Letter

Optimal measurement of field properties with quantum sensor networks

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We consider a quantum sensor network of qubit sensors coupled to a field $f(\boldsymbol{x};\boldsymbol{\theta})$ analytically parameterized by the vector of parameters $\boldsymbol{\theta}$. The qubit sensors are fixed at positions $\boldsymbol{x}_1,\dots,\boldsymbol{x}_d$. While the functional form of $f(\boldsymbol{x};\boldsymbol{\theta})$ is known, the parameters $\boldsymbol{\theta}$ are not. We derive saturable bounds on the precision of measuring an arbitrary analytic function $q(\boldsymbol{\theta})$ of these parameters and construct the optimal protocols that achieve these bounds. Our results are obtained from a combination of techniques from quantum information theory and duality theorems for linear programming. They can be applied to many problems, including optimal placement of quantum sensors, field interpolation, and the measurement of functionals of parametrized fields.

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I. INTRODUCTION

It is well established that entangled probes in quantum metrology can be used to obtain more accurate measurements than unentangled probes [1–6]. In particular, while measurements of a single parameter using d unentangled probes asymptotically obtain a mean squared error (MSE) from the true value of order $O(1/d)$, using d maximally entangled probes, each coupled independently to the parameter, one obtains an MSE of order $O(1/d^2)$—the so-called Heisenberg limit [1,7]. More recently, understanding the role of entanglement and generalizing this scaling advantage to the measurement of multiple parameters at once or functions of those parameters has been an area of keen interest [6,8–23] due to a wide array of practical applications [24–30]. Importantly, optimal bounds and protocols have been derived for measuring analytic functions of independent parameters, each coupled to a qubit sensor in a so-called quantum sensor network [15]. The problem of directly measuring a spatially dependent field of known form, possibly with extra noise sources, has also been considered [18].

In this paper, we consider the following very general problem that is relevant for many technological applications of quantum sensor networks. A set of quantum sensors at positions $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_d\}$ is locally probing a physical field $f(\boldsymbol{x};\boldsymbol{\theta})$, which depends on a set of parameters $\boldsymbol{\theta}\in\mathbb{R}^k$, where we have used boldface to denote vectors. We assume that we know the functional form of $f(\boldsymbol{x};\boldsymbol{\theta})$ but we do not know the values of the parameters $\boldsymbol{\theta}$. For instance, these parameters may be the positions of several known charges and $f(\boldsymbol{x};\boldsymbol{\theta})$ one of the components of the resulting electric field. Our objective is to measure a function of the parameters $q(\boldsymbol{\theta})$. This could be, for instance, the field value $q(\boldsymbol{\theta})=f(\boldsymbol{x}_0;\boldsymbol{\theta})$ at a position $\boldsymbol{x}_0$ without sensor or the spatial average $q(\boldsymbol{\theta})=\int_R d\boldsymbol{x}\, f(\boldsymbol{x};\boldsymbol{\theta})$ over some region R of interest. In the following, we derive saturable bounds on the precision for measuring $q(\boldsymbol{\theta})$ using quantum entanglement. The setup is depicted in Fig. 1.

As a more concrete example, consider a network of three quantum sensors that are locally coupled to a field $f(\boldsymbol{x};\theta_1,\theta_2)$ parametrized by $\boldsymbol{\theta}=(\theta_1,\theta_2)$. The field amplitudes at the positions of the sensors shall be $f_1(\boldsymbol{\theta})=\theta_1$, $f_2(\boldsymbol{\theta})=\theta_2$, $f_3(\boldsymbol{\theta})=\theta_1+\theta_2$, respectively, where we have introduced the shorthand notation $f_i(\boldsymbol{\theta})=f(\boldsymbol{x}_i;\boldsymbol{\theta})$. Assume we want to measure the value of $q(\theta_1,\theta_2)=\theta_1$. One possible strategy is to simply use the first sensor to measure $f_1(\boldsymbol{\theta})$. On the other hand, we could also measure $\frac{1}{2}(f_1(\boldsymbol{\theta})-f_2(\boldsymbol{\theta})+f_3(\boldsymbol{\theta}))$, thereby potentially gaining accuracy by harnessing entanglement between the individual sensors. In fact, there are infinitely many variations of the second strategy, and we eventually expect some of them to be superior to the first strategy.

