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Simple $\mathcal{D}$-module components of local cohomology modules

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ABSTRACT

For a projective variety $V \subset \mathbb{P}^n_k$ over a field of characteristic zero, with homogeneous ideal I in $A = k[x_0, \ldots, x_n]$, we consider the local cohomology modules $H^i_I(A)$. These have a structure of holonomic $\mathcal{D}$-module over A, and we investigate their filtration by simple $\mathcal{D}$-modules. In case V is nonsingular, we can describe completely the simple $\mathcal{D}$-module components of $H^i_I(A)$ for all i, in terms of the Betti numbers of V.

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1. Introduction

The local cohomology groups $H^i_Y(X,\mathcal{F})$ of an Abelian sheaf $\mathcal{F}$ on a topological space X, with support in a closed subset Y, were introduced by Grothendieck in 1961 in his Harvard and Paris seminars [8], [9]. If X is the spectrum of a ring A, and Y is the closed subset associated to an ideal I, and $\mathcal{F}$ is the sheaf of sections of an A-module M, these groups are denoted $H^i_I(M)$. The latter can be computed algebraically as the derived functors of the functor $\Gamma_I(\bullet)$ that to each A-module M associates the submodule of elements with support in I, that is, that are annihilated by some power of I. In the 50 years since their introduction, these groups and modules have found wide application in algebraic geometry and in commutative algebra.

Even when the ring A is Noetherian and the module M finitely generated, the local cohomology modules $H^i_I(M)$ are rarely finitely generated. If A is a local ring with maximal ideal $\mathfrak{m}$, then at least the modules $H^i_{\mathfrak{m}}(M)$ are *cofinite*, meaning that they satisfy the descending chain condition, or equivalently, that $\operatorname{Hom}_A(k, H^i_{\mathfrak{m}}(M))$ is a finite dimensional k-vector space, where $k = A/\mathfrak{m}$ is the residue field of A. Grothendieck asked whether for any ideal I, the modules $H^i_I(M)$ might be *I-cofinite* in the sense that $\operatorname{Hom}_A(A/I, H^i_I(M))$ is finitely generated. This turns out not to be so in general [4], though an analogous property does hold in the derived category. Thus the search for finiteness properties of these modules has remained elusive.

In another direction, Grothendieck observed, in a letter to Atiyah in 1963 [7], that the ordinary cohomology of a nonsingular algebraic variety over the complex numbers could be computed purely algebraically using the de Rham complex of algebraic differential forms. This led to a theory of algebraic de Rham cohomology and homology for schemes of finite type over a field of characteristic zero. Grothendieck also asked for a theory of sheaves with integrable connection (now called $\mathcal{D}$-modules) to explain the Gauss–Manin connection on sheaves of relative cohomology. The details of these theories have been worked out, with all the usual functorial properties, purely algebraically, in a paper of the first author [5].

Meanwhile, in the early 1970s, as part of their study of systems of partial differential equations, Sato and Kashiwara in Kyoto developed an extensive theory of holomorphic $\mathcal{D}$-modules on complex manifolds. It is already clear from Kashiwara's thesis [11] that local cohomology modules of $\mathcal{D}$-modules are again $\mathcal{D}$-modules. This applies in particular to the structure sheaf, which carries a natural structure of $\mathcal{D}$-module. Thus, if A is a polynomial ring or a power series over a field k of characteristic zero, and I is an ideal in A, then the local cohomology modules $H^i_I(A)$ are finitely generated as $\mathcal{D}$-modules, where $\mathcal{D}$ is the (non-commutative) ring of differential operators over A. Moreover, they are *holonomic* $\mathcal{D}$-modules, which implies that they are of finite length as $\mathcal{D}$-modules, and have a finite composition series whose factors are simple $\mathcal{D}$-modules.

This extra structure on the local cohomology modules that arise in commutative algebra was not noticed by algebraists until Lyubeznik, in a seminal paper [12, Theorem 3.4],

exploited it to show, for instance, that the local cohomology modules $H^i_I(A)$ can have only finitely many associated primes.

Our basic question in this paper is, if $V \subset \mathbb{P}^n_k$ is an algebraic variety, with homogeneous ideal I in the polynomial ring $A = k[x_0, \dots, x_n]$, knowing that the local cohomology modules $H^i_I(A)$ are holonomic $\mathcal{D}$-modules, what are their simple $\mathcal{D}$-module components, and what can we learn about the geometry of V from them?

If V is a variety of codimension r, it is easy to see that $H^i_I(A) = 0$ for $i < r$. In his thesis [15], Ogus found conditions for the cofiniteness and vanishing of those modules in the "non-critical" range $i > r$, in terms of the algebraic de Rham cohomology of V, but said nothing about the large "critical" case $H^r_I(A)$ where r is the codimension. Generalizing his method, we compute in Section 4 the de Rham cohomology groups $H^j_{DR}(M)$ for the $\mathcal{D}$-module $M = H^i_I(A)$, for each i and j, in terms of the algebraic de Rham cohomology of V. This allows us to recover the result of Ogus on the vanishing and cofiniteness of $H^i_I(A)$ in the non-critical range $i > r$.

To interpret this information, in Section 5, which is the technical heart of our paper, we prove that for a holonomic $\mathcal{D}$-module M over a power series ring $R = k[\![x_1, \dots, x_n]\!]$, the dimension of the top de Rham cohomology module $H^n_{DR}(M)$ is equal to the largest number of copies of the injective envelope E of k over R whose direct sum E^m can appear as a quotient $\mathcal{D}$-module of M. This result, while apparently dual in some way to the elementary statement that $H^0_{DR}(M)$ gives the rank of the largest free $\mathcal{D}$-module R^m that can appear as a submodule of M, is not at all easy to prove. We use an extension of methods employed by van den Essen [20–22], which he used to show that the de Rham cohomology modules $H^j_{DR}(M)$ of a holonomic $\mathcal{D}$-module are all finite dimensional.

Our main result is that if V is a nonsingular subvariety of $\mathbb{P}^n_k$ of codimension r, then its critical local cohomology module $H^r_I(A)$ has a simple sub-$\mathcal{D}$-module with support along V, with quotient a direct sum E^m of copies of E, where m is determined by the Betti numbers of V in the sense of algebraic de Rham cohomology (Theorem 6.4). In particular, if V is a nonsingular curve of genus g, then $m = 2g$, so that for a rational curve $V \subset \mathbb{P}^n_k$, the $\mathcal{D}$-module $H^{n-1}_I(A)$ is simple. The only other result we know giving the simple $\mathcal{D}$-module composition of $H^r_i(A)$ for a projective variety V is the theorem of Raicu [16], which shows that for V the d-uple embedding of another projective space in $\mathbb{P}^n_k$, the corresponding $\mathcal{D}$-module is simple. His result, proved by representation theory, is recovered by ours.

In the non-critical case $i > r$, once one knows that a module $H^i_I(A)$ is cofinite, it is easy to see that as a $\mathcal{D}$-module it is a direct sum of copies of E, the injective hull of k over A. As a by-product of our method we also compute the number of copies of E in these direct sums, sometimes called Lyubeznik numbers. These results have been previously obtained by other authors, mostly in the context of complex-analytic spaces (see Remark 4.9). We include them here for completeness.

Experts in $\mathcal{D}$-module theory may also be able to find analogues of our main result in the complex-analytic category and perhaps even recover our result from them.

However, the importance of our work is that we develop the theory entirely algebraically. We make no use of the extensive theory of analytic $\mathcal{D}$-modules, of perverse sheaves and intersection cohomology, nor of the Riemann–Hilbert correspondence that relates holomorphic holonomic $\mathcal{D}$-modules to algebraic holonomic $\mathcal{D}$-modules with regular singularities.

2. The language of algebraic $\mathcal{D}$-modules

Let k be a field of characteristic zero, and let R be either the polynomial ring $k[x_1, \dots, x_n]$ or the formal power series ring $k[\![x_1, \dots, x_n]\!]$. Let $\mathcal{D}$ be the ring of differential operators $R < \partial_1, \dots, \partial_n >$, where ∂_i is the partial derivative $\partial/\partial x_i$. This is a non-commutative ring with the relations $\partial_i x_i = x_i \partial_i + 1$ for each i. An R-module M, together with a left action of $\mathcal{D}$ on M, will be called a $\mathcal{D}$-module. We will use the books of Björk [2] and Hotta et al. [6] as our basic references.

One can define the *dimension* of a finitely generated $\mathcal{D}$-module. It is an integer between n and $2n$. The $\mathcal{D}$-modules with minimal dimension n are called *holonomic* $\mathcal{D}$-modules. They are of finite length as $\mathcal{D}$-modules and therefore have a filtration whose quotients are simple $\mathcal{D}$-modules. (For the polynomial ring case, see [2, 1.5.3], where these modules are also called modules in the *Bernstein class*. For the power series case, see [2, 2.7.13 and the remarks just before 3.3.1].)

