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The zero surface tension limit of three-dimensional interfacial Darcy flow ☆

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Abstract

We study the zero surface tension limit of three-dimensional interfacial Darcy flow. We start with a proof of well-posedness of three-dimensional interfacial Darcy flow for any positive value of the surface tension coefficient. The primary tool for this well-posedness proof is an energy estimate. The time of existence for these solutions will, in general, go to zero with the surface tension parameter. However, in the case that a stability condition is satisfied by the initial data, we prove an additional energy estimate, establishing that the time of existence can be made uniform in the surface tension parameter. Then, an additional estimate allows the limit to be taken as surface tension vanishes, demonstrating that three-dimensional interfacial Darcy flow without surface tension is the limit of three-dimensional interfacial Darcy flow with surface tension as surface tension vanishes. This provides a new proof of existence of solutions for the problem without surface tension.

Keywords: Surface tension; Darcy flow; Well-posedness; Asymptotics; Porous media; Muskat problem

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1. Introduction

We consider a sharp-interface model of two-phase Darcy flow in three spatial dimensions. The fluid velocities are given by Darcy's Law, which models flow in a porous medium [9]. The interface is the boundary between the lower fluid region $\Omega_1(t)$ and the upper fluid region $\Omega_2(t)$, where t is the temporal variable. To be precise, in the bulk of each fluid, the fluid velocity is given by

$$\mathbf{v}_i(x, y, z, t) = -\frac{b^2}{12\nu_i}\nabla(p_i + \rho_i g z), \tag{1.1}$$

for $i \in \{1, 2\}$ and $(x, y, z) \in \Omega_i(t)$. The fluids are taken to be incompressible as well, so that

$$\operatorname{div}(\mathbf{v}_i) = 0. \tag{1.2}$$

The constant b is a physical parameter related to the porosity and permeability of the medium, and g is the constant acceleration due to gravity. The constants ν_i and ρ_i are the viscosity and density, respectively, of fluid i. Of course, $\mathbf{v}_i$ and p_i are the velocity and pressure of fluid i. The fluids under consideration are driven by gravity and by surface tension; the gravity is clearly present in (1.1). The surface tension enters through the Laplace-Young jump condition for the pressure across the interface; we will see this below in Section 2.3.

This interfacial problem is well-posed in the case that surface tension is accounted for at the interface [15], and also in the case of zero surface tension (this is the Muskat problem) if the Saffman-Taylor stability condition is satisfied [23]. In the case in which this stability condition is violated, it has been shown in the case of two-dimensional case that analytic solutions exist [24], and the zero surface tension limit can then be studied for these solutions. Dai, Tanveer, and Siegel, and separately Ceniceros and Hou, have shown that the solutions without surface tension are not the limit of solutions with surface tension as surface tension vanishes when the Saffman-Taylor condition is violated [25], [26], [12], [13].

The second author previously studied the two-dimensional case in [3]. The lines of the argument are the same, but many details are more difficult in the three-dimensional case. In the two-dimensional case, the interface is one-dimensional, and the interface can be described by its tangent angle and arclength element; furthermore, the arclength element was taken to depend only on time. In the present setting, we instead study the mean curvature of the interface and the first fundamental form of the free surface, and it is not possible to insist that the first fundamental form be independent of the spatial variable. As a result, instead of becoming semilinear, the problem with surface tension in the present case is only quasilinear. Furthermore, one of the primary ideas of the argument is to approximate the Birkhoff-Rott integral with simpler singular integrals; the Birkhoff-Rott integral is a singular integral which gives the velocity of the interface. In the case of a one-dimensional interface, the Birkhoff-Rott integral is approximated by a Hilbert transform, and the remainder from making this approximation is very smooth. In the present case, we approximate the Birkhoff-Rott integral with Riesz transforms, and the remainder is smooth, but not nearly as smooth as in the previous case. This does complicate the argument, although we are able to deal with the complication. As a result, we will rely on the parabolic smoothing that is available in the problem to simplify the energy estimates. In the case of the one-dimensional interface in [3], we noted the presence of the parabolic smoothing, but we did not make use of it.

The present work continues the use of a method which has its roots in the numerical works of Hou, Lowengrub, and Shelley (HLS) [16], [17]. In these papers, to remove the stiffness from numerical computations of interfacial Darcy flow with surface tension and interfacial Euler flow with surface tension, HLS used an arclength parameterization and introduced a small-scale decomposition of the problem, identifying the most singular terms in the evolution equations. These ideas were subsequently used analytically, including an extension by the second author and Masmoudi for interfacial Euler and Darcy flows in three spatial dimensions [2], [5], [6], and by Cordoba, Cordoba, and Gancedo for the three-dimensional interfacial Darcy problem [14]. The generalization of the arclength parameterization used by HLS to a two-dimensional free surface used in these works is an isothermal parameterization; this is discussed in Section 2.1 below.

Subsequently, the extension of the HLS to analysis for three-dimensional fluids was brought back to numerical studies, with Ambrose, Siegel, and Tlupova making an extension of all of these works to develop a non-stiff numerical method for 3D interfacial Darcy flow [7], [8]. These ideas have also been used in numerical analysis, as Ceniceros and Hou have shown that a version of the HLS numerical method is convergent [11], and in [4], Ambrose, Liu, and Siegel have shown that a version of the method of [8] is convergent.

The plan of the paper is as follows: in Section 2, we describe the equations of motion for the 3D interfacial Darcy flow problem. This includes specifying our isothermal parameterization and our small-scale decomposition; the small-scale decomposition requires making decompositions of singular integral operators. In Section 3, we provide some lemmas giving bounds for useful operators, and use these to make some preliminary estimates. In Section 4, we prove well-posedness of the problem with surface tension, finding existence on a time interval which depends badly on the surface tension parameter. In Section 5, in the case that the Saffman-Taylor stability condition is satisfied, we extend the time of existence of solutions for a time interval which is independent of surface tension, and pass to the limit as surface tension vanishes.

2. The equations of motion

We study a surface $\mathbf{X}(\vec{\alpha}, t) = (x(\vec{\alpha}, t), y(\vec{\alpha}, t), z(\vec{\alpha}, t))$, where $\vec{\alpha} = (\alpha, \beta) \in \mathbb{R}^2$ is the spatial parameter of the surface, and t is time. We use the following frame of unit tangent and normal vectors at each point of the surface:

$$\hat{\mathbf{t}}^1 = \frac{\mathbf{X}_\alpha}{|\mathbf{X}_\alpha|}, \qquad \hat{\mathbf{t}}^2 = \frac{\mathbf{X}_\beta}{|\mathbf{X}_\beta|}, \qquad \hat{\mathbf{n}} = \hat{\mathbf{t}}^1 \times \hat{\mathbf{t}}^2.$$

We write the velocity of the free surface using its normal and tangential components,

$$\mathbf{X}_t = U\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2. \tag{2.1}$$

2.1. *The tangential velocities and choice of parameterization*

While the normal velocity is determined by the fluid dynamics, the tangential velocities are not. Instead, the tangential velocities can be chosen arbitrarily, as these only determine the parameterization of the free surface. That is, the location of the free surface will not be changed, no matter our choice of V_1 and V_2. Our choice of parameterization is to always maintain a global isothermal parameterization. To this end, we introduce the first fundamental coefficients of the free surface:

$$E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha, \qquad F = \mathbf{X}_\alpha \cdot \mathbf{X}_\beta, \qquad G = \mathbf{X}_\beta \cdot \mathbf{X}_\beta. \tag{2.2}$$

Then, our choice is to have

$$E = G, \qquad F = 0, \tag{2.3}$$

for all $\vec{\alpha}$ and for all t. For any surface, such a parameterization always exists locally, and in some cases globally. Fortunately the case we consider has been proved to have a global isothermal parameterization. In the sequel, we consider the case in which the interface between the two fluids is asymptotic to the flat plane at infinity, and in this case, it is known that the isothermal parameterization exists globally [14]. The tangential velocities are then found by using (2.1) together with the time derivative of (2.3), $E_t = G_t$ and $F_t = 0$. The corresponding calculation is given in detail in [5]. As a result, we find that the tangential velocities V_1 and V_2 should satisfy the equations

$$\left(\frac{V_1}{\sqrt{E}}\right)_\alpha - \left(\frac{V_2}{\sqrt{E}}\right)_\beta = \frac{U(L-N)}{E}, \tag{2.4}$$

$$\left(\frac{V_1}{\sqrt{E}}\right)_\beta + \left(\frac{V_2}{\sqrt{E}}\right)_\alpha = \frac{2UM}{E}. \tag{2.5}$$

Here, L, M, and N are the coefficients of the second fundamental form of the free surface,

$$L = -\mathbf{X}_\alpha \cdot \hat{\mathbf{n}}_\alpha = \mathbf{X}_{\alpha\alpha} \cdot \hat{\mathbf{n}}, \qquad N = -\mathbf{X}_\beta \cdot \hat{\mathbf{n}}_\beta = \mathbf{X}_{\beta\beta} \cdot \hat{\mathbf{n}}, \qquad M = -\mathbf{X}_\alpha \cdot \hat{\mathbf{n}}_\beta = -\mathbf{X}_\beta \cdot \hat{\mathbf{n}}_\alpha = \mathbf{X}_{\alpha\beta} \cdot \hat{\mathbf{n}}. \tag{2.6}$$

Then, if V_1 and V_2 satisfy (2.4) and (2.5), and if the initial surface satisfies (2.3), then the surface will satisfy (2.3) at positive times as well.

2.2. Geometric identities

We will differentiate the normal and tangential vectors many times in the sequel, and therefore, formulas for these derivatives in the context of the isothermal parameterization will be helpful. We have the following:

$$\hat{\mathbf{n}}_\alpha = -\frac{L}{E^{1/2}}\hat{\mathbf{t}}^1 - \frac{M}{E^{1/2}}\hat{\mathbf{t}}^2, \tag{2.7}$$

$$\hat{\mathbf{n}}_\beta = -\frac{M}{E^{1/2}}\hat{\mathbf{t}}^1 - \frac{N}{E^{1/2}}\hat{\mathbf{t}}^2, \tag{2.8}$$

$$\hat{\mathbf{t}}^1_\alpha = -\frac{E_\beta}{2E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}}, \tag{2.9}$$

$$\hat{\mathbf{t}}^1_\beta = \frac{E_\alpha}{2E}\hat{\mathbf{t}}^2 + \frac{M}{E^{1/2}}\hat{\mathbf{n}}, \tag{2.10}$$

$$\hat{\mathbf{t}}^2_\alpha = \frac{E_\beta}{2E}\hat{\mathbf{t}}^1 + \frac{M}{E^{1/2}}\hat{\mathbf{n}}, \tag{2.11}$$

$$\hat{\mathbf{t}}^2_\beta = -\frac{E_\alpha}{2E}\hat{\mathbf{t}}^1 + \frac{N}{E^{1/2}}\hat{\mathbf{n}}. \tag{2.12}$$

2.3. The normal velocity and the fluid dynamics

We mentioned above that the normal component of the velocity of the free surface, U, is determined by the fluid dynamics. There are two equivalent descriptions of the relevant fluid dynamics which can lead us to the appropriate formula for U: one directly from potential theory, and one which considers the Biot-Savart law for recovering the velocity from the vorticity (and thus using potential theory indirectly). We focus on the calculation which uses potential theory directly.

From (1.1), we see that the velocity is equal to the gradient of a potential in each phase, with the potential, ϕ_i, given by

$$\phi_i = -\frac{b^2}{12\nu_i}(p_i + \rho_i g z),$$

for $i \in \{1,2\}$; recall that this means that this equation holds for all $(x, y, z) \in \Omega_i(t)$. Combining equations (1.1) and (1.2) with this definition of ϕ_i, we see that $\Delta\phi_i = 0$, for $i \in \{1,2\}$. The normal component of the velocity of the free surface must be the same when calculated from above and below (this is one of the boundary conditions for the problem); we recall that the normal derivative of ϕ_i is the normal velocity of the interface. While there is no jump in the normal derivative, there is of course a jump between ϕ_1 and ϕ_2 at the free surface, and we give the name μ to this jump:

$$\mu = \frac{b^2}{12}\left(-\frac{p_1}{\nu_1} + \frac{p_2}{\nu_2}\right) - \frac{b^2}{12}\left(\frac{\rho_1 g z}{\nu_1} - \frac{\rho_2 g z}{\nu_2}\right). \tag{2.13}$$

Since there is no jump in the normal derivative across the interface, we may express ϕ_i by means of a double-layer potential. The source strength for the double-layer potential is μ, the jump in potential; we write the double-layer potential suppressing the time dependence:

$$\phi_i(x, y, z) = \pm\frac{1}{2\pi}\iint_{\mathbb{R}^2} \mu(\vec{\alpha})\frac{(x, y, z) - (x(\vec{\alpha}), y(\vec{\alpha}), z(\vec{\alpha}))}{|(x, y, z) - (x(\vec{\alpha}), y(\vec{\alpha}), z(\vec{\alpha}))|^2} \cdot \hat{\mathbf{n}}(\vec{\alpha})\, d\vec{\alpha}; \tag{2.14}$$

again, this holds for $(x, y, z) \in \Omega_i(t)$.

If we add the two potentials at the free surface, we get

$$\phi_1 + \phi_2 = \frac{b^2}{12}\left(-\frac{p_1}{\nu_1} - \frac{p_2}{\nu_2}\right) - \frac{b^2}{12}\left(\frac{\rho_1 g z}{\nu_1} + \frac{\rho_2 g z}{\nu_2}\right). \tag{2.15}$$

The equations (2.13) and (2.15) can be solved for p_1 and p_2 (we omit the algebra). Then, we can subtract p_1 and p_2, and make use of the Laplace-Young jump condition (the jump in pressure across the interface is proportional to the mean curvature of the interface, with the constant of proportionality being a material parameter depending on the chemical makeup of the two fluids). Then, the resulting equation can be solved for μ:

$$\mu = \tau\kappa - A_\nu(\phi_1 + \phi_2) - Rz, \tag{2.16}$$

where τ is the non-negative, constant coefficient of surface tension, and A_ν and R are given by the formulas

$$A_\nu = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2}, \qquad R = \frac{b^2 g(\rho_1 - \rho_2)}{6(\nu_1 + \nu_2)}.$$

Notice that (2.16) is actually an integral equation for μ, as the expressions for ϕ_1 and ϕ_2 in (2.14) involve μ under the integral. We could restate this as

$$\mu + A_\nu \mathfrak{D}\mu = \tau\kappa - Rz, \tag{2.17}$$

where $\mathfrak{D}$ is the resulting integral operator. To be more precise, given the surface $\mathbf{X}$, the operator $\mathfrak{D}$ applied to μ is given by the integral on the right-hand side of (2.14).

