a standard argument shows that $\lambda \neq \mu$.

We claim that that either $\mu - \lambda \in \langle \Delta^+ \rangle_{\mathbb{Z}_{>0}}$ or $\lambda - \mu \in \langle \Delta^+ \rangle_{\mathbb{Z}_{>0}}$. Indeed, assume $\lambda - \mu \notin \langle \Delta^+ \rangle_{\mathbb{Z}_{>0}}$. Then the weight space M_μ must be a subspace of $U(\mathfrak{b})M_\lambda$ as otherwise the sequence would split. Therefore $\mu - \lambda \in \langle \Delta^+ \rangle_{\mathbb{Z}_{>0}}$.

If $\mu - \lambda \in \langle \Delta^+ \rangle_{\mathbb{Z}_{>0}}$, then $d(\lambda) - d(\mu) \in \mathbb{Z}_{\leq 0}$. If $\lambda - \mu \in \langle \Delta^+ \rangle_{\mathbb{Z}_{>0}}$, then M is isomorphic to a quotient of $M_n(\lambda)$ for some n. Therefore $m(\lambda, \mu) \neq 0$. By Corollary 4.13 $\lambda - \mu$ is a sum of simple positive roots, and hence $d(\lambda) = d(\mu)$. □

Corollary 4.20. *If $M \in \mathcal{OLA}$ is indecomposable, then $d(\nu) - d(\nu') \in \mathbb{Z}$ for any two weights of M.*

We say that a simple module $L(\lambda) \in \mathcal{OLA}$ *has degree d* if $d(\lambda) = d$.

Lemma 4.21. *Let $M \in \mathcal{OLA}$ have a simple constituent of degree $d \in \mathbb{C}$ an let the degree of every simple constituent of M belong to $d + \mathbb{Z}_{\leq 0}$. Then there exists a unique submodule $N \subset M$ such that any simple constituent of N has degree d, and every simple constituent of M/N has degree lying in $d + \mathbb{Z}_{<0}$.*

Proof. Let N be some maximal (possibly zero) submodule of M whose simple subquotients have degree d. We claim that the degrees of all simple subquotients of M/N lie in $d + \mathbb{Z}_{<0}$. Indeed, if we assume the contrary, then at some level of the socle filtration of M there is a simple constituent $L(\mu)$ of degree $d' = d + l$ for $l \in \mathbb{Z}_{<0}$, and there is a simple constituent $L(\lambda)$ of degree d at the next level with a nontrivial extension of $L(\lambda)$ by $L(\mu)$. This contradicts Lemma 4.19. □

Corollary 4.22. *Let $M \in \mathcal{OLA}$ satisfy the condition of Lemma 4.21. Then M has an exhausting canonical filtration*

$$0 = D_0(M) \subset D_1(M) \subset D_2(M) \subset \dots \tag{4.1}$$

such that all simple constituents of $D_i(M)/D_{i-1}(M)$ have degree $d - i + 1$.

We define $\mathcal{OLA}(\mathfrak{s})$ as the category of $\mathfrak{s}$-modules which satisfy conditions (i)-(iii) of Section 3 for the Borel subalgebra $\mathfrak{b} \cap \mathfrak{s}$ of $\mathfrak{s}$ where $\mathfrak{b}$ is our fixed perfect Borel subalgebra of $\mathfrak{g}$. Next, we denote by $\mathcal{OLA}^d$ the full subcategory of $\mathcal{OLA}$ consisting of all objects whose simple constituents have degree d. Obviously, $\mathcal{OLA}^d$ is a Serre subcategory of $\mathcal{OLA}$. For any $M \in \mathcal{OLA}^d$ we set

$$M^+ := \bigoplus_{d(\mu)=d} M_\mu.$$

Then clearly M^+ is an object of $\mathcal{OLA}(\mathfrak{s})$. Furthermore $(\cdot)^+ : \mathcal{OLA}^d \to \mathcal{OLA}(\mathfrak{s})$ is an exact faithful functor.

Lemma 4.23. *For any objects M and $L(\lambda)$ of $\mathcal{OLA}^d$, the multiplicity $[M : L(\lambda)]$ equals the multiplicity $[M^+ : L_{\mathfrak{s}}(\lambda)]$ in $\mathcal{OLA}(\mathfrak{s})$, $L_{\mathfrak{s}}(\lambda)$ being a simple $\mathfrak{s}$-module with $\mathfrak{b} \cap \mathfrak{s}$-highest weight λ.*

Proof. The proof is similar to the proof of Lemma 4.16 and we leave it to the reader. □

5. $\mathcal{OLA}$ as a highest weight category

In this section we show that $\mathcal{OLA}$ is a highest weight category according to Definition 3.1 of [CPS]. In particular, this requires introducing standard objects parametrized by the eligible weights, as well as specifying an interval-finite partial order on eligible weights.

5.1. **Standard objects.** Consider the endofunctor Φ in the category $\mathfrak{g}$-mod

$$\Phi(M) := \varinjlim \Phi_n(M),\ \Phi_n(M) := M^{\mathfrak{g}_{n,n}}.$$

The restriction of Φ to $\mathcal{OLA}$ is the identity functor.

Recall also that, if $\Gamma_{\mathfrak{h}}(M)$ stands for the largest $\mathfrak{h}$-semisimple submodule of a $\mathfrak{g}$-module M, then $\Gamma_{\mathfrak{h}}$ is a well-defined endofunctor on the category $\mathfrak{g}$-mod.

