



On Existence and Uniqueness to Homogeneous Boltzmann Flows of Monatomic Gas Mixtures

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Abstract

We solve the Cauchy problem for the full non-linear homogeneous Boltzmann system of equations describing multi-component monatomic gas mixtures for binary interactions in three dimensions. More precisely, we show the existence and uniqueness of the vector value solution by means of an existence theorem for ODE systems in Banach spaces under the transition probability rates assumption corresponding to hard potentials rates in the interval $(0, 1]$, with an angular section modeled by an integrable function of the angular transition rates modeling binary scattering effects. The initial data for the vector valued solutions needs to be a vector of non-negative measures with finite total number density, momentum and strictly positive energy, as well as to have a finite $L^1_{k_*}(\mathbb{R}^3)$ -integrability property corresponding to a sum across each species of k_* -polynomial weighted norms depending on the corresponding mass fraction parameter for each species as much as on the intermolecular potential rates, referred as to the scalar polynomial moment of order k_* . The rigorous existence and uniqueness results rely on a new angular averaging lemma adjusted to vector values solution that yield a Povzner estimate with constants that decay with the order of the corresponding dimensionless scalar polynomial moment. In addition, such initial data yields global generation of such scalar polynomial moments at any order as well as their summability of moments to obtain estimates for corresponding scalar exponentially decaying high energy tails, referred as to scalar exponential moments associated to the system solution. Such scalar polynomial and exponential moments propagate as well.

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1. Introduction

We consider a mixture of I monatomic gases, labeled with $\mathcal{A}_1, \dots, \mathcal{A}_I$. In the kinetic theory framework, each species of the mixture $\mathcal{A}_i$ is statistically described with its own distribution function $f_i := f_i(t, x, v)$ that, in general, depends on time $t \geq 0$, space position $x \in \mathbb{R}^3$ and velocity of molecules $v \in \mathbb{R}^3$ (in this manuscript we restrict ourselves to the spatially homogeneous case, that is, we drop dependence on space position x). The distribution function f_i changes due to binary interactions (or collisions) with other particles. In the mixture setting, these particles can belong to other species $\mathcal{A}_j$, $j \neq i$. Therefore, the evolution of each f_i involves not only the particle-particle interaction of specie $\mathcal{A}_i$, but also interactions between $\mathcal{A}_i$ and $\mathcal{A}_j$, $j \neq i$.

In the mixture framework, the evolution of each distribution function f_i describing the mixture component $\mathcal{A}_i$ is governed by the Boltzmann-like equation, which traditionally introduces a collision operator as a measure of its change. Now, one has multi-species collision operators and their transition probabilities, or cross sections, between the different distribution functions describing each component of the mixture [21]. Since all species are considered simultaneously in a system of species with binary interactions, one is led to introduce a vector valued set of distribution functions $\mathbb{F} = [f_i]_{1 \leq i \leq I}$, whose evolution is governed by a vector of collision operators and whose i -th component (that describes precisely evolution of f_i) is $[\mathbb{Q}(\mathbb{F})]_i = \sum_{j=1}^I \mathcal{Q}_{ij}(f_i, f_j)$. In this formula, operator $\mathcal{Q}_{ij}(f_i, f_j)$ describes the influence of species $\mathcal{A}_j$ for the distribution function f_j on species $\mathcal{A}_i$ with the distribution function f_i . Note that summation over all $j = 1, \dots, I$ is in the spirit of taking into account the influence of all species $\mathcal{A}_j$, $j = 1, \dots, I$, on the considered species $\mathcal{A}_i$.

From a mathematical viewpoint, the challenging situation occurs when masses of species molecules are not equal (i.e. $m_i \neq m_j$). In such a situation, underlying binary collisions between molecules lose some symmetry properties, which can dramatically change their mathematical treatment; for instance, in order to study diffusion asymptotics when one needs to show the compactness of a part of

linearized Boltzmann operator [9]. In the mixture framework, a linear system of linearized Boltzmann equations has been recently studied in [11], corresponding to the perturbative setting of our model when the non-linear system is linearized near Maxwellian states corresponding to each species. In this case authors showed the existence, uniqueness, positivity and exponential trend to equilibrium.

In this work, we give the first existence and uniqueness result for the non-linear system of spatially homogeneous Boltzmann equations for multi-species mixtures with binary interactions in a suitable Banach space. We also emphasize that our approach for solving the Cauchy problem for the Boltzmann equation with variable hard potentials relies on some specific conditions on the initial moments, without requesting entropy boundedness. The hard potentials assumption correspond to collision cross sections related to the species $\mathcal{A}_i$ and $\mathcal{A}_j$ proportional to the local relative speed with a power exponent $\gamma_{ij} \in (0, 1]$, and L^1 -integrable angular part b_{ij} , as function of the scattering direction.

In addition, the existence and uniqueness of a vector value solution $\mathbb{F}(t, v)$ needs to assume that initially its scalar zero and second moment (i.e. the scalar number density and energy of the mixture) are strictly positive and finite, and additionally that this function has at least an upper bounded k_* -polynomial moments, where $k_* := \max\{\bar{k}, 2 + 2\bar{\gamma}\}$, for $\bar{k} = \max_{1 \leq i, j \leq I} \{k_*^{ij}\}$ and $\bar{\gamma} = \max_{1 \leq i, j \leq I} \gamma_{ij}$, is sufficiently large to ensure the prevail of the polynomial moments of loss term with respect to those same moments of the gain term. Each k_*^{ij} depends on the angular transition rate b_{ij} as well as on the two-body mass fraction $r_{ij} := m_i/(m_i + m_j)$ associated to each component on the vector solution. All these parameters are defined in the next Sect. 2 dedicated to notation, preliminaries and main results.

The result is obtained following the general ODE theory that studies differential equations in suitable Banach spaces [17]. In the context of the (single) Boltzmann equation, this theory was proposed as a main tool in [10] for solving the Cauchy problem with hard spheres in three dimensions and constant angular transition probability kernel. However, the notes [10] do not completely verify all conditions of general ODE theory for the Boltzmann equation. This was the motivation for [3] to revise the application of ODE theory from [17] in the case of the Boltzmann equation with more general hard potentials and an integrable angular cross section, and in particular, to provide a complete proof of the sub-tangent condition.

One very interesting new aspect from this approach is that the ODE flow in a suitable Banach space without imposing initial bounded entropy condition yields an alternative approach that allows for a rather general theory for gathering estimates where one can apply a rather general result in order to find solutions to the Cauchy problem for Boltzmann type flows where there is no classical entropy that is dissipated, or even some conservation laws that may not be satisfied. Such problems have already been solved for polymer kinetic problems [1], the quantum Boltzmann equation for bosons in very low temperature [5], and more recently in the study of weak wave turbulence models for stratified flows [15].

After proving the existence and uniqueness of the vector value solution $\mathbb{F}$ to the Boltzmann system, we turn to the study of generation and propagation of scalar polynomial and exponential moments of its solution $\mathbb{F}$.

The techniques we use in this manuscript are adaptations or extensions of results that have been developed for scalar Boltzmann type equations models.

In the case of the classical Boltzmann equation for the single elastic monatomic gas model, polynomial moments have been exclusively considered, for instance, in [12, 23] for hard potentials where propagation and generation of such moments was proved. At about the same time, Bobylev introduced in [6] the concept of exponential moment as a measure of the distribution solution tail, referred as to *tail temperature*, by showing that solutions to the Boltzmann equation for monatomic gases, modeled by elastic hard spheres (i.e. power exponent $\gamma = 1$) in three dimensions with a constant angular dependent cross-section as a function of the scattering direction, have inverse Maxwellian weighted moments, globally in time, whose tail decay rate depend on moments of the initial data. His proof consists in showing that infinite sums of renormalized polynomial moments are summable whose limit is proportional to a L^1 -Gaussian weighted norm for the unique probability density function solving the initial value problem associated to the Boltzmann equation, whose rate depends on the initial data that must also be integrable with a Gaussian weight. These techniques of understanding moments summability in order to obtain high energy tail behavior for the solution of the Boltzmann equation were extended to inelastic interactions with stochastic heating sources, shear flows or self-similarity scalings to obtain non-equilibrium statistical stationary (NESS) states [8] where the exponential rates did not necessarily correspond to Gaussian weighted moments.

This concept in the elastic case was further extended by [14] to collision kernels for hard potentials (i.e. $\gamma \in (0, 1]$) for any angular section with L^{1+} -integrability. Further, generation of exponential moments of order $\gamma/2$ with bounded angular section were shown in [18].

By this point it had become clear that the study of general forms of exponential moments resulted as a by-product of the analysis of polynomial moments (or tails), and so a spur of work arose for the improvement of conditions and results that will allow to estimate, globally in time. These results were extended to collision kernels for hard potentials with $\gamma \in (0, 2]$ for any angular section with just L^1 -integrability by a new approach using partial sums summability techniques, rather than using summability studies by power series associated to renormalized moments as proposed in [6, 8, 14, 18]. The generation results were improved to obtain exponential moments of order γ , while Gaussian moments were propagated for any initial data that would have that property, independent of γ . All these results were extended to the angular non-cutoff regime (lack of angular integrability) in [16, 22] still for hard potentials with $\gamma \in (0, 2]$, and in [7, 19] for pseudo-Maxwellian and Maxwellian case ($\gamma = 0$). In the later referenced work, these non-Gaussian tailed moments are called Mittag-Leffler moments as in fact the summability of partial sums is shown to converge to an L^1 -Mittag-Leffler function weighted norm for the unique probability density function solving the initial value problem associated to the Boltzmann equation, whose order and rate depend on the initial data as much as on the order of singularity in the angular section.

A very important tool for the success of summability properties for polynomial moments relies on the fact that such moments are both created and propagated

depending on how moments of the collision operator can be estimated: the positive part of the (gain) collision operator must have a decay rate with respect to the moment order while the negative part of such moments prevails in the dynamics, when sufficiently many moments are taken into account.

This is indeed a key step, arising as a consequence of an *angular averaged Povzner lemma*. In the case of single gas components, these estimates are based on integration of the collision operator against polynomial test functions on the pre-collisional velocities in the sphere. While these objects were originally introduced by Povzner [20] in the 1960s, a sharper form that uses the conservation of energy and angular averaging was introduced in [6] for the case of hard spheres in three dimensions with a constant angular cross section, where the polynomial test functions are proportional to even powers of the velocity magnitude. Later this technique was extended in [8] for the inelastic collision with heating sources, in [14] to the elastic case with hard potentials with L^{1+} integrable angular cross section, as well as in [2] for the case with just L^1 integrable angular cross section. Further, the approach was enlarged to hard spheres with non-integrable angular cross section in [16, 22] for hard potentials. *All of these estimates were developed for the mono-component model.*

Hence, the angular averaged Povzner lemma is our starting point in the case of mixtures as well. However, it requires a subtle modification of the polynomial weight that define *the scalar moment for the mixture*, to be defined in (2.1) next section, that renormalizes the polynomial test function from just even powers of the magnitude of the velocity vector to a dimensionless bracket form independent of mass density units, as the mono-component treatment to obtain moment estimates from [6] for the elastic case, or from [14] for inelastic hard sphere interactions, can not be directly extended to the mixture case, when masses are possibly different.

These facts enticed us to introduce a new approach that relies on a way of rewriting collisional rules and scalar polynomial moments in such a dimensionless, independent of mass density units form that it is very convenient to obtain a convex combination form between the conserved local quantities for a binary interaction, namely, local center of mass and energy. As a consequence, we conclude that averaging over the S^2 -sphere yields decay properties as the a function of the moment order for as long as the angular kernel is L^1 -integrable on S^2 . In particular, these decay properties will be significantly influenced by the disparateness of the species masses. It will be shown that as much as renormalized species masses deviate from each other, the decay rate will do so even more slowly.

The paper is organized as follows: in Sect. 2 we introduce notation and preliminaries, and state the main results, namely the Existence and Uniqueness Theorem for the vector value solution of the homogeneous Boltzmann system, and then generation and propagation of both scalar polynomial and exponential moments. Then in Sect. 3 we describe in details kinetic model that we use. Section 4 contains two preliminary Lemmas that we need for further work, including the Povzner lemma. Sections 5, 6 and 7 are devoted to proofs of our main results. A final Appendix contains some auxiliary calculations relevant to our estimates.

2. Notation, Preliminaries and Main Results

2.1. Notation and Preliminaries

In this paper, we consider a mixture of I gases, and we label its components with $\mathcal{A}_1, \dots, \mathcal{A}_I$. Each component of the mixture $\mathcal{A}_i, i = 1, \dots, I$, is described with its own distribution function, denoted with $f_i := f_i(t, v) \geq 0$, that, in this manuscript, depends on time $t > 0$ and velocity $v \in \mathbb{R}^3$. Fixing some $i \in \{1, \dots, I\}$, distribution function f_i satisfy Boltzmann like equation, which now, in the mixture context, has to take into account influence of all other components of the mixture on species $\mathcal{A}_i$. In the kinetic theory style, this is achieved by defining collision operator Q_{ij} for each $j = 1, \dots, I$ that measures interaction between species $\mathcal{A}_i$ that we fixed and all the others $\mathcal{A}_j, j = 1, \dots, I$, including itself $\mathcal{A}_i$. If the species $\mathcal{A}_j$ are described with distribution functions f_j , then the evolution of f_i is described via

$$\partial_t f_i(t, v) = \sum_{j=1}^I Q_{ij}(f_i, f_j)(t, v), \quad i = 1, \dots, I. \quad (2.1)$$

The form of Q_{ij} , for distribution functions f and g and any $i, j = 1, \dots, I$, is given by the non-local bilinear form

$$Q_{ij}(g, h)(v) = \int_{\mathbb{R}^3} \int_{S^2} \left(\frac{1}{J} g(v'_{ij}) h(v'_{*ij}) - g(v) h(v_*) \right) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_*, \quad (2.2)$$

where pre-collisional quantities v'_{ij} and v'_{*ij} depend on post-collisional ones v, v_* and parameter σ , as much as on the masses m_i and m_j mass of colliding particles of species $\mathcal{A}_i$ and $\mathcal{A}_j$, respectively, in the following manner:

$$\begin{aligned} v'_{ij} &= \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} |v - v_*| \sigma, \\ v'_{*ij} &= \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} |v - v_*| \sigma. \end{aligned} \quad (2.3)$$

The collisional rules (2.3) can be written in scattering direction coordinates (or in a center of mass reference framework) by introducing the velocity of center of mass V_{ij} and relative velocity u of the two colliding particles

$$V_{ij} := \frac{m_i v + m_j v_*}{m_i + m_j}, \quad u := v - v_*, \quad (2.4)$$

as follows:

$$v'_{ij} = V_{ij} + \frac{m_j}{m_i + m_j} |u| \sigma, \quad v'_{*ij} = V_{ij} - \frac{m_i}{m_i + m_j} |u| \sigma; \quad (2.5)$$

equivalently, introducing the two-body mass fraction parameter $r_{ij} = \frac{m_i}{m_i + m_j} \in (0, 1)$, associated to one of the particles, say m_i , without loss of generality, we get

$$v'_{ij} = V_{ij} + (1 - r_{ij}) |u| \sigma, \quad v'_{*ij} = V_{ij} - r_{ij} |u| \sigma. \quad (2.6)$$

Remark 1. For simplicity of notation, from now on, we will eliminate subindices i, j from $v'_{ij}, v'_{*ij}, V_{ij}$ and r_{ij} .

The transition probability rates or collision cross section terms $\mathcal{B}_{ij}$ are positive functions supposed to satisfy the following micro-reversibility assumptions

$$\mathcal{B}_{ij}(v, v_*, \sigma) = \mathcal{B}_{ij}(v', v'_*, \sigma') = \mathcal{B}_{ji}(v_*, v, -\sigma), \quad (2.7)$$

where $\sigma = u' / |u'|$ and $u' = v' - v'_*$ (note that then $\sigma' = u / |u|$).

The factor in the positive non-local binary term $\mathcal{J} = |\det J_{(v', v'_*, \sigma') / (v, v_*, \sigma)}|$ is the absolute value of determinant of the Jacobian associated to the exchange of velocity variables transformation (2.3) from pre to post for the given binary interaction. The Jacobian of this transformation can be easily computed by passing to the scattering direction coordinates i.e by considering the following mappings: $(v', v'_*, \sigma') \mapsto (u', V', \sigma') \mapsto (|u'|, \frac{u'}{|u'|}, V', \sigma') \mapsto (|u|, \frac{u}{|u|}, V, \sigma) \mapsto (u, V, \sigma) \mapsto (v, v_*, \sigma)$, with the notation (2.4) and using Remark 1. The first mapping is of unit Jacobian from definition of u and V , the second one is passage from Cartesian to spherical coordinates for u' . Since from the collisional rules of (2.3) it follows that $|u'| = |u|$ and $V' = V$, the passage from primes to non-primes described in the third mapping is of unit Jacobian. Then we pass from spherical to Cartesian coordinates for u and finally go back to the original variables (v, v_*, σ) . Thus, the Jacobian is computed as the decomposition of the mentioned mappings,

$$\mathcal{J} = 1 \cdot \frac{1}{|u'|^2} \cdot 1 \cdot |u|^2 \cdot 1 = 1,$$

since $|u'| = |u|$. Therefore, each Q_{ij} from (2.2) simply becomes

$$Q_{ij}(g, h)(v) = \int_{\mathbb{R}^3} \int_{S^2} (g(v')h(v'_*) - g(v)h(v_*)) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_*. \quad (2.8)$$

Since we consider a mixture as a whole, it will be convenient to introduce the following vector notation: we put all distribution functions $f_i, i = 1, \dots, I$ into vector of distribution functions

$$\mathbb{F} = [f_i]_{1 \leq i \leq I}. \quad (2.9)$$

Moreover, a vector value collision operator is defined as

$$\mathbb{Q}(\mathbb{F}) = \left[\sum_{j=1}^I Q_{ij}(f_i, f_j) \right]_{1 \leq i \leq I}. \quad (2.10)$$

Then the system of Boltzmann equations (2.1) can be written in a vector form

$$\partial_t \mathbb{F}(t, v) = \mathbb{Q}(\mathbb{F})(t, v). \quad (2.11)$$

Definition 2.1. (*Bracket forms for the mixture's dimensionless polynomial moments independent of mass density units*) Let $\mathbb{F} = [f_i]_{1 \leq i \leq I}$ be a suitable vector value distribution function. Let the mixture's bracket forms be denoted by

$$\langle v \rangle_i := \sqrt{1 + \frac{m_i}{\sum_{j=1}^I m_j}} |v|^2, \quad v \in \mathbb{R}^3. \quad (2.12)$$

Scalar polynomial moments independent of mass density units of order $q \geq 0$ for $\mathbb{F}$ is defined with

$$m_q[\mathbb{F}](t) = \sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t, v) \langle v \rangle_i^q dv. \quad (2.13)$$

In particular, we define a scalar polynomial moment of zero order for each species $\mathcal{A}_i$ as

$$m_{0,i}[\mathbb{F}](t) = \int_{\mathbb{R}^3} f_i(t, v) dv, \quad i = 1, \dots, I,$$

having in mind that $\sum_{i=1}^I m_{0,i}[\mathbb{F}] = m_0[\mathbb{F}]$.

Scalar exponential moment, or exponential weighted L^1 -forms, for $\mathbb{F}$ of a rate $\alpha := (\alpha_1, \dots, \alpha_I)$, $\alpha_i > 0$, and an order $s := (s_1, \dots, s_I) > 0$, $0 < s_i \leq 2$, is defined by

$$\mathcal{E}_s[\mathbb{F}](\alpha, t) = \sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t, v) e^{\alpha_i \langle v \rangle_i^{s_i}} dv. \quad (2.14)$$

The case $s_i = 2, \forall i$, is referred to as an inverse Maxwellian (or Gaussian) moment, otherwise these are super exponential moments (some authors referred as stretched exponentials though this concept usually refers to exponential times).

Remark 2. It can be noticed that both dimensionless polynomial and exponential moments for the mixture are defined as a sum of the resulting moments corresponding to each species independent of mass density units. In particular, when $\mathbb{F}$ solves the Boltzmann system of equations (2.11), then $m_{0,i}[\mathbb{F}]$ is interpreted as number density of the species $\mathcal{A}_i$, for any $i = 1, \dots, I$, while the zeroth scalar moment $m_0[\mathbb{F}]$ is the total number density of the mixture. Moreover, the second scalar moment $m_2[\mathbb{F}]$ represents total energy of the mixture.

Remark 3. If, for given exponential moments individually for each species $\mathcal{A}_i$, we seek for the maximum value of both their rate and order, i.e.

$$\hat{\alpha} = \max_{1 \leq i \leq I} \alpha_i, \quad \hat{s} = \max_{1 \leq i \leq I} s_i, \quad (2.15)$$

then

$$\mathcal{E}_s[\mathbb{F}](\alpha, t) \leq \sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t, v) e^{\hat{\alpha} \langle v \rangle_i^{\hat{s}}} dv =: \mathcal{E}_{\hat{s}}[\mathbb{F}](\hat{\alpha}, t).$$

Therefore, finiteness of the exponential moment $\mathcal{E}_s[\mathbb{F}](\alpha, t)$ is a consequence of the finiteness of $\mathcal{E}_{\hat{s}}[\mathbb{F}](\hat{\alpha}, t)$, with $\hat{\alpha}$ and $\hat{s}$ as in (2.15), for any time $t \geq 0$.

2.1.1. Functional Space We work in L^1 space weighted polynomially in velocity v and summed over all species, that is,

$$L_k^1 = \left\{ \mathbb{F} = [f_i]_{1 \leq i \leq I} \text{ measurable} : \sum_{i=1}^I \int_{\mathbb{R}^3} |f_i(v)| \langle v \rangle_i^k dv < \infty, k \geq 0 \right\}, \quad (2.16)$$

where the polynomial weight was defined in (2.12) by $\langle v \rangle_i = \left(1 + \frac{m_i}{\sum_{j=1}^I m_j} |v|^2 \right)^{1/2}$; its associated norm is

$$\|\mathbb{F}\|_{L_k^1} = \sum_{i=1}^I \int_{\mathbb{R}^3} |f_i(v)| \langle v \rangle_i^k dv. \quad (2.17)$$

Note that if $\mathbb{F} \geq 0$, then its norm in L_k^1 is precisely its polynomial moment of order k , i.e. $\|\mathbb{F}\|_{L_k^1} := m_k[\mathbb{F}]$.

Sometimes we will consider species separately, i.e., fix some component of the mixture $\mathcal{A}_i$. We define a space together with its norm as

$$L_{k,i}^1 = \left\{ g \text{ measurable} : \int_{\mathbb{R}^3} |g(v)| \langle v \rangle_i^k dv < \infty, k \geq 0 \right\},$$

$$\|g\|_{L_{k,i}^1} = \int_{\mathbb{R}^3} |g(v)| \langle v \rangle_i^k dv.$$

Note that the norm of $\mathbb{F}$ in L_k^1 is related to the norm of its components f_i in the space $L_{k,i}^1$ via $\|\mathbb{F}\|_{L_k^1} = \sum_{i=1}^I \|f_i\|_{L_{k,i}^1}$.