We can differentiate (2.16) with respect to each of α and β. When doing so, we must remember that ϕ_1 and ϕ_2 are functions defined not only on the interface; thus, for instance, we have $\partial_\alpha \phi_1 = \nabla\phi_1 \cdot \mathbf{X}_\alpha$. We find the following, upon differentiating (2.16):

$$\mu_\alpha = \tau\kappa_\alpha - A_\nu \left(\nabla\phi_1 \cdot \mathbf{X}_\alpha + \nabla\phi_2 \cdot \mathbf{X}_\alpha\right) - Rz_\alpha, \tag{2.18}$$

$$\mu_\beta = \tau\kappa_\beta - A_\nu \left(\nabla\phi_1 \cdot \mathbf{X}_\beta + \nabla\phi_2 \cdot \mathbf{X}_\beta\right) - Rz_\beta. \tag{2.19}$$

To find the limiting values of $\nabla\phi_i$ for $i \in \{1, 2\}$, the gradient of (2.14) can be taken, and then the Plemelj formulas can be used (see [10] or [21] for discussion of the Plemelj formulas). Carrying out this calculation, we find the following limiting values for the gradients of the potentials:

$$\nabla\phi_1 = \mathbf{W} + \frac{\mu_\alpha}{2\sqrt{E}}\hat{\mathbf{t}}^1 + \frac{\mu_\beta}{2\sqrt{E}}\hat{\mathbf{t}}^2, \tag{2.20}$$

$$\nabla\phi_2 = \mathbf{W} - \frac{\mu_\alpha}{2\sqrt{E}}\hat{\mathbf{t}}^1 - \frac{\mu_\beta}{2\sqrt{E}}\hat{\mathbf{t}}^2. \tag{2.21}$$

We will give the definition of $\mathbf{W}$, the Birkhoff-Rott integral, shortly. Using (2.20) and (2.21) with (2.18) and (2.19), we find the following:

$$\mu_\alpha = \tau\kappa_\alpha - 2A_\nu\sqrt{E}\mathbf{W} \cdot \hat{\mathbf{t}}^1 - Rz_\alpha, \tag{2.22}$$

$$\mu_\beta = \tau\kappa_\beta - 2A_\nu\sqrt{E}\mathbf{W} \cdot \hat{\mathbf{t}}^2 - Rz_\beta. \tag{2.23}$$

The Birkhoff-Rott integral is

$$\mathbf{W}(\vec{\alpha}) = -\frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} (\mu'_\alpha \mathbf{X}'_\beta - \mu'_\beta \mathbf{X}'_\alpha) \times \frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3}\, d\vec{\alpha}'. \tag{2.24}$$

In the integrand in (2.24), functions followed by a prime are evaluated at $\vec{\alpha}'$, and unprimed functions are evaluated at $\vec{\alpha}$. Notice from (2.20) and (2.21) that we can find the normal velocity of the free surface by taking the dot product with the normal vector:

$$U = \mathbf{W} \cdot \hat{\mathbf{n}}. \tag{2.25}$$

The principal value integral in (2.24) makes sense as long as the surface is non-self-intersecting. As in prior works by the authors and collaborators [1], [5], [6], [18], we follow the works of Wu in assuming that a chord-arc condition is satisfied [29], [30]. We assume that there exists a constant $\bar{d} > 0$ such that the initial surface satisfies

$$\frac{|\mathbf{X}(\alpha,\beta,0)-\mathbf{X}(\alpha',\beta',0)|^2}{(\alpha-\alpha')^2+(\beta-\beta')^2} > \bar{d}, \qquad \forall(\alpha,\beta)\neq(\alpha',\beta'). \tag{2.26}$$

We will then endeavor to solve the initial value problem satisfying the bound

$$\frac{|\mathbf{X}(\alpha,\beta,t)-\mathbf{X}(\alpha',\beta',t)|^2}{(\alpha-\alpha')^2+(\beta-\beta')^2} > \frac{\bar{d}}{2}, \qquad \forall(\alpha,\beta)\neq(\alpha',\beta'), \tag{2.27}$$

for $t > 0$.

Remark 1. While the normal component of the velocity does not jump at the free surface, the tangential velocity does have a jump. Since the velocities are given by the gradient of a potential in the bulk of the two fluid regions, the fluids are irrotational in the bulk. However, since there is a jump in velocity at the free surface, the vorticity is actually measure-valued; that is, the vorticity can be expressed by means of a Dirac mass supported on the free surface, and thus the free surface is a vortex sheet. The alternative derivation of the Birkhoff-Rott integral and the formula (2.25) which we mentioned above makes use of this structure of the vorticity, and uses the Biot-Savart law to find the velocity from the vorticity. The interested reader might consult [22] or [10] for details of the derivation of the Birkhoff-Rott integral for a vortex sheet in the case of three-dimensional fluids.

Remark 2. We have mentioned above that (2.17) is an integral equation for μ, and thus equations (2.22), (2.23) are a system of integral equations. All of these integral equations are solvable. For existence and regularity of μ, see Lemmas 3.12 and 3.13 below.

2.4. Evolution of E and κ

With the free surface being parameterized by an isothermal parameterization, the formula for the mean curvature can be written as

$$\kappa = \frac{L+N}{2E}. \tag{2.28}$$

Our primary estimates will be energy estimates for κ. The evolution equation for κ can be inferred from (2.1), using (2.28) with (2.2) and (2.6). For the moment, a convenient way to write the evolution equation for the curvature is

$$(\sqrt{E}\kappa)_t = \frac{\Delta U}{2\sqrt{E}} + \frac{V_1}{\sqrt{E}}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{\sqrt{E}}(\sqrt{E}\kappa)_\beta + \frac{UM^2}{\sqrt{E}} + \frac{L}{2\sqrt{E}}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2\sqrt{E}}\left(\frac{V_2}{\sqrt{E}}\right)_\beta. \tag{2.29}$$

(Detailed calculations leading to (2.29) and related formulas in this section may be found in [5].) Of course, to specify κ_t, we would need to use (2.29) together with an evolution equation for E.

We can infer an evolution equation for E from (2.1), using the definition $E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha$, or alternatively we could use the isothermal parameterization, $E = G = \mathbf{X}_\beta \cdot \mathbf{X}_\beta$. These considerations yield the following:

$$E_t = 2\sqrt{E}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2 E_\beta}{2E}\right) = 2\sqrt{E}\left(V_{2,\beta} - \frac{UN}{\sqrt{E}} + \frac{V_1 E_\alpha}{2E}\right). \tag{2.30}$$

If $\mathbf{X}$ is parameterized according to (2.3), then we have the following two derivatives which will be used for iteration in Section 4.2. On one hand, since $E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha = \mathbf{X}_\beta \cdot \mathbf{X}_\beta$ and $\mathbf{X}_\alpha \cdot \mathbf{X}_\beta = 0$, we have

$$\Delta E = 2(\mathbf{X}_{\alpha\beta} \cdot \mathbf{X}_{\alpha\beta}) - 2(\mathbf{X}_{\alpha\alpha} \cdot \mathbf{X}_{\beta\beta}). \tag{2.31}$$

On the other hand,

$$\Delta \mathbf{X} = \left(\Delta \mathbf{X} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} = (L + N)\hat{\mathbf{n}} = 2E\kappa\hat{\mathbf{n}}.$$

2.5. *The Birkhoff-Rott integral and its consequences*

The most singular term on the right-hand side of (2.29) is the term which includes ΔU. We seek a more useful expression for ΔU, and in light of (2.25), it will be helpful to find more useful expressions for $\mathbf{W}$ and its derivatives. To this end, we will be rewriting $\mathbf{W}$ and its derivatives in terms of well-understood singular integral operators such as Riesz transforms. The Riesz transforms H_1 and H_2 are singular integral operators defined as

$$(H_1 f)(\alpha, \beta) = \frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} \frac{f(\alpha', \beta')(\alpha - \alpha')}{\left((\alpha - \alpha')^2 + (\beta - \beta')^2\right)^{3/2}}\, d\alpha' d\beta', \tag{2.32}$$

$$(H_2 f)(\alpha, \beta) = \frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} \frac{f(\alpha', \beta')(\beta - \beta')}{\left((\alpha - \alpha')^2 + (\beta - \beta')^2\right)^{3/2}}\, d\alpha' d\beta'. \tag{2.33}$$

These have Fourier symbol $\hat{H}_i(\xi) = i\xi_i/|\xi|$. If the surface $\mathbf{X}$ were flat, that is if $\mathbf{X}(\alpha, \beta) = (\alpha, \beta, 0)$, then we see from (2.24) that the Birkhoff-Rott integral $\mathbf{W}$ could be expressed exactly with Riesz transforms. When $\mathbf{X}$ is not flat, we will instead see that Riesz transforms give the leading-order part of $\mathbf{W}$. More information about Riesz transforms can be found, for instance, in [27] or [20]. We also take this opportunity to define some additional integral operators with weakly singular kernels,

$$(G_{11} f)(\alpha, \beta) = \frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} \frac{f(\alpha', \beta')(\alpha - \alpha')^2}{\left((\alpha - \alpha')^2 + (\beta - \beta')^2\right)^{3/2}}\, d\alpha' d\beta', \tag{2.34}$$

$$(G_{12} f)(\alpha, \beta) = (G_{21} f)(\alpha, \beta) = \frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} \frac{f(\alpha', \beta')(\alpha - \alpha')(\beta - \beta')}{\left((\alpha - \alpha')^2 + (\beta - \beta')^2\right)^{3/2}}\, d\alpha' d\beta', \tag{2.35}$$

$$(G_{22}f)(\alpha,\beta) = \frac{1}{4\pi}\mathrm{PV}\iint\limits_{\mathbb{R}^2} \frac{f(\alpha',\beta')(\beta-\beta')^2}{\left((\alpha-\alpha')^2+(\beta-\beta')^2\right)^{3/2}}\,d\alpha' d\beta'. \tag{2.36}$$

We begin with the following formula, which can be found simply by applying the Laplacian to U and using geometric identities as appropriate:

$$\Delta U = \left[(\mathbf{W}_\alpha\cdot\hat{\mathbf{n}})_\alpha + (\mathbf{W}_\beta\cdot\hat{\mathbf{n}})_\beta\right] + \left((\mathbf{W}\cdot\hat{\mathbf{t}}^1)\left(-\frac{L}{E^{1/2}}\right) + (\mathbf{W}\cdot\hat{\mathbf{t}}^2)\left(-\frac{M}{E^{1/2}}\right)\right)_\alpha + \left((\mathbf{W}\cdot\hat{\mathbf{t}}^1)\left(-\frac{M}{E^{1/2}}\right) + (\mathbf{W}\cdot\hat{\mathbf{t}}^2)\left(-\frac{N}{E^{1/2}}\right)\right)_\beta. \tag{2.37}$$

Our immediate goal is to rewrite the most singular terms here, which are the terms in the square brackets on the right-hand side of (2.37), in terms of Riesz transforms.

We follow the development of the Birkhoff-Rott integral as given in Section 2.2 of [2]. Our goal in this section is to extract the most singular terms from the Birkhoff-Rott integral, allowing us to find a useful expression for the quantity ΔU in (2.29).

To begin with, we develop an expression for $\Delta\mu$ and $\mu_{\alpha\beta}$. First, we differentiate μ_α with respect to α, finding simply

$$\mu_{\alpha\alpha} = \tau\kappa_{\alpha\alpha} - 2A_\nu E^{1/2}\mathbf{W}\cdot\hat{\mathbf{t}}^1_\alpha - 2A_\nu E^{1/2}\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}^1 - \frac{A_\nu E_\alpha}{E^{1/2}}\mathbf{W}\cdot\hat{\mathbf{t}}^1 - Rz_{\alpha\alpha}. \tag{2.38}$$

As part of our effort to rewrite this, we rewrite $z_{\alpha\alpha}$. We denote $n_3 = \hat{\mathbf{n}}\cdot(0,0,1)$, and we have

$$z_{\alpha\alpha} = \left(E^{1/2}\hat{\mathbf{t}}^1\cdot(0,0,1)\right)_\alpha.$$

Applying the α-derivative, this becomes

$$z_{\alpha\alpha} = \frac{E_\alpha z_\alpha}{2E} + E^{1/2}(\hat{\mathbf{t}}^1_\alpha\cdot\hat{\mathbf{n}})n_3 + (\hat{\mathbf{t}}^1_\alpha\cdot\hat{\mathbf{t}}^2)z_\beta.$$

Using the identity (2.9), this becomes

$$z_{\alpha\alpha} = \frac{E_\alpha z_\alpha}{2E} - \frac{E_\beta z_\beta}{2E} + Ln_3. \tag{2.39}$$

We have the corresponding formula for $z_{\beta\beta}$

$$z_{\beta\beta} = \frac{E_\beta z_\beta}{2E} - \frac{E_\alpha z_\alpha}{2E} + Nn_3.$$

Continuing, we combine (2.38) with (2.9) and (2.39):

$$\mu_{\alpha\alpha} = \tau\kappa_{\alpha\alpha} + A_1 := \tau\kappa_{\alpha\alpha} - 2A_\nu LU + \frac{A_\nu E_\beta}{E^{1/2}}\mathbf{W}\cdot\hat{\mathbf{t}}^2 - \frac{A_\nu E_\alpha}{E^{1/2}}\mathbf{W}\cdot\hat{\mathbf{t}}^1 - 2A_\nu E^{1/2}\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}^1 - \frac{RE_\alpha z_\alpha}{2E} + \frac{RE_\beta z_\beta}{2E} - RLn_3.$$

We make the same calculation of $\mu_{\beta\beta}$, finding the following:

$$\mu_{\beta\beta} = \tau\kappa_{\beta\beta} + A_2 := \tau\kappa_{\beta\beta} - 2A_\nu NU - \frac{A_\nu E_\beta}{E^{1/2}}\mathbf{W}\cdot\hat{\mathbf{t}}^2 + \frac{A_\nu E_\alpha}{E^{1/2}}\mathbf{W}\cdot\hat{\mathbf{t}}^1 - 2A_\nu E^{1/2}\mathbf{W}_\beta\cdot\hat{\mathbf{t}}^2 + \frac{RE_\alpha z_\alpha}{2E} - \frac{RE_\beta z_\beta}{2E} - RNn_3.$$

Adding the above two equations, it follows that

$$\mu_{\alpha\alpha} + \mu_{\beta\beta} = \tau\kappa_{\alpha\alpha} + \tau\kappa_{\beta\beta} - 4A_\nu UE\kappa - 2RE\kappa n_3 - 2A_\nu\sqrt{E}\left(\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}^1 + \mathbf{W}_\beta\cdot\hat{\mathbf{t}}^2\right). \tag{2.40}$$

Using the identity (2.10), we have a formula for $z_{\alpha\beta}$:

$$z_{\alpha\beta} = \frac{E_\beta z_\alpha}{2E} + \frac{E_\alpha z_\beta}{2E} + Mn_3.$$

Moreover, we have the following expression for $\mu_{\alpha\beta}$:

$$\mu_{\alpha\beta} = \tau\kappa_{\alpha\beta} + A_3 := \tau\kappa_{\alpha\beta} - 2A_\nu MU - \frac{A_\nu E_\alpha}{E^{1/2}}\mathbf{W}\cdot\hat{\mathbf{t}}^2 - \frac{A_\nu E_\beta}{E^{1/2}}\mathbf{W}\cdot\hat{\mathbf{t}}^1 - 2A_\nu E^{1/2}\mathbf{W}_\beta\cdot\hat{\mathbf{t}}^1 - \frac{RE_\beta z_\alpha}{2E} - \frac{RE_\alpha z_\beta}{2E} - RMn_3.$$

We now introduce three integral operators. Given some function $\mathcal{F}$, we define $\mathcal{K}[\mathbf{X}]\mathcal{F}$, $\mathcal{J}[\mathbf{X}]\mathcal{F}$, and $\mathcal{J}_1[\mathbf{X}]\mathcal{F}$:

$$\mathcal{K}[\mathbf{X}]\mathcal{F}(\vec{\alpha}) = \frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} \mathcal{F}(\vec{\alpha}') \times K(\vec{\alpha},\vec{\alpha}')\, d\vec{\alpha}', \tag{2.41}$$

$$\mathcal{J}[\mathbf{X}]\mathcal{F}(\vec{\alpha}) = \frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} \mathcal{F}(\vec{\alpha}') \times J(\vec{\alpha},\vec{\alpha}')\, d\vec{\alpha}',$$

$$\mathcal{L}_i[\mathbf{X}]\mathcal{F}(\vec{\alpha}) = \frac{1}{4\pi}\mathrm{PV}\iint_{\mathbb{R}^2} \mathcal{F}(\vec{\alpha}') \times L_i(\vec{\alpha},\vec{\alpha}')\, d\vec{\alpha}'. \tag{2.42}$$