Let now M be a $\mathfrak{g}$-module such that the elements of $\mathfrak{n}$ act locally nilpotently on $\Gamma_{\mathfrak{h}}(M)$. Then $\Phi \circ \Gamma_{\mathfrak{h}}(M)$ is an object of $\mathcal{OLA}$, and for any X in $\mathcal{OLA}$ we have a canonical isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(X, \Phi \circ \Gamma_{\mathfrak{h}}(M)) = \mathrm{Hom}_{\mathfrak{g}}(X, M). \tag{5.1}$$

Let $\mathrm{Ext}^i_{\mathfrak{g},\mathfrak{h}}$ denote the ext-group in the category $\mathcal{C}_{\mathfrak{g},\mathfrak{h}}$ of $\mathfrak{g}$-modules semisimple over $\mathfrak{h}$. As $\mathcal{OLA}$ is clearly a Serre subcategory in $\mathcal{C}_{\mathfrak{g},\mathfrak{h}}$, the equality

$$\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, N) = \mathrm{Ext}^1_{\mathcal{OLA}}(M, N) \tag{5.2}$$

holds for any two objects M, N of $\mathcal{OLA}$. Moreover, if X is an object of $\mathcal{C}_{\mathfrak{g},\mathfrak{h}}$ with locally nilpotent action of the elements of $\mathfrak{n}$ and $N = \Phi(X)$, we have $\mathrm{Hom}_{\mathfrak{g}}(M, X/N) = 0$ and hence an embedding

$$\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, N) \hookrightarrow \mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, X). \tag{5.3}$$

For any eligible weight $\lambda \in \langle \tilde{I} \rangle_{\mathbb{C}}$ let

$$\tilde{W}(\lambda) := \Gamma_{\mathfrak{h}}(\mathrm{Coind}^{\mathfrak{g}}_{\bar{\mathfrak{b}}} \mathbb{C}_\lambda).$$

We define the *standard object* $W(\lambda)$ by setting $W(\lambda) := \Phi(\tilde{W}(\lambda))$. Since the elements of $\mathfrak{n}$ act locally nilpotently on $\tilde{W}(\lambda)$, we conclude that $W(\lambda)$ is an object in $\mathcal{OLA}$.

Lemma 5.1. *(a) The $\mathfrak{g}$-module $W(\lambda)$ is indecomposable with simple socle $L(\lambda)$;*
(b) $\dim \mathrm{Hom}_{\mathfrak{g}}(M_n(\lambda), W(\mu)) = \delta_{\lambda,\mu}$ *for sufficiently large n;*
(c) $\mathrm{Ext}^1_{\mathcal{OLA}}(M_n(\lambda), W(\mu)) = 0$ *for sufficiently large n.*

Proof. As we already pointed out, the elements of $\mathfrak{n}$ act locally nilpotently on $\tilde{W}(\lambda)$. Therefore, by (5.1) and Frobenius reciprocity we have

$$\mathrm{Hom}_{\mathfrak{g}}(L(\lambda), W(\mu)) = \mathrm{Hom}_{\mathfrak{g}}(L(\lambda), \mathrm{Coind}^{\mathfrak{g}}_{\bar{\mathfrak{b}}} \mathbb{C}_\mu) = \mathrm{Hom}_{\bar{\mathfrak{b}}}(L(\lambda), \mathbb{C}_\mu).$$

Now (a) follows from the isomorphism of $\bar{\mathfrak{b}}$-modules $L(\lambda)/\bar{\mathfrak{b}}L(\lambda) \simeq \mathbb{C}_\lambda$.

Let us prove (b). We have

$$\mathrm{Hom}_{\mathfrak{g}}(M_n(\lambda), W(\mu)) = \mathrm{Hom}_{\mathfrak{g}}(M_n(\lambda), \mathrm{Coind}^{\mathfrak{g}}_{\bar{\mathfrak{b}}} \mathbb{C}_\mu) = \mathrm{Hom}_{\bar{\mathfrak{b}}}(M_n(\lambda), \mathbb{C}_\mu),$$

and (b) follows from the isomorphism of $\bar{\mathfrak{b}}$-modules $M_n(\lambda)/\bar{\mathfrak{n}}M_n(\lambda) \simeq \mathbb{C}_\lambda$.

Next, we prove (c). By (5.2) and (5.3) it suffices to show that

$$\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M_n(\lambda), \tilde{W}(\lambda)) = 0.$$

We use Shapiro's Lemma:

$$\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M_n(\lambda), \tilde{W}(\lambda)) = \mathrm{Ext}^1_{\bar{\mathfrak{b}},\mathfrak{h}}(M_n(\lambda), \mathbb{C}_\mu).$$

Since $M_n(\lambda)$ is free over the nilpotent ideal $\bar{\mathfrak{m}}_n$, we have $\mathrm{Ext}^1_{\mathfrak{h}+\bar{\mathfrak{m}}_n,\mathfrak{h}}(M_n(\lambda), \mathbb{C}_\mu) = 0$. Therefore

$$\mathrm{Ext}^1_{\bar{\mathfrak{b}},\mathfrak{h}}(M_n(\lambda), \mathbb{C}_\mu) = \mathrm{Ext}^1_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n},\mathfrak{h}\cap\mathfrak{g}_{n,n}}(\mathbb{C}_\lambda, \mathbb{C}_\mu).$$

For sufficiently large n, we have $\lambda|_{\mathfrak{h}\cap\mathfrak{g}_{n,n}} = \mu|_{\mathfrak{h}\cap\mathfrak{g}_{n,n}} = 0$. This implies

$$\mathrm{Ext}^1_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n},\mathfrak{h}\cap\mathfrak{g}_{n,n}}(\mathbb{C}_\lambda, \mathbb{C}_\mu) = \mathrm{Ext}^1_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n},\mathfrak{h}\cap\mathfrak{g}_{n,n}}(\mathbb{C}, \mathbb{C}) = 0.$$

□

Lemma 5.2. *If* $\mathrm{Ext}^1_{\mathcal{OLA}}(L(\lambda), W(\mu)) \neq 0$ *or* $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(L(\lambda), \tilde{W}(\mu)) \neq 0$, *then* $\mu <_{fin} \lambda$.

Proof. Claim (c) of Lemma 5.1 implies the existence of the surjective map

$$\mathrm{Hom}_{\mathfrak{g}}(N(\lambda), W(\mu)) \to \mathrm{Ext}^1_{\mathcal{OLA}}(L(\lambda), W(\mu))$$

where $N(\lambda)$ is the kernel of the canonical projection $M_n(\lambda) \to L(\lambda)$. The $\mathfrak{g}$-module $N(\lambda)$ has finite length and all simple constituents $L(\nu)$ of $N(\lambda)$ satisfy $\nu <_{fin} \lambda$. Hence $\mu <_{fin} \lambda$. The statement for $\tilde{W}(\mu)$ is similar. □

Corollary 5.3. *If* $\lambda^+_{fin} = \{\lambda\}$, *then* $W(\lambda)$ *is injective in* $\mathcal{OLA}$.