Finally, since we use bracket forms $\langle \cdot \rangle$ defined in (2.12), the monotonicity property holds, i.e.

$$\|f_i\|_{L_{k_1,i}^1} \leq \|f_i\|_{L_{k_2,i}^1} \text{ and } \|\mathbb{F}\|_{L_{k_1}^1} \leq \|\mathbb{F}\|_{L_{k_2}^1}, \text{ whenever } 0 \leq k_1 \leq k_2. \quad (2.18)$$

2.2. Main Results

We study the Cauchy problem for the system of spatially homogeneous Boltzmann equations for the mixture of gases $\mathcal{A}_1, \dots, \mathcal{A}_I$:

$$\begin{cases} \partial_t \mathbb{F}(t, v) = \mathbb{Q}(\mathbb{F})(t, v), & t > 0, v \in \mathbb{R}^3, \\ \mathbb{F}(0, v) = \mathbb{F}_0(v), \end{cases} \quad (2.19)$$

where $\mathbb{F}$ is a vector of distribution functions $\mathbb{F} = [f_i]_{1 \leq i \leq I}$, f_i being a distribution function of the component $\mathcal{A}_i$, $i = 1, \dots, I$, as defined in (2.9), and $\mathbb{Q}(\mathbb{F}) = \left[\sum_{j=1}^I \mathcal{Q}_{ij}(f_i, f_j) \right]_{1 \leq i \leq I}$ is a collision operator introduced in (2.8, 2.10).

We consider the particular case when the transition probability terms $\mathcal{B}_{ij}$, $i, j = 1, \dots, I$ are assumed to take the form

$$\mathcal{B}_{ij}(v, v_*, \sigma) = |u|^{\gamma_{ij}} b_{ij}(\sigma \cdot \hat{u}), \quad \gamma_{ij} \in (0, 1], \text{ and } b_{ij}(\sigma \cdot \hat{u}) \in L^1(S^2; d\sigma),$$

(2.20)

where $u := v - v_*$, $\hat{u} := u/|u|$. This form of cross-section corresponds to variable hard potentials with an integrable angular part. In the mixture setting, both potential γ_{ij} and angular kernel b_{ij} may depend on species $\mathcal{A}_i$ and $\mathcal{A}_j$. In order to satisfy micro-reversibility assumptions (2.7), it is supposed that

$$\gamma_{ij} = \gamma_{ji}, \quad \text{and} \quad b_{ij}(\sigma \cdot \hat{u}) = b_{ji}(\sigma \cdot \hat{u})$$

for any choice $i, j = 1, \dots, I$. Moreover, let $\overline{\gamma}$ denote the maximum value of potentials γ_{ij} , i.e.

$$\overline{\gamma} = \max_{1 \leq i, j \leq I} \gamma_{ij}. \quad (2.21)$$

2.2.1. Povzner Lemma by Angular Averaging The essential ingredient of this manuscript is the Povzner lemma obtained by averaging in the scattering angle representation of the collision kernel, originally introduced in [6, 8], for the case of elastic and inelastic collisions; it estimates the positive contribution of the collision operator after integration against $\sigma \in S^2$, that is crucial for all further proofs.

Lemma 2.2. (Povzner lemma by angular averaging for the mixing model) *Let the angular part $b_{ij}(\sigma \cdot \hat{u})$ of the cross-section be integrable in σ variable (that is $b_{ij} \in L^1(S^2; d\sigma)$), $\hat{u}$ being the normalized relative velocity $u = v - v_*$. Let v' and v'_* be functions of v, v_*, σ as in (2.3), with $m_i, m_j > 0$. Then the following estimate holds for any fixed i, j :*

$$\int_{S^2} \left(\langle v' \rangle_i^k + \langle v'_* \rangle_j^k \right) b_{ij}(\sigma \cdot \hat{u}) d\sigma \leq \mathcal{C}_{\frac{k}{2}}^{ij} \left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right)^{\frac{k}{2}}, \quad (2.22)$$

where constant $\mathcal{C}_{\frac{k}{2}}^{ij}$ tends to zero as k grows, and moreover,

$$\mathcal{C}_{\frac{k}{2}}^{ij} - \|b_{ij}\|_{L^1(d\sigma)} < 0, \quad \text{for any } k \geq k_*^{ij}, \quad 1 \leq i, j \leq I, \quad (2.23)$$

where each k_*^{ij} depends on b_{ij} and r_{ij} .

The proof of Lemma 2.2 genuinely reflects a difference between single and multicomponent gas, with an emphasis on writing collisional rules in a convex combination form for mixtures, in contrast to symmetric or “half-half” writing for the single component gas. It turns out that the single component case—due to symmetry—has a lot of room for estimates and further simplification, as presented in [8] as for example. For mixtures, this is not the case any longer, and writing should be exact as much as possible; we use Taylor expansion of second order with a remainder in the integral form, and estimates are done only in the remainder.

A very important consequence of the Povzner lemma is the ability to estimate moments of the collision operator. In particular, averaging over the sphere yields decay properties of the gain term polynomial moment with respect to its order. This decay allows polynomial moments of loss term to prevail in dynamics, when

sufficiently many moments are taken into account. In a single component gas, it suffices to take $2+$ order of polynomial moment, that is, slightly more than the energy, to obtain this property [8]. Mixtures bring great novelty in this aspect, too: decay properties of the constant issuing from the Povzner lemma strongly depend on the two-body mass fraction parameter r_{ij} . We study this issue in detail in the case $b_{ij} \in L^\infty(S^2; d\sigma)$ when it is possible to explicitly calculate the constant $\mathcal{C}_{k/2}^{ij}$ from (2.23). It will be shown that when $r_{ij} = 1/2$ (which corresponds to $m_i = m_j$), we recover the same decay properties of the constant $\mathcal{C}_{k/2}^{ij}$ as in the case of single gas component. However, when mixtures are considered, we observe that as much as r_{ij} deviates from $1/2$, the larger k_*^{ij} that ensures (2.23) is, or larger and larger order of moment that guarantees prevail of loss term moment is.

2.2.2. Existence and Uniqueness Theory In this manuscript, we discuss existence and uniqueness for the vector value solution $\mathbb{F}$ to the initial value problem (2.19) of space homogeneous Boltzmann equations for monatomic gas mixtures, with transition probabilities (or collision kernels) associated to species $\mathcal{A}_i$ and $\mathcal{A}_j$, $i, j \in \{1, \dots, I\}$ having hard potential growth of order $|u|^{\gamma_{ij}}$ for $\gamma_{ij} \in (0, 1]$ and an integrable angular part b_{ij} , with an initial total mixture number density and energy bounded below (i.e. the initial data can not be singular measure), and have at least a k_* (scalar) polynomial moments

$$k_* \geq \max\{\bar{k}, 2 + 2\bar{\gamma}\} \quad \text{for } \bar{k} = \max_{1 \leq i, j \leq I} \{k_*^{ij}\} \text{ and } \bar{\gamma} = \max_{1 \leq i, j \leq I} \gamma_{ij}, \quad (2.24)$$

chosen to ensure the inequality (2.23) holds for any $i, j = 1, \dots, I$.

Such a study fits into an abstract framework of ODE theory in Banach spaces, which can be found in [17]. For the Boltzmann equation, the application of this theory was clarified in [3], after being recognized in [10]. The formulation of theorem that we apply in this manuscript is given in Appendix A. As for the choice of Banach space, it is known that the natural Banach space for solving the Boltzmann equation is L^1 polynomially weighted, or in mixture setting space, L_k^1 defined in (2.16). More precisely, here we take $k = 2$, because the norm in that space is related to energy whose conservation is exploited.

In order to apply Theorem A.1, we need to find an invariant region $\Omega \subset L_2^1$ in which collision operator $\mathbb{Q} : \Omega \rightarrow L_2^1$ will satisfy (i) Hölder continuity, (ii) Sub-tangent and (iii) one-sided Lipschitz conditions.

To that end, we first study the map $\mathcal{L}_{\bar{\gamma}, k_*} : [0, \infty) \rightarrow \mathbb{R}$, defined with

$$\mathcal{L}_{\bar{\gamma}, k_*}(x) = -Ax^{1+\frac{\bar{\gamma}}{k_*}} + Bx,$$

where A and B are positive constants, $\bar{\gamma} \in (0, 1]$ and k_* defined in (2.24). This map has only one root, denoted with $x_{\bar{\gamma}, k_*}^*$, at which $\mathcal{L}_{\bar{\gamma}, k_*}$ changes from positive to negative. Thus, for any $x \geq 0$, we may write

$$\mathcal{L}_{\bar{\gamma}, k_*}(x) \leq \max_{0 \leq x \leq x_{\bar{\gamma}, k_*}^*} \mathcal{L}_{\bar{\gamma}, k_*}(x) =: \mathcal{L}_{\bar{\gamma}, k_*}^*.$$

Define

$$C_{k_*} := x_{\overline{\gamma}, k_*}^* + \mathcal{L}_{\overline{\gamma}, k_*}^*. \quad (2.25)$$

Now, we are in position to define the bounded, convex and closed subset $\Omega \subset L_2^1$:

$$\begin{aligned} \Omega = \Big\{ & \mathbb{F}(t, \cdot) \in L_2^1 : \mathbb{F} \geq 0 \text{ in } v, \sum_{i=1}^I \int_{\mathbb{R}^3} m_i v f_i(t, v) dv = 0, \\ & \exists c_0, C_0, c_2, C_2, C_{2+\varepsilon} > 0, \text{ and } C_0 < c_2, \text{ such that } \forall t \geq 0, \\ & c_0 \leq m_0[\mathbb{F}](t) \leq C_0, \quad c_2 \leq m_2[\mathbb{F}](t) \leq C_2, \\ & m_{2+\varepsilon}[\mathbb{F}](t) \leq C_{2+\varepsilon}, \text{ for } \varepsilon > 0, \\ & m_{k_*}[\mathbb{F}](t) \leq C_{k_*}, \text{ with } C_{k_*} \text{ from (2.25)} \Big\}, \end{aligned}$$

where

$$m_{2+\varepsilon}[\mathbb{F}](t) = \|\mathbb{F}\|_{L_{2+\varepsilon}^1} = \sum_{i=1}^I \int_{\mathbb{R}^3} |f_i(t, v)| \langle v \rangle_i^{2+\varepsilon} dv$$

for any $\varepsilon > 0$, which can be arbitrarily small.

Then, the existence and uniqueness theory of a vector value $\mathbb{F}$ solution to the Cauchy problem (2.19) fits into the study of ODE in a Banach space $(L_2^1, \|\cdot\|_{L_2^1})$ and its bounded, convex and closed subset Ω . The collision operator $\mathbb{Q}$ is viewed as a map $\mathbb{Q} : \Omega \rightarrow L_2^1$. We will show that it satisfies Hölder continuity, sub-tangent and one-sided Lipschitz conditions, which will enable us to prove the following Theorem:

Theorem 2.3. (Existence and Uniqueness) *Assume that $\mathbb{F}(0, v) = \mathbb{F}_0(v) \in \Omega$. Then the Boltzmann system (2.19) for the cross section (2.20) has the unique solution in $\mathcal{C}([0, \infty), \Omega) \cap \mathcal{C}^1((0, \infty), L_2^1)$.*

Remark 4. Let us point out that for the existence and uniqueness result no conditions on initial entropy are necessary. However, if the initial data has finite entropy, then the entropy inequality implies that it will remain bounded for all times. Let us give a sketch of the proof. The definition of the entropy and entropy inequality is taken from [13], Proposition 1.

Definition 2.4. (Mixture entropy and entropy production) Let $\mathbb{F}$ be a vector value distribution function as in (2.9). The (mixture) entropy is defined as

$$\eta(t) = \sum_{i=1}^I \int_{\mathbb{R}^3} f_i \log f_i dv, \quad (2.26)$$

while the (mixture) entropy production is given with

$$D(\mathbb{F}) = \sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \log f_i dv. \quad (2.27)$$

Then the following Proposition holds:

Proposition 1. (Entropy inequality or the first part of the H-theorem, [13]) *Let us assume that the cross section terms $\mathcal{B}_{ij}$, $1 \leq i, j \leq I$, are positive almost everywhere and that $\mathbb{F} \geq 0$ is such that both collision operator $\mathbb{Q}(\mathbb{F})$ and entropy production are well defined. Then the entropy production is non-positive, i.e. $D(\mathbb{F}) \leq 0$.*

As an immediate consequence, we get from the Boltzmann equation that $\partial_t \eta \leq 0$; in other words, $\eta(t) \leq \eta(0)$ for any $t \geq 0$. Therefore, we conclude that the entropy inequality implies that mixture entropy remains bounded at any time if initially so.

2.2.3. Generation and Propagation of Polynomial Moments The second part of the manuscript is devoted to the study of generation and propagation of scalar polynomial moments associated to the solution of the Boltzmann system (2.19) for the cross section (2.20), that initially belongs to Ω .

First, in the following Lemma, we derive from the Boltzmann system (2.19) an ordinary differential inequality for polynomial moment of order k , $m_k[\mathbb{F}](t)$, for large enough k , that relies on the Povzner estimate from Lemma 2.2, uniformly in each pair i, j :

Lemma 2.5. (Ordinary differential inequality for polynomial moments) *Let $\mathbb{F} = [f_i]_{i=1, \dots, I}$ be a solution of the Boltzmann system (2.19) with the cross section (2.20)–(2.21). Then the polynomial moment (2.13) satisfies the following Ordinary Differential Inequality:*

$$\frac{d}{dt} m_k[\mathbb{F}](t) = \sum_{i=1}^I [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^k dv \leq -A_k m_k[\mathbb{F}](t)^{1+\frac{\bar{\gamma}}{k}} + B_k m_k[\mathbb{F}](t), \quad (2.28)$$

for large enough k to ensure (2.24), and some positive constants A_k and B_k .

The proof of this Lemma follows from comparison principles for ODE's, which yields the generation and propagation estimates stated in the following Theorem, which is proved in Sect. 6:

Theorem 2.6. (Generation and propagation of polynomial moments) *Let $\mathbb{F}$ be a solution of the Boltzmann system (2.19) with a cross section (2.20)–(2.21) and an initial data $\mathbb{F}(0, v) = \mathbb{F}_0(v) \in \Omega$. Then,*

1. (Generation) *There is a constant $\mathfrak{C}^m$ such that for any $k > k_*$ defined in (2.24),*

$$m_k[\mathbb{F}](t) \leq \mathfrak{C}^m \left(1 - e^{-\frac{\bar{\gamma} B_k t}{k}} \right)^{-\frac{k}{\bar{\gamma}}}, \quad \forall t > 0, \quad (2.29)$$

where constants $\mathfrak{C}^m$ depend on A_k, B_k from (2.28) and $\bar{\gamma}$.

2. (Propagation) *Moreover, if $m_k[\mathbb{F}](0) < \infty$, then*

$$m_k[\mathbb{F}](t) \leq \max\{\mathfrak{C}^m, m_k[\mathbb{F}](0)\}, \quad (2.30)$$

for all $t \geq 0$.

Finally, we show that, under the assumed conditions on the collision kernel form (2.20), the renormalized series of moments is summable depending on the moments of the initial data, yielding the following result on generation and propagation of exponential, or Mittag-Leffler moments:

2.2.4. Generation and Propagation of Exponential Moments With bounds on polynomial moment at hand, one can deal with exponential moments. We prove the following Theorem:

Theorem 2.7. (Generation and propagation of exponential moments) *Let $\mathbb{F}$ be a solution of the Boltzmann system (2.19) with a cross section (2.20)–(2.21) and an initial data $\mathbb{F}(0, v) = \mathbb{F}_0(v) \in \Omega$. Then,*

(a) (Generation) *There exist constants $\alpha > 0$ and $\mathfrak{B}^\mathcal{E} > 0$ such that*

$$\mathcal{E}_{\overline{v}}[\mathbb{F}](\alpha \min \{t, 1\}, t) \leq \mathfrak{B}^\mathcal{E}, \quad \forall t \geq 0.$$

(b) (Propagation) *Let $0 < s \leq 2$. Suppose that there exists a constant $\alpha_0 > 0$, such that*

$$\mathcal{E}_s[\mathbb{F}](\alpha_0, 0) \leq M_0 < \infty. \quad (2.31)$$

Then there exist constants $0 < \alpha \leq \alpha_0$ and $\mathfrak{C}^\mathcal{E} > 0$ such that

$$\mathcal{E}_s[\mathbb{F}](\alpha, t) \leq \mathfrak{C}^\mathcal{E}, \quad \forall t \geq 0. \quad (2.32)$$

3. Kinetic Model

3.1. Study of Collision Process

In our setting molecules are assumed to interact via elastic collisions. Let us fix two colliding molecules; one of the species $\mathcal{A}_i$ having mass m_i and pre-collisional velocity v' and the another one belonging to the species $\mathcal{A}_j$ with mass m_j and pre-collisional velocity v'_* (note that we here immediately adopted the simplicity of notation pointed out in Remark 1). If the post-collisional velocities are denoted with v and v_* , respectively, then the momentum and kinetic energy during the collision are conserved:

$$\begin{aligned} m_i v' + m_j v'_* &= m_i v + m_j v_*, \\ m_i |v'|^2 + m_j |v'_*|^2 &= m_i |v|^2 + m_j |v_*|^2. \end{aligned} \quad (3.1)$$

As usual, we parametrize these equations with a parameter $\sigma \in S^2$, in order to express pre-collisional velocities in terms of post-collisional ones:

$$\begin{aligned} v' &= \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} |v - v_*| \sigma, \\ v'_* &= \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} |v - v_*| \sigma. \end{aligned} \quad (3.2)$$

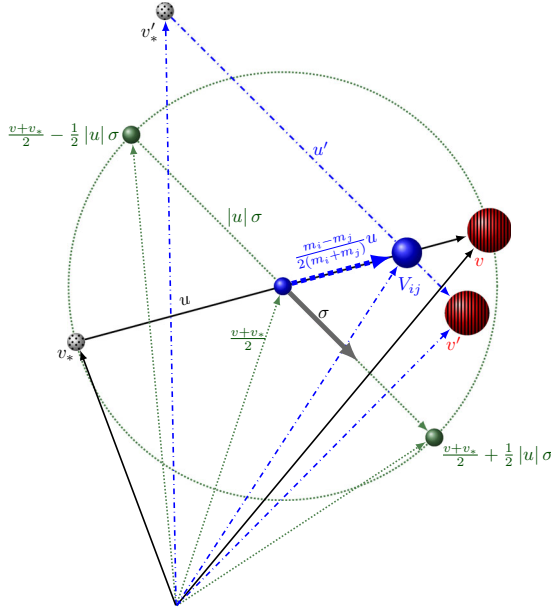


Fig. 1. Illustration of the collision transformation, with notation $V_{ij} := \frac{m_i v + m_j v_*}{m_i + m_j}$, $u := v - v_*$, $u' := v' - v'_*$. The displacement of the center of mass with respect to a single component elastic binary interaction is given by $(r_{ij} - \frac{1}{2})u = \frac{m_i - m_j}{2(m_i + m_j)}u$, if $m_i > m_j$. Solid lines denote vectors after collision, or given data. Dash-dotted vectors represent primed (pre-collisional) quantities that can be calculated from the given data, and compared to the case $m_i = m_j$, represented by dotted vectors. The dashed vector direction is the displacement along the direction of the relative velocity u proportional to the half difference of relative masses, (which clearly vanishes for $m_i = m_j$, reducing the model to a classical collision). Note that the scattering direction σ is preserved as the pre-collisional relative velocity u' keeps the same magnitude as the post-collisional u , u' is parallel the reference elastic pre-collisional relative velocity $|u|\sigma$

Note that if $m_i = m_j$, then the collisional rules simplify and take the usual single component gas form

$$v' = \frac{v + v_*}{2} + \frac{1}{2} |v - v_*| \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{1}{2} |v - v_*| \sigma. \quad (3.3)$$

Figure 1 illustrates the collision transformation (3.2) and aims at explaining its difference with respect to the collision transformation (3.3) when masses are equal. Namely, for given v, v_*, σ and m_i, m_j , we calculate center of mass $V = \frac{m_i v + m_j v_*}{m_i + m_j}$, and velocities v' and v'_* according to (3.2). One can notice that the magnitude of the relative velocity does not change during the collision, i.e. $|v - v_*| = |v' - v'_*|$, as it is when masses are the same. Difference comes with the vector of center of mass: the vector of center of mass for equal masses $\frac{v+v_*}{2}$ displaces by adding a quantity that is proportional to the difference of masses $m_i - m_j$ and thus is peculiar to the

mixture case. More precisely,

$$V = \frac{v + v_*}{2} + \frac{m_i - m_j}{2(m_i + m_j)}u,$$

with $u := v - v_*$.

3.2. Collision Operators

Collision operators Q_{ij} , as defined in (2.8), describe binary interactions between molecules of species $\mathcal{A}_i$ and $\mathcal{A}_j$, $i, j = 1, \dots, I$. Fix the species $\mathcal{A}_i$ for any $i = 1, \dots, I$, and let its distribution function be g . On the other hand, let distribution function h describe species $\mathcal{A}_j$.

Note that each Q_{ij} for a fix (i, j) -pair has its corresponding counterpart, Q_{ji} , that describes interaction of molecules of species $\mathcal{A}_j$ with molecules of species $\mathcal{A}_i$:

$$Q_{ji}(h, g)(v) = \int_{\mathbb{R}^3} \int_{S^2} (h(w')g(w'_*) - h(v)g(v_*)) \mathcal{B}_{ji}(v, v_*, \sigma) d\sigma dv_*, \quad (3.4)$$

where pre-collisional velocities w' and w'_* now differ from the previous ones given in (3.2) by an exchange of mass $m_i \leftrightarrow m_j$, i.e.

$$\begin{aligned} w' &= \frac{m_j v + m_i v_*}{m_i + m_j} + \frac{m_i}{m_i + m_j} |v - v_*| \sigma, \\ w'_* &= \frac{m_j v + m_i v_*}{m_i + m_j} - \frac{m_j}{m_i + m_j} |v - v_*| \sigma. \end{aligned} \quad (3.5)$$

When $m_i = m_j$, although primed velocities are the same, Q_{ij} and Q_{ji} still defer, because of the cross section.