In (2.42), we take $i \in \{1, 2\}$. The kernel K is given by

$$\begin{aligned} K(\vec{\alpha},\vec{\alpha}') = {} & \frac{\mathbf{X}(\vec{\alpha}) - \mathbf{X}(\vec{\alpha}')}{|\mathbf{X}(\vec{\alpha}) - \mathbf{X}(\vec{\alpha}')|^3} - \frac{\mathbf{X}_\alpha(\vec{\alpha}')(\alpha-\alpha') + \mathbf{X}_\beta(\vec{\alpha}')(\beta-\beta')}{E^{3/2}(\vec{\alpha}')|\vec{\alpha}-\vec{\alpha}'|^3} \\ & - \frac{\frac{1}{2}\mathbf{X}_{\alpha\alpha}(\vec{\alpha}')(\alpha-\alpha')^2 + \frac{1}{2}\mathbf{X}_{\beta\beta}(\vec{\alpha}')(\beta-\beta')^2 + \mathbf{X}_{\alpha\beta}(\vec{\alpha}')(\alpha-\alpha')(\beta-\beta')}{E^{3/2}(\vec{\alpha}')|\vec{\alpha}-\vec{\alpha}'|^3} \\ & + \frac{3}{4}\frac{\left(E_\alpha(\vec{\alpha}')(\alpha-\alpha') + E_\beta(\vec{\alpha}')(\beta-\beta')\right)\left(\mathbf{X}_\alpha(\vec{\alpha}')(\alpha-\alpha') + \mathbf{X}_\beta(\vec{\alpha}')(\beta-\beta')\right)}{E^{5/2}(\vec{\alpha}')|\vec{\alpha}-\vec{\alpha}'|^3}. \end{aligned} \tag{2.43}$$

The kernels J, L_1, and L_2 are somewhat simpler:

$$J(\vec{\alpha},\vec{\alpha}') = \frac{\mathbf{X}(\vec{\alpha})-\mathbf{X}(\vec{\alpha}')}{|\mathbf{X}(\vec{\alpha})-\mathbf{X}(\vec{\alpha}')|^3} - \frac{\mathbf{X}_\alpha(\vec{\alpha}')(\alpha-\alpha') + \mathbf{X}_\beta(\vec{\alpha}')(\beta-\beta')}{E^{3/2}(\vec{\alpha}')|\vec{\alpha}-\vec{\alpha}'|^3},$$

$$L_1(\vec{\alpha},\vec{\alpha}') = (D_\alpha + D_{\alpha'})J(\vec{\alpha},\vec{\alpha}'), \tag{2.44}$$

$$L_2(\vec{\alpha},\vec{\alpha}') = (D_\beta + D_{\beta'})J(\vec{\alpha},\vec{\alpha}'). \tag{2.45}$$

We mention that the operator $\mathcal{K}[\mathbf{X}]$ is an error from making a second-order Taylor expansion of the kernel $\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}$ which is present in the Birkhoff-Rott integral. The operator $\mathcal{J}[X]$ is similar but is simpler, because it is the error from making only the first-order Taylor expansion. These kernels are similar to kernels of convolution type; for a kernel of convolution type, applying $D_\alpha + D_{\alpha'}$ or $D_\beta + D_{\beta'}$ would annihilate the kernel. For all of these we have smoothing properties, and these are expressed in the results of Section 3 below.

We have the following formula for $\mathbf{W}$:

$$\mathbf{W} = H_1\left(\frac{\mu_\alpha}{2E^{1/2}}\hat{\mathbf{n}}\right) + H_2\left(\frac{\mu_\beta}{2E^{1/2}}\hat{\mathbf{n}}\right) + \mathcal{J}[\mathbf{X}]g, \tag{2.46}$$

where we have introduced the notation

$$g = \mu_\beta \mathbf{X}_\alpha - \mu_\alpha \mathbf{X}_\beta. \tag{2.47}$$

We now differentiate (2.46) to find formulas for $\mathbf{W}_\alpha$ and $\mathbf{W}_\beta$. To begin with, we write the following, which makes use of (2.7):

$$\begin{aligned}\mathbf{W}_\alpha = {}& H_1\left(\frac{\mu_{\alpha\alpha}}{2E^{1/2}}\hat{\mathbf{n}}\right) + H_2\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\hat{\mathbf{n}}\right) - H_1\left(\frac{\mu_\alpha L}{2E}\hat{\mathbf{t}}^1\right) - H_1\left(\frac{\mu_\alpha M}{2E}\hat{\mathbf{t}}^2\right) - H_2\left(\frac{\mu_\beta L}{2E}\hat{\mathbf{t}}^1\right)\\ &- H_2\left(\frac{\mu_\beta M}{2E}\hat{\mathbf{t}}^2\right) - H_1\left(\frac{\mu_\alpha E_\alpha}{4E^{3/2}}\hat{\mathbf{n}}\right) - H_2\left(\frac{\mu_\beta E_\alpha}{4E^{3/2}}\hat{\mathbf{n}}\right) + D_\alpha \mathcal{J}[\mathbf{X}]g.\end{aligned}$$

We continue to rewrite this; for now, we pull the vectors outside of the Riesz transforms, incurring commutators:

$$\begin{aligned}\mathbf{W}_\alpha = {}& H_1\left(\frac{\mu_{\alpha\alpha}}{2E^{1/2}}\right)\hat{\mathbf{n}} + H_2\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\right)\hat{\mathbf{n}} - H_1\left(\frac{\mu_\alpha L}{2E}\right)\hat{\mathbf{t}}^1 - H_1\left(\frac{\mu_\alpha M}{2E}\right)\hat{\mathbf{t}}^2 - H_2\left(\frac{\mu_\beta L}{2E}\right)\hat{\mathbf{t}}^1\\ &- H_2\left(\frac{\mu_\beta M}{2E}\right)\hat{\mathbf{t}}^2 - H_1\left(\frac{\mu_\alpha E_\alpha}{4E^{3/2}}\right)\hat{\mathbf{n}} - H_2\left(\frac{\mu_\beta E_\alpha}{4E^{3/2}}\right)\hat{\mathbf{n}} + \Xi_1 + D_\alpha \mathcal{J}[\mathbf{X}]g.\end{aligned}$$

The collection Ξ_1 is defined as

$$\begin{aligned}\Xi_1 = {}& [H_1,\hat{\mathbf{n}}]\left(\frac{\mu_{\alpha\alpha}}{2E^{1/2}}\right) + [H_2,\hat{\mathbf{n}}]\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\right) - [H_1,\hat{\mathbf{t}}^1]\left(\frac{\mu_\alpha L}{2E}\right) - [H_1,\hat{\mathbf{t}}^2]\left(\frac{\mu_\alpha M}{2E}\right)\\ &- [H_2,\hat{\mathbf{t}}^1]\left(\frac{\mu_\beta M}{2E}\right) - [H_2,\hat{\mathbf{t}}^2]\left(\frac{\mu_\beta N}{2E}\right) - [H_1,\hat{\mathbf{n}}]\left(\frac{\mu_\alpha E_\alpha}{4E^{3/2}}\right) - [H_2,\hat{\mathbf{n}}]\left(\frac{\mu_\beta E_\alpha}{4E^{3/2}}\right).\end{aligned} \tag{2.48}$$

Notice that Ξ_1 includes second derivatives of μ; we decompose it further such that the remainder only includes at most first order derivatives of μ. That is,

$$\Xi_1 = \tau[H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right) + \tau[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right) + R_1, \tag{2.49}$$

with

$$R_1 = [H_1,\hat{\mathbf{n}}]A_1 + [H_2,\hat{\mathbf{n}}]A_3 - [H_1,\hat{\mathbf{t}}^1]\left(\frac{\mu_\alpha L}{2E}\right) - [H_1,\hat{\mathbf{t}}^2]\left(\frac{\mu_\alpha M}{2E}\right)$$
$$- [H_2,\hat{\mathbf{t}}^1]\left(\frac{\mu_\beta M}{2E}\right) - [H_2,\hat{\mathbf{t}}^2]\left(\frac{\mu_\beta N}{2E}\right) - [H_1,\hat{\mathbf{n}}]\left(\frac{\mu_\alpha E_\alpha}{4E^{3/2}}\right) - [H_2,\hat{\mathbf{n}}]\left(\frac{\mu_\beta E_\alpha}{4E^{3/2}}\right).$$

Furthermore, notice that we can write $D_\alpha\mathcal{J}[\mathbf{X}]g$ as follows:

$$D_\alpha\mathcal{J}[\mathbf{X}]g = \mathcal{J}[\mathbf{X}]g_\alpha + \mathcal{L}_1[\mathbf{X}]g.$$

Similarly, we differentiate (2.46) with respect to β, making use of (2.8), finding the following formula for $\mathbf{W}_\beta$:

$$\mathbf{W}_\beta = H_1\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\hat{\mathbf{n}}\right) + H_2\left(\frac{\mu_{\beta\beta}}{2E^{1/2}}\hat{\mathbf{n}}\right) - H_1\left(\frac{\mu_\alpha M}{2E}\hat{\mathbf{t}}^1\right) - H_1\left(\frac{\mu_\alpha N}{2E}\hat{\mathbf{t}}^2\right) - H_2\left(\frac{\mu_\beta M}{2E}\hat{\mathbf{t}}^1\right)$$
$$- H_2\left(\frac{\mu_\beta N}{2E}\hat{\mathbf{t}}^2\right) - H_1\left(\frac{\mu_\alpha E_\beta}{4E^{3/2}}\hat{\mathbf{n}}\right) - H_2\left(\frac{\mu_\beta E_\beta}{4E^{3/2}}\hat{\mathbf{n}}\right) + D_\beta\mathcal{J}[\mathbf{X}]g.$$

As before, we rewrite this by pulling the vectors through the Riesz transform, yielding the following:

$$\mathbf{W}_\beta = H_1\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\right)\hat{\mathbf{n}} + H_2\left(\frac{\mu_{\beta\beta}}{2E^{1/2}}\right)\hat{\mathbf{n}} - H_1\left(\frac{\mu_\alpha M}{2E}\right)\hat{\mathbf{t}}^1 - H_1\left(\frac{\mu_\alpha N}{2E}\right)\hat{\mathbf{t}}^2 - H_2\left(\frac{\mu_\beta M}{2E}\right)\hat{\mathbf{t}}^1$$
$$- H_2\left(\frac{\mu_\beta N}{2E}\right)\hat{\mathbf{t}}^2 - H_1\left(\frac{\mu_\alpha E_\beta}{4E^{3/2}}\right)\hat{\mathbf{n}} - H_2\left(\frac{\mu_\beta E_\beta}{4E^{3/2}}\right)\hat{\mathbf{n}} + \Xi_2 + D_\beta\mathcal{J}[\mathbf{X}]g.$$

Of course, the collection Ξ_2 is defined as

$$\Xi_2 = [H_1,\hat{\mathbf{n}}]\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\right) + [H_2,\hat{\mathbf{n}}]\left(\frac{\mu_{\beta\beta}}{2E^{1/2}}\right) - [H_1,\hat{\mathbf{t}}^1]\left(\frac{\mu_\alpha M}{2E}\right) - [H_1,\hat{\mathbf{t}}^2]\left(\frac{\mu_\alpha N}{2E}\right)$$
$$- [H_2,\hat{\mathbf{t}}^1]\left(\frac{\mu_\beta M}{2E}\right) - [H_2,\hat{\mathbf{t}}^2]\left(\frac{\mu_\beta N}{2E}\right) - [H_1,\hat{\mathbf{n}}]\left(\frac{\mu_\alpha E_\beta}{4E^{3/2}}\right) - [H_2,\hat{\mathbf{n}}]\left(\frac{\mu_\beta E_\beta}{4E^{3/2}}\right). \tag{2.50}$$

We plug in the formulas for $\mu_{\alpha\beta}$ and $\mu_{\beta\beta}$, and this becomes

$$\Xi_2 = \tau[H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right) + \tau[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right) + R_2, \tag{2.51}$$

with

$$R_2 = [H_1, \hat{\mathbf{n}}]A_3 + [H_2, \hat{\mathbf{n}}]A_2 - [H_1, \hat{\mathbf{t}}^1]\left(\frac{\mu_\alpha M}{2E}\right) - [H_1, \hat{\mathbf{t}}^2]\left(\frac{\mu_\alpha N}{2E}\right) \\ - [H_2, \hat{\mathbf{t}}^1]\left(\frac{\mu_\beta M}{2E}\right) - [H_2, \hat{\mathbf{t}}^2]\left(\frac{\mu_\beta N}{2E}\right) - [H_1, \hat{\mathbf{n}}]\left(\frac{\mu_\alpha E_\beta}{4E^{3/2}}\right) - [H_2, \hat{\mathbf{n}}]\left(\frac{\mu_\beta E_\beta}{4E^{3/2}}\right).$$

Furthermore, as before, we can rewrite $D_\beta \mathcal{J}[\mathbf{X}]g$:

$$D_\beta \mathcal{J}[\mathbf{X}]g = \mathcal{J}[\mathbf{X}]g_\beta + \mathcal{L}_2[\mathbf{X}]g. \tag{2.52}$$

2.6. *Our small-scale decomposition*

As described in Section 1, we follow the philosophy of the numerical works [16], [17], and the extensions of these in analytical works such as [1], [2], [5], [6]. This requires making a so-called *small-scale decomposition*, in which we rewrite the evolution equations to isolate the most singular terms. In light of (2.29), in which the evolution of κ is given in terms of ΔU, we will now be making detailed calculations to find the leading-order part of ΔU. Our goal of this section is to arrive at (2.58) below.

