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The May-Milgram filtration and $\mathcal{E}_k$-cells

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We describe an $\mathcal{E}_k$-cell structure on the free $\mathcal{E}_{k+1}$-algebra on a point, and more generally describe how the May-Milgram filtration of $\Omega^m\Sigma^m S^k$ lifts to a filtration of the free $\mathcal{E}_{k+m}$-algebra on a point by iterated pushouts of free $\mathcal{E}_k$-algebras.

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1 Introduction

Cell structures on topological spaces have many uses, and operadic cell structures play similar roles in the study of algebras over an operad; they appear when defining model category structures on categories of algebras over operads, and in the case of the little k-cubes operad $\mathcal{E}_k$ they have found applications to homological stability [12, 8, 9].

Despite the usefulness of such $\mathcal{E}_k$-cell structures, few have been described explicitly. In this paper we give an $\mathcal{E}_k$-cell decomposition of an $\mathcal{E}_k$-algebra weakly equivalent to the free $\mathcal{E}_{k+1}$-algebra on a point. We generalize this to a filtration of the free $\mathcal{E}_{k+m}$-algebra on a point by $\mathcal{E}_k$-algebras, and explain how this can be thought of as a lift of the May-Milgram filtration of the iterated based loop space $\Omega^m\Sigma^m S^k$.

We shall state our results without giving definitions (which appear in Section 2), with the exception of that of a cell attachment in the category $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top})$. This definition uses the free $\mathcal{E}_k$-algebra functor $\mathbf{E}_k = \mathbf{F}^{\mathcal{E}_k}$, which is left adjoint to the forgetful functor $\mathcal{U}^{\mathcal{E}_k}\colon \mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top}) \to \mathsf{Top}$ sending an $\mathcal{E}_k$-algebra to its underlying space. Let $\mathbf{A}$ be an $\mathcal{E}_k$-algebra in Top, and let $e\colon \partial D^d \to \mathcal{U}^{\mathcal{E}_k}(\mathbf{A})$ be a map of topological spaces. To attach a d-dimensional $\mathcal{E}_k$-cell to $\mathbf{A}$, we take the adjoint map $\mathbf{E}_k(\partial D^d) \to \mathbf{A}$ of e and take the pushout of the following diagram in $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top})$:

$$\begin{array}{ccc} \mathbf{E}_k(\partial D^d) & \longrightarrow & \mathbf{A} \\ \downarrow & & \downarrow \\ \mathbf{E}_k(D^d) & \longrightarrow & \mathbf{A} \cup_e^{\mathcal{E}_k} D^d. \end{array}$$

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To make this homotopy invariant we need to require that $\mathbf{A}$ is cofibrant, or derive the cell-attachment construction.

An $\mathcal{E}_k$-algebra is called *cellular* if it is built by iterated $\mathcal{E}_k$-cell attachments (such algebras are always cofibrant). Giving an $\mathcal{E}_k$-cell structure on $\mathbf{A}$ is giving a weak equivalence between $\mathbf{A}$ and a cellular $\mathcal{E}_k$-algebra. Note that the following colimit is also a homotopy colimit:

Theorem 1.1 $\mathbf{E}_{k+1}(*)$ *is weakly equivalent as an* $\mathcal{E}_k$*-algebra to a cellular* $\mathcal{E}_k$*-algebra with exactly one cell in dimensions divisible by* k *and no other cells. That is, it is weakly equivalent to the colimit* $\mathrm{colim}_{r\in\mathbb{N}}\,\mathbf{A}_r$ *of algebras* $\mathbf{A}_r$ *obtained by setting* $\mathbf{A}_{-1} = \varnothing$ *and taking iterated pushouts in* $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top})$

$$\begin{array}{ccc} \mathbf{E}_k(\partial D^{rk}) & \longrightarrow & \mathbf{A}_r \\ \downarrow & & \downarrow \\ \mathbf{E}_k(D^{rk}) & \longrightarrow & \mathbf{A}_{r+1}. \end{array}$$

An $\mathcal{E}_k$-cell structure on $\mathbf{A}$ induces an ordinary cell structure on its k-fold delooping $B^k\mathbf{A}$ (see Remark 2.6). The one induced on $B^k\mathbf{E}_{k+1}(*) \cong \Omega\Sigma S^k$ by the $\mathcal{E}_k$-cell structure of Theorem 1.1 is that coming from the James construction [11]. The filtration coming from the James construction was generalized by May and Milgram [14, 15] to a filtration on $\Omega^m\Sigma^m S^k$, and we will construct a filtration of $\mathbf{E}_{k+m}(*)$ by $\mathcal{E}_k$-algebras which deloops to the May-Milgram filtration on $B^k\mathbf{E}_{k+m}(*) \simeq \Omega^m\Sigma^m S^k$.

To state this result precisely, we need to introduce some notation. Let $\mathbb{I}$ denote the open interval $(0,1)$, $F_r(\mathbb{I}^m)$ the space of *ordered configurations* of r points in $\mathbb{I}^m$, and $C_r(\mathbb{I}^m) := F_r(\mathbb{I}^m)/\mathfrak{S}_r$ the space of *unordered configurations* of r points in $\mathbb{I}^m$. Furthermore, let $\phi_{m,r}$ denote the vector bundle $F_r(\mathbb{I}^m) \times_{\mathfrak{S}_r} \mathbb{R}^{r-1} \to C_r(\mathbb{I}^m)$ with $\mathbb{R}^{r-1}$ the representation of the symmetric group $\mathfrak{S}_r$ given by the orthogonal complement to the trivial representation in the permutation representation with its usual metric. This vector bundle inherits a Riemannian metric. For $E \to B$ a vector bundle with Riemannian metric, let $D(E)$ denote its unit disk bundle, $S(E)$ denote its unit sphere bundle and kE denote its k-fold Whitney sum.

Theorem 1.2 $\mathbf{E}_{k+m}(*)$ *is weakly equivalent as an* $\mathcal{E}_k$*-algebra to the colimit* $\mathrm{colim}_{r\in\mathbb{N}}\,\mathbf{A}_r$ *of algebras* $\mathbf{A}_r$ *obtained by setting* $\mathbf{A}_{-1} = \varnothing$ *and taking iterated pushouts in* $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top})$

$$\begin{array}{ccc} \mathbf{E}_k(S(k\phi_{m,r+1})) & \longrightarrow & \mathbf{A}_r \\ \downarrow & & \downarrow \\ \mathbf{E}_k(D(k\phi_{m,r+1})) & \longrightarrow & \mathbf{A}_{r+1}. \end{array}$$

This implies that the homotopy cofiber of $\mathbf{A}_r \to \mathbf{A}_{r+1}$ in $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top})$ is the free $\mathcal{E}_k$-algebra on the Thom space of $k\phi_{m,r+1}$ viewed as a based space. These Thom spaces and their corresponding Thom spectra are well-studied, e.g. being related to Brown-Gitler spectra when $k = 2$ [5, 4]. When $m = 1$, $C_r(\mathbb{I}^m)$ is contractible and the vector bundle $k\phi_{m,r+1}$ has dimension kr. Thus the sphere bundle is homotopy equivalent to ∂D^{rk} and hence Theorem 1.1 is a consequence of Theorem 1.2.

In Corollary 4.14 we give a configuration space model $\mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(*)$ for the $\mathcal{E}_k$-algebras $\mathbf{A}_r$. The space $\mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(*)$ is given by the spaces of configuration spaces of points in $\mathbb{I}^m \times \mathbb{I}^k$ such that each subset $\{x\} \times \mathbb{I}^k$ contains at most r points. We use this in Theorem 5.2 to prove that the k-fold delooping of $\mathbf{A}_r$ is homotopy equivalent to the rth stage of the May-Milgram filtration of $\Omega^m\Sigma^m S^k$.

Remark 1.3 Our results bear a resemblance to the Dunn–Lurie additivity theorem [6] [13, Theorem 5.1.2.2]. This result says that $\mathcal{E}_{k+m} \simeq \mathcal{E}_k \otimes \mathcal{E}_m$ for a suitable tensor product of operads, and our result says that $\mathbf{E}_{k+m}(*)$ can be obtained as $\mathcal{E}_k$-algebra from the cardinality filtration on $\mathbf{E}_m(*)$, twisted by the vector bundles $k\phi_{m,r+1}$. It would be interesting to know whether it is possible to deduce Theorem 1.2 from the additivity theorem.

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Contents

2 Recollections of homotopy theory for algebras over an operad

We work in the setting of [8] and use similar notation when possible.

Assumption 2.1 S is a simplicially enriched complete and cocomplete category with closed symmetric monoidal structure such that the tensor product $\otimes$ commutes with sifted colimits.

Assumption 2.2 S comes equipped with a cofibrantly generated model structure which is both simplicial and monoidal, and such that the monoidal unit $\mathbb{1}$ is cofibrant.

The first version of [8] required that homotopy equivalences are weak equivalences but this is in fact always the case by Proposition 9.5.16 of [10].

When G is a symmetric monoidal category, then we may endow the category S^{G} of functors $\mathsf{G} \to \mathsf{S}$ with the Day convolution tensor product; this will also be symmetric monoidal. Similarly, the category S_* of pointed objects in S with smash product inherits these properties.

Example 2.3 The examples of S most relevant to this paper are: (i) the category sSet of simplicial sets with the Quillen model structure and cartesian product, and (ii) the category Top of compactly generated weakly Hausdorff spaces with the Quillen model structure and cartesian product (see [18] for more details about the point-set topology).

Let FB_∞ denote the category of (possibly empty) finite sets and bijections, then the objects of the category $(\mathsf{S}^{\mathsf{G}})^{\mathsf{FB}_\infty}$ of functors $\mathsf{FB}_\infty \to \mathsf{S}^{\mathsf{G}}$ are called *symmetric sequences*. In addition to the Day convolution tensor product, $(\mathsf{S}^{\mathsf{G}})^{\mathsf{FB}_\infty}$ admits a composition product $\circ$ (which is rarely symmetric); for $\mathcal{X}, \mathcal{Y} \in (\mathsf{S}^{\mathsf{G}})^{\mathsf{FB}_\infty}$ the evaluation of the composition product $\mathcal{X} \circ \mathcal{Y}$ on the set $\{1, 2, \ldots, r\}$ is given by

$$\mathcal{X} \circ \mathcal{Y}(r) = \bigsqcup_{d \geq 0} \mathcal{X}(d) \otimes_{\mathfrak{S}_d} \left(\bigsqcup_{k_1 + \ldots + k_d = r} \mathfrak{S}_r \times_{\mathfrak{S}_{k_1} + \ldots + \mathfrak{S}_{k_d}} \mathcal{Y}(k_1) \otimes \ldots \otimes \mathcal{Y}(k_d) \right).$$

A (symmetric) *operad* is a unital monoid with respect to this composition product. An $\mathcal{O}$-algebra $\mathbf{A}$ is an object $A \in \mathsf{S}$ with a left $\mathcal{O}$-module structure on A considered as a symmetric sequence concentrated in cardinality 0. Equivalently we can use the associated monad on S, for which we also use the notation $\mathcal{O}$,

$$\mathcal{O}(X) := \bigsqcup_{r \geq 0} \mathcal{O}(r) \otimes_{\mathfrak{S}_r} X^{\otimes r},$$

and define an $\mathcal{O}$-algebra to be an algebra over this monad. The category $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$ of $\mathcal{O}$-algebras is both complete and cocomplete.

A *free* $\mathcal{O}$-algebra is one of the form $\mathcal{O}(X)$ with $\mathcal{O}$-algebra structure maps induced by the monad multiplication and unit. We use the notation $\mathbf{F}^{\mathcal{O}} \colon \mathsf{S}^{\mathsf{G}} \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$ for the free $\mathcal{O}$-algebra functor, which is the left adjoint to the forgetful functor $\mathcal{U}^{\mathcal{O}} \colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}}) \to \mathsf{S}^{\mathsf{G}}$ sending an algebra to its the underlying object. Note $\mathcal{O} = \mathcal{U}^{\mathcal{O}}\mathbf{F}^{\mathcal{O}}$.