3.3. Weak Form of Collision Operator

Testing the collision operator against some suitable test functions $\psi(v)$ and $\phi(v)$ yields

$$\begin{aligned} & \int_{\mathbb{R}^3} Q_{ij}(g, h)(v) \psi(v) dv \\ &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} g(v) h(v_*) (\psi(v') - \psi(v)) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_* dv, \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^3} Q_{ji}(h, g)(v) \phi(v) dv \\ &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} h(v_*) g(v) (\phi(v'_*) - \phi(v_*)) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_* dv, \end{aligned}$$

where now v' and v'_* are denoting the post-collisional velocities as defined by (3.2). Therefore, looking at these two integrals pairwise, meaning that each time when Q_{ij} is considered we add his pair Q_{ji} , we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (Q_{ij}(g, h)(v)\psi(v) + Q_{ji}(h, g)(v)\phi(v)) \, dv \\ &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} g(v)h(v_*) (\psi(v') + \phi(v'_*) - \psi(v) - \phi(v_*)) \\ & \quad \mathcal{B}_{ij}(v, v_*, \sigma) \, d\sigma \, dv_* \, dv, \end{aligned} \quad (3.6)$$

with v' and v'_* are now given by the post-collisional velocities as defined by (3.2).

Some choice of test function leads to annihilation of the weak form. Namely, from the conservation laws during collision process, we see

$$\int_{\mathbb{R}^3} Q_{ij}(g, h)(v) \, dv = 0, \quad (3.7)$$

as well as

$$\int_{\mathbb{R}^3} \left(Q_{ij}(g, h)(v) \left(\frac{m_i v}{m_i |v|^2} \right) + Q_{ji}(h, g)(v) \left(\frac{m_j v}{m_j |v|^2} \right) \right) \, dv = 0. \quad (3.8)$$

Therefore, if we consider distribution function $\mathbb{F} = [f_i]_{1 \leq i \leq I}$, then the weak form (3.6) yields

$$\begin{aligned} & 2 \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} Q_{ij}(f_i, f_j)(v) \psi_i(v) \, dv \\ &= \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f_i(v) f_j(v_*) \\ & \quad \times (\psi_i(v') + \psi_j(v'_*) - \psi_i(v) - \psi_j(v_*)) \mathcal{B}_{ij}(v, v_*, \sigma) \, d\sigma \, dv_* \, dv. \end{aligned} \quad (3.9)$$

3.4. Conservation Laws

Weak forms of collision operator imply some its conservative properties. More precisely, for any suitable $\mathbb{F}$, (3.7) implies

$$\int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \, dv = 0, \quad \text{for any } i = 1, \dots, I, \quad (3.10)$$

and moreover, from (3.9), choosing $\psi_\ell(x) = m_\ell |x|^2$ and $\psi_\ell(x) = m_\ell x$, $x \in \mathbb{R}^3$, one has

$$\sum_{i=1}^I [\mathbb{Q}(\mathbb{F})]_i m_i |v|^2 \, dv = 0 \quad (3.11)$$

and

$$\sum_{i=1}^I [\mathbb{Q}(\mathbb{F})]_i m_i v \, dv = 0$$

for any time $t \geq 0$.

If $\mathbb{F}$ is a solution to the Boltzmann system (2.19), then these properties imply conservation laws for the number density of each species $\mathcal{A}_i$, $i = 1, \dots, I$, and the total energy of the mixture. Indeed,

$$\partial_t m_{0,i}[\mathbb{F}](t) = 0, \quad \forall i = 1, \dots, I, \quad \partial_t m_2[\mathbb{F}](t) = 0. \quad (3.12)$$

4. Proof of Povzner Lemma 2.2

The proof of Povzner lemma 2.2 by angular averaging for the mixing model leads us to obtain estimates for the quantity $\langle v'_i \rangle_i^k + \langle v'_* \rangle_j^k$ integrated over the sphere S^2 , which represents the gain part of (3.9) for $\psi_i(x) = \langle x \rangle_i^k$. The usual techniques used in [2], for example, cannot be directly adapted when $m_i \neq m_j$. This becomes clear when one writes local kinetic energies of each colliding molecule pair. When $m_i \neq m_j$, these energies can be written as a certain convex combination, while the single component case (or in the same fashion when $m_i = m_j$) corresponds to the “middle” of this convex combination, or to the “halves” (see Remark 5 below). The single component situation (or when $m_i = m_j$) is therefore “symmetric”, in a sense, and the techniques for proof of a sharper Povzner lemma by angular averaging, as done by [6] or [14], cannot be extended to the mixture case in a straight forward form.

Indeed, in the mixture setting when $m_i \neq m_j$, the proof of the Povzner lemma 2.2 in the cases of a non-linear gas mixture system uses a non-trivial modification of a powerful energy identity in scattering angle coordinates. This identity is needed in order to compute moment estimates that clearly show positive moments from the gain collision operator part are dominated by the moments of the corresponding loss part, which yields a very sharp estimate sufficient to obtain not only moments propagation and generation, but also their scaled summability that prove propagation and generation of exponential moment estimates as well. An energy identity in scattering angle coordinates was first developed in [6,8] for the elastic and inelastic case for scalar Boltzmann binary models. While such an identity is rather easy in the elastic single species setting, where local energies are just the sum of the collision invariant $|v|^2$ and just its interacting counterpart $|v_*|^2$, in the mixing case under consideration the problem becomes highly non-trivial and the local energies to be estimated now depend on binary sums of $\langle v \rangle_i^2$ and $\langle v_* \rangle_j^2$ and their corresponding post collisional sum of $\langle v' \rangle_i^2$ and $\langle v'_* \rangle_j^2$.

Lemma 4.1. (Energy identity in scattering direction coordinates for the (i, j) -pair of colliding particles.) *Consider any (i, j) -pair of interacting velocities v and v_* corresponding to particles masses m_i and m_j , respectively, with i, j fixed. Let their*

local micro energy be $E_{ij} = \langle v \rangle_i^2 + \langle v_* \rangle_j^2$, with $\langle v \rangle_i^2$ and $\langle v_* \rangle_j^2$ defined according to (2.12), and recall the two-body mass fraction parameter $r_{ij} = \frac{m_i}{m_i + m_j}$ introduced in (2.5).

Then, there exists a couple of functions $p_{ij} = p_{ij}(v, v_*, m_i, m_j)$ and $q_{ij} = q_{ij}(v, v_*, m_i, m_j)$ such that $p_{ij} + q_{ij} = E_{ij}$ and the following representation holds:

$$\langle v'_{ij} \rangle_i^2 = p_{ij} + \lambda_{ij} \sigma \cdot \hat{V}_{ij}, \quad \langle v'_{*ij} \rangle_j^2 = q_{ij} - \lambda_{ij} \sigma \cdot \hat{V}_{ij}, \quad (4.1)$$

where $\lambda_{ij} := 2\sqrt{r_{ij}(1-r_{ij})(sE_{ij}-1)((1-s_{ij})E_{ij}-1)}$ with $s_{ij} = s_{ij}(v, v_*, m_i, m_j) \in [0, 1]$. In particular, this representation preserves the local energy identity

$$\langle v'_{ij} \rangle_i^2 + \langle v'_{*ij} \rangle_j^2 = p_{ij} + q_{ij} = E_{ij} = \langle v \rangle_i^2 + \langle v_* \rangle_j^2. \quad (4.2)$$

Moreover, the following inequalities hold:

$$p_{ij} + \lambda_{ij} \leq E_{ij}, \quad q_{ij} + \lambda_{ij} \leq E_{ij}, \quad (4.3)$$

for any velocities $v, v_* \in \mathbb{R}^3$ and any masses $m_i, m_j > 0$.

As we mentioned earlier in Remark 1, we eliminate subindex ij from E_{ij} , p_{ij} , q_{ij} , λ_{ij} , s_{ij} as we did in Remark 1 for v'_{ij} , v'_{*ij} , V_{ij} and r_{ij} .

Proof of Lemma 4.1. As anticipated, we represent the exchange of coordinates at the interaction using the center of mass and relative velocity reference frame (2.3) [with its symmetric form (3.5)] where the angular integration is performed in the scattering direction corresponding to the post-collisional relative velocity $\sigma = \hat{u}'$. Thus, let's denote with V the vector of center-of-mass and with u the relative velocity as in (2.4):

$$V = \frac{m_i v + m_j v_*}{m_i + m_j}, \quad u = v - v_*.$$

Then, taking the squares of the magnitudes of the post-collisional velocities given in (3.2), one obtains

$$\begin{aligned} |v'|^2 &= |V|^2 + \frac{m_j^2}{(m_i + m_j)^2} |u|^2 + \frac{2m_j}{m_i + m_j} |u| |V| \sigma \cdot \hat{V}, \\ |v'_*|^2 &= |V|^2 + \frac{m_i^2}{(m_i + m_j)^2} |u|^2 - \frac{2m_i}{m_i + m_j} |u| |V| \sigma \cdot \hat{V}, \end{aligned}$$

where $\hat{V}$ denotes the unit vector of V . Passing to $\langle \cdot \rangle$ bracket forms from (2.12) implies

$$\begin{aligned} \langle v'_i \rangle^2 &= 1 + \frac{m_i}{\sum_{\ell=1}^I m_\ell} |V|^2 + \frac{m_i m_j^2}{(m_i + m_j)^2 \sum_{\ell=1}^I m_\ell} |u|^2 \\ &\quad + \frac{2m_i m_j}{(m_i + m_j) \sum_{\ell=1}^I m_\ell} |u| |V| \sigma \cdot \hat{V}, \\ \langle v'_j \rangle^2 &= 1 + \frac{m_j}{\sum_{\ell=1}^I m_\ell} |V|^2 + \frac{m_j m_i^2}{(m_i + m_j)^2 \sum_{\ell=1}^I m_\ell} |u|^2 \\ &\quad - \frac{2m_i m_j}{(m_i + m_j) \sum_{\ell=1}^I m_\ell} |u| |V| \sigma \cdot \hat{V}. \end{aligned} \quad (4.4)$$

Let us introduce the total energy E of two colliding particles in $\langle \cdot \rangle$ bracket forms, which is conserved during collision process by (3.1):

$$E := \langle v \rangle_i^2 + \langle v_* \rangle_j^2 = \langle v' \rangle_i^2 + \langle v'_* \rangle_j^2.$$

Using the above Eq. (4.4), the energy E can be written in $u - V$ notation as well:

$$E = 2 + \frac{m_i + m_j}{\sum_{\ell=1}^I m_\ell} |V|^2 + \frac{m_i m_j}{(m_i + m_j) \sum_{\ell=1}^I m_\ell} |u|^2. \quad (4.5)$$

The aim is to represent the squares of the post-collisional velocities $\langle v' \rangle_i^2$ and $\langle v'_* \rangle_j^2$ as a scalar convex combination of different “parts” of the energy E . This is achieved by introducing two quantities:

(i) the parameter $r \in (0, 1)$, that distributes masses in the following convex pair:

$$r = \frac{m_i}{m_i + m_j} \quad \text{and} \quad 1 - r = \frac{m_j}{m_i + m_j}; \quad (4.6)$$

(ii) the function $s \in [0, 1]$ that convexly partitions the energy E into two components, one related to $|u|^2$ and another to $|V|^2$, using the above identity (4.5) as follows:

$$sE = 1 + \frac{m_i m_j}{(m_i + m_j) \sum_{\ell=1}^I m_\ell} |u|^2 \quad \text{and} \quad (1 - s)E = 1 + \frac{m_i + m_j}{\sum_{\ell=1}^I m_\ell} |V|^2. \quad (4.7)$$

Finally, each of the post-collisional quantities, $\langle v' \rangle_i^2$ and $\langle v'_* \rangle_j^2$, as written in the representation as in (4.4), can be recast through the energy E and the dot product between the center of mass vector V and the scattering direction σ as follows:

$$\begin{aligned} \langle v' \rangle_i^2 &= r(1 - s)E + (1 - r)sE + 2\sqrt{r(1 - r)(sE - 1)((1 - s)E - 1)} \sigma \cdot \hat{V}, \\ \langle v'_* \rangle_j^2 &= rsE + (1 - r)(1 - s)E - 2\sqrt{r(1 - r)(sE - 1)((1 - s)E - 1)} \sigma \cdot \hat{V}, \end{aligned} \quad (4.8)$$

which yields the important relation that expresses the post-collisional local micro energy E as a rotation of factors of E and $V \cdot \sigma$, while preserving the local energy itself. Indeed, denoting

$$\begin{aligned} p &= r(1-s)E + (1-r)sE, \\ q &= E - p = rsE + (1-r)(1-s)E, \\ \lambda &= 2\sqrt{r(1-r)(sE-1)((1-s)E-1)} \\ &= 2\sqrt{r(1-r)} \frac{\sqrt{m_i m_j}}{\sum_{\ell=1}^I m_\ell} |u| |V|, \end{aligned}$$

$p+q = E$ and the representation (4.1) clearly follows, while preserving the binary micro energy relation (4.2):

$$\langle v'_i \rangle_i^2 = p + \lambda \sigma \cdot \hat{V}, \quad \langle v'_j \rangle_j^2 = q - \lambda \sigma \cdot \hat{V}.$$

This completes the proof of the energy identities (4.1) and (4.2).

Moreover, it follows that

$$\frac{1}{E} (p + \lambda) \leq \left(\sqrt{r(1-s)} + \sqrt{(1-r)s} \right)^2 \leq 1,$$

since

$$\max_{\substack{0 \leq r \leq 1 \\ 0 \leq s \leq 1}} \left(\sqrt{r(1-s)} + \sqrt{(1-r)s} \right) = 1.$$

Similarly,

$$\frac{1}{E} (q + \lambda) \leq \left(\sqrt{rs} + \sqrt{(1-r)(1-s)} \right)^2 \leq 1$$

uniformly in any (i, j) -pair, which concludes the proof of Lemma. $\square$

Remark 5. Let us elaborate on the difference between writing kinetic energies (4.4) when $m_i \neq m_j$ versus $m_i = m_j$. In order to be more precise, we will put a bar on a quantity when assuming the same masses. For instance, the total energy of the two colliding particles of the same masses m_i is

$$\bar{E} = \langle v \rangle_i^2 + \langle v_* \rangle_i^2 = 2 + \frac{2m_i}{\sum_{j=1}^I m_j} |V|^2 + \frac{m_i}{2 \sum_{j=1}^I m_j} |u|^2.$$

When $m_i = m_j$, we have the parameter $r = 1/2$, and consequently, for $\bar{p} := p(v, v_*, m_i, m_i)$, $\bar{q} := q(v, v_*, m_i, m_i)$ and $\bar{\lambda}$ give us

$$\bar{p} = \bar{q} = \frac{1}{2} \bar{E}, \quad \bar{\lambda} = \frac{m_i}{\sum_{j=1}^I m_j} |u| |V|,$$

which gives the squares of the magnitudes of the post-collisional velocities when $m_i = m_j$:

$$\begin{aligned}\langle v'_i \rangle^2 &= \bar{E} \left(\frac{1}{2} + \frac{m_i}{\sum_{j=1}^I m_j} \frac{|u| |V|}{\bar{E}} \sigma \cdot \hat{V} \right), \\ \langle v'_* \rangle^2 &= \bar{E} \left(\frac{1}{2} - \frac{m_i}{\sum_{j=1}^I m_j} \frac{|u| |V|}{\bar{E}} \sigma \cdot \hat{V} \right).\end{aligned}\quad (4.9)$$

Now, the difference between (4.8) as a convex combination writing in the mixture setting and (4.9) as its special “middle point”, or “half” case in single component case (or mixture for $m_i = m_j$), is clear.

Another important aspect to be pointed out is the comparison of inequalities (4.3) in the case $m_i \neq m_j$ versus $m_i = m_j$. When $m_i = m_j$, by simply performing Young’s inequality we get

$$\frac{\bar{\lambda}}{\bar{E}} \leq \frac{1}{2}, \quad (4.10)$$

which yields

$$\frac{1}{\bar{E}} (\bar{p} + \bar{\lambda}) = \frac{1}{\bar{E}} (\bar{q} + \bar{\lambda}) \leq 1.$$

This inequality is an analogue of (4.3) for $m_i = m_j$. Note that when masses are the same we can make use of the Young inequality, while in the case of different masses, we have to be more precise, since both $\frac{1}{\bar{E}} (\bar{p} + \bar{\lambda})$ and $\frac{1}{\bar{E}} (\bar{q} + \bar{\lambda})$ attain 1 as a maximal value for some values of their arguments, and therefore there is no room for any inequality. In particular, this inequality will be of decisive importance for the success of the Povzner lemma that will guarantee decay of the gain term with respect to the number of moments.

Proof of Povzner lemma 2.2. In order to compute the angular average estimate (2.22) we use the representation (4.1) and (4.2) from the energy identity Lemma 4.1 raised to power $k/2$. Then, the left hand side integral of (2.22) becomes

$$\begin{aligned}& \int_{S^2} \left(\langle v'_i \rangle^k + \langle v'_* \rangle^k \right) b_{ij}(\sigma \cdot \hat{u}) \, d\sigma \\ &= \int_{S^2} \left(\left(p + \lambda \sigma \cdot \hat{V} \right)^{\frac{k}{2}} + \left(q - \lambda \sigma \cdot \hat{V} \right)^{\frac{k}{2}} \right) b_{ij}(\sigma \cdot \hat{u}) \, d\sigma.\end{aligned}\quad (4.11)$$

Now we use polar coordinates for σ and $\hat{V}$ with zenith $\hat{u}$. Namely, denoting with θ the angle between σ and $\hat{u}$, we decompose σ as

$$\begin{aligned}\sigma &= \cos \theta \, \hat{u} + \sin \theta \, \omega, \text{ with } \hat{u} \cdot \omega = 0 \text{ and} \\ \omega &= (\cos \varphi, \sin \varphi), \quad \theta \in [0, \pi), \varphi \in [0, 2\pi).\end{aligned}\quad (4.12)$$

In the same fashion we decompose $\hat{V}$, by denoting with $\alpha \in [0, \pi)$ the angle between $\hat{V}$ and $\hat{u}$:

$$\hat{V} = \cos \alpha \, \hat{u} + \sin \alpha \, \Phi, \text{ where } \Phi \in S^1 \text{ with } \hat{u} \cdot \Phi = 0.$$

Then the scalar product $\sigma \cdot \hat{V}$ becomes

$$\sigma \cdot \hat{V} = \cos \theta \cos \alpha + \Phi \cdot \omega \sin \theta \sin \alpha.$$

Defining $\tau := \cos \theta$ and expressing $\sin \theta = \sqrt{1 - \tau^2}$, since $\sin \theta \geq 0$ on the range of θ , this scalar product reads as

$$\sigma \cdot \hat{V} = \tau \cos \alpha + \Phi \cdot \omega \sqrt{1 - \tau^2} \sin \alpha =: \mu = \mu(\tau, \alpha, \Phi \cdot \omega). \quad (4.13)$$

In the integral (4.11), we first express σ in its polar coordinates (4.12) and then change variables $\theta \mapsto \tau = \cos \theta$, which yields

$$\begin{aligned} & \int_{S^2} \left((p + \lambda \sigma \cdot \hat{V})^{\frac{k}{2}} + (q - \lambda \sigma \cdot \hat{V})^{\frac{k}{2}} \right) b_{ij}(\sigma \cdot \hat{u}) \, d\sigma \\ &= \int_0^{2\pi} \int_0^\pi \left((p + \lambda \sigma \cdot \hat{V})^{\frac{k}{2}} + (q - \lambda \sigma \cdot \hat{V})^{\frac{k}{2}} \right) b_{ij}(\cos \theta) \sin \theta \, d\theta \, d\varphi \\ &= \int_0^{2\pi} \int_{-1}^1 \left((p + \lambda \mu)^{\frac{k}{2}} + (q - \lambda \mu)^{\frac{k}{2}} \right) b_{ij}(\tau) \, d\tau \, d\varphi. \end{aligned}$$

For $k \geq 4$, Taylor expansion of $(p + \lambda \mu)^{k/2}$ and $(q - \lambda \mu)^{k/2}$ around $\mu = 0$ up to second order yields:

$$\begin{aligned} (p + \lambda \mu)^{\frac{k}{2}} &= p^{\frac{k}{2}} + \frac{k}{2} p^{\frac{k}{2}-1} \lambda \mu + \frac{k}{2} \left(\frac{k}{2} - 1 \right) \lambda^2 \mu^2 \int_0^1 (1-z) (p + \lambda \mu z)^{\frac{k}{2}-2} \, dz, \\ (q - \lambda \mu)^{\frac{k}{2}} &= q^{\frac{k}{2}} - \frac{k}{2} q^{\frac{k}{2}-1} \lambda \mu + \frac{k}{2} \left(\frac{k}{2} - 1 \right) \lambda^2 \mu^2 \int_0^1 (1-z) (q - \lambda \mu z)^{\frac{k}{2}-2} \, dz. \end{aligned}$$

For $2 < k < 4$, we stop at the first order and proceed similarly.