In contrast to previous work [15], where one considers estimating a given function $F(f_1(\boldsymbol{\theta}),\dots,f_d(\boldsymbol{\theta}))$ of independent local field amplitudes $f_1(\boldsymbol{\theta}),\dots,f_d(\boldsymbol{\theta})$, we consider here the problem of estimating a function of the parameters, $q(\theta_1,\dots,\theta_k)$, instead. Due to the correlation of the local field amplitudes, there are many measurement strategies that need to be considered and compared in terms of accuracy. In this paper, we determine the optimal protocol for this very general setup.

In applications, one often measures field amplitudes that depend on the same set of parameters. Therefore, by allowing for the estimation of quantities that depend on measurements of correlated field amplitudes, this work addresses many problems of practical interest left unsolved by previous work. These applications include optimal spatial sensor placement and field interpolation. As a physically motivated example, we explicitly demonstrate how our protocol may be applied to a toy version of the field interpolation problem [31]. In addition to finding the optimal attainable variance and a corresponding protocol for a wide class of problems of practical significance, another primary contribution of our work is the use of optimization duality theorems in the derivation of quantum Cramér–Rao bounds, a technical approach we anticipate being of use beyond the scope of this specific problem.

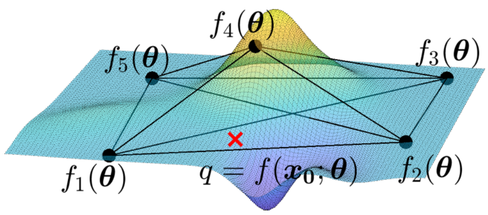

FIG. 1. At each position $\boldsymbol{x}_i$, a quantum sensor (black dots) is coupled to a field $f(\boldsymbol{x};\boldsymbol{\theta})$, whose functional form is known, but the parameters $\boldsymbol{\theta}$ are not. The protocols presented here utilize entanglement to obtain the highest accuracy allowed by quantum mechanics in estimating the quantity $q(\boldsymbol{\theta})$. One example problem is to estimate the field value $q = f(\boldsymbol{x}_0;\boldsymbol{\theta})$ at a location $\boldsymbol{x}_0$ (red cross) without a sensor.

II. PROBLEM SETUP

We formally consider a quantum sensor network as a collection of d quantum subsystems, called sensors, each associated with a Hilbert space $\mathcal{H}_i$ [12,32]. The full Hilbert space is $\mathcal{H} = \bigotimes_{i=1}^d \mathcal{H}_i$. We imprint a collection of field amplitudes $f(\boldsymbol{\theta}) = (f_1(\boldsymbol{\theta}), \ldots, f_d(\boldsymbol{\theta}))^T$ onto a quantum state, represented by an initial density matrix $\rho_{\rm in}$, through the unitary evolution $\rho_{\rm f} = U(f)\rho_{\rm in}U(f)^\dagger$. Here, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^T$ is a set of independent unknown parameters. To be specific, we consider qubit sensors and a unitary evolution generated by the Hamiltonian

$$\hat{H} = \hat{H}_{\rm c}(t) + \sum_{i=1}^{d} \frac{1}{2} f_i(\boldsymbol{\theta})\hat{\sigma}_i^z, \tag{1}$$

with $\hat{\sigma}_i^{x,y,z}$ the Pauli operators acting on qubit i and $f_i(\boldsymbol{\theta}) = f(\boldsymbol{x}_i, \boldsymbol{\theta})$ the local field amplitude at the position of the ith sensor. Our results apply to more general quantum sensor networks (see Outlook). The term $\hat{H}_{\rm c}(t)$ is a time-dependent control Hamiltonian that we choose, which may include coupling to ancilla qubits. This time-dependent control is not necessary to achieve an optimal protocol [10,33] but one may use such control to design optimal protocols with simpler requirements on the choice of input state $\rho_{\rm in}$ [10].