Any $\mathcal{O}$-algebra admits a canonical presentation as a reflexive coequalizer of free $\mathcal{O}$-algebras:

$$\mathbf{F}^{\mathcal{O}}(\mathcal{O}(\mathcal{U}^{\mathcal{O}}(\mathbf{A}))) \rightrightarrows \mathbf{F}^{\mathcal{O}}(\mathcal{U}^{\mathcal{O}}(\mathbf{A})) \longrightarrow \mathbf{A},$$

the top map coming from the $\mathcal{O}$-algebra structure map $\mathcal{O}(\mathcal{U}^{\mathcal{O}}(\mathbf{A})) \to \mathcal{U}^{\mathcal{O}}(\mathbf{A})$, and the bottom map coming from the natural transformation $\mathbf{F}^{\mathcal{O}}\mathcal{O} \to \mathbf{F}^{\mathcal{O}}$ induced by the monad multiplication. Thus free $\mathcal{O}$-algebras generate the category of $\mathcal{O}$-algebras under sifted colimits, and the category of right $\mathcal{O}$-module functors $\mathsf{C} \to \mathsf{D}$ preserving sifted colimits is equivalent to the category of functors $\mathsf{Alg}_{\mathcal{O}}(\mathsf{C}) \to \mathsf{D}$ preserving sifted colimits; one constructs the latter from the former using the canonical presentation, and one constructs the former from the latter by evaluating on free $\mathcal{O}$-algebras.

For $X \in \mathsf{S}^{\mathsf{G}}$ and $g \in \mathsf{G}$, the evaluation $X \mapsto X(g) \in \mathsf{S}$ has a left adjoint; given $Y \in \mathsf{S}$ we denote its image under this left adjoint by Y^g. Given a map $\partial D^d \to \mathcal{U}^{\mathcal{O}}(\mathbf{A})(g)$ (where ∂D^d stands for $\partial D^d \otimes \mathbb{1}$, the copowering of ∂D^d with the monoidal unit), we obtain by adjunction first a map $\partial D^{g,d} \to \mathcal{U}^{\mathcal{O}}(\mathbf{A})$ and then a map $\mathbf{F}^{\mathcal{O}}(\partial D^{g,d}) \to \mathbf{A}$. An $\mathcal{O}$*-cell attachment* is defined to be the following pushout in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$

(1)
$$\begin{array}{ccc} \mathbf{F}^{\mathcal{O}}(\partial D^{g,d}) & \longrightarrow & \mathbf{A} \\ \downarrow & & \downarrow \\ \mathbf{F}^{\mathcal{O}}(D^{g,d}) & \longrightarrow & \mathbf{A} \cup^{\mathcal{O}} D^{g,d}. \end{array}$$

Explicitly this pushout may be constructed as the following reflexive coequalizer

$$\mathbf{F}^{\mathcal{O}}(\mathcal{O}(\mathcal{U}^{\mathcal{O}}(\mathbf{A})) \cup D^{g,d}) \rightrightarrows \mathbf{F}^{\mathcal{O}}(\mathcal{U}^{\mathcal{O}}(\mathbf{A}) \cup D^{g,d}) \longrightarrow \mathbf{A} \cup^{\mathcal{O}} D^{g,d}.$$

The left vertical map in (1) is a cofibration, so cell attachments are homotopy-invariant when $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$ is left proper. In general we need to derive the construction; we will momentarily explain when this can be done using a monadic bar resolution.

Using the copowering of S^{G} over sSet, any operad in simplicial sets gives rise to an operad in S^{G}, and using the strong monoidal functor $\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{sSet}$ so does any operad in compactly-generated weakly Hausdorff topological spaces. We shall restrict our attention to operads $\mathcal{O}$ in sSet which are Σ-cofibrant, i.e. for all $r \geq 0$ the $\mathfrak{S}_r$-action on $\mathcal{O}(r)$ is free. We may attempt to define a model structure on $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$ by declaring the (trivial) fibrations and weak equivalences to be those of underlying objects. If it exists, this is called the projective model structure.

Assumption 2.4 The projective model structure exists on $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$.

When this assumption is satisfied, the projective model structure will be a cofibrantly generated model structure with generating (trivial) cofibrations obtained by applying $\mathcal{O}$ to the generating (trivial) cofibrations of the model structure on S^{G}. Since S^{G} is a simplicial and monoidal model category, it is automatic that the forgetful functor $U^{\mathcal{O}}\colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}}) \to \mathsf{S}^{\mathsf{G}}$ preserves (trivial) cofibrations, cf. Lemma 9.5 of [8]. When $\mathcal{O}$ is a Σ-cofibrant operad in simplicial sets, the projective model structure exists in the settings of Example 2.3, cf. Section 9.2 of [8].

When $\mathcal{U}^{\mathcal{O}}(\mathbf{A}) \in \mathsf{S}^{\mathsf{G}}$ is cofibrant, we may use the monadic bar resolution to find an explicit cofibrant replacement of $\mathbf{A}$ and thus compute derived functors.

Definition 2.5 The *monadic bar resolution* is the augmented simplicial object $B_\bullet(\mathbf{F}^{\mathcal{O}}, \mathcal{O}, \mathbf{A})$ with p-simplices given by $\mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{A})))$ for $p \geq 0$ and $\mathbf{A}$ for $p = -1$. The face maps and augmentation are induced by the monad multiplication and the $\mathcal{O}$-algebra structure on $\mathbf{A}$, and the degeneracies by the unit of the monad.

This is a special case of the *two-sided monadic bar construction*, which is used throughout the paper. It takes as input a monad T, a right T-functor F and a T-algebra $\mathbf{A}$ with underlying object A, and has p-simplices given by $B_\bullet(F, T, \mathbf{A}) = F(T^p(A))$. The face maps and degeneracy maps are similar to above, for details see e.g. Section 9 of [14].

Let $|-|$ denote the (thin) geometric realization, and introduce the notation $B(\mathbf{F}^{\mathcal{O}}, \mathcal{O}, \mathbf{A}) := |B_\bullet(\mathbf{F}^{\mathcal{O}}, \mathcal{O}, \mathbf{A})|$. Note that here we take geometric realization in the categories of $\mathcal{O}$-algebras, but $\mathcal{U}^{\mathcal{O}}$ commutes with geometric realization by Section 8.3.3 of [8]. The augmentation induces a map $B(\mathbf{F}^{\mathcal{O}}, \mathcal{O}, \mathbf{A}) \to \mathbf{A}$, which is always a weak equivalence using an extra degeneracy argument. It is a free simplicial resolution in the sense of Definition 8.16 of [8] when the bar construction is Reedy cofibrant. Because $\mathcal{O}$ is Σ-cofibrant, this is the case when $\mathcal{U}^{\mathcal{O}}(\mathbf{A})$ is cofibrant, using the Reedy cofibrancy criterion of Lemma 9.14 of [8].

Remark 2.6 In [12], $\mathcal{O}$-algebra cell attachments were defined using partial algebras. The formula in Definition 3.1 of [12], written in our notation, is $|[p] \mapsto \mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathbf{A}) \cup D^d)|$. This may be obtained by inserting $B(\mathbf{F}^{\mathcal{O}}, \mathcal{O}, \mathbf{A})$ into the underived formula for cell attachment. We explained above that this gives derived cell attachment when $\mathcal{U}^{\mathcal{O}}(\mathbf{A})$ is cofibrant, but in Top and sSet this assumption is unnecessary. In sSet, every object is cofibrant. In Top, we use that geometric realization sends levelwise weak equivalences between proper simplicial spaces to weak equivalences, even if the simplicial spaces are not levelwise cofibrant. This allows us to cofibrantly replace $\mathbf{A}$ in the category of $\mathcal{O}$-algebras (which will be cofibrant in topological spaces because $\mathcal{O}$ is Σ-cofibrant). Thus results of [12] apply: in particular, Proposition 6.12 of [12] implies that an $\mathcal{E}_k$-cell structure on an $\mathcal{E}_k$-algebra in topological spaces deloops to an ordinary cell structure on the k-fold delooping. The reason for this is that delooping preserves homotopy pushouts and $\mathbf{E}_k(\partial D^d) \to \mathbf{E}_k(D^d)$ deloops to $\partial D^d \hookrightarrow D^d$.

3 Rank completion

We shall define a rank completion filtration in the case that we are working in a category of functors S^{G} where G has a notion of rank, and the operad $\mathcal{O}$ and $\mathcal{O}$-algebra $\mathbf{A}$ satisfy mild conditions. Later in this paper, $\mathcal{O}$ will be $\mathcal{E}_k$ and G will be $\mathbb{N}$; the rank function will be used to keep track of the number of points in a configuration, and though we shall not use this, G can be used to record group actions on configurations.

Assumption 3.1 G is a symmetric monoidal groupoid equipped with strong monoidal functor $\kappa\colon \mathsf{G} \to \mathbb{N}$, which we call a *rank functor*.

Let $\mathsf{G}_{\leq r}$ denote the full subcategory on G on those objects g such that $\kappa(g) \leq r$, and G_r denote the full subcategory on objects g such that $\kappa(g) = r$. Precomposition gives restriction functors $(\leq r)^*$ and $(r)^*$ participating in adjunctions

$$\mathsf{S}^{\mathsf{G}_{\leq r}} \underset{(\leq r)^*}{\overset{(\leq r)_*}{\rightleftarrows}} \mathsf{S}^{\mathsf{G}}, \qquad \mathsf{S}^{\mathsf{G}_r} \underset{(r)^*}{\overset{(r)_*}{\rightleftarrows}} \mathsf{S}^{\mathsf{G}}.$$

There are further relative restriction and extension functors between $\mathsf{S}^{\mathsf{G}_r}$, $\mathsf{S}^{\mathsf{G}_{\leq r}}$ for different r, participating in analogous adjunctions. The functors $(\leq r)^*$ and $(r)^*$ are themselves left adjoints; though we will not use their right adjoints, we will use that $(\leq r)^*$ and $(r)^*$ commute with colimits.

It follows from the formula for Day convolution that $\mathsf{S}^{\mathsf{G}_{\leq r}}$ inherits a symmetric monoidal tensor product, an alternative expression for which is given by $X \otimes Y = (\leq r)^*((\leq r)_*(X) \otimes (\leq r)_*(Y))$. This makes visible that $(\leq r)^*$ is strong monoidal and simplicial. In particular, the functor $(\leq r)^*$ takes $\mathcal{O}$-algebras in S^{G} to $\mathcal{O}$-algebras in $\mathsf{S}^{\mathsf{G}_{\leq r}}$. Its left adjoint $(\leq r)_*$ in general does not. However, we may use the canonical presentation of $\mathcal{O}$-algebras explained in the previous section to construct a left adjoint $(\leq r)_*^{\mathrm{alg}} \colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r}}) \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$ to $(\leq r)^* \colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}}) \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r}})$. Explicitly it is the following reflexive coequalizer

$$\mathbf{F}^{\mathcal{O}}((\leq r)_*\mathcal{O}(\mathcal{U}^{\mathcal{O}}(\mathbf{A}))) \rightrightarrows \mathbf{F}^{\mathcal{O}}((\leq r)_*\mathcal{U}^{\mathcal{O}}(\mathbf{A})) \longrightarrow (\leq r)_*^{\mathrm{alg}}(\mathbf{A}).$$

It is defined uniquely up to isomorphism by demanding that $(\leq r)_*^{\mathrm{alg}} F^{\mathcal{O}}(X) = F^{\mathcal{O}}((\leq r)_*(X))$ and that it preserves sifted colimits.

Definition 3.2 We define the *rth rank completion functor* $\mathbf{T}_r \colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}}) \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$ to be $(\leq r)_*^{\mathrm{alg}}(\leq r)^*$.

This functor underlies the monad associated to the adjunction $(\leq r)_*^{\mathrm{alg}} \dashv (\leq r)^*$ and has a right adjoint. The counit gives a natural transformation $\mathbf{T}_r \Rightarrow \mathrm{id}$, and the commutative diagram of groupoids

$$\begin{array}{ccccccc} & & & \mathsf{G} & & & \\ & \nearrow & \uparrow & \nwarrow & \nwarrow & & \\ \mathsf{G}_{\leq 0} & \longrightarrow & \mathsf{G}_{\leq 1} & \longrightarrow & \mathsf{G}_{\leq 2} & \longrightarrow & \cdots, \end{array}$$

gives rise to a tower of natural transformations of functors $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}}) \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$

$$\begin{array}{ccccccc} & & & \mathrm{id} & & & \\ & \nearrow & \uparrow & \nwarrow & \nwarrow & & \\ \mathbf{T}_0 & \longrightarrow & \mathbf{T}_1 & \longrightarrow & \mathbf{T}_2 & \longrightarrow & \cdots. \end{array}$$

Since colimits are computed objectwise and the map $(g)^*\mathbf{T}_r(\mathbf{A}) \to (g)^*\mathbf{A}$ is the identity as soon as $r \geq \kappa(g)$, the natural transformation $\mathrm{colim}_{r \in \mathbb{N}} \mathbf{T}_r \Rightarrow \mathrm{id}$ is a natural isomorphism.