Proof. By Corollary 4.18(a), it suffices to check that $\mathrm{Ext}^1_{\mathcal{OLA}}(L(\mu), W(\lambda)) = 0$ for every eligible weight μ. Thus, the statement is an immediate corollary of Lemma 5.2. □

5.2. **Injective objects.** Let us prove now that $\mathcal{OLA}$ has enough injective objects. Recall that $\mathfrak{s}$ denotes the subalgebra generated by $\mathfrak{h}$ and by all root spaces corresponding to finite roots. Let $L_{\mathfrak{s}}(\mu)$ be the simple $\mathfrak{b}\cap\mathfrak{s}$-highest weight module in the category $\bar{\mathcal{O}}_{\mathfrak{s}}$ studied in [N]. Since μ is almost dominant, $L_{\mathfrak{s}}(\mu)$ has an (indecomposable) injective envelope $I_{\mathfrak{s}}(\mu)$, see [N]. Furthermore, let

$$\tilde{W}_{\mathfrak{s}}(\nu) := \Gamma_{\mathfrak{h}}(\mathrm{Coind}^{\mathfrak{s}}_{\mathfrak{s}\cap\bar{\mathfrak{b}}}\,\mathbb{C}_\mu).$$

It follows from [N] that $I_{\mathfrak{s}}(\mu)$ has a finite filtration

$$0 = I_{\mathfrak{s}}(\mu)^0 \subset I_{\mathfrak{s}}(\mu)^1 \subset \cdots \subset I_{\mathfrak{s}}(\mu)^k = I_{\mathfrak{s}}(\mu),$$

such that $I_{\mathfrak{s}}(\mu)^i/I_{\mathfrak{s}}(\mu)^{i-1} \simeq \tilde{W}_{\mathfrak{s}}(\mu_i)$ with $\mu_1 = \mu$ and $\mu_i >_{fin} \mu$ for $i > 1$.

Set $\bar{\mathfrak{p}} := \bar{\mathfrak{b}} + \mathfrak{s}$ and

$$\tilde{I}(\mu) := \Gamma_{\mathfrak{h}}(\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{s}\cap\bar{\mathfrak{b}}}\, I_{\mathfrak{s}}(\mu)).$$

Since $\tilde{W}(\nu) \simeq \Gamma_{\mathfrak{h}}(\mathrm{Coind}^{\mathfrak{g}}_{\bar{\mathfrak{p}}}\,\mathbb{C}_\nu)$, we obtain that $\tilde{I}(\mu)$ has a finite filtration

$$0 = \tilde{I}(\mu)^0 \subset \tilde{I}(\mu)^1 \subset \cdots \subset \tilde{I}(\mu)^k = \tilde{I}(\mu), \tag{5.4}$$

such that $\tilde{I}(\mu)^i/\tilde{I}(\mu)^{i-1} \simeq \tilde{W}(\mu_i)$ with $\mu_1 = \mu$ and $\mu_i >_{fin} \mu$ for $i > 1$.

Now, consider $L(\lambda)$ for an arbitrary eligible weight. If $\lambda \notin \mu^+_{fin}$ then $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(L(\lambda), \tilde{I}(\mu)) = 0$ by Lemma 5.2, while $\lambda \in \mu^+_{fin}$ implies $d(\lambda) = d(\mu)$. Therefore Shapiro's lemma implies

$$\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(L(\lambda), \tilde{I}(\mu)) = \mathrm{Ext}^1_{\bar{\mathfrak{p}},\mathfrak{h}}(L(\lambda), I_{\mathfrak{s}}(\mu)).$$

Furthermore, there is an isomorphism of $\mathfrak{s}$-modules $L(\lambda) = L_{\mathfrak{s}}(\lambda) \oplus \bar{\mathfrak{r}}L(\lambda)$ where $\bar{\mathfrak{r}}$ is the nil-radical of $\bar{\mathfrak{p}}$. We have $\mathrm{Ext}^1_{\bar{\mathfrak{p}},\mathfrak{h}}(\bar{\mathfrak{r}}L(\lambda), I_{\mathfrak{s}}(\mu)) = 0$ as $d(\nu) < d(\mu)$ for any weight ν of $\bar{\mathfrak{r}}L(\lambda)$. Consequently,

$$\mathrm{Ext}^1_{\bar{\mathfrak{p}},\mathfrak{h}}(L(\lambda), I_{\mathfrak{s}}(\mu)) = \mathrm{Ext}^1_{\mathfrak{s},\mathfrak{h}}(L_{\mathfrak{s}}(\lambda), I_{\mathfrak{s}}(\mu)) = 0.$$

As a result, we obtain $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(L(\lambda), \tilde{I}(\mu)) = 0$ for any λ, and hence $I(\mu) := \Phi(\tilde{I}(\mu))$ is an injective object in $\mathcal{OLA}$ with socle $L(\mu)$.

Proposition 5.4. *For any eligible weight μ, the injective module $I(\mu)$ admits a finite filtration*

$$0 = I(\mu)^0 \subset I(\mu)^1 \subset \cdots \subset I(\mu)^k = I(\mu),$$

such that $I(\mu)^i / I(\mu)^{i-1} \simeq W(\mu_i)$ with $\mu_1 = \mu$ and $\mu_i >_{fin} \mu$ for $i > 1$.

Proof. The idea is to apply Φ to (5.4). If n is sufficiently large, there is an isomorphism of $\mathfrak{g}_{n,n}$-modules:

$$\tilde{W}(\mu_i) \simeq \Gamma_{\mathfrak{h}}(\mathrm{Hom}_{\mathbb{C}}(\mathbf{S}(\mathfrak{b}/(\mathfrak{g}_{n,n} \cap \mathfrak{b})), \tilde{W}_{\mathfrak{g}_{n,n}}(0)))$$