Let $\Omega_{R/k}$ be the module of differentials over R, generated by $dx_1, \dots, dx_n$, and let $\Omega^i_{R/k}$ be its exterior power. If M is a $\mathcal{D}$-module, the actions of ∂_i on M give rise to a complex $M \otimes_R \Omega^\bullet$ of R-modules and k-linear maps, called the *de Rham complex* of M. Its cohomology groups will be denoted by $H^i_{DR}(M)$. If M is a holonomic $\mathcal{D}$-module, then $H^i_{DR}(M)$ are finite-dimensional k-vector spaces. (In the polynomial ring case, the proof is not difficult [2, 1.6.1]. In the power series case, however the question is difficult, and was left as an open problem in Björk's book. It was proved later by van de Essen [22, 2.2] as a consequence of his inductive result that if M is holonomic, then for a suitable choice of coordinates, $M/\partial_n M$ will also be holonomic over the power series ring in $n-1$ variables.)

Our interest in $\mathcal{D}$-modules comes from the following theorem of Kashiwara.

Theorem 2.1. *If M is a holonomic $\mathcal{D}$-module over the polynomial ring or the power series ring R as above and if I is an ideal of R, then the local cohomology modules $H^i_I(M)$ have natural structures of holonomic $\mathcal{D}$-modules.*

Proof. (See also [12, 2.2].) One first shows that if M is holonomic over R and $f \in R$, then the localized module M_f is also holonomic. (For the polynomial ring case, see for instance [3, 3.4.1]. For the power series case, see [2, 3.4.1]). Then if $f_1, \dots, f_s$ is a set of generators of I, we can compute the local cohomology modules $H^i_I(M)$ from the Čech complex formed of the localizations of M at products of the f_i. Since kernels, images, and quotients of holonomic modules are holonomic, it follows that the $H^i_I(M)$ are holonomic. □

Examples 2.2.

(1) The ring R itself is a holonomic $\mathcal{D}$-module and is in fact simple. To see this, we note that any element of R generates R as a $\mathcal{D}$-module. Indeed, just differentiate enough times so that the element becomes a unit, then multiply by the other elements of R.
(2) Another important example is $E = H^n_{\mathfrak{m}}(R)$, where $\mathfrak{m} = (x_1, \dots, x_n)$. This is an injective hull of k over R, and is also a simple $\mathcal{D}$-module. As in (1) above, any element generates the whole module. Just multiply by enough x_i to arrive at the socle $x_1^{-1} \cdots x_n^{-1}$, then differentiate to get any other monomial in E.
(3) The de Rham cohomology of the $\mathcal{D}$-module R is equal to k in degree 0 and 0 otherwise. This is a consequence of the algebraic Poincaré lemma [5, II, 7.1].
(4) The de Rham cohomology of the $\mathcal{D}$-module E is k in degree n, and 0 otherwise. Look first at the case $n = 1$, when $E = A_x/A$ is the k-vector space generated by the negative powers of x. Consider the map $\varphi : R = k[x] \longrightarrow E$ defined by $\varphi(x^\ell) = \ell! x^{-\ell-1}$. This is an isomorphism of k vector spaces (where we use the convention that $0! = 1$). Note that by construction $\partial\varphi = -\varphi x$. Thus φ gives an isomorphism from the complex $R \xrightarrow{x} R$ to the complex $E \xrightarrow{\partial} E$. Taking the tensor product (over k) of this isomorphism of complexes over $k[x_i]$, we obtain an isomorphism of the Koszul complex for $R = k[x_1, \dots, x_n]$ with respect to $x_1, \dots, x_n$ and the Rham complex for E. Hence $H^n_{DR}(E) = k$, and the others are zero. For the case when R is a power series ring, notice that the complex $E \otimes \Omega^\bullet$ is the same as in the polynomial ring case.
(5) If M is a holonomic $\mathcal{D}$-module whose support, as an R-module, is at the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, then M is the direct sums of a finite number of copies of E. This follows for example from Kashiwara's equivalence [6, 1.6.1, 1.6.4], or one can prove it directly as in [12, 2.4.a].
(6) For any holonomic $\mathcal{D}$-module M, its zeroth de Rham cohomology $H^0_{DR}(M)$ has dimension equal to the rank t of the largest trivial sub-$\mathcal{D}$-module $M_0 = R^t$ of M. Indeed, if $m \in H^0_{DR}(M)$, then the natural map $R \to M$ defined by $a \mapsto am$ is an injective $\mathcal{D}$-module homomorphism. One of our main results, Theorem 5.1, is a non-trivial analogous statement about the last de Rham cohomology group $H^n_{DR}(M)$.

3. Algebraic de Rham cohomology and homology

In this section we recall the basic definitions and properties of algebraic de Rham cohomology and homology that we will use in this paper. Our basic references will be Grothendieck [7] and Hartshorne [5].

Let Y be a closed subscheme of a scheme X smooth and of finite type of dimension n over an algebraically closed field k of characteristic zero. Let $\Omega^\bullet_{X/k}$ be the de Rham complex with k-linear maps

$$\mathcal{O}_X \to \Omega^1_X \xrightarrow{d} \Omega^2_X \to \cdots \to \Omega^n_X \,.$$

We define the algebraic de Rham *homology* of Y to be

$$H_i^{DR}(Y) = \mathrm{H}_Y^{2n-i}(X, \Omega_X^\bullet)\,,$$

namely the local hyper-cohomology with support in Y of the complex $\Omega_X^\bullet$ [5, II.3].

We define the algebraic de Rham *cohomology* of Y by passing to the formal completion $\mathfrak{X}$ of X along Y and taking hyper-cohomology

$$H_{DR}^i(Y) = \mathrm{H}^i(\mathfrak{X}, \hat{\Omega}_{\mathfrak{X}}^\bullet)\,,$$

of the formal completion of $\Omega_X^\bullet$ along Y [5, II.1]. The main properties of these groups are summarized in the following theorem:

Theorem 3.1. *Let Y be a scheme of finite type over k, embeddable in a scheme X smooth over k.*

(1) The groups $H_i^{DR}(Y)$ and $H_{DR}^i(Y)$ are independent of the embedding of Y in a smooth scheme X.
(2) The groups $H_i^{DR}(Y)$ and $H_{DR}^i(Y)$ are finite-dimensional k-vector spaces.
(3) The groups $H_i^{DR}(Y)$ and $H_{DR}^i(Y)$ are all zero for $i < 0$ and $i > 2d$, where $d = \dim Y$.
(4) If Y is proper over k, then $H_i^{DR}(Y) \cong H_{DR}^i(Y)'$, where $'$ denotes the dual k-vector space.
(5) If Y is smooth over k, then $H_i^{DR}(Y) \cong H_{DR}^{2d-i}(Y)$.
(6) If Z is a closed subset of Y, then there is a long exact sequence of homology

$$\cdots \longrightarrow H_i^{DR}(Z) \longrightarrow H_i^{DR}(Y) \longrightarrow H_i^{DR}(Y - Z) \longrightarrow H_{i-1}^{DR}(Z) \longrightarrow \cdots$$

(7) If $k = \mathbb{C}$, then $H_{DR}^i(Y) \cong H^i(Y^{an}, \mathbb{C})$, the usual complex cohomology of the associated complex-analytic space Y^{an}, and $H_i^{DR}(Y)$ calculates the Borel–Moore homology of Y^{an}.

Proof. These are all in [5]. Item (1) is II.1.4 and II.3.2; item (2) is II.6.1; item (3) is II.7.2.; item (4) is II.5.1; item (5) is II.3.4; item (6) is II.3.3; and finally, item (7) is IV.1.1 and 1.2. □

Example 3.2.

(1) If $Y = \mathbb{A}_k^n$, the affine n-space, then by definition $H_{DR}^i(Y)$ is just the de Rham cohomology $H_{DR}^i(R)$ of the polynomial ring $R = k[x_1, \ldots, x_n]$ as a $\mathcal{D}$-module, which is k in degree 0 and 0 otherwise (see Example 2.2 (3)). Since Y is smooth over k its homology $H_i^{DR}(Y)$ is k in degree $2n$ and 0 otherwise (see Theorem 3.1 (5)).

(2) If $Y = \mathbb{P}^n_k$, we can show inductively that $H^i_{DR}(Y) = k$ for each i even, $0 \leq i \leq 2n$, and that $H^i_{DR}(Y) = 0$ for each i odd. The same is true for homology. Just start with $\mathbb{P}^0_k = \mathbb{A}^0_k$, which has k in degree 0, and use the long exact sequence of Theorem 3.1(6) with $Z = \mathbb{P}^{n-1}_k \subset Y = \mathbb{P}^n_k$ and $Y - Z = \mathbb{A}^n_k$ to find first the homology, and then use Theorem 3.1(4) to obtain the cohomology of $\mathbb{P}^n_k$.

Proposition 3.3. (*Lichtenbaum theorem for algebraic de Rham cohomology*)

(1) Let Y be a scheme of dimension d over k. Then $H^{2d}_{DR}(Y) \neq 0$ if and only if at least one irreducible component of Y is proper over k.
(2) For Y any scheme, $H^{DR}_0(Y) \neq 0$ if and only if Y has at least one connected component that is proper over k.