The functor $(\leq r)^*$ obviously preserves fibrations and weak equivalences, so $(\leq r)_*^{\mathrm{alg}}$ is a left Quillen functor. However, $(\leq r)^*$ also preserves cofibrations, as these are retracts of iterated pushouts along free $\mathcal{O}$-algebra maps, which are preserved by $(\leq r)^*$. Hence

$\mathbf{T}_r = (\leq r)_*^{\mathrm{alg}}(\leq r)^*$ preserves trivial cofibrations between cofibrant objects, and thus admits a left derived functor by precomposition with a functorial cofibrant replacement. Moreover, as explained above, when $\mathcal{U}^{\mathcal{O}}(\mathbf{A})$ is cofibrant we may use a monadic bar resolution to cofibrantly replace $\mathbf{A}$. As a composition of two left adjoints, $\mathbf{T}_r$ commutes with geometric realization. Thus we obtain the following formula for $\mathbf{T}_r^{\mathbb{L}}(\mathbf{A})$:

$$\mathbf{T}_r^{\mathbb{L}}(\mathbf{A}) = B(\mathbf{F}^{\mathcal{O}}(\leq r)_*, \mathcal{O}, (\leq r)^*\mathcal{U}^{\mathcal{O}}(\mathbf{A})) = B(\mathbf{F}^{\mathcal{O}}(\leq r)_*(\leq r)^*, \mathcal{O}, \mathcal{U}^{\mathcal{O}}(\mathbf{A})), \tag{2}$$

the latter equality following from the fact that $(\leq r)^*$ commutes with $\mathcal{O}$.

We next restrict our attention to a setting where the underlying object of $\mathbf{F}^{\mathcal{O}}(X)$ agrees with X in rank $\leq r$ up to homotopy, for those X which are concentrated in rank r. To see when this occurs, note that for any operad $\mathcal{O}$ and X concentrated in rank r, $\mathcal{O}(X)$ is isomorphic to $\mathcal{O}(0)$ in rank 0 and $\mathcal{O}(1) \otimes X$ in rank r. Hence the following assumption suffices:

Assumption 3.3 $\mathcal{O}$ is a *non-unitary* operad in simplicial sets, i.e. $\mathcal{O}(0) = \varnothing$, and $\mathcal{O}(1) \simeq *$.

Definition 3.4 We say $X \in \mathsf{S}^{\mathsf{G}}$ is *reduced* if it is concentrated in rank > 0, that is, $X(g)$ is initial when $\kappa(g) = 0$.

The horizontal maps in the following proposition are obtained from the identity morphisms of $(r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_r^{\mathbb{L}}(\mathbf{A})$ and $(r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_{r+1}^{\mathbb{L}}(\mathbf{A})$ respectively, the vertical maps from the natural transformation $\mathbf{T}_r \Rightarrow \mathbf{T}_{r+1}$.

Proposition 3.5 *For reduced* $\mathbf{A}$ *there is a homotopy cocartesian square in* $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}})$

$$\begin{array}{ccc} \mathbf{F}^{\mathcal{O}}((r+1)_*(r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_r^{\mathbb{L}}(\mathbf{A})) & \longrightarrow & \mathbf{T}_r^{\mathbb{L}}(\mathbf{A}) \\ \downarrow & & \downarrow \\ \mathbf{F}^{\mathcal{O}}((r+1)_*(r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_{r+1}^{\mathbb{L}}(\mathbf{A}))) & \longrightarrow & \mathbf{T}_{r+1}^{\mathbb{L}}(\mathbf{A}), \end{array}$$

where we remark that $(r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_{r+1}^{\mathbb{L}}(\mathbf{A}) \cong (r+1)^*\mathcal{U}^{\mathcal{O}}(\mathbf{A})$.

Proof This diagram is obtained by applying $(\leq r+1)_*^{\mathrm{alg}}$ to a diagram in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}}_{\leq r+1})$, a functor which preserves homotopy cocartesian squares as it is a left Quillen functor. Hence it suffices to prove that the following is homotopy cocartesian in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r+1}})$:

$$\begin{array}{ccc} \mathbf{F}^{\mathcal{O}}((r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_r^{\mathbb{L}}(\mathbf{A})) & \longrightarrow & (\leq r)_*^{\mathrm{alg}}(\leq r)^*\mathbf{A} \\ \downarrow & & \downarrow \\ \mathbf{F}^{\mathcal{O}}((r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{A}) & \longrightarrow & (\leq r+1)^*\mathbf{A}, \end{array}$$

where $\mathbf{F}^{\mathcal{O}}$ now denotes the free algebra functor $\mathsf{S}^{\mathsf{G}_{\leq r+1}} \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r+1}})$, $(\leq r)_*$ and $(r+1)_*$ denote the left adjoints to $(\leq r)^* \colon \mathsf{S}^{\mathsf{G}_{\leq r+1}} \to \mathsf{S}^{\mathsf{G}_{\leq r}}$ and $(\leq r+1)^* \mathsf{S}^{\mathsf{G}_{\leq r+1}} \to \mathsf{S}^{\mathsf{G}_{r+1}}$ respectively, and $(\leq r)_*^{\mathrm{alg}}$ denotes the left adjoint to $(\leq r)^* \colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r+1}}) \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r}})$.

The result now follows from the next lemma: substitute in its statement

$$\begin{aligned} X &\rightsquigarrow (r+1)^* \mathcal{U}^{\mathcal{O}} \mathbf{T}_r^{\mathbb{L}}(\mathbf{A}) \\ Y &\rightsquigarrow (r+1)^* \mathcal{U}^{\mathcal{O}} \mathbf{A} \\ \mathbf{A} &\rightsquigarrow (\leq r)_*^{\mathrm{alg}} (\leq r)^* \mathbf{A} \\ \mathbf{B} &\rightsquigarrow \mathbf{A}. \end{aligned}$$

Verifying condition (ii) uses that $\mathcal{O}(1) \simeq *$. □

Lemma 3.6 *Suppose $\mathbf{A}, \mathbf{B} \in \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r+1}})$ are cofibrant in $\mathsf{S}^{\mathsf{G}_{\leq r+1}}$ and reduced, and $X, Y \in \mathsf{S}^{\mathsf{G}_{\leq r+1}}$ are cofibrant and concentrated in rank $r+1$. Then a commutative square*

$$\begin{array}{ccc} \mathbf{F}^{\mathcal{O}}(X) & \longrightarrow & \mathbf{A} \\ \downarrow & & \downarrow \\ \mathbf{F}^{\mathcal{O}}(Y) & \longrightarrow & \mathbf{B}, \end{array}$$

is homotopy cartesian in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r+1}})$ if the following two conditions hold:

(i) the map $(\leq r)^* \mathbf{A} \to (\leq r)^* \mathbf{B}$ is a weak equivalence,

(ii) the commutative square

$$\begin{array}{ccc} (r+1)^* X & \longrightarrow & (r+1)^* \mathcal{U}^{\mathcal{O}} \mathbf{A} \\ \downarrow & & \downarrow \\ (r+1)^* Y & \longrightarrow & (r+1)^* \mathcal{U}^{\mathcal{O}} \mathbf{B} \end{array}$$

is homotopy cocartesian.

Proof We may assume without loss of generality that $\mathbf{A}$ and $\mathbf{B}$ are cofibrant in $\mathsf{S}^{\mathsf{G}_{\leq r+1}}$ and $X \to Y$ is a cofibration between cofibrant objects. We can factor the commutative square as

$$\begin{array}{ccccc} \mathbf{F}^{\mathcal{O}}(X) & \longrightarrow & B(\mathbf{F}^{\mathcal{O}}, \mathcal{O}, \mathcal{U}^{\mathcal{O}}(\mathbf{A})) & \xrightarrow{\simeq} & \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}^{\mathcal{O}}(Y) & \longrightarrow & B(\mathbf{F}^{\mathcal{O}}, \mathcal{O}, \mathcal{U}^{\mathcal{O}}(\mathbf{B})) & \xrightarrow{\simeq} & \mathbf{B}, \end{array}$$

where the horizontal maps are weak equivalences because $\mathbf{A}$ and $\mathbf{B}$ are cofibrant in $\mathsf{S}^{\mathsf{G}_{\leq r+1}}$.

The left square is the geometric realization of the following square of simplicial objects

$$\begin{array}{ccc} \big([p] \mapsto \mathbf{F}^{\mathcal{O}}(X)\big) & \longrightarrow & \big([p] \mapsto \mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{A})))\big) \\ \downarrow & & \downarrow \\ \big([p] \mapsto \mathbf{F}^{\mathcal{O}}(Y)\big) & \longrightarrow & \big([p] \mapsto \mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{B})))\big). \end{array}$$

All these simplicial objects are Reedy cofibrant; this is evident for the left entries, and for the right entries follows from another application of Lemma 9.14 of [8]. Geometric realization of Reedy cofibrant simplicial objects is a homotopy colimit, and thus commutes with homotopy pushouts. In particular, a levelwise homotopy cocartesian diagram of Reedy cofibrant simplicial objects geometrically realizes to a homotopy cocartesian diagram. It thus suffices to prove that each of the levels

$$\begin{array}{ccc} \mathbf{F}^{\mathcal{O}}(X) & \longrightarrow & \mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{A}))) \\ \downarrow & & \downarrow \\ \mathbf{F}^{\mathcal{O}}(Y) & \longrightarrow & \mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{B}))) \end{array}$$

is homotopy cocartesian.

Since $X \to Y$ is a cofibration between cofibrant objects, the map from the homotopy pushout to the bottom-right corner is given by

$$\mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{A})) \cup_X Y) \longrightarrow \mathbf{F}^{\mathcal{O}}(\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{B}))).$$

This is a weak equivalence in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}_{\leq r+1}})$ if and only if the map on underlying objects is. Since $\mathbf{F}^{\mathcal{O}}$ preserves weak equivalences between cofibrant objects, it suffices to prove that the map

$$\mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{A})) \cup_X Y \longrightarrow \mathcal{O}^p(\mathcal{U}^{\mathcal{O}}(\mathbf{B}))$$

is a weak equivalence. Indeed, both objects are cofibrant since $X \mapsto \mathcal{O}(X)$ preserves cofibrant objects, as do pushouts along a cofibration.

We do this by induction over p. For $p = 0$, we observe that since X and Y are concentrated in rank $r+1$, we have a commutative diagram

$$\begin{array}{ccc} (\leq r)^*\mathcal{U}^{\mathcal{O}}(\mathbf{A}) & \longrightarrow & (\leq r)^*\mathcal{U}^{\mathcal{O}}(\mathbf{B}) \\ \downarrow \cong & & \| \\ (\leq r)^*\mathcal{U}^{\mathcal{O}}(\mathbf{A}) \cup_X Y & \longrightarrow & (\leq r)^*\mathcal{U}^{\mathcal{O}}(\mathbf{B}). \end{array}$$

Thus for ranks $\leq r$ the result follows from assumption (i). In rank $r+1$, the case $p=0$ follows from assumption (ii). This completes the proof of the initial case.

To prove the induction step, it suffices to prove the following statement: if Z, W are reduced and X, Y are concentrated in degree $r+1$, then if (i) $(\leq r)^*Z \to (\leq r)^*W$ is a weak equivalence and (ii) the commutative square

$$\begin{array}{ccc} (r+1)^*X & \longrightarrow & (r+1)^*Z \\ \downarrow & & \downarrow \\ (r+1)^*Y & \longrightarrow & (r+1)^*W \end{array}$$

is homotopy cocartesian, then (i') $(\leq r)^*\mathcal{O}(Z) \to (\leq r)^*\mathcal{O}(W)$ is a weak equivalence and (ii') the commutative square

$$\begin{array}{ccc} (r+1)^*X & \longrightarrow & (r+1)^*\mathcal{O}(Z) \\ \downarrow & & \downarrow \\ (r+1)^*Y & \longrightarrow & (r+1)^*\mathcal{O}(W) \end{array} \tag{3}$$

is homotopy cocartesian.