where $\tilde{W}_{\mathfrak{g}_{n,n}}(0)$ is the obvious analogue of $\tilde{W}(0)$. Moreover, $\mathbf{S}(\mathfrak{b}/(\mathfrak{g}_{n,n} \cap \mathfrak{b}))$ is an object of $\tilde{\mathbb{T}}_{\mathfrak{g}_{n,n}}$. Since $\mathrm{Ext}^1_{\mathfrak{g}_{n,n},\mathfrak{h}_{n,n}}(L, \tilde{W}_{\mathfrak{g}_{n,n}}(0))$ for any object L of $\mathcal{OLA}_{\mathfrak{g}_{n,n}}$, we get $\mathrm{Ext}^1_{\mathfrak{g}_{n,n},\mathfrak{h}_{n,n}}(\mathbb{C}, \tilde{W}(\mu_i)) = 0$. Hence $\Phi_n = \mathrm{Hom}_{\mathfrak{g}_{n,n}}(\mathbb{C}, \cdot)$ induces a filtration

$$0 = \Phi_n(\tilde{I}(\mu)^0) \subset \Phi_n(\tilde{I}(\mu)^1) \subset \cdots \subset \Phi_n(\tilde{I}(\mu)^k) = \Phi_n(\tilde{I}(\mu)),$$

such that $\Phi_n(\tilde{I}(\mu)^i)/\Phi_n(\tilde{I}(\mu)^{i-1}) \simeq \Phi_n(\tilde{W}(\mu_i))$ with $\mu_1 = \mu$ and $\mu_i >_{fin} \mu$ for $i > 1$. The statement follows by passing to the direct limit. □

Proposition 5.5. *For any $\lambda \in \mathbb{C}\tilde{I}$, the module $W(\lambda)$ has a finite injective resolution $R^\cdot(\lambda)$ of length not greater than $|\lambda^+_{fin}|$ and satisfying the following properties:*

1. *if $I(\mu)$ appears in $R^\cdot(\lambda)$ then $\mu \geq_{fin} \lambda$;*
2. *the multiplicity of $I(\lambda)$ in $R^\cdot(\lambda)$ equals 1;*
3. *the multiplicity of $I(\mu)$ in $R^\cdot(\lambda)$ is finite for every μ.*

Proof. Immediate consequence of Proposition 5.4. □

Corollary 5.6. *(a) If $\mathrm{Ext}^i_{\mathcal{OLA}}(L(\lambda), W(\mu)) \neq 0$ then $\mu \leq_{fin} \lambda$;*
(b) $\dim \mathrm{Ext}^i_{\mathcal{OLA}}(L(\lambda), W(\mu)) < \infty$ for all $i \geq 0$;
(c) $\mathrm{Ext}^i_{\mathcal{OLA}}(L(\lambda), W(\mu)) = 0$ for $i > |\mu^+_{fin}|$.

Proposition 5.7. *(Analogue of BGG reciprocity) The multiplicity $(I(\mu) : W(\lambda))$ equals $m(\lambda, \mu)$.*

Proof. Follows from the identity

$$[M_n(\lambda) : L(\mu)] = \dim \operatorname{Hom}_{\mathfrak{g}}(M_n(\lambda), I(\mu)) = (I(\mu) : W(\lambda)),$$

where the second equality is a consequence of Lemma 5.1, (c). □

5.3. **Jordan–Hölder multiplicities for standard objects.** Now we calculate the multiplicities of $[W(\lambda) : L(\nu)]$. We start by computing $\Phi_n(W(\lambda))$.

Recall the Lie subalgebra $\mathfrak{s}_n \subset \mathfrak{k}_n$. Consider the $\mathfrak{s}_n$-module

$$R(n,p) := \begin{cases} \bigoplus_{\mu\in\mathcal{P},\,|\mu|=p} \mathbb{S}_\mu(V_n^L)^* \boxtimes \mathbb{S}_\mu(V_n^R) \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \bigoplus_{\mu\in\mathcal{P},\,|\mu|=p} \mathbb{S}_{2\mu}(V_n^*) \text{ for } \mathfrak{g} = \mathfrak{o}(\infty) \\ \bigoplus_{\mu\in\mathcal{P},\,|\mu|=p} \mathbb{S}_{(2\mu)'}(V_n^*) \text{ for } \mathfrak{g} = \mathfrak{sp}(\infty) \end{cases}$$

where V_n, V_n^L and V_n^R are introduced in the preamble to Section 4, $\mathcal{P}$ stands for the set of all partitions, and the superscript $'$ indicates conjugating a partition (transposing the corresponding Young diagram). Fix a decomposition of $\mathfrak{h}_n$-modules $\bar{\mathfrak{b}} \cap \mathfrak{k}_n = (\bar{\mathfrak{b}} \cap \mathfrak{s}_n) \oplus \mathfrak{z}_n$ and set $\mathfrak{z}_n R(n,p) = 0$ in order to define a $\bar{\mathfrak{b}} \cap \mathfrak{k}_n$-module structure on $R(n,p)$.

Lemma 5.8. *For sufficiently large n there is an isomorphism of $\mathfrak{k}_n$-modules*

$$\Phi_n(W(\lambda)) \simeq \bigoplus_{p\geq 0} \operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{k}_n}^{\mathfrak{k}_n}(R(n,p)\otimes \mathbb{C}_\lambda).$$

Proof. First, we have isomorphisms of $\mathfrak{k}_n \oplus \mathfrak{g}_{n,n}$-modules

$$\operatorname{Coind}_{\bar{\mathfrak{b}}}^{\mathfrak{g}} \mathbb{C}_\lambda \simeq \operatorname{Coind}_{\bar{\mathfrak{b}}\cap(\mathfrak{k}_n\oplus\mathfrak{g}_{n,n})}^{\mathfrak{k}_n\oplus\mathfrak{g}_{n,n}} \operatorname{Hom}_{\mathbb{C}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda) \simeq$$

$$\operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{k}_n}^{\mathfrak{k}_n} \operatorname{Hom}_{\mathbb{C}}(\mathbf{S}(\mathfrak{r}_n), \operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n}}^{\mathfrak{g}_{n,n}} \mathbb{C}_\lambda),$$

where the structure of $\bar{\mathfrak{b}} \cap (\mathfrak{k}_n \oplus \mathfrak{g}_{n,n})$-module on $\operatorname{Hom}_{\mathbb{C}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda)$ comes from the isomorphism $\mathfrak{r}_n \simeq \mathfrak{g}/(\mathfrak{k}_n + \mathfrak{g}_{n,n} + \bar{\mathfrak{b}})$.