First, we take inner product of $\mathbf{W}_\alpha$ with $\hat{\mathbf{t}}^1$ and $\mathbf{W}_\beta$ with $\hat{\mathbf{t}}^2$, finding

$$W_\alpha \cdot \hat{\mathbf{t}}^{\mathbf{1}} = -H_1\left(\frac{\mu_\alpha L}{2E}\right) - H_2\left(\frac{\mu_\beta L}{2E}\right) + \Xi_1 \cdot \hat{\mathbf{t}}^{\mathbf{1}} + D_\alpha \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{1}},$$
$$W_\beta \cdot \hat{\mathbf{t}}^{\mathbf{2}} = -H_1\left(\frac{\mu_\alpha N}{2E}\right) - H_2\left(\frac{\mu_\beta N}{2E}\right) + \Xi_2 \cdot \hat{\mathbf{t}}^{\mathbf{2}} + D_\beta \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{2}}.$$

So, it follows that

$$W_\alpha \cdot \hat{\mathbf{t}}^{\mathbf{1}} + W_\beta \cdot \hat{\mathbf{t}}^{\mathbf{2}} = -H_1(\kappa\mu_\alpha) - H_2(\kappa\mu_\beta) + \Xi_1 \cdot \hat{\mathbf{t}}^{\mathbf{1}} + D_\alpha \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{1}} + \Xi_2 \cdot \hat{\mathbf{t}}^{\mathbf{2}} + D_\beta \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{2}}. \tag{2.53}$$

We use this expression in (2.40), finding the following:

$$\mu_{\alpha\alpha} + \mu_{\beta\beta} = \tau\kappa_{\alpha\alpha} + \tau\kappa_{\beta\beta} - 4A_\nu U E\kappa - 2RE\kappa n_3 \\ - 2A_\nu\sqrt{E}\left(-H_1(\kappa\mu_\alpha) - H_2(\kappa\mu_\beta) + \Xi_1 \cdot \hat{\mathbf{t}}^{\mathbf{1}} + D_\alpha \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{1}} + \Xi_2 \cdot \hat{\mathbf{t}}^{\mathbf{2}} + D_\beta \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{2}}\right). \tag{2.54}$$

We also take the inner product with $\hat{\mathbf{n}}$:

$$W_\alpha \cdot \hat{\mathbf{n}} = H_1\left(\frac{\mu_{\alpha\alpha}}{2E^{1/2}}\right) + H_2\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\right) - H_1\left(\frac{\mu_\alpha E_\alpha}{4E^{3/2}}\right) - H_2\left(\frac{\mu_\beta E_\alpha}{4E^{3/2}}\right) + \Xi_1 \cdot \hat{\mathbf{n}} + D_\alpha \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{n}}$$
$$= H_1\left(\frac{\mu_{\alpha\alpha}}{2E^{1/2}}\right) + H_1\left(\frac{\mu_{\beta\beta}}{2E^{1/2}}\right) - H_1\left(\frac{\mu_\alpha E_\alpha}{4E^{3/2}}\right) - H_1\left(\frac{\mu_\beta E_\beta}{4E^{3/2}}\right) + \Xi_1 \cdot \hat{\mathbf{n}} + D_\alpha \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{n}},$$

and

$$W_\beta\cdot\hat{\mathbf{n}}=H_1\left(\frac{\mu_{\alpha\beta}}{2E^{1/2}}\right)+H_2\left(\frac{\mu_{\beta\beta}}{2E^{1/2}}\right)-H_1\left(\frac{\mu_\alpha E_\beta}{4E^{3/2}}\right)-H_2\left(\frac{\mu_\beta E_\beta}{4E^{3/2}}\right)+\Xi_2\cdot\hat{\mathbf{n}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}$$
$$=H_2\left(\frac{\mu_{\alpha\alpha}}{2E^{1/2}}\right)+H_2\left(\frac{\mu_{\beta\beta}}{2E^{1/2}}\right)-H_2\left(\frac{\mu_\alpha E_\alpha}{4E^{3/2}}\right)-H_2\left(\frac{\mu_\beta E_\beta}{4E^{3/2}}\right)+\Xi_2\cdot\hat{\mathbf{n}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}.$$

Using the facts that $\Lambda=H_1\partial_\alpha+H_2\partial_\beta$ and $H_1\partial_\beta=H_2\partial_\alpha$, we have

$$(W_\alpha\cdot\hat{\mathbf{n}})_\alpha+(W_\beta\cdot\hat{\mathbf{n}})_\beta=\Lambda\left(\frac{\Delta\mu}{2E^{1/2}}\right)-\Lambda\left(\frac{\mu_\alpha E_\alpha+\mu_\beta E_\beta}{4E^{3/2}}\right)$$
$$+\left(\Xi_1\cdot\hat{\mathbf{n}}\right)_\alpha+\left(\mathcal{J}[\mathbf{X}]g_\alpha\cdot\hat{\mathbf{n}}\right)_\alpha+\left(\Xi_2\cdot\hat{\mathbf{n}}\right)_\beta+\left(\mathcal{J}[\mathbf{X}]g_\beta\cdot\hat{\mathbf{n}}\right)_\beta+\left(\mathcal{L}_1[\mathbf{X}]g\cdot\hat{\mathbf{n}}\right)_\alpha+\left(\mathcal{L}_2[\mathbf{X}]g\cdot\hat{\mathbf{n}}\right)_\beta. \tag{2.55}$$

We write the partial derivatives of g as

$$g_\alpha=\tau\kappa_{\alpha\beta}\mathbf{X}_\alpha-\tau\kappa_{\alpha\alpha}\mathbf{X}_\beta+g_1,\quad g_\beta=\tau\kappa_{\beta\beta}\mathbf{X}_\alpha-\tau\kappa_{\alpha\beta}\mathbf{X}_\beta+g_2,$$

with

$$g_1=A_3\mathbf{X}_\alpha-A_1\mathbf{X}_\beta+\mu_\beta\mathbf{X}_{\alpha\alpha}-\mu_\alpha\mathbf{X}_{\alpha\beta},\quad g_2=A_2\mathbf{X}_\alpha-A_3\mathbf{X}_\beta+\mu_\beta\mathbf{X}_{\alpha\beta}-\mu_\alpha\mathbf{X}_{\beta\beta}.$$

Using the facts $\Lambda H_1=-\partial_\alpha$ and $\Lambda H_2=-\partial_\beta$, the first term on the right-hand side of (2.55) is the following:

$$\Lambda\left(\frac{\Delta\mu}{2E^{1/2}}\right)=\tau\Lambda\left(\frac{\Delta\kappa}{2E^{1/2}}\right)-\Lambda\left(2A_\nu U\sqrt{E}\kappa+R\sqrt{E}\kappa n_3\right)-A_\nu\kappa\Delta\mu-A_\nu(\kappa_\alpha\mu_\alpha+\kappa_\beta\mu_\beta)$$
$$-A_\nu\Lambda\left(\Xi_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{J}[\mathbf{X}]g_\alpha\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\Xi_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+\mathcal{J}[\mathbf{X}]g_\beta\cdot\hat{\mathbf{t}}^{\mathbf{2}}+\mathcal{L}_1[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{L}_2[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right). \tag{2.56}$$

We rewrite the second term on the right-hand side of (2.56) as

$$-\Lambda\left(2A_\nu U\sqrt{E}\kappa+R\sqrt{E}\kappa n_3\right)$$
$$=-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa-2A_\nu\sqrt{E}\kappa\Lambda U$$
$$-\left(\Lambda\left(2A_\nu U\sqrt{E}\kappa+R\sqrt{E}\kappa n_3\right)-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa-2A_\nu\sqrt{E}\kappa\Lambda U\right).$$

Using the fact that $(H_1^2+H_2^2)f=-f$, we have

$$-2A_\nu\kappa\sqrt{E}\Lambda U=-2A_\nu\kappa\sqrt{E}\left(H_1\left(\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}\right)+H_2\left(\mathbf{W}_\beta\cdot\hat{\mathbf{n}}\right)+H_1\left(\mathbf{W}\cdot\hat{\mathbf{n}}_\alpha\right)+H_2\left(\mathbf{W}\cdot\hat{\mathbf{n}}_\beta\right)\right)$$
$$=A_\nu\kappa\Delta\mu-\frac{A_\nu\kappa\mu_\alpha E_\alpha}{2E}-\frac{A_\nu\kappa\mu_\beta E_\beta}{2E}-2A_\nu\kappa\sqrt{E}H_1\left(\Xi_1\cdot\hat{\mathbf{n}}+D_\alpha\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}+\mathbf{W}\cdot\hat{\mathbf{n}}_\alpha\right)$$
$$-2A_\nu\kappa\sqrt{E}H_2\left(\Xi_2\cdot\hat{\mathbf{n}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}+\mathbf{W}\cdot\hat{\mathbf{n}}_\beta\right).$$

Using this with (2.56) yields the following:

$$\Lambda\left(\frac{\Delta\mu}{2E^{1/2}}\right)=\tau\Lambda\left(\frac{\Delta\kappa}{2E^{1/2}}\right)-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa-A_\nu(\kappa_\alpha\mu_\alpha+\kappa_\beta\mu_\beta)$$
$$-A_\nu\Lambda\left(\Xi_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{J}[\mathbf{X}]g_\alpha\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\Xi_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+\mathcal{J}[\mathbf{X}]g_\beta\cdot\hat{\mathbf{t}}^{\mathbf{2}}+\mathcal{L}_1[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{L}_2[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)$$
$$-\left(\Lambda\left(2A_\nu U\sqrt{E}\kappa+R\sqrt{E}\kappa n_3\right)-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa-2A_\nu\sqrt{E}\kappa\Lambda U\right)$$
$$-\frac{A_\nu\kappa\mu_\alpha E_\alpha}{2E}-\frac{A_\nu\kappa\mu_\beta E_\beta}{2E}-2A_\nu\kappa\sqrt{E}H_1\left(\Xi_1\cdot\hat{\mathbf{n}}+D_\alpha\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}+\mathbf{W}\cdot\hat{\mathbf{n}}_\alpha\right)$$
$$-2A_\nu\kappa\sqrt{E}H_2\left(\Xi_2\cdot\hat{\mathbf{n}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}+\mathbf{W}\cdot\hat{\mathbf{n}}_\beta\right).$$

Plugging in (2.49) and (2.51) for the tangential parts of Ξ_1 and Ξ_2 above yields the following:

$$\Lambda\left(\frac{\Delta\mu}{2E^{1/2}}\right)=\tau\Lambda\left(\frac{\Delta\kappa}{2E^{1/2}}\right)-\tau A_\nu\Lambda\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)$$
$$-\tau A_\nu\Lambda\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa$$
$$-A_\nu(\kappa_\alpha\mu_\alpha+\kappa_\beta\mu_\beta)-\tau A_\nu\Lambda\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha-\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha-\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)$$
$$-A_\nu\Lambda\left(R_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{J}[\mathbf{X}]g_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+R_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+\mathcal{J}[\mathbf{X}]g_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+\mathcal{L}_1[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{L}_2[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)$$
$$-\left(\Lambda\left(2A_\nu U\sqrt{E}\kappa+R\sqrt{E}\kappa n_3\right)-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa-2A_\nu\sqrt{E}\kappa\Lambda U\right)$$
$$-\frac{A_\nu\kappa\mu_\alpha E_\alpha}{2E}-\frac{A_\nu\kappa\mu_\beta E_\beta}{2E}-2A_\nu\kappa\sqrt{E}H_1\left(\Xi_1\cdot\hat{\mathbf{n}}+D_\alpha\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}+\mathbf{W}\cdot\hat{\mathbf{n}}_\alpha\right)$$
$$-2A_\nu\kappa\sqrt{E}H_2\left(\Xi_2\cdot\hat{\mathbf{n}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{n}}+\mathbf{W}\cdot\hat{\mathbf{n}}_\beta\right).$$

Using the fact $\Lambda\partial_\alpha=H_1\Delta$ and $\Lambda\partial_\beta=H_2\Delta$, the second term on the right-hand side of (2.55) is

$$-\Lambda\left(\frac{\mu_\alpha E_\alpha+\mu_\beta E_\beta}{4E^{3/2}}\right)=-\frac{E_\alpha}{4E^{3/2}}H_1\Delta\mu-\frac{E_\beta}{4E^{3/2}}H_2\Delta\mu-\left[\Lambda,\frac{E_\alpha}{4E^{3/2}}\right]\mu_\alpha-\left[\Lambda,\frac{E_\beta}{4E^{3/2}}\right]\mu_\beta.$$

Using the equation (2.40), we have

$$-\Lambda\left(\frac{\mu_\alpha E_\alpha+\mu_\beta E_\beta}{4E^{3/2}}\right)$$
$$=-\frac{E_\alpha}{4E^{3/2}}\tau H_1\Delta\kappa-\frac{E_\beta}{4E^{3/2}}\tau H_2\Delta\kappa-\left[\Lambda,\frac{E_\alpha}{4E^{3/2}}\right]\mu_\alpha-\left[\Lambda,\frac{E_\beta}{4E^{3/2}}\right]\mu_\beta$$
$$-\left(\frac{E_\alpha}{4E^{3/2}}H_1+\frac{E_\beta}{4E^{3/2}}H_2\right)\left(-4A_\nu UE\kappa-2RE\kappa n_3-2A_\nu\sqrt{E}\left(\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}^1+\mathbf{W}_\beta\cdot\hat{\mathbf{t}}^2\right)\right).$$

We are ready to conclude that

$$(W_\alpha \cdot \hat{\mathbf{n}})_\alpha + (W_\beta \cdot \hat{\mathbf{n}})_\beta = \tau\Lambda\left(\frac{\Delta\kappa}{2E^{1/2}}\right) - \tau A_\nu \Lambda\left([H_1, \hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}} + [H_2, \hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}}\right)$$
$$- \tau A_\nu \Lambda\left([H_1, \hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}} + [H_2, \hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) - \frac{E_\alpha}{4E^{3/2}}\tau H_1 \Delta\kappa - \frac{E_\beta}{4E^{3/2}}\tau H_2 \Delta\kappa$$
$$- \left(2A_\nu U\sqrt{E} + R\sqrt{E}n_3\right)\Lambda\kappa - A_\nu(\kappa_\alpha\mu_\alpha + \kappa_\beta\mu_\beta)$$
$$- \tau A_\nu \Lambda\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha - \kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}} + \mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha - \kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}}\right)$$
$$+ \tau\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha - \kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha + \tau\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha - \kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) + R_3,$$

where R_3 is given by

$$R_3 = -A_\nu\Lambda\left(R_1 \cdot \hat{\mathbf{t}}^{\mathbf{1}} + \mathcal{J}[\mathbf{X}]g_1 \cdot \hat{\mathbf{t}}^{\mathbf{1}} + R_2 \cdot \hat{\mathbf{t}}^{\mathbf{2}} + \mathcal{J}[\mathbf{X}]g_2 \cdot \hat{\mathbf{t}}^{\mathbf{2}} + \mathcal{L}_1[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{1}} + \mathcal{L}_2[\mathbf{X}]g \cdot \hat{\mathbf{t}}^{\mathbf{2}}\right)$$
$$- \left(\Lambda\left(2A_\nu U\sqrt{E}\kappa + R\sqrt{E}\kappa n_3\right) - \left(2A_\nu U\sqrt{E} + R\sqrt{E}n_3\right)\Lambda\kappa - 2A_\nu\sqrt{E}\kappa\Lambda U\right)$$
$$- \frac{A_\nu\kappa\mu_\alpha E_\alpha}{2E} - \frac{A_\nu\kappa\mu_\beta E_\beta}{2E} - 2A_\nu\kappa\sqrt{E}H_1\left(\Xi_1 \cdot \hat{\mathbf{n}} + D_\alpha\mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{n}} + \mathbf{W}\cdot\hat{\mathbf{n}}_\alpha\right)$$
$$- 2A_\nu\kappa\sqrt{E}H_2\left(\Xi_2 \cdot \hat{\mathbf{n}} + D_\beta\mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{n}} + \mathbf{W}\cdot\hat{\mathbf{n}}_\beta\right) - \left[\Lambda, \frac{E_\alpha}{4E^{3/2}}\right]\mu_\alpha - \left[\Lambda, \frac{E_\beta}{4E^{3/2}}\right]\mu_\beta$$
$$- \left(\frac{E_\alpha}{4E^{3/2}}H_1 + \frac{E_\beta}{4E^{3/2}}H_2\right)\left(-4A_\nu UE\kappa - 2RE\kappa n_3 - 2A_\nu\sqrt{E}\left(\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}^1 + \mathbf{W}_\beta\cdot\hat{\mathbf{t}}^2\right)\right)$$
$$+ \left(\Xi_1\cdot\hat{\mathbf{n}}\right)_\alpha + \left(\mathcal{J}[\mathbf{X}]g_1\cdot\hat{\mathbf{n}}\right)_\alpha + \left(\Xi_2\cdot\hat{\mathbf{n}}\right)_\beta + \left(\mathcal{J}[\mathbf{X}]g_2\cdot\hat{\mathbf{n}}\right)_\beta + \left(\mathcal{L}_1[\mathbf{X}]g\cdot\hat{\mathbf{n}}\right)_\alpha + \left(\mathcal{L}_2[\mathbf{X}]g\cdot\hat{\mathbf{n}}\right)_\beta.$$