Proof. This is a straightforward consequence of the result of Theorem 3.1, together with the Mayer–Vietoris sequences and the exact sequences of a birational morphism [5, 4.1, 4.2, 4.4, 4.5]. □

Proposition 3.4. *Let $\mathcal{C}$ be a nonsingular projective curve of genus g over k. Then the dimension of the de Rham cohomology groups are $h^0_{DR}(\mathcal{C}) = h^2_{DR}(\mathcal{C}) = 1$ and $h^1_{DR}(\mathcal{C}) = 2g$. The homology groups are the same.*

Proof. In this case, the de Rham complex is just $\mathcal{O}_{\mathcal{C}} \xrightarrow{d} \Omega^1_{\mathcal{C}}$.

There is a spectral sequence

$$E_1^{pq} = H^q(\mathcal{C}, \Omega^p) \Longrightarrow E^n = H^n_{DR}(\mathcal{C}).$$

The first d_1-map is $H^0(\mathcal{O}_{\mathcal{C}}) \to H^0(\Omega^1_{\mathcal{C}})$, which is zero, because the only global sections of $\mathcal{O}_{\mathcal{C}}$ are constants, and their derivative is zero. Hence $H^0_{DR}(\mathcal{C}) = k$. By duality, see Theorem 3.1(4) and (5), we see also that $H^2_{DR}(\mathcal{C}) = k$. Hence the other d_1-map $H^1(\mathcal{O}_{\mathcal{C}}) \to H^1(\Omega^1_{\mathcal{C}})$ must also be zero and so $H^1_{DR}(\mathcal{C}) \cong H^1(\mathcal{O}_{\mathcal{C}}) \otimes H^0(\Omega^1_{\mathcal{C}})$, which has dimension $2g$. The result for homology then follows from Theorem 3.1(5). □

Proposition 3.5. *Let $\mathcal{C}$ be an integral projective curve over k. For each singular point $P \in \mathcal{C}$, let n_P be the number of branches of $\mathcal{C}$ at P, that is, the number of points of the normalization $\tilde{\mathcal{C}}$ of $\mathcal{C}$ lying over P. Then $h^0_{DR}(\mathcal{C}) = h^2_{DR}(\mathcal{C}) = 1$ and $h^1_{DR}(\mathcal{C}) = 2g + \sum_{P \in \mathcal{C}}(n_P - 1)$, where g is the genus of the normalization $\tilde{\mathcal{C}}$. The homology groups are the same.*

Proof. We use the exact sequence of homology for a proper birational morphism [5, II,4.5] applied to the projection $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$. Let Z be the singular locus of $\mathcal{C}$, and let Z' be its inverse image in $\tilde{\mathcal{C}}$. Then we have

$$\cdots \longrightarrow H^{DR}_i(Z') \longrightarrow H^{DR}_i(Z) \oplus H^{DR}_i(\tilde{\mathcal{C}}) \longrightarrow H^{DR}_i(\mathcal{C}) \longrightarrow H^{DR}_{i-1}(Z') \longrightarrow \cdots$$

Since $\tilde{\mathcal{C}}$ is smooth and projective, its homology (Proposition 3.4) has dimension $h_i^{DR}(\tilde{\mathcal{C}}) = 1, 2g, 1$ for $i = 0, 1, 2$ respectively. The homology of Z and Z' is in degree 0 only, and is just the number of points in each. Thus

$$h_1^{DR}(\mathcal{C}) = h_1^{DR}(\tilde{\mathcal{C}}) + \#Z' - \#Z\,,$$

which gives the result. The same holds for the cohomology of $\mathcal{C}$ by Theorem 3.1(4). □

Remark 3.6. Of course Proposition 3.4 and Proposition 3.5 could have been proved by using the comparison theorem (see Theorem 3.1(7)) and the well-known results about the cohomology of compact Riemann surfaces, but we wished to keep our exposition purely algebraic.

4. Local cohomology of a projective variety

Now we come to the main subject of our investigation. Let V be a closed subscheme of the projective space $\mathbb{P}_k^n$ over an algebraically closed field k of characteristic zero. Let V have codimension r. Let $A = k[x_0, \dots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}_k^n$ and let I be the homogeneous ideal of V in A. We propose to investigate the local cohomology modules $H_I^i(A)$. We keep these notations throughout this section.

Proposition 4.1. (*The non-critical case*) *Let V be an equidimensional closed subscheme of $\mathbb{P}_k^n$ of codimension r. Let I be the homogeneous ideal of V in $A = k[x_0, \dots, x_n]$.*

(1) $H_I^i(A) = 0$ for $i < r$ and for $i > n + 1$.
(2) If V is a set-theoretic complete intersection in $\mathbb{P}_k^n$, then $H_I^i(A) = 0$ for $i > r$.
(3) If V is a local complete intersection scheme, then for all $r < i \leq n + 1$, $H_I^i(A)$ has support at the irrelevant maximal ideal $\mathfrak{m} = (x_0, \dots, x_n)$ of A.

Proof. Since V has codimension r, the ideal I has height r and hence contains a regular sequence of length r for A. The first part of assertion (1) now follows from the characterization of I-depth in terms of local cohomology and the second part holds because the dimension of the ring is $n + 1$. For (2), notice that in this case there is an ideal $(f_1, \dots, f_r)$ generated by r elements having the same radical as I, so computing local cohomology using the Čech complex we obtain $H_I^i(A) = 0$ for $i > r$. Assertion (3) is a result of Ogus [15, 4.1, 4.3] proved using a local version of (2). □

Next we will make use of the $\mathcal{D}$-module structure on the local cohomology modules $H_i^i(A)$.

Proposition 4.2. *Let $X = \mathbb{A}^{n+1}$ be* Spec A*, let I be an ideal of A, and let $Y =$* Spec (A/I)*. Then there is a spectral sequence*

$$E_2^{pq} = H^p_{DR}(H^q_I(A)) \Longrightarrow H^{DR}_{2n+2-p-q}(Y)$$

relating the de Rham cohomology of the $\mathcal{D}$-modules $H^i_I(A)$ to the algebraic de Rham homology of the scheme Y.

Proof. We compute the algebraic de Rham homology of Y using its embedding in X, so that by definition (see Section 3)

$$H_i^{DR}(Y) = \mathrm{H}_Y^{2n+2-i}(X, \Omega_X^\bullet)\,.$$

(Note the shift by $2n+2$ since X has dimension $n+1$). Then we use the spectral sequence of local hyper-cohomology of $\Omega_X^\bullet$, which is

$$E_1^{pq} = H^q_Y(X, \Omega^p_X) \Longrightarrow E^{p+q} = \mathrm{H}_Y^{p+q}(X, \Omega_X^\bullet)\,.$$

Since Ω^p_X is a free $\mathcal{O}_X$-module, for each q we can write

$$E_1^{pq} = H^q_I(A) \otimes \Omega^p_X\,.$$

Thus for q fixed, the row E_1^{pq} with differentials d_1^{pq} becomes the de Rham complex of the $\mathcal{D}$-module $H^q_I(A)$, and its homology the E_2^{pq} terms, become the de Rham cohomology $H^p_{DR}(H^q_I(A))$. Thus the spectral sequence of the proposition is the same spectral sequence, but starting with the E_2 page. □

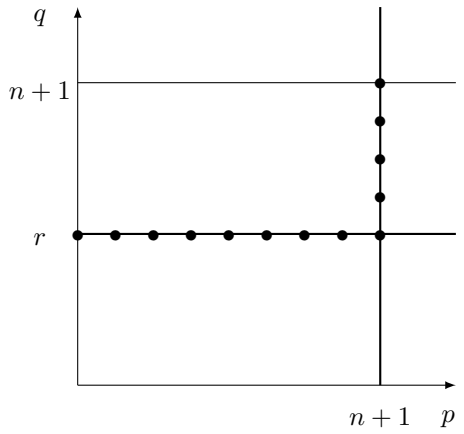
And now, we will see that when V is a local complete intersection, the spectral sequence of Proposition 4.2 degenerates.

Theorem 4.3. *Let V be a local complete intersection in $\mathbb{P}^n_k$, equidimensional of codimension r, and let $Y \subset X = \mathbb{A}^{n+1}$ be defined by the homogeneous ideal I of V. Then*

(1) $H^j_{DR}(H^r_I(A)) = H^{DR}_{2n+2-j-r}(Y)$ for $0 \le j \le n+1$, and
(2) $H^{n+1}_{DR}(H^i_I(A)) = H^{DR}_{n+1-i}(Y)$ for $r \le i \le n+1$.

All other values of $H^j_{DR}(H^i_I(A))$ are zero, as are all other values of $H_i^{DR}(Y)$.

Proof. First of all, $H^i_I(A) = 0$ for $i < r$ by Proposition 4.1(1). Next, for $i > r$ we know that $H^i_I(A)$ has support at the maximal ideal by Proposition 4.1(3), and hence is isomorphic to a direct sum of copies of E by Example 2.2(5). In that case, $H^j_{DR}(H^i_I(A)) = 0$ except for $j = n+1$, by Example 2.2(4). Thus the only possible non-zero initial terms of the spectral sequence of Proposition 4.2 are for $q = r$ and $0 \le p \le n+1$, or for $p = n+1$ and any $r \le q \le n+1$. This gives a curios L-shaped spectral sequence.