Our goal is to estimate a given function of the parameters $q(\boldsymbol{\theta})$ at their (unknown) true value, which we denote as $\boldsymbol{\theta}'$. The estimate of this quantity $q(\boldsymbol{\theta}')$ is based on measurements of the final state $\rho_{\rm f}$, specified by a set of operators $\{\hat{\Pi}_\xi\}$ that constitute a positive operator-valued measure (POVM) with $\int d\xi\ \hat{\Pi}_\xi = 1$. We repeat this experiment many times and estimate $q(\boldsymbol{\theta}')$ via an estimator $\tilde{q}$ obtained from the data. On a more technical level, we assume that the sensor placements allow us to obtain an estimate of $\boldsymbol{\theta}'$, which ensures the problem is solvable [34]. This assumption implies that the number d of quantum sensors should be larger than k. (See Outlook for cases where we can violate this assumption.) The choice of initial state $\rho_{\rm in}$, control Hamiltonian $\hat{H}_{\rm c}(t)$ and POVM $\{\hat{\Pi}_\xi\}$ defines a *protocol to estimate* $q(\boldsymbol{\theta}')$.

Before proceeding, let us fix some notation. We emphasize that $\boldsymbol{\theta}$ is treated as a variable, with unknown true value (given by the physical fields) denoted $\boldsymbol{\theta}'$. Thus $q(\boldsymbol{\theta})$ is a function, whereas $q(\boldsymbol{\theta}')$ is a specific number obtained by evaluating the function at the true value $\boldsymbol{\theta}'$. We derive our bounds as functions of this general $\boldsymbol{\theta}$ for wherever $q(\boldsymbol{\theta})$ is analytic, but importantly the ultimate bound depends on evaluation at the true value $\boldsymbol{\theta}'$. We use indices $i, j = 1, \ldots, d$ to label quantum sensors and $m, n = 1, \ldots, k$ to label parameters.

The MSE of the estimate $\tilde{q}$ from the true value $q(\boldsymbol{\theta}')$ is given by

$$\mathcal{M} = \mathbb{E}[(\tilde{q} - q(\boldsymbol{\theta}'))^2] = \text{Var}\ \tilde{q} + (\mathbb{E}[\tilde{q}] - q(\boldsymbol{\theta}'))^2, \tag{2}$$

where the first and second terms are the variance and estimate bias, respectively. We define the optimal protocol to measure $q(\boldsymbol{\theta}')$ as the one that minimizes $\mathcal{M}$ given a fixed amount of total time t. To determine the optimal protocol, we first derive lower bounds on $\mathcal{M}$ using techniques from quantum information theory. We then construct specific protocols that saturate these bounds.

III. MSE BOUND

In this section, we derive a saturable lower bound on $\mathcal{M}$ that can be achieved in time t [35]. To derive our bound, we begin with the following result on single-parameter estimation from Ref. [33]. If the unitary evolution of the quantum state is controlled by a single parameter q, then

$$\mathcal{M} \geqslant \frac{1}{\mathcal{F}_Q} \geqslant \frac{1}{t^2||\hat{h}_q||_{\rm s}^2}, \tag{3}$$

where $\mathcal{F}_Q$ is the quantum Fisher information, $\hat{h}_q = \partial\hat{H}/\partial q$ is the generator with $\gamma_{\rm max}$ ($\gamma_{\rm min}$) its largest (smallest) eigenvalue, and $||\hat{h}_q||_{\rm s} = \gamma_{\rm max} - \gamma_{\rm min}$ is the seminorm of $\hat{h}_q$. The first inequality is the quantum Cramér–Rao bound [36–39].

It is not obvious that Eq. (3) may be applied to the problem of estimating $q(\boldsymbol{\theta})$ as we have $k > 1$ parameters controlling the evolution of the state. However, we circumvent this issue by considering an infinite set of imaginary scenarios, each corresponding to a choice of artificially fixing $k-1$ degrees of freedom and leaving only $q(\boldsymbol{\theta})$ free to vary. Under any such choice, our final quantum state depends on a single parameter, and we can apply Eq. 3 to the imaginary scenario under consideration.

We note that any such imaginary scenario requires giving ourselves information that we do not have in reality. However, additional information can only result in a lower value of $\mathcal{M}$. Therefore, any lower bound on $\mathcal{M}$ derived from any of the imaginary scenarios is also a lower bound for estimating the function $q(\boldsymbol{\theta})$. For a bound derived this way to be saturable, there must be some choice(s) of artificially fixing $k-1$ degrees of freedom that does not give us *any* useful information about $q(\boldsymbol{\theta})$, and thus yields the sharpest possible bound. This is, in fact, the case. In our analysis below, the existence of such a choice becomes self-evident since we present a protocol that achieves the tightest bound. However, in the Supplemental Material, we prove that such a choice exists purely on information theoretic grounds [40].