Recall that the result of application of Φ_n depends only on the restriction to $\mathfrak{g}_{n,n}$. Therefore

$$\Phi_n(\operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{k}_n}^{\mathfrak{k}_n} \operatorname{Hom}_{\mathbb{C}}(\mathbf{S}(\mathfrak{r}_n), \operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n}}^{\mathfrak{g}_{n,n}} \mathbb{C}_\lambda)) \simeq \operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{k}_n}^{\mathfrak{k}_n} \Phi_n(\operatorname{Hom}_{\mathbb{C}}(\mathbf{S}(\mathfrak{r}_n), \operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n}}^{\mathfrak{g}_{n,n}} \mathbb{C}_\lambda)).$$

Furthermore, we have

$$\Phi_n(\operatorname{Hom}_{\mathbb{C}}(\mathbf{S}(\mathfrak{r}_n), \operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n}}^{\mathfrak{g}_{n,n}} \mathbb{C}_\lambda)) \simeq$$

$$\operatorname{Hom}_{\mathfrak{g}_{n,n}}(\mathbf{S}(\mathfrak{r}_n), \operatorname{Coind}_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n}}^{\mathfrak{g}_{n,n}} \mathbb{C}_\lambda) \simeq \operatorname{Hom}_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda).$$

Since $\mathbf{S}(\mathfrak{r}_n)$ is a direct sum of objects from $\mathbb{T}_{\mathfrak{g}_{n,n}}$ and $\mathbb{C}_\lambda$ is a trivial $\mathfrak{g}_{n,n}$-module, we have

$$\operatorname{Hom}_{\bar{\mathfrak{b}}\cap\mathfrak{g}_{n,n}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda) \simeq \operatorname{Hom}_{\mathfrak{g}_{n,n}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda).$$

Next, we observe the following $\mathfrak{s}_n \oplus \mathfrak{g}_{n,n}$-module isomorphism

$$\mathfrak{r}_n \simeq \begin{cases} V_n^L \boxtimes (\bar{V}_n)_* \oplus (V_n^R)^* \boxtimes \bar{V}_n \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ V_n \boxtimes \bar{V}_n \text{ for } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty). \end{cases}$$

To finish the proof we have to show that

$$\mathrm{Hom}_{\mathfrak{g}_{n,n}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda) \simeq \bigoplus_{p \geq 0} R(n,p) \otimes \mathbb{C}_\lambda. \tag{5.5}$$

If $\mathfrak{g} = \mathfrak{sl}(\infty)$ then

$$\mathbf{S}(V_n^L \boxtimes (\bar{V}_n)_* \oplus (V_n^R)^* \boxtimes \bar{V}_n) = \bigoplus_{\mu,\nu \in \mathcal{P}} (\mathbb{S}_\mu(V_n^L) \boxtimes \mathbb{S}_\mu((\bar{V}_n)_*)) \otimes (\mathbb{S}_\nu(V_n^R)^* \boxtimes \mathbb{S}_\nu(\bar{V}_n)).$$

Since

$$\mathrm{Hom}_{\mathfrak{g}_{n,n}}(\mathbb{S}_\mu((\bar{V}_n)_*) \otimes \mathbb{S}_\nu(\bar{V}_n), \mathbb{C}) = \begin{cases} \mathbb{C} \text{ for } \mu = \nu \\ 0 \text{ for } \mu \neq \nu, \end{cases}$$

we obtain

$$\mathrm{Hom}_{\mathfrak{g}_{n,n}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda) = \bigoplus_{\mu \in \mathcal{P}} \mathbb{S}_\mu(V_n^L)^* \otimes \mathbb{S}_\mu(V_n^R) \otimes \mathbb{C}_\lambda.$$

If $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ then

$$\mathbf{S}(V_n \boxtimes \bar{V}_n) = \bigoplus_{\nu \in \mathcal{P}} \mathbb{S}_\nu(V_n) \boxtimes \mathbb{S}_\nu(\bar{V}_n).$$

Since

$$\mathrm{Hom}_{\mathfrak{g}_{n,n}}(\mathbb{S}_\nu(\bar{V}_n), \mathbb{C}) = \begin{cases} \begin{cases} \mathbb{C} \text{ if } \nu = 2\mu \\ 0 \text{ otherwise} \end{cases} & \text{for } \mathfrak{g} = \mathfrak{o}(\infty) \\ \begin{cases} \mathbb{C} \text{ if } \nu = (2\mu)' \\ 0 \text{ otherwise} \end{cases} & \text{for } \mathfrak{g} = \mathfrak{sp}(\infty), \end{cases}$$

we obtain

$$\mathrm{Hom}_{\mathfrak{g}_{n,n}}(\mathbf{S}(\mathfrak{r}_n), \mathbb{C}_\lambda) = \begin{cases} \bigoplus_{\mu \in \mathcal{P}} \mathbb{S}_{2\mu}(V_n^*) \otimes \mathbb{C}_\lambda \text{ for } \mathfrak{g} = \mathfrak{o}(\infty), \\ \bigoplus_{\mu \in \mathcal{P}} \mathbb{S}_{(2\mu)'}(V_n^*) \otimes \mathbb{C}_\lambda \text{ for } \mathfrak{g} = \mathfrak{sp}(\infty). \end{cases}$$

In both cases we have now established (5.5), and the statement follows. □

Let $W_{\mathfrak{s}}(\lambda)$ be a standard object in the category $\mathcal{OLA}(\mathfrak{s})$: its definition is the obvious analogue of the definition of $W(\lambda)$. Next, we define the $\mathfrak{s}$-modules $R(\infty, k)$ by setting

$$R(\infty, k) := \begin{cases} \bigoplus_{\mu \in \mathcal{P},\, |\mu| = k} \mathbb{S}_\mu(V_*^L) \boxtimes \mathbb{S}_\mu(V^R) \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty) \\ \bigoplus_{\mu \in \mathcal{P},\, |\mu| = k} \mathbb{S}_{2\mu}(V_*) \text{ for } \mathfrak{g} = \mathfrak{o}(\infty) \\ \bigoplus_{\mu \in \mathcal{P},\, |\mu| = k} \mathbb{S}_{(2\mu)'}(V_*) \text{ for } \mathfrak{g} = \mathfrak{sp}(\infty), \end{cases}$$

where

$$V_*^L := \varinjlim (V_n^L)^*, \; V^R := \varinjlim V_n^R.$$

We are now ready to describe the canonical filtration (4.1) of the standard objects $W(\lambda)$. Let $\Gamma_{\mathfrak{h}_n}$ denote the endofunctor of $\mathfrak{h}_n$-semisimple vectors on the category $\mathfrak{k}_n$-mod. Define the $\mathfrak{k}_n$-module

$$S(n,p,\lambda) := \Gamma_{\mathfrak{h}_n}(\mathrm{Coind}_{\mathfrak{b}\cap\mathfrak{k}_n}^{\mathfrak{k}_n}(R(n,p)\otimes\mathbb{C}_\lambda)).$$