Now, let us analyze the integrands. By the Young inequality, for λ , the following estimates hold:

$$\pm \lambda \leq q - 1 \leq q, \quad \text{and} \quad \pm \lambda \leq p - 1 \leq p. \quad (4.14)$$

We recall definition of p and q , and get

$$p = (r(1-s) + (1-r)s) E, \quad q = (rs + (1-r)(1-s)) E$$

for $r \in (0, 1)$ and $s \in [0, 1]$. Considering r as parameter, for both coefficients, the maximum with respect to variable s is achieved on the boundary, i.e. for either $s = 0$ or $s = 1$, and moreover the following estimate holds for both coefficients:

$$r(1-s) + (1-r)s \leq \max\{r, (1-r)\}, \quad rs + (1-r)(1-s) \leq \max\{r, (1-r)\}.$$

Denoting

$$\bar{r} = \max\{r, 1-r\}, \quad (4.15)$$

we conclude on the upper bound for both p and q :

$$p \leq \bar{r} E, \quad q \leq \bar{r} E.$$

Moreover, for p and q it holds that

$$p = r(1-s)E + (1-r)sE \geq \underline{r}E \quad \text{and} \quad q \geq \underline{r}E,$$

where we have denoted

$$\underline{r} = \min\{r, 1-r\}. \quad (4.16)$$

Taking into account the inequalities above, one has

$$p + \lambda\mu z \leq p + q\mu z = E - q(1-\mu z) \leq E(1-\underline{r}(1-|\mu|z)),$$

and similarly,

$$q - \lambda\mu z \leq E(1-\underline{r}(1-|\mu|z)).$$

Therefore,

$$\begin{aligned} & (p + \lambda\mu)^{\frac{k}{2}} + (q - \lambda\mu)^{\frac{k}{2}} \\ & \leq p^{\frac{k}{2}} + q^{\frac{k}{2}} + \frac{k}{2}\mu \left(p^{\frac{k}{2}} + q^{\frac{k}{2}} \right) \\ & \quad + k \left(\frac{k}{2} - 1 \right) \bar{r}^2 \mu^2 E^{\frac{k}{2}} \int_0^1 (1-z)(1-\underline{r}(1-|\mu|z))^{\frac{k}{2}-2} dz. \end{aligned}$$

Then

$$\int_0^{2\pi} \int_{-1}^1 \left((p + \lambda\mu)^{\frac{k}{2}} + (q - \lambda\mu)^{\frac{k}{2}} \right) b_{ij}(\tau) d\tau d\varphi \leq P_1 + P_2 + P_3,$$

with

$$\begin{aligned} P_1 &:= \left(p^{\frac{k}{2}} + q^{\frac{k}{2}} \right) \int_0^{2\pi} \int_{-1}^1 b_{ij}(\tau) d\tau d\varphi, \\ P_2 &:= \frac{k}{2} \left(p^{\frac{k}{2}} + q^{\frac{k}{2}} \right) \int_0^{2\pi} \int_{-1}^1 \mu b_{ij}(\tau) d\tau d\varphi, \\ P_3 &:= k \left(\frac{k}{2} - 1 \right) \bar{r}^2 E^{\frac{k}{2}} \int_0^{2\pi} \int_{-1}^1 \mu^2 \\ & \quad \times \left(\int_0^1 (1-z)(1-\underline{r}(1-|\mu|z))^{\frac{k}{2}-2} dz \right) b_{ij}(\tau) d\tau d\varphi. \end{aligned}$$

Term P_1 . Introducing constant $\tilde{C}_n$, we have

$$\tilde{C}_n = \bar{r}^n, \quad 0 < \bar{r} < 1, \quad (4.17)$$

which clearly decays in n , so we get

$$P_1 = \|b_{ij}\|_{L^1(d\sigma)} \left(p^{\frac{k}{2}} + q^{\frac{k}{2}} \right) \leq \|b_{ij}\|_{L^1(d\sigma)} 2\tilde{C}_{\frac{k}{2}} E^{\frac{k}{2}}.$$

Term P_2 . Taking into account definition of μ from (4.13), the parity arguments yield

$$P_2 = \frac{k}{2} \left(p^{\frac{k}{2}} + q^{\frac{k}{2}} \right) \int_0^{2\pi} \int_{-1}^1 \tau \cos \alpha b_{ij}(\tau) d\tau d\varphi,$$

after bounding $\cos \alpha \leq 1$. Using the estimate above for P_1 and the fact that $\tau \cos \alpha \leq 1$, we finally obtain

$$P_2 \leq \|b_{ij}\|_{L^1(d\sigma)} k \tilde{C}_{\frac{k}{2}} E^{\frac{k}{2}}.$$

Since the constant $\tilde{C}_{\frac{k}{2}}$ has power decay in k , the constant $k \tilde{C}_{\frac{k}{2}}$ also decreases in k .

Term P_3 . We can compute explicitly the integral with respect to z as

$$\begin{aligned} & \int_0^1 (1-z)(1-a(1-Az))^{n-2} dz \\ &= \frac{1}{a^2 A^2} \frac{1}{n(n-1)} \left((1+a(A-1))^n - (1-a)^n - aA(1-a)^{n-1}n \right) \end{aligned}$$

for any $0 < a < 1$ and $A > 0$. If $A = 0$, then we easily obtain

$$\int_0^1 (1-z)(1-a)^{n-2} dz = \frac{1}{2}(1-a)^{n-2}.$$

In our case, $a = \underline{r}$ and $A = |\mu|$, μ being a function of variables of integration τ and φ defined in (4.13) that satisfies $|\mu| \leq 1$, and thus P_3 becomes

$$\begin{aligned} P_3 &= 2 \frac{\bar{r}^2}{\underline{r}^2} E^{\frac{k}{2}} \int_0^{2\pi} \int_{-1}^1 \left((1 + \underline{r}(|\mu| - 1))^{\frac{k}{2}} \right. \\ &\quad \left. - (1 - \underline{r})^{\frac{k}{2}} - \underline{r}|\mu|(1 - \underline{r})^{\frac{k}{2}-1} \frac{k}{2} \right) b_{ij}(\tau) d\tau d\varphi \\ &=: P_{3_1} + P_{3_2} + P_{3_3}, \\ &\leq E^{\frac{k}{2}} \left(\check{C}_{\frac{k}{2}}^{b_{ij}} + \|b_{ij}\|_{L^1(d\sigma)} \left(\bar{C}_{\frac{k}{2}} + \hat{C}_{\frac{k}{2}} \right) \right), \end{aligned}$$

where we have denoted

$$\begin{aligned} P_{3_1} &:= 2 \frac{\bar{r}^2}{\underline{r}^2} E^{\frac{k}{2}} \int_0^{2\pi} \int_{-1}^1 (1 + \underline{r}(|\mu| - 1))^{\frac{k}{2}} b_{ij}(\tau) d\tau d\varphi, \\ P_{3_2} &:= -2 \frac{\bar{r}^2}{\underline{r}^2} (1 - \underline{r})^{\frac{k}{2}} E^{\frac{k}{2}} \int_0^{2\pi} \int_{-1}^1 b_{ij}(\tau) d\tau d\varphi, \\ P_{3_3} &:= -\frac{\bar{r}^2}{\underline{r}} k (1 - \underline{r})^{\frac{k}{2}-1} E^{\frac{k}{2}} \int_0^{2\pi} \int_{-1}^1 |\mu| b_{ij}(\tau) d\tau d\varphi. \end{aligned}$$

Term P_{3_1} . We rewrite term P_{3_1} as

$$P_{3_1} = \check{C}_{\frac{k}{2}}^{b_{ij}} E^{\frac{k}{2}}$$

by introducing the constant $\check{C}_n^{b_{ij}}$:

$$\check{C}_n^{b_{ij}} = 2 \frac{\bar{r}^2}{r^2} \int_0^{2\pi} \int_{-1}^1 (1 + r(|\mu| - 1))^n b_{ij}(\tau) d\tau d\varphi. \quad (4.18)$$

In order to study its properties, we first note that $1 + r(|\mu| - 1) \leq 1$, since $|\mu| \leq 1$, and the equality holds only when $|\mu| = 1$ (or $\sigma = \{\pm \hat{V}\}$). Therefore, the sequence of functions

$$A_n(x) := (1 + r(x - 1))^n$$

decreases monotonically in n and tends to 0 as $n \rightarrow \infty$ for every $x \in (0, 1)$ up to a set of measure zero. Finally, we conclude by monotone convergence Theorem that $\check{C}_{\frac{k}{2}}^{b_{ij}} \searrow 0$ as $k \rightarrow \infty$.

When $b_{ij} \in L^\infty(S^2; d\sigma)$, we can obtain the explicit decay rate of the constant $\check{C}_{\frac{k}{2}}^{b_{ij}}$, since in this case the integral (4.18) significantly simplifies. The rate will be calculated in the Remark 6 below.

Term P_{3_2} . For the term P_{3_2} we immediately obtain

$$P_{3_2} = \|b_{ij}\|_{L^1(d\sigma)} \bar{C}_{\frac{k}{2}} E^{\frac{k}{2}},$$

with the constant

$$\bar{C}_n = -2 \frac{\bar{r}^2}{r^2} (1 - r)^n.$$

Term P_{3_3} . We first estimate the term P_{3_3} using $|\mu| \leq 1$ to get

$$\begin{aligned} P_{3_3} &\leq \frac{\bar{r}^2}{r} k (1 - r)^{\frac{k}{2}-1} E^{\frac{k}{2}} \int_0^{2\pi} \int_{-1}^1 b_{ij}(\tau) d\tau d\varphi \\ &\leq \|b_{ij}\|_{L^1(d\sigma)} \hat{C}_{\frac{k}{2}} E^{\frac{k}{2}}, \end{aligned}$$

and the constant is defined with

$$\hat{C}_n = 2 \frac{\bar{r}^2}{r} n (1 - r)^{n-1}. \quad (4.19)$$

Gathering estimates for P_1 , P_2 and P_3 completes the proof of (2.22) with

$$\mathcal{C}_n^{ij} = \|b_{ij}\|_{L^1(d\sigma)} \left((2n + 2) \check{C}_n + \bar{C}_n + \hat{C}_n \right) + \check{C}_n^{b_{ij}}, \quad n > 2,$$

and $\mathcal{C}_n^{ij} = \|b_{ij}\|_{L^1(d\sigma)} 2\check{C}_n$, if $1 < n \leq 2$. Thus, the constant $\mathcal{C}_{\frac{k}{2}}^{ij}$ issuing from the Povzner lemma satisfies $\mathcal{C}_{\frac{k}{2}}^{ij} \rightarrow 0$ as $k \rightarrow \infty$, and so there exists $k_*^{ij} = k_*^{ij}(r_{ij}, b_{ij})$ for which $\mathcal{C}_{\frac{k}{2}}^{ij} < \|b_{ij}\|_{L^1(d\sigma)}$, for $k > k_*^{ij}$. $\square$

Remark 6. (The case $b_{ij} \in L^\infty(S^2; d\sigma)$) When the angular kernel is assumed to be bounded, some calculations are simpler. Pulling out the L^∞ norm of b_{ij} , we have

$$\int_0^{2\pi} \int_{-1}^1 b_{ij}(\tau) d\tau d\varphi \leq 4\pi \|b_{ij}\|_{L^\infty(d\sigma)},$$

and so terms P_1 and P_{3_2} become

$$P_1 \leq 8\pi \|b_{ij}\|_{L^\infty(d\sigma)} \tilde{C}_{\frac{k}{2}} E^{\frac{k}{2}}, \quad P_{3_2} = 4\pi \|b_{ij}\|_{L^\infty(d\sigma)} \bar{C}_{\frac{k}{2}} E^{\frac{k}{2}}.$$

Moreover, when b_{ij} is assumed to be bounded, the starting integral (4.11) does not depend on $\sigma \cdot \hat{u}$ anymore, so we may take $\hat{V}$ instead of $\hat{u}$ as a zenith of polar coordinates in (4.12), which amounts to taking $\alpha = 0$ in (4.13), which implies that $\mu = \tau$. In this case, thanks to the parity arguments, term P_2 vanishes, and term P_{3_3} can be explicitly calculated without using any estimate, so we get

$$\begin{aligned} P_{3_3} &= -\frac{\bar{r}^2}{\underline{r}} k (1 - \underline{r})^{\frac{k}{2}-1} E^{\frac{k}{2}} \int_0^{2\pi} \int_{-1}^1 |\tau| b_{ij}(\tau) d\tau d\varphi \\ &= -2\pi \|b_{ij}\|_{L^\infty(d\sigma)} \hat{C}_{\frac{k}{2}} E^{\frac{k}{2}}, \end{aligned}$$

with the constant $\hat{C}_{\frac{k}{2}}$ from (4.19).

Finally, let us compute explicitly the constant $\check{C}_n^{b_{ij}}$ from (4.18) when $b_{ij}(\sigma \cdot \hat{u}) \in L^\infty(S^2; d\sigma)$. Namely, pulling out the L^∞ norm of b_{ij} from the integral and using $\mu = \tau$, we get

$$\begin{aligned} \check{C}_n^{b_{ij}} &= 2 \frac{\bar{r}^2}{\underline{r}^2} \|b_{ij}\|_{L^\infty(d\sigma)} \int_0^{2\pi} \int_{-1}^1 (1 + \underline{r}(|\tau| - 1))^n d\tau d\varphi \\ &= 8\pi \frac{\bar{r}^2}{\underline{r}^3} \|b_{ij}\|_{L^\infty(d\sigma)} \left(\frac{1}{n+1} - \frac{(1-\underline{r})^{n+1}}{n+1} \right), \end{aligned}$$

which shows its decay rate.

To summarize, the constant from the Povzner lemma in the case of bounded angular part reads as

$$\mathcal{C}_n^{ij} = 4\pi \|b_{ij}\|_{L^\infty(d\sigma)} \mathcal{C}_n^\infty(r), \quad (4.20)$$

where we have denoted

$$\mathcal{C}_n^\infty(r) = 2\tilde{C}_n + \bar{C}_n - \frac{1}{2}\hat{C}_n + 2\frac{\bar{r}^2}{\underline{r}^3} \left(\frac{1}{n+1} - \frac{(1-\underline{r})^{n+1}}{n+1} \right), \quad n > 2, \quad (4.21)$$

and $\mathcal{C}_n^\infty(r) = 2\tilde{C}_n$ if $1 < n \leq 2$, recalling (4.16) and (4.15). Moreover, it satisfies $\mathcal{C}_{\frac{k}{2}}^{ij} < 4\pi \|b_{ij}\|_{L^\infty(d\sigma)}$, or, equivalently, $\mathcal{C}_n^\infty(r) < 1$ for sufficiently large k_* depending on r_{ij} and b_{ij} .

4.1. Study of the Povzner Constant for $b_{ij}(\sigma \cdot \hat{u}) \in L^\infty(S^2; d\sigma)$

In this section we study in detail the constant (4.20) from the Povzner lemma 2.2 in the case of bounded angular part. More precisely, we study its normalized part (4.21)

$$\mathcal{C}_n^\infty(r) = 2\bar{r}^n - 2\frac{\bar{r}^2}{\underline{r}^2}(1-r)^n - \frac{\bar{r}^2}{\underline{r}}n(1-r)^{n-1} + 2\frac{\bar{r}^2}{\underline{r}^3}\left(\frac{1}{n+1} - \frac{(1-\underline{r})^{n+1}}{n+1}\right) \quad (4.22)$$

for $n > 2$ and $\mathcal{C}_n^\infty(r) = 2\bar{r}^n$ if $1 < n \leq 2$, with $\bar{r} = \max\{r, 1-r\}$ and $\underline{r} = \min\{r, 1-r\}$, and elaborate more on its decay rate in n depending on r .

First, taking $r = \frac{1}{2}$ we expect to recover the same properties as for the single gas when decay rate of the Povzner constant [3] was $\frac{2}{n+1}$, that monotonically decreases and tends to zero in $n > 1$. In our case,

$$\mathcal{C}_n^\infty\left(\frac{1}{2}\right) = \begin{cases} \frac{4}{n+1} - \left(\frac{1}{2}\right)^n \left(n + \frac{2}{n+1}\right), & \text{if } n > 2, \\ 2\left(\frac{1}{2}\right)^n, & \text{if } 1 < n \leq 2, \end{cases}$$

which keeps the same properties as for the single gas, as is illustrated in Fig. 2.

For general $r \in (0, 1)$, decay properties of the constant issuing from the Povzner lemma (4.21) strongly depend on r or the degree to which species masses m_i , $i = 1, \dots, I$ are disparate. It is clear that, since $0 < r < 1$, the constant $\mathcal{C}_n^\infty(r)$ will tend to zero as n goes to infinity. Here we are interested in a more subtle question: determine n_* such that it holds that $\mathcal{C}_n^\infty(r) < 1$ for $n \geq n_*$ and any fixed $0 < r < 1$. Converging $\mathcal{C}_n^\infty(r)$ in n towards zero for any $0 < r < 1$ ensures the existence of such n_* . It can be observed that n_* grows as much as r is deviated from $\frac{1}{2}$, since the constants in $\mathcal{C}_n^\infty(r)$ with a power decay rate will decay more slowly as r deviates from $\frac{1}{2}$. This behavior is illustrated in Fig. 3. We can reformulate the problem as follows: for some fixed value of n determine the interval of r for which it holds that $\mathcal{C}_n^\infty(r) < 1$, as is illustrated in Fig. 4.

5. Proof of Existence and Uniqueness Theorem 2.3

Before proving Theorem 2.3, we first study a property of the collision operator that is a consequence of the Povzner Lemmas 2.23 and B.1.

Lemma 5.1. *Let $\mathbb{F} = [f_i]_{i=1,\dots,I} \in \Omega$ and k_* as defined in (2.24). Then, the following estimate holds:*

$$\sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{k_*} dv \leq -A_{k_*} m_{k_*}[\mathbb{F}](t)^{1+\frac{\bar{\gamma}}{k_*}} + B_{k_*} m_{k_*}[\mathbb{F}](t), \quad (5.1)$$

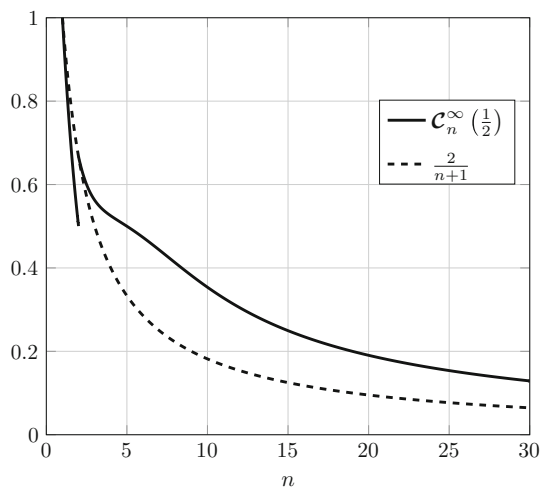


Fig. 2. Comparison of the Povzner constant for $r = \frac{1}{2}$ in the mixture setting and the single component gas for $n > 1$

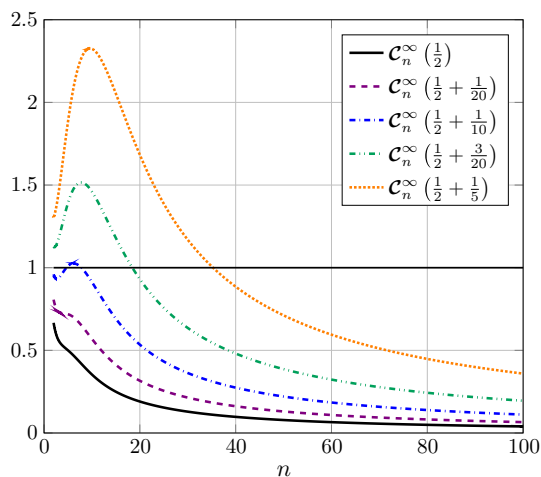


Fig. 3. Constant $\mathcal{C}_n^\infty(r)$ from Povzner Lemma 2.2 for some fixed value of $r =: r_*$. This figure illustrates the non-monotonic behavior in n variable, and the growth of n needed to ensure that $\mathcal{C}_n^\infty(r_*) < 1$ caused by a deviation of r with respect to $\frac{1}{2}$

with positive constants

$$A_{k_*} = \min_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(\mathrm{d}\sigma)} - c_{\frac{k_*}{2}}^{ij} \right) \frac{c_{lb}}{\max_{1 \leq i \leq I} m_i} (IC_0)^{-\frac{\overline{\gamma}}{k_*}},$$

$$B_{k_*} = 2 C_2 \max_{1 \leq i, j \leq I} \left(\left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} c_{\frac{k_*}{2}}^{ij} \right) \sum_{\ell=1}^{\lfloor \frac{k_*+1}{2} \rfloor} \binom{k_*}{\ell}, \quad (5.2)$$

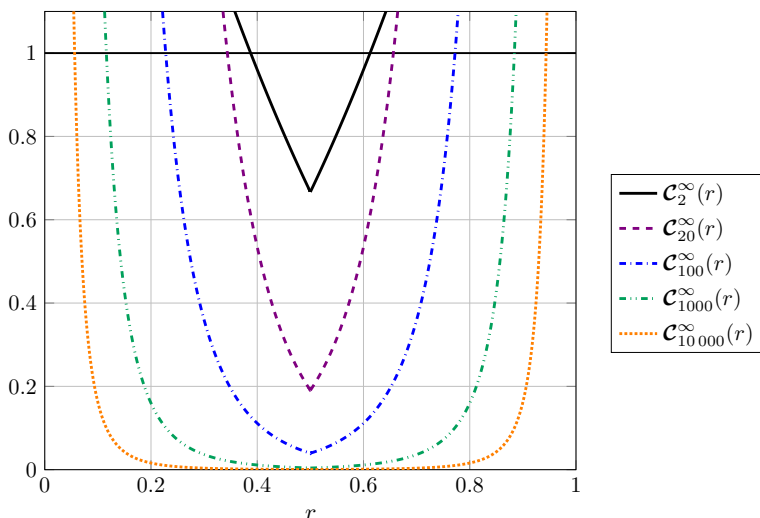


Fig. 4. Constant $\mathcal{C}_n^\infty(r)$ from Povzner Lemma 2.2 for some fixed value of $n =: n_*$. This figure illustrates the interval of r for which it holds $\mathcal{C}_{n_*}^\infty(r) < 1$

where C_0 and C_2 are from the characterization of the set Ω , c_{lb} is from the lower bound (B.4), and $\mathcal{C}_{\frac{k_*}{2}}^{ij}$ is a constant from the Povzner Lemma 2.2 with $k_* > \bar{k}$, as defined in (2.24), ensuring the property (2.23) for any pair (i, j) that yields positivity of the constant A_{k_*} .

Remark 7. It is important to notice that the strict positivity of the constant A_{k_*} can be viewed as a **coercive condition** that secures global in time solutions, without the need to require boundedness of entropy.