More formally, consider a basis $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_k\}$ such that, without loss of generality, $\boldsymbol{\alpha}_1 = \nabla q(\boldsymbol{\theta}') =: \boldsymbol{\alpha}$. We then consider any choice of the remaining basis vectors. For any such choice, let $\boldsymbol{\alpha}_n$ correspond to a function $q_n(\boldsymbol{\theta}) = \boldsymbol{\alpha}_n \cdot \boldsymbol{\theta}$. Therefore, if we consider a particular choice of basis, we are also considering a corresponding set of functions $\{q_1(\boldsymbol{\theta}) = q(\boldsymbol{\theta}), q_2(\boldsymbol{\theta}), \cdots, q_k(\boldsymbol{\theta})\}$. We suppose we are given the values

$\{q_n(\boldsymbol{\theta}')\}_{n\geqslant 2}$, fixing $k-1$ degrees of freedom. The resulting problem is now determined by a single parameter, and Eq. (3) applies.

The derivative of H with respect to q, while holding $q_2, \ldots, q_k$ fixed, is

$$\hat{h}_q = \left.\frac{\partial \hat{H}}{\partial q}\right|_{q_2,\ldots,q_k} = \sum_{i=1}^{d} \frac{1}{2}(\nabla f_i(\boldsymbol{\theta}') \cdot \boldsymbol{\beta})\hat{\sigma}_i^z, \tag{4}$$

where $\boldsymbol{\beta} = (\frac{\partial\theta_1}{\partial q}, \ldots, \frac{\partial\theta_k}{\partial q})|_{q_2,\ldots,q_k}$. Using the chain rule, we find that $\boldsymbol{\beta}$ satisfies $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 1$.

As we show formally in the Supplemental Material [40], every $\boldsymbol{\beta} \in \mathbb{R}^k$ in Eq. (4) corresponds to a valid choice of the $k-1$ dimensional subspace spanned by $\{\boldsymbol{\alpha}_n\}_{n\geqslant 2}$. Therefore, since $\hat{h}_q$ depends on $\{\boldsymbol{\alpha}_n\}_{n\geqslant 2}$ only through β, the tightest bound on $\mathcal{M}$ is found by optimizing over arbitrary choices of $\boldsymbol{\beta}$ subject to the constraint $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 1$.

To formulate the corresponding optimization problem, define the matrix G by

$$G_{im}(\boldsymbol{\theta}') = \frac{\partial f_i}{\partial \theta_m}(\boldsymbol{\theta}'). \tag{5}$$

We emphasize that G depends on the true value of the parameters $\boldsymbol{\theta}'$. Utilizing $||\frac{1}{2}\hat{\sigma}_i^z||_s = 1$, we write the seminorm of $\hat{h}_q$ as

$$||\hat{h}_q||_s = \sum_{i=1}^{d} |\nabla f_i(\boldsymbol{\theta}') \cdot \boldsymbol{\beta}| = ||G(\boldsymbol{\theta}')\boldsymbol{\beta}||_1, \tag{6}$$

with $||\boldsymbol{x}||_1 = \sum_{i=1}^{d} |x_i|$ the L^1 or Manhattan norm. Therefore, for any $\boldsymbol{\beta}$ satisfying $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 1$, we have

$$\mathcal{M} \geqslant \frac{1}{t^2||\hat{h}_q||_s^2} = \frac{1}{t^2||G(\boldsymbol{\theta}')\boldsymbol{\beta}||_1^2}. \tag{7}$$

To obtain the sharpest bound, we must solve what we refer to as the bound problem for $G(\boldsymbol{\theta}')$ and $\boldsymbol{\alpha}$:

Bound problem: *Given a nonzero vector $\boldsymbol{\alpha} \in \mathbb{R}^k$ and a real $d \times k$ matrix G, compute $u = \max_{\boldsymbol{\beta}} \frac{1}{||G\boldsymbol{\beta}||_1}$ under the condition $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 1$.*

This is a linear programming problem and can in general be solved in time that is polynomial in d and k (see, e.g., Ref. [41]). Hereafter, we refer to the resulting sharpest bound as the bound.

