



Global Weak Solutions to Compressible Navier–Stokes–Vlasov–Boltzmann Systems for Spray Dynamics

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Communicated by G.-Q. Chen

Abstract. This work focus on the construction of weak solutions to a kinetic-fluid system of partial differential–integral equations modeling the evolution of particles droplets in a compressible fluid. The system is given by a coupling between the standard isentropic compressible Navier–Stokes equations for the macroscopic description of a gas fluid flow, and a Vlasov–Boltzmann type equation governing the evolution of spray droplets modeled as particles with varying radius. We establish the existence of global weak solutions with finite energy, whose density of gas satisfies the renormalized mass equation. The proof combines techniques inspired by the work of Feireisl et al. (J Math Fluid Mech 3:358–392, 2001) on the weak solutions of the compressible Navier–Stokes equations in a coupled system to the kinetic problem for the spray droplets by extending techniques of Leger and Vasseur (J Hyperbolic Differ Equ 6(1):185–206, 2009) developed for the incompressible fluid-kinetic system.

Mathematics Subject Classification. 35Q35, 76D05, 82C40, 35H10.

Keywords. Navier–Stokes–Vlasov–Boltzmann equations, Compressible flow, Weak solutions.

1. Introduction

A large variety models describing sprays dynamics, introduced by Williams [28], are obtained by coupling a of fluid mechanics equation and a kinetic one describing the spray as perfect bubbles. In such a system models, the gas surrounding the spray is described by classical fluid macroscopic quantities: its density $\rho(t, x) \geq 0$ and velocity $\mathbf{u}(t, x)$. Depending on the physical properties of such gas fluid, the evolution of those quantities are usually ruled by the Navier–Stokes or Euler Equations compressible flows. Because air flow viscosity is an important component for spray dynamics, the fluid model is the associated to the compressible Navier Stokes framework.

The spray droplet evolution is assumed to be given by independent distributed continuum random variables described by a distribution function $f = f(t, x, \mathbf{v}, r) \geq 0$ given by the probability of finding a droplet with center at position x , with radius r , time t , moving with velocity $\mathbf{v}$. Depending on physical properties of the droplets, the evolution of f is governed by a kinetic equation given by a Vlasov-linear Boltzmann model, were the non-local Boltzmann operator models collisions and breakup.

In such a system models, the coupling comes from drag force in the fluid equation and the acceleration in the Vlasov term of kinetic equation, as the fluid a dense phase and the droplets in a disperse phase strongly interact on each other.

More specifically we consider an spray model given by the following Navier–Stokes–Vlasov–Boltzmann system of equations for droplet particles dispersed in a compressible viscous fluid

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (1.1)$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} - \lambda \nabla \operatorname{div} \mathbf{u} = \mathbf{F}_r(t, x), \quad (1.2)$$

$$f_t + \xi \cdot \nabla_x f + \operatorname{div}_\xi(Ff) = Q(f), \quad (1.3)$$

for (x, ξ, r, t) in $\Omega \times \mathbb{R}^3 \times [a, b] \times [0, \infty)$, where $\Omega \subset \mathbb{R}^3$, ρ is the density of the fluid, $\mathbf{u}$ is the velocity of the fluid, $p = \rho^\gamma$ is the pressure for some $\gamma > 1$. The viscosity coefficients μ and λ satisfy the relationship

$$\mu > 0, \quad \lambda + \frac{\mu}{3} \geq 0.$$

The probability density distribution function $f(x, \xi, r, t)$ associated droplet particles depends on the physical position $x \in \Omega$, the velocity of particle $\xi \in \mathbb{R}^3$, the radius of a particle $r \in [a, b]$, and the time $t \in [0, T]$, where $a, b > 0$ are the constants. Its observables are the spray thermodynamic quantities that are obtained by their statistical moments, defined by

the average (zero moment) of the gas particle probability density is

$$\mathbf{n}(t, x) = \int_a^b \int_{\mathbb{R}^3} r f \, d\xi \, dr, \quad (1.4)$$

and the kinetic current (first moment) is

$$\mathbf{j}(t, x) = \int_a^b \int_{\mathbb{R}^3} r \xi f \, d\xi \, dr. \quad (1.5)$$

The particle-fluid interaction is determined through the drag force exerted by the air fluid onto the spray particles, associated to the vector F in the spray equation (1.3) modeling the droplets acceleration. This force is typically given by the well known Stokes' law,

$$F(x, \xi, r, t) = \frac{9\mu}{2\rho_l} \frac{\mathbf{u} - \xi}{r^2}, \quad (1.6)$$

μ is the dynamic viscosity scale, and ρ_l is the mass density scale associated to the compressible fluid system (1.1, 1.2). Without loss of generality we take $\rho_l = \frac{9\mu}{2}$ throughout the paper.

The right hand side term in the momentum associated to the compressible fluid equation (1.2), is modeled by

$$\mathbf{F}_r(t, x) = - \int_a^b \int_{\mathbb{R}^3} \frac{4}{3} \rho_l r^3 f F \, d\xi \, dr. \quad (1.7)$$

The nonlocal kinetic particle interaction operator $Q(f)$ takes into account the complex phenomena happening at the level of the droplet particles, such as the interaction laws and breakup. Assuming that droplets keep the same velocities before and after breaking, the kinetic spray operator is determined by

$$Q(f)(x, \xi, r, t) = -\nu f(x, \xi, r, t) + \nu \int_{r > r^*} B(r^*, r) f(x, \xi, r^*, t) \, dr^*, \quad (1.8)$$

where $\nu \geq 0$ is the fragmentation rate and $B = B(r^*, r) \geq 0$ is related to the transition probability of ending up with droplet particles of radius r out of the breakup process of droplet particles of radius r^* . This is a typical structure of the breakage model kernel.

The fluid-particle system (1.1)–(1.8) arises in many applications such as sprays, aerosols, and more general two phase flows where one phase (disperse) can be considered as a suspension of particles onto the other one (dense) regarded as a bulk fluid. These type of systems, either (1.1)–(1.8) or its variants, have been used in the modeling of phenomena ranging from solid grain sedimentation by external forces, fuel-droplets in combustion theory (such as in the study of engines), chemical engineering, bio-sprays in medicine, waste water treatment, to pollutants in the air. We refer [1, 4, 6, 9, 10, 14, 25, 26, 28] to the reader for more physical background, applications and discussions of the fluid-particle systems. From the mathematical viewpoint, Leger and Vasseur [18] have shown the existence of global weak solutions to a related of an incompressible version of Vlasov–Boltzmann–Navier–Stokes equations.

The aim of this current paper is to establish the existence of global weak solutions to the system (1.1)–(1.8), or equivalently to

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (1.9)$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} - \lambda \nabla \operatorname{div} \mathbf{u} = - \int_a^b \int_{\mathbb{R}^3} r(\mathbf{u} - \xi) f \, d\xi \, dr, \quad (1.10)$$

$$f_t + \xi \cdot \nabla_x f + \operatorname{div}_\xi \left(\frac{(\mathbf{u} - \xi) f}{r^2} \right) = Q(f), \quad (1.11)$$

subject to the following initial data:

$$\rho|_{t=0} = \rho_0(\mathbf{x}) \geq 0, \quad (\rho \mathbf{u})|_{t=0} = \mathbf{m}_0(x), \quad f|_{t=0} = f_0(x, \xi, r), \quad (1.12)$$

where $Q(f)$ is given by (1.8).

The collision operator $Q(f)$ satisfies the following **hypotheses A**:

I. $B \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$, $B \geq 0$, and $B(r, r^*) = 0$ if $r \geq r^*$.
for all $(r, r^*) \in \mathbb{R}^+ \times \mathbb{R}^+$.

II. $\int_a^b B(r, r^*) \, dr = \int_{R(a)}^{R(b)} B(r, r^*) \, dr$, with

$$R(r) = \sqrt[3]{r^{*3} - r^3} \quad \text{and} \quad 0 \leq a \leq b \leq \frac{r^*}{\sqrt[3]{2}}.$$

III. $\int_0^{\frac{r^*}{\sqrt[3]{2}}} B(r, r^*) \, dr = \int_{\frac{r^*}{\sqrt[3]{2}}}^{r^*} B(r, r^*) \, dr = 1$, which without loss of generality, both integrals to be one by renormalization.

In order to solve the initial value problem for system (1.9)–(1.12) with assumptions (I–III), our strategy consists in combining a regularization method for solving the fluid system using the compressible Navier–Stokes system recently developed by Feireisl et al. [12], in an iteration that couples the air fluid equation to the initial value problem of the Vlasov-linear Boltzmann for the droplet particle evolution. For this coupling, we adapt the approach proposed by Leger and Vasseur [18] for the solving the system associated to the same kinetic equation coupled to a fluid given by the incompressible Navier–Stokes system.

The manuscript is organized as follows. In the Sect. 2 we introduce some fundamentals and prove, for a fixed droplet particle distribution $f(x, \xi, r, t)$, the basic a priori momentum and energy identities for the compressible Navier Stokes’ equation.

In Sect. 3, we first introduce the two level ε, δ -regularization technique from [12] to system (1.9)–(1.12) by adding an ε -viscous term to the mass equation and an ε -modification of the momentum equation that preserves the energy identities for fixed $f(x, \xi, r, t)$ derived in Sect. 2, and a δ -modification that modify the pressure law. In addition, we employ techniques from [12], where each ε, δ -regularized Navier Stokes (1.9–1.10) part is solved uniquely by a k -finite dimensional approximating model, introduced in [12, 13]. Then for each $u_{\mathbf{k}}^{\varepsilon, \delta}$, we finally solve the Vlasov-linear-Boltzmann equation (1.11) using the approach of [18], whole solution is an approximating $f_{\mathbf{k}}^{\varepsilon, \delta}$. This iteration is shown to construct unique solutions $(\rho_{\mathbf{k}}^{\varepsilon, \delta}, u_{\mathbf{k}}^{\varepsilon, \delta}, f_{\mathbf{k}}^{\varepsilon, \delta})$ to the $\varepsilon, \delta, \mathbf{k}$ -approximating system to (1.9–1.10–1.11) by means of a fixed point argument in a Banach space, where initial data is modified by introducing the parameter $\underline{\rho} > 0$ that keep our the $\rho_{\mathbf{k}}^{\varepsilon, \delta}$ estimates bounded below from vacuum uniformly in ε, δ and $\mathbf{k}$. In addition, we show that the unique solutions $(\rho_{\mathbf{k}}^{\varepsilon, \delta}, u_{\mathbf{k}}^{\varepsilon, \delta}, f_{\mathbf{k}}^{\varepsilon, \delta})$ for the $\varepsilon, \delta, \mathbf{k}$ -approximating system, satisfy momentum and energy identities, uniformly in ε, δ and $\mathbf{k}$, and the approximating density $\rho_{\mathbf{k}}^{\varepsilon, \delta}$ is bounded below by $\underline{\rho} > 0$ uniformly in ε and $\mathbf{k}$.