Proposition 5.9. *There are isomorphisms of $\mathfrak{g}$-modules*

$$W(\lambda) \simeq \varinjlim \big(\bigoplus_{p\geq 0} S(n,p,\lambda)\big),$$

$$D_k(W(\lambda)) \simeq \varinjlim \big(\bigoplus_{0\leq p\leq k-1} S(n,p,\lambda)\big),$$

and

$$(D_{k+1}(W(\lambda))/D_k(W(\lambda)))^+ \simeq R(\infty,k)\otimes W_{\mathfrak{s}}(\lambda).$$

Proof. The first isomorphism follows from Lemma 5.8 and the identity $\Phi_n\circ\Gamma_{\mathfrak{h}} = \Gamma_{\mathfrak{h}_n}\circ\Phi_n$.

To verify the existence of the second isomorphism, we first observe that

$$\mathrm{Hom}_{\mathfrak{k}_n}(S(n,p,\lambda), S(n+1,q,\lambda)) = 0 \text{ if } p>q,$$

as follows from a direct comparison of supports. Hence $W(\lambda)$ has an ascending exhaustive filtration $0=F_0\subset F_1\subset F_2\subset\dots$ with

$$F_p/F_{p-1}\simeq \varinjlim S(n,p-1,\lambda).$$

We claim that $F_p = D_p(W(\lambda))$. To prove this it suffices to check that $\varinjlim S(n,p,\lambda)$ is an object of $\mathcal{OLA}^{d+p}$, where $d=d(\lambda)$. Indeed, $S(n,p,\lambda)$ has a filtration with quotients isomorphic to $W_{\mathfrak{k}_n}(\lambda+\gamma)$ for all weights γ of $R(n,p)$. Note that $d(\gamma)=p$. If

$$[\varinjlim S(n,p,\lambda) : L(\mu)]\neq 0,$$

then there exists a weight γ of $R(n,p)$ such that $[W_{\mathfrak{k}_n}(\lambda+\gamma) : L_{\mathfrak{k}_n}(\mu)]\neq 0$ for all sufficiently large n. Since the character of $W_{\mathfrak{k}_n}(\lambda)$ coincides with the character of $M_{\mathfrak{k}_n}(\lambda)$, by the same argument as in Lemma 4.12 we obtain that $\lambda+\gamma=\mu$ or $\lambda+\gamma-\mu$ is a sum of positive finite roots. Hence $d(\mu)=d(\lambda+\gamma)=d+p$.

Finally, let's establish the third isomorphism. Define the functor $T:\mathcal{O}^d_{\mathfrak{k}_n}\to\mathcal{O}_{\mathfrak{s}_n}$ by setting

$$T_d(N) := \oplus_{\nu\in\operatorname{supp} N,\, d(\nu)=d} N_\nu.$$

Then $M^+ = \varinjlim (T_d\circ\Phi_n(M))$ for $M\in\mathcal{OLA}^d$. In particular,

$$(D_{k+1}(W(\lambda))/D_k(W(\lambda)))^+ = \varinjlim (T_{d+k}(S(n,k,\lambda)) \simeq \varinjlim R(n,k)\otimes W_{\mathfrak{s}}(\lambda) = R(\infty,k)\otimes W_{\mathfrak{s}}(\lambda).$$

$\square$

Before stating the main result of this subsection we need to introduce some further notation. Consider the $\mathfrak{s}$-module

$$R := \bigoplus_{k\geq 0} R(\infty, k),$$

and denote by $\mathcal{R}$ (respectively, $\mathcal{R}_k$) the support of R (respectively, $R(\infty, k)$).

If $\mathfrak{g} = \mathfrak{sl}(\infty)$, then all $\gamma \in \mathcal{R}_k$ are of the form $\gamma^L + \gamma^R$, where $\gamma^L = \sum_{i\in\mathbb{Z}_{>0}} a_i\varepsilon_i$ and $\gamma^R = \sum_{i\in\mathbb{Z}_{<0}} b_i\varepsilon_i$ for some $a_i \in \mathbb{Z}_{\leq 0}, b_i \in \mathbb{Z}_{\geq 0}$ such that $-\sum a_i = \sum b_i = k$. For $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, every $\gamma \in \mathcal{R}_k$ can be written uniquely in the form $\gamma = \sum_{i>0} a_i\varepsilon_i$ with non-positive integers a_i such that $\sum a_i = -2k$.

Let μ be a partition. By $K(\mu, \gamma)$ we denote the multiplicity of a weight γ in the $\mathfrak{sl}(\infty)$-module $\mathbb{S}_\mu(V)$. If γ is also a partition then $K(\mu, \gamma)$ are Kostka numbers by definition. In fact, $K(\mu, \gamma)$ are always Kostka numbers as $K(\mu, \gamma) = K(\mu, w(\gamma))$ for $w \in \mathcal{W}$, and for any given $\gamma \in \operatorname{supp} \mathbb{S}_\mu(V)$ there is a suitable $w \in \mathcal{W}$ for which $w(\gamma)$ is a partition. By $\mathcal{P}_{ev}$ we denote the set of even partitions and by $\mathcal{P}'_{ev}$ the set of all partitions whose conjugates are even partitions.