Proof. We start with the weak form (3.9). Taking test function $\psi_i(x) = \langle v \rangle_i^{k_*}$ and cross section (2.20), we have

$$\sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{k_*} dv \quad (5.3)$$

$$\begin{aligned} &= \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} \langle v \rangle_i^{k_*} Q_{ij}(f_i, f_j) dv \\ &= \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |v - v_*|^{\gamma_{ij}} f_i(v) f_j(v_*) \\ &\quad \times \left(\langle v \rangle_i^{k_*} + \langle v_* \rangle_j^{k_*} - \langle v \rangle_i^{k_*} - \langle v_* \rangle_j^{k_*} \right) b_{ij}(\sigma \cdot \hat{u}) d\sigma dv_* dv, \end{aligned} \quad (5.4)$$

where this collisional rules are (3.2). The primed quantities integrated over sphere S^2 are estimated via Povzner lemma. Indeed, by Lemma 2.2, (5.3) becomes

$$\begin{aligned} \sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{k_*} dv &\leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_i(v) f_j(v_*) |v - v_*|^{\gamma_{ij}} \\ &\times \left(\mathcal{C}_{\frac{k_*}{2}}^{ij} \left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right)^{\frac{k_*}{2}} - \|b_{ij}\|_{L^1(d\sigma)} \left(\langle v \rangle_i^{k_*} + \langle v_* \rangle_j^{k_*} \right) \right) dv_* dv, \quad (5.5) \end{aligned}$$

where $\mathcal{C}_{\frac{k_*}{2}}^{ij}$ is a constant from Povzner lemma 2.2 with $k_* \geq \bar{k} = \max_{1 \leq i, j \leq I} \{k_*^{ij}\}$ chosen large enough to ensure (2.23) uniformly in i, j -pairs. On the one hand, we use polynomial inequalities from Lemmas C.1 and C.2 to get

$$\begin{aligned} &\left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right)^{\frac{k_*}{2}} \\ &\leq (\langle v \rangle_i + \langle v_* \rangle_j)^{k_*} \\ &\leq \langle v \rangle_i^{k_*} + \langle v_* \rangle_j^{k_*} + \sum_{\ell=1}^{\ell_{k_*}} \binom{k_*}{\ell} \left(\langle v \rangle_i^\ell \langle v_* \rangle_j^{k_*-\ell} + \langle v \rangle_i^{k_*-\ell} \langle v_* \rangle_j^\ell \right), \\ &\leq \langle v \rangle_i^{k_*} + \langle v_* \rangle_j^{k_*} + \left(\langle v \rangle_i \langle v_* \rangle_j^{k_*-1} + \langle v \rangle_i^{k_*-1} \langle v_* \rangle_j \right) \left(\sum_{\ell=1}^{\ell_{k_*}} \binom{k_*}{\ell} \right), \end{aligned}$$

with $\ell_{k_*} = \lfloor \frac{k_*+1}{2} \rfloor$, and therefore

$$\begin{aligned} \sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{k_*} dv &\leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_i(v) f_j(v_*) |v - v_*|^{\gamma_{ij}} \\ &\times \left\{ - \left(\|b_{ij}\|_{L^1(d\sigma)} - \mathcal{C}_{\frac{k_*}{2}}^{ij} \right) \left(\langle v \rangle_i^{k_*} + \langle v_* \rangle_j^{k_*} \right) \right. \\ &\left. + \mathcal{C}_{\frac{k_*}{2}}^{ij} \left(\sum_{\ell=1}^{\ell_{k_*}} \binom{k_*}{\ell} \right) \left(\langle v \rangle_i \langle v_* \rangle_j^{k_*-1} + \langle v \rangle_i^{k_*-1} \langle v_* \rangle_j \right) \right\} dv_* dv. \quad (5.6) \end{aligned}$$

On the other hand we use upper and lower bound of the non-angular cross section $|v - v_*|^{\gamma_{ij}}$. For the upper bound, from (B.2) it follows that

$$|v - v_*|^{\gamma_{ij}} \leq \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \left(\langle v \rangle_i^{\gamma_{ij}} + \langle v_* \rangle_j^{\gamma_{ij}} \right) \leq \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \left(\langle v \rangle_i^{\bar{\gamma}} + \langle v_* \rangle_j^{\bar{\gamma}} \right)$$

for $\bar{\gamma} = \max_{1 \leq i, j \leq I} \gamma_{ij} \in (0, 1]$. For the lower bound, we use Lemma B.1, but we first check that all assumptions are satisfied from the fact that $\mathbb{F} \in \Omega$. Indeed, bounds on m_0 imply

$$c_0 \min_{1 \leq i \leq I} m_i \leq \sum_{i=1}^I \int_{\mathbb{R}^3} m_i f_i dv \leq C_0 \max_{1 \leq i \leq I} m_i.$$

From the other side, bounds on m_2 yield

$$(c_2 - C_0) \sum_{j=1}^I m_j \leq \sum_{i=1}^I \int_{\mathbb{R}^3} m_i |v|^2 f_i \, dv \leq (C_2 - c_0) \sum_{j=1}^I m_j.$$

Therefore, for constants c and C , from the assumptions of Lemma B.1, we can choose

$$c := \min \left\{ c_0 \min_{1 \leq i \leq I} m_i, (c_2 - C_0) \sum_{j=1}^I m_j \right\},$$

$$C := \max \left\{ C_0 \max_{1 \leq i \leq I} m_i, (C_2 - c_0) \sum_{j=1}^I m_j \right\}.$$

Note that positivity of c is guaranteed by the definition of the set Ω . Finally, since it can be estimated that

$$\sum_{i=1}^I \int_{\mathbb{R}^3} m_i |v|^{2+\varepsilon} f_i \, dv \leq m_{2+\varepsilon} \left(\sum_{j=1}^I m_j \right)^{1+\frac{\varepsilon}{2}} \max_{1 \leq i \leq I} m_i^{-\frac{\varepsilon}{2}},$$

we can choose

$$B := C_{2+\varepsilon} \left(\sum_{j=1}^I m_j \right)^{1+\frac{\varepsilon}{2}} \max_{1 \leq i \leq I} m_i^{-\frac{\varepsilon}{2}}.$$

Then (B.4) implies

$$\sum_{i=1}^I \int_{\mathbb{R}^3} f_i(v) |v - v_*|^{\gamma_{ij}} \, dv \geq \frac{1}{\max_{1 \leq i \leq I} m_i} c_{lb} \langle v_* \rangle_j^{\overline{\gamma}}$$

and

$$\sum_{j=1}^I \int_{\mathbb{R}^3} f_j(v_*) |v - v_*|^{\gamma_{ij}} \, dv \geq \frac{1}{\max_{1 \leq j \leq I} m_j} c_{lb} \langle v \rangle_i^{\overline{\gamma}}.$$

With these estimates, (5.6) becomes

$$\sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{k_*} \, dv \leq -D_{k_*} m_{k_*+\overline{\gamma}} + E_{k_*} (m_{1+\overline{\gamma}} m_{k_*-1} + m_{k_*-1+\overline{\gamma}} m_1),$$

where D_{k_*} and E_{k_*} are positive constants

$$D_{k_*} = \min_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} - c_{\frac{k_*}{2}}^{ij} \right) \frac{c_{lb}}{\max_{1 \leq i \leq I} m_i},$$

$$E_{k_*} = \max_{1 \leq i, j \leq I} \left(\left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} c_{\frac{k_*}{2}}^{ij} \right) \sum_{\ell=1}^{\ell_{k_*}} \binom{k_*}{\ell}.$$

In particular, D_{k_*} is positive since, by assumption, $k_* \geq \bar{k}$ defined in (2.24) large enough ensuring (2.23) for the constant $\mathcal{C}_{\frac{k_*}{\bar{k}}}^{ij}$ from Povzner lemma (2.22).

Arriving at moment notation, we can use the monotonicity of moments (2.18), together with an estimate on m_2 from the characterization of set Ω , to get the following estimate:

$$\sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{k_*} dv \leq -D_{k_*} m_{k_*+\bar{\gamma}} + 2E_{k_*} C_2 m_{k_*}.$$

It remains to use a control from below derived in (C.3) for the highest order moment $m_{k_*+\bar{\gamma}}$, taking $k = k_*$, $\lambda = \bar{\gamma}$ and $Cm_0 = C_0$ there, to get

$$m_{k_*+\bar{\gamma}} \geq (IC_0)^{-\frac{\bar{\gamma}}{k_*}} m_{k_*}^{1+\frac{\bar{\gamma}}{k_*}},$$

which yields the final estimate (5.1). $\square$

We turn to the proof of the Existence and Uniqueness Theorem 2.3. Our proof follows the one given in [3] for the single Boltzmann equation. In particular, our aim is to apply Theorem A.1 from a general ODE theory in Banach spaces. In order to do this, we first show that the collision operator is a mapping $\mathbb{Q} : \Omega \rightarrow L_2^1$. Indeed, take any $\mathbb{F} \in \Omega$. Then,

$$\begin{aligned} \|\mathbb{Q}(\mathbb{F})\|_{L_2^1} &= \sum_{i=1}^I \int_{\mathbb{R}^3} |[\mathbb{Q}(\mathbb{F})]_i(v)| \langle v \rangle_i^2 dv \\ &\leq \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} |Q_{ij}(f_i, f_j)(v)| \langle v \rangle_i^2 dv. \end{aligned} \quad (5.7)$$

The absolute value $|Q_{ij}(f_i, f_j)(v)|$ is written with the help of a sign function and the shorter notation

$$|Q_{ij}(f_i, f_j)(v)| = Q_{ij}(f_i, f_j)(v) s_{ij}(v), \quad s_{ij}(v) := \text{sign}(Q_{ij}(f_i, f_j)(v)).$$

Then $s_{ij}(v) \langle v \rangle_i^2$ in (5.7) are viewed as test functions, so the weak form (3.9) implies

$$\begin{aligned} \|\mathbb{Q}(\mathbb{F})\|_{L_2^1} &\leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f_i(v) f_j(v_*) \mathcal{B}_{ij}(v, v_*, \sigma) \\ &\quad \times \left(s_{ij}(v') \langle v' \rangle_i^2 + s_{ji}(v'_*) \langle v'_* \rangle_j^2 - s_{ij}(v) \langle v \rangle_i^2 - s_{ji}(v_*) \langle v_* \rangle_j^2 \right) \\ &\quad \times d\sigma dv_* dv. \end{aligned}$$

Since the sign function is upper bounded by 1, we obtain

$$\begin{aligned} \|\mathbb{Q}(\mathbb{F})\|_{L_2^1} &\leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f_i(v) f_j(v_*) \mathcal{B}_{ij}(v, v_*, \sigma) \\ &\quad \times \left(\langle v' \rangle_i^2 + \langle v'_* \rangle_j^2 + \langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right) d\sigma dv_* dv. \end{aligned}$$

Using the conservation of energy (3.1), together with the form of the cross section (2.20), implies

$$\begin{aligned} \|\mathbb{Q}(\mathbb{F})\|_{L_2^1} &\leq \sum_{i=1}^I \sum_{j=1}^I \|b_{ij}\|_{L^1(d\sigma)} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_i(v) f_j(v_*) |v - v_*|^{\gamma_{ij}} \\ &\quad \times \left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right) dv_* dv. \end{aligned}$$

Finally, using upper bound (B.3), we obtain the estimate in terms of norms:

$$\begin{aligned} \|\mathbb{Q}(\mathbb{F})\|_{L_2^1} &\leq \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) \\ &\quad \times \sum_{i=1}^I \sum_{j=1}^I \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_i(v) f_j(v_*) \langle v \rangle_i^{\bar{\gamma}} \langle v_* \rangle_j^{\bar{\gamma}} \left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right) dv_* dv \\ &= 2 \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) \left(\|\mathbb{F}\|_{L_{2+\bar{\gamma}}^1} \|\mathbb{F}\|_{L_{\bar{\gamma}}^1} \right). \end{aligned}$$

Since $\mathbb{F} \in \Omega$, the right hand side is bounded, and therefore $\mathbb{Q}(\mathbb{F}) \in L_2^1$.

The next task is to show that the mapping $\mathbb{F} \mapsto \mathbb{Q}(\mathbb{F})$, when restricted to Ω , satisfies (i) Hölder continuity, (ii) the sub-tangent and (iii) the one-sided Lipschitz conditions. Indeed, the proof is divided into proofs of these three properties.

Assume that $\mathbb{F}, \mathbb{G} \in \Omega$ and cross section $\mathcal{B}_{ij}$ are given in (2.20). Then, the following three properties hold:

(i) Hölder continuity condition

$$\|\mathbb{Q}(\mathbb{F}) - \mathbb{Q}(\mathbb{G})\|_{L_2^1} \leq C_H \|\mathbb{F} - \mathbb{G}\|_{L_2^1}^{\frac{1}{2}}; \quad (5.8)$$

(ii) Sub-tangent condition

$$\lim_{h \rightarrow 0^+} \frac{\text{dist}(\mathbb{F} + h\mathbb{Q}(\mathbb{F}), \Omega)}{h} = 0,$$

where

$$\text{dist}(\mathbb{H}, \Omega) = \inf_{\omega \in \Omega} \|\mathbb{H} - \omega\|_{L_2^1};$$

(iii) One-sided Lipschitz condition

$$[\mathbb{Q}(\mathbb{F}) - \mathbb{Q}(\mathbb{G}), \mathbb{F} - \mathbb{G}] \leq C_L \|\mathbb{F} - \mathbb{G}\|_{L_2^1},$$

where, by Remark 9,

$$\begin{aligned} &[\mathbb{Q}(\mathbb{F}) - \mathbb{Q}(\mathbb{G}), \mathbb{F} - \mathbb{G}] \\ &= \lim_{h \rightarrow 0^-} \frac{\left(\|\mathbb{F} - \mathbb{G}\| + h (\mathbb{Q}(\mathbb{F}) - \mathbb{Q}(\mathbb{G}))\|_{L_2^1} - \|\mathbb{F} - \mathbb{G}\|_{L_2^1} \right)}{h} \\ &\leq \sum_{i=1}^I \int_{\mathbb{R}^3} ([\mathbb{Q}(\mathbb{F})]_i(v) - [\mathbb{Q}(\mathbb{G})]_i(v)) \text{sign}(f_i(v) - g_i(v)) \langle v \rangle_i^2 dv. \end{aligned}$$

Constants C_H and C_L depend on $\|b_{ij}\|_{L^1(d\sigma)}$, the number of species I and their masses m_i , $i = 1, \dots, I$, and constants from characterization of the set Ω .

Proof of (i) Hölder continuity condition. Let $\mathbb{F} = [f_i]_{1 \leq i \leq I}$ and $\mathbb{G} = [g_i]_{1 \leq i \leq I}$ belong to Ω . We need to estimate the following expression

$$I_H := \|\mathbb{Q}(\mathbb{F}) - \mathbb{Q}(\mathbb{G})\|_{L^1_2} = \sum_{i=1}^I \int_{\mathbb{R}^3} \left| \sum_{j=1}^I (Q_{ij}(f_i, f_j) - Q_{ij}(g_i, g_j)) \right| \langle v \rangle_i^2 dv. \quad (5.9)$$

Using the binary structure of collision operator (2.1), it follows that

$$Q_{ij}(f_i, f_j) - Q_{ij}(g_i, g_j) = \frac{1}{2} (Q_{ij}(f_i - g_i, f_j + g_j) + Q_{ij}(f_i + g_i, f_j - g_j)). \quad (5.10)$$

Therefore, using properties of absolute value, (5.9) becomes

$$I_H \leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} (|Q_{ij}(f_i - g_i, f_j + g_j)| + |Q_{ij}(f_i + g_i, f_j - g_j)|) \langle v \rangle_i^2 dv. \quad (5.11)$$

The absolute value of collision operator will be written with the help of a sign function, using $|\cdot| = \cdot \operatorname{sign}(\cdot)$. Since, at the end, all sign functions will be bounded by 1, we will not go deeply into details of its structure. Thus, let us for the moment denote

$$\operatorname{sign}(Q_{ij}(f_i - g_i, f_j + g_j)) = s_{ij}^{-+}, \quad \operatorname{sign}(Q_{ij}(f_i + g_i, f_j - g_j)) = s_{ij}^{+-}.$$

Then, (5.11) becomes

$$I_H \leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} (Q_{ij}(f_i - g_i, f_j + g_j) s_{ij}^{-+} \langle v \rangle_i^2 + Q_{ij}(f_i + g_i, f_j - g_j) s_{ij}^{+-} \langle v \rangle_i^2) dv. \quad (5.12)$$

Now we use the weak form (3.6), and in order to do thus, we have to match pairs. Indeed, we notice that the pair for ij -th element of the first sum is the ji -th element of the second sum. That is, (3.6) implies, after dropping the sign function,

$$\begin{aligned} & \int_{v \in \mathbb{R}^3} (Q_{ij}(f_i - g_i, f_j + g_j) s_{ij}^{-+} \langle v \rangle_i^2 + Q_{ji}(f_j + g_j, f_i - g_i) s_{ji}^{+-} \langle v \rangle_j^2) dv \\ & \leq \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \\ & \quad \times (\langle v \rangle_i^2 + \langle v_* \rangle_j^2 + \langle v \rangle_i^2 + \langle v_* \rangle_j^2) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_* dv \\ & = 2 \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \\ & \quad \times (\langle v \rangle_i^2 + \langle v_* \rangle_j^2) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_* dv; \end{aligned}$$

the last equality is due to the conservation law at the microscopic level (3.1). Therefore, (5.12) becomes

$$I_H \leq \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \\ \times \left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_* dv.$$

Now we use the form of cross section (2.20). Inequality (B.2) yields the following upper bound of the previous expression:

$$I_H \leq \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) \\ \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3} |f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \\ \times \left(\langle v \rangle_i^{2+\bar{\gamma}} + \langle v \rangle_i^2 \langle v_* \rangle_j^{\bar{\gamma}} + \langle v_* \rangle_j^2 \langle v \rangle_i^{\bar{\gamma}} + \langle v_* \rangle_j^{2+\bar{\gamma}} \right) dv_* dv \\ \leq \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) \left(\|\mathbb{F} - \mathbb{G}\|_{L_{2+\bar{\gamma}}^1} \|\mathbb{F} + \mathbb{G}\|_{L_0^1} \right. \\ \left. + \|\mathbb{F} - \mathbb{G}\|_{L_2^1} \|\mathbb{F} + \mathbb{G}\|_{L_{\bar{\gamma}}^1} + \|\mathbb{F} - \mathbb{G}\|_{L_{\bar{\gamma}}^1} \|\mathbb{F} + \mathbb{G}\|_{L_2^1} \right. \\ \left. + \|\mathbb{F} - \mathbb{G}\|_{L_0^1} \|\mathbb{F} + \mathbb{G}\|_{L_{2+\bar{\gamma}}^1} \right).$$

Monotonicity of the norm (2.18) yields

$$I_H \leq 2 \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) \\ \times \|\mathbb{F} - \mathbb{G}\|_{L_{2+\bar{\gamma}}^1} \left(\|\mathbb{F} + \mathbb{G}\|_{L_2^1} + \|\mathbb{F} + \mathbb{G}\|_{L_{2+\bar{\gamma}}^1} \right).$$

By the interpolation inequality (C.2), it follows that

$$I_H \leq 2I \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) \\ \times \|\mathbb{F} - \mathbb{G}\|_{L_2^1}^{1/2} \|\mathbb{F} - \mathbb{G}\|_{L_{2+2\bar{\gamma}}^1}^{1/2} \left(\|\mathbb{F} + \mathbb{G}\|_{L_2^1} + \|\mathbb{F} + \mathbb{G}\|_{L_{2+\bar{\gamma}}^1} \right). \quad (5.13)$$

Then we can bound term by term to get

$$\|\mathbb{F} - \mathbb{G}\|_{L_{2+2\bar{\gamma}}^1}^{1/2} \leq \|\mathbb{F}\|_{L_{2+2\bar{\gamma}}^1}^{1/2} + \|\mathbb{G}\|_{L_{2+2\bar{\gamma}}^1}^{1/2} \leq 2C_{2+2\bar{\gamma}}^{1/2},$$

and in the same fashion,

$$\|\mathbb{F} + \mathbb{G}\|_{L_2^1} \leq 2C_2, \quad \|\mathbb{F} + \mathbb{G}\|_{L_{2+\bar{\gamma}}^1} \leq 2C_{2+\bar{\gamma}},$$

since both $\mathbb{F}$ and $\mathbb{G}$ belong to Ω . Therefore, (5.13) becomes

$$I_H \leq 8 \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) C_{2+2\bar{\gamma}}^{1/2} (C_2 + C_{2+\bar{\gamma}}) \|\mathbb{F} - \mathbb{G}\|_{L_2^1}^{1/2},$$

which concludes the proof of Hölder continuity. $\square$

Proof of (ii) sub-tangent condition. In order to prove the sub-tangent condition, we first observe that, since we are in cut-off case, it is possible to split collision operator $\mathbb{Q}(\mathbb{F})$ into a gain and a loss term. Namely,

$$[\mathbb{Q}(\mathbb{F})]_i = [\mathbb{Q}^+(\mathbb{F})]_i - f_i(v) [\nu(\mathbb{F})]_i,$$

where $\mathbb{Q}^+$ is a positive operator, and collision frequency $\nu(\mathbb{F})$, for any component $1 \leq i \leq I$, reads as

$$[\nu(\mathbb{F})]_i = \sum_{j=1}^I \iint_{\mathbb{R}^3 \times S^2} f_j(v_*) \mathcal{B}_{ij}(v, v_*, \sigma) d\sigma dv_* \geq 0.$$

In our case, $\nu(\mathbb{F})$ is finite whenever $\mathbb{F} \in \Omega$. Indeed, for the cross section (2.20)–(2.21), and since $|v - v_*|^{\gamma_{ij}} \leq |v - v_*|^{\bar{\gamma}}$ for $|v - v_*| \geq 1$ and $|v - v_*|^{\bar{\gamma}} \leq |v|^{\bar{\gamma}} + |v_*|^{\bar{\gamma}}$,

$$\begin{aligned} 0 \leq [\nu(\mathbb{F})]_i(v) &\leq \left(\max_{1 \leq i, j \leq I} \|b_{ij}\|_{L^1(d\sigma)} \right) \sum_{j=1}^I \int_{\mathbb{R}^3} f_j(v_*) |v - v_*|^{\gamma_{ij}} dv_* \\ &\leq \left(\max_{1 \leq i, j \leq I} \|b_{ij}\|_{L^1(d\sigma)} \right) \left(\sum_{j=1}^I \int_{|v-v_*|<1} f_j(v_*) dv_* \right. \\ &\quad \left. + \sum_{j=1}^I \int_{|v-v_*|\geq 1} f_j(v_*) |v - v_*|^{\bar{\gamma}} dv_* \right) \\ &\leq \left(\max_{1 \leq i, j \leq I} \|b_{ij}\|_{L^1(d\sigma)} \right) \left(C_0 + |v|^{\bar{\gamma}} C_0 + \left(\frac{\sum_{i=1}^I m_i}{\min_{1 \leq j \leq I} m_j} \right)^{\bar{\gamma}/2} \|\mathbb{F}\|_{L^1_{\bar{\gamma}}} \right) \\ &\leq K (1 + |v|^{\bar{\gamma}}), \end{aligned}$$

where

$$K = \left(\max_{1 \leq i, j \leq I} \|b_{ij}\|_{L^1(d\sigma)} \right) \left(2C_0 + \left(\frac{\sum_{i=1}^I m_i}{\min_{1 \leq j \leq I} m_j} \right) C_2 \right). \quad (5.14)$$

Proposition 2. Fix $\mathbb{F} \in \Omega$. Then, for any $\varepsilon > 0$, there exists $h_1 > 0$ such that $B(\mathbb{F} + h\mathbb{Q}(\mathbb{F}), h\varepsilon) \cap \Omega \neq \emptyset$ for any $0 < h < h_1$.

Proof. Set $\chi_R(v)$ as the characteristic function of the ball of radius $R > 0$ and introduce the truncated function $\mathbb{F}_R(t, v) = \chi_R(v)\mathbb{F}(t, v)$. Let

$$\mathbb{W}_R = \mathbb{F} + h\mathbb{Q}(\mathbb{F}_R). \quad (5.15)$$

The idea of the proof is to find R such that on the one hand, $\mathbb{W}_R \in \Omega$, and on the other hand, $\mathbb{W}_R \in B(\mathbb{F} + h\mathbb{Q}(\mathbb{F}), h\varepsilon)$, with h explicitly calculated.