Finally, we study in Sect. 4 the limiting process that yields a global weak solution to (1.9–1.10–1.11), by first performing the limit $k \rightarrow \infty$, next the limit $\varepsilon \rightarrow 0$, and last the limit $\delta \rightarrow 0$ obtaining a limiting triplet (ρ, u, f) whose initial data has $\rho(x, 0) \geq \underline{\rho} > 0$ for an arbitrary $\underline{\rho} > 0$. So the existence of solutions in then proved for any initial data who density ρ may vanish locally.

2. A Priori Estimates

In this section, we derive some fundamental a priori estimates for each equation on the system (1.9)–(1.11). They are crucial to show the existence of weak solutions upon passing to the limits in the regularized approximation scheme.

We first recall the notation of renormalized solutions, [12, 13, 19]. In fact, multiplying (1.9) by $b'(\rho)$ we deduce

$$h(\rho)_t + \operatorname{div}(h(\rho)\mathbf{u}) + (h'(\rho)\rho - h(\rho))\operatorname{div}\mathbf{u} = 0 \quad (2.1)$$

for any differentiable function h . Thus, we give the following definition.

Definition 2.1. Equation (1.9) is satisfied in the renormalized sense, more specifically, Eq. (2.1) holds in the distributional sense, for any $h \in C^1(\mathbb{R})$ such that

$$h'(z) = 0 \quad \text{for all } |z| \geq M,$$

for some constant $M > 0$.

Here, for the sake of simplicity we will consider the case of bounded domain with periodic boundary conditions, namely $\Omega = \mathbb{T}^3$. In this paper, we assume that

$$\begin{cases} \rho_0 \geq 0 \text{ almost everywhere in } \Omega, \mathbf{m}_0 \in L^2(\Omega), \\ \mathbf{m}_0 = 0 \text{ almost everywhere on } \{\rho_0 = 0\}, \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega), \\ f_0 \in L^\infty \cap L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+), \quad r^3|\xi|^3 f_0 \in L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+). \end{cases} \quad (2.2)$$

Definition 2.2. The triplet $(\rho, \mathbf{u}, f)$ is a global weak solution to problem (1.9)–(2.2) if, for any $T > 0$, the following properties hold,

- i. $\rho \geq 0$, $\rho \in C([0, T]; L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0, T; H_0^1(\Omega))$, $\rho|\mathbf{u}|^2 \in L^\infty(0, T; L^1(\Omega))$;
- ii. $f(t, x, \xi, r) \geq 0$, for any $(t, x, \xi, r) \in (0, T) \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^+$;
- iii. $f \in L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+) \cap L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))$;
- iv. $r^3|\xi|^3 f \in L^\infty(0, T; L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))$;
- v. Equation (1.9) is satisfied in the renormalized sense.
- vi. For any $\varphi \in C^1([0, T] \times \Omega)$, for almost everywhere t , the following identity holds

$$\begin{aligned} & - \int_{\Omega} \mathbf{m}_0 \cdot \varphi(0, x) dx + \int_0^t \int_{\Omega} \left(-\rho \mathbf{u} \cdot \partial_t \varphi - (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi - \rho^\gamma \nabla \varphi \right. \\ & \left. + \mu \nabla \mathbf{u} \cdot \nabla \varphi + \lambda \operatorname{div} \mathbf{u} \operatorname{div} \varphi + \int_{\mathbb{R}^3} r f(\mathbf{u} - \xi) \cdot \varphi d\xi dr \right) dx dt = 0; \end{aligned} \quad (2.3)$$

- vii. For any $\phi \in C^1([0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$ with compact support with respect to x, ξ , and r , such that $\phi(T, \cdot, \cdot, \cdot) = 0$, the following identity holds

$$\begin{aligned} & - \int_0^T \int_{\Omega} \int_{\mathbb{R}^3} f \left(\partial_t \phi + \xi \cdot \nabla_x \phi + \frac{(\mathbf{u} - \xi)}{r^2} \cdot \nabla_\xi \phi \right) dx d\xi ds \\ & = \int_{\Omega} \int_{\mathbb{R}^3} f_0 \phi(0, \cdot, \cdot) dx d\xi + \int_0^T \int_{\Omega} Q(f) \phi dx dt; \end{aligned} \quad (2.4)$$

- viii. The energy inequality

$$\begin{aligned} & \int_{\Omega} \rho |\mathbf{u}|^2 dx + \int_{\Omega} \int_{\mathbb{R}^3} f(1 + |\xi|^2) d\xi dx + 2\mu \int_0^T \int_{\Omega} |\nabla \mathbf{u}|^2 dx dt + 2\lambda \int_0^T \int_{\Omega} |\operatorname{div} \mathbf{u}|^2 dx dt \\ & \leq \int_{\Omega} \frac{|\mathbf{m}_0|^2}{\rho_0} dx + \int_{\Omega} \int_{\mathbb{R}^3} (1 + |\xi|^2) f_0 d\xi dx \end{aligned} \quad (2.5)$$

holds for almost everywhere $t \in [0, T]$.

Our main result on existence of global weak solutions reads as follows.

Theorem 2.1. *Under the assumption (2.2), for any $\gamma > \frac{3}{2}$, there exists a global weak solution $(\rho, \mathbf{u}, f)$ to the initial value problem (1.9)–(1.12) for any $T > 0$.*

We start now to gather estimates for the momentum equation. Multiplying (1.10) by $\mathbf{u}$, integrating over Ω , and using (1.9), we deduce that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{1}{2} \left(\rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx + \mu \int_{\Omega} |\nabla \mathbf{u}|^2 dx + \lambda \int_{\Omega} |\operatorname{div} \mathbf{u}|^2 dx \\ &= - \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r f(\mathbf{u} - \xi) \cdot \mathbf{u} d\xi dx dr. \end{aligned} \quad (2.6)$$

Meanwhile, multiplying the Vlasov–Boltzmann equation (1.11) by $r^3 \frac{|\xi|^2}{2}$, taking integration with respects to r, ξ, x , and using integration by parts, one obtains

$$\begin{aligned} & \frac{d}{dt} \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} \frac{1}{2} r^3 |\xi|^2 f d\xi dx dr - \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r(\mathbf{u} - \xi) \xi f d\xi dx dr \\ &= \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^2 Q(f) d\xi dx dr. \end{aligned} \quad (2.7)$$

Thus, from (2.6) and (2.7), the following energy equality holds

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx + \frac{d}{dt} \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^2 f d\xi dx dr \\ &+ 2\mu \int_{\Omega} |\nabla \mathbf{u}|^2 dx + 2\lambda \int_{\Omega} |\operatorname{div} \mathbf{u}|^2 dx + 2 \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r f(\mathbf{u} - \xi)^2 d\xi dx dr = 0, \end{aligned} \quad (2.8)$$

where we used the following equality

$$\int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^2 Q(f) d\xi dx dr = 0.$$

In fact, the last identity is obtained from the following Lemma 2.1 (setting $p = 2$), that uses the properties II–V on $Q(f)$ from hypotheses A.

Lemma 2.1. *Under the properties II–V on $Q(f)$ from hypotheses A, then for any $p \geq 1$, we have*

$$\int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p Q(f) d\xi dx dr = 0. \quad (2.9)$$

Proof

$$\begin{aligned} & \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p Q(f) d\xi dx dr = -\nu \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f(x, \xi, r, t) d\xi dx dr \\ &+ \nu \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} \int_{r^* > r} r^3 |\xi|^p B(r^*, r) f(x, \xi, r^*, t) dr^* d\xi dx dr \\ &= -\nu \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f(x, \xi, r, t) d\xi dx dr \\ &+ \nu \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} |\xi|^p \left(\int_{r^* > r} r^3 B(r^*, r) dr \right) f(x, \xi, r^*, t) dr^* d\xi dx. \end{aligned}$$

From following [18], one can see that the properties II–V on $Q(f)$ yield

$$\int_{r^* > r} r^3 B(r^*, r) dr = (r^*)^3,$$

so replacing in the second term one obtains a symmetrization property yielding the zero integral, hence yield (2.9) holds. $\square$

Next we estimate the transport Vlasov–Boltzmann equation (1.11) multiplying by r^3 and integrating with respects to r, ξ, x , and using integration by parts, one obtains that

$$\frac{d}{dt} \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 f(x, \xi, r, t) d\xi dx dr = 0. \quad (2.10)$$

In fact, this was proved in [18]. Using (2.8) and (2.10), one obtains the following energy identity

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx + \frac{d}{dt} \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 (|\xi|^2 + 1) f d\xi dx dr \\ + 2\mu \int_{\Omega} |\nabla \mathbf{u}|^2 dx + 2\lambda \int_{\Omega} |\operatorname{div} \mathbf{u}|^2 dx + 2 \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r f (\mathbf{u} - \xi)^2 d\xi dx dr = 0. \end{aligned} \quad (2.11)$$

3. Regularization

In order to prove Theorem 2.1, motivated by the techniques developed by Feireisl et al. [12] and the work of Feireisl [13], we first regularize the system (1.8)–(1.11) by perturbing both the mass and momentum equations, (1.9) and (1.10) respectively, by adding ε -viscous terms and the δ -modified pressure as follows (while for simplicity we will not denote the solutions $(\rho, \mathbf{u}, f)$ dependance on the parameters ε and δ in this section, we will referred to the dependance to these parameters by solutions triplets $(\rho^{\varepsilon, \delta}, \mathbf{u}^{\varepsilon, \delta}, f^{\varepsilon, \delta}) = (\rho, \mathbf{u}, f)$ when is needed for clarification.)