Recalling the expansion of ΔU in (2.37), we rewrite the final two terms on the right-hand side:

$$\left((\mathbf{W}\cdot\hat{\mathbf{t}}^1)\left(-\frac{L}{E^{1/2}}\right) + (\mathbf{W}\cdot\hat{\mathbf{t}}^2)\left(-\frac{M}{E^{1/2}}\right)\right)_\alpha$$
$$= -L\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\alpha - L_\alpha\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right) - M\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\alpha - M_\alpha\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right),$$
$$\left((\mathbf{W}\cdot\hat{\mathbf{t}}^1)\left(-\frac{M}{E^{1/2}}\right) + (\mathbf{W}\cdot\hat{\mathbf{t}}^2)\left(-\frac{N}{E^{1/2}}\right)\right)_\beta$$
$$= -M\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\beta - M_\beta\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right) - N\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\beta - N_\beta\left(\frac{(\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right).$$

We calculate as follows:

$$\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\alpha + \left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\beta = -\frac{H_1(\kappa\mu_\alpha)}{\sqrt{E}} - \frac{H_2(\kappa\mu_\beta)}{\sqrt{E}} + \frac{2UM}{E}$$

$$+\frac{\Xi_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+D_\alpha\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\Xi_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{2}}}{\sqrt{E}},$$

since by the geometric identities (2.10) and (2.11), we have

$$\mathbf{W}\cdot\left(\frac{\hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\alpha+\mathbf{W}\cdot\left(\frac{\hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\beta=\frac{2UM}{E}.$$

Then using the further identities

$$M_\alpha=L_\beta-E_\beta\kappa,\qquad M_\beta=N_\alpha-E_\alpha\kappa,$$
$$L_\alpha+N_\alpha=2\kappa_\alpha E+2E_\alpha\kappa,\qquad L_\beta+N_\beta=2\kappa_\beta E+2E_\beta\kappa,$$

we may complete our expansion of ΔU:

$$\begin{aligned}\Delta U&=\tau\Lambda\left(\frac{\Delta\kappa}{2E^{1/2}}\right)-\tau A_\nu\Lambda\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)\\
&-\tau A_\nu\Lambda\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)-\frac{E_\alpha}{4E^{3/2}}\tau H_1\Delta\kappa-\frac{E_\beta}{4E^{3/2}}\tau H_2\Delta\kappa\\
&-\tau A_\nu\Lambda\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha-\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha-\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)\\
&-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa-A_\nu(\kappa_\alpha\mu_\alpha+\kappa_\beta\mu_\beta)\\
&+\tau\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha-\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha+\tau\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha-\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)+R_3\\
&-M\left(-\frac{H_1\left(\kappa\mu_\alpha\right)}{\sqrt{E}}-\frac{H_2\left(\kappa\mu_\beta\right)}{\sqrt{E}}+\frac{2UM}{E}+\frac{\Xi_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+D_\alpha\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\Xi_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{2}}}{\sqrt{E}}\right)\\
&-L\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\alpha-N\left(\frac{(\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\beta-\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)(2\kappa_\alpha E+E_\alpha\kappa)-\left(\frac{\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)\left(2\kappa_\beta E+E_\beta\kappa\right).\end{aligned}\tag{2.57}$$

Recalling that we are working toward an expression for the evolution of the mean curvature, κ, we plug all these expressions into (2.29), while also rewriting $(\sqrt{E}\kappa)_\alpha=\sqrt{E}\kappa_\alpha-\kappa E_\alpha/2\sqrt{E}$. These considerations yield the following:

$$\begin{aligned}\frac{(\sqrt{E}\kappa)_t}{\sqrt{E}}&=\tau\frac{\Lambda}{2E}\left(\frac{\Delta\kappa}{2E^{1/2}}\right)-\tau A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)\\
&-\tau A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)-\frac{E_\alpha}{8E^{5/2}}\tau H_1\Delta\kappa-\frac{E_\beta}{8E^{5/2}}\tau H_2\Delta\kappa\\
&-\tau A_\nu\frac{\Lambda}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha-\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha-\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)-\left(\frac{2A_\nu U+Rn_3}{2\sqrt{E}}\right)\Lambda\kappa\end{aligned}$$

$$
\begin{aligned}
&-\frac{A_\nu(\kappa_\alpha\mu_\alpha+\kappa_\beta\mu_\beta)}{2E}+\frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha-\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha\\
&+\frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha-\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)+\frac{R_3}{2E}+\frac{V_1-\mathbf{W}\cdot\hat{\mathbf{t}}^1}{2E^{3/2}}(\kappa E_\alpha)+\left(\frac{V_1-\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)\kappa_\alpha\\
&+\frac{V_2-\mathbf{W}\cdot\hat{\mathbf{t}}^2}{2E^{3/2}}(\kappa E_\beta)+\left(\frac{V_2-\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)\kappa_\beta+\frac{UM^2}{E}+\frac{MH_1(\kappa\mu_\alpha)}{2E}+\frac{MH_2\left(\kappa\mu_\beta\right)}{2E}-\frac{2UM^2}{E^2}\\
&-\frac{\Xi_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+D_\alpha\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\Xi_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{2}}}{2E^{3/2}}M\\
&+\frac{L}{2E}\left(\frac{V_1-\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\alpha+\frac{N}{2E}\left(\frac{V_2-\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\beta.
\end{aligned}
$$

Since $\kappa_t=(\sqrt{E}\kappa)_t/\sqrt{E}+E_t\kappa/2E$, using the evolution equation (2.30), we conclude that the evolution of κ is given by the following:

$$
\begin{aligned}
\kappa_t=&\ \tau\frac{\Lambda}{2E}\left(\frac{\Delta\kappa}{2E^{1/2}}\right)-\tau A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)\\
&-\tau A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}+[H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)-\frac{E_\alpha}{8E^{5/2}}\tau H_1\Delta\kappa-\frac{E_\beta}{8E^{5/2}}\tau H_2\Delta\kappa\\
&-\tau A_\nu\frac{\Lambda}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha-\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha-\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)\\
&+\frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha-\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha+\frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha-\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)+Q_1+Q_2,
\end{aligned}
\tag{2.58}
$$

where Q_1 and Q_2 are the following:

$$
Q_1=-\left(\frac{2A_\nu U+Rn_3}{2\sqrt{E}}\right)\Lambda\kappa+\left(\frac{V_1-\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}-\frac{A_\nu\mu_\alpha}{2E}\right)\kappa_\alpha+\left(\frac{V_2-\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}-\frac{A_\nu\mu_\beta}{2E}\right)\kappa_\beta,
\tag{2.59}
$$

$$
\begin{aligned}
Q_2=&\ \frac{R_3}{2E}+\frac{V_1-\mathbf{W}\cdot\hat{\mathbf{t}}^1}{2E^{3/2}}(\kappa E_\alpha)+\frac{V_2-\mathbf{W}\cdot\hat{\mathbf{t}}^2}{2E^{3/2}}(\kappa E_\beta)++\frac{UM^2}{E}+\frac{MH_1(\kappa\mu_\alpha)}{2E}+\frac{MH_2\left(\kappa\mu_\beta\right)}{2E}\\
&-\frac{2UM^2}{E^2}-\frac{\Xi_1\cdot\hat{\mathbf{t}}^{\mathbf{1}}+D_\alpha\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{1}}+\Xi_2\cdot\hat{\mathbf{t}}^{\mathbf{2}}+D_\beta\mathcal{J}[\mathbf{X}]g\cdot\hat{\mathbf{t}}^{\mathbf{2}}}{2E^{3/2}}M\\
&+\frac{L}{2E}\left(\frac{V_1-\mathbf{W}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\alpha+\frac{N}{2E}\left(\frac{V_2-\mathbf{W}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\beta+\kappa\left(\frac{V_{1,\alpha}}{\sqrt{E}}-\frac{UL}{E}+\frac{V_2E_\beta}{2E^{3/2}}\right).
\end{aligned}
\tag{2.60}
$$

3. Preliminary estimates and useful formulas

In this section we state a number of useful lemmas. Some of these lemmas are given without proof if they have appeared clearly in other works or are standard analysis facts. For others of these lemmas, we do provide a proof. To begin with, Lemmas 3.1, 3.2, 3.3, 3.4, and 3.5 all appear in Section 6 of [5], and thus we omit the proofs of these. In this section, we will need the definitions of the Riesz transforms H_i and the weakly singular integral operators G_{ij}, which are given above in (2.32), (2.33), (2.34), (2.35), and (2.36), many times. Before giving our first estimates, we make a remark on regularity.

Remark 3. Throughout the sequel we will be estimating the unknowns and quantities related to the unknowns in Sobolev spaces H^s and related spaces (such as H^{s-1} and so on). In all instances, the index s should be understood to be taken *sufficiently large*. This means that there exists an absolute constant $s_0 \in \mathbb{N}$ such that as long as s is taken to satisfy $s \geq s_0$, all of our estimates go through. We do not count the minimum such value of s_0.

Lemma 3.1. *Let $\mathcal{F}$ be in $H^{s-3/2}$, $\mathbf{X} \in H^{s+1}$. Let $\mathbf{X}$ satisfy (2.27). Recall the definition of the operator $\mathcal{K}[\mathbf{X}]$ given in (2.41) and (2.43); then $\mathcal{K}[\mathbf{X}]\mathcal{F}(\alpha, \beta)$ is in H^s with*

$$\|\mathcal{K}[\mathbf{X}]\mathcal{F}\|_s \leq C(1 + \|\mathbf{X}\|_{s+1})^2 \|\mathcal{F}\|_{s-3/2}.$$

Lemma 3.2. *If $f \in H^{s+1}$ and $g \in H^{s-1}$, then $[G_{ij}, f]g$ is in H^{s+1}, with the estimate*

$$\|[G_{ij}, f]g\|_{s+1} \leq \|f\|_{s+1}\|g\|_{s-1}.$$

Lemma 3.3. *Let $\mathcal{F}$ be in $H^{s-1/2}$, $\mathbf{X} \in H^{s+1}$. Let $\mathbf{X}$ satisfy (2.27). For $i \in \{1, 2\}$, recall the definitions of the operators $\mathcal{L}_i[\mathbf{X}]$ given in (2.42) and (2.44), (2.45); then for $i \in \{1, 2\}$, $\mathcal{L}_i[\mathbf{X}]\mathcal{F}(\alpha, \beta)$ is in H^s with*

$$\|\mathcal{L}_i[\mathbf{X}]\mathcal{F}\|_s \leq C(1 + \|\mathbf{X}\|_{s+1})^2 \|\mathcal{F}\|_{s-1/2}, \qquad i = 1, 2.$$

Lemma 3.4. *If $f \in H^{s+1}$ and $g \in H^s$, then $[f, H_i]g$ is in H^{s+1}, with the estimate*

$$\|[f, H_i]g\|_{s+1} \leq \|f\|_{s+1}\|g\|_s.$$

Lemma 3.5. *If $\hat{\mathbf{n}} \in H^s$ and $g \in H^{s-2}$, then $[\hat{\mathbf{n}}, H_i]g \cdot \hat{\mathbf{n}}$ is in H^s, with the estimate*

$$\|[\hat{\mathbf{n}}, H_i]g \cdot \hat{\mathbf{n}}\|_s \leq \|\hat{\mathbf{n}}\|_s \|g\|_{s-2}.$$

We next give a related commutator estimate, but the exact form of this did not appear in [5], and thus we include a short proof.

Lemma 3.6. *Let $s > d/2$. If $f \in H^{s+1}$ and $g \in H^s$, then $[\Lambda, f]g$ is in H^s, with the estimate*

$$\|[\Lambda, f]g\|_s \leq \|f\|_{s+1}\|g\|_s.$$

Proof. Notice that

$$[\Lambda, f]g = \Lambda(fg) - f\Lambda g = -[f, H_1]g_\alpha - [f, H_2]g_\beta + H_1(gf_\alpha) + H_2(gf_\beta). \tag{3.1}$$

By Lemma 3.4, we have

$$\| -[f, H_1]g_\alpha - [f, H_2]g_\beta\|_s \leq \|f\|_{s+1}\|g\|_s.$$

When $s > d/2$, we know the Sobolev space H^s is algebraic, that is

$$\|H_1(gf_\alpha)\|_s \leq \|f_\alpha\|_s\|g\|_s \leq \|f\|_{s+1}\|g\|_s.$$

This completes the proof of the lemma. □

Next, Lemmas 3.7, 3.8, 3.9, and 3.10 express standard Sobolev estimates and are noted without further proof.

Lemma 3.7. *If $f \in H^s$ and $g \in H^s$, then $\Lambda(fg) - f\Lambda g - g\Lambda f$ is in H^s, with the estimate*

$$\|\Lambda(fg) - f\Lambda g - g\Lambda f\|_s \leq \|f\|_s\|g\|_s.$$

Lemma 3.8. *For $s > 0$, then*

$$\|[\Lambda^s, f]g\|_0 \leq C\left(\|\nabla f\|_{L^\infty}\|g\|_{s-1} + \|f\|_s\|g\|_{L^\infty}\right). \tag{3.2}$$

Lemma 3.9. *For $s \geq 0$, $f, g \in H^s$ then*

$$\|fg\|_s \leq c(\|f\|_0^{1/2}\|f\|_d^{1/2}\|g\|_s + \|f\|_s\|g\|_0^{1/2}\|g\|_d^{1/2})$$

Lemma 3.10. *For $0 < m < s$, and $f \in H^s$, then*

$$\|\Lambda^m f\|_0 \leq \|\Lambda^s f\|^{m/s}\|f\|^{1-m/s} \tag{3.3}$$

Next we have a lemma about the isothermal parameterization, which is related to the Gauss equation and Gauss's *Theorema egregium*.

Lemma 3.11. *If $\mathbf{X} \in H^{s+1}$ then $E - 1$ is also in H^{s+1}.*

Proof. By the equation (2.31), we have

$$\Delta E = 2(\mathbf{X}_{\alpha\beta} \cdot \mathbf{X}_{\alpha\beta}) - 2(\mathbf{X}_{\alpha\alpha} \cdot \mathbf{X}_{\beta\beta}). \tag{3.4}$$

The right-hand side of (2.31) is in $H^{s-1} \bigcap L^1$ when $\mathbf{X} \in H^{s+1}$. This completes the proof. □

Next we have a lemma on the solvability of our integral equations. We do not include the proof as the solvability is well-established in the related works [14], [30].

Lemma 3.12. *If $\mathbf{X} \in H^{s+1}$, $\kappa \in H^s$, then μ is well-defined and belongs to H^0.*

Then, having shown that μ exists and is in L^2, it is possible to establish higher regularity.

Lemma 3.13. *If $\mathbf{X} \in H^{s+1}$, $E \in H^{s+1}$ and $\kappa \in H^s$, then μ is well-defined and belongs to H^s. Furthermore, we have $\mathbf{W} \cdot \hat{\mathbf{t}}^i \in H^s$.*

We also do not prove this lemma here, as it is similar to Lemma 7 of [2]; see also the discussion in [14] as to why this invertibility is uniform for surfaces $\mathbf{X}$ with bounded Sobolev norm.

Next we give a lemma to establish the regularity of the normal velocity, in terms of the regularity of the surface.