Step 1. We first show that it is possible to find an h_1 such that $\mathbb{W}_R$ remains non-negative for as long $0 < h < h_1$. Indeed, for any $\mathbb{F} \in \Omega$, its truncation is $\mathbb{F}_R \in \Omega$ as well. Since $\mathbb{Q}^+$ is a positive operator, we have

$$[\mathbb{W}_R]_i = f_i + h[\mathbb{Q}^+(\mathbb{F}_R)]_i - h[\mathbb{F}_R]_i[\nu(\mathbb{F}_R)]_i \geq f_i \left(1 - hK \left(1 + R^{\bar{\gamma}}\right)\right) \geq 0$$

for any $0 < h < \frac{1}{K(1+R^{\bar{\gamma}})}$, and $1 \leq i \leq I$, with K from (5.14).

Step 2. Since $\mathbb{F}_R \in \Omega$, we use conservative properties of the collision operator detailed in (3.10) and (3.11) to obtain

$$\sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F}_R)]_i \, dv = 0, \quad \sum_{i=1}^I [\mathbb{Q}(\mathbb{F}_R)]_i \langle v \rangle_i^2 \, dv = 0.$$

From (5.15), we get

$$m_0[\mathbb{W}_R] = m_0[\mathbb{F}], \quad m_2[\mathbb{W}_R] = m_2[\mathbb{F}],$$

independently of R , which yields all needed lower and upper bounds on these quantities.

Step 3. Finally, we need to show that $L_{k_*}^1$ norm of $\mathbb{W}_R$ is bounded.

Let the map $\mathcal{L}_{\bar{\gamma}, k_*} : [0, \infty) \rightarrow \mathbb{R}$ be defined with $\mathcal{L}_{\bar{\gamma}, k_*}(x) = -A_{k_*}x^{1+\frac{\bar{\gamma}}{k_*}} + B_{k_*}x$, where $\bar{\gamma} \in (0, 1]$ and k_* is as defined in (2.24) such that it yields the positivity of constants A_{k_*} and B_{k_*} ; it has only one root, denoted with $x_{\bar{\gamma}, k_*}^*$, at which $\mathcal{L}_{\bar{\gamma}, k_*}$ changes from positive to negative. Thus, for any $x \geq 0$, we may write

$$\mathcal{L}_{\bar{\gamma}, k_*}(x) \leq \max_{0 \leq x \leq x_{\bar{\gamma}, k_*}^*} \mathcal{L}_{\bar{\gamma}, k_*}(x) =: \mathcal{L}_{\bar{\gamma}, k_*}^*.$$

Now, Lemma 5.1 implies

$$\sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{k_*} \, dv \leq \mathcal{L}_{\bar{\gamma}, k_*}(m_{k_*}[\mathbb{F}]) \leq \mathcal{L}_{\bar{\gamma}, k_*}^*.$$

Define

$$\xi_{\bar{\gamma}, k_*} := x_{\bar{\gamma}, k_*}^* + \mathcal{L}_{\bar{\gamma}, k_*}^*.$$

For any $\mathbb{F} \in \Omega$ we have two possibilities: either $m_{k_*}[\mathbb{F}] \leq x_{\bar{\gamma}, k_*}^*$ or $m_{k_*}[\mathbb{F}] > x_{\bar{\gamma}, k_*}^*$. For the former, it follows that

$$m_{k_*}[\mathbb{W}_R] \leq x_{\bar{\gamma}, k_*}^* + h \left(\sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F}_R)]_i \langle v \rangle_i^{k_*} \, dv \right) \leq x_{\bar{\gamma}, k_*}^* + h \mathcal{L}_{\bar{\gamma}, k_*}^* \leq \xi_{\bar{\gamma}, k_*},$$

where we have assumed, without loss of generality, that $h \leq 1$. For the latter, we choose $R = R(\mathbb{F})$ sufficiently large such that $m_{k_*}[\mathbb{F}_R] > x_{\bar{\gamma}, k_*}^*$, and therefore,

$$\mathcal{L}_{\bar{\gamma}, k_*}(m_{k_*}[\mathbb{F}_R]) \leq 0.$$

As a consequence,

$$m_{k_*}[\mathbb{W}_R] \leq x_{\bar{\gamma}, k_*}^* \leq \xi_{\bar{\gamma}, k_*}.$$

Therefore, we constructed a constant C_{k_*} from a characterization of the set Ω , that is, $\xi_{\bar{\gamma}, k_*}$.

The conclusion is that $\mathbb{W}_R \in \Omega$ for any $0 < h < h_*$, where

$$h_* = \min \left\{ 1, \frac{1}{K(1 + R(\mathbb{F})^{\bar{\gamma}})} \right\},$$

and K is from (5.14).

Now, Hölder estimate (5.8) implies that

$$h^{-1} \|\mathbb{F} + h\mathbb{Q}(\mathbb{F}) - \mathbb{W}_R\|_{L_2^1} = \|\mathbb{Q}(\mathbb{F}) - \mathbb{Q}(\mathbb{F}_R)\|_{L_2^1} \leq C_H \|\mathbb{F} - \mathbb{F}_R\|_{L_2^1}^{\frac{1}{2}} \leq \varepsilon$$

for $R := R(\varepsilon)$ sufficiently large. Then, for this choice of R , $\mathbb{W}_R \in B(\mathbb{F} + h\mathbb{Q}(\mathbb{F}), h\varepsilon)$.

Finally, choosing $R = \max\{R(\mathbb{F}), R(\varepsilon)\}$ and h_1 as

$$h_1 = \min \left\{ 1, \frac{1}{K(1 + R^{\bar{\gamma}})} \right\}, \quad (5.16)$$

with c given in (5.14), one concludes that $\mathbb{W}_R \in B(\mathbb{F} + h\mathbb{Q}(\mathbb{F}), h\varepsilon) \cap \Omega$. $\square$

Once the Proposition 2 is proved, it immediately follows that

$$h^{-1} \text{dist}(\mathbb{F} + h\mathbb{Q}(\mathbb{F}), \Omega) \leq \varepsilon, \quad \forall 0 < h < h_1,$$

with h_1 from (5.16), which concludes the proof of the tangency condition. $\square$

Proof of (iii) one-sided Lipschitz condition. From definition and representation (5.10), we have

$$\begin{aligned} I_L &:= [\mathbb{Q}(\mathbb{F}) - \mathbb{Q}(\mathbb{G}), \mathbb{F} - \mathbb{G}] \\ &\leq \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} (Q_{ij}(f_i, f_j) - Q_{ij}(g_i, g_j)) \text{sign}(f_i(v) - g_i(v)) \langle v \rangle_i^2 \, dv \\ &= \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} (Q_{ij}(f_i - g_i, f_j + g_j) + Q_{ij}(f_i + g_i, f_j - g_j)) \\ &\quad \text{sign}(f_i(v) - g_i(v)) \langle v \rangle_i^2 \, dv. \end{aligned}$$

Changing $i \leftrightarrow j$ in the second integral, we precisely obtain the binary structure of the weak form (3.6) that yields

$$\begin{aligned}
 I_L &\leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} \left(Q_{ij}(f_i - g_i, f_j + g_j) \operatorname{sign}(f_i(v) - g_i(v)) \langle v \rangle_i^2 \right. \\
 &\quad \left. + Q_{ji}(f_j + g_j, f_i - g_i) \operatorname{sign}(f_j(v) - g_j(v)) \langle v \rangle_j^2 \right) dv \\
 &= \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} \mathcal{B}_{ij}(v, v_*, \sigma) (f_i(v) - g_i(v)) (f_j(v_*) + g_j(v_*)) \\
 &\quad \times \left(\operatorname{sign}(f_i(v') - g_i(v')) \langle v' \rangle_i^2 + \operatorname{sign}(f_j(v'_*) - g_j(v'_*)) \langle v'_* \rangle_j^2 \right. \\
 &\quad \left. - \operatorname{sign}(f_i(v) - g_i(v)) \langle v \rangle_i^2 - \operatorname{sign}(f_j(v_*) - g_j(v_*)) \langle v_* \rangle_j^2 \right) d\sigma dv_* dv.
 \end{aligned}$$

Using the upper bound of the sign function, one has

$$\begin{aligned}
 I_L &\leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} \mathcal{B}_{ij}(v, v_*, \sigma) \\
 &\quad \times \left(|f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \left(\langle v' \rangle_i^2 + \langle v'_* \rangle_j^2 \right) \right. \\
 &\quad \left. - |f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \langle v \rangle_i^2 \right. \\
 &\quad \left. + |f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \langle v_* \rangle_j^2 \right) d\sigma dv_* dv.
 \end{aligned}$$

Then, conservation of energy implies

$$\begin{aligned}
 I_L &\leq \sum_{i=1}^I \sum_{j=1}^I \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} \mathcal{B}_{ij}(v, v_*, \sigma) \\
 &\quad \times |f_i(v) - g_i(v)| (f_j(v_*) + g_j(v_*)) \langle v_* \rangle_j^2 d\sigma dv_* dv.
 \end{aligned}$$

Now, specifying the collision cross section (2.20) and using (B.3), we get

$$|v - v_*|^{\gamma_{ij}} \leq \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \langle v \rangle_i^{\gamma_{ij}} \langle v_* \rangle_j^{\gamma_{ij}} \leq \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \langle v \rangle_i^{\overline{\gamma}} \langle v_* \rangle_j^{\overline{\gamma}},$$

and we obtain

$$I_L \leq \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) \|\mathbb{F} - \mathbb{G}\|_{L^1_{\overline{\gamma}}} \|\mathbb{F} + \mathbb{G}\|_{L^1_{2+\overline{\gamma}}}.$$

Thanks to the monotonicity of norms (2.18),

$$\|\mathbb{F} - \mathbb{G}\|_{L^1_{\overline{\gamma}}} \leq \|\mathbb{F} - \mathbb{G}\|_{L^1_2},$$

and we finally obtain

$$I_L \leq 2 \max_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right) C_{2+\overline{\gamma}} \|\mathbb{F} - \mathbb{G}\|_{L^1_2},$$

which completes the proof of one-sided Lipschitz condition. $\square$

6. Proof of Theorem 2.6 (Generation and Propagation of Polynomial Moments)

The proof consists of several steps. First, once the existence and uniqueness of vector value solution $\mathbb{F}$ to the Boltzmann system (2.19) is proven, we can derive from the Boltzmann system an ordinary differential inequality for the scalar polynomial moment $m_k[\mathbb{F}](t)$. Then, the comparison principle for ODEs will yield estimates that guarantee both generation and propagation of these polynomial moments.

Step 1. (Ordinary Differential Inequality for the Polynomial Moment.)

Lemma 6.1. *Let $\mathbb{F} = [f_i]_{i=1,\dots,I}$ be a solution of the Boltzmann system (2.19). Then the polynomial moment (2.13) satisfies the following Ordinary Differential Inequality:*

$$\frac{d}{dt} m_k[\mathbb{F}](t) \leq -A_k m_k[\mathbb{F}](t)^{1+\frac{\bar{\gamma}}{k}} + B_k m_k[\mathbb{F}](t), \quad (6.1)$$

for $k \geq k_*$ as defined in (2.24), with positive constants A_k and B_k as defined in Lemma 5.1, Eq. (5.2), after replacing k_* by $k \geq k_*$.

Proof. Consider the i -th equation of the Boltzmann system (2.19):

$$\partial_t f_i(t, v) = \sum_{j=1}^I Q_{ij}(f_i, f_j)(t, v), \quad i = 1, \dots, I.$$

Integration with respect to velocity v with weight $\langle v \rangle_i^k$, $k \geq 0$, and summation over all species $i = 1, \dots, I$ yields

$$\frac{d}{dt} m_k[\mathbb{F}](t) = \sum_{i=1}^I \sum_{j=1}^I \int_{\mathbb{R}^3} \langle v \rangle_i^k Q_{ij}(f_i, f_j)(t, v) dv, \quad (6.2)$$

after recalling definition (2.13) of polynomial moment. Using results from Lemma 5.1 for $k \geq k_*$ as defined in (2.24), we conclude the estimate (6.1). $\square$

Step 2. (Comparison Principle.) The starting point is the inequality (6.1). We associate to it an ODE of Bernoulli type as follows:

$$y'(t) = -a y(t)^{1+c} + b y(t), \quad (6.3)$$

whose solution will be an upper bound for $m_k[\mathbb{F}](t)$. Indeed, the solution to (6.3) reads as

$$y(t) = \left(\frac{a}{b} \left(1 - e^{-c b t} \right) + y(0)^{-c} e^{-c b t} \right)^{-\frac{1}{c}}. \quad (6.4)$$

Step 3. (Generation of Polynomial Moments.) Dropping initial data in (6.4) yields

$$y(t) \leq \left(\frac{a}{b} \left(1 - e^{-c b t} \right) \right)^{-\frac{1}{c}}, \quad \forall t > 0.$$

Setting $y(t) := \mathfrak{m}_k[\mathbb{F}](t)$, $a := A_k$, $b := B_k$ and $c := \bar{\gamma}/k$ implies the generation estimate (2.29) with

$$\mathfrak{C}^{\mathfrak{m}} = \left(\frac{A_k}{B_k} \right)^{-\frac{k}{\bar{\gamma}}}, \quad \text{for any } k \geq k_*.$$

Remark 8. For later purposes, we also derive the following inequality by approximating the last result: for $t < 1$, we may write

$$\begin{aligned} (1 - e^{-c b t})^{-\frac{1}{c}} &= (c b t)^{-\frac{1}{c}} \left(1 + \frac{b}{2} t + o(t) \right) \\ &\leq (c b)^{-\frac{1}{c}} e^{\frac{b}{2} t} t^{-\frac{1}{c}} \leq (c b)^{-\frac{1}{c}} e^{\frac{b}{2}} t^{-\frac{1}{c}}. \end{aligned}$$

On the other hand, for $t \geq 1$, it follows that

$$(1 - e^{-c b t})^{-\frac{1}{c}} \leq (1 - e^{-c b})^{-\frac{1}{c}}.$$

Therefore,

$$y(t) \leq \left(\frac{a}{b} \right)^{-\frac{1}{c}} \begin{cases} (c b)^{-\frac{1}{c}} e^{\frac{b}{2}} t^{-\frac{1}{c}}, & t < 1 \\ (1 - e^{-c b})^{-\frac{1}{c}}, & t \geq 1. \end{cases} \quad (6.5)$$

In other words, plugging in $y(t) := \mathfrak{m}_k[\mathbb{F}](t)$, $a := A_k$, $b := B_k$ and $c := \bar{\gamma}/k$ yields

$$\mathfrak{m}_k[\mathbb{F}](t) \leq \mathfrak{B}^{\mathfrak{m}} \max\{1, t^{-\frac{k}{\bar{\gamma}}}\}, \quad \forall t > 0, \quad (6.6)$$

where the constant is

$$\mathfrak{B}^{\mathfrak{m}} = \mathfrak{C}^{\mathfrak{m}} \max \left\{ \left(\frac{\bar{\gamma}}{B_k k} \right)^{-\frac{k}{\bar{\gamma}}} e^{\frac{B_k}{2}}, \left(1 - e^{-\frac{\bar{\gamma}}{B_k k}} \right)^{-\frac{k}{\bar{\gamma}}} \right\}, \quad \text{for any } k \geq k_*.$$

Step 4. (Propagation of Polynomial Moments.) For the propagation result, when $y(0)$ is assumed to be finite, we first notice that $y(t)$ is a monotone function of t , which approaches $y(0)$ as $t \rightarrow 0$ on the one hand, and converges to $(a/b)^{-1/c}$ when $t \rightarrow \infty$ on the other hand. Therefore,

$$y(t) \leq \max\{y(0), (a/b)^{-1/c}\}$$

for all $t \geq 0$. Again, taking $y(t) := \mathfrak{m}_k[\mathbb{F}](t)$, $a := A_k$, $b := B_k$ and $c := \bar{\gamma}/k$ for any $k \geq k_*$ implies the propagation estimate (7.1).

7. Generation and Propagation of Exponential Moments

Let $\mathbb{F}$ be a solution of the Boltzmann system (2.19). In this section we prove both generation and propagation of an exponential moment (2.14) related to $\mathbb{F}$. The proof strongly relies on the generation and propagation of polynomial moments stated in Theorem 2.29. Moreover, it uses polynomial moment ODI, but written in a slightly different manner than in Sect. 6.1, which we make precise in the following Lemma:

Lemma 7.1. *Let $\mathbb{F}$ be a solution of the Boltzmann system (2.19). Then there exist positive constants K_1 and K_2 such that the following two polynomial moments ODI hold:*

- *ODI needed for propagation of exponential moments:*

$$\begin{aligned} \frac{d}{dt} m_{sk}[\mathbb{F}](t) &\leq -K_1 m_{sk+\bar{\gamma}}[\mathbb{F}](t) \\ &+ K_2 \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{\frac{sk}{2}}^{ij} \right) \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} (m_{s\ell+\bar{\gamma}}[\mathbb{F}](t) m_{sk-s\ell}[\mathbb{F}](t) \\ &+ m_{sk-s\ell+\bar{\gamma}}[\mathbb{F}](t) m_{s\ell}[\mathbb{F}](t)). \end{aligned} \quad (7.1)$$

- *ODI needed for generation of exponential moments:*

$$\begin{aligned} \frac{d}{dt} m_{\bar{\gamma}k}[\mathbb{F}](t) &\leq -K_1 m_{\bar{\gamma}k+\bar{\gamma}}[\mathbb{F}](t) \\ &+ K_2 \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{\frac{\bar{\gamma}k}{2}}^{ij} \right) \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} (m_{\bar{\gamma}\ell+\bar{\gamma}}[\mathbb{F}](t) m_{\bar{\gamma}k-\bar{\gamma}\ell}[\mathbb{F}](t) \\ &+ m_{\bar{\gamma}k-\bar{\gamma}\ell+\bar{\gamma}}[\mathbb{F}](t) m_{\bar{\gamma}\ell}[\mathbb{F}](t)). \end{aligned} \quad (7.2)$$

Proof. We briefly point out that the main steps in the proofs are an adaption of the proof given in [22]. Let us consider the polynomial moment

$$m_{\delta q}[\mathbb{F}](t) =: m_{\delta q}, \quad 0 < \delta \leq 2, \quad q \geq 0, \quad \text{with } \delta q > k_*,$$

with k_* as defined in (2.24), and derive an ODI for it starting from (5.5) so that $\mathcal{C}_{\frac{\delta q}{2}}^{ij} < \|b_{ij}\|_{L^1(d\sigma)}$ holds uniformly for any pair $i, j = 1, \dots, I$, with $\mathcal{C}_n^{ij}$ being the constant from Povzner lemma 2.2. Once we derive it, (7.1) will follow, setting $\delta := s$, and (7.2) will follow with $\delta := \bar{\gamma}$. Indeed, from (5.5) we get that

$$\begin{aligned} m'_{\delta q} &= \sum_{i=1}^I \int_{\mathbb{R}^3} [\mathbb{Q}(\mathbb{F})]_i \langle v \rangle_i^{\delta q} dv \leq \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^I \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_i(v) f_j(v_*) |v - v_*|^{\gamma_{ij}} \\ &\times \left(\mathcal{C}_{\frac{\delta q}{2}}^{ij} \left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right)^{\frac{\delta q}{2}} - \|b_{ij}\|_{L^1(d\sigma)} \left(\langle v \rangle_i^{\delta q} + \langle v_* \rangle_j^{\delta q} \right) \right) dv_* dv. \end{aligned}$$

Before applying Lemma C.1, we first estimate, since $(\delta/2) \leq 1$,

$$\left(\langle v \rangle_i^2 + \langle v_* \rangle_j^2 \right)^{\frac{\delta q}{2}} \leq \left(\langle v \rangle_i^\delta + \langle v_* \rangle_j^\delta \right)^q,$$

and then apply it, which gives us

$$\begin{aligned} \left(\langle v \rangle_i^\delta + \langle v_* \rangle_j^\delta \right)^q &\leq \langle v \rangle_i^{\delta q} + \langle v_* \rangle_j^{\delta q} \\ &\quad + \sum_{\ell=1}^{\ell_q} \binom{q}{\ell} \left(\langle v \rangle_i^{\delta \ell} \langle v_* \rangle_j^{\delta q - \delta \ell} + \langle v \rangle_i^{\delta q - \delta \ell} \langle v_* \rangle_j^{\delta \ell} \right), \end{aligned}$$

with $\ell_q = \lfloor \frac{q+1}{2} \rfloor$. The bound from above and below of the non-angular part of the cross-section, $|v - v_*|^{\gamma_{ij}}$, is used as in Sect. 6.1. This implies a polynomial moment ODI:

$$\mathbf{m}'_{\delta q}(t) \leq -K_1 \mathbf{m}_{\delta q + \bar{\gamma}} + \mathcal{C}_{\frac{\delta q}{2}}^{ij} K_2 \sum_{\ell=1}^{\ell_q} \binom{q}{\ell} (\mathbf{m}_{\delta \ell + \bar{\gamma}} \mathbf{m}_{\delta q - \delta \ell} + \mathbf{m}_{\delta q - \delta \ell + \bar{\gamma}} \mathbf{m}_{\delta \ell}),$$

where K_1 and K_2 are positive constants, since $\delta q \geq k_*$, with k_* as defined in (2.24), to get

$$\begin{aligned} K_1 &= \min_{1 \leq i, j \leq I} \left(\|b_{ij}\|_{L^1(d\sigma)} - \mathcal{C}_{\frac{\delta q}{2}}^{ij} \right) \frac{c_{lb}}{\max_{1 \leq i \leq I} m_i}, \\ K_2 &= \frac{1}{2} \left(\max_{1 \leq i, j \leq I} \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \right), \end{aligned}$$

which completes the proof. $\square$

8. Proof of Theorem 2.7 (b) (Propagation of Exponential Moments)

Using the Taylor series of an exponential function, one can represent the exponential moment as

$$\mathcal{E}_s[\mathbb{F}](\alpha, t) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \mathbf{m}_{sk}[\mathbb{F}](t).$$

We will show that the exponential rate is $\alpha = \alpha(k_*)$, that is, it depends on the k_* parameter defined in (2.24).

We consider its partial sum as shifted by $\bar{\gamma}$ one, namely,

$$\mathcal{E}_s^n[\mathbb{F}](\alpha, t) = \sum_{k=0}^n \frac{\alpha^k}{k!} \mathbf{m}_{sk}[\mathbb{F}](t), \quad \mathcal{E}_{s; \bar{\gamma}}^n[\mathbb{F}](\alpha, t) = \sum_{k=0}^n \frac{\alpha^k}{k!} \mathbf{m}_{sk + \bar{\gamma}}[\mathbb{F}](t). \quad (8.1)$$

In order to have lighter writing, we will drop from moment notation dependence on t and α , and the relation to $\mathbb{F}$, and we will instead write

$$\mathcal{E}_s^n[\mathbb{F}](\alpha, t) =: \mathcal{E}_s^n, \quad \mathcal{E}_{s; \bar{\gamma}}^n[\mathbb{F}](\alpha, t) =: \mathcal{E}_{s; \bar{\gamma}}^n, \quad \mathbf{m}_{sk + \bar{\gamma}}[\mathbb{F}](t) =: \mathbf{m}_{sk + \bar{\gamma}}.$$

When it will be important to highlight dependence on t and α , we will also, for example, write $\mathcal{E}_s^n(\alpha, t)$ instead of $\mathcal{E}_s^n$.