The ε, δ regularized Navier–Stokes system is given by

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{u}) &= \varepsilon \Delta \rho, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho^\gamma + \delta \nabla \rho^\beta - \mu \Delta \mathbf{u} - \lambda \nabla \operatorname{div} \mathbf{u} - \varepsilon \nabla \mathbf{u} \cdot \nabla \rho + \mathbf{n} \mathbf{u} &= \mathbf{j}, \\ f_t + \xi \cdot \nabla_x f + \operatorname{div}_\xi \left(\frac{(\mathbf{u} - \xi) f}{r^2} \right) &= Q(f), \end{aligned} \quad (3.1)$$

where

$$\mathbf{n}(t, x) = \int_a^b \int_{\mathbb{R}^3} r f d\xi dr, \quad \mathbf{j} = \int_a^b \int_{\mathbb{R}^3} r \xi f d\xi dr,$$

and $Q(f)$ is given by (1.8).

The initial data is denoted by $(\rho_0, \mathbf{u}_0, f_0)$ and assume to be uniform in the ε, δ parameters and to satisfy

$$\begin{aligned} \rho(0) &= \rho_0(x) \in C^{2+\nu}(\bar{\Omega}), \quad 0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho}, \\ (\rho \mathbf{u})(0) &= \mathbf{m}_0, \quad \mathbf{m}_0 = (m_0^1, m_0^2, m_0^3), \quad \text{where } m_0^i \in C^2(\bar{\Omega}), \\ f(0) &= f_0(x, \xi, r), \quad f_0 \geq 0, \quad f_0 \in L^\infty(\Omega \times \mathbb{R}^3 \times R^+) \cap L^1(\Omega \times \mathbb{R}^3 \times R^+) \\ &\text{and it is compactly supported with respects to } r, \xi. \end{aligned} \quad (3.2)$$

In order to solve this ε, δ regularized Navier–Stokes part of spray fluid system (1.8)–(1.11), we need to show that first moment $j(x, t)$ is bounded in $L^p(0, T; L^q(\Omega))$, for some $p, q > 1$, where of the $j(x, t)$, the solution for Vlasov–Boltzmann transport equation kinetic equation (1.11), is a source term in the ε, δ -regularized momentum equation of Navier–Stokes part of system.

Following arguments introduced by Feireisl et al. [12] and Feireisl [13] for just fluid systems models the compressible by means of these type of ε, δ regularizations of viscosity and pressure terms in Navier–Stokes part, we introduce the approximate by finite dimensional spaces that will yield a sequence of solution triplets with enough compactness to converge solutions of the ε, δ regularization of the spray fluid system (1.8)–(1.11).

In order to accomplish this goal, we start defining the following finite dimensional Banach space $X_k = \operatorname{span}\{e_1, e_2, \dots, e_k\}$, for $n \in \mathbb{N}$, and each e_i is an orthogonal basis of $L^2(\Omega)$, which is also an orthogonal basis of $H^2(\Omega)$.

In particular, e_i could be chosen by $-\Delta e_i = \lambda_i e_i$, that is eigenfunction of the Laplace operator acting over the domain Ω .

Thus, without loss of generality, we consider an infinite sequence of finite dimensional spaces

$$X_k = \text{span}\{e_i\}_{i=1}^k, \quad k = 1, 2, 3, \dots, \quad (3.3)$$

and will construct a sequences of triplets $(\rho_k^{\varepsilon, \delta}, \mathbf{u}_k^{\varepsilon, \delta}, f_k^{\varepsilon, \delta}) = (\rho_k, \mathbf{u}_k, f_k)$ solutions of the following k, ε, δ -approximate problem as follows.

Step 1: Starting from $\mathbf{u}_{k-1}$ given in $C([0, T]; X_{k-1})$, where $X_{k-1} = \text{span}\{e_1, e_2, \dots, e_{k-1}\}$ solve the following initial value problem for the Vlasov–Boltzmann transport equation (1.11).

For any $f_0 \in L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+) \cap L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$ with $f_0 \geq 0$, and $\text{supp} f_0 \subset \Omega \times \mathbb{R}^3$, solve the Vlasov–Boltzmann transport equation

$$\begin{aligned} \partial_t f_k + \xi \cdot \nabla_x f_k + \text{div}_\xi \left(\frac{\mathbf{u}_{k-1} - \xi}{r^2} f_k \right) &= Q(f_k)(x, \xi, r, t), \quad \forall t > 0, \\ f_k(x, \xi, r, 0) &= f_0(x, \xi, r) \quad \text{for all } (x, \xi, r) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^+. \end{aligned} \quad (3.4)$$

and show the the first moment $j_k(x, t) = \int (\xi, r) f_k(x, \xi, r, t) d\xi dr$ associated to is bounded in $L^\infty(0, T; L^2(\Omega))$.

Step 2: For any initial data density-velocity pair $(\rho_k, \mathbf{u}_k)(x, 0)$ satisfying $\rho_k \in L^\gamma(X_k)$, $\mathbf{u}_k \in L^2(X_k)$ and $\nabla \mathbf{u}_k \in L^2(X_k)$, there is a unique weak k, ε - approximate solution triple $\rho_k \in L^\infty([0, T]; L^\gamma(X_k))$, $\mathbf{u}_k \in L^\infty([0, T]; L^2(X_k))$ and $\nabla \mathbf{u}_k \in L^2([0, T]; L^2(X_k))$ satisfying the integral equation

$$\begin{aligned} \int_\Omega \rho \mathbf{u}_k(t) \cdot \varphi \, dx - \int_\Omega \mathbf{m}_0 \cdot \varphi \, dx &= \int_0^t \int_\Omega (\mu \Delta \mathbf{u}_k + \lambda \nabla \text{div} \mathbf{u}_k) \varphi \, dx \, dt \\ &+ \int_0^t \int_\Omega (\varepsilon \nabla \mathbf{u}_k \cdot \nabla \rho - \text{div}(\rho \mathbf{u}_k \otimes \mathbf{u}_k) - \nabla \rho^\gamma - \delta \nabla \rho^\beta - \mathbf{n} \mathbf{u}_k + j) \varphi \, dx \, dt \end{aligned} \quad (3.5)$$

for any test function $\varphi \in X_k$.

The goal in the rest of this section is to prove the following Proposition that secures the existence of a k approximating problems associated to the $\varepsilon \delta$ -regularized system

Proposition 3.1. *For any initial data $(\rho_0, \mathbf{u}_0)(x, 0)$ with $\rho_0 \in L^\gamma(\Omega)$, $\mathbf{u}_0 \in L^2(\Omega)$ and $\nabla \mathbf{u}_0 \in L^2(\Omega)$, and $f_0 \in L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+) \cap L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$, there exists a unique weak solution to the spray fluid system (3.4)–(3.5) denoted by the triplet $(\rho_k^{\varepsilon, \delta}, \mathbf{u}_k^{\varepsilon, \delta}, f_k^{\varepsilon, \delta}) = (\rho_k, \mathbf{u}_k, f_k)$ in the spaces $L^\infty([0, T]; L^\gamma(X_k) \times L^\infty([0, T]; L^2(X_k)) \times (f_0 \in L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+) \cap L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))$.*

In addition the triplet components are uniformly bounded in the k and ε and δ parameters.

The proof of Proposition 3.1 is rather elaborated and will be done in several parts that gather the necessary estimates to complete it.

We start proving or recalling the following results. First, Propositions 3.2 and 3.3 will be sufficient to complete Step 1. After that we prove all sufficient steps to complete the existence of a regularization triplet $(\rho_k^{\varepsilon, \delta}, \mathbf{u}_k^{\varepsilon, \delta}, f_k^{\varepsilon, \delta})$ in a series of Propositions from Proposition 3.4 to Proposition refProposition at the first level, as much as Lemmas 3.1 to 3.2, that will yield a complete proof of Proposition 3.1.

The first result towards addressing the Step 1 of the k -iteration argument, was mostly developed by Leger and Vasser [18], when applied to the coupling with incompressible Navier Stokes. We recall the following in this coming Proposition 3.2 whose proof can be found in [18].

Proposition 3.2. *For any given $\mathbf{u} \in C([0, T], C(\Omega))$, there exist a unique non-negative weak solution to the kinetic problem (3.4) for any $T > 0$, provided the initial data satisfies*

$$f_0 \in L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+) \cap L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$$

and

$$f_0 \geq 0, \quad \text{supp} f_0 \subset \Omega \times \mathbb{R}^3,$$

that is, $f(x, \xi, r, t)$ satisfies

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^+ \times \mathbb{R}^6} f \left(\varphi_t + \xi \cdot \nabla_x \varphi - \frac{\mathbf{u} - \xi}{r^2} \cdot \nabla_\xi \varphi \right) dx d\xi dr + \int_0^T \int_{\mathbb{R}^+ \times \mathbb{R}^6} Q(f) \varphi dx d\xi dr \\ & + \int_0^T \int_{\mathbb{R}^+ \times \mathbb{R}^6} f^0 \varphi(0, x, \xi, r) dx d\xi dr = 0 \end{aligned} \quad (3.6)$$

for any test function $\varphi(t, x, \xi, r)$.

Moreover, this non-negative weak solution satisfies the following estimates:

$$\begin{aligned} f & \in L^\infty(0, T; L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)), \\ f & \in L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)), \\ f & \in C([0, T]; W^{-1,p}(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)), \quad \text{for any } 1 \leq p \leq \infty, \\ \text{supp}(f) & \subset \Omega \times \mathbb{R}^3 \text{ for a.e. } t \in [0, T]. \end{aligned} \quad (3.7)$$

The next step is to secure that the weak solution $f_k(x, \xi, r, t)$ constructed in Proposition 3.2 has its kinetic first moment $j_k(x, t) \in L^\infty(0, T; L^2(\Omega))$.