There are no non-trivial d_2 maps so the spectral sequence degenerates and gives the isomorphism of the theorem. □

Corollary 4.4. *Let V, I, Y be as in Theorem 4.3. Then $H^j_{DR}(H^r_I(A)) = 0$ for $j < r$.*

Proof. This is because Y, being a scheme of dimension $n - r + 1$, has no homology in degrees $> 2n - 2r + 2$, by Theorem 3.1(3). □

Corollary 4.5. *Let V, I, Y be as in Theorem 4.3. For $i > r$ the $\mathcal{D}$-module $H^i_I(A)$ is isomorphic to E^{m_i} where $m_i = \dim_k H^{DR}_{n+1-i}(Y)$.*

Proof. We have seen in the proof of Theorem 4.3 that for $i > r$ we have $H^i_I(A) \cong E^{m_i}$ for some m_i. Since $H^j_{DR}(E) = 0$ for $j \neq n+1$ and k for $j = n+1$, this number m_i is the dimension of $H^{n+1}_{DR}(H^i_I(A))$, which is equal to the dimension of $H^{DR}_{n+1-i}(Y)$, according to the theorem. □

Our next task is to relate the algebraic de Rham homology of Y to that of V.

Proposition 4.6. *With the hypotheses of Theorem 4.3, assume furthermore that V is connected, of dimension $d \geq 1$. Then*

(1) $H^{DR}_0(Y) = H^{DR}_1(Y) = 0$,
(2) $H^{DR}_2(Y) \cong H^{DR}_1(V)$,
(3) $H^{DR}_{2d+2}(Y) \cong H^{DR}_{2d}(V)$, *and*
(4) for $3 \leq i \leq 2d+1$ *the homology of Y is determined by the exact sequence*

$$0 = H^{DR}_{2d+1}(V) \xrightarrow{h} H^{DR}_{2d-1}(V) \to H^{DR}_{2d+1}(Y) \to H^{DR}_{2d}(V) \to \dots$$
$$\to H^{DR}_3(Y) \to H^{DR}_2(V) \xrightarrow{h} H^{DR}_0(V) \to 0$$

where h denotes cap-product with the hyperplane class.

Proof. We use a method of proof similar to [5, II, 3.2] but with homology instead of cohomology. The first step is to compare the homology of Y to that of $Y - P$, where P corresponds to $\mathfrak{m} = (x_0, \dots, x_n)$ in A. Since P only has homology in degree zero, the exact sequence of Theorem 3.1(6) gives an exact sequence

$$0 \to H_1^{DR}(Y) \to H_1^{DR}(Y - P) \to H_0^{DR}(P) \to H_0^{DR}(Y) \to H_0^{DR}(Y - P) \to 0\,,$$

and isomorphisms

$$H_i^{DR}(Y) \cong H_i^{DR}(Y - P) \quad \text{for } \ i \geq 2$$

Next we note that $Y - P$ is isomorphic to the geometric vector bundle $\mathbb{V}(\mathcal{O}_V(-1))$ minus its zero section, so we can apply the Thom–Gysin sequence [5, II, 7.9.3] to obtain a long exact sequence

$$\dots \to H_i^{DR}(V) \xrightarrow{h} H_{i-2}^{DR}(V) \to H_i^{DR}(Y - P) \to H_{i-1}^{DR}(V) \to \dots$$

where h is the cap-product in homology [5, II, 7.4]. From the last terms of this sequence it follows that $H_0^{DR}(Y - P) = 0$ and $H_1^{DR}(Y - P) \cong H_0^{DR}(V) = k$, since V is connected (see also Proposition 3.3). Since $H_0^{DR}(P) = k$ and $H_0^{DR}(Y) = 0$ by Proposition 3.3, the earlier sequence now implies that $H_1^{DR}(Y) = 0$. Since V has dimension ≥ 1, it follows that the cap product $h : H_2^{DR}(V) \to H_0^{DR}(V)$ is surjective, and so $H_2^{DR}(Y) \cong H_2^{DR}(Y - P) \cong H_1^{DR}(V)$. (To see that h is surjective, note that it is dual to the cup-product $H_{DR}^0(V) \to H_{DR}^2(V)$, and the image of the generator of $H_{DR}^0(V)$ is the hyperplane class in $H_{DR}^2(V)$, which, having self-intersection equal to the degree of V must be non-zero.) Now using $H_i^{DR}(Y) \cong H_i^{DR}(Y - P)$ for $i \geq 3$ gives the desired assertions (3) and (4). □

Corollary 4.7. [*Ogus,* [15, 4.4]] *Let V be a local complete intersection in $\mathbb{P}_k^n$, equidimensional of codimension r, connected, of dimension $d \geq 1$, with homogeneous ideal I in A. Then the groups $H_I^i(A)$ are zero for all $i > r$ if and only if the restriction maps*

$$H_{DR}^j(\mathbb{P}^n) \longrightarrow H_{DR}^j(V)$$

are isomorphisms for all $j < n - r$.

Proof. By Theorem 4.3 it follows that the $H_I^i(A) = 0$ for all $i > r$ is equivalent to $H_j^{DR}(Y) = 0$ for all $j < n + 1 - r$. This in turn, by Proposition 4.6, is equivalent to saying that the cap-product $h : H_j^{DR}(V) \to H_{j-2}^{DR}(V)$ is an isomorphism for all $j < n - r$ and surjective for $j = n - r$. By duality (see Theorem 3.1(4)) this is equivalent to saying that the cup-product $H_{DR}^{j-2}(V) \to H_{DR}^j(V)$ is an isomorphism for all $j < n - r$ and injective for $j = n - r$. Beginning with $H_{DR}^{-1}(V) = 0$ and $H_{DR}^0(V) = k$, and using the fact that the cohomology of projective space is 0 in odd degrees and k in even degrees generated by the hyperplane class $h \in H_{DR}^2(\mathbb{P}^n)$ (see Example 3.2(2)), our calculation is

equivalent to saying that the restriction map $H^j_{DR}(\mathbb{P}^n) \to H^j_{DR}(V)$ is an isomorphism for all $j < n - r$. □

As an illustration of these results, we gather together our conclusions for a nonsingular variety.

Theorem 4.8. *Let V be a nonsingular irreducible variety in $\mathbb{P}^n_k$ of codimension r and dimension $d = n-r \geq 1$, with homogeneous ideal I in A. Then writing $b_i = \dim H^{DR}_i(V)$ for the Betti numbers of V, we have*

(1) (Non-critical case) $H^i_I(A) = E^{m_i}$ with $m_i = b_{n-i} - b_{n-i-2}$ for $r < i < n$ and $H^i_I(A) = 0$ for $i \geq n$.
(2) (Critical case $i = r$) $H^j_{DR}(H^r_I(A)) = 0$ for $j < r$, and has dimension $b_{n+d-j} - b_{n+d-j+2}$ for $r \leq j \leq n$, and $b_d - b_{d-2}$ for $j = n+1$.

Proof. Since V is nonsingular of dimension d, the hard Lefschetz theorem tell us that capping with the hyperplane class h gives a map

$$h : H^{DR}_i(V) \longrightarrow H^{DR}_{i-2}(V)$$

that is surjective for $i \leq d+1$ and injective for $i \geq d+1$. Therefore by Proposition 4.6, we find that

$$h^{DR}_i(Y) = \begin{cases} 0 & \text{for} \quad i = 0, 1 \\ b_1 & \text{for} \quad i = 2 \\ b_{i-1} - b_{i-3} & \text{for} \quad 3 \leq i \leq d+1 \\ b_{i-2} - b_i & \text{for} \quad d+1 < i \leq 2d \\ b_{2d-1} & \text{for} \quad i = 2d+1 \\ b_{2d} & \text{for} \quad i = 2d+2\,. \end{cases}$$

Substituting these values in the statement of Theorem 4.3 gives the desired assertions. □

Remark 4.9. Many of the results of this section concerning the non-critical case $i > r$ are not new. What is new are the results concerning the $\mathcal{D}$-module structure of the critical case $H^r_I(A)$, especially Theorem 4.3(1) and Theorem 4.8(2).

The understanding of the relationship between cofiniteness and vanishing of the local cohomology modules $H^i_I(A)$ for $i > r$ and the algebraic de Rham cohomology of the projective variety V goes back to Ogus [15]. This connection is acknowledged in the last paragraph of Lyubeznik's paper [12], just after he has defined some new numerical invariants of a local ring, commonly called *Lyubeznik numbers*. The study of these numbers has led to several results analogous to ours for complex analytic spaces.

García-López and Sabbah [10] give a result similar to our Corollary 4.5 for an isolated singularity of a complex analytic space, in terms of local topological invariants.

Blickle and Bondu [1] give a similar result for a point P in a complex analytic space Y under the condition that $Y - P$ is an intersection homology manifold. This condition is probably equivalent to Ogus's condition on the DR-depth [15, 4.1], which is in fact *equivalent* to the local cohomology modules $H^i_I(X)$ being cofinite for $i > r$.

Lyubeznik, Singh, and Walther [14, 3.1] give another analogue of our Corollary 4.5 over $\mathbb{C}$, taking as *hypothesis* that these local cohomology groups have support at $\mathfrak{m}$, and computing the m_i in terms of the complex singular cohomology of $\mathbb{C}^n \setminus Y$.

Switala [17] in a recent paper about Lyubeznik numbers recovers independently our Theorem 4.8(1) for the vertex of the cone over a nonsingular projective variety. His argument is similar to ours, but uses cohomology instead of homology.