Proposition 5.10. *(a) If $\mathfrak{g} = \mathfrak{sl}(\infty)$, then*

$$[W(\lambda) : L(\nu)] = \sum_{\mu\in\mathcal{P},\gamma\in\mathcal{R}} K(\mu, -\gamma^L)K(\mu, \gamma^R)m(\lambda + \gamma, \nu).$$

(b) If $\mathfrak{g} = \mathfrak{o}(\infty)$, then

$$[W(\lambda) : L(\nu)] = \sum_{\mu\in\mathcal{P}_{ev},\gamma\in\mathcal{R}} K(\mu, -\gamma)m(\lambda + \gamma, \nu).$$

(c) If $\mathfrak{g} = \mathfrak{sp}(\infty)$, then

$$[W(\lambda) : L(\nu)] = \sum_{\mu\in\mathcal{P}'_{ev},\gamma\in\mathcal{R}} K(\mu, -\gamma)m(\lambda + \gamma, \nu).$$

Proof. The proposition follows from Proposition 5.9 and Lemma 4.23. Indeed, let $d(\nu) = d(\lambda) + k$. Then

$$[W(\lambda) : L(\nu)] = [(D_{k+1}(W(\lambda))/D_k(W(\lambda)))^+ : L_\mathfrak{s}(\nu)] = [R(\infty, k) \otimes W_\mathfrak{s}(\lambda) : L_\mathfrak{s}(\nu)].$$

Since $R(\infty, k) \otimes W_\mathfrak{s}(\lambda)$ has a filtration with quotients isomorphic to $W_\mathfrak{s}(\lambda + \gamma)$ where γ runs over $\mathcal{R}_k$, the multiplicity $(R(\infty, k) \otimes W_\mathfrak{s}(\lambda) : W_\mathfrak{s}(\lambda + \gamma))$ equals the multiplicity $c_k(\gamma)$ of the weight γ in $R(\infty, k)$. Therefore

$$[R(\infty, k) \otimes W_\mathfrak{s}(\lambda) : L_\mathfrak{s}(\nu)] = \sum_\gamma c_k(\gamma)[W_\mathfrak{s}(\lambda + \gamma) : L(\nu)] = \sum_\gamma c_k(\gamma)m(\lambda + \gamma, \nu). \tag{5.6}$$

The statement now follows from an explicit calculation of $c_k(\gamma)$:

$$c_k(\gamma) = \begin{cases} \sum_{\mu\in\mathcal{P},\, |\mu|=k} K(\mu, -\gamma^L)K(\mu,\gamma^R) \text{ for } \mathfrak{g} = \mathfrak{sl}(\infty), \\ \sum_{\mu\in\mathcal{P}_{ev},\, |\mu|=2k} K(\mu, -\gamma) \text{ for } \mathfrak{g} = \mathfrak{o}(\infty), \\ \sum_{\mu\in\mathcal{P}'_{ev},\, |\mu|=2k} K(\mu, -\gamma) \text{ for } \mathfrak{g} = \mathfrak{sp}(\infty). \end{cases}$$

□

5.4. **Highest weight category.** We are now ready to define a new partial order $\leq_{inf}$ on the set of eligible weights. This is the partial order needed for the structure of highest weight category on $\mathcal{OLA}$. We write $\mu \lhd_{inf} \nu$ if one of the following holds:

(i) $\mu = \nu + \gamma$ for some $\gamma \in \mathcal{R}$,

(ii) $\mu \leq_{fin} \nu$.

By definition, the partial order $\leq_{inf}$ is the reflexive and transitive closure of the relation $\lhd_{inf}$.

Remark 5.11. Note that $\mu \leq_{inf} \nu$ whenever $\mu \leq_{fin} \nu$. Furthermore, $\mu \leq_{inf} \nu$ implies $d(\mu) \leq d(\nu)$. Finally, it is a consequence of the formula (5.6) that

$$[W(\lambda) : L(\nu)] \neq 0 \;\;\Rightarrow\;\; \nu \leq_{inf} \lambda. \tag{5.7}$$

The condition (5.7) justifies introducing the partial order $\leq_{inf}$ as the inequality $\nu \leq_{fin} \lambda$ does not necessarily hold when $[W(\lambda) : L(\nu] \neq 0$.

Lemma 5.12. *The order $\leq_{inf}$ is interval-finite.*

Proof. Let $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. Then we can take $\rho = \sum_{i\geq 1} -i\varepsilon_i$. For an eligible weight λ, set $\tilde{\lambda} = \lambda + \rho$ and write $\tilde{\lambda} = \sum_{i\geq 1} \tilde{\lambda}_i\varepsilon_i$. Let $i \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$ be such that

$$\operatorname{Re} \tilde{\lambda}_j \geq m \;\text{ for all } j \leq i. \tag{5.8}$$

We claim that if $\kappa \leq_{inf} \mu$ and (5.8) holds for κ that it also holds for μ. Indeed, it suffices to check this in two situations:

- $\tilde{\mu} = s_\alpha(\tilde{\kappa})$ for some reflection $s_\alpha \in \mathcal{W}$ such that $\tilde{\mu} - \tilde{\kappa} \in \langle\alpha\rangle_{\mathbb{Z}_{>0}}$.
- $\tilde{\mu} = \tilde{\kappa} - \gamma$ for some $\gamma \in \mathcal{R}$.

In both cases the checking is straightforward and we leave it to the reader.

Now we note that for any eligible λ and μ there exists $n \in \mathbb{Z}_{>0}$ such that condition(5.8) holds for both λ and μ whenever $i > n$ and $m = -i$. Then, if $\lambda \leq_{inf} \kappa \leq_{inf} \mu$ we have $\tilde{\lambda}_i = \tilde{\kappa}_i = \tilde{\mu}_i = -i$ for any $i > n$. Therefore, in order to check that for fixed λ and μ there are at most finitely many κ satisfying $\lambda \leq_{inf} \kappa \leq_{inf} \mu$, it suffices to establish that there are at most finitely many possibilities for the restriction $\kappa|_{\mathfrak{h}_n}$. But this follows from the well-known interval-finiteness of the standard weight order for the finite-dimensional reductive Lie algebra $\mathfrak{k}_n$.

In the case of $\mathfrak{sl}(\infty)$ we apply the same argument to the weights λ^L and λ^R separately. □

Finally, the implication (5.7), Lemma 5.12 and Corollary 4.18(a) yield the following.