Deducing (i) from (i') is done by noting that $(\leq r)^*$ commutes with $\mathcal{O}$ and $\mathcal{O}$ preserves weak equivalences between cofibrant objects. To deduce (ii') from (i) and (ii), we use the formula

$$(r+1)^*\mathcal{O}(Z) = \bigsqcup_{n\geq 1} \mathcal{O}(n) \otimes_{\mathfrak{S}_n} \left(\bigsqcup_{\substack{1\leq r_1,\ldots,r_n\leq r+1 \\ \sum r_i = r+1}} (r_1)_*(r_1)^*Z \otimes \cdots \otimes (r_n)_*(r_n)^*Z \right)$$

and a similar one for W. To restrict the r_i to positive integers, we used that $\mathcal{O}$ is non-unitary, and that Z and W are reduced. From this expression, we see that (3) is a coproduct of two commutative diagrams. The first is

$$\begin{array}{ccc} \mathbb{i} & \longrightarrow & (r+1)^*\mathcal{O}((\leq r)_*(\leq r)^*Z) \\ \downarrow & & \downarrow \\ \mathbb{i} & \longrightarrow & (r+1)^*\mathcal{O}((\leq r)_*(\leq r)^*W), \end{array}$$

where $\mathbb{i}$ is the initial object, which is homotopy cocartesian because the right map is a weak equivalence as a consequence of (i). The second is

$$\begin{array}{ccc} (r+1)^*X & \longrightarrow & \mathcal{O}(1) \otimes (r+1)^*Z \\ \downarrow & & \downarrow \\ (r+1)^*Y & \longrightarrow & \mathcal{O}(1) \otimes (r+1)^*W, \end{array}$$

which homotopy cocartesian by (ii) since $\mathcal{O}(1) \simeq *$. □

We thus get a sequence of maps

$$\mathbf{T}_0^{\mathbb{L}}(\mathbf{A}) \longrightarrow \mathbf{T}_1^{\mathbb{L}}(\mathbf{A}) \longrightarrow \mathbf{T}_2^{\mathbb{L}}(\mathbf{A}) \longrightarrow \cdots$$

whose homotopy colimit is naturally weakly equivalent to $\mathbf{A}$ and whose homotopy cofibers we understand. This is the *rank completion filtration*.

When we can make sense of homology, e.g. in one of the settings mentioned in Section 10.1 of [8], we get a corresponding spectral sequence converging conditionally to the homology of $\mathcal{U}^{\mathcal{O}}(\mathbf{A})$. The E^1-page will be rather unwieldy, and we believe the following spectral sequence may be more useful:

Remark 3.7 Let $(-)_+$ denote the monad whose underlying functor takes the coproduct with the terminal object (so that algebras over it are pointed objects). As $\mathcal{O}$ is a *non-unitary* operad in simplicial sets, cf. Assumption 3.3, there is a canonical map of monads from $\mathcal{O}$ to $(-)_+$ which can be viewed as an augmentation of $\mathcal{O}$. This augmentation is given on $X \in \mathsf{S}^{\mathsf{G}}$ by the map $\mathcal{O}(X) = \bigsqcup_{n\geq 1} \mathcal{O}(n) \otimes_{\mathfrak{S}_n} X^{\otimes n} \to X_+$ which on the summand $\mathcal{O}(1) \otimes X$ is the map $\mathcal{O}(1) \otimes X \to * \otimes X = X$ and on the summands $\mathcal{O}(n) \otimes_{\mathfrak{S}_n} X^{\otimes n}$ for $n \geq 2$ is the unique map to the terminal object.

Taking indecomposables with respect to this augmentation, we obtain the $\mathcal{O}$-indecomposables functor $Q^{\mathcal{O}}\colon \mathsf{Alg}_{\mathcal{O}}(\mathsf{S}^{\mathsf{G}}) \to \mathsf{S}_*^{\mathsf{G}}$ determined uniquely up to isomorphism by demanding that $Q^{\mathcal{O}} F^{\mathcal{O}} \cong (-)_+$ and that $Q^{\mathcal{O}}$ commutes with sifted colimits. Applying its left-derived functor $Q^{\mathcal{O}}_{\mathbb{L}}$ to the diagram in the previous proposition, we get that if $\mathbf{A}$ is reduced there is a homotopy cocartesian square in $\mathsf{S}_*^{\mathsf{G}}$:

$$\begin{array}{ccc} (r+1)_*(r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_r^{\mathbb{L}}(\mathbf{A})_+ & \longrightarrow & Q^{\mathcal{O}}_{\mathbb{L}}(\mathbf{T}_r^{\mathbb{L}}(\mathbf{A})) \\ \downarrow & & \downarrow \\ (r+1)_*(r+1)^*\mathcal{U}^{\mathcal{O}}\mathbf{T}_{r+1}^{\mathbb{L}}(\mathbf{A})_+ & \longrightarrow & Q^{\mathcal{O}}_{\mathbb{L}}(\mathbf{T}_{r+1}^{\mathbb{L}}(\mathbf{A})). \end{array}$$

When we can make sense of homology, we can define $\mathcal{O}$-homology by $H^{\mathcal{O}}_{g,d}(\mathbf{A}) := \tilde{H}_d((g)^* Q^{\mathcal{O}}_{\mathbb{L}}(\mathbf{A}))$. The result of the previous discussion is a conditionally convergent spectral sequence (suppressing the filtration degree, so in particular the p in $E^1_{p,q}$ refers to rank)

$$E^1_{p,q} = H_{p+q}((p)^*\mathbf{T}_p^{\mathbb{L}}(\mathbf{A}), (p)^*\mathbf{T}_{p-1}^{\mathbb{L}}(\mathbf{A})) \Longrightarrow H^{\mathcal{O}}_{p,p+q}(\mathbf{A}),$$

where it may be helpful to recall that $(p)^*\mathbf{T}_p^{\mathbb{L}}(\mathbf{A}) \cong (p)^*\mathbf{A}$.

4 An $\mathcal{E}_k$-algebraic analogue of the May-Milgram filtration

To deduce our results, we specialize the results of the previous section to $\mathcal{O} = \mathcal{E}_k$, the non-unital little k-cubes operad. Recall that $\mathbb{I}$ denotes the open interval $(0,1)$, and let $\mathrm{Emb}^{\mathrm{rect}}(\bigsqcup_n \mathbb{I}^k, \mathbb{I}^k)$ denote the space of ordered n-tuples of rectilinear embeddings $\mathbb{I}^k \to \mathbb{I}^k$ with disjoint image (that is, they are a composition of translation and dilation by positive real numbers in each of the k directions).

Definition 4.1 The *non-unital little k-cubes operad* $\mathcal{E}_k$ has topological space $\mathcal{E}_k(n)$ of n-ary operations given by

$$\mathcal{E}_k(n) := \begin{cases} \varnothing & \text{if } n = 0, \\ \mathrm{Emb}^{\mathrm{rect}}(\bigsqcup_n \mathbb{I}^k, \mathbb{I}^k) & \text{if } n > 0, \end{cases}$$

with symmetric group $\mathfrak{S}_n$ permuting the n-tuples. The unit in $\mathcal{E}_k(1)$ is the identity map $\mathbb{I}^k \to \mathbb{I}^k$, and composition is induced by composition of embeddings.

This satisfies Assumption 3.3 and hence gives rise to an operad in S^{G}, all of whose objects are concentrated on the monoidal unit of G. $\mathcal{E}_k$-algebras in S^{G} are algebras over this operad, and we shall adopt the shorter notation $\mathbf{E}_k$ for the free $\mathcal{E}_k$-algebra functor $\mathbf{F}^{\mathcal{E}_k}$. (If $\mathsf{S} = \mathsf{Top}$ and $\mathsf{G} = *$, as a consequence of our conventions these are algebras over the operad $|\mathrm{Sing}(\mathcal{E}_k)|$ in topological spaces.)

We shall take $\mathsf{G} = \mathbb{N}$, with $\kappa\colon \mathbb{N} \to \mathbb{N}$ the identity functor. Let

$$\mathcal{U}^{\mathcal{E}_{k+m}}_{\mathcal{E}_k}\colon \mathsf{Alg}_{\mathcal{E}_{k+m}}(\mathsf{S}^{\mathbb{N}}) \longrightarrow \mathsf{Alg}_{\mathcal{E}_k}(\mathsf{S}^{\mathbb{N}})$$

denote the forgetful functor induced by the map of operads $\mathcal{E}_k \to \mathcal{E}_{k+m}$ given by sending a cube $e\colon \mathbb{I}^k \to \mathbb{I}^k$ to $e \times \mathrm{id}_{\mathbb{I}^m}\colon \mathbb{I}^k \times \mathbb{I}^m \to \mathbb{I}^k \times \mathbb{I}^m$. For the sake of brevity we will often write $\mathcal{U}$ for $\mathcal{U}^{\mathcal{E}_{k+m}}_{\mathcal{E}_k}$.

We are interested in free algebras on a point, which we will consider concentrated in rank 1. In this section we will more generally study $\mathbf{E}_{k+m}(X)$ for $X \in \mathsf{S}^{\mathbb{N}}$ satisfying a similar condition:

Assumption 4.2 $X \in \mathsf{S}^{\mathbb{N}}$ is concentrated in rank 1, i.e. $X(g)$ is initial unless $g = 1$ (so in particular reduced), and X is cofibrant.

We will give an elementary geometric model for the $\mathcal{E}_k$-algebra $\mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X))$, and use Proposition 3.5 to describe $\mathcal{U}\mathbf{E}_{k+m}(X)$ up to weak equivalence as a colimit of iterated pushouts along maps of free $\mathcal{E}_k$-algebras. The following definition was mentioned in the introduction:

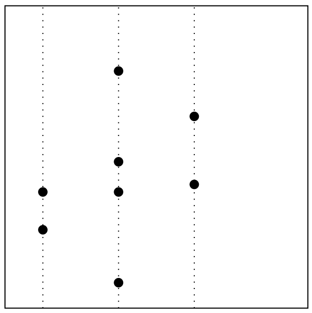

Figure 1: An element of $F_8(\mathbb{I}^k \times \mathbb{I}^m)$ for $k = 1$, $m = 1$ (suppressing the labels on the point for the sake of clarity) which is in $F_8^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ when $r \geq 4$, but not when $r < 4$.

Definition 4.3 For a manifold M and $n \geq 1$, the topological space $F_n(M)$ of *ordered configurations of n points in M* is given by $\{(m_1, \ldots, m_n) \mid m_i \neq m_j \text{ if } i \neq j\} \subset M^n$. For $n = 0$ we define $F_n(M) = \varnothing$.

We choose to define $F_0(M)$ to be empty since we work with non-unital $\mathcal{E}_k$-algebras. For $n > 0$, the topological space $F_n(M)$ is homeomorphic to the space of embeddings of the set $\{1, \ldots, n\}$ into M, and precomposition by permutations of $\{1, \ldots, n\}$ defines a $\mathfrak{S}_n$-action on the space $F_n(M)$. Taking the singular simplicial set, these assemble to a symmetric sequence $F(M)$ in sSet. For $M = \mathbb{I}^k \times \mathbb{I}^m$, this is a left $\mathcal{E}_{k+m}$-module, where the action is given by composition of embeddings.

Just like we used the enrichment of copowering of $\mathsf{S}^\mathbb{N}$ over sSet to make the operad $\mathcal{E}_k$ in sSet into an operad in $\mathsf{S}^\mathbb{N}$, we use it to make the left $\mathcal{E}_{k+m}$-module $F(\mathbb{I}^k \times \mathbb{I}^m)$ in sSet into a left $\mathcal{E}_{k+m}$-module in $\mathsf{S}^\mathbb{N}$. Analogously to the free $\mathcal{E}_k$-algebra construction, we can take the composition product of $F(\mathbb{I}^k \times \mathbb{I}^m) \in (\mathsf{S}^\mathbb{N})^\mathsf{G}$ with an object $X \in \mathsf{S}^\mathbb{N}$ considered as a symmetric sequence concentrated in cardinality 0. We refer to this as "applying" $F(\mathbb{I}^k \times \mathbb{I}^m)$ to X. The resulting object $F(\mathbb{I}^k \times \mathbb{I}^m)(X) \in \mathsf{S}^\mathbb{N}$ comes endowed with an $\mathcal{E}_{k+m}$-algebra structure. This construction is natural in X, and thus we obtain a functor $\mathbf{F}(\mathbb{I}^k \times \mathbb{I}^m)\colon \mathsf{S}^\mathbb{N} \to \mathsf{Alg}_{\mathcal{E}_{k+m}}(\mathsf{S}^\mathbb{N})$.