5. $\mathcal{D}$-modules over the power series ring

Theorem 5.1. *Let A be the power series ring $k[\![x_1, \dots, x_n]\!]$, let E be an injective hull of k over A, and let M be a holonomic $\mathcal{D}$-module. If $m = \dim_k H^n_{DR}(M)$, then there is a surjective homomorphism of $\mathcal{D}$-modules*

$$M \longrightarrow E^m \longrightarrow 0\,.$$

Proof. Recall that $H^n_{DR}(M)$ is a finite dimensional k-vector space (see Section 2). Observe that since $H^n_{DR}(M)$ is the homology of the last term of the de Rham complex, it is simply $M/(\partial_1, \dots, \partial_n)M$. Choose a linear map from $M/(\partial_1, \dots, \partial_n)M$ to k and compose it with the canonical epimorphism from M to $M/(\partial_1, \dots, \partial_n)M$ to obtain a map ψ from M to k. Because of Proposition 5.13 below, ψ is a continuous map in the sense of Definition 5.3. Therefore, by Proposition 5.4, the map ψ corresponds to an A-linear map φ from M to E. The correspondence in Proposition 5.4 depends on the choice of a k-linear projection π of E to its socle k. We now choose π to be the projection of E to $E/(\partial_1, \dots, \partial_n)E$, which is isomorphic to the socle of E. Then we have a diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad\varphi\quad} & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ M/(\partial_1, \dots, \partial_n)M & \longrightarrow & k = E/(\partial_1, \dots, \partial_n)E \end{array}$$

which shows that φ maps the kernel of ψ to the kernel of π. Hence

$$\varphi((\partial_1, \dots, \partial_n)M) \subset (\partial_1, \dots, \partial_n)E\,.$$

Now according to Proposition 5.5 the map φ is not only A-linear, but is also a map of $\mathcal{D}$-modules. Further, observe that since E is a simple $\mathcal{D}$-module and φ is not zero, φ

is surjective. Similarly, if we take m linearly independent maps from $M/(\partial_1, \dots, \partial_n)M$ to k they give us, m surjective $\mathcal{D}$-modules maps from M to E. The direct sum of these is the required surjective map from M to E^m. This completes the proof of Theorem 5.1, subject to Propositions 5.4, 5.5, 5.13 below. □

Corollary 5.2. *Let A be the power series ring $k[\![x_1, \dots, x_n]\!]$, let E be an injective hull of k over A, and let M be a holonomic $\mathcal{D}$-module. Then*

$$\dim_k H^n_{DR}(M) = \dim_k \operatorname{Hom}_{\mathcal{D}}(M, E)\,.$$

Proof. By the theorem we have

$$\dim_k H^n_{DR}(M) \leq \dim_k \operatorname{Hom}_{\mathcal{D}}(M, E)\,.$$

Conversely, if $\dim_k \operatorname{Hom}_{\mathcal{D}}(M, E) = s$, then there is a surjective map from M to E^s. Apply the de Rham cohomology functor. Then

$$H^n_{DR}(M) \longrightarrow H^n_{DR}(E^s) \longrightarrow 0$$

Since $H^n_{DR}(E) = k$, we have

$$\dim_k H^n_{DR}(M) \geq \dim_k H^n_{DR}(E^s) = s\,. \quad \square$$

Now we are ready to prove the three propositions that are the main ingredients in the proof of Theorem 5.1.

Definition 5.3. Let $(A, \mathfrak{m}, k)$ be a local ring that contains its residue field k. Let M be an A-module. A k-linear homomorphism ψ of M to k is called *continuous* if for every finitely generated submodule N of M there is an integer s such that $\psi(\mathfrak{m}^s N) = 0$. We denote the A-module of continuous linear homomorphisms by $\operatorname{Homcont}_k(M, k)$.

The following proposition appears in [9,18] but we give the proof for convenience.

Proposition 5.4. [9, IV, Remarque 5.5] *Let $(A, \mathfrak{m}, k)$ be a local ring that contains its residue field k, and E an injective hull of k. For any A-module M, the Matlis dual* $\operatorname{Hom}_A(M, E)$ *is isomorphic as a k-vector space to the module* $\operatorname{Homcont}_k(M, k)$.

Proof. Choose a k-linear projection π of E to its socle k. Then for any $\varphi \in \operatorname{Hom}_A(M, E)$, composing with π, we obtain a k-linear homomorphism ψ from M to k. Let us show that ψ is continuous. For any finitely generated submodule N of M the image $\varphi(N)$ is a finitely generated submodule of E and therefore is an A-module of finite length. Thus there exists an integer s such that $\mathfrak{m}^s\varphi(N) = 0$. It follows that $\varphi(\mathfrak{m}^s N) = 0$ and thus $\psi(\mathfrak{m}^s N) = 0$. Hence ψ is continuous in the sense of Definition 5.3. We have thus

constructed a k-linear map Λ from $\mathrm{Hom}_A(M,E)$ to $\mathrm{Homcont}_k(M,k)$. Now we show that Λ is an isomorphism. If $M = k$, it is obvious. For an A-module of finite length, the statement follows by induction on the length of the module and short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and the fact that $\mathrm{Hom}_A(M,E)$ and $\mathrm{Homcont}_k(M,k)$ are contravariant exact functors. Next, if M is a finitely generated A-module, then every homomorphism of either $\mathrm{Hom}_A(M,E)$ or $\mathrm{Homcont}_k(M,k)$ factors through $M/\mathfrak{m}^\ell M$ for some ℓ, hence we have

$$\begin{aligned}\mathrm{Hom}_A(M,E) &= \varinjlim \mathrm{Hom}_A(M/\mathfrak{m}^\ell M, E) \simeq \varinjlim \mathrm{Homcont}_k(M/\mathfrak{m}^\ell M, k)\\ &= \mathrm{Homcont}_k(M,k)\,.\end{aligned}$$

For M an arbitrary A-module, think of M as the direct limit of its finitely generated submodules, $M = \varinjlim M_\ell$. Thus we have

$$\begin{aligned}\mathrm{Hom}_A(M,E) &= \mathrm{Hom}_A(\varinjlim M_\ell, E) = \varprojlim \mathrm{Hom}_A(M_\ell, E)\\ &\simeq \varprojlim \mathrm{Homcont}_k(M_\ell,k) = \mathrm{Homcont}_k(\varinjlim M_\ell, k)\\ &= \mathrm{Homcont}_k(M,k)\,. \quad \square\end{aligned}$$

Proposition 5.5. *Let A be the power series ring $k[\![x_1,\dots,x_n]\!]$, let E be an injective hull of k over A, let M be a $\mathcal{D}$-module, and let $\varphi : M \longrightarrow E$ be an A-linear map such that*

$$\varphi(\partial M) \subset \partial E\,,$$

where $\partial = (\partial_1,\dots,\partial_n)$. Then φ is also $\mathcal{D}$-linear.

Proof. We must show that $\partial_i\varphi(m) = \varphi(\partial_i m)$ for all i and for all $m \in M$. Observe that both sides are elements of ∂E. The left hand side because it is a ∂_i of something, the right hand side because of our hypothesis that $\varphi(\partial M) \subset \partial E$. Next we note that the map $\partial E \longrightarrow E^n$, sending $e \in \partial E$ to $(x_1 e,\dots,x_n e) \in E^n$, is injective. Therefore it sufficient to prove that for every j,

$$x_j\partial_i\varphi(m) = x_j\varphi(\partial_i m) \quad \forall m \in M\,. \tag{$*$}$$

We claim that statement $(*)$ is equivalent to showing

$$\partial_i\varphi(x_j m) = \varphi(\partial_i(x_j m)) \quad \forall m \in M\,.$$

If $i \neq j$ the claim is clear because x_j and ∂_i commute and φ is A-linear. If $i = j$, then we use the equation $x_i\partial_i = \partial_i x_i - 1$ in the ring of differential operators. Indeed, notice that the left hand side of $(*)$ is

$$x_i\partial_i(\varphi(m)) = \partial_i\varphi(x_i m) - \varphi(m)$$

while the right hand side of $(*)$ is

$$x_i\varphi(\partial_i m) = \varphi(x_i\partial_i m) = \varphi(\partial_i(x_i m) - m) = \varphi(\partial_i(x_i m)) - \varphi(m)\,.$$

Now after canceling $\varphi(m)$ we obtain the desired claim. We have thus replaced the original problem for $m \in M$ by the same problem for $x_j m$. Repeating the same procedure it is sufficient to show

$$\partial_i\varphi(\alpha m) = \varphi(\partial_i(\alpha m)) \quad \forall m \in M$$

for all monomials $\alpha \in A$ of any high degree we like. To conclude notice that both sides are zero for degree of α sufficiently large. Indeed, the left hand side is clearly zero since $\varphi(\alpha m) = \alpha\varphi(m)$ and $\varphi(m) \in E$. The right hand side is zero because using the product rule we have

$$\varphi(\partial_i(\alpha m)) = \varphi(\partial_i(\alpha)m) + \varphi(\alpha\partial_i m) = \partial_i(\alpha)\varphi(m) + \alpha\varphi(\partial_i m)$$

and both $\partial_i(\alpha)$ and α have sufficiently high degree. □

Remark 5.6. The statement and the proof of Proposition 5.5 also hold over a polynomial ring or its localization at the maximal ideal.