Lemma 3.14. *If $\mathbf{X} \in H^{s+1}$, $E \in H^{s+1}$ and $\kappa \in H^s$, then $U = \mathbf{W} \cdot \hat{\mathbf{n}} \in H^{s-1}$.*

Proof. We sketch the proof. Recall the equation (2.46); then we have

$$U = \mathbf{W} \cdot \hat{\mathbf{n}} = H_1\left(\frac{\mu_\alpha}{2\sqrt{E}}\right) + H_2\left(\frac{\mu_\beta}{2\sqrt{E}}\right) + \mathcal{J}[\mathbf{X}]g \cdot \hat{\mathbf{n}} + \left[H_1, \hat{\mathbf{n}}\right]\left(\frac{\mu_\alpha}{2\sqrt{E}}\right) \cdot \hat{\mathbf{n}} + \left[H_2, \hat{\mathbf{n}}\right]\left(\frac{\mu_\beta}{2\sqrt{E}}\right) \cdot \hat{\mathbf{n}},$$

where

$$\mathcal{J}[\mathbf{X}]g = \mathcal{K}[\mathbf{X}]g + G_{11}\left(\frac{g \times \mathbf{X}_{\alpha\alpha}}{2E^{3/2}} - \frac{3(g \times \mathbf{X}_\alpha)E_\alpha}{4E^{5/2}}\right) + G_{12}\left(\frac{g \times \mathbf{X}_{\alpha\beta}}{2E^{3/2}} - \frac{3(g \times \mathbf{X}_\alpha)E_\beta + 3(g \times \mathbf{X}_\beta)E_\alpha}{4E^{5/2}}\right) + G_{22}\left(\frac{g \times \mathbf{X}_{\beta\beta}}{2E^{3/2}} - \frac{3(g \times \mathbf{X}_\beta)E_\beta}{4E^{5/2}}\right).$$

By Lemma 3.1, we have

$$\|\mathcal{K}[\mathbf{X}]g\|_s \le C(1 + \|\mathbf{X}\|_{s+1})^2 \|g\|_{s-3/2} \le C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}).$$

The operators G_{ij} are of order -1; therefore

$$\left\| G_{11}\left(\frac{g \times \mathbf{X}_{\alpha\alpha}}{2E^{3/2}} - \frac{3(g \times \mathbf{X}_\alpha)E_\alpha}{4E^{5/2}}\right)\right\|_s \le c\left\|\frac{g \times \mathbf{X}_{\alpha\alpha}}{2E^{3/2}} - \frac{3(g \times \mathbf{X}_\alpha)E_\alpha}{4E^{5/2}}\right\|_{s-1} \le C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}, \|E\|_{s+1}).$$

It is similar to estimate terms involving G_{12} and G_{22}. We have established

$$\|\mathcal{J}[\mathbf{X}]g\|_s \le C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}, \|E\|_{s+1}).$$

Applying Lemma 3.5 and the estimate on μ completes the proof of the lemma. □

Remark 4. If we substitute g to f and assume that $f \in H^s$, and we calculate that $D_\alpha \mathcal{J}[\mathbf{X}]f = \mathcal{J}[\mathbf{X}]D_\alpha f + \mathcal{L}_1[\mathbf{X}]f$ and apply Lemma 3.3, then we see that $\mathcal{J}[\mathbf{X}]f \in H^{s+1}$.

Next we have a lemma on the tangential velocities. Notice that $L, M, N \in H^{s-1}$ if $X \in H^{s+1}$.

Lemma 3.15. *If $\mathbf{X} \in H^{s+1}$, $E \in H^{s+1}$ and $\kappa \in H^s$, then $V_i \in H^s$.*

Proof. We apply ∂_α to both sides of equation (2.4) and we apply ∂_β to both sides of equation (2.5), and then add those equations. It follows that

$$\Delta\left(\frac{V_1}{\sqrt{E}}\right)=\left(\frac{U(L-N)}{E}\right)_\alpha+\left(\frac{2UM}{E}\right)_\beta. \tag{3.5}$$

By Lemma 3.14 we know that $U\in H^{s-1}$, and thus the right-hand side of (3.5) is in H^{s-2}. We therefore have $V_1\in H^s$. Similarly, we have $V_2\in H^s$ since

$$\Delta\left(\frac{V_2}{\sqrt{E}}\right)=-\left(\frac{U(L-N)}{E}\right)_\beta+\left(\frac{2UM}{E}\right)_\alpha.$$

This completes the proof of the lemma. $\square$

We are now in a position to make estimates for a large number of related quantities.

Lemma 3.16. *If* $\mathbf{X}\in H^{s+1}$, $E\in H^{s+1}$ *and* $\kappa\in H^s$, *then we have the following estimates*

$$\begin{aligned}
\|Q_1\|_{s-1}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}),\\
\|A_i\|_{s-1}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}),\quad i=1,2,3,\\
\|g_i\|_{s-1}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1}),\quad i=1,2,\\
\|R_i\|_{s}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}),\quad i=1,2,\\
\|\Xi_i\|_{s-1}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}),\quad i=1,2, && (3.6)\\
\|\Xi_i\cdot\hat{\mathbf{n}}\|_{s}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}),\quad i=1,2, && (3.7)\\
\|R_3\|_{s-3/2}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}),\\
\|Q_2\|_{s-3/2}&\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}).
\end{aligned}$$

Proof. The estimate for Q_1 is based on the regularity of L, N, U, V_i, $\mathbf{W}\cdot\hat{\mathbf{t}}^i$ and μ. The estimate for $A_i\in H^{s-1}$ directly follows from the regularity of L, N, U, E and $\mathbf{W}\cdot\hat{\mathbf{t}}^i$. The estimate for g_i is based on the estimates of A_i and μ. The estimate for $R_i\in H^s$ for $i=1,2$ follows from Lemma 3.4 with $f\in H^s$ and $g\in H^{s-1}$. It also implies $\Xi_i\in H^s$ by Lemma 3.4. Based on the estimates on R_1 and R_2, to establish $\Xi_i\cdot\hat{\mathbf{n}}\in H^s$, we only need to estimate $[H_i,\hat{\mathbf{n}}]g\cdot\hat{\mathbf{n}}$ with $g\in H^{s-2}$, and we are able to do so by Lemma 3.5.

Now we estimate R_3. There are several terms which need attention. First, since $g\in H^{s-1}$, by Lemma 3.3, we have $\mathcal{L}_i[\mathbf{X}]g\in H^{s-1/2}$, moreover $(\mathcal{L}_i[\mathbf{X}]g\cdot\hat{\mathbf{t}}^i)_\alpha\in H^{s-3/2}$ and $\left(\mathcal{L}_i[\mathbf{X}]g\cdot\hat{\mathbf{n}}\right)_\beta\in H^{s-3/2}$. Second, $\mathcal{J}[\mathbf{X}]g_i\in H^s$ is the same as $\mathcal{J}[\mathbf{X}]g\in H^s$. And by Lemma 3.7, we have the following estimate:

$$\begin{aligned}
&\left\|\left(\Lambda\left(2A_\nu U\sqrt{E}\kappa+R\sqrt{E}\kappa n_3\right)-\left(2A_\nu U\sqrt{E}+R\sqrt{E}n_3\right)\Lambda\kappa-2A_\nu\sqrt{E}\kappa\Lambda U\right)\right\|_{s-1}\\
&\qquad\le C(\|\kappa\|_s,\|\mathbf{X}\|_{s+1},\|E\|_{s+1}).
\end{aligned}$$

Finally, for other terms which comprise R_3, we omit the details but conclude that they are in H^{s-1}.

We also omit the details of proof of the estimate $\|Q_2\|_{s-3/2} \le C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}, \|E\|_{s+1})$ since they are similar to the details of the estimate of R_3. □

Finally, we have one more lemma on regularity of the surface $\mathbf{X}$; the proof of this may be found in [5].

Lemma 3.17. *If $\mathbf{X} \in H^{s+1}$ and $\kappa \in H^s$ and $\mathbf{X}$ is parameterized according to (2.3), then $\mathbf{X} \in H^{s+2}$.*

4. Well-posedness with surface tension

In this section, we provide a complete proof that the three-dimensional interfacial Darcy flow problem is well-posed, for any fixed, positive value of the surface tension parameter, τ. In Section 4.1, we give an *a priori* estimate for the linearized system. In Section 4.2, we begin the proof of well-posedness, setting up an iterated system of evolution equations. We prove that this system has solutions, and we also prove estimates for the growth of the solutions. In Section 4.4, we use these estimates to allow us to take the limit of the iterates, finding a solution of the physical problem. We also discuss uniqueness and continuous dependence on the initial data.

We first study the well-posedness of the linearized Cauchy problem for κ. More precisely, we will consider the linear Cauchy problem for η:

$$\begin{aligned}
\eta_t = {} & \tau \frac{\Lambda}{2E}\left(\frac{\Delta\eta}{2E^{1/2}}\right) - \tau A_\nu \frac{\Lambda}{2E}\left([H_1, \hat{\mathbf{n}}]\left(\frac{\eta_{\alpha\alpha}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}} + [H_2, \hat{\mathbf{n}}]\left(\frac{\eta_{\alpha\beta}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}}\right) \\
& - \tau A_\nu \frac{\Lambda}{2E}\left([H_1, \hat{\mathbf{n}}]\left(\frac{\eta_{\alpha\beta}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}} + [H_2, \hat{\mathbf{n}}]\left(\frac{\eta_{\beta\beta}}{2E^{1/2}}\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) - \frac{E_\alpha}{8E^{5/2}}\tau H_1 \Delta\eta - \frac{E_\beta}{8E^{5/2}}\tau H_2 \Delta\eta \\
& - \tau A_\nu \frac{\Lambda}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\eta_{\alpha\beta}\mathbf{X}_\alpha - \eta_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}} + \mathcal{J}[\mathbf{X}]\left(\eta_{\beta\beta}\mathbf{X}_\alpha - \eta_{\alpha\beta}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) \\
& + \frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\eta_{\alpha\beta}\mathbf{X}_\alpha - \eta_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha + \frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\eta_{\beta\beta}\mathbf{X}_\alpha - \eta_{\alpha\beta}\mathbf{X}_\beta\right)\cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) + Q_1 + Q_2,
\end{aligned} \tag{4.1}$$

with initial condition $\eta(t, \vec{x})|_{t=0} = \eta_0$, where $\tau > 0$, Q_1 and Q_2 are nonhomogenous terms. Notice that the relationship between $\hat{\mathbf{t}}^i$, $\hat{\mathbf{n}}$ and $\mathbf{X}$ is the same as before.

4.1. *The* a priori *estimate*

We now prove an estimate for solutions of the linearized system, keeping in mind Remark 3.

Theorem 4.1. *Suppose that $E \in C([0, T], H^{s+1} \bigcap C^1([0, T], H^{s-1})$ and $E \ge C_0 > 0$ for some constant C_0, $\mathbf{X} \in C([0, T], H^{s+1} \bigcap C^1([0, T], H^{s-1})$ and $\kappa \in C([0, T], H^s \bigcap C^1([0, T], H^{s-3})$. Assume that $\mathbf{X}$ satisfies (2.27). We assume that the initial data $\eta_0 \in H^s$ and the nonhomogenous terms $Q_1 \in L^2([0, T], H^{s-1})$ and $Q_2 \in L^2([0, T], H^{s-3/2})$. Then there exists a unique solution $\eta \in C([0, T'], H^s)$ of (4.1) with initial data $\eta|_{t=0} = \eta_0$, and there exists a constant $m > 0$ such that the following estimate holds:*

$$\|\eta\|_s + e^{C\tau t}\int_0^t \frac{\tau m\|\Lambda^{s+3/2}\eta\|_0^2}{5}ds \le e^{C\tau t}\left(\|\eta_0\|_s + \int_0^t \frac{5\|Q_1+Q_2\|_{s-3/2}^2}{m\tau}ds\right). \quad (4.2)$$

Proof. The well-posedness for the linear Cauchy problem for (4.1) will be achieved by classical steps, such as approximation, demonstration of existence of approximate solutions, passing to the limit, and demonstrating uniqueness. The main step, which is what we will now demonstrate, is to perform energy estimates; restated, other than the energy estimates, the details are routine. We define the energy

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1, \quad (4.3)$$

with $\mathcal{E}_0 = \frac{1}{2}\|\eta\|_0^2$ and $\mathcal{E}_1 = \frac{1}{2}\|\Lambda^s\eta\|_0^2$. To begin with, we take the time derivative of $\mathcal{E}_0$:

$$\frac{d\mathcal{E}_0}{dt} = \iint \eta\eta_t d\alpha d\beta.$$

Since s is sufficiently large, using the evolution of η and preliminary estimates in Section 3, we immediately find

$$\frac{d\mathcal{E}_0}{dt} \le C\left(\mathcal{E} + \|Q_1+Q_2\|_{s-3/2}^2\right).$$

We next take the time derivative of $\mathcal{E}_1$.

$$\frac{d\mathcal{E}_1}{dt} = \tau\iint \left(\Lambda^s\eta\right)\Lambda^s\left(\frac{\Lambda}{2E}\left(\frac{\Delta\eta}{2E^{1/2}}\right)\right)d\alpha d\beta + \tau\iint\left(\Lambda^{s+3/2}\eta\right)\Lambda^{s-3/2}Q_3 d\alpha d\beta$$
$$+ \iint\left(\Lambda^{s+3/2}\eta\right)\Lambda^{s-3/2}(Q_1+Q_2)d\alpha d\beta, \quad (4.4)$$

where Q_3 is given by

$$Q_3 = -A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\eta_{\alpha\alpha}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}} + [H_2,\hat{\mathbf{n}}]\left(\frac{\eta_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)$$
$$- A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\eta_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}} + [H_2,\hat{\mathbf{n}}]\left(\frac{\eta_{\beta\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right) - \frac{E_\alpha}{8E^{5/2}}H_1\Delta\eta - \frac{E_\beta}{8E^{5/2}}H_2\Delta\eta$$
$$- A_\nu\frac{\Lambda}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\eta_{\alpha\beta}\mathbf{X}_\alpha - \eta_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}} + \mathcal{J}[\mathbf{X}]\left(\eta_{\beta\beta}\mathbf{X}_\alpha - \eta_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)$$
$$+ \frac{1}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\eta_{\alpha\beta}\mathbf{X}_\alpha - \eta_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha + \frac{1}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\eta_{\beta\beta}\mathbf{X}_\alpha - \eta_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right)_\beta. \quad (4.5)$$

We first deal with the first term on the right-hand side of (4.4). Noting that $\Delta = -\Lambda^2$, we find the following:

$$\iint \left(\Lambda^s \eta\right) \Lambda^s \left(\frac{\Lambda}{2E}\left(\frac{\Delta\eta}{2E^{1/2}}\right)\right) d\alpha d\beta = -\iint \left(\Lambda^s \eta\right) \Lambda^s \left(\frac{\Lambda}{2E}\left(\frac{\Lambda^2\eta}{2E^{1/2}}\right)\right) d\alpha d\beta$$
$$= -\iint \left(\Lambda^{s+3/2}\eta\right) \Lambda^{s-3/2}\left(\frac{\Lambda^3\eta}{4E^{3/2}}\right) d\alpha d\beta$$
$$-\iint \left(\Lambda^{s+3/2}\eta\right) \Lambda^{s-3/2}\left(\frac{1}{4E}\left[\Lambda, \frac{1}{E^{1/2}}\right]\Lambda^2\eta\right) d\alpha d\beta$$
$$= -\iint \frac{1}{4E^{3/2}}\left(\Lambda^{s+3/2}\eta\right) \Lambda^{s+3/2}\eta d\alpha d\beta - \iint \left(\Lambda^{s+3/2}\eta\right)\left[\Lambda^{s-3/2}, \frac{1}{4E^{3/2}}\right]\Lambda^3\eta d\alpha d\beta$$
$$-\iint \left(\Lambda^{s+3/2}\eta\right) \Lambda^{s-3/2}\left(\frac{1}{4E}\left[\Lambda, \frac{1}{E^{1/2}}\right]\Lambda^2\eta\right) d\alpha d\beta.$$

By Lemma 3.8, for sufficiently large s, we have

$$\left\|\left[\Lambda^{s-3/2}, \frac{1}{4E^{3/2}}\right]\Lambda^3\eta\right\|_0 \le c\left(\left\|\nabla\left(\frac{1}{4E^{3/2}}\right)\right\|_{L^\infty} \|\Lambda^3\eta\|_{s-5/2} + \left\|\frac{1}{4E^{3/2}}\right\|_{s-3/2} \|\Lambda^3\eta\|_{L^\infty}\right)$$
$$\le c\|\eta\|_{s+1/2} \le c(\|\eta\|_0 + \|\Lambda^{s+1/2}\eta\|_0). \quad (4.6)$$

By Lemma 3.10, $\|\Lambda^{s+1/2}\eta\|_0 \le c\|\Lambda^{s+3/2}\eta\|_0^{\frac{s+1/2}{s+3/2}} \|\eta\|_0^{\frac{1}{s+3/2}}$. Using this bound, Young's inequality, and (4.6), we have

$$\left|-\iint \left(\Lambda^{s+3/2}\eta\right)\left[\Lambda^{s-3/2}, \frac{1}{4E^{3/2}}\right]\Lambda^3\eta d\alpha d\beta\right|$$
$$\le c\|\eta\|_0^{\frac{1}{s+3/2}} \|\Lambda^{s+3/2}\eta\|_0^{\frac{2s+2}{s+3/2}} + c\|\eta\|_0\|\Lambda^{s+3/2}\eta\|_0 \le \frac{\|\Lambda^{s+3/2}\eta\|_0^2}{\varpi} + C\mathcal{E},$$

where ϖ is a constant and will be chosen later.