Definition 4.4 We let $F_n^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ denote the subspace of $F_n(\mathbb{I}^k \times \mathbb{I}^m)$ consisting of ordered configurations $\eta = (m_1, \ldots, m_n)$ such that for all $x \in \mathbb{I}^k$ the intersection $\eta \cap (\{x\} \times \mathbb{I}^m)$ has cardinality at most r.

Since the condition defining $F_n^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ is invariant under the $\mathfrak{S}_n$-action, these topological spaces may be assembled into a symmetric sequence $F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m) \subset$

$F(\mathbb{I}^k \times \mathbb{I}^m)$ in sSet and by the copowering also in $\mathsf{S}^{\mathbb{N}}$. The left $\mathcal{E}_{k+m}$-module structure on $F(\mathbb{I}^k \times \mathbb{I}^m)$ does not restrict. However, using the map of operads $\mathcal{E}_k \to \mathcal{E}_{k+m}$ induced by the inclusion $\mathbb{I}^k \to \mathbb{I}^k \times \mathbb{I}^m$ on the first k coordinates, we get a left $\mathcal{E}_k$-module structure on $F(\mathbb{I}^k \times \mathbb{I}^m)$ which *does* restrict and application of this symmetric sequence gives a functor $\mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)\colon \mathsf{S}^{\mathbb{N}} \to \mathsf{Alg}_{\mathcal{E}_k}(\mathsf{S}^{\mathbb{N}})$.

As we assumed that X is cofibrant, we can use a monadic bar resolution to give an explicit formula for $\mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) \in \mathsf{Alg}_{\mathcal{E}_k}(\mathsf{S}^{\mathbb{N}})$:

$$\mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) = B(\mathbf{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^*\mathcal{U}\mathbf{E}_{k+m}(X)).$$

We take this specific model for the domain of the map in the following proposition:

Proposition 4.5 *There are weak equivalences*

$$\alpha_r\colon \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) \longrightarrow \mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X),$$

of $\mathcal{E}_k$-algebras, which fit into commutative diagrams for $r \geq 0$

$$\begin{array}{ccc} \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) & \longrightarrow & \mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(X)) \\ \downarrow \alpha_r & & \downarrow \alpha_{r+1} \\ \mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X) & \longrightarrow & \mathbf{F}^{[r+1]}(\mathbb{I}^k \times \mathbb{I}^m)(X). \end{array} \tag{4}$$

Let us start by defining the maps:

Lemma 4.6 *There are maps $\alpha_r\colon \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) \longrightarrow \mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X)$ of $\mathcal{E}_k$-algebras making Diagram (4) commute.*

Proof The map $\mathcal{E}_{k+m} \to F(\mathbb{I}^k \times \mathbb{I}^m)$ which sends a cube to its center is a homotopy equivalence of left $\mathcal{E}_{k+m}$-modules in symmetric sequences, so we have an induced weak equivalence $\mathbf{E}_{k+m}(X) \to \mathbf{F}(\mathbb{I}^k \times \mathbb{I}^m)(X)$ of $\mathcal{E}_{k+m}$-algebras. To define α_r, we first insert this weak equivalence into the right entry of the bar construction

$$\begin{array}{c} |B_\bullet(\mathbf{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^*\mathcal{U}\mathbf{E}_{k+m}(X))| \\ \downarrow \simeq \\ |B_\bullet(\mathbf{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^*\mathcal{U}\mathbf{F}(\mathbb{I}^k \times \mathbb{I}^m)(X))|. \end{array}$$

The assumption that X is concentrated in rank 1 gives us an isomorphism

$$(\leq r)^*\mathcal{U}\mathbf{F}(\mathbb{I}^k \times \mathbb{I}^m)(X)) \cong \mathcal{U}((\leq r)^*\mathbf{F}(\mathbb{I}^k \times \mathbb{I}^m))((\leq r)^*X).$$

Here $(\leq r)^* F(\mathbb{I}^k \times \mathbb{I}^m)$ is an object in the truncated symmetric sequence category (functors from the category of possibly empty finite sets of cardinality $\leq r$ into sSet with tensor product the restriction of the composition product) and $(\leq r)^* \mathbf{F}(\mathbb{I}^k \times \mathbb{I}^m)$ is the functor given by tensoring with $(\leq r)^* F(\mathbb{I}^k \times \mathbb{I}^m)$.

Because $\otimes$ commutes with colimits in each variable and geometric realization, the target is obtained by applying the symmetric sequence

$$|B_\bullet(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m})|$$

in Top to X (as always, via Sing and the simplicial copowering). We define a map of left $\mathcal{E}_k$-modules in symmetric sequences in Top

$$a_r \colon |B_\bullet(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m}))| \longrightarrow F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$$

by describing an augmentation from $B_\bullet(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m}))$ to $F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$. The 0-simplices of the former are given by

$$\mathcal{E}_k(\leq r)_*(\leq r)^* F(\mathbb{I}^{k+m}) = \bigsqcup_{n\geq 1} \mathcal{E}_k(n) \times_{\mathfrak{S}_n} \bigsqcup_{1\leq k_1,\ldots,k_n \leq r} F_{k_i}(\mathbb{I}^k \times \mathbb{I}^m)$$

where the rank of each component is $k_1 + ... + k_n$. Given a collection of embeddings $e_i \colon \mathbb{I}^k \to \mathbb{I}^k$ and configurations $\xi_i \in F_{k_i}(\mathbb{I}^k \times \mathbb{I}^m)$, we may take the union of the images $(e_i \times \mathrm{id}_{\mathbb{I}^m})(\xi_i)$ in $\mathbb{I}^k \times \mathbb{I}^m$ and obtain an ordered configuration of $k_1 + \ldots + k_m$ points such that no subset $\{x\} \times \mathbb{I}^m$ contains more than r points. This map is easily seen to be compatible with the left $\mathcal{E}_k$-module structures. That the diagram commutes is clear from the definition. □

We next prove that each a_r is a weak homotopy equivalence, using a microfibration argument.

Definition 4.7 A map $\pi \colon E \to B$ of topological spaces is a *microfibration* if for each $i \geq 0$ and commutative diagram

$$\begin{array}{ccc} D^i \times \{0\} & \xrightarrow{h} & E \\ \downarrow & & \downarrow \pi \\ D^i \times [0,1] & \xrightarrow{H} & B, \end{array}$$

there exists an $\epsilon > 0$ and a partial lift $\tilde{H} \colon D^i \times [0,\epsilon] \to E$, i.e. $\pi \circ \tilde{H} = H|_{D^i \times [0,\epsilon]}$ and $\tilde{H}|_{D^i \times \{0\}} = h$.

Lemma 4.8 (Lemma 2.2 of [19]) *If $\pi \colon E \to B$ is a microfibration with weakly contractible fibers, then π is a weak homotopy equivalence.*

Our strategy is to prove that a_r is a microfibration with weakly contractible fibers. To do this, we use the following lemma in point-set topology.

Lemma 4.9 *Let $X_\bullet$ be a levelwise Hausdorff simplicial space. Let*

$$X_{1,\bullet} \subset X_{2,\bullet} \subset \cdots \subset X_\bullet$$

be an $\mathbb{N}_{>0}$-indexed sequence of simplicial subspaces such that: (i) $X_{s,p} \subset X_p$ is compact for all s, p, and (ii) each point $x \in X_p$ has an open neighborhood contained in some $X_{s,p}$. If C is compact, then any continuous map $C \to |X_\bullet|$ factors as $C \to |X_{s,\bullet}| \to |X_\bullet|$ for some s.

Proof The strategy is to first identify $|X_\bullet|$ with the sequential colimit $\operatorname{colim}_s |X_{s,\bullet}|$ and then show that this particular sequential colimit commutes with maps out of the compact space C.

The inclusions $X_{s,p} \to X_p$ induce a continuous bijection $\operatorname{colim}_s X_{s,p} \to X_p$. To show it is a homeomorphism we need to prove it is open: $V \subset \operatorname{colim}_s X_{s,p}$ being open means that all $V \cap X_{s,p}$ are open, and by the hypothesis for all $x \in V$, V contains an open neighborhood of x in X_p, which means it is open in X_p. Since colimits of simplicial spaces are computed levelwise, $\operatorname{colim}_s X_{s,\bullet} \to X_\bullet$ is an isomorphism of simplicial spaces. Since geometric realization commutes with filtered colimits (it has a right adjoint when working with compactly generated weakly Hausdorff spaces), the canonical map $\operatorname{colim}_s |X_{s,\bullet}| \to |X_\bullet|$ is a homeomorphism.

In CGWH spaces, maps out of a compact space commute with sequential colimits of closed inclusions by Lemma 3.6 of [18]. Thus we shall verify that each map $|X_{s,\bullet}| \to |X_{s+1,\bullet}|$ is a closed inclusion, using its description as a colimit of the maps of skeleta:

$$\begin{array}{ccccc}
\mathrm{sk}_0|X_{s,\bullet}| & \longrightarrow & \mathrm{sk}_1|X_{s,\bullet}| & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathrm{sk}_0|X_{s+1,\bullet}| & \longrightarrow & \mathrm{sk}_1|X_{s+1,\bullet}| & \longrightarrow & \cdots .
\end{array}$$

We claim all maps in this diagram are closed inclusions. All maps are clearly continuous injections and a continuous injection between compact Hausdorff spaces is always a closed inclusion, so it suffices to prove that each space is compact Hausdorff. They are compact because each $\mathrm{sk}_p|X_{s,\bullet}|$ is a quotient of the compact space $\bigsqcup_{k\leq p} \Delta^k \times X_{s,k}$. They are Hausdorff because we may freely add degeneracies to write $\mathrm{sk}_p|X_{s,\bullet}|$ as the geometric realization of a levelwise Hausdorff simplicial space and apply Theorem 1.1 of [16]. Furthermore, from the construction it is clear each square is a pullback square.

The result then follows from the following result about CGWH spaces, Lemma 3.9 of [18]: given a commutative diagram

$$\begin{array}{ccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\ B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots \end{array}$$

with all maps closed inclusions and all squares pullbacks, the induced map $\operatorname{colim}_s A_s \to \operatorname{colim}_s B_s$ is also a closed inclusion. $\square$

Lemma 4.10 *The map $a_r\colon |B_\bullet(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m})| \to F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ is a microfibration.*

Proof Fixing a cardinality n, we need to prove that the component $a_r(n)$ is a microfibration. Suppose we are given a commutative diagram

$$\begin{array}{ccc} D^i \times \{0\} & \xrightarrow{h} & |B_\bullet(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m}))|(n) \\ \downarrow & & \downarrow{\scriptstyle a_r(n)} \\ D^i \times [0,1] & \xrightarrow{H} & F^{[r]}_n(\mathbb{I}^k \times \mathbb{I}^m). \end{array}$$

Since $D^i \times [0,1]$ is compact, there exists a $\delta > 0$ such that H factors over the compact subspace of configurations ξ where

(a) the points in ξ have distance $\geq \delta$ from each other,

(b) for all closed cubes $C \subset \mathbb{R}^k$ with equal sides of length $< \delta$, the set $C \times \mathbb{I}^m$ contains at most r points of ξ.

Let us abbreviate $B_p(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m}))(n)$ by X_p, and by X_p^δ the subspace of X_p of elements whose image under $\alpha_r(n)$ satisfies (a) and (b).

Let $\rho_p\colon X_p^\delta \to (0,\infty)$ be the minimum of the distances from the points in the image ξ to the boundaries of the images of the cubes. Then $X_\bullet^\delta$ is a simplicial space with a sequence of continuous functions $\rho_p\colon X_p^\delta \to (0,\infty)$ such that $\rho_{p+1} \circ s_i = \rho_p$ and $\rho_{p-1} \circ d_i \geq \rho_p$. For each integer $s \geq 1$, the subspaces $X_{s,p}^\delta := \rho_p^{-1}([1/s, \infty)) \subset X_p^\delta$ assemble to a simplicial space $X_{s,\bullet}^\delta$.