Theorem 5.7. ([2, 3.3.19], [20, Prop. 1]) *Let A be the power series ring $k[\![x_1, \ldots, x_n]\!]$ and let M be a holonomic $\mathcal{D}$-module over A. Then there exists a nonzero element $g \in A$ such that $M[g^{-1}]$ is a holonomic $\mathcal{D}$-module that is finitely generated as an $A[g^{-1}]$-module. Furthermore, after a linear change of variables, we may assume that $g(x_1, 0, \ldots, 0) \neq 0$, and in that case we can take g to be a Weierstraß polynomial*

$$x_1^r + a_1 x_1^{r-1} + \ldots + a_r\,,$$

with $a_i \in k[\![x_2, \ldots, x_n]\!]$. In this situation we say that M is x_1-regular.

Theorem 5.8. [21, Thm. I] *Let A be the power series ring $k[\![x_1, \ldots, x_n]\!]$ and let M be a holonomic $\mathcal{D}$-module over A that is x_1-regular. Then $M/\partial_1(M)$ is a holonomic $\mathcal{D}$-module over the ring $k[\![x_2, \ldots, x_n]\!]$.*

Remark 5.9. Van den Essen showed by an example [20] that for an arbitrary holonomic $\mathcal{D}$-module M, the quotient $M/\partial_1(M)$ need not to be a holonomic $\mathcal{D}$-module over the ring $k[\![x_2, \ldots, x_n]\!]$. However, with the extra condition that M is x_1-regular, this holds.

Lemma 5.10. *Let A be the power series ring $k[\![x_1, \ldots, x_n]\!]$, let M be a holonomic $\mathcal{D}$-module that is x_1-regular, and let e be any element of M. Then there exists a differential operator of the form*

$$P = a_0 + a_1\partial_1 + \ldots + a_r\partial_1^r \qquad a_i \in A$$

where a_r has a pure power of x_1, and such that

$$P(b) \cdot e \subset \partial_1(M) \quad \forall b \in A\,.$$

Proof. While not given exactly in this form, our statement and proof are based on a careful reading of [21]. According to Theorem 5.7 there is an element g such that $M[g^{-1}]$ is finitely generated as an $A[g^{-1}]$-module, and furthermore g can be taken to be a Weierstraß polynomial in x_1. Let $x = x_1$ and $\partial = \partial_1$. Let e be an element of M. Then $e, \partial e, \partial^2 e, \ldots, \partial^i e, \ldots$ are linearly dependent over $A[g^{-1}]$. Therefore, there exists an integer r and elements $c_i \in A[g^{-1}]$ such that

$$\partial^r e = \sum_0^{r-1} c_i \partial^i e\,.$$

Clearing denominators we can write

$$g^s \partial^r e = \sum_0^{r-1} d_i \partial^i e\,,$$

where $d_i \in A$. So we can consider the differential operator

$$Q = \sum_0^r d_i \partial^i\,,$$

using the d_i above for $0 \leq i \leq r-1$ and $d_r = -g^s$. By construction $Q(e) = 0$ in M and d_r is a Weierstraß polynomial in x. There exist $a_i \in A$ such that the differential operator Q can be written as

$$Q = a_0 - \partial a_1 + \ldots + (-1)^r \partial^r a_r\,.$$

Let P be the differential operator

$$P = a_0 + a_1\partial + \ldots + a_r\partial^r\,.$$

Then for every $b \in A$ we claim there is an equality of differential operators

$$bQ = P(b) + \partial R\,, \tag{1}$$

where R is another differential operator and $P(b)$ means P acting on $b \in A$. Then, if we apply the two operators defined in (1) to e we have,

$$0 = bQ(e) = P(b)e + \partial R(e)\,.$$

This implies that for all $b \in A$, $P(b)e \subset \partial(M)$, thus establishing the desired conclusion.

To complete the proof we need to prove the equality of differential operators in (1). By linearity it suffices to show the claim for $Q = (-1)^i \partial^i a_i$ and $P = a_i \partial^i$. We need to show that $bQ - P(b) = \partial R_i$ for some operator R_i. To prove the claim we show by induction on i a stronger claim, more precisely, that for all $f, g \in A$

$$(-1)^i f \partial^i g = g\partial^i(f) + \partial R_i\,,$$

for some operator R_i. If $i = 0$, we can take $R_0 = 0$ because $fg = gf$. If $i = 1$ we have

$$f\partial + \partial(f) = \partial f\,.$$

Since $\partial(f) \in A$ we have $g\partial(f) = \partial(f)g$. Thus take $R_1 = -\partial fg$. Let $i \geq 2$ and assume that the statement holds for $i - 1$. We have

$$\begin{aligned}(-1)^i f\partial^i g &= (-1)^i f\partial(\partial^{i-1} g) = (-1)^i [\partial f - \partial(f)](\partial^{i-1} g)\\ &= (-1)^i \partial f \partial^{i-1} g + (-1)^{i-1}\partial(f)\partial^{i-1} g\end{aligned}$$

By induction hypothesis we have

$$(-1)^{i-1}\partial(f)\partial^{i-1} g = g\partial^{i-1}(\partial(f)) + \partial R_{i-1}$$

thus substituting in the previous equation we obtain

$$(-1)^i f\partial^i g = g\partial^{i-1}(\partial(f)) + \partial R_{i-1} + (-1)^i \partial f \partial^{i-1} g = g\partial^i(f) + \partial R_i$$

where $R_i = R_{i-1} + (-1)^i f\partial^{i-1} g$, which proves the claim. □

The following lemma is our key technical result.

Proposition 5.11. *Let A be the power series ring $k[\![x_1, \ldots, x_n]\!]$, let $B = k[\![x_2, \ldots, x_n]\!]$, let $x = x_1$, and let $\partial = \partial_1$. Let P be a differential operator of the form*

$$P = \sum_{i=0}^{r} a_i \partial^i \qquad a_i \in A\,,$$

where a_r has a pure power of x. Then there exist integers s and ℓ_0 such that every $f \in A$ can be written in the form

$$f = \sum_{i=0}^{s-1} e_i x^i + \sum_{\ell \geq \ell_0} b_\ell P(x^\ell)$$

where $e_i, b_\ell \in B$. *Furthermore, if* $f \in Ax^m$ *for some* $m \geq s$ *then*

$$e_i \in \mathfrak{m}_B^{\rho(m)} \quad \forall\ 0 \leq i \leq s-1\,,$$

where ρ *is a function that tends to infinity with* m.

Proof. We first show that there exist integers ℓ_0 and t such that for all $\ell \geq \ell_0$

$$P(x^\ell) = \sum_{i=0}^{\infty} c_{\ell,i} x^i$$

where $c_{\ell,i} \in B$ satisfy the following conditions

(i)	$c_{\ell,i} = 0$	for $i < \ell - r$
(ii)	$c_{\ell,i} \in \mathfrak{m}_B$	for $\ell - r \leq i < \ell - r + t$
(iii)	$c_{\ell,i}$ is a unit in B	for $i = \ell - r + t$.

(There is no restriction on $c_{\ell,i}$ for $i > \ell - r + t$.) Condition (i) is clear because x^ℓ has degree ℓ and ∂^r is the highest differential in P. To prove (ii) and (iii), write for $\ell \geq r$

$$P(x^\ell) = a_0 x^\ell + \ell a_1 x^{\ell-1} + 2\binom{\ell}{2} a_2 x^{\ell-2} + \ldots + r!\binom{\ell}{r} a_r x^{\ell-r}\,.$$

Let t be the least power of x whose coefficient is a unit in B among all the power series

$$\{a_0 x^r, a_1 x^{r-1}, \ldots, a_r\}\,.$$

Notice that t exists because by hypothesis a_r has a pure power of x. Let $\lambda_i \in k$ be the constant term of the coefficient of x^t in $a_i x^{r-i}$. By construction, at least one of the λ_i is non zero. The constant term of the coefficient of $x^{\ell+t-r}$ in $P(x^\ell)$ is

$$g(\ell) = \lambda_0 + \ell\lambda_1 + \ldots + r!\binom{\ell}{r}\lambda_r\,.$$

Since $g(\ell)$ is a non zero polynomial in ℓ, it has at most finitely many zeros. Choose ℓ_0 such that $g(\ell) \neq 0$ for all $\ell \geq \ell_0$. Now for any $\ell \geq \ell_0$ write $P(x^\ell) = \sum_{i=0}^{\infty} c_{\ell,i} x^i$, where $c_{\ell,i} \in B$. Then by construction the $c_{\ell,i}$ will satisfy the condition (ii) and (iii), namely, the first one that is a unit is $c_{\ell,\ell-r+t}$.