Proposition 3.3. *If $\mathbf{u}_k \in C([0, T]; X_k)$, then there exist operators $n_k = N(\mathbf{u}_k)$, $j = L(\mathbf{u}_k) : C([0, T]; X_k) \rightarrow C([0, T]; C(\Omega))$ satisfying*

i) (Lipschitz estimate for the kinetic density)

$$\|n_k^1 - n_k^2\|_{L^\infty(0, T; L^\infty(\Omega))} \leq C(a, b, T) \|\mathbf{u}_k^1 - \mathbf{u}_k^2\|_{L^2(0, T; L^2(\Omega))}. \quad (3.8)$$

ii) (Lipschitz estimate for the mean velocity)

$$\|j_k^1 - j_k^2\|_{L^\infty(0, T; L^\infty(\Omega))} \leq C(a, b, T) \|\mathbf{u}_k^1 - \mathbf{u}_k^2\|_{L^2(0, T; L^2(\Omega))}, \quad (3.9)$$

for any $\mathbf{u}_k^1, \mathbf{u}_k^2$ in the following set

$$M_L = \{\mathbf{u}_k \in C([0, T]; X_k); \|u\|_{C([0, T]; X_k)} \leq L, \quad t \in [0, T]\}.$$

Proof Following the strategy in [18], one can construct a sequence of solutions verifying

$$\begin{cases} \partial_t f_k + \xi \cdot \nabla f_k + \text{div}_\xi \left(\frac{\mathbf{u}_{k-1} - \xi}{r^2} f_n \right) = -\nu f_k(x, \xi, r, t) \\ + \nu \int_{r > r^*} B(r^*, r) f_{k-1}(x, \xi, r^*, t) dr^* \\ f_k(x, \xi, r, 0) = f_0(x, \xi, r). \end{cases} \quad (3.10)$$

as follows. First, we need to write the following ODEs:

$$\begin{cases} \frac{dx}{dt} = \xi; \\ \frac{d\xi}{dt} = \frac{\mathbf{u}_{k-1} - \xi}{r^2}; \\ x(0) = x; \\ \xi(0) = \xi, \end{cases} \quad (3.11)$$

then, by the characteristic method, we have the following solution to (3.10)

$$\begin{aligned} f_k(t, x, \xi, r) & = e^{-\int_0^t (\nu - \frac{3}{r^2}) ds} f_0(x(0, t, x, \xi), \xi(0, t, \tau), r) \\ & + \nu \int_0^t \int_{\mathbb{R}^+} e^{-\int_0^t (\nu - \frac{3}{r^2}) ds} B(r, r^*) f_{k-1}(\tau, x(\tau, t, x, \xi), r^*) dr^* d\tau. \end{aligned} \quad (3.12)$$

So taking the limits as $k \rightarrow \infty$, one obtains the weak solutions to (3.4) by the standard argument of weak convergence as in [18]. However, we need to use (3.12) to derive some new estimates due to the compressible fluids and the coupling to the kinetic equations. Let f_k^1 and f_k^2 be two solutions to (3.10)

corresponding to $\mathbf{u}_{k-1}^1$ and $\mathbf{u}_{k-1}^2$ respectively, and f^1 and f^2 be two weak solutions to (3.4) corresponding to $\mathbf{u}^1$ and $\mathbf{u}^2$ respectively. Letting $Y(t, x, \xi) = (x, \xi)$, we have

$$\begin{aligned} \|f_k^1 - f_k^2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} &\leq C(T)\|Y_1 - Y_2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} \\ &\quad + C(T) \int_0^t \|f_k^1 - f_k^2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} ds. \end{aligned} \quad (3.13)$$

In addition,

$$f_k \rightarrow f \text{ in } L^p(0, T; L^p(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)) \quad \text{and } f \in L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)), \quad (3.14)$$

hence, letting $k \rightarrow \infty$ in (3.13), yields

$$\begin{aligned} \|f^1 - f^2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} &\leq C(T)\|Y_1 - Y_2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} \\ &\quad + C(T) \int_0^t \|f^1 - f^2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} ds. \end{aligned} \quad (3.15)$$

However, for the current model we need to control the characteristic ODE's of the transport flow depending on $\mathbf{u}_k(x, t)$, that we estimate as follows.

The first term above, after using (3.11) with $\mathbf{u}_{k-1}$, can be estimated by

$$\begin{aligned} \|Y_1 - Y_2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} &\leq C \left(\int_0^t \left\| \frac{\mathbf{u}_{k-1}^1 - \mathbf{u}_{k-1}^2}{r^2} \right\|_{L^\infty(\Omega)} ds \right. \\ &\quad \left. + \int_0^t (1 + \left\| \frac{\mathbf{u}_{k-1}^1}{r^2} \right\|_{W^{1,\infty}(\Omega)}) \|Y_1 - Y_2\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} ds \right), \end{aligned}$$

and so by the Gronwall inequality, we obtain

$$\|Y_1 - Y_2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} \leq C(a, b, T) \int_0^t \|\mathbf{u}_{k-1}^1 - \mathbf{u}_{k-1}^2\|_{L^2(\Omega)} ds. \quad (3.16)$$

In addition, by (3.15) and (3.16),

$$\|f_k^1 - f_k^2\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))} \leq C(a, b, T) \|\mathbf{u}_{k-1}^1 - \mathbf{u}_{k-1}^2\|_{L^2(0,T;\Omega)}. \quad (3.17)$$

Thus, letting $\mathbf{n}_k = N(\mathbf{u}_{k-1})$ and $j_k = L(\mathbf{u}_{k-1})$, it follows from (3.17) that

$$\begin{aligned} \|\mathbf{n}_k^1 - \mathbf{n}_k^2\|_{L^\infty(0,T;L^\infty(\Omega))} &= \|N(\mathbf{u}_{k-1}^1) - N(\mathbf{u}_{k-1}^2)\|_{L^\infty(0,T;L^\infty(\Omega))} \\ &\leq C(a, b, T) \|\mathbf{u}_{k-1}^1 - \mathbf{u}_{k-1}^2\|_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

and

$$\begin{aligned} \|j_k^1 - j_k^2\| &= \|L(\mathbf{u}_{k-1}^1) - L(\mathbf{u}_{k-1}^2)\|_{L^\infty(0,T;L^\infty(\Omega))} \\ &\leq C(a, b, T) \|\mathbf{u}_{k-1}^1 - \mathbf{u}_{k-1}^2\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Hence, estimates (3.8) and (3.9) hold the proof which completes the of Proposition 3.3. $\square$

Hence, gathering the result from Proposition 3.2 and estimates from Proposition 3.3 we have completed **Step 1** that provides enough estimates to accomplish **Step 2**, which culminate in the proof of Proposition 3.1.

For **Step 2** of the iteration, it is natural to obtain an energy identity for the kinetic part (1.11) of k, δ, ε -approximate compressible fluid kinetic system. The following proposition yields such identity.

Proposition 3.4. [*Kinetic energy conservation*] If $\mathbf{u} \in C([0, T]; X_{k-1})$, any weak solution f of (3.4) satisfies the following identity:

$$\begin{aligned} &\int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_k d\xi dr dx - \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_0 d\xi dr dx \\ &= 2 \int_0^t \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r (\mathbf{u}_{k-1} - \xi) f_k \xi d\xi dr dx dt. \end{aligned}$$

Proof For any $\mathbf{u} \in L^r(0, T; L^{N+p}(\Omega))$, we first denote it by $\mathbf{u}_{k-1} := u$ in the proof as we will use it later on in the approximation of the ε, δ regularization of the spray fluid system.

Using $1 + |\xi|^2$ to multiply on both sides of (3.10), and taking integration by parts, we have

$$\begin{aligned} & \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_k d\xi dr dx - \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_k^0 d\xi dr dx \\ &= 2 \int_0^t \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r (\mathbf{u}_{k-1} - \xi) f_k \xi d\xi dr dx dt \\ & \quad - \nu \int_0^t \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_k \xi d\xi dr dx dt \\ & \quad + \nu \int_0^t \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} \int_{r>r^*} r^3 (1 + |\xi|^2) f_{k-1}(x, \xi, r^*, t) dr^* \xi d\xi dr dx dt. \end{aligned} \quad (3.18)$$

Letting $k \rightarrow \infty$ in (3.18), by the convergence from (3.14) and Fubini's theorem, the conclusion follows. $\square$

Lemma 3.1. *Let $\mathbf{u} \in L^r(0, T; L^{N+p}(\Omega))$ be fixed with any $1 \leq r \leq \infty$ and $p \geq 1$. Assume that $f_0 \in L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+) \cap L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$, $r^3 |\xi|^p f_0 \in L^1(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$, then the solution $f(x, \xi, r, t)$ of (3.4) has the following estimate*

$$\begin{aligned} & \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f d\xi dx dr \\ & \leq p C_{T,N,b} \left(\left(\int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f_0 d\xi dx dr \right)^{\frac{1}{N+p}} + (\|f_0\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + 1) \|\mathbf{u}_{k-1}\|_{L^r(0,T;L^{N+p}(\Omega))} \right)^{N+p}, \end{aligned} \quad (3.19)$$

for any $0 \leq t \leq T$.

Proof For any $\mathbf{u} \in L^r(0, T; L^{N+p}(\Omega))$, as in the previous Proposition's proof, denote it by $\mathbf{u}_{k-1} := u$. Then, for any $p \geq 1$, multiplying $r^3 |\xi|^p$ on both sides of kinetic equation (3.10), we have

$$\begin{aligned} & \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f_k d\xi dx dr - \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f_k^0 d\xi dx dr \\ &= p \int_0^t \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r (\mathbf{u}_{k-1} - \xi) f_k |\xi|^{p-1} \cdot \frac{\xi}{|\xi|} d\xi dx dr dt \\ & \quad - \nu \int_0^t \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 |\xi|^p f_k \xi d\xi dr dx dt \\ & \quad + \nu \int_0^t \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} \int_{r>r^*} r^3 |\xi|^p f_{k-1}(x, \xi, r^*, t) dr^* \xi d\xi dr dx dt. \end{aligned} \quad (3.20)$$

Therefore, letting $k \rightarrow \infty$ in (3.20), (3.14) and the Fubini's theorem yields

$$\begin{aligned} & \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f_k d\xi dx dr - \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f_k^0 d\xi dx dr \\ & \quad + p \int_0^t \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r |\xi|^p f d\xi dx dr dt = \int_0^T I(t) dt, \end{aligned} \quad (3.21)$$

with

$$I(t) = p \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r |\xi|^{p-1} f \mathbf{u}_{k-1} \cdot \frac{\xi}{|\xi|} d\xi dx dr. \quad (3.22)$$