This satisfies the hypotheses of Lemma 4.9 (condition (i) of that lemma is the reason we use $X_\bullet^\delta$ instead of $X_\bullet$, and uses that only the interiors of cubes need to be disjoint, not their closures). Hence the map h factors over some stage $|X_{s,\bullet}^\delta|$ with $\delta \geq \frac{1}{s} > 0$ such that for all $d \in D^i$, the configuration $H(d,0)$ is given by a ξ which satisfies the properties

(a) the points in ξ have distance $\geq \frac{1}{s}$ from each other,

(b) for all closed cubes $C \subset \mathbb{R}^k$ with equal sides of length $< \frac{1}{s}$, the set $C \times \mathbb{I}^m$ contains at most r points of ξ,

(c) the points in ξ have distance $\geq \frac{1}{s}$ to the boundaries of the images of the cubes in $h(d)$.

By continuity of the map H, there is an $\epsilon > 0$ such that for all $d \in D^i$ and $t \in [0,\epsilon]$, the configuration $H(d,t)$ is within distance $\frac{1}{3s}$ of $H(d,0)$. The partial lift is given by the map which assigns to $(d,t) \in D^i \times [0,\epsilon]$ the element of $|X_\bullet|$ represented by configuration $H(d,t)$ inside the cubes coming from the unique non-degenerate representative of $h(d)$. To see this is well-defined, note that (c) implies a point in $H(d,t)$ remains within the same cubes of $h(d)$ as the corresponding point in $H(d,0)$.

To see it is continuous, note that the movement of the points in the configuration can be described by recording their displacements by an element $\Delta(d,t)$ of $\left([-\frac{1}{3s},\frac{1}{3s}]^{k+m}\right)^n$ (each $\Delta(d,0)$ equals 0). That is, $\Delta(d,t)$ is defined by $H(d,t) = H(d,0) + \Delta(d,t)$.

There is a simplicial map

$$\left([-\frac{1}{3s},\frac{1}{3s}]^{k+m}\right)^n \times X^{1/s}_{s,\bullet} \longrightarrow X_\bullet$$

obtained by applying the displacement to the configuration. This is continuous, and well-defined because whenever we move points in the configuration of an element of $X^{1/s}_{s,p}$ at most $\frac{1}{3s}$ in any of the directions, they do not (a) collide with each other, (b) have more than r points in a subset $\{x\} \times \mathbb{I}^m$, and (c) cross boundaries of cubes. That the lift is continuous then follows by observing that it can be realized as a composition of continuous maps

$$\begin{aligned} D^i \times [0,\epsilon] &\xrightarrow{\Delta \times h} \left([-\frac{1}{3s},\frac{1}{3s}]^{k+m}\right)^n \times |X^{1/s}_{s,\bullet}| \\ &\xrightarrow{\cong} \left|\left([-\frac{1}{3s},\frac{1}{3s}]^{k+m}\right)^n \times X^{1/s}_{s,\bullet}\right| \\ &\longrightarrow |X_\bullet|. \square \end{aligned}$$

We now prove that the fibers of $a_r(n)$ are weakly contractible, so Lemma 4.8 implies that $a_r(n)$ is a weak homotopy equivalence.

Lemma 4.11 *The fibers of the map* $a_r \colon |B_\bullet(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m})| \to F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ *are weakly contractible.*

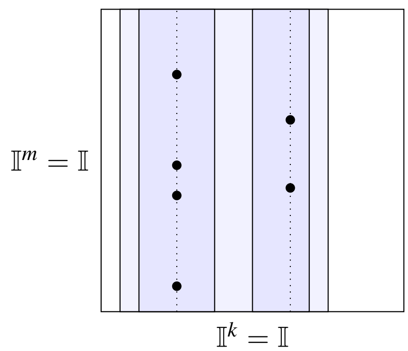

Figure 2: An element of X_1 in the case $k = m = 1$, $r = 4$, and $n = 6$. There are two innermost cubes and one outermost cube.

Proof Fix a configuration $\xi \in F_n^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$. As above, the fiber $\epsilon^{-1}(\xi)$ is given by the geometric realization of the subsimplicial space of $X_\bullet$ with underlying configuration ξ. Call this simplicial space $X_\bullet(\xi)$. Another application of Lemma 9.14 of [8] tells us this is Reedy cofibrant.

Let $\xi' \in \mathrm{Sym}_n(\mathbb{I}^k) := (\mathbb{I}^k)^n/\mathfrak{S}_n$ be the configuration with multiplicities obtained by projecting ξ onto $\mathbb{I}^k$. The p-simplices of $X_\bullet(\xi)$ are given by $p+1$ levels of nested k-dimensional cubes such that all points of ξ' are contained in an innermost cube and all cubes except the outermost ones contain at most r points of ξ' counted with multiplicity. Let $(e_1, e_2, \ldots e_l) \in \mathcal{E}_k(l)$ be a collection of cubes such that every cube e_i contains exactly one point of ξ' (counted without multiplicity) and every point of ξ' (counted without multiplicity) is in one of the cubes e_i. Let $X_\bullet(\xi, e)$ denote the subsimplicial space of $X_\bullet(\xi)$ where we require that if a cube contains a point of ξ', then it contains the corresponding e_i. This is also Reedy cofibrant. Thus, since the inclusion $X_\bullet(\xi, e) \hookrightarrow X_\bullet(\xi)$ induces a levelwise homotopy equivalence, it induces a weak equivalence on geometric realizations. View $X_\bullet(\xi, e)$ as an augmented simplicial space by adding a point in degree -1. There is an extra degeneracy $X_p(\xi, e) \to X_{p+1}(\xi, e)$ given by inserting e_i in the innermost cubes, and hence $|X_\bullet(\xi, e)|$ is contractible.

□

Proof of Proposition 4.5 By combining Lemmas 4.8, 4.10, and 4.11, we see that the map $a_r \colon |B_\bullet(\mathcal{E}_k(\leq r)_*, \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m})| \to F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ is a weak equivalence. Since a_r is a map of symmetric sequences and all of the symmetric group actions are free, it is weak equivalence of symmetric sequences. The result follows because applying a weak equivalence between Σ-cofibrant symmetric sequences in sSet to a cofibrant object is a weak equivalence by Lemma 9.1 of [8], and geometric realization preserves weak equivalences between Reedy cofibrant simplicial objects. ◻

The inclusion $\mathbb{I}^k \times F_{r+1}(\mathbb{I}^m) \hookrightarrow F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m)$ given by $(x, \xi) \mapsto x \times \xi$, i.e. sending each point $m_i \in \xi$ to $x \times m_i$, has image given by the complement of $F^{[r]}_{r+1}(\mathbb{I}^k \times \mathbb{I}^m)$ in $F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m)$. Let $\varphi_{m,r}$ denote the trivial vector bundle over $\mathbb{I}^k \times F_r(\mathbb{I}^m)$ given by $\mathbb{I}^k \times F_r(\mathbb{I}^m) \times \mathbb{R}^{r-1} \to \mathbb{I}^k \times F_r(\mathbb{I}^m)$, with $\mathfrak{S}_r$ acting diagonally and with $\mathbb{R}^{r-1}$ the orthogonal complement to the trivial representation in the permutation representation with its usual metric. The vector bundle $\varphi_{m,r}$ can be thought of as the $\mathfrak{S}_r$-equivariant analogue of $\phi_{m,r}$ from the introduction.

Lemma 4.12 *The normal bundle of $\mathbb{I}^k \times F_{r+1}(\mathbb{I}^m)$ in $F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m)$ is $\mathfrak{S}_{r+1}$-equivariantly isomorphic to $k\varphi_{m,r+1}$.*

Proof The normal bundle to $\mathbb{I}^k \times F_{r+1}(\mathbb{I}^m)$ is the orthogonal complement in $T(F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m))$ to the tangent bundle $T(\mathbb{I}^k \times F_{r+1}(\mathbb{I}^m))$. The former is the restriction of $T(\mathbb{I}^k \times \mathbb{I}^m)^{\oplus r+1} \cong (T\mathbb{I}^k)^{\oplus r+1} \oplus (T\mathbb{I}^m)^{\oplus r+1}$, and the latter is the restriction of $T\mathbb{I}^k \oplus (T\mathbb{I}^m)^{\oplus r+1}$. The inclusion is the diagonal on the first term and the identity on the second, and equivariant for the $\mathfrak{S}_{r+1}$-action. Thus the normal bundle is $\mathfrak{S}_{r+1}$-equivariantly isomorphic to the restriction of the orthogonal complement of the diagonal $T\mathbb{I}^k \subset (T\mathbb{I}^k)^{\oplus r+1}$. This is isomorphic to a k-fold Whitney sum of the trivial $\mathbb{R}^r$-bundle, with $\mathfrak{S}_{r+1}$-action given by the standard representation. ◻

The vector bundle $\varphi_{m,r+1}$ inherits a Riemannian metric, and we let $S(k\varphi_{m,r+1})$ be the sphere bundle with fiber over $(x, \xi) \in \mathbb{I}^k \times F_{r+1}(\mathbb{I}^m)$ those vectors of length $\frac{1}{2}d(y, \partial I^k)$. This bounds a disk bundle $D(k\varphi_{m,r+1})$, and both are clearly isomorphic to the unit sphere and disk bundles. Using the exponential map, we obtain the horizontal maps in the following commutative diagram of $\mathfrak{S}_{r+1}$-spaces:

$$\begin{array}{ccc} S(k\varphi_{m,r+1}) & \longrightarrow & F^{[r]}_{r+1}(\mathbb{I}^k \times \mathbb{I}^m) \\ \downarrow & & \downarrow \\ D(k\varphi_{m,r+1}) & \longrightarrow & F^{[r+1]}_{r+1}(\mathbb{I}^k \times \mathbb{I}^m) = F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m). \end{array}$$

Proposition 4.13 *For all $r \geq 0$, there is a zigzag of homotopy cocartesian squares*

$$\begin{array}{ccccc} \mathbf{E}_k((r+1)_* S(k\varphi_{m,r+1}) \otimes_{\mathfrak{S}_{r+1}} X^{\otimes r+1}) & \longrightarrow & \cdots & \xleftarrow{\simeq} & \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{E}_k((r+1)_* D(k\varphi_{m,r+1}) \otimes_{\mathfrak{S}_{r+1}} X^{\otimes r+1}) & \cdots & \cdots & \longrightarrow & \mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(X)). \end{array}$$

Proof The above result implies that for cofibrant X, we have a homotopy cocartesian square

$$\begin{array}{ccc} S(k\varphi_{m,r+1}) \otimes_{\mathfrak{S}_{r+1}} X^{\otimes r+1} & \longrightarrow & (r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X) \\ \downarrow & & \downarrow \\ D(k\varphi_{m,r+1}) \otimes_{\mathfrak{S}_{r+1}} X^{\otimes r+1} & \longrightarrow & (r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{F}^{[r+1]}(\mathbb{I}^k \times \mathbb{I}^m)(X). \end{array}$$

By Proposition 4.5, we have a commutative diagram with horizontal maps weak equivalences

$$\begin{array}{ccc} (r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X) & \xleftarrow[\alpha_r]{\simeq} & (r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) \\ \downarrow & & \downarrow \\ (r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{F}^{[r+1]}(\mathbb{I}^k \times \mathbb{I}^m)(X) & \xleftarrow[\alpha_{r+1}]{\simeq} & (r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(X)). \end{array}$$

Since applying $\mathbf{E}_k$ and $(r+1)_*$ preserves homotopy cocartesian squares, doing so gives us the left and middle squares. For the right square, specialize Proposition 3.5 to $\mathcal{O} = \mathcal{E}_k$ and $\mathbf{A} = \mathcal{U}\mathbf{E}_{k+m}(X)$ to obtain a homotopy cocartesian square

$$\begin{array}{ccc} \mathbf{E}_k((r+1)_*(r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X))) & \longrightarrow & \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(X)) \\ \downarrow & & \downarrow \\ \mathbf{E}_k((r+1)_*(r+1)^* \mathcal{U}^{\mathcal{E}_k} \mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(X))) & \longrightarrow & \mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(X)). \end{array}$$

□

We now deduce Theorem 1.2 from this by taking $\mathsf{S} = \mathsf{Top}$ and $X = (1)^*(*)$; we need to resolve the issue that Proposition 4.13 only provides zigzags.