To continue set $s = \ell_0 - r + t$. Then we show that every $f \in A$ can be written in the form

$$f = \sum_{i=0}^{s-1} e_i x^i + \sum_{\ell \geq \ell_0} b_\ell P(x^\ell)$$

where $e_i, b_\ell \in B$. We claim, by induction on k, that

$$f \equiv f_k = \sum_{i=0}^{s-1} e_{i,k} x^i + \sum_{\ell \geq \ell_0} b_{\ell,k} P(x^\ell) \qquad \mod \mathfrak{m}_B^k \tag{2}$$

for suitable $e_{i,k}$ and $b_{\ell,k}$ in B and furthermore $e_{i,k} \equiv e_{i,k+1}$ and $b_{\ell,k} \equiv b_{\ell,k+1}$ modulo $\mathfrak{m}_B^k$. First observe that for all $\ell \geq \ell_0$

$$P(x^\ell) \equiv \alpha_\ell x^{\ell-r+t} \mod \mathfrak{m}_B\,, \tag{3}$$

where α_ℓ is a unit in A. Here we use that $c_{\ell,i} \in \mathfrak{m}_B$ for all $0 \leq i < \ell - r + t$. If the original power series f is

$$f = \sum_{i=0}^{\infty} \beta_i x^i$$

with $\beta_i \in B$, then we define f_1 using $e_{i,1} = \beta_i$ for $0 \leq i \leq s-1$ and $b_{\ell,1} = \beta_{\ell-r+t}(\alpha_\ell)^{-1}$ for $\ell \geq \ell_0$. Thus $f \equiv f_1 \mod \mathfrak{m}_B$ and the claim follows for $k = 1$. For the $k+1$ step, we consider $f - f_k$. Notice that the coefficients $\gamma_{i,k}$ of $f - f_k$ as a power series in x are all in $\mathfrak{m}_B^k$. Therefore when we use $P(x^\ell)$ to adjust the $\ell - r + t$ coefficient of $f - f_k$, we have by (3)

$$\gamma_{\ell-r+t,k} P(x^\ell) \equiv \gamma_{\ell-r+t} \alpha_\ell x^{\ell-r+t} \mod \mathfrak{m}_B^{k+1}\,.$$

Define $f_{k+1} = f_k + \sum_{\ell \geq \ell_0} \gamma_{\ell-r+t,k}(\alpha_\ell)^{-1} P(x^\ell)$. Then $f \equiv f_{k+1} \mod \mathfrak{m}_B^{k+1}$. Writing f_{k+1} in the form (2) we obtain the coefficients $e_{i,k+1}$ and $b_{\ell,k+1}$ and observe that, by construction, they are congruent to the coefficients $e_{i,k}$ and $b_{\ell,k}$ modulo $\mathfrak{m}_B^k$. Now the desired assertion follows by passing to the limit: namely $e_i = \lim e_{i,k}$ and $b_\ell = \lim b_{\ell,k}$ as k goes to infinity.

To explain the 'Furthermore' statement, suppose that $f \in Ax^m$. Then f, as a power series in x, begins in degree $\geq m$. We claim that f_k starts in degree $\geq m - kt$ and the coefficients of x^j for $m - kt \leq j < m - (k-1)t$ are in $\mathfrak{m}_B^k$. Recall that $P(x^\ell)$ as a power series in x begins in degree $\geq \ell - r$ but its first unit coefficient is in degree $\ell - r + t$. We prove the claim by induction on k. For $k = 1$ recall that $b_{\ell,1} = \beta_{\ell-r+t}(\alpha_\ell)^{-1}$ and the first β_i that can be different from zero is β_m, hence the first $P(x^\ell)$ we are using is for $\ell = m + r - t$ and that can only start in degree $\ell - r = (m + r - t) - r = m - t$. Furthermore, the coefficients of x^j for $m - t \leq j < m$ of $P(x^{m+r-t})$ as power series in x are in $\mathfrak{m}_B$ by condition (ii) on the $c_{\ell,i}$ above, or, by the fact that $f_1 \equiv f \mod \mathfrak{m}_B$. By the construction used to build f_k from f_{k-1} we see that f_k satisfies our claim. Now

the coefficients of f are obtained by taking the limits of the coefficients of the f_k. Since s is fixed and m can be taken as large as we like, we have that e_i for $0 \leq i \leq s-1$ are contained in $\mathfrak{m}_B^{\rho(m)}$ where $\rho(m)$ is a function that tends to infinity with m, approximately equal to $(m-s)/t$. □

Remark 5.12. A result similar to this was proved by van den Essen [21] but without the "Furthermore" statement, which is crucial to our proof.

Proposition 5.13. *Let A be the power series ring $k[\![x_1, \ldots, x_n]\!]$, let M be a holonomic $\mathcal{D}$-module, and let N be a finitely generated submodule of M. Then there exists an integer r such that*

$$\mathfrak{m}^r N \subset (\partial_1, \ldots, \partial_n)M\,.$$

Proof. We may assume that N is generated by one element e. By Theorem 5.7 we can make a change of variables so that M is x_1-regular. Let $x = x_1$ and $\partial = \partial_1$. By Lemma 5.10 there exists a differential operator of the form given in the lemma such that $P(a)e \subset \partial M$ for all $a \in A$. We apply Proposition 5.11 to this differential operator. By Proposition 5.11 there exists an integer s such that for all $f \in Ax^m$ with $m \geq s$ we have

$$f \cdot e \subset \mathfrak{m}_B^{\rho(m)} E + \partial M\,, \tag{4}$$

where $B = k[\![x_2, \ldots, x_n]\!]$ and $E = B(e, xe, \ldots, x^{s-1}e)$, because $P(x^\ell)e \subset \partial M$ for all ℓ by Lemma 5.10.

We show the statement by induction on the number of variables n. If $n = 1$ then $B = k$ and thus $\mathfrak{m}_B = 0$. Therefore, by (4) for all $f \in Ax^m$ with $m \geq s$ we have

$$f \cdot e \subset \partial M\,,$$

hence we obtain

$$\mathfrak{m}^s N \subset \partial M\,,$$

which is the desired assertion for $n = 1$. If $n \geq 2$, we apply Theorem 5.8, which says $\overline{M} = M/\partial M$ is a holonomic $\mathcal{D}$-module over B. Let $\overline{E}$ be the image of E in $\overline{M}$, then $\overline{E}$ is a finitely generated submodule of $\overline{M}$. By the induction hypothesis there exists an integer t such that

$$\mathfrak{m}_B^t \overline{E} \subset (\partial_2, \ldots, \partial_n)\overline{M}\,.$$

This implies that

$$\mathfrak{m}_B^t E \subset (\partial_1, \ldots, \partial_n)M\,. \tag{5}$$

By Proposition 5.11 we can take m_0 large enough so that $m_0 \geq s$ and $\rho(m_0) \geq t$. Take r to be $r = m_0 + \rho(m_0)$. We claim that

$$\mathfrak{m}_A^r N = \mathfrak{m}_A^r e \subset (\partial_1, \dots, \partial_n)M\,.$$

Write any monomial $\alpha \in \mathfrak{m}_A^r$ as $x^i\gamma$. Notice that either $i \geq m_0$ or $\gamma \in \mathfrak{m}_B^j$ with $j \geq \rho(m_0) \geq t$. In the first case $\alpha \in x^{m_0}A$, hence by (4) (notice (4) applies because $m_0 \geq s$)

$$\alpha e \subset \mathfrak{m}_B^{\rho(m_0)} E + \partial M \subset (\partial_1, \dots, \partial_n)M$$

where the last inclusion hold by (5) and because we chose m_0 in such a way $\rho(m_0) \geq t$. In the second case, the claim follows directly by (5). □

6. Applications

We want to apply the results of Section 5 to study the critical local cohomology module $H_I^r(A)$, where V is a variety in $\mathbb{P}_k^n$ of codimension r, with homogeneous ideal $I \subset A = k[x_0, \dots, x_n]$. To do this we need to pass to the completion.

If M is a $\mathcal{D}$-module over the polynomial ring $A = k[x_0, \dots, x_n]$, we can consider the completion $M \otimes_A \hat{A}$ where $\hat{A} = k[\![x_0, \dots, x_n]\!]$. It has a natural structure of $\mathcal{D}$-module over $\hat{A}$. Furthermore, there is a natural map on the de Rham cohomology groups: $H_{DR}^i(M) \longrightarrow H_{DR}^i(\hat{M})$. Unfortunately, even for a holonomic $\mathcal{D}$-module over A, the map on de Rham cohomology may fail to be an isomorphism (see Example 6.1). It would be nice to have general conditions under which these maps are isomorphisms. For instance, if M is also a graded A-module, and the ∂_i act as graded k-linear maps of degree -1, then are the completion maps φ^i isomorphisms? We do not know, so we will settle for a more limited criterion.

Example 6.1. Let $A = k[x]$ and let M be a free A-module of rank one. We denote its generator by $e \in M$, so that the elements of M are written ae for $a \in A$. To give M a structure of $\mathcal{D}$-module we can take ∂e to be anything we like. So for example, let $\partial e = x^2 e$. Then for any power of x, $\partial(x^n e) = x^n \partial e + n x^{n-1} e = (x^{n+2} + n x^{n-1})e$, and we can extend to all of M by linearity. Now it is clear that the map $\partial : M \longrightarrow M$ is injective, so that $H_{DR}^0(M) = 0$. On the other hand, the image of ∂ is a k-vector subspace of codimension 2, so $H_{DR}^1(M) = M/\partial M$ has dimension 2. This shows that Theorem 5.1 is false for M, since any nonzero $\mathcal{D}$-module homomorphism of M to E would have to be surjective, which is impossible since M is finitely generated as an A-module.