The idea of the proof is to show that the partial sum $\mathcal{E}_s^n$ is bounded uniformly in time t and n . To this end, we first derive ordinary differential inequality (ODI) for it.

ODI for $\mathcal{E}_s^n$. Taking derivative with respect to time t of (8.1), we get

$$\frac{d}{dt} \mathcal{E}_s^n = \sum_{k=0}^{k_0-1} \frac{\alpha^k}{k!} m'_{sk} + \sum_{k=k_0}^n \frac{\alpha^k}{k!} m'_{sk},$$

where k_0 is an index that will be determined later on. We use a polynomial moment ODE (7.1) for the second term, which yields

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_s^n &\leq \sum_{k=0}^{k_0-1} \frac{\alpha^k}{k!} m'_{sk} - K_1 \sum_{k=k_0}^n \frac{\alpha^k}{k!} m_{sk+\bar{\gamma}} \\ &\quad + K_2 \sum_{k=k_0}^n \left(\max_{1 \leq i, j \leq I} c_{\frac{sk}{2}}^{ij} \right) \frac{\alpha^k}{k!} \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} (m_{s\ell+\bar{\gamma}} m_{sk-s\ell} + m_{sk-s\ell+\bar{\gamma}} m_{s\ell}) \\ &=: S_0 - K_1 S_1 + K_2 S_2. \end{aligned} \quad (8.2)$$

We estimate each sum S_0 , S_1 and S_2 separately.

Term S_0 . Propagation of polynomial moment (2.30) ensures a bound on m_{sk} uniformly in time, which implies from (6.1) a bound on its derivative, i.e. there exists a constant c_{k_0} such that

$$m_{sk}, m'_{sk} \leq c_{k_0} \quad \text{for all } k \in \{0, 1, \dots, k_0\}. \quad (8.3)$$

For S_0 , this yields

$$S_0 \leq c_{k_0} \sum_{k=0}^{k_0-1} \frac{\alpha^k}{k!} \leq c_{k_0} e^\alpha \leq 2 c_{k_0} \quad (8.4)$$

for α small enough to satisfy

$$e^\alpha \leq 2. \quad (8.5)$$

Term S_1 . We complete first the term S_1 to introduce a shifted partial sum $\mathcal{E}_{s;\bar{\gamma}}^n$ by means of

$$S_1 = \sum_{k=k_0}^n \frac{\alpha^k}{k!} D_k m_{sk+\bar{\gamma}} = \mathcal{E}_{s;\bar{\gamma}}^n - \sum_{k=0}^{k_0-1} \frac{\alpha^k}{k!} D_k m_{sk+\bar{\gamma}}.$$

From the bound (8.3) we can estimate $m_{sk+\bar{\gamma}}$ as well:

$$m_{sk+\bar{\gamma}} \leq c_{k_0}, \quad k = 0, \dots, k_0 - 1,$$

which together with considerations for the term S_0 yields

$$S_1 \geq \mathcal{E}_{s;\bar{\gamma}}^n - 2c_{k_0}. \quad (8.6)$$

Term S_2 . Term S_2 can be separated into two terms, namely

$$S_2 = \sum_{k=k_0}^n \left(\max_{1 \leq i, j \leq I} \mathbf{c}_{\frac{sk}{2}}^{ij} \right) \frac{\alpha^k}{k!} \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} (\mathbf{m}_{s\ell+\bar{\gamma}} \mathbf{m}_{sk-s\ell} + \mathbf{m}_{sk-s\ell+\bar{\gamma}} \mathbf{m}_{s\ell}) =: S_{2_1} + S_{2_2}.$$

Their treatment is the same, so as let perform an estimate on S_{2_1} . Rearranging, we can write

$$S_{2_1} = \sum_{k=k_0}^n \left(\max_{1 \leq i, j \leq I} \mathbf{c}_{\frac{sk}{2}}^{ij} \right) \sum_{\ell=1}^{\ell_k} \frac{\alpha^\ell \mathbf{m}_{s\ell+\bar{\gamma}}}{\ell!} \frac{\alpha^{k-\ell} \mathbf{m}_{sk-s\ell}}{(k-\ell)!} \leq \left(\max_{1 \leq i, j \leq I} \mathbf{c}_{\frac{sk_0}{2}}^{ij} \right) \mathcal{E}_{s;\bar{\gamma}}^n \mathcal{E}_s^n;$$

the last inequality is due to the decreasing property of $\mathbf{c}_k^{ij}$ in $k \geq k_*$, uniformly for any i, j , with k_* defined in (2.24). Therefore, we can estimate

$$S_2 \leq 2 \left(\max_{1 \leq i, j \leq I} \mathbf{c}_{\frac{sk_0}{2}}^{ij} \right) \mathcal{E}_{s;\bar{\gamma}}^n \mathcal{E}_s^n. \quad (8.7)$$

Finally, the desired ODI for $\mathcal{E}_s^n$ is obtained from (8.2) by gathering all estimates: (8.4), (8.6) and (8.7). Namely,

$$\frac{d}{dt} \mathcal{E}_s^n \leq -K_1 \mathcal{E}_{s;\bar{\gamma}}^n + 2c_{k_0}(1 + K_1) + 2K_2 \left(\max_{1 \leq i, j \leq I} \mathbf{c}_{\frac{sk_0}{2}}^{ij} \right) \mathcal{E}_{s;\bar{\gamma}}^n \mathcal{E}_s^n. \quad (8.8)$$

Bound on $\mathcal{E}_s^n$. For each $n \in \mathbb{N}$ we define

$$T_n := \sup\{t \geq 0 : \mathcal{E}_s^n(\alpha, \tau) \leq 4M_0, \forall \tau \in [0, t]\},$$

where M_0 is a bound on initial data in (2.31). We will show that $\mathcal{E}_s^n(t)$ is uniformly bounded in t and n by proving that $T_n = \infty$ for all $n \in \mathbb{N}$.

The sequence T_n is well-defined and positive. Indeed, since $\alpha \leq \alpha_0$, at time $t = 0$ we have that

$$\mathcal{E}_s^n(\alpha, 0) = \sum_{k=0}^n \frac{\alpha^k}{k!} \mathbf{m}_{sk}(0) \leq \sum_{k=0}^n \frac{\alpha_0^k}{k!} \mathbf{m}_{sk}(0) \leq \mathcal{E}_s(\alpha_0, 0) < 4M_0$$

uniformly in n , by assumption (2.31). Since each term $\mathbf{m}_{sk}(t)$ is a continuous function of t , so is $\mathcal{E}_s^n(\alpha, t)$. Therefore, $\mathcal{E}_s^n(\alpha, t) < 4M_0$ on some time interval $[0, t_n)$, $t_n > 0$. Thus T_n is well-defined and positive for every $n \in \mathbb{N}$.

For $t \in [0, T_n]$ it follows that $\mathcal{E}_s^n(\alpha, t) \leq 4M_0$, which, from (8.8), implies that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_s^n &\leq -\mathcal{E}_{s;\bar{\gamma}}^n \left(K_1 - 8K_2 \left(\max_{1 \leq i, j \leq I} \mathbf{c}_{\frac{sk_0}{2}}^{ij} \right) M_0 \right) \\ &\quad + 2c_{k_0}(1 + K_1). \end{aligned} \quad (8.9)$$

Since $\mathcal{C}_{\frac{sk_0}{2}}^{ij}$, for any i, j , converges to zero as $\frac{sk_0}{2} > k_*$ goes to infinity, we can choose $k_0 > 2\frac{k_*}{s}$ such that

$$K_1 - 8 K_2 \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{k_*}^{ij} \right) M_0 > \frac{K_1}{2},$$

or, equivalently,

$$K_1 < 16 K_2 \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{k_*}^{ij} \right) M_0, \quad (8.10)$$

with K_1 depending on k_* as defined in (2.24). Hence, (8.9) becomes

$$\frac{d}{dt} \mathcal{E}_s^n \leq -\frac{K_1}{2} \mathcal{E}_{s;\bar{\gamma}}^n + 2c_{k_*} (1 + K_1). \quad (8.11)$$

The next step consists in finding a lower bound for $\mathcal{E}_{s;\bar{\gamma}}^n$ in terms of $\mathcal{E}_s^n$. Indeed, we can estimate

$$\begin{aligned} \mathcal{E}_{s;\bar{\gamma}}^n &= \sum_{k=0}^n \frac{\alpha^k}{k!} m_{s k + \bar{\gamma}} \geq \sum_{k=0}^n \frac{\alpha^k}{k!} \sum_{i=1}^I \int_{\{(v)_i \geq \alpha^{-1/2}\}} f_i(t, v) \langle v \rangle_i^{s k + \bar{\gamma}} dv \\ &\geq \alpha^{-\bar{\gamma}/2} \left(\mathcal{E}_s^n - \sum_{k=0}^n \frac{\alpha^k}{k!} \sum_{i=1}^I \int_{\{(v)_i < \alpha^{-1/2}\}} f_i(t, v) \langle v \rangle_i^{s k} dv \right) \\ &\geq \alpha^{-\bar{\gamma}/2} \left(\mathcal{E}_s^n - \sum_{k=0}^n \frac{\alpha^{k(1-\frac{s}{2})}}{k!} m_0(0) \right) \geq \alpha^{-\bar{\gamma}/2} \left(\mathcal{E}_s^n - m_0(0) e^{\alpha^{1-\frac{s}{2}}} \right). \end{aligned}$$

Plugging this result into (8.11) yields

$$\frac{d}{dt} \mathcal{E}_s^n \leq -\frac{K_1}{2} \alpha^{-\bar{\gamma}/2} \mathcal{E}_s^n + \frac{K_1}{2} \alpha^{-\bar{\gamma}/2} m_0(0) e^{\alpha^{1-\frac{s}{2}}} + 2c_{k_0} (1 + K_1).$$

By the maximum principle for ODEs, it follows that

$$\begin{aligned} \mathcal{E}_s^n(\alpha, t) &\leq \max \left\{ \mathcal{E}_s^n(\alpha, 0), m_0(0) e^{\alpha^{1-\frac{s}{2}}} + \frac{4c_{k_0} (1 + K_1)}{K_1 \alpha^{-\bar{\gamma}/2}} \right\} \\ &\leq M_0 + m_0(0) e^{\alpha^{1-\frac{s}{2}}} + \alpha^{\bar{\gamma}/2} \frac{4c_{k_0} (1 + K_1)}{K_1} \end{aligned} \quad (8.12)$$

for any $t \in [0, T_n]$. On the other hand, since $s \leq 2$, the following limit property holds:

$$m_0(0) e^{\alpha^{1-\frac{s}{2}}} + \alpha^{\bar{\gamma}/2} \frac{4c_{k_0} (1 + K_1)}{K_1} \rightarrow m_0(0), \quad \text{as } \alpha \rightarrow 0,$$

and $m_0(0) < \mathcal{E}_s^n(\alpha_0, 0)$ for any α_0 , and therefore, by (2.31), $m_0(0) < M_0$. Thus, we can choose a sufficiently small $\alpha = \alpha_1$ such that

$$m_0(0) e^{\alpha^{1-\frac{s}{2}}} + \alpha^{\bar{\gamma}/2} \frac{4c_{k_*} (1 + K_1)}{K_1} < 3M_0 \quad (8.13)$$

for any $s \leq 2$ and $K_1 = K_1(k_*)$ from (8.10). In this case, inequality (8.12) implies the following strict inequality:

$$\mathcal{E}_s^n(\alpha, t) < 4M_0, \quad (8.14)$$

for any $t \in [0, T_n]$ and $0 < \alpha(k_*) \leq \alpha_1$, with α depending on k_* defined in (2.24). **Conclusion I.** If k_0 is chosen such that (8.11) holds, and the choice of α is such that $0 < \alpha \leq \alpha_0$ and (8.5), (8.13) are satisfied, which amounts to taking $\alpha = \min\{\alpha_0, \ln 2, \alpha_1\}$, then we have strict inequality (8.14), $\mathcal{E}_s^n(\alpha, t) < 4M_0$, which holds on the closed interval $[0, T_n]$ uniformly in n . Because of the continuity of $\mathcal{E}_s^n(\alpha, t)$ with respect to time t , this strict inequality actually holds on a slightly larger time interval $[0, T_n + \varepsilon]$, $\varepsilon > 0$. This contradicts the maximality of T_n , unless $T_n = +\infty$. Therefore, $\mathcal{E}_s^n(\alpha, t) \leq 4M_0$ for all $t \geq 0$ and $n \in \mathbb{N}$. Thus, letting $n \rightarrow \infty$, we conclude that

$$\mathcal{E}_s[\mathbb{F}](\alpha, t) = \lim_{n \rightarrow \infty} \mathcal{E}_s^n[\mathbb{F}](\alpha, t) \leq 4M_0, \quad \forall t \geq 0,$$

i.e. the solution $\mathbb{F}$ to a system of Boltzmann equations with finite initial exponential moment of order s and rate α_0 will propagate exponential moments of the same order s and a rate α that satisfies $\alpha = \min\{\alpha_0, \ln 2, \alpha_1\}$. It is also very interesting to note that the rate α depends on the k_* parameter from (2.24), which depends on uniformity in the i, j pairs, upper bounds for the intermolecular potentials γ_{ij} and for controls of the κ_*^{ij} as defined in (2.23) in the Povzner Lemma 2.2.

9. Proof of Theorem 2.7 (a) (Generation of Exponential Moments)

We consider an exponential moment of order $\bar{\gamma}$ and rate αt , where α depends on k_* from (2.24) for the solution $\mathbb{F}$ of the Boltzmann system, namely

$$\mathcal{E}_{\bar{\gamma}}[\mathbb{F}](\alpha t, t) = \sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t, v) e^{\alpha t \langle v \rangle_i^{\bar{\gamma}}} dv = \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} m_{\bar{\gamma}k}[\mathbb{F}](t).$$

Consider its partial sum, and a shifted one at that, to get

$$\mathcal{E}_{\bar{\gamma}}^n[\mathbb{F}](\alpha t, t) = \sum_{k=0}^n \frac{(\alpha t)^k}{k!} m_{\bar{\gamma}k}[\mathbb{F}](t), \quad \mathcal{E}_{\bar{\gamma}; \bar{\gamma}}^n[\mathbb{F}](\alpha t, t) = \sum_{k=0}^n \frac{(\alpha t)^k}{k!} m_{\bar{\gamma}k + \bar{\gamma}}[\mathbb{F}](t).$$

As usual, we will most of the time lighten notation by omitting explicit dependence on time t and the relation to $\mathbb{F}$, and write

$$\mathcal{E}_{\bar{\gamma}}^n[\mathbb{F}](\alpha t, t) =: \mathcal{E}_{\bar{\gamma}}^n, \quad \mathcal{E}_{\bar{\gamma}; \bar{\gamma}}^n[\mathbb{F}](\alpha t, t) =: \mathcal{E}_{\bar{\gamma}; \bar{\gamma}}^n.$$

Fix α and $\bar{\gamma}$ and define

$$\bar{T}_n := \sup \left\{ t \in [0, 1] : \mathcal{E}_{\bar{\gamma}}^n[\mathbb{F}](\alpha t, t) \leq 4\bar{M}_0 \right\}.$$

$\bar{T}_n$ is well defined. Indeed, taking $\bar{M}_0 := \sum_{i=1}^I f_i(t, v) \langle v \rangle_i^2 dv = \sum_{i=1}^I f_i(0, v) \langle v \rangle_i^2 dv$ for $t = 0$, we get $\mathcal{E}_{\bar{\gamma}}^n(0, 0) \leq \mathcal{E}_{\bar{\gamma}}(0, 0) = m_0(0) < 4\bar{M}_0$. By the continuity of partial sum $\mathcal{E}_{\bar{\gamma}}^n$ with respect to t , $\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t) \leq 4\bar{M}_0$ on a slightly larger time interval $t \in [0, t_n)$, $t_n > 0$, and thus $\bar{T}_n > 0$.

ODI for $\mathcal{E}_{\bar{\gamma}}^n$. Taking the time derivative of $\mathcal{E}_{\bar{\gamma}}^n$ yields

$$\frac{d}{dt} \mathcal{E}_{\bar{\gamma}}^n = \alpha \sum_{k=1}^n \frac{(\alpha t)^{k-1}}{(k-1)!} m_{\bar{\gamma}k} + \sum_{k=0}^{k_0-1} \frac{(\alpha t)^k}{k!} m'_{\bar{\gamma}k} + \sum_{k=k_0}^n \frac{(\alpha t)^k}{k!} m'_{\bar{\gamma}k}.$$

For the first term we simply re-index the sum and use the definition of a shifted partial sum, and for the last one we use polynomial moment ODI (7.2), which, taken together, implies

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_{\bar{\gamma}}^n &\leq \alpha \mathcal{E}_{\bar{\gamma}; \bar{\gamma}}^n + \sum_{k=0}^{k_0-1} \frac{(\alpha t)^k}{k!} m'_{\bar{\gamma}k} - K_1 \sum_{k=k_0}^n \frac{(\alpha t)^k}{k!} m_{\bar{\gamma}k+\bar{\gamma}} \\ &\quad + K_2 \sum_{k=k_0}^n \frac{(\alpha t)^k}{k!} \left(\max_{1 \leq i, j \leq I} c_{\frac{\bar{\gamma}k}{2}}^{ij} \right) \sum_{\ell=1}^{\ell_k} \\ &\quad \times \binom{k}{\ell} (m_{\bar{\gamma}\ell+\bar{\gamma}} m_{\bar{\gamma}k-\bar{\gamma}\ell} + m_{\bar{\gamma}k-\bar{\gamma}\ell+\bar{\gamma}} m_{\bar{\gamma}\ell}) \\ &=: \alpha \mathcal{E}_{\bar{\gamma}; \bar{\gamma}}^n + S_0 - K_1 S_1 + K_2 (S_{2_1} + S_{2_2}). \end{aligned} \quad (9.1)$$

Term S_0 . From polynomial moment generation estimate (6.6) we can bound a polynomial moment of any order, as well as its derivative, by means of (6.1). In particular,

$$m_{\bar{\gamma}k} \leq \mathfrak{B}^m \max_{t>0} \{1, t^{-k}\}, \quad m'_{\bar{\gamma}k} \leq B_{\bar{\gamma}k} \mathfrak{B}^m \max_{t>0} \{1, t^{-k}\}.$$

Denote

$$\bar{c}_{k_0} := \max_{k \in \{0, \dots, k_0-1\}} \{\mathfrak{B}^m, B_{\bar{\gamma}k} \mathfrak{B}^m\}.$$

For S_0 , taking $t \leq 1$, we have $m'_{\bar{\gamma}k} \leq \bar{c}_{k_0} t^{-k}$, and therefore

$$S_0 := \sum_{k=0}^{k_0-1} \frac{(\alpha t)^k}{k!} m'_{\bar{\gamma}k} \leq \bar{c}_{k_0} \sum_{k=0}^{k_0-1} \frac{\alpha^k}{k!} \leq \bar{c}_{k_0} e^\alpha \leq 2\bar{c}_{k_0}$$

for α such that

$$e^\alpha \leq 2. \quad (9.2)$$

Term S_1 . Using the boundedness of $m_{\bar{\gamma}k+\bar{\gamma}}$, we can write

$$S_1 := \sum_{k=k_0}^n \frac{(\alpha t)^k}{k!} m_{\bar{\gamma}k+\bar{\gamma}} = \mathcal{E}_{\bar{\gamma}; \bar{\gamma}}^n - \sum_{k=0}^{k_0-1} \frac{(\alpha t)^k}{k!} m_{\bar{\gamma}k+\bar{\gamma}} \geq \mathcal{E}_{\bar{\gamma}; \bar{\gamma}}^n - 2\bar{c}_{k_0} \frac{1}{t}$$

for α chosen as in (9.2).

Term S_2 . Terms S_{2_1} and S_{2_2} are treated in the same fashion. We will detail the calculation for S_{2_1} . We first reorganize the terms in sum and get

$$\begin{aligned} S_{2_1} &:= \sum_{k=k_0}^n \frac{(\alpha t)^k}{k!} \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{\frac{\bar{y}k}{2}}^{ij} \right) \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} m_{\bar{y}\ell+\bar{y}} m_{\bar{y}k-\bar{y}\ell} \\ &= \sum_{k=k_0}^n \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{\frac{\bar{y}k}{2}}^{ij} \right) \sum_{\ell=1}^{\ell_k} \frac{(\alpha t)^\ell m_{\bar{y}\ell+\bar{y}}}{\ell!} \frac{(\alpha t)^{k-\ell} m_{\bar{y}k-\bar{y}\ell}}{(k-\ell)!} \\ &\leq \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{\frac{\bar{y}k_0}{2}}^{ij} \right) \mathcal{E}_{\bar{y};\bar{y}}^n \mathcal{E}_{\bar{y}}^n, \end{aligned}$$

since constant $\mathcal{C}_{\frac{\bar{y}k}{2}}^{ij}$ decays with respect to k for any i, j and large enough $k_0 \geq 2\frac{k_*}{\bar{y}}$, with k_* from (2.24) to ensure (2.23), and therefore $\mathcal{C}_{\frac{\bar{y}k}{2}}^{ij} \leq \mathcal{C}_{k_*}^{ij}$. Gathering all estimates together, (9.1) becomes

$$\frac{d}{dt} \mathcal{E}_{\bar{y}}^n \leq \alpha \mathcal{E}_{\bar{y};\bar{y}}^n + 2\bar{c}_{k_0} - K_1 \left(\mathcal{E}_{\bar{y};\bar{y}}^n - 2\bar{c}_{k_0} \frac{1}{t} \right) + K_2 \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{k_*}^{ij} \right) \mathcal{E}_{\bar{y};\bar{y}}^n \mathcal{E}_{\bar{y}}^n, \quad (9.3)$$

for α satisfying (9.2).