By Lemma 3.6, we have

$$\left\|\Lambda^{s-3/2}\left(\frac{1}{4E}\left[\Lambda, \frac{1}{\sqrt{E}}\right]\Lambda^2\eta\right)\right\|_0 \le c\|\Lambda^2\eta\|_{s-3/2} \le c(\|\eta\|_0 + \|\Lambda^{s+1/2}\eta\|_0).$$

So by Young's inequality,

$$\left|-\iint \left(\Lambda^{s+3/2}\eta\right) \Lambda^{s-3/2}\left(\frac{1}{4E}\left[\Lambda, \frac{1}{\sqrt{E}}\right]\Lambda^2\eta\right) d\alpha d\beta\right| \le \frac{\|\Lambda^{s+3/2}\eta\|_0^2}{\varpi} + C\mathcal{E}. \quad (4.7)$$

We make our first conclusion that

$$\iint \left(\Lambda^s \eta\right) \Lambda^s \left(\frac{\Lambda}{2E}\left(\frac{\Delta^2\eta}{2E^{1/2}}\right)\right) d\alpha d\beta \le -\iint \frac{1}{4E^{3/2}}\left(\Lambda^{s+3/2}\eta\right)^2 d\alpha d\beta + \frac{2\|\Lambda^{s+3/2}\eta\|_0^2}{\varpi} + C\mathcal{E}. \quad (4.8)$$

Now we consider the second term on the right-hand side of (4.4). That is, we consider $\|Q_3\|_{s-3/2}$. First, it is easy to say that $\| -\frac{E_\alpha}{8E^{5/2}}H_1\Delta\eta - \frac{E_\beta}{8E^{5/2}}H_2\Delta\eta\|_{s-3/2} \le c(\|\eta\|_0 + \|\Lambda^{s+1/2}\eta\|_0)$. Otherwise, there are many similar terms comprising Q_3, and we only give details of bounding a couple representative examples of these, in particular $\|[H_i, \hat{\mathbf{n}}]f\|_{s-1/2}$ and $\|\mathcal{J}[\mathbf{X}]f\|_{s-1/2}$. By Lemma 3.4, $\|[H_i, \hat{\mathbf{n}}]f\|_{s-1/2} \le \|\hat{\mathbf{n}}\|_{s-1/2}\|f\|_{s-3/2}$ and by previous discussion, $\|\mathcal{J}[\mathbf{X}]f\|_{s-1/2} \le c(1 + \|\mathbf{X}\|_{s+1/2})^2\|f\|_{s-3/2}$. Here, f is in terms of second derivatives of η in Q_3, so $\|f\|_{s-3/2} \le \|\eta\|_{s+1/2}$. We conclude that

$$\|Q_3\|_{s-3/2} \le c(\|\eta\|_0 + \|\Lambda^{s+1/2}\eta\|_0). \tag{4.9}$$

So by Young's inequality again,

$$\left|\iint \left(\Lambda^{s+3/2}\eta\right)\Lambda^{s-3/2}Q_3 d\alpha d\beta\right| \le \frac{\|\Lambda^{s+3/2}\eta\|_0^2}{\varpi} + C\mathcal{E}.$$

Finally, by the Hölder inequality,

$$\left|\iint \left(\Lambda^{s+3/2}\eta\right)\Lambda^{s-3/2}(Q_1 + Q_2) d\alpha d\beta\right| \le \frac{\tau\|\Lambda^{s+3/2}\eta\|_0^2}{\varpi} + \frac{\varpi\|Q_1 + Q_2\|_{s-3/2}^2}{\tau}.$$

Now we make the conclusion that

$$\frac{d\mathcal{E}}{dt} \le -\tau \iint \frac{1}{4E^{3/2}}\left(\Lambda^{s+3/2}\eta\right)^2 d\alpha d\beta + \frac{4\tau\|\Lambda^{s+3/2}\eta\|_0^2}{\varpi} + C\tau\mathcal{E} + \frac{\varpi\|Q_1 + Q_2\|_{s-3/2}^2}{\tau}. \tag{4.10}$$

We know $E > 0$, and $E \in L^\infty$ when $s > 1$. Then there exists $m > 0$ such that $-\frac{1}{4E^{3/2}} \le -m$. Now we take $\varpi = 5/m$. Then

$$\frac{d\mathcal{E}}{dt} + \frac{\tau m\|\Lambda^{s+3/2}\eta\|_0^2}{5} \le C\tau\mathcal{E} + \frac{5\|Q_1 + Q_2\|_{s-3/2}^2}{m\tau}. \tag{4.11}$$

By Gronwall's inequality, it follows that

$$\mathcal{E}(t) + e^{C\tau t}\int_0^t \frac{\tau m\|\Lambda^{s+3/2}\eta\|_0^2}{5} ds \le e^{C\tau t}\left(\mathcal{E}(0) + \int_0^t \frac{5\|Q_1 + Q_2\|_{s-3/2}^2}{m\tau} ds\right). \quad \square \tag{4.12}$$

Remark 5. This above estimate is not uniform in τ, since $1/\tau \to \infty$ as $\tau \to 0$. Thus the time of existence is going to 0 as τ goes to 0.

4.2. The iterated system

Now we devote ourselves to deal with the nonlinear system for $\mathbf{X}$ and κ. We take the initial data $\mathbf{X}_0 \in H^{s+2}$. We may take this surface to have a global isothermal parametrization [14]. Assume that there exists $c_0 > 0$ such that $E_0 > c_0$.

We will construct $(\mathbf{X}^l, E^l, \kappa^l)$ by an iteration method, given an initial $\mathbf{X}^0$, E^0, and κ^0. We take these to be given by, and calculated from, the initial data $\mathbf{X}_0$. Then $E^0 = E_0 = \mathbf{X}^0_\alpha \cdot \mathbf{X}^0_\alpha = \mathbf{X}^0_\beta \cdot \mathbf{X}^0_\beta$, and $\mathbf{X}^0_\alpha \cdot \mathbf{X}^0_\beta = 0$. The second fundamental coefficients L^0, N^0, and M^0 are given by the equation (2.6) and κ^0 is given by the equation (2.28); then, $\kappa^0 \in H^s$. Solving the equation (2.16), we get the solution μ^0, and $\mathbf{W}^0$ is then defined by the Birkhoff-Rott Integral (2.24). Moreover, $U^0 = \mathbf{W}^0 \cdot \hat{\mathbf{n}}^0$. Then (V_1^0, V_2^0) is determined by solving the following system:

$$\left(\frac{V_1^0}{\sqrt{E^0}}\right)_\alpha - \left(\frac{V_2^0}{\sqrt{E^0}}\right)_\beta = \frac{U^0(L^0 - N^0)}{E^0},$$
$$\left(\frac{V_1^0}{\sqrt{E^0}}\right)_\beta + \left(\frac{V_2^0}{\sqrt{E^0}}\right)_\alpha = \frac{2U^0 M^0}{E^0}.$$

Assume that we have already constructed $(\mathbf{X}^l, \kappa^l, E^l)$. We next determine μ^l by solving (2.16). Then we use the Birkhoff-Rott integral (2.24) to get $\mathbf{W}^l$. We denote the iterated quantities related to $\mathbf{X}^l$ as follows:

$$\hat{\mathbf{t}}^{1,l} = \frac{\mathbf{X}^l_\alpha}{|\mathbf{X}^l_\alpha|}, \qquad \hat{\mathbf{t}}^{2,l} = \frac{\mathbf{X}^l_\beta}{|\mathbf{X}^l_\beta|}, \qquad \hat{\mathbf{n}}^l = \frac{\mathbf{X}^l_\alpha \times \mathbf{X}^l_\beta}{|\mathbf{X}^l_\alpha \times \mathbf{X}^l_\beta|}$$
$$L^l = \mathbf{X}^l_{\alpha\alpha} \cdot \hat{\mathbf{n}}^l, \qquad N^l = \mathbf{X}^l_{\beta\beta} \cdot \hat{\mathbf{n}}^l, \qquad M^l = \mathbf{X}^l_{\alpha\beta} \cdot \hat{\mathbf{n}}^l.$$

We also let $U^l = \mathbf{W}^l \cdot \hat{\mathbf{n}}^l$.

Now we describe how we find the next iterates. To begin with, we construct κ^{l+1} to solve the linear Cauchy problem:

$$\begin{aligned}
\kappa_t^{l+1} = {} & \tau \frac{\Lambda}{2E^l}\left(\frac{\Delta\kappa^{l+1}}{2\sqrt{E^l}}\right) - \tau A_\nu \frac{\Lambda}{2E^l}\left([H_1, \hat{\mathbf{n}}^l]\left(\frac{\kappa^{l+1}_{\alpha\alpha}}{2\sqrt{E^l}}\right)\cdot \hat{\mathbf{t}}^{1,l} + [H_2, \hat{\mathbf{n}}^l]\left(\frac{\kappa^{l+1}_{\alpha\beta}}{2\sqrt{E^l}}\right)\cdot \hat{\mathbf{t}}^{1,l}\right) \\
& - \tau A_\nu \frac{\Lambda}{2E^l}\left([H_1, \hat{\mathbf{n}}^l]\left(\frac{\kappa^{l+1}_{\alpha\beta}}{2\sqrt{E^l}}\right)\cdot \hat{\mathbf{t}}^{2,l} + [H_2, \hat{\mathbf{n}}^l]\left(\frac{\kappa^{l+1}_{\beta\beta}}{2\sqrt{E^l}}\right)\cdot \hat{\mathbf{t}}^{2,l}\right) \\
& - \frac{E_\alpha}{8(E^l)^{5/2}}\tau H_1 \Delta\kappa^{l+1} - \frac{E^l_\beta}{8(E^l)^{5/2}}\tau H_2 \Delta\kappa^{l+1} \\
& - \tau A_\nu \frac{\Lambda}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\kappa^{l+1}_{\alpha\beta}\mathbf{X}^l_\alpha - \kappa^{l+1}_{\alpha\alpha}\mathbf{X}^l_\beta\right)\cdot \hat{\mathbf{t}}^{1,l} + \mathcal{J}[\mathbf{X}^l]\left(\kappa^{l+1}_{\beta\beta}\mathbf{X}^l_\alpha - \kappa^{l+1}_{\alpha\beta}\mathbf{X}^l_\beta\right)\cdot \hat{\mathbf{t}}^{2,l}\right) \\
& + \frac{\tau}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\kappa^{l+1}_{\alpha\beta}\mathbf{X}^l_\alpha - \kappa^{l+1}_{\alpha\alpha}\mathbf{X}^l_\beta\right)\cdot \hat{\mathbf{t}}^{1,l}\right)_\alpha + \frac{\tau}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\kappa^{l+1}_{\beta\beta}\mathbf{X}^l_\alpha - \kappa^{l+1}_{\alpha\beta}\mathbf{X}^l_\beta\right)\cdot \hat{\mathbf{t}}^{2,l}\right) \\
& + Q_1^l + Q_2^l \qquad (4.13)
\end{aligned}$$

with initial condition $\kappa^{l+1}(t,\vec{x})|_{t=0} = \kappa_0$, with $\tau > 0$, and where Q_1^l and Q_2^l are functions of $\mathbf{X}^l, \kappa^l, E^l, \mu^l, V_1^l, V_2^l, U^l, L^l, N^l, M^l, \hat{\mathbf{t}}^{1,l}, \hat{\mathbf{t}}^{2,l}$, and $\hat{\mathbf{n}}^l$, given by (2.59) and (2.60) respectively.

Secondly, we determine (V_1^{l+1}, V_2^{l+1}) by solving the elliptic system

$$\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right)_\alpha - \left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right)_\beta = \frac{U^l(L^l - N^l)}{E^l}, \tag{4.14}$$

$$\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right)_\beta + \left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right)_\alpha = \frac{2U^l M^l}{E^l}, \tag{4.15}$$

which enforces the isothermal parameterization.

Let $\widetilde{\mathbf{X}}^{l+1}$ be given by the solution of the initial value problem

$$\widetilde{\mathbf{X}}_t^{l+1} = U^l \hat{\mathbf{n}}^l + V_1^{l+1}\hat{\mathbf{t}}^{1,l} + V_2^{l+1}\hat{\mathbf{t}}^{2,l}, \quad \widetilde{\mathbf{X}}^{l+1}|_{t=0} = \mathbf{X}_0. \tag{4.16}$$

We have one more intermediate variable $\widehat{\mathbf{X}}^{l+1}$, which is given by solving the elliptic equation

$$\Delta\widehat{\mathbf{X}}^{l+1} - \widehat{\mathbf{X}}^{l+1} = 2\kappa^l \widetilde{\mathbf{X}}_\alpha^{l+1} \times \widetilde{\mathbf{X}}_\beta^{l+1} - \widetilde{\mathbf{X}}^{l+1}. \tag{4.17}$$

Now we are ready to construct $\mathbf{X}^{l+1}$ by solving the following elliptic equation

$$\Delta\mathbf{X}^{l+1} - \mathbf{X}^{l+1} = 2\kappa^l \widehat{\mathbf{X}}_\alpha^{l+1} \times \widehat{\mathbf{X}}_\beta^{l+1} - \widetilde{\mathbf{X}}^{l+1}. \tag{4.18}$$

Finally, we define E^{l+1} also by solving the following elliptic equation:

$$\Delta E^{l+1} - E^{l+1} = 2(\mathbf{X}_{\alpha\beta}^{l+1}\cdot\mathbf{X}_{\alpha\beta}^{l+1} - \mathbf{X}_{\alpha\alpha}^{l+1}\cdot\mathbf{X}_{\beta\beta}^{l+1}) - \frac{1}{2}(\mathbf{X}_\alpha^{l+1}\cdot\mathbf{X}_\alpha^{l+1} + \mathbf{X}_\beta^{l+1}\cdot\mathbf{X}_\beta^{l+1}). \tag{4.19}$$

4.3. Estimates for the iteration

As we now know that the iterated solutions exist, we provide estimates for the solutions and related quantities at each step, keeping in mind Remark 3 on regularity.