Thanks to Hölder's inequality, $I(t)$ can be controlled as follows

$$I(t) \leq p \|\mathbf{u}_{k-1}\|_{L^s(\Omega)} \left(\int_{\Omega} \left(\int_a^b \int_{\mathbb{R}^3} r |\xi|^{p-1} f \, d\xi \, dr \right)^{s'} dx \right)^{\frac{1}{s}}, \quad (3.23)$$

where

$$\frac{1}{s} + \frac{1}{s'} \leq 1.$$

Further, since for any $R > 0$, then the integrand in the above estimate from (3.23) is controlled by

$$\begin{aligned} \int_a^b \int_{\mathbb{R}^3} r |\xi|^{p-1} f \, d\xi \, dr &= \int_a^b \int_{|\xi| \leq R} r |\xi|^{p-1} f \, d\xi \, dr + \int_a^b \int_{|\xi| \geq R} r |\xi|^{p-1} f \, d\xi \, dr \\ &\leq b^2 \|f\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} \frac{R^{N+p-1}}{N+p-1} + \frac{1}{a^2 R} \int_a^b \int_{|\xi| \geq R} r^3 |\xi|^p f \, d\xi \, dr. \end{aligned} \quad (3.24)$$

Then, taking $s = N + p$ in (3.23), and

$$R = \left(\int_a^b \int_{\mathbb{R}^3} r^3 |\xi|^p f \, d\xi \, dr \right)^{\frac{1}{N+p}} > 0$$

in (3.24), one obtains the following estimate for $I(t)$, defined in (3.22),

$$I(t) \leq p \|\mathbf{u}_{k-1}\|_{L^{N+p}} \left(\frac{1}{a^2} + \frac{b^2}{N+p-1} \|f\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} \right) \left(\int_a^b \int_{\mathbb{R}^3} r^3 |\xi|^p f \, d\xi \, dr \right)^{\frac{N+p-1}{N+p}}. \quad (3.25)$$

We end the proof by observing that from (3.21) and (3.25), it follows

$$\begin{aligned} &\int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f \, d\xi \, dx \, dr \\ &\leq p C_{T,N,a,b} \left(\left(\int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^3 f_0 \, d\xi \, dx \, dr \right)^{\frac{1}{N+p}} + (\|f_0\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + 1) \|\mathbf{u}_{k-1}\|_{L^r(0,T;L^{N+p}(\Omega))} \right)^{N+p} \end{aligned}$$

for any $0 \leq t \leq T$, which completes the estimate stated in Lemma 3.1. $\square$

By now we have gathered enough information to obtain estimates for the zero moment (1.4) and first moment (1.5) of the solutions of the Vlasov–Boltzmann equations for the spray (disperse) part of the system.

We estimate these quantities in the following Lemma 3.2, that may be similar to the variation of the classical moment regularity, so called averaging Lemmas applied to Boltzmann type equations, see by Lions and Perthame [21]. Our proof closely follows the argument by Hamdache [17].

Lemma 3.2. *Under the hypothesis of Lemma 3.1, for any $p \geq 1$, $0 \leq t \leq T$, we have*

$$\|\mathbf{n}_k\|_{L^{\frac{N+p}{N}}(\Omega)} \leq C_{N,b,T} (\|f_k\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + 1) \left(\int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f_k \, d\xi \, dx \, dr \right)^{\frac{N}{N+p}}, \quad (3.26)$$

and

$$\|j_k\|_{L^{\frac{N+p}{N+1}}(\Omega)} \leq C_{N,b,T} (\|f_k\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + 1) \left(\int_a^b \int_{\Omega} \int_{\mathbb{R}^3} r^3 |\xi|^p f_k \, d\xi \, dx \, dr \right)^{\frac{N+1}{N+p}}. \quad (3.27)$$

Proof For any $R > 0$, we can estimate n as follows

$$\begin{aligned} n(t, x) &= \int_a^b \int_{\mathbb{R}^3} r f \, d\xi \, dr = \int_a^b \int_{|\xi| \leq R} r f \, d\xi \, dr + \int_a^b \int_{|\xi| \geq R} r f \, d\xi \, dr \\ &\leq bR^N \|f\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + \frac{1}{a^2 R^p} \int_a^b \int_{|\xi| \geq R} r^3 |\xi|^p f \, d\xi \, dr. \end{aligned} \quad (3.28)$$

Taking

$$R = \left(\int_a^b \int_{\mathbb{R}^3} r^3 |\xi|^p f \, d\xi \, dr \right)^{\frac{1}{N+p}}$$

which is finite by Lemma 3.1, depending only on the initial data, yields

$$n(t, x) \leq C_{N,b} \left(\|f\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + \frac{1}{a^2} \right) \left(\int_a^b \int_{\mathbb{R}^3} r^3 |\xi|^p f \, d\xi \, dr \right)^{\frac{N}{N+p}},$$

and since the estimate 3.19 is uniform in $[0, T]$, thus

$$\|n(t, x)\|_{L^\infty(0,T;L^{\frac{N+p}{N}}(\Omega))} \leq C_{N,b,T} \left(\|f\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + \frac{1}{a^2} \right) \left(\int_\Omega \int_a^b \int_{\mathbb{R}^3} r^3 |\xi|^p f \, d\xi \, dr \, dx \right)^{\frac{N}{N+p}}.$$

We can also use the same arguments to show

$$\|j\|_{L^\infty(0,T;L^{\frac{N+p}{N+1}}(\Omega))} \leq C_{N,b,T} (\|f\|_{L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)} + 1) \left(\int_\Omega \int_a^b \int_{\mathbb{R}^3} r^3 |\xi|^p f \, d\xi \, dr \, dx \right)^{\frac{N+1}{N+p}}.$$

□

Next, we observe that looking at the eigenfunctions of the Laplace operator

$$-\Delta e_i = \lambda_i e_i \quad \text{in } \Omega$$

have bounded solutions, then

$$\mathbf{u} \in L^2(0, T; L^\infty(\Omega)).$$

In particular, such estimate allows us to apply Lemma 3.1 to obtain

$$\int_a^b \int_\Omega \int_{\mathbb{R}^3} r^3 |\xi|^5 f \, d\xi \, dx \, dr < \infty, \quad (3.29)$$

provided the initial data satisfies

$$\int_a^b \int_\Omega \int_{\mathbb{R}^3} r^3 |\xi|^p f_0 \, d\xi \, dx \, dr < \infty,$$

for any $p \geq 5$.

Therefore, Applying Lemma 3.2 to get estimate to the corresponding first moment of the solution of the kinetic equation to (3.29) with $p = 5$ and $N = 3$, we obtain

$$\mathbf{n} = N(\mathbf{u}) \in L^\infty(0, T; L^{\frac{8}{3}}(\Omega)), \quad j = L(\mathbf{u}) \in L^\infty(0, T; L^2(\Omega)), \quad (3.30)$$

and satisfy the estimates (3.8) and (3.9). As a consequence we are able to solve the following regularized compressible Navier–Stokes part by using the estimate on the first kinetic moment $j(t, x)$ of the system

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{u}) &= \varepsilon \Delta \rho, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho^\gamma + \delta \nabla \rho^\beta - \mu \Delta \mathbf{u} - \lambda \nabla \operatorname{div} \mathbf{u} - \varepsilon \nabla \mathbf{u} \cdot \nabla \rho + N(\mathbf{u}) \mathbf{u} &= j, \end{aligned} \quad (3.31)$$

with the initial data (3.2).

In fact, we notice that $n\mathbf{u}$ is a good term for the compressible Navier–Stokes equations because $n(t, x) \geq 0$ is on the left side of the momentum equation and so t is an absorbing term that stabilized the momentum flow dynamics. Another advantage is that the right hand side $j(x, t)$ is bounded in $L^\infty(0, T; L^2(\Omega))$.

Thus the weak solution $(\rho, \mathbf{u})$ to (3.31) can be constructed following the now classical approach in Feireisl et al. [12] and Feireisl [13] for fluid equations. In fact, we can find the approximate solutions $\mathbf{u}_k \in C([0, T]; X_k)$ satisfy the integral equation (3.5), transcribed here for easier reading,

$$\begin{aligned} \int_{\Omega} \rho \mathbf{u}_k(t) \cdot \varphi \, dx - \int_{\Omega} \mathbf{m}_0 \cdot \varphi \, dx &= \int_0^t \int_{\Omega} (\mu \Delta \mathbf{u}_k + \lambda \nabla \operatorname{div} \mathbf{u}_k) \varphi \, dx \, dt \\ &+ \int_0^T \int_{\Omega} (\varepsilon \nabla \mathbf{u}_k \cdot \nabla \rho - \operatorname{div}(\rho \mathbf{u}_k \otimes \mathbf{u}_k) - \nabla \rho^\gamma - \delta \nabla \rho^\beta - n \mathbf{u}_k + j) \varphi \, dx \, dt \end{aligned} \quad (3.32)$$

for any test function $\varphi \in X_k$.

Then, in order to show that (3.5) is solvable, we follow the same arguments as in [12, 13], and introduce the following two operators that are crucial to apply fixed point arguments later by generating an Ordinary Differential Inequality (ODI) in a suitable Banach space.