Corollary 5.13. *The category $\mathcal{OLA}$ is a highest weight category according to Definition 3.1 in* [CPS], *with standard objects $W(\lambda)$ and partial order $\leq_{inf}$.*

5.5. **Blocks of $\mathcal{OLA}$.** Recall that $\langle\tilde{I}\rangle_{\mathbb{C}}$ is the set of eligible weights. Let $Q = \langle\Delta\rangle_{\mathbb{Z}}$ denote the root lattice. For $\kappa \in \langle\tilde{I}\rangle_{\mathbb{C}}/Q$ we define $\mathcal{OLA}_\kappa$ as the full subcategory of $\mathcal{OLA}$ consisting of modules M with $\operatorname{supp} M \subset \kappa$. Then obviously

$$\mathcal{OLA} = \Pi_{\kappa\in\langle\tilde{I}\rangle_{\mathbb{C}}/Q}\mathcal{OLA}_\kappa.$$

The following theorem claims that blocks of $\mathcal{OLA}$ are "maximal possible" as two simple objects of $\mathcal{OLA}$ are in different blocks if and only if their supports are not linked by elements of the root lattice. This result is a generalization of the description of blocks of the category $\mathbb{T}_{\mathfrak{g}}$ [DPS], and is in sharp contrast with the description of blocks in the classical BGG category $\mathcal{O}$.

Theorem 5.14. *The subcategory $\mathcal{OLA}_\kappa$ is indecomposable for any $\kappa \in \langle\tilde{I}\rangle_{\mathbb{C}}/Q$.*

Proof. We start by noticing that $\langle\mathcal{R}_1\rangle_{\mathbb{Z}} = Q$. Hence it suffices to prove that for any $\lambda \in \langle\tilde{I}\rangle_{\mathbb{C}}$ and any $\gamma \in \mathcal{R}_1$, the simple modules $L(\lambda)$ and $L(\lambda+\gamma)$ belong to the same block. This follows immediately from (5.6) with $k = 1$ since $[W(\lambda) : L(\lambda)] = [W(\lambda) : L(\lambda+\gamma)] = 1$ and $W(\lambda)$ is indecomposable. $\square$

A block $\mathcal{OLA}_\kappa$ is *integral* if it contains $L(\lambda)$ for some $\lambda \in \langle\tilde{I}\rangle_{\mathbb{Z}}$ (equivalently, such that $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for any $\alpha \in \Delta$).

Corollary 5.15. *The integral blocks of $\mathcal{OLA}$ are parametrized by $\mathbb{Z}$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, and by $\mathbb{Z}/2\mathbb{Z}$ for $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$.*

6. Annihilators in $U(\mathfrak{g})$ of objects of $\mathcal{OLA}$

In this short final section we discuss the annihilators in $U(\mathfrak{g})$ of the objects of $\mathcal{OLA}$. We restrict ourselves to the case $\mathfrak{g} = \mathfrak{sl}(\infty)$. Recall that, according to Theorem 7.1 in [PP2], the primitive ideals of $U(\mathfrak{sl}(\infty))$ are parametrized by quadruples (x, y, Y_l, Y_r) where x, y run over $\mathbb{Z}_{\geq 0}$ and Y_l, Y_r are arbitrary partitions. The parameter x comes from the characteristic pro-variety of the ideal [PP1] and is called *rank*, while the parameter y is the *Grassmann number*. In the paper [PP3] an algorithm for computing the annihilator of an arbitrary simple highest weight $\mathfrak{sl}(\infty)$-module is presented. A significant difference with the case of a finite-dimensional Lie algebra is that the annihilators of most simple highest weight $\mathfrak{sl}(\infty)$-modules equal zero in $U(\mathfrak{sl}(\infty))$.

Furthermore, it is a direct observation based on Theorem 7.1 in [PP2] that, for a simple object $L(\lambda)$ of $\mathcal{OLA}$ the annihilator $\operatorname{Ann}_{U(\mathfrak{g})}L(\lambda)$ is nonzero and has the form $I(x, 0, Y_l, Y_r)$ for some x, Y_l and Y_r. In particular, the annihilators of simple objects of $\mathcal{OLA}$ have Grassmann number equal to zero.

Corollary 6.1. *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. If M is a finitely generated object of $\mathcal{OLA}$, then* $\mathrm{Ann}_{U(\mathfrak{g})} M \neq 0$.

Proof. By Proposition 4.17, any finitely generated module in $\mathcal{OLA}$ has finite length. By the above observation, the annihilator in $U(\mathfrak{g})$ of any simple module in $\mathcal{OLA}$ is nonzero. Finally, it is an exercise to check, using Theorem 5.3 in [PP2], that the intersection of finitely many primitive ideals of $U(\mathfrak{g})$ is nonzero. □

We conjecture that Corollary 6.1 holds for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, but in these cases the algorithm for computing the primitive ideal of a simple highest weight module is still in progress.

If $L(\lambda) \in \mathcal{OLA}$ is integrable, then $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda) = I(0, 0, \lambda^1, \lambda^2)$ where λ^1 and λ^2 are the two partitions comprising λ, see Subsection 4.1. Moreover, a simple module $L(\lambda) \in \mathcal{OLA}$ is not integrable precisely when $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda) = I(x, 0, Y_l, Y_r)$ for $x \neq 0$. This follows from a result of A. Sava [S] but also from a direct application of the algorithm of [PP2]. In fact, all primitive ideals of the form $I(x, 0, Y_l, Y_r)$ are annihilators of simple objects of $\mathcal{OLA}$. Indeed, the reader will verify immediately using Theorem 7.1 in [PP3] that, given $x \in \mathbb{Z}_{\geq 0}$ and partitions $Y_l = (y^l_1, y^l_2, \ldots, y^l_k)$, $Y_r = (y^r_1, y^r_2, \ldots, y^r_s)$, we have

$$\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda) = I(x, 0, Y_l, Y_r)$$

for $\lambda := \lambda^L + \lambda^R$, $\lambda^L = \sum_{i=1}^{x} a_i \varepsilon_i + \sum_{i=1}^{k} y^l_i \varepsilon_{x+i}$, $\lambda^R = -\sum_{i=1}^{s} y^r_{s+1-i} \varepsilon_{-i}$, where $a_1, \ldots a_x$ are complex numbers satisfying the conditions $a_i \notin \mathbb{Z}$, $a_i - a_j \notin \mathbb{Z}$ for all i, j.

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