Proof of Theorem 1.2 We start with an elementary homotopy-theoretic observation. Given a commutative diagram of topological spaces

$$\begin{array}{ccccc} S(k\phi_{m,r+1}) & \longrightarrow & X & \xleftarrow{\simeq} & X' \\ \downarrow & & \downarrow & & \downarrow \\ D(k\phi_{m,r+1}) & \longrightarrow & Y & \xleftarrow{\simeq} & Y' \end{array}$$

with decorated arrows weak equivalences, we can find maps $S(k\phi_{m,r+1}) \to X'$ and

$D(k\phi_{m,r+1}) \to Y'$ such that in the following diagram

$$\begin{array}{ccccccc} S(k\phi_{m,r+1}) & \xrightarrow{\ \ } & X & \xrightarrow{\ \ } & X' & \xleftarrow{\simeq} & \\ \downarrow & & & & \downarrow & & \downarrow \\ D(k\phi_{m,r+1}) & & Y & & Y' & \xrightarrow{\ \ } & \xleftarrow{\simeq} \end{array}$$

the outer square commutes and the triangles commute up to homotopy. To prove this, first homotope $S(k\phi_{m,r+1}) \to X$ until a lift exists (which is possible since the domain has the homotopy type of a CW-complex). Because $S(k\phi_{m,r+1}) \hookrightarrow D(k\phi_{m,r+1})$ admits the structure of a NDR-pair, we may extend this to a homotopy of commutative diagrams. At this point it suffices to find a lift in the commutative diagram

$$\begin{array}{ccc} S(k\phi_{m,r+1}) & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ D(k\phi_{m,r+1}) & \longrightarrow & Y, \end{array}$$

which exists as $(D(k\phi_{m,r+1}), S(k\phi_{m,r+1}))$ is homotopy equivalent to a CW pair.

Given this observation, we prove by induction over r that we may construct $\mathbf{A}_r \simeq \mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(*))$ by iterated pushouts along free algebras, obtaining in the process maps between the $\mathbf{A}_r$ satisfying $\operatorname{colim}_r \mathbf{A}_r = \operatorname{hocolim}_r \mathbf{A}_r \simeq \mathbf{F}^{\mathcal{E}k+m}(*)$.

The initial case is $\mathbf{A}_{-1} = \varnothing$. For the induction step, let us assume we have produced $\mathbf{A}_r$ as in the statement of Theorem 1.2 with a weak equivalence $\beta_r \colon \mathbf{A}_r \to \mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(*))$. Using the observation in the diagram of Proposition 4.13, we may assume we have a homotopy cocartesian commutative diagram

$$\begin{array}{ccc} S(k\phi_{m,r+1}) & \longrightarrow & (r+1)^*\mathcal{U}^{\mathcal{E}_k}\mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(*)) \\ \downarrow & & \downarrow \\ D(k\phi_{m,r+1}) & \longrightarrow & (r+1)^*\mathcal{U}^{\mathcal{E}_k}\mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(*)). \end{array}$$

Applying the observation again to lift along α_r, we may assume we have a homotopy cocartesian commutative diagram

$$\begin{array}{ccccc} S(k\phi_{m,r+1}) & \longrightarrow & (r+1)^*\mathcal{U}^{\mathcal{E}_k}\mathbf{A}_r & \xrightarrow{(r+1)^*\mathcal{U}^{\mathcal{E}_k}\beta_r} & (r+1)^*\mathcal{U}^{\mathcal{E}_k}\mathbf{T}^{\mathbb{L}}_r(\mathcal{U}\mathbf{E}_{k+m}(*)) \\ \downarrow & & & & \downarrow \\ D(k\phi_{m,r+1}) & & \longrightarrow & & (r+1)^*\mathcal{U}^{\mathcal{E}_k}\mathbf{T}^{\mathbb{L}}_{r+1}(\mathcal{U}\mathbf{E}_{k+m}(*)). \end{array}$$

We take adjoints and define $\mathbf{A}_{r+1}$ as the pushout fitting in a commutative diagram

$$\begin{array}{ccccc}
\mathbf{E}_k((r+1)_* S(k\phi_{m,r+1})) & \longrightarrow & \mathbf{A}_r & \xrightarrow{\beta_r} & \mathbf{T}_r^{\mathbb{L}}(\mathcal{U}\mathbf{E}_{k+m}(*)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{E}_k((r+1)_* D(k\phi_{m,r+1})) & \longrightarrow & \mathbf{A}_{r+1} & \xrightarrow{\beta_{r+1}} & \mathbf{T}_{r+1}^{\mathbb{L}}(\mathcal{U}\mathbf{E}_{k+m}(*)).
\end{array}$$

The outer and left squares are homotopy cocartesian, the former by construction and the latter as a homotopy pushout. Thus the right square is also homotopy cocartesian, and hence the map $\beta_{r+1}\colon \mathbf{A}_{r+1} \to \mathbf{T}_{r+1}^{\mathbb{L}}(\mathcal{U}\mathbf{F}^{\mathcal{E}_{k+m}}(*))$ is a weak equivalence. □

Combining the last sentence of this proof with Proposition 4.5, we get the description of $\mathbf{A}_r$ announced in the introduction.

Corollary 4.14 *There are weak equivalences of $\mathcal{E}_k$-algebras*

$$\mathbf{A}_r \xrightarrow{\beta_r} \mathbf{T}_r^{\mathbb{L}}(\mathcal{U}\mathbf{E}_{k+m}(*)) \xrightarrow{\alpha_r} \mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(*).$$

Remark 4.15 It is plausible that Theorem 1.2 may be deduced from results analogous to those in [8]. One would need an CW approximation theorem for $\mathcal{E}_k$-algebras in $\mathsf{Top}^{\mathbb{N}}$, and verify that one may desuspend the identification of $\Sigma^k Q_{\mathbb{L}}^{\mathcal{E}_k}(\mathbf{F}^{\mathcal{E}_{k+m}}(*))$ with the k-fold bar construction with respect to the canonical augmentation.

5 Relation to the May-Milgram filtration

We now explain the relationship between the results in the previous section and the May-Milgram filtration on $\Omega^m\Sigma^m S^k$.

Definition 5.1 Given a based topological space (X, x_0), let $C(M;X)$ be the quotient of $\bigsqcup_{n\geq 0} F_n(M) \times_{\mathfrak{S}_n} X^n$ by the relation that $(m_1, \dots m_n; x_1, \dots x_n)$ is equivalent to $(m_1, \dots m_{n-1}; x_1, \dots x_{n-1})$ if $x_n = x_0$. We call this the *configuration space of unordered points in M with labels in X.*

When $X = S^0$, we recover ordinary unordered configuration spaces and drop X from the notation. Work of Milgram and May implies that $\Omega^m\Sigma^m X$ has the weak homotopy type of $C(\mathbb{I}^m; X)$ when X is connected [15, 14]. The rth stage $\mathcal{M}_r(C(\mathbb{I}^m;X))$ of the May-Milgram filtration of $C(\mathbb{I}^m;X) \simeq \Omega^m\Sigma^m X$ is defined to be the image of $F_r(\mathbb{I}^m) \times_{\mathfrak{S}_i} X^i$ in $C(\mathbb{I}^m;X)$.

In this paper, we use only the case $X = S^k$. In that case, $\Omega^m \Sigma^m S^k$ is weakly equivalent to the k-fold delooping of $C(\mathbb{I}^k \times \mathbb{I}^m) = \mathbf{F}(\mathbb{I}^k \times \mathbb{I}^m)(*) \simeq \mathbf{E}_{k+m}(*)$. Let us denote the $\mathcal{E}_k$-algebra $\mathbf{F}^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(*)$ by $C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$.

Theorem 5.2 *The k-fold delooping of $C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ is homotopy equivalent to the rth stage in the May-Milgram filtration of $\Omega^m \Sigma^m S^k$.*

To prove this, we need to consider a generalization of $C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ where points can vanish if they enter certain regions.

Definition 5.3 Let M be a manifold and $N \subset M$ a subspace. Let $C^{[r]}(M \times \mathbb{I}^m)$ denote the subspace of $C(M \times \mathbb{I}^m)$ of configurations ξ where $\xi \cap \left(\{x\} \times \mathbb{I}^m\right)$ has cardinality $\leq r$ for all $x \in M$. Let $C^{[r]}((M, N) \times \mathbb{I}^m)$ be the quotient of $C^{[r]}(M \times \mathbb{I}^m)$ by the equivalence relation that $\xi \sim \xi'$ if $\xi \cap ((M \setminus N) \times \mathbb{I}^m) = \xi' \cap (((M \setminus N) \times \mathbb{I}^m)$.

We drop the superscript for $r = \infty$ and drop the $- \times \mathbb{I}^m$ for $m = 0$. There are two configuration space models for $\Omega^m \Sigma^m S^k = \Omega^m S^{k+m}$. The first is a special case of May's approximation theorem from [14], building on the work of Milgram in [15], and the second is a specialization of Proposition 2 of [3].

Theorem 5.4 (May) *For $k > 0$, $C(\mathbb{I}^m; S^k)$ is weakly homotopy equivalent to $\Omega^m \Sigma^m S^k$.*

Theorem 5.5 (Bödigheimer) *For $k > 0$, $C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$ is weakly homotopy equivalent to $\Omega^m \Sigma^m S^k$.*

We will relate these two models of $\Omega^m \Sigma^m S^k$, and compare filtrations of these spaces. The topological space $C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$ is filtered by the $C^{[r]}((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$. From now on, we view S^k as $\mathbb{R}^k / (\mathbb{R}^k \setminus \mathbb{I}^k)$ with base point given by the image of $\mathbb{R}^k \setminus \mathbb{I}^k$. Define a map ρ by

$$\begin{aligned} \rho \colon C(\mathbb{I}^m; S^k) &\longrightarrow C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m) \\ ((m_1; x_1), \ldots, (m_r, x_r)) &\longmapsto (x_1 \times m_1, \ldots, x_r \times m_r), \end{aligned}$$

where $m_i \in \mathbb{I}^m$ and $x_i \in \mathbb{R}^k / (\mathbb{R}^k \setminus \mathbb{I}^k) = S^k$. This inclusion has image consisting of those configurations with at most one point in each fiber of $\mathbb{R}^k \times \mathbb{I}^m \to \mathbb{I}^m$. We denote its restrictions by $\rho_r \colon \mathcal{M}_r(C(\mathbb{I}^m; S^k)) \to C^{[r]}((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$.

Lemma 5.6 *The maps ρ and ρ_r are homotopy equivalences.*

Proof The strategy is to scale the configurations so that in each fiber of $\mathbb{R}^k \times \mathbb{I}^m \to \mathbb{I}^m$ all but at most one point is pushed into $\mathbb{R}^k \setminus \mathbb{I}^k$. To do so, we pick a continuous function $\eta\colon C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m) \to (0, \infty)$ with the property that for all $\xi \in C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$ and $x \in \mathbb{I}^m$, there is at most one point in ξ that is within distance $\eta(\xi)$ of $0 \times x \in \mathbb{R}^k \times \mathbb{I}^m$. Let $\phi_t^R\colon \mathbb{R}^k \to \mathbb{R}^k$ be a continuous family of maps, depending on $t \in [0,1]$ and $R > 0$, such that:

- $\phi_0^R = \mathrm{id}$,
- $\phi_t^R|_{(\phi_t^R)^{-1}(\mathbb{I}^k)}$ a homeomorphism onto its image,
- $\phi_t^R(\mathbb{R}^k \setminus \mathbb{I}^k) \subset \mathbb{R}^k \setminus \mathbb{I}^k$ and $\phi_1^R(y) \in \mathbb{R}^k \setminus \mathbb{I}^k$ if $||y|| > R$.

Then we define

$$\begin{aligned} H\colon [0,1] \times C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m) &\longrightarrow C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m) \\ (t, \xi) &\longmapsto (\mathrm{id} \times \phi_t^{\eta(\xi)})_*(\xi), \end{aligned}$$

where the subscript $*$ means induced map on configuration spaces. For $t = 1$, all but at most one point in each fiber are pushed into $\mathbb{R}^k \setminus \mathbb{I}^k$ (where these points vanish). In particular, we can regard it as a continuous map

$$h\colon C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m) \longrightarrow C(\mathbb{I}^m; S^k).$$

The homotopy H then provides a homotopy from $\rho \circ h$ to the identity on $C((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k)$, and since it preserves the subspace $C(\mathbb{I}^m; S^k)$ also a homotopy from $h \circ \rho$ to the identity on $C(\mathbb{I}^m; S^k)$. Thus ρ is a homotopy equivalence.