In our case, the iteration map for a fixed point argument is constructed as follows. For any given $\mathbf{u} \in C([0, T]; X_k)$, ρ is a solution to the following problem

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho, \\ \rho_0 \in C^\infty(\mathbb{T}^3), \quad \rho_0 \geq \underline{\rho} > 0. \end{cases} \quad (3.33)$$

First, we introduce the operator $\mathcal{S}$ as follows

$$\mathcal{S} : C([0, T]; X_k) \rightarrow C([0, T]; C(\Omega)), \rho = \mathcal{S}(\mathbf{u}),$$

and recall the following two Propositions that can be found in [12]

Proposition 3.5. *If $0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho}$, $\rho_0 \in C^\infty(\Omega)$, $\mathbf{u} \in C([0, T]; X_k)$, then there exists an operator $\mathcal{S} : C([0, T]; X_k) \rightarrow C([0, T]; C(\Omega))$ satisfying*

- i) $\rho = \mathcal{S}(\mathbf{u})$ is a unique solution to the problem (3.33).
- ii) *Density bounds:*

$$0 < \underline{\rho} e^{-\int_0^T \|\operatorname{div} \mathbf{u}\|_{L^\infty} dt} \leq \rho(x, t) \leq \bar{\rho} e^{\int_0^T \|\operatorname{div} \mathbf{u}\|_{L^\infty} dt}, \quad \text{for any } x \in \Omega, \quad t \geq 0. \quad (3.34)$$

- iii) *Lipchitz condition:*

$$\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{C([0, T]; C(\Omega))} \leq TC(\rho_0, \varepsilon, L) \|\mathbf{u}_1 - \mathbf{u}_2\|_{C([0, T]; X_k)}, \quad (3.35)$$

for any $\mathbf{u}_1, \mathbf{u}_2$ in the following set

$$M_L = \{\mathbf{u} \in C([0, T]; X_k); \|\mathbf{u}\|_{C([0, T]; X_k)} \leq L, \quad t \in [0, T]\}.$$

In addition, for any given function $\rho \in C^1(\Omega)$ with $\rho \geq \underline{\rho} > 0$, we introduce an operator $\mathcal{M}$ for fixed t , satisfying

$$\mathcal{M}[\rho] : X_k \rightarrow X_k^*, \quad \langle \mathcal{M}[\rho] \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} \, dx, \quad \text{for any } \mathbf{u}, \mathbf{v} \in X_k,$$

and we recall from [12], (page 363–364) the following proposition describing the properties of $\mathcal{M}$:

Proposition 3.6. *For any given function $\rho \in C^0(0, T; C^1(\Omega))$ with $\rho \geq \underline{\rho} > 0$, where $\underline{\rho}$ is a constant,*

- i) $\|\mathcal{M}[\rho]\|_{\mathcal{L}(X_k, X_k^*)} \leq C(k) \|\rho\|_{L^1}$.
- ii) $\|\mathcal{M}[\rho]\|_{\mathcal{L}(X_k, X_k^*)} \geq \inf_{x \in \Omega} \rho$
- iii) *If $\inf_{x \in \Omega} \rho \geq \underline{\rho} > 0$, then the operator is invertible with*

$$\|\mathcal{M}^{-1}[\rho]\|_{\mathcal{L}(X_k^*, X_k)} \leq \underline{\rho}^{-1},$$

where $\mathcal{L}(X_k^*, X_k)$ is the set of bounded linear mappings from X_k^* to X_k .

iv) $\mathcal{M}^{-1}[\rho]$ is Lipschitz continuous in X_k^* in the sense

$$\|\mathcal{M}^{-1}[\rho_1] - \mathcal{M}^{-1}[\rho_2]\|_{\mathcal{L}(X_k^*, X_k)} \leq C(n, \underline{\rho}) \|\rho_1 - \rho_2\|_{L^1(\Omega)} \quad (3.36)$$

for all $\rho_1, \rho_2 \in C^0(0, T; L^1(\Omega))$ such that $\rho_1, \rho_2 \geq \underline{\rho} > 0$.

The proofs of these two propositions can be found on [12] (page 363) and (page 363–364) respectively. They are sufficient in order to show the needed compactness for the existence of a fixed point solution set.

We apply the strategy of [12] to the problem under consideration, namely the existence of solutions to the coupled compressible fluid equation to the gas kinetic equation, done through the gas density $\mathbf{n}$ defined by (1.4) and gas current j defined by (1.5).

Indeed, making use of the operators $\mathcal{M}[\rho]$, $\rho = \mathcal{S}(\mathbf{u}_k)$, $\mathbf{n} = N(\mathbf{u}_k)$ and $j = L(\mathbf{u}_k)$, we rewrite (3.5) as the following ordinary differential equation on the finite-dimensional space X_k :

$$\begin{aligned} \frac{d}{dt} (\mathcal{M}[\mathcal{S}(\mathbf{u}_k)(t)] \mathbf{u}_k(t)) &= \mathcal{N}(\mathcal{S}(\mathbf{u}_k), N(\mathbf{u}_k), L(\mathbf{u}_k), \mathbf{u}_k), \quad t > 0, \\ \mathcal{M}[\mathcal{S}(\mathbf{u}_k)(0)] \mathbf{u}_k(0) &= \mathcal{M}[\rho_0] \mathbf{u}_0, \end{aligned} \quad (3.37)$$

where

$$\begin{aligned} \mathcal{N}(\mathcal{S}(\mathbf{u}_k), N(\mathbf{u}_k), L(\mathbf{u}_k), \mathbf{u}_k), \varphi t &= \int_{\Omega} (\mu \Delta \mathbf{u}_k + \lambda \nabla \operatorname{div} \mathbf{u}_k + \varepsilon \nabla \mathbf{u}_k \cdot \nabla \rho) \cdot \varphi \, dx \\ &\quad - \int_{\Omega} (\operatorname{div}(\rho \mathbf{u}_k \otimes \mathbf{u}_k) + \nabla \rho^\gamma + \delta \nabla \rho^\beta + \mathbf{n} \mathbf{u}_k - j) \cdot \varphi \, dx, \end{aligned}$$

for all $\varphi \in X_k$. Integrating (3.37) over $(0, t)$, we can write the problem as the following nonlinear problem:

$$\mathbf{u}_k(t) = \mathcal{M}^{-1}[\mathcal{S}(\mathbf{u}_k)(t)] (\mathcal{M}[\rho_0] \mathbf{u}_0 + \int_0^t \mathcal{N}(\mathcal{S}(\mathbf{u}_k), N(\mathbf{u}_k), L(\mathbf{u}_k), \mathbf{u}_k)(s) ds). \quad (3.38)$$

Since $\mathcal{N}(\mathcal{S}(\mathbf{u}_k), N(\mathbf{u}_k), L(\mathbf{u}_k), \mathbf{u}_k)$ is a Liptzchiz function, as all its argument from (3.8), (3.9), (3.35) and (3.36), this equation can be solved with the fixed-point theorem of Banach, at least on a small time $0 < T' \leq T$. Thus, we obtained a unique $\mathbf{u}_k \in C^0(0, T'; X_k)$.

In order to extend the existence final time in order to get $T' = T$, it is enough to show there exists uniform estimates on solution triplet $(\rho_k, \mathbf{u}_k, f_k)$ in suitable functional spaces defined over the finite dimensional space X_k .

Indeed, the following definition of a suitable energy functional and subsequent proposition provide the global in time existence of solutions to the approximation system (3.1)–(3.2).

Definition 3.1. (*The Energy Functional*) The natural energy functional associated to the triplet $(\rho_k, \mathbf{u}_k, f_k)$ solution to the approximation system (3.1)–(3.2) is given by

$$\begin{aligned} E(t) := E(\rho_k, \mathbf{u}_k, f_k)(t) &:= \int_{\Omega} \left(\frac{1}{2} \rho_k |\mathbf{u}_k|^2 + \frac{\rho_k^\gamma}{\gamma - 1} + \frac{\delta}{\beta - 1} \rho_k^\beta \right) dx \\ &\quad + \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_k \, d\xi \, dr \, dx, \end{aligned}$$

The corresponding initial energy is

$$E_0 := E(0) = \int_{\Omega} \left(\frac{\mathbf{m}_0^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} + \frac{\delta}{\beta - 1} \rho_0^\beta \right) dx + \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_0 \, d\xi \, dr \, dx. \quad (3.39)$$

The desired estimates will follow from the following result.

Proposition 3.7. (The Energy Inequality) *Let the triplet $(\rho_k, \mathbf{u}_k, f_k)$ be the solution to system (3.1)–(3.2) constructed above, then for any $T > 0$, the $(\rho_k, \mathbf{u}_k, f_k)$ satisfies the following energy inequality*

$$\begin{aligned} E(t) + \mu \int_0^T \int_{\Omega} |\nabla \mathbf{u}_k|^2 dx dt + \lambda \int_0^T \int_{\Omega} |\operatorname{div} \mathbf{u}_k|^2 dx dt \\ + \varepsilon \int_0^T \int_{\Omega} (\gamma \rho_k^{\gamma-2} + \delta \beta \rho_k^{\beta-2}) |\nabla \rho_k|^2 dx dt \leq E_0. \end{aligned} \quad (3.40)$$

Proof First, taking $\varphi = \mathbf{u}_k$ in (3.5), one obtains the following identity corresponding to the regularized Navier–Stokes part (3.31)

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho_k |\mathbf{u}_k|^2 + \frac{\rho_k^{\gamma}}{\gamma-1} + \frac{\delta}{\beta-1} \rho_k^{\beta} \right) dx \\ + \mu \int_{\Omega} |\nabla \mathbf{u}_k|^2 dx + \lambda \int_{\Omega} |\operatorname{div} \mathbf{u}_k|^2 dx + \varepsilon \int_{\Omega} (\gamma \rho_k^{\gamma-2} + \delta \beta \rho_k^{\beta-2}) |\nabla \rho_k|^2 dx \\ + \int_{\Omega} \mathbf{n}_k |\mathbf{u}_k|^2 dx = \int_{\Omega} j_k \mathbf{u}_k dx, \end{aligned} \quad (3.41)$$

for any $t \in [0, T']$. Next, applying Proposition 3.4, and adding (3.41), we obtain the following L^2 energy identity for the whole system that includes the kinetic equation (3.4):

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} \left(\frac{1}{2} \rho_k |\mathbf{u}_k|^2 + \frac{\rho_k^{\gamma}}{\gamma-1} + \frac{\delta}{\beta-1} \rho_k^{\beta} \right) dx + \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_k d\xi dr dx \right) \\ + \mu \int_{\Omega} |\nabla \mathbf{u}_k|^2 dx + \lambda \int_{\Omega} |\operatorname{div} \mathbf{u}_k|^2 dx + \varepsilon \int_{\Omega} (\gamma \rho_k^{\gamma-2} + \delta \beta \rho_k^{\beta-2}) |\nabla \rho_k|^2 dx \\ + \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r f_k |\mathbf{u}_k - \xi|^2 d\xi dr dx = 0 \end{aligned}$$

on $[0, T']$.

Integrating with respect to t , we deduce the following energy identity

$$\begin{aligned} E(\rho_k, \mathbf{u}_k, f_k)(t) + \mu \int_0^{T_k} \int_{\Omega} |\nabla \mathbf{u}_k|^2 dx dt + \lambda \int_0^{T_k} \int_{\Omega} |\operatorname{div} \mathbf{u}_k|^2 dx dt \\ + \varepsilon \int_0^{T_k} \int_{\Omega} (\gamma \rho_k^{\gamma-2} + \delta \beta \rho_k^{\beta-2}) |\nabla \rho_k|^2 dx dt \\ + \int_0^{T_k} \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r f_k |\mathbf{u}_k - \xi|^2 d\xi dr dx dt = E_0, \end{aligned}$$

on $[0, T']$, where the total energy $E(t) = E(\rho_k, \mathbf{u}_k, f_k)(t)$ and its initial form E_0 were defined in (3.39) and (3.39), respectively.