IV. OPTIMAL PROTOCOL

We now turn to the problem of providing a protocol that saturates this bound. For clarity of presentation, we develop this protocol in the case that both the field $f(\boldsymbol{\theta})$ and the objective $q(\boldsymbol{\theta})$ are linear in the parameters $\boldsymbol{\theta}$; that is, $f(\boldsymbol{\theta}) = G\boldsymbol{\theta}$, with $\boldsymbol{\theta}$-independent G, and $q(\boldsymbol{\theta}) = \boldsymbol{\alpha}\cdot\boldsymbol{\theta}$. However, the existence of an asymptotically optimal protocol can be proven in the more general case that $f(\boldsymbol{\theta})$ and $q(\boldsymbol{\theta})$ are analytic in the neighborhood of the true value $\boldsymbol{\theta}'$ [40].

Similar to the approach taken in Ref. [15], this generalization ultimately amounts to using a two-step protocol. In the first step, one spends an asymptotically negligible time t_1 estimating the values of the parameters $\boldsymbol{\theta}$. Then one linearizes $f(\boldsymbol{\theta})$ and $q(\boldsymbol{\theta})$ about this estimate $\tilde{\boldsymbol{\theta}}$ and spends the remaining time $t_2 = t - t_1$ estimating the resulting linearized objective. (Note, asymptotically, $t_2 \sim t$.) Therefore, while we leave the rigorous analysis of this generalization to the Supplemental Material [40], the analytic case reduces to the linear case considered here, and therefore the principle insights are made most readily apparent in this context.

For the linear case, we propose an explicit protocol to measure q and show that it saturates the bound and thus is optimal. The optimal protocol measures the linear combination

$$\lambda(\boldsymbol{f}) = \boldsymbol{w}\cdot\boldsymbol{f}, \tag{8}$$

where $\boldsymbol{f}$ is the vector of local field amplitudes. The vector $\boldsymbol{w} \in \mathbb{R}^d$ is chosen such that $\tilde{\lambda}(\boldsymbol{f}) = \tilde{q}(\boldsymbol{\theta})$ is an unbiased estimator of $q(\boldsymbol{\theta}')$, and will be optimized to saturate the bound. (We note that, for $d > k$, there are many choices of $\boldsymbol{w}$ that satisfy $\lambda = q$.)

For the estimator $\tilde{\lambda}$ to be unbiased, we must have $\mathbb{E}[\tilde{q}] = q(\boldsymbol{\theta}') = \boldsymbol{\alpha}\cdot\boldsymbol{\theta}'$. This is achieved by choosing $\boldsymbol{w}$ to satisfy the *consistency condition:*

$$G^T\boldsymbol{w} = \boldsymbol{\alpha}. \tag{9}$$

Indeed, this implies

$$\mathbb{E}[\tilde{q}] = \mathbb{E}[\boldsymbol{w}\cdot f] = (G\boldsymbol{\theta}')^T\boldsymbol{w} = \boldsymbol{\theta}'\cdot\left(G^T\boldsymbol{w}\right) = \boldsymbol{\alpha}\cdot\boldsymbol{\theta}'. \tag{10}$$

We prove in the Supplemental Material that, under our assumption that we can estimate $\boldsymbol{\theta}'$, Eq. 9 may always be satisfied for some $\boldsymbol{w}$, and therefore our protocol is valid.

For any such choice of $\boldsymbol{w}$, we use the optimal linear protocol of Ref. [10]—which for completeness, we summarize in the Supplemental Material [40]—to measure $\lambda(\boldsymbol{f})$. The variance obtained by this protocol is

$$\mathrm{Var}\ \tilde{q} = \frac{||\boldsymbol{w}||_\infty^2}{t^2}, \tag{11}$$

where $||\boldsymbol{w}||_\infty = \max_i|w_i|$. Since we are dealing with an unbiased estimator, the MSE coincides with the variance of the estimator in Eq. 11. To find $\boldsymbol{w}$ with the lowest possible value of $||\boldsymbol{w}||_\infty$ (i.e., the smallest variance), we must solve what we refer to as the protocol problem:

Protocol problem: *Given a nonzero vector $\boldsymbol{\alpha} \in \mathbb{R}^k$ and a real $d \times k$ matrix G, compute $u' = \min_{\boldsymbol{w}} ||\boldsymbol{w}||_\infty$ under the condition $G^T\boldsymbol{w} = \boldsymbol{\alpha}$.*

This, again, can be efficiently solved by generic linear programming algorithms [41,42] or special-purpose algorithms [43–45].