Bound on $\mathcal{E}_{\bar{y}}^n$. Consider $t \in [0, \bar{T}_n]$. On this interval, $\mathcal{E}_{\bar{y}}^n(\alpha t, t) \leq 4\bar{M}_0$, and $\bar{T}_n \leq 1$ yields $t^{-1} \geq 1$, which implies, for (9.3), the following estimate:

$$\frac{d}{dt} \mathcal{E}_{\bar{y}}^n \leq -\mathcal{E}_{\bar{y};\bar{y}}^n \left(-\alpha + K_1 - K_2 \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{k_*}^{ij} \right) 4\bar{M}_0 \right) + \frac{2\bar{c}_{k_*}}{(1+K_1)t}.$$

Since $\mathcal{C}_{\frac{\bar{y}k_0}{2}}^{ij}$ converges to zero as $k_0 \geq 2\frac{k_*}{\bar{y}}$ uniformly i, j , choosing such a large k_0 and a small enough α such that

$$-\alpha + K_1 - K_2 \left(\max_{1 \leq i, j \leq I} \mathcal{C}_{k_*}^{ij} \right) 4\bar{M}_0 > \frac{K_1}{2},$$

with $K_1 = K_1(k_*)$, yields

$$\frac{d}{dt} \mathcal{E}_{\bar{y}}^n \leq -\frac{K_1}{2} \mathcal{E}_{\bar{y};\bar{y}}^n + \frac{K_3}{t}$$

for $K_3(k_*) := 2\bar{c}_{k_*} (1 + K_1(k_*))$. Finally, the shifted moment can be bounded as follows:

$$\mathcal{E}_{\bar{y};\bar{y}}^n(\alpha t, t) = \sum_{k=1}^{n+1} \frac{(\alpha t)^k m_{\bar{y}k}(t)}{k!} \frac{k}{\alpha t} \geq \frac{1}{\alpha t} \sum_{k=2}^n \frac{(\alpha t)^k m_{\bar{y}k}(t)}{k!} \geq \frac{\mathcal{E}_{\bar{y}}^n(\alpha t, t) - \bar{M}_0}{\alpha t},$$

which yields

$$\frac{d}{dt} \mathcal{E}_{\bar{\gamma}}^n \leq -\frac{K_1}{2\alpha t} \left(\mathcal{E}_{\bar{\gamma}}^n - \bar{M}_0 - \frac{2\alpha}{K_1} K_3 \right).$$

Now we choose α small enough so that

$$\bar{M}_0 + \frac{2\alpha}{K_1} K_3 < 2\bar{M}_0, \quad \text{or, equivalently} \quad \alpha = \alpha(k_*) < \frac{K_1(k_*)\bar{M}_0}{2K_3(k_*)},$$

which implies

$$\frac{d}{dt} \mathcal{E}_{\bar{\gamma}}^n(\alpha t, t) \leq -\frac{K_1}{2\alpha t} \left(\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t) - 2\bar{M}_0 \right).$$

As in [22], integrating this differential inequality with an integrating factor $t^{\frac{K_1}{2\alpha}}$ yields

$$\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t) \leq \max \left\{ \mathcal{E}_{\bar{\gamma}}^n(0, 0), 2\bar{M}_0 \right\} \leq 2\bar{M}_0, \quad \forall t \in [0, \bar{T}_n], \quad (9.4)$$

since $\mathcal{E}_{\bar{\gamma}}^n(0, 0) = m_0(0) < 2\bar{M}_0$.

Conclusion II. From (9.4), the following bound on $\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t)$ holds:

$$\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t) \leq 2\bar{M}_0 < 4\bar{M}_0, \quad \forall t \in [0, \bar{T}_n].$$

Exploring the continuity of the partial sum $\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t)$, this inequality holds on a slightly larger interval, which contradicts the maximality of $\bar{T}_n$, unless $\bar{T}_n = 1$. Therefore, we can conclude that $\bar{T}_n = 1$ for all $n \in \mathbb{N}$, or, in other words, that

$$\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t) \leq 4\bar{M}_0, \quad \forall t \in [0, 1], \quad \forall n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$, we conclude that

$$\mathcal{E}_{\bar{\gamma}}^n(\alpha t, t) \leq 4\bar{M}_0, \quad \forall t \in [0, 1]. \quad (9.5)$$

In particular, for time $t = 1$, (9.5) can be seen as an initial condition for propagation (2.31), and thus the exponential moment of the order $\bar{\gamma}$ and a rate $0 < \bar{\alpha} \leq \alpha(k_*)$ stays uniformly bounded for all $t > 1$ for k_* , as in (2.24).

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Appendix A. Existence and Uniqueness Theory for ODE in Banach Spaces

Theorem A.1. Let $E := (E, \|\cdot\|)$ be a Banach space, $\mathcal{S}$ be a bounded, convex and closed subset of E , and $\mathcal{Q} : \mathcal{S} \rightarrow E$ be an operator satisfying the following properties:

(a) *The Hölder continuity condition*

$$\|\mathcal{Q}[u] - \mathcal{Q}[v]\| \leq C \|u - v\|^\beta, \quad \beta \in (0, 1), \quad \forall u, v \in \mathcal{S};$$

(b) *The Sub-tangent condition*

$$\lim_{h \rightarrow 0^+} \frac{\text{dist}(u + h\mathcal{Q}[u], \mathcal{S})}{h} = 0, \quad \forall u \in \mathcal{S};$$

(c) *The One-sided Lipschitz condition*

$$[\mathcal{Q}[u] - \mathcal{Q}[v], u - v] \leq C \|u - v\|, \quad \forall u, v \in \mathcal{S},$$

where $[\varphi, \phi] = \lim_{h \rightarrow 0^-} h^{-1} (\|\phi + h\varphi\| - \|\phi\|)$.

Then the equation

$$\partial_t u = \mathcal{Q}[u], \quad \text{for } t \in (0, \infty), \quad \text{with initial data } u(0) = u_0 \text{ in } \mathcal{S},$$

has a unique solution in $C([0, \infty), \mathcal{S}) \cap C^1((0, \infty), E)$.

The proof of this Theorem on ODE flows on Banach spaces can be found in the unpublished notes [10] or in [3].

Remark 9. In Sect. 5, we will concentrate on $E := L_2^1$. Therefore, for the one-sided Lipschitz condition, we will use the following inequality:

$$[\varphi, \phi] \leq \sum_{i=1}^I \int_{\mathbb{R}^3} \varphi_i(v) \text{sign}(\phi_i(v)) \langle v \rangle_i^2 dv,$$

for $\varphi = [\varphi_i]_{1 \leq i \leq I}$ and $\phi = [\phi_i]_{1 \leq i \leq I}$, as pointed out in [3].

Appendix B. Upper and Lower Bound of the Cross Section

In this section, we derive an upper and lower estimate for the non-angular part of the cross section, $|v - v_*|^{\gamma_{ij}}$, $\gamma_{ij} \in (0, 1]$, with $1 \leq i, j \leq I$. First, for the upper estimate, by triangle inequality, we have

$$\begin{aligned} \sqrt{\frac{m_i}{\sum_{i=1}^I m_i}} \sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v - v_*| &\leq \min \left\{ \sqrt{\frac{m_i}{\sum_{i=1}^I m_i}}, \sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} \right\} |v - v_*| \\ &\leq \min \left\{ \sqrt{\frac{m_i}{\sum_{i=1}^I m_i}}, \sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} \right\} (|v| + |v_*|) \end{aligned}$$

$$\begin{aligned}
&\leq \sqrt{\frac{m_i}{\sum_{i=1}^I m_i}} |v| + \sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v_*| \\
&\leq \sqrt{1 + \frac{m_i}{\sum_{i=1}^I m_i}} |v|^2 + \sqrt{1 + \frac{m_j}{\sum_{i=1}^I m_i}} |v_*|^2.
\end{aligned}
\tag{B.1}$$

Therefore,

$$|v - v_*|^{\gamma_{ij}} \leq \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \left(\langle v \rangle_i^{\gamma_{ij}} + \langle v_* \rangle_j^{\gamma_{ij}} \right) \tag{B.2}$$

for $\gamma_{ij} \in (0, 1]$, and any $i, j \in \{1, \dots, I\}$.

From (B.1) it also follows that

$$\begin{aligned}
&\sqrt{\frac{m_i}{\sum_{i=1}^I m_i}} \sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v - v_*| \\
&\leq \sqrt{\frac{m_i}{\sum_{i=1}^I m_i}} |v| + \sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v_*| \\
&= \left(\frac{m_i}{\sum_{i=1}^I m_i} |v|^2 + \frac{m_j}{\sum_{i=1}^I m_i} |v_*|^2 + 2 \frac{\sqrt{m_i m_j}}{\sum_{i=1}^I m_i} |v| |v_*| \right)^{1/2} \\
&\leq \langle v \rangle_i \langle v_* \rangle_j.
\end{aligned}$$

Therefore,

$$|v - v_*|^{\gamma_{ij}} \leq \left(\frac{\sum_{i=1}^I m_i}{\sqrt{m_i m_j}} \right)^{\gamma_{ij}} \langle v \rangle_i^{\gamma_{ij}} \langle v_* \rangle_j^{\gamma_{ij}} \tag{B.3}$$

for $\gamma_{ij} \in (0, 1]$ and $1 \leq i, j \leq I$.

Then, for the lower estimate, we use the ideas of Lemma 2.1 in [4] to prove the next Lemma. Note that here functions F do not need to be solutions of the Boltzmann problem. Moreover, this lower bound may not hold for F being a singular measure, since the estimate degenerates as c goes to zero.

Lemma B.1. *Let $\gamma_{ij} \in [0, 2]$, for any $i, j \in \{1, \dots, I\}$, and assume that $0 \leq \{F(t) = [f_1(t) \dots f_I(t)]^T\}_{t \geq 0} \subset L_2^1$ satisfies*

$$\begin{aligned}
c &\leq \sum_{i=1}^I \int_{\mathbb{R}^3} m_i f_i(t, v) dv \leq C, \quad c \leq \sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t, v) m_i |v|^2 dv \leq C, \\
&\sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t, v) m_i v dv = 0
\end{aligned}$$

for some positive constants c and C . Assume also the boundedness of the moment

$$\sum_{i=1}^I \int_{\mathbb{R}^3} f_i(t, v) m_i |v|^{2+\varepsilon} dv \leq B, \quad \varepsilon > 0.$$

Then, there exists a constant c_{lb} characterized in (B.11) such that

$$\sum_{i=1}^I \int_{\mathbb{R}^3} m_i f_i(t, w) |v - w|^{\gamma_{ij}} dw \geq c_{lb} \langle v \rangle_j^{\bar{\gamma}} \quad (\text{B.4})$$

for any $j \in \{1, \dots, I\}$, with $\bar{\gamma} = \max_{1 \leq i, j \leq I} \gamma_{ij}$.

Proof. Case $\gamma_{ij} = 0$ is trivial, so take $\gamma_{ij} \in (0, 2]$ for any $i, j, = 1, \dots, I$. Let us denote the open ball centered at the origin and of radius $r > 0$ with $B(0, r) \subset \mathbb{R}^3$. We consider separately cases when $v \in B(0, r)$ and $v \in B(0, r)^c$, with r to be chosen later on depending on constants c, C , and γ_{ij} .

For $v \in B(0, r)^c$ we first consider the whole domain $\mathbb{R}^3$, and write, by the Young inequality, for any $v \in \mathbb{R}^3$ and $\gamma_{ij} \in (0, 2]$,

$$\sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v - w|^{\gamma_{ij}} dw \geq \sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) (\tilde{c} |v|^{\gamma_{ij}} - |w|^{\gamma_{ij}}) dw,$$

where $\tilde{c} = \min_{1 \leq i, j \leq I} (\min\{1, 2^{1-\gamma_{ij}}\})$. Since

$$\begin{aligned} & \sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |w|^{\gamma_{ij}} dw \\ & \leq \sum_{i=1}^I m_i \int_{B(0,1)} f_i(t, w) dw + \sum_{i=1}^I m_i \int_{B(0,1)^c} f_i(t, w) |w|^2 dw \leq 2C, \end{aligned}$$

we obtain that, for any $v \in \mathbb{R}^3$,

$$\begin{aligned} & \sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v - w|^{\gamma_{ij}} dw \\ & \geq \tilde{c} \sum_{i=1}^I |v|^{\gamma_{ij}} m_i \int_{\mathbb{R}^3} f_i(t, w) dw - 2C. \end{aligned} \quad (\text{B.5})$$

Since, for any $i, j = 1, \dots, I$ and $v \in \mathbb{R}^3$, we have the following lower bound:

$$|v|^{\gamma_{ij}} = |v|^{\gamma_{ij}} \left(\mathbb{1}_{|v|<1}(v) + \mathbb{1}_{|v|\geq 1}(v) \right) \geq |v|^{\bar{\gamma}} + 1,$$

where

$$\bar{\gamma} = \max_{1 \leq i, j \leq I} \gamma_{ij}.$$

Therefore, (B.5) becomes

$$\begin{aligned} \sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v - w|^{\gamma_{ij}} dw &\geq \tilde{c} c |v|^{\bar{\gamma}} + \tilde{c} c - 2C \\ &\geq \tilde{c} c \left(\sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v| \right)^{\bar{\gamma}} + \tilde{c} c - 2C \end{aligned}$$

for every $j \in \{1, \dots, I\}$. Since here $v \in B(0, r)^c$, we choose r in a such a way as to ensure that

$$\tilde{c} c \left(\sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v| \right)^{\bar{\gamma}} + \tilde{c} c - 2C \geq \frac{\tilde{c} c}{2} \left(\sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v| \right)^{\bar{\gamma}},$$

which amounts to choosing

$$r := r_* = \left(\frac{2C}{\tilde{c}c} \right)^{\frac{1}{\bar{\gamma}}}, \quad (\text{B.6})$$

since $C \geq c$ by assumption, and $\tilde{c} \leq 1$. Therefore, for $v \in B(0, r^*)^c$, we have

$$\sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v - w|^{\gamma_{ij}} dw \geq \frac{\tilde{c} c}{2} \left(\sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v| \right)^{\bar{\gamma}} \quad (\text{B.7})$$

for any $j \in \{1, \dots, I\}$.

On the other hand, let us study the case $v \in B(0, r^*)$. First note that for any $R > 0$,

$$\begin{aligned} &\sum_{i=1}^I m_i \int_{|v-w| \leq R} f_i(t, w) |v - w|^2 dw \\ &= \sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v - w|^2 dw - \sum_{i=1}^I m_i \int_{|v-w| \geq R} f_i(t, w) |v - w|^2 dw \\ &\geq c |v|^2 + c - \sum_{i=1}^I m_i \int_{|v-w| \geq R} f_i(t, w) |v - w|^2 dw \\ &\geq c(1 + |v|^2) - \frac{1}{R^\varepsilon} \sum_{i=1}^I m_i \int_{|v-w| \geq R} f_i(t, w) |v - w|^{2+\varepsilon} dw. \end{aligned} \quad (\text{B.8})$$

Next, we have

$$\begin{aligned} \sum_{i=1}^I m_i \int_{|v-w| \geq R} f_i(t, w) |v - w|^{2+\varepsilon} dw &\leq 2^{1+\varepsilon} \max\{C, B\} (1 + |v|^{2+\varepsilon}) \\ &\leq 2^{1+\varepsilon} \max\{C, B\} (1 + |v|^2)^{\frac{2+\varepsilon}{2}} \leq 2^{1+\varepsilon} \max\{C, B\} (1 + r_*^2)^{\frac{2+\varepsilon}{2}}. \end{aligned}$$

Choosing $R := R(r_*, c, C, B) > 0$ sufficiently large such that

$$\begin{aligned} \frac{1}{R^\varepsilon} 2^{1+\varepsilon} \max\{C, B\} \left(1 + r_*^2\right)^{\frac{2+\varepsilon}{2}} &\leq \frac{c}{2}, \quad \text{or } R \\ &\geq \left(2^{2+\varepsilon} \left(\frac{\max\{C, B\}}{c}\right) \left(1 + r_*^2\right)^{\frac{2+\varepsilon}{2}}\right)^{\frac{1}{\varepsilon}}, \end{aligned} \quad (\text{B.9})$$

from (B.8) we have

$$\sum_{i=1}^I m_i \int_{|v-w| \leq R} f_i(t, w) |v-w|^2 \, dw \geq \frac{c}{2} \quad \forall v \in B(0, r_*).$$

Moreover, for this choice of R , for any $\gamma_{ij} \in (0, 2]$ we have

$$\begin{aligned} \sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v-w|^{\gamma_{ij}} \, dw &\geq \sum_{i=1}^I m_i \int_{|v-w| \leq R} f_i(t, w) |v-w|^{\gamma_{ij}} \, dw \\ &\geq \sum_{i=1}^I R^{\gamma_{ij}-2} m_i \int_{|v-w| \leq R} f_i(t, w) |v-w|^2 \, dw. \end{aligned}$$

Since $R \geq 1$, we can bound $R^{\gamma_{ij}-2} \geq R^{(\min_{1 \leq i, j \leq I} \gamma_{ij})-2}$, which yields the estimate

$$\sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v-w|^{\gamma_{ij}} \, dw \geq \frac{c}{2R^{2-\min_{1 \leq i, j \leq I} \gamma_{ij}}} \quad \forall v \in B(0, r_*). \quad (\text{B.10})$$

Finally, summarizing (B.7) and (B.10),

$$\begin{aligned} \sum_{i=1}^I m_i \int_{\mathbb{R}^3} f_i(t, w) |v-w|^{\gamma_{ij}} \, dw &\geq \frac{c}{2R^{2-\min_{1 \leq i, j \leq I} \gamma_{ij}}} \mathbb{1}_{B(0, r_*)}(v) \\ &\quad + \frac{\tilde{c} c}{2} \left(\sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v| \right)^{\bar{\gamma}} \mathbb{1}_{B(0, r_*)^c}(v) \\ &\geq \frac{\tilde{c} c}{2R^{2-\min_{1 \leq i, j \leq I} \gamma_{ij}}} \left(\mathbb{1}_{B(0, r_*)}(v) + \left(\sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v| \right)^{\bar{\gamma}} \mathbb{1}_{B(0, r_*)^c}(v) \right). \end{aligned}$$

Then there exists a constant c_{lb} such that

$$\frac{\tilde{c} c}{2R^{2-\min_{1 \leq i, j \leq I} \gamma_{ij}}} \left(\mathbb{1}_{B(0, r_*)}(v) + \left(\sqrt{\frac{m_j}{\sum_{i=1}^I m_i}} |v| \right)^{\bar{\gamma}} \mathbb{1}_{B(0, r_*)^c}(v) \right) \geq c_{lb} \langle v \rangle_j^{\bar{\gamma}}$$

for any $j \in \{1, \dots, I\}$. In fact, one may even construct c_{lb} in order to ensure the last inequality. For example, c_{lb} can take the value

$$c_{lb} = \frac{c}{2} \tilde{c} \left(2^{2+\varepsilon} \left(\frac{\max\{C, B\}}{c} \right) \left(1 + \left(\frac{2C}{\tilde{c}} \right)^{\frac{2}{\gamma}} \right)^{\frac{2+\varepsilon}{2}} \right)^{\frac{-2+\min_{1 \leq i, j \leq I} \gamma_{ij}}{\varepsilon}} \\ \times \left(1 + \frac{\max_{1 \leq j \leq I} m_j}{\sum_{i=1}^I m_i} \left(\frac{2C}{\tilde{c}} \right)^2 \right)^{-\bar{\gamma}/2}, \quad (\text{B.11})$$

by taking into account (B.6) and (B.9). $\square$

Appendix C. Some Technical Results

Lemma C.1. (Polynomial inequality I, Lemma 2 from [8].) *Assume $p > 1$, and let $n_p = \lfloor \frac{p+1}{2} \rfloor$. Then, for all $x, y > 0$, the following inequality holds:*

$$(x+y)^p - x^p - y^p \leq \sum_{n=1}^{n_p} \binom{p}{n} (x^n y^{p-n} + x^{p-n} y^n).$$

Lemma C.2. (Polynomial inequality II.) *Let $b+1 \leq a \leq \frac{p+1}{2}$. Then, for any $x, y \geq 0$,*

$$x^a y^{p-a} + x^{p-a} y^a \leq x^b y^{p-b} + x^{p-b} y^b.$$

Proof. This Lemma is a modified version of Lemma A.1 from [22]. Indeed, the proof is the same; one just needs to observe that $a-b \geq 0$ and $p-a-b \geq 0$, and therefore that

$$(y^{a-b} - x^{a-b}) x^b y^b (y^{p-a-b} - x^{p-a-b}) \geq 0$$

for any $x, y \geq 0$. $\square$

Lemma C.3. (Interpolation inequality.) *Let $k = \alpha k_1 + (1-\alpha)k_2$, $\alpha \in (0, 1)$, $0 < k_1 \leq k \leq k_2$. Then, for any $g \in L_{k,i}^1$*

$$\|g\|_{L_{k,i}^1} \leq \|g\|_{L_{k_1,i}^1}^\alpha \|g\|_{L_{k_2,i}^1}^{1-\alpha}. \quad (\text{C.1})$$

We can extend this interpolation inequality for vector functions $\mathbb{G} = [g_i]_{1 \leq i \leq I}$. Namely, under the same assumptions,

$$\|\mathbb{G}\|_{L_k^1} \leq I \|\mathbb{G}\|_{L_{k_1}^1}^\alpha \|\mathbb{G}\|_{L_{k_2}^1}^{1-\alpha}. \quad (\text{C.2})$$

Lemma C.4. (Jensen's inequality.) Let $f(x)$ be positive and integrable in $\mathbb{R}^d$ and G a convex function. Then

$$G\left(\frac{1}{\int f(x) \, dx} \int f(x)g(x) \, dx\right) \leq \frac{1}{\int f(x) \, dx} \int f(x)G(g(x)) \, dx$$

for any positive function g .

We apply this lemma specifying that $g(x) = \langle x \rangle_i^k$ and $G(x) = x^{1+\frac{\lambda}{k}}$, $\lambda \in (0, 1]$ and $k \geq 1$. This implies

$$\int_{\mathbb{R}^3} f_i(v) \langle v \rangle_i^{k+\lambda} \, dv \geq \left(\int_{\mathbb{R}^3} f_i(v) \, dv \right)^{-\frac{\lambda}{k}} \left(\int_{\mathbb{R}^3} f_i(v) \langle v \rangle_i^k \, dv \right)^{1+\frac{\lambda}{k}}.$$

If, additionally, we have an upper bound on the zero order scalar polynomial moment, that is, if it holds that

$$\int_{\mathbb{R}^3} f_i(v) \, dv = m_{0,i}[\mathbb{F}] \leq m_0[\mathbb{F}] \leq C m_0,$$

then

$$\int_{\mathbb{R}^3} f_i(v) \langle v \rangle_i^{k+\lambda} \, dv \geq C m_0^{-\frac{\lambda}{k}} \left(\int_{\mathbb{R}^3} f_i(v) \langle v \rangle_i^k \, dv \right)^{1+\frac{\lambda}{k}}.$$

Summing over $i = 1, \dots, I$, after some manipulation we get a control from below for the moment $m_{k+\lambda}[\mathbb{F}]$. Indeed, we get that

$$m_{k+\lambda}[\mathbb{F}] \geq (I C m_0)^{-\frac{\lambda}{k}} m_k[\mathbb{F}]^{1+\frac{\lambda}{k}}. \quad (\text{C.3})$$

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