In particular, since both terms

$$\begin{aligned} \varepsilon \int_0^{T_k} \int_{\Omega} (\gamma \rho_k^{\gamma-2} + \delta \beta \rho_k^{\beta-2}) |\nabla \rho_k|^2 dx dt \\ \text{and} \\ \int_0^T \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r f_k |\mathbf{u}_k - \xi|^2 d\xi dr dx dt, \end{aligned}$$

are non-negative, then the energy inequality (3.40) naturally. $\square$

The energy inequality (3.40), together with estimate (3.34), yield the following uniform bounds in k and ε and δ , for the the components of the triplet solutions to system (3.1)–(3.2),

$$\begin{aligned}\|\mathbf{u}_k\|_{L^\infty(0,T;L^2(\Omega))} &\leq C_0 < \infty, \\ \|\rho_k\|_{L^\infty(0,T;L^\gamma(\Omega))} &\leq C_0 < \infty, \\ \|\nabla \mathbf{u}_k\|_{L^2(0,T;L^2(\Omega))} &\leq C_0 < \infty,\end{aligned}\tag{3.42}$$

where C_0 only depends on the initial data through the energy relation evaluated on the data.

To end, noting that the $L^\infty(X_k)$ and $L^2(X_k)$ –norms are equivalent on the finite dimensional space X_k , then

$$\sup_{t \in [0, T_k]} (\|\mathbf{u}_k\|_{L^\infty} + \|\nabla \mathbf{u}_k\|_{L^\infty}) \leq C_0(E_0).$$

As a consequence of this observation, the existence and uniqueness in the time interval $[0, T']$ is uniform in time, and by bootstrapping arguments, the existence and uniqueness extends to $[0, T]$ for all $T > 0$.

Hence, the global in time existence and uniqueness proof of a weak solution triplet $(\rho_k^{\varepsilon, \delta}, \mathbf{u}_k^{\varepsilon, \delta}, f_k^{\varepsilon, \delta}) = (\rho_k, \mathbf{u}_k, f_k)$ to the k approximation of the ε, δ regularization (3.1)–(3.2) system for any $T > 0$ completes the proof of Proposition 3.1.

4. Recover Weak Solutions by a k, ε, δ -Limiting Process

In order to complete Theorem 2.1, we need to recover weak solutions to (1.8)–(1.11). To this end, we study the passage to the limit behavior in the following order, as $k \rightarrow \infty$, next $\varepsilon \rightarrow 0$ and finally $\delta \rightarrow 0$, for the unique solutions constructed as in Proposition 3.1. Here we use the triplet $(\rho_k, \mathbf{u}_k, f_k)$ to denote such solution, where we still omit ε and δ for notation simplicity,

Using the bound from the energy inequality (3.40), the following uniformly estimates hold

$$\|\sqrt{\rho_k} \mathbf{u}_k\|_{L^\infty(0,T;L^2(\Omega))} \leq C_0 < \infty, \tag{4.1}$$

$$\|\rho_k\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C_0 < \infty, \tag{4.2}$$

$$\|\nabla \mathbf{u}_k\|_{L^2(0,T;L^2(\Omega))} \leq C_0 < \infty, \tag{4.3}$$

$$\delta \int_{\Omega} \frac{1}{\beta - 1} \rho_k^\beta dx \leq C_0 < \infty, \quad \text{for any } t \in (0, T), \tag{4.4}$$

$$\varepsilon \int_0^T \int_{\Omega} (\gamma \rho_k^{\gamma-2} + \delta \beta \rho_k^{\beta-2}) |\nabla \rho_k|^2 dx dt \leq C_0 < \infty, \tag{4.5}$$

$$\int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_k d\xi dr dx \leq C_0 < \infty, \quad \text{for any } t \in (0, T), \tag{4.6}$$

where $C_0 = C_0(E_0)$ only depends on the initial data through the energy relation evaluated on the data, as given in (3.39).

Then a consequence we can show the following Lemma.

Lemma 4.1. *There exists a constant C independent on index k , and regularization parameters ε and δ such that*

$$\|\mathbf{n}_k(t)\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{4.7}$$

$$\|j_k(t)\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))} \leq C. \tag{4.8}$$

Proof By (4.3), we have

$$\|\mathbf{u}_k\|_{L^2(0,T;L^6(\Omega))} \leq C,$$

where $C = C(E_0)$ is uniform in k, ε and δ ; and hence $\mathbf{u}_k$ is also uniformly bounded in $L^2(0, T; L^6(\Omega))$. Therefore, taking $N = p = 3$ in Lemmas 3.1 and 3.2, then (4.7) and (4.8) follow. $\square$

The next step is to show that the limit in k for the sequence of solution $(\rho_k, \mathbf{u}_k, f_k)$ exists in the following sense.

Proposition 4.1. *Let the solutions of $(\rho_k, \mathbf{u}_k, f_k)$ constructed in Proposition 3.7, then for any $\gamma > \frac{3}{2}$,*

$$\begin{aligned}\rho_k &\rightarrow \rho \quad \text{in } L^1((0, T) \times \Omega) \quad \text{and } C([0, T]; L_{weak}^\gamma(\Omega)), \\ \mathbf{u}_k &\rightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; W_0^{1,2}(\Omega)), \\ \rho_k \mathbf{u}_k &\rightarrow \rho \mathbf{u} \quad \text{in } C([0, T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega)),\end{aligned}$$

and

$$\rho_k^\gamma \rightarrow \rho^\gamma \quad \text{in } L^{\frac{\gamma+\theta}{\gamma}}((0, T) \times \Omega) \quad \text{for some } 0 < \theta < \frac{\gamma}{3}.$$

Remark 4.1. The proof of this proposition follows from techniques developed by Lions [20] and Feireisl and collaborators [11–13] applied to the compressible Navier–Stokes equations with the external forces. They are crucial for the limiting process of the solution to the whole fluid-kinetic system. In the sake of completeness we write some of these estimates in the actual larger system context.

The uniform estimate (4.10) holds for solutions to the compressible Navier–Stokes equations, even with the external force if it belongs to $L^p(0, T; L^q(\Omega))$ for some $p, q > 1$. For the more detail, we refer the readers to [11–13, 20].

Thus, the first step consist in controlling the uniform estimate of the force term in k , δ and ε , namely

$$-\int_a^b \int_{\mathbb{R}^3} r(\mathbf{u}_k - \xi) f_k d\xi dr = -\mathbf{n}_k \mathbf{u}_k + j_k, \quad (4.9)$$

which has been proved to be bounded in $L^p(0, T; L^q(\Omega))$ for some $p, q > 1$, uniformly in k , δ and ε . In fact, we can obtain the control

$$\|j_k - \mathbf{n}_k \mathbf{u}_k\|_{L^2(0, T; L^{\frac{3}{2}}(\Omega))} \leq C \|j_k\|_{L^\infty(0, T; L^{\frac{3}{2}}(\Omega))} + C \|\mathbf{n}_k\|_{L^\infty(0, T; L^2(\Omega))} \|\mathbf{u}_k\|_{L^2(0, T; L^6(\Omega))},$$

that allow us to conclude that $j_k - \mathbf{n}_k \mathbf{u}_k$ is uniformly bounded in $L^2(0, T; L^{\frac{3}{2}}(\Omega))$.

Note that $-\int_a^b \int_{\mathbb{R}^3} r(\mathbf{u}_k - \xi) f_k d\xi dr$ is bounded in $L^2(0, T; L^{\frac{3}{2}}(\Omega))$, we can apply the argument in [11–13, 20] to (3.1). We obtain the following estimate in Lemma 4.2.

Lemma 4.2. *For any $\gamma > \frac{3}{2}$, there exists a constant $0 < \theta < \frac{\gamma}{3}$, depending on γ , such that*

$$\int_0^T \int_{\Omega} (a \rho_k^{\gamma+\theta} + \delta \rho_k^{\beta+\theta}) dx dt \leq C < \infty, \quad (4.10)$$

where $C > 0$ is uniformly on n , ε and δ .

With above convergence of Proposition 4.1 in hand, we are ready to pass to the limits for the Navier–Stokes part as $k \rightarrow \infty$. We could use the similar arguments to handle the other limits with respects to ε and δ . For more details on the weak stability of the compressible Navier–Stokes equations, we refer the readers to [12, 13, 19].

The next lines focus on the stability of weak solutions to the kinetic equation (3.4). By (3.7) follows the convergence of $f_k - f$ in the following weak* topology, independently of the parameters ε and δ , as the boundedness of the unknowns depend on the energy functional on the initial data, so they are controlled independently on ε and δ , that is

$$f_k \rightharpoonup f \quad L^\infty(0, T; L^p(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)) - \text{weak}^*, \text{ for any } 1 < p \leq \infty, . \quad (4.11)$$

Similarly, letting $\varphi(x)$ be a smooth compactly supported test function, we can estimate

$$\begin{aligned}
\int_{\Omega} (j_k - j) dx &:= \int_{\Omega} (j_k - \int_a^b \int_{\mathbb{R}^3} r \xi f d\xi dr) \varphi(x) dx \\
&\leq \iiint r (f_k - f) (1 + |\xi|) \varphi(x) d\xi dr dx \\
&= \iiint \left(r^{\frac{2}{3}} (f_k - f)^{\frac{2}{3}} (1 + |\xi|)^{\frac{4}{3}} \varphi^{\frac{2}{3}}(x) \right) \left(r^{\frac{1}{3}} (f_k - f)^{\frac{1}{3}} (1 + |\xi|)^{-\frac{1}{3}} \varphi^{\frac{1}{3}}(x) \right) d\xi dr dx \quad (4.12) \\
&\leq 2 \left(\iiint r (f_k - f) (1 + |\xi|^2) \varphi(x) d\xi dr dx \right)^{\frac{2}{3}} \left(\iiint r (f_k - f) \frac{\varphi(x)}{1 + |\xi|} d\xi dr dx \right)^{\frac{1}{3}} \\
&= 2C \left(\iiint r (f_k - f) \frac{\varphi(x)}{1 + |\xi|} d\xi dr dx \right)^{\frac{1}{3}},
\end{aligned}$$

where we used (4.6) and the estimate

$$\begin{aligned}
&\left(\iiint r (f_k - f) (1 + |\xi|^2) \varphi(x) d\xi dr dx \right)^{\frac{2}{3}} \\
&\leq \left(2 \iiint r (1 + |\xi|^2) f_k d\xi dr dx \right)^{\frac{2}{3}} \leq C(C_0).
\end{aligned}$$

Thus, the last term in (4.12) converges to zero as k goes to infinity since f_k converges to f weakly in $L^2(0, T; L^2(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))$ and

$$\frac{r\varphi(x)}{1 + |\xi|} \in L^2_{loc}(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+),$$

both independently of the parameters ε and δ

It follows that

$$j_k \rightharpoonup j \quad \text{weakly in } L^\infty(0, T; L^p(\Omega)) \quad \text{for any } 1 \leq p \leq \frac{3}{2}. \quad (4.13)$$

independently of the parameters ε and δ , as well.