To show that the optimal protocol from solving this problem saturates the bound, we now show that the bound problem and protocol problem are equivalent in that $u = u'$. For this, we utilize the strong duality theorem for linear programming [43,46] [47]. It states that, for linear programming problems like the protocol problem, there is a dual problem whose solution is identical to the original problem. In our case, we have the following dual problem:

Dual protocol problem: *Given a nonzero vector $\boldsymbol{\alpha} \in \mathbb{R}^k$ and a real $d \times k$ matrix G, compute $u'' = \max_{\boldsymbol{v}} \boldsymbol{\alpha}\cdot\boldsymbol{v}$ under the condition $||G\boldsymbol{v}||_1 \leqslant 1$.*

The strong duality theorem then implies $u'' = u'$. Additionally, there is a correspondence between the two solution vectors $\boldsymbol{w^0}$ and $\boldsymbol{v^0}$, so, given the solution vector to one problem, we can find the solution vector to the other [43,46]. We now prove the following theorem.

Theorem. *Let u and u' be the solutions to the bound and protocol problems, respectively. Then $u = u'$.*

Proof. By the strong duality theorem, the solution of the dual protocol problem satisfies $u'' = \max_{\boldsymbol{v}} \boldsymbol{\alpha} \cdot \boldsymbol{v} = u'$. Let the corresponding solution vector of the dual protocol problem be $\boldsymbol{v^0}$. Define $\boldsymbol{\beta^0} := \boldsymbol{v^0}/u'$. We have $\boldsymbol{\alpha} \cdot \boldsymbol{\beta^0} = u'/u' = 1$, thus $\boldsymbol{\beta^0}$ satisfies the constraint of the bound problem. To prove the theorem, we show that $u' \leqslant u$ and $u \leqslant u'$. On the one hand, provided $||G\boldsymbol{\beta^0}||_1 \neq 0$, the condition $||G\boldsymbol{v^0}||_1 \leqslant 1$ of the dual problem implies

$$u' \leqslant \frac{1}{||G\boldsymbol{\beta}^0||_1} \leqslant \max_{\boldsymbol{\beta}} \frac{1}{||G\boldsymbol{\beta}||_1} = u. \tag{12}$$

On the other hand, for *any* $\boldsymbol{\beta}$ satisfying the constraint $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$ of the bound problem, and for the optimal $\boldsymbol{w} = \boldsymbol{w^0}$ of the protocol problem satisfying $||\boldsymbol{w^0}||_\infty = u'$, Hölder's inequality yields

$$1 = \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = (G^T \boldsymbol{w^0})^T \boldsymbol{\beta} = \boldsymbol{w^0} \cdot (G\boldsymbol{\beta}) \leqslant ||\boldsymbol{w^0}||_\infty ||G\boldsymbol{\beta}||_1$$
$$\Rightarrow \frac{1}{||G\boldsymbol{\beta}||_1} \leqslant ||\boldsymbol{w^0}||_\infty = u' \text{ for all } \boldsymbol{\beta}. \tag{13}$$

This shows that $u' \geqslant 1/||G\boldsymbol{\beta}||_1$ for all $\boldsymbol{\beta}$, thus $u' \geqslant u$, which completes the proof. As a byproduct, we learn from Eq. (12) that $\boldsymbol{\beta}^0$ maximizes $1/||G\boldsymbol{\beta}||_1$, and so is the solution vector of the bound problem.

Theorem IV implies that the protocol measuring λ with optimal $\boldsymbol{w}$ saturates the bound.