Since ρ, h and H preserve the filtration, this also proves the ρ_r are homotopy equivalences. $\square$

Thus $C^{[r]}((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$ is homotopy equivalent to the rth stage of the May-Milgram filtration. We claim that the k-fold delooping of $C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ is $C^{[r]}((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$.

The k-fold bar construction of an augmented E_k-algebra is defined in full generality in Section 13.1 of [8]. We will specialize it to the $\mathcal{E}_k$-algebra $C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ in Top, with its canonical augmentation to $*$, and make a minor modification to the "grids" for the sake of computational convenience, replacing $[0,1]$ with $[1/4, 3/4]$ in the following definition:

Definition 5.7 We write $\mathcal{P}_k(p_1, \ldots, p_k) \subset \prod_j \mathbb{R}^{p_j+1}$ for the subspace of k-tuples

$$\{1/4 < t_0^j < \cdots < t_{p_j}^j < 3/4\}_{1 \leq j \leq k}.$$

We make $[p_1, \ldots, p_k] \mapsto \mathcal{P}_k(p_1, \ldots, p_k)$ into a k-fold semi-simplicial space by defining the ith face map in the jth direction by forgetting t_i^j.

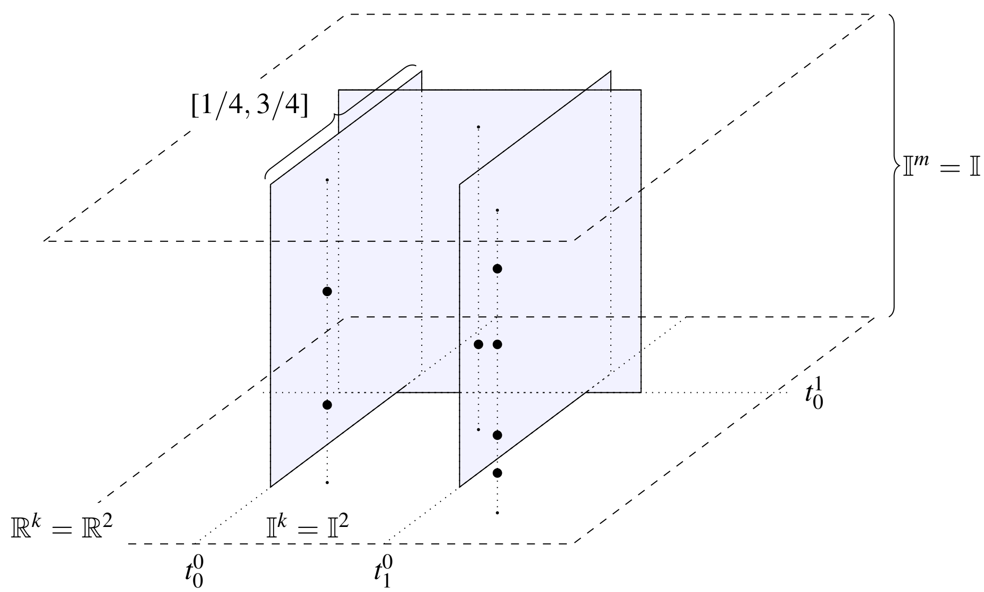

Figure 3: An element of $X_{1,0}$ for $r = 4$, $k = 2$, and $m = 1$. Points in the configuration disappear when they leave the cube $\mathbb{I}^2 \times \mathbb{I}$, cannot hit the walls $\{t_i^0\} \times [1/4, 3/4] \times \mathbb{I}$ for $i = 0, 1$ or $[1/4, 3/4] \times \{t_0^1\} \times \mathbb{I}$, and every vertical line segment can contain at most 4 points.

Definition 5.8 $B^{E_k}_{\bullet,\cdots,\bullet}(C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m))$ is the k-fold semi-simplicial space with $(p_1, \ldots, p_k)$-simplices given by the subspace of

$$(\{t_i^j\}, \xi) \in \mathcal{P}_k(p_1, \ldots, p_k) \times C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$$

such that ξ is contained in $\prod_j [t_0^j, t_{p_j}^j] \times \mathbb{I}^m$ and ξ is disjoint from $[1/4, 3/4]^{j-1} \times \{t_i^j\} \times [1/4, 3/4]^{k-j} \times \mathbb{I}^m$ for $1 \leq j \leq k$ and $0 \leq i \leq p_j$.

For $0 < i < p_j$, the ith face map in the jth direction is given by the corresponding face map on $\mathcal{P}_k$ and the identity on $C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$. The 0th face map in the jth direction is given by the corresponding face map on $\mathcal{P}_k$ and by deleting all particles in ξ which have jth coordinate $< t_1^j$. Similarly, the p_jth face map in the jth direction is given by the corresponding face map on $\mathcal{P}_k$ and by deleting all particles in ξ which have jth coordinate $> t_{p_j-1}^j$.

Definition 5.9 The k-fold delooping of $C^r(\mathbb{I}^k \times \mathbb{I}^m)$ is the pointed topological space given by

$$B^k C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m) := ||B^{E_k}_{\bullet,\cdots,\bullet}(C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m))||.$$

Combined with Lemma 5.6, the following proposition completes the proof of Theorem 5.2.

Proposition 5.10 *There is a zig-zag of weak equivalences of $\mathcal{E}_m$-algebras*

$$B^k C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m) \xleftarrow{||f_\bullet||} ||X_{\bullet,\dots,\bullet}|| \xrightarrow{\epsilon} C^{[r]}\left((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m\right).$$

Proof We start by defining the augmented k-fold semi-simplicial topological space $X_{\bullet,\cdots,\bullet}$: its topological space of $(p_1,\dots,p_k)$-simplices

$$X_{p_1,\dots,p_k} \subset \mathcal{P}_k(p_1,\dots,p_k) \times C^{[r]}\left((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m\right)$$

is the subspace of $(\{t_i^j\}, \xi)$ such that ξ is disjoint from $[1/4, 3/4]^{j-1} \times \{t_i^j\} \times [1/4,3/4]^{k-j} \times \mathbb{I}^m$ for each $1 \leq j \leq k$ and $0 \leq i \leq p_j$. This is augmented over $C^{[r]}\left((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m\right)$. The ith face map in the jth direction is given by forgetting t_i^j and the augmentation forgets all t_i^j's.

We denote the map

$$||X_{\bullet,\cdots,\bullet}|| \longrightarrow C^{[r]}\left((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m\right)$$

by ϵ. To show this is a weak equivalence, we prove it is a microfibration with weakly contractible fibers and invoke Lemma 4.8. For $\xi \in C^{[r]}\left((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m\right)$, let $S_\xi^j \subseteq \mathbb{R}$ be the subspace of $t \in (1/4, 3/4)$ such that

$$\xi \cap \left([1/4,3/4]^{j-1} \times \{t\} \times [1/4,3/4]^{k-j} \times \mathbb{I}^m\right) = \varnothing.$$

The fiber $\epsilon^{-1}(\xi)$ is the thick geometric realization of a k-fold semi-simplicial space with space of $(p_1,\cdots,p_k)$-simplices homotopy equivalent to the product of sets of order preserving-maps from $\{0,\dots,p_j\}$ to $\pi_0(S_\xi^j)$ for $1 \leq j \leq k$, which is product of simplices. Since levelwise weak equivalences induce weak equivalences on thick geometric realizations (see e.g. Theorem 2.2 of [7]), the fibers of ϵ are weakly contractible.

The proof that ϵ is a microfibration is similar to that of Lemma 4.10. The key fact is that if

$$\xi \cap \left([1/4,3/4]^{j-1} \times \{t^j\} \times [1/4,3/4]^{k-j} \times \mathbb{I}^m\right) = \varnothing,$$

the same will be true for nearby configurations (this is why we use $[1/4, 3/4]$ instead of $\mathbb{I}$, otherwise new points could appear and hit the forbidden regions immediately).

We will next construct a k-fold semi-simplicial map

$$f_\bullet \colon X_{\bullet,\dots,\bullet} \longrightarrow B^{E_k}_{\bullet,\cdots,\bullet}(C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m))$$

and prove its thick geometric realization is a weak equivalence. The map $f_{p_1,\dots,p_k}$ is defined on a $(p_1,\dots,p_k)$-simplex $(\{t_i^j\},\xi)$ by deleting from a configuration $\xi \in$

$C^{[r]}\left((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m\right)$ those points outside $\prod_{j=1}^k [t_0^j, t_{p_j}^j]$, and interpreting the remaining configuration as an element of $C^{[r]}\left(\mathbb{I}^k \times \mathbb{I}^m\right)$. Since we are taking thick geometric realizations, to prove $||f_\bullet||$ is a weak equivalence, it suffices to prove each $f_{p_1,\cdots,p_k}$ is a weak homotopy equivalence.

To prove this, we first observe that the inclusion of the subspace $X'_{p_1,\ldots,p_k}$ of $X_{p_1,\ldots,p_k}$ of those $(\{t_i^j\}, \xi)$ such that for all $1 \leq j \leq k$ we have

$$\xi \cap \left([1/4, 3/4]^{j-1} \times ([1/4, t_0^j] \cup [t_{p_j}^j, 3/4]) \times [1/4, 3/4]^{k-j} \times \mathbb{I}^m\right) = \varnothing,$$

is a homotopy equivalence. In other words, in $X'_{p_1,\cdots,p_k}$ all points in ξ lie either in $\prod_{j=1}^k [t_0^j, t_{p_j}^j]$ or have one of their first k coordinates $< 1/4$ or $> 3/4$.

Thus it suffices to prove that the inclusion

$$g_{p_1,\cdots,p_k} \colon B^{E_k}_{p_1,\cdots,p_k}(C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)) \longrightarrow X'_{p_1,\ldots,p_k},$$

which regards a configuration in $C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$ as one in $C^{[r]}((\mathbb{R}^k, \mathbb{R}^k \setminus \mathbb{I}^k) \times \mathbb{I}^m)$, is a homotopy equivalence. Then the composition $f_{p_1,\ldots,p_k} \circ g_{p_1,\cdots,p_k}$ is the identity on $B^{E_k}_{p_1,\cdots,p_k}(C^{[r]}(\mathbb{I}^k \times \mathbb{I}^m))$. A homotopy from $g_{p_1,\ldots,p_k} \circ f_{p_1,\cdots,p_k}$ to the identity on $X'_{p_1,\ldots,p_k}$ is given as follows: it is the identity on the points in ξ in $\prod_{j=1}^k [t_0^j, t_{p_j}^j]$ and pushes the remaining points linearly outwards from $(1/2, \cdots, 1/2)$ until all are in the regions $\mathbb{R}^k \setminus \mathbb{I}^k$ where they vanish. □

Remark 5.11 Snaith showed that the May-Milgram filtration stably splits [17]. However, its lift to a filtration of $\mathbf{E}_{k+m}(*)$ of $\mathcal{E}_k$-algebras *does not* split after taking suspension spectra. Such a splitting would imply that $C_2(\mathbb{I}^2 \times \mathbb{I}^1) \simeq \mathbb{R}P^2$ stably splits off a copy of $F_2^{[1]}(\mathbb{I}^2 \times \mathbb{I}^1)/\mathfrak{S}_2 \simeq \mathbb{R}P^1$.

However, this filtration *does* split after stabilizing in a different manner. Recall $Q_{\mathbb{L}}^{\mathcal{E}_k}$ denotes the derived indecomposables functor of Remark 3.7. Basterra-Mandell showed that derived indecomposables can be considered as stabilization of an algebra over an operad [1], and derived indecomposables of an $\mathcal{E}_k$-algebra may be computed using its k-fold bar construction, see [2] or Chapter 13 of [8]. Thus the induced filtration of the stabilization $Q_{\mathbb{L}}^{\mathcal{E}_k}(\Sigma^\infty \mathbf{E}_{k+m}(*)_+)$ agrees with the suspension spectra of the May-Milgram filtration and hence splits by the work of Snaith [17].

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