Similarly, we have that

$$\mathbf{n}_k = \iint r f_k d\xi dr \rightharpoonup \mathbf{n} = \iint r f d\xi dr \quad \text{weakly in } L^2(0, T; L^2_{loc}(\Omega)). \quad (4.14)$$

By (3.7) again, f_k is uniformly bounded in $L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))$. Relying on this, we can show the following uniform bounds. With (4.14), we have the weak convergence of $Q(f_k)$.

Lemma 4.3. *If (3.7), then $Q(f_k)$ is uniformly bounded in*

$$L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap L^\infty(0, T; L^p(\Omega \times \mathbb{R}^3 \times \mathbb{R}^+))$$

for any $p \geq 1$, and, independently of the parameters ε and δ ,

$$\int_a^b \int_{\mathbb{R}^3} Q(f_k) d\xi dr \rightharpoonup \int_a^b \int_{\mathbb{R}^3} Q(f) d\xi dr \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \quad (4.15)$$

Proof

$$\begin{aligned}
\|Q(f_k)\|_{L^\infty} &\leq \nu \|f_k(x, \xi, r, t)\|_{L^\infty} + \nu \|f_k(x, \xi, r, t)\|_{L^\infty} \int_{r>r^*} B(r^*, r) dr^* \\
&\leq (\nu + C\nu) \|f_k(x, \xi, r, t)\|_{L^\infty},
\end{aligned}$$

where we used a fact

$$\int_{r>r^*} B(r^*, r) dr^* \leq C.$$

Similarly,

$$\begin{aligned}\|Q(f_k)\|_{L^1} &\leq \nu \|f_k(x, \xi, r, t)\|_{L^1} + \nu \left\| \int_{r>r^*} B(r^*, r) f_k(t, x, \xi, r^*) dr^* \right\|_{L^1} \\ &\leq \nu \|f_k(x, \xi, r, t)\|_{L^1} + \nu \|f_k(x, \xi, r, t)\|_{L^1} \int_{r>r^*} B(r, r^*) dr^* \\ &\leq (\nu + c\nu) \|f_k(x, \xi, r, t)\|_{L^1}.\end{aligned}$$

For any smooth $\varphi(x)$,

$$\begin{aligned}&\int_{\Omega} \left(\int_a^b \int_{\mathbb{R}^3} Q(f_k)(x, \xi, r, t) d\xi dr - \int_a^b \int_{\mathbb{R}^3} Q(f_k)(x, \xi, r, t) d\xi dr \right) \varphi(x) dx \\ &\leq \frac{\nu}{a} \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r(f_k - f) d\xi dr \varphi(x) dx + \frac{\nu}{a} \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} \int_{r>r^*} r B(r^*, r) (f_k - f) dr^* d\xi dr dx \\ &\leq \frac{C\nu}{a} \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r(f_k - f) d\xi dr \varphi(x) dx \rightarrow 0, \quad \text{as } k \rightarrow \infty,\end{aligned}$$

independently of the parameters ε and δ By (4.14), we have (4.15). $\square$

The last task is to handle the convergence of the right-hand side of (3.5)

$$\int_a^b \int_{\mathbb{R}^3} r \mathbf{u}_k f_k d\xi dr.$$

In order to prove this convergence, we follow a rather similar argument from [23], after we invoke the following compactness lemma from [20].

Lemma 4.4. *Let g^n and h^n converge weakly to g and h respectively in $L^{p_1}(0, T; L^{p_2}(\Omega))$ and $L^{q_1}(0, T; L^{q_2}(\Omega))$ where $1 \leq p_1, q_1 \leq +\infty$,*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

We assume in addition that

$$\frac{\partial g^n}{\partial t} \text{ is bounded in } L^1(0, T; W^{-m,1}(\Omega)), \text{ for some } m \geq 0 \text{ independently of } n$$

and

$$\|h^n - h^n(\cdot + \xi, t)\|_{L^{q_1}(0, T; L^{q_2}(\Omega))} \rightarrow 0, \quad \text{as } |\xi| \rightarrow 0, \quad \text{uniformly in } n.$$

Then, $g^n h^n$ converges to gh in the sense of distributions on $\Omega \times (0, T)$.

Indeed, we first recall that

$$(\mathbf{n}_k)_t = -\operatorname{div}_x(j_k),$$

and so $(\mathbf{n}_k)_t$ is bounded in $L^\infty(0, T; W^{-1,1}(\Omega))$. Next, since $\nabla \mathbf{u}_k$ is bounded in $L^2(0, T; L^2(\Omega))$, we can apply a Lemma 4.4, to obtain the distributional convergence for the macroscopic current of the spray droplets

$$\mathbf{n}_k \mathbf{u}_k \rightarrow \mathbf{n} \mathbf{u} \text{ in the sense of distributions.} \quad (4.16)$$

Similarly, we are able to show, as $k \rightarrow \infty$,

$$\int_{\Omega} \int_a^b \int_{\mathbb{R}^3} \frac{\mathbf{u}_k - \xi}{r^2} f_k \phi d\xi dr dx \rightarrow \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} \frac{\mathbf{u} - \xi}{r^2} f \phi d\xi dr dx \quad (4.17)$$

for any $\phi \in C^1([0, T] \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$ with compact support, independently of the parameters ε and δ .

Then, from Proposition 4.1, and limit results from (4.13), (4.15), (4.16) and (4.17), we are ready to pass to the limits in the weak formulation of the Navier–Stokes and in the weak formulation of kinetic equation.

Indeed, set the weak formulation for any sufficiently smooth, compactly supported functions $\varphi(t, x)$ and $\phi(t, x, v, r)$,

$$\begin{aligned} \int_{\Omega} \rho_k \mathbf{u}_k(t) \cdot \varphi \, dx - \int_{\Omega} \mathbf{m}_0 \cdot \varphi \, dx &= \int_0^t \int_{\Omega} (\mu \Delta \mathbf{u}_k + \lambda \nabla \operatorname{div} \mathbf{u}_k) \varphi \, dx \, dt \\ &+ \int_0^t \int_{\Omega} \left(\varepsilon \nabla \mathbf{u}_k \cdot \nabla \rho_k - \operatorname{div}(\rho_k \mathbf{u}_k \otimes \mathbf{u}_k) - \nabla \rho_k^\gamma - \delta \nabla \rho_k^\beta - \mathbf{n}_k \mathbf{u}_k + j_k \right) \varphi \, dx \, dt, \end{aligned}$$

and

$$\begin{aligned} &- \int_0^t \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} f_k \left(\partial_t \phi + \xi \cdot \nabla_x \phi + \frac{(\mathbf{u}_k - \xi)}{r^2} \cdot \nabla_{\xi} \phi \right) \, dx d\xi \, dr ds \\ &= \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} f_0 \phi(0, \cdot, \cdot, \cdot) \, dx d\xi \, dr + \int_0^t \int_a^b \int_{\Omega} \int_{\mathbb{R}^3} Q(f_k) \phi \, d\xi \, dx \, dr \, dt', \end{aligned}$$

respectively.

As it was stressed, all bounds and k -convergence limits calculated in this section are independent on ε and δ . Thus, we can pass into the limits as $k \rightarrow \infty$, $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$ at the same time.

Thus, all convergence results in this section allow us to recover the weak formulations (2.3)–(2.4) by passing into the limits as $k \rightarrow \infty$ first, and then proceed to the $\varepsilon \rightarrow 0$ convergence, and last the $\delta \rightarrow 0$ one.

Therefore, passing to the limits in (3.40) with respects to $k \rightarrow \infty$, $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$, the control of energy inequality (2.5) is obtained from the following Lemma.

Lemma 4.5. *If $(\rho, \mathbf{u})$ is the weak limit of $(\rho_k, \mathbf{u}_k)$ as k goes to infinity, then*

$$\begin{aligned} &\int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx + \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f \, d\xi \, dr \, dx \\ &\quad + \mu \int_0^T \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx \, dt + \lambda \int_0^T \int_{\Omega} |\operatorname{div} \mathbf{u}|^2 \, dx \, dt \\ &\leq \int_{\Omega} \left(\frac{\mathbf{m}_0^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx + \int_{\Omega} \int_a^b \int_{\mathbb{R}^3} r^3 (1 + |\xi|^2) f_0 \, d\xi \, dr \, dx. \end{aligned} \tag{4.18}$$

In addition, the same conclusion holds true as the limits $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$.

Proof By the weak convergence and energy convexity, estimates (4.18) follow by passing to the limit from (3.40) with respect to $k \rightarrow \infty$.

Finally, since all estimates are uniformly for both ε and δ , then the corresponding limiting problem, as both parameters tend to zero, yield a solution to the problem posed in Theorem 2.1. $\square$

Thus, we have completed the proof of our main result Theorem 2.1.

Acknowledgements. I. M. Gamba acknowledges support from NSF Grant DMS-1413064 and DMS-1715515. C. Yu acknowledges support from NSF Grant DMS-1540162. This work was also partially funded by NSF RNMS-1107465. The authors thank the support from the Institute of Computational Engineering and Sciences (ICES) at the University of Texas Austin.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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(accepted: May 23, 2020; published online: September 4, 2020)