As an instructive example, we return to the toy model presented in the Introduction. Consider three sensors coupled to local field amplitudes $f_1(\boldsymbol{\theta}) = \theta_1$, $f_2(\boldsymbol{\theta}) = \theta_2$, and $f_3(\boldsymbol{\theta}) = \theta_1 + \theta_2$. Our objective is $q(\boldsymbol{\theta}) = \theta_1$, so $\boldsymbol{\alpha} = (1, 0)^T$. We have

$$G^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{14}$$

First, consider the bound problem. The constraint $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 1$ implies $\boldsymbol{\beta} = (1, b)^T$ with arbitrary b. The maximum of $1/||G\boldsymbol{\beta}||_1$ is achieved for $\boldsymbol{\beta}^0 = (1, 0)^T$, yielding $u = 1/2$. For the protocol problem, the constraint in Eq. (9) gives $w_1 + w_2 = 1$ and $w_2 + w_3 = 0$. The corresponding minimal value of $||\boldsymbol{w}||_\infty$ is $u' = 1/2$ for $\boldsymbol{w^0} = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^T$. Finally, for the dual protocol problem, the constraint $||G\boldsymbol{v}||_1 \leqslant 1$ implies $|v_1| + |v_2| + |v_1 + v_2| \leqslant 1$. The solution vector is $\boldsymbol{v^0} = (1/2, 0)^T$, which yields $u'' = \boldsymbol{\alpha} \cdot \boldsymbol{v^0} = 1/2$. This explicit example demonstrates that $u = u' = u''$. Furthermore, as noted in the proof of Theorem IV, $\boldsymbol{\beta}^0 = \boldsymbol{v^0}/u'$.

V. APPLICATIONS

Having derived optimal bounds and protocols saturating them, we now discuss some applications. We begin by considering the same example as above and show that, remarkably, our results in this case indicate that the best entangled and best unentangled weighting strategies need not be the same. With or without entanglement, we estimate $q(\boldsymbol{\theta}) = \theta_1$ by measuring a linear combination $\boldsymbol{w} \cdot \boldsymbol{f}$ with the constraints $w_1 + w_3 = 1$, $w_2 + w_3 = 0$. Without entanglement, our only option is to measure each component of $\boldsymbol{f}$ independently in parallel for time t, yielding a total MSE for $q(\boldsymbol{\theta})$ of $||\boldsymbol{w}||_2^2/t^2$. In stark contrast, for the entangled case, the MSE is given by $||\boldsymbol{w}||_\infty^2/t^2$. It is easy to see that minimizing the Euclidean and supremum norm of $\boldsymbol{w}$, subject to our constraints, does not yield the same result: Without entanglement, $\boldsymbol{w} = (\frac{2}{3}, -\frac{1}{3}, \frac{1}{3})^T$ is optimal, yielding an MSE of $\frac{2}{3t^2}$. With entanglement, $\boldsymbol{w} = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^T$ is optimal, with MSE of $\frac{1}{4t^2}$. This simple example shows that, to achieve the optimal result with entanglement, one cannot in general use the weights $\boldsymbol{w}$ that are optimal without entanglement.

Our results are practically relevant for any situation where one knows the functional form of the field of interest $f(\boldsymbol{x}; \boldsymbol{\theta})$ and seeks to determine some quantity dependent on the parameters of that field. Examples include functionals of the form $q(\boldsymbol{\theta}) = \int_R d\boldsymbol{x}\ k(\boldsymbol{x}) f(\boldsymbol{x}; \boldsymbol{\theta})$ with any kernel $k(\boldsymbol{x})$ and region of integration R. The examples from the Introduction correspond to $k(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$ and $k(\boldsymbol{x}) = 1$. Since the $\boldsymbol{\theta}$ dependence of $f(\boldsymbol{x}, \boldsymbol{\theta})$ is analytic, this amounts to evaluating an analytic function $q(\boldsymbol{\theta})$.

As it is of clear physical relevance, we explicitly consider a simple, one-dimensional version of the former case with $k(x) = \delta(x - x_0)$, namely, field interpolation. Consider the situation of k particles at positions $x \in \{z_1, \cdots, z_k\}$ with unknown charges specified by the parameters $\{\theta_1, \cdots, \theta_k\}$ (and true values given by $\{\theta_1', \cdots, \theta_k'\}$). Suppose we seek to determine the magnitude of (one component of) the electric field $q(\boldsymbol{\theta})$ at $x = x_0$ using $d \geqslant k$ sensors at positions $x \in \{s_1, \cdots s_d\}$. We then have

$$q(\boldsymbol{\theta}) = \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{k} \frac{\theta_n}{(z_n - x_0)^2}, \tag{15}$$