Now let consider $\hat{M} = M \otimes_A \hat{A}$. We claim that there is another generator of $\hat{M}$, call it ue, with $u \in \hat{A}$ a unit, for which $\partial(ue) = 0$. To find u, we want $\partial(ue) = 0$, hence $u\partial e + \partial u \cdot e = 0$. We need to solve the differential equation $\partial u = -x^2 u$. Just take $u = e^{-\frac{x^3}{3}}$, which is a unit in $\hat{A}$. Thus $\hat{M}$ is isomorphic to the standard $\mathcal{D}$-module structure on $\hat{A}$ with $H_{DR}^0(\hat{M}) = 1$ and $H_{DR}^1(\hat{M}) = 0$. So we see that the passage to the completion does not preserve de Rham cohomology.

Theorem 6.2. *Let $A = k[x_0, \dots, x_n]$, let $I \subset A$ be a homogeneous ideal, and let M be a local cohomology module $M = H^i_I(A)$ with its $\mathcal{D}$-module structure. Then the completion maps*

$$H^j_{DR}(M) \longrightarrow H^j_{DR}(\hat{M})$$

where $\hat{M} = M \otimes_A \hat{A}$, are isomorphisms for all j. Furthermore $\hat{M}$ is also holonomic over $\hat{A}$.

Proof. The last statement follows from Theorem 2.1 and the fact that

$$\hat{M} = H^i_I(A) \otimes_A \hat{A} = H^i_{\hat{I}}(\hat{A})\,.$$

We will pull ourselves up by our bootstraps using the earlier results of Section 4. First of all, since the local cohomology $H^i_I(A)$ can be computed from the Čech complex of localizations of A at product of the f_i, where $\{f_1, \dots, f_s\}$ is a set of generators of I, and a short exact sequence of modules gives a long exact sequence of de Rham cohomology, we reduce to the case where $I = (f)$ is generated by a single homogeneous polynomial $f \in A$. In this case there is only one non-zero local cohomology, namely, $M = H^1_f(A) = A_f/A$. Let V be the corresponding hypersurface in $\mathbb{P}^n_k$, and let $Y \subset \mathbb{A}^{n+1}_k$ be the affine cone over V, namely, the affine subscheme of $\operatorname{Spec} A$ defined by f. Then the hypotheses of Theorem 4.3 are satisfied and hence by that theorem we have that for each j

$$H^j_{DR}(M) = H^{DR}_{2n+1-j}(Y)$$

the algebraic de Rham homology of Y.

The same proof as for Proposition 4.2 and Theorem 4.3, carried out over the formal power series ring $\hat{A} = k[\![x_0, \dots, x_n]\!]$, will show that

$$H^j_{DR}(\hat{M}) = H^{DR}_{2n+1-j}(Y')$$

where $Y' \subset \operatorname{Spec} \hat{A}$ is the subscheme defined by the same polynomial $f \in \hat{A}$. Here we use the local theory of algebraic de Rham cohomology and homology ([5, III]). Therefore we just need to show that the natural maps $H^{DR}_i(Y) \longrightarrow H^{DR}_i(Y')$ are isomorphisms for all i. This is a question purely in the theory of algebraic de Rham homology, which we prove next. □

Proposition 6.3. (*Strong excision for homology*) *Let I be a homogeneous ideal in $A = k[x_0, \dots, x_n]$ and let Y be the affine scheme in $\operatorname{Spec} A = \mathbb{A}^{n+1}_k$ defined by I. Let $\hat{A} = k[\![x_0, \dots, x_n]\!]$, let $\hat{I} = I\hat{A}$, and let $Y' \subset \operatorname{Spec} \hat{A}$ be defined by $\hat{I}$. Then there are natural isomorphisms of de Rham homology*

$$H^{DR}_i(Y) \longrightarrow H^{DR}_i(Y')$$

for each i.

Proof. Let $V \subset \mathbb{P}^n_k$ be the projective scheme defined by I. Then according to [5, III, 3.2] there is an exact sequence

$$\dots \to H_{i+1}(Y') \to H_i(V) \to H_{i-2}(V) \to H_i(Y') \to \dots$$

We have established in Proposition 4.6(4) the same sequence with $H_i(Y)$ in place of $H_i(Y')$. Note that this does not depend on the special hypotheses of Theorem 4.3 and Proposition 4.6. Since there are compatible maps between these sequences, we conclude that $H_i(Y) \xrightarrow{\sim} H_i(Y')$ for all i. □

Theorem 6.4. (*Main Theorem*) *Let V be a nonsingular variety of codimension r in the projective space $\mathbb{P}^n_k$. Let I be the homogeneous ideal of V in $A = k[x_0, \dots, x_n]$. Then the local cohomology module $M = H^r_I(A)$ has a simple sub-$\mathcal{D}$-module M_0 with support on Y, the cone over V, and the quotient M/M_0 is a direct sum of $b_d - b_{d-2}$ copies of E, the injective hull of k over A, where $d = \dim V$, and b_i are the Betti numbers $b_i = \dim H^{DR}_i(V)$.*

Proof. By Corollary 4.5, $H^{n+1}_{DR}(H^r_I(A)) = H^{DR}_{n+1-r}(Y)$. On the other hand, since V is nonsingular and $\dim V = d = n - r$, by Theorem 4.8 the dimension of this homology group is $b_d - b_{d-2}$.

Next, letting $M = H^r_I(A)$, we apply Theorem 6.2 to see that $H^j_{DR}(M) = H^j_{DR}(\hat{M})$ for each j. Therefore, by Theorem 5.1 and Corollary 5.2, we have a surjective map $\hat{M} \longrightarrow E^m$ with $m = b_d - b_{d-2}$, and $\dim \operatorname{Hom}_{\mathcal{D}}(\hat{M}, E) = m$. Composing with the natural map $M \longrightarrow \hat{M}$ we obtain a map $M \longrightarrow E^m$, which must be surjective since $H^{n+1}_{DR}(M) \cong H^{n+1}_{DR}(\hat{M})$.

On the other hand, since Y has only one singular point at P, it follows from the general theory [6, Section 3.4], that if we take the simple $\mathcal{D}_Y$-module $\mathcal{O}_Y$ on the smooth part $Y - P$ of Y, then M contains a simple $\mathcal{D}$-module, the *minimal extension* of $\mathcal{O}_Y$ to X in the sense of [6, 3.4.2] and that the quotient M/M_0 has support at P. That quotient must be a sum of copies of E (see Example 2.2 (5)), and therefore is equal to the quotient E^m found above. □

Corollary 6.5. *If $V = \mathcal{C}$ is a nonsingular curve of genus g in $\mathbb{P}^n_k$, then there is just one nonzero local cohomology group $M = H^{n-1}_I(A)$. It has a simple sub-$\mathcal{D}$-module $M_0 \subset M$ supported on the cone over $\mathcal{C}$, and the quotient M/M_0 is isomorphic to E^{2g}. In particular, if $\mathcal{C}$ is a nonsingular rational curve, then $M = H^{n-1}_I(A)$ is a simple $\mathcal{D}$-module.*

Proof. Indeed, $\dim H_i(V) = 1, 2g, 1$ for $i = 0, 1, 2$, respectively, hence, $m = b_1 - b_{-1} = 2g$. □

Corollary 6.6. *If V is any embedding of a projective space $\mathbb{P}^d$ in another projective space $\mathbb{P}^n$, then the local cohomology group $M = H^{n-d}_I(A)$ is a simple $\mathcal{D}$-module.*

Proof. Indeed, the homology of V has dimension 1 in even degree and dimension zero in odd degrees, so in any case $b_d - b_{d-2} = 0$. □

Remark 6.7. In the special case of the Veronese embedding of $\mathbb{P}^d_k$ in $\mathbb{P}^n_k$, this was proved by an entirely different method, using representation theory, by Raicu [16].

Remark 6.8. For a singular projective variety $V \subset \mathbb{P}^n_k$ of codimension r, think of a stratification of V by locally closed nonsingular subvarieties. Then we expect $M = H^r_I(A)$ to have one simple sub-$\mathcal{D}$-module corresponding to the smooth part of V, and a succession of contributions coming from the strata of the singular locus, and finally a quotient that is equal to $\dim H^{DR}_{d+1}(Y)$ copies of E as before.

In particular, if $V = \mathcal{C}$ is an integral curve, then we expect one component for the smooth part of $\mathcal{C}$, then at each singular point P, $n_P - 1$ copies of the injective hull of the line cone over P (in the notation of Proposition 3.5), and then $h_1(\mathcal{C})$ copies of E.

7. Added in proof

After this paper was posted on the arXiv, Switala and Zhang ([19]) picked up our hints on duality (see Introduction) and graded $\mathcal{D}$-modules (see beginning of Section 6) to study the Matlis dual $D(M)$ of a graded $\mathcal{D}$-module M. They show that if $H^i_{dR}(M)$ is finite dimensional, then $H^{n-i}_{dR}(D(M))$ is the vector space dual of $H^i_{dR}(M)$. This allows them to give alternate proofs of our key results Theorem 5.1 and Theorem 6.4 without using the difficult analytic methods of van den Essen. Furthermore, Lyubeznik used Switala's duality theory to generalize our Corollary 5.2 ([13]).

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