which is linear in the unknown parameters $\{\theta_n\}$. Similarly, the fields measured by the sensors, given by $f_i(\boldsymbol{\theta}) = (1/4\pi\epsilon_0) \sum_{n=1}^{k} (\theta_n/(z_n - s_i)^2)$, are also linear in the parameters. Our protocol then applies quite simply to this situation with

$$G(\boldsymbol{\theta}')_{\text{in}} = \frac{1}{4\pi\epsilon_0 (z_n - s_i)^2} \tag{16}$$

and the elements of $\boldsymbol{\alpha}$ given by $\boldsymbol{\alpha}_n = 1/[4\pi\epsilon_0 (z_n - x_0)^2]$. One can then straightforwardly solve the bound problem, the protocol problem, or the dual protocol problem given the particular locations of charges and sensors via analytic or numeric methods.

Our findings are also relevant for determining the optimal placement of sensors in space, i.e., determining the best locations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ in the control space X in which they reside. For example, if the sensors are confined to a plane, then $X = \mathbb{R}^2$. This problem clearly consists of two parts: (1) evaluating the best possible MSE for any chosen set of sensor locations and (2) optimizing the result over possible locations. The MSE amounts to the cost function in usual optimization problems. Our results solve this first part as it would be used in the inner loop of a numerical optimization algorithm. The full problem, involving also the second part, is a high dimensional optimization in a space of dimension $d \times \dim(X)$. Therefore, in general, one expects that finding the global optimal placement could be quite challenging. However, even finding a local optimum in this space is clearly of practical use.

VI. OUTLOOK

While we assumed that we can obtain an individual estimate of the true value $\boldsymbol{\theta}'$ of the parameters, one could imagine situations where this assumption is not satisfied. Some such systems are underdetermined and not uniquely solvable, but in some cases we can reparametrize $\boldsymbol{\theta} \to \boldsymbol{\theta}^*$ to satisfy the assumption. For example, if two parameters in the initial parametrization always appear as a product $\theta_1\theta_2$ in both f and q, we cannot individually estimate θ_1 or θ_2. However, we can reparametrize $\theta_1\theta_2 \to \theta_1^*$ and thus satisfy our initial assumption.

Our work applies to physical settings beyond qubit sensors—that is, any situation where Eq. 3, may be applied our results should hold, provided we use the corresponding seminorm for the particular coupling. One example is using a collection of d Mach-Zehnder interferometers where the role of local fields is played by interferometer phases [11,48–52]. Here the limiting resource is the number of photons N available to distribute among interferometers and not the total time t. We note, however, there are subtleties that we do not consider here when only the average number of photons is known [53]. The optimal variance for measuring a linear combination of local field values in this setting is conjectured in Ref. [10]. Under the assumption that this conjecture is correct, we may replace Eq. (11) with $\mathcal{M} = \frac{\|\boldsymbol{w}\|_\infty^2}{N^2}$ and otherwise everything remains the same as the qubit sensor case. One could also consider the entanglement-enhanced continuous-variable protocol of Ref. [14] for measuring linear combinations of field-quadrature displacements. A variation of this protocol has been experimentally implemented in Ref. [20]. We expect our bound and protocol could be extended to all the scenarios just described or even to the hybrid case where some local fields couple to qubits, some to Mach-Zehnder interferometers, and some to field quadratures. The ultimate attainable limit in such physical settings remains an open question, however.

One could consider the case $d < k$ provided the d sensors are not required to be at fixed locations. For instance, if one had access to continuously movable sensors in a 1D control space X, by the Riesz representation theorem [46], one could encode any linear functional of $f(\boldsymbol{x}; \boldsymbol{\theta})$ by moving the sensors according to a particular corresponding velocity schedule. As a simple example, one can consider evaluating the integral of some function of (one component of) a magnetic field over one-dimensional physical space by moving a qubit sensor through the field and measuring the accumulated phase. One could also consider variations of this work in the context of semiparametric estimation [54]. We leave further exploration of such schemes to future work.

Finally, we emphasize that our protocol requires the use of highly entangled pure states (such as the Greenberger–Horne–Zeilinger (GHZ) state) and does not consider the effects of decoherence or noise. Provided decoherence times are long, our results are applicable, but, beyond this limit, analyzing our protocols in such open systems (or designing different, more noise-robust protocols) remains an interesting and important question.

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