Lemma 4.2. *The iterates $(\mathbf{X}^l, E^l, \kappa^l)$ are defined for all l and there exists $T > 0$ and positive constants C_0, C_1, C_2, C_3 and C_4, such that for all l, for all $t \in [0, T]$, (2.27) is satisfied and we have the following bounds:*

$$E^l \geq C_0 > 0, \quad |\mathbf{X}_\alpha^l \times \mathbf{X}_\beta^l| \geq C_0 > 0, \tag{4.20}$$

$$\|\kappa^l\|_{C^0([0,T];H^s)} \leq C_1, \tag{4.21}$$

$$\|\mathbf{X}^l\|_{C^0([0,T];H^{s+1})} + \|E^l\|_{C^0([0,T];H^{s+1})} \leq C_2, \tag{4.22}$$

$$\|\partial_t\kappa^l\|_{C^0([0,T];H^{s-3})} \leq C_3, \tag{4.23}$$

$$\|\partial_t\mathbf{X}^l\|_{C^0([0,T];H^{s-1})} + \|\partial_t E^l\|_{C^0([0,T];H^{s-1})} \leq C_4. \tag{4.24}$$

Proof. We proceed by induction. We will determine C_0, C_1, C_2, C_3, and C_4 as we go. Given the above initial data, the conclusions are immediately true for $l = 0$ for any $C_0 \le c_0$. Assume that $(\mathbf{X}^l, E^l, \kappa^l)$ exists, satisfying (4.20), (4.21) (4.22), (4.23) and (4.24). By Lemma 3.16, we have $\|Q_1^l\|_{L^2([0,T],H^{s-1})} \le C(C_1, C_2)$ and $\|Q_2^l\|_{L^2([0,T],H^{s-3/2})} \le C(C_1, C_2)$. Furthermore Theorem 4.1 shows that the solution of (4.13), κ^{l+1}, is in $C^0([0, T]; H^s)$ and then $\partial_t\kappa^{l+1}$ in $C^0([0, T]; H^{s-3})$ for some $T > 0$. Moreover, by the energy estimate (4.2), the estimate of κ^{l+1} satisfies

$$\|\kappa^{l+1}(t)\|_s \le e^{C(C_1,C_2)\tau t}\|\kappa_0\|_s + 5te^{C(C_1,C_2)\tau t}C(C_1, C_2)/m\tau.$$

Hence taking $C_1 = 2\|\kappa_0\|_s$, we may take T small enough (independent of l) such that

$$\|\kappa^{l+1}(t)\|_{L^\infty([0,T];H^s)} \le C_1.$$

Using Lemma 3.15, we have

$$\|V_1\|_{L^\infty([0,T];H^s)} + \|V_2\|_{L^\infty([0,T];H^s)} \le C(C_1, C_2).$$

By the evolution equation (4.16), $\widetilde{\mathbf{X}}^{l+1} \in C^0([0, T]; H^{s-1})$ and

$$\|\widetilde{\mathbf{X}}^{l+1}\|_{L^\infty([0,T];H^{s-1})} \le \|\mathbf{X}_0\|_{s-1} + C(C_1, C_2)T. \tag{4.25}$$

Hence we may take T small enough such that $\|\widetilde{\mathbf{X}}^{l+1}\|_{L^\infty([0,T];H^{s-1})} \le 2\|\mathbf{X}_0\|_{s-1}$. By elliptic equations (4.17) and (4.18), we then have $\mathbf{X}^{l+1} \in C^0([0, T]; H^{s+1})$ and

$$\|\widehat{\mathbf{X}}^{l+1}\|_{L^\infty([0,T];H^s)} \le C(\|\mathbf{X}_0\|_{s-1}, \|\kappa_0\|_s), \tag{4.26}$$

$$\|\mathbf{X}^{l+1}\|_{L^\infty([0,T];H^{s+1})} \le C(\|\mathbf{X}_0\|_{s-1}, \|\kappa_0\|_s). \tag{4.27}$$

Moreover by elliptic equation (4.19), we have $E^{l+1} \in C^0([0, T]; H^{s+1})$ and the estimate

$$\|E^{l+1}\|_{L^\infty([0,T];H^{s+1})} \le C(\|\mathbf{X}_0\|_{s-1}, \|\kappa_0\|_s).$$

So we now take $C_2 = \max\{2C(\|\mathbf{X}_0\|_{s-1}, \|\kappa_0\|_s), \|\mathbf{X}_0\|_{s+1}\}$, the estimate (4.22) holds. This choice of C_2 implies that T can be chosen in (4.25) to be independent of l.

We know that by the evolution equations (4.16) and (4.13),

$$\|\partial_t\widetilde{\mathbf{X}}^{l+1}\|_{L^\infty([0,T];H^{s-3})} \le C(C_1, C_2)$$

and

$$\|\partial_t\kappa^{l+1}(t)\|_{s-3} \le C(C_1, C_2)(1 + \|\kappa^{l+1}(t)\|_s) \le C(C_1, C_2)(1 + C_1).$$

Taking C_3, such that $C_3 \ge C(C_1, C_2)(1 + C_1)$, we hold the estimate (4.23). Taking the time derivative of (4.17) and (4.18), it follows that

$$\|\partial_t\mathbf{X}^{l+1}\|_{L^\infty([0,T];H^{s-1})} \le C(C_1, C_2, C_3). \tag{4.28}$$

And, taking the time derivative of (4.19), it follows that

$$\|\partial_t E^{l+1}\|_{L^\infty([0,T];H^{s-1})} \leq C(C_1, C_2, C_3).$$

So we take C_4, such that $C_4 \geq C(C_1, C_2, C_3)$, then getting the estimate (4.24).

Notice that if s is sufficiently large enough such that $H^{s-1} \subset L^\infty$, then

$$|E^{l+1}(t)| \geq E_0 - \int_0^t \partial_s E^{l+1} ds \geq c_0 - tC_4,$$

and

$$|\mathbf{X}_\alpha^{l+1}(t) \times \mathbf{X}_\beta^{l+1}(t)| \geq |\mathbf{X}_\alpha^{l+1}(0) \times \mathbf{X}_\beta^{l+1}(0)| - \int_0^t \partial_s \mathbf{X}_\alpha^{l+1}(s) \times \mathbf{X}_\beta^{l+1}(s) ds \geq c_0 - tC_4C_2.$$

So we can take T small enough such that $c_0 - TC_4 \geq C_0$ and $c_0 - TC_4C_2 \geq C_0$. Similarly, since s is sufficiently large and since we may take T sufficiently small, the estimate (4.28) and the initial condition (2.26) combine to imply (2.27). This completes the lemma. □

4.4. The limit of the iterated system

In this section, we prove that we have a Cauchy sequence, which implies the convergence of the iterative procedure. We will prove existence of a limit in a low norm; regularity of the limit follows primarily using the uniform bound in the high norm. The main result of this section is the following lemma.

Lemma 4.3. *The sequence $(\mathbf{X}^l, \kappa^l, E^l)$ is Cauchy sequence in the space*

$$C^0([0,T]; H^3),\ C^0([0,T]; H^2) \quad \text{and} \quad C^0([0,T]; H^3). \tag{4.29}$$

Proof. In the proof, we use some of the same variable names over again but with different meaning. We denote $(\delta\mathbf{X}, \delta\kappa, \delta E) = (\mathbf{X}^{l+1} - \mathbf{X}^l, \kappa^{l+1} - \kappa^l, E^{l+1} - E^l)$, $\delta\mu = \mu^{l+1} - \mu^l$, $\delta U = U^{l+1} - U^l$, and so on. We define an energy functional

$$D_l = \frac{1}{2}\|\delta\mathbf{X}\|_3^2 + \frac{1}{2}\|\delta\kappa\|_0^2 + \frac{1}{2}\|\Lambda^2\delta\kappa\|_0^2 + \frac{1}{2}\|\delta E\|_3^2. \tag{4.30}$$

As in Lemma 3.14, we have

$$\|\delta U\|_1^2 \leq CD_l. \tag{4.31}$$

Furthermore, by the definition of (4.14) and (4.15), we have

$$\|\delta V_i\|_2^2 \leq CD_{l-1}. \tag{4.32}$$

We now get the equations for the difference of the l-th and $(l+1)$-st iterates:

$$
\begin{aligned}
\delta\kappa_t = {}& \tau\frac{\Lambda}{2E^l}\left(\frac{\Delta\delta\kappa}{2\sqrt{E^l}}\right) - \tau A_\nu \frac{\Lambda}{2E^l}\left([H_1,\hat{\mathbf{n}}^l]\left(\frac{\delta\kappa_{\alpha\alpha}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{1,l} + [H_2,\hat{\mathbf{n}}^l]\left(\frac{\delta\kappa_{\alpha\beta}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{1,l}\right)\\
&- \tau A_\nu \frac{\Lambda}{2E^l}\left([H_1,\hat{\mathbf{n}}^l]\left(\frac{\delta\kappa_{\alpha\beta}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{2,l} + [H_2,\hat{\mathbf{n}}^l]\left(\frac{\delta\kappa_{\beta\beta}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{2,l}\right)\\
&- \frac{E^l_\alpha}{8(E^l)^{5/2}}\tau H_1\Delta\delta\kappa - \frac{E^l_\beta}{8(E^l)^{5/2}}\tau H_2\Delta\delta\kappa\\
&- \tau A_\nu\frac{\Lambda}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\delta\kappa_{\alpha\beta}\mathbf{X}^l_\alpha - \delta\kappa_{\alpha\alpha}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{1,l} + \mathcal{J}[\mathbf{X}^l]\left(\delta\kappa_{\beta\beta}\mathbf{X}^l_\alpha - \delta\kappa_{\alpha\beta}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{2,l}\right)\\
&+ \frac{\tau}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\delta\kappa_{\alpha\beta}\mathbf{X}^l_\alpha - \delta\kappa_{\alpha\alpha}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{1,l}\right)_\alpha + \frac{\tau}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\delta\kappa_{\beta\beta}\mathbf{X}^l_\alpha - \delta\kappa_{\alpha\beta}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{2,l}\right)\\
&+ F^l(\kappa^l,\mathbf{X}^l,E^l) - F^{l-1}(\kappa^l,\mathbf{X}^{l-1},E^{l-1}) + Q^l_1 - Q^{l-1}_1 + Q^l_2 - Q^{l-1}_2.
\end{aligned}
$$

Notice that we are using both $F^l(\kappa^l,\mathbf{X}^l,E^l)$ and $F^{l-1}(\kappa^l,\mathbf{X}^{l-1},E^{l-1})$ above; we only write out the formula for $F^l(\kappa^l,\mathbf{X}^l,E^l)$, as the other formula is the same except that all quantities except the curvature term just make use of the $(l-1)$-st iterates. The formula for F^l is

$$
\begin{aligned}
F^l(\kappa^l,\mathbf{X}^l,E^l) = {}& \tau\frac{\Lambda}{2E^l}\left(\frac{\Delta\kappa^l}{2\sqrt{E^l}}\right) - \tau A_\nu \frac{\Lambda}{2E^l}\left([H_1,\hat{\mathbf{n}}^l]\left(\frac{\kappa^l_{\alpha\alpha}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{1,l} + [H_2,\hat{\mathbf{n}}^l]\left(\frac{\kappa^l_{\alpha\beta}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{1,l}\right)\\
&- \tau A_\nu \frac{\Lambda}{2E^l}\left([H_1,\hat{\mathbf{n}}^l]\left(\frac{\kappa^l_{\alpha\beta}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{2,l} + [H_2,\hat{\mathbf{n}}^l]\left(\frac{\kappa^l_{\beta\beta}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^{2,l}\right)\\
&- \frac{E_\alpha}{8(E^l)^{5/2}}\tau H_1\Delta\kappa^l - \frac{E^l_\beta}{8(E^l)^{5/2}}\tau H_2\Delta\kappa^l\\
&- \tau A_\nu\frac{\Lambda}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\kappa^l_{\alpha\beta}\mathbf{X}^l_\alpha - \kappa^l_{\alpha\alpha}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{1,l} + \mathcal{J}[\mathbf{X}^l]\left(\kappa^l_{\beta\beta}\mathbf{X}^l_\alpha - \kappa^l_{\alpha\beta}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{2,l}\right)\\
&+ \frac{\tau}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\kappa^l_{\alpha\beta}\mathbf{X}^l_\alpha - \kappa^l_{\alpha\alpha}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{1,l}\right)_\alpha + \frac{\tau}{2E^l}\left(\mathcal{J}[\mathbf{X}^l]\left(\kappa^l_{\beta\beta}\mathbf{X}^l_\alpha - \kappa^l_{\alpha\beta}\mathbf{X}^l_\beta\right)\cdot\hat{\mathbf{t}}^{2,l}\right).
\end{aligned}
\tag{4.33}
$$

First, similarly to (4.11), we have

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int \delta\kappa^2 + (\Lambda^2\delta\kappa)^2 d\alpha d\beta + \frac{\tau m\|\Lambda^{7/2}\delta\kappa\|_0^2}{5} \le C\tau\int \delta\kappa^2 + (\Lambda^2\delta\kappa)^2 d\alpha d\beta\\
&+ \frac{5(\|F^l(\kappa^l,\mathbf{X}^l,E^l) - F^{l-1}(\kappa^l,\mathbf{X}^{l-1},E^{l-1})\|_{1/2}^2 + \|Q^l_1 + Q^l_2 - Q^{l-1}_1 - Q^{l-1}_2\|_{1/2}^2)}{m\tau}.
\end{aligned}
\tag{4.34}
$$

With s large enough, as in prior estimates of Q_1, Q_2, we can conclude that

$$
\|F^l(\kappa^l,\mathbf{X}^l,E^l) - F^{l-1}(\kappa^l,\mathbf{X}^{l-1},E^{l-1})\|_{1/2}^2 + \|Q^l_1 + Q^l_2 - Q^{l-1}_1 - Q^{l-1}_2\|_{1/2}^2 \le CD_{l-1}.
\tag{4.35}
$$

By Gronwall's inequality, it follows that

$$\frac{1}{2}\|\delta\kappa(t)\|^2 + \frac{1}{2}\|\Lambda^2\delta\kappa(t)\|^2 \leq \frac{5Ce^{C\tau t}}{m\tau}\int_0^t D_{l-1}(s)ds. \tag{4.36}$$

Now we discuss $\mathbf{X}^{l+1} - \mathbf{X}^l$ and $E^{l+1} - E^l$. First, by the equation (4.19), we have

$$\|\delta E\|_3^2 \leq C\|\delta \mathbf{X}\|_3^2. \tag{4.37}$$

Then by equation (4.17) and (4.18), we have

$$\begin{aligned}\|\delta\mathbf{X}\|_3 &\leq \|2\kappa^l\widehat{\mathbf{X}}_\alpha^{l+1}\times\widehat{\mathbf{X}}_\beta^{l+1} - 2\kappa^{l-1}\widehat{\mathbf{X}}_\alpha^l\times\widehat{\mathbf{X}}_\beta^l\|_1^2 + \|\widetilde{\mathbf{X}}^{l+1} - \widetilde{\mathbf{X}}^l\|_1^2\\ &\leq C\|\kappa^l - \kappa^{l-1}\|_1^2 + C\|\widehat{\mathbf{X}}^{l+1} - \widehat{\mathbf{X}}^l\|_2^2 + C\|\widetilde{\mathbf{X}}^{l+1} - \widetilde{\mathbf{X}}^l\|_1^2\\ &\leq C\|\kappa^l - \kappa^{l-1}\|_1^2 + C\|\widetilde{\mathbf{X}}^{l+1} - \widetilde{\mathbf{X}}^l\|_1^2,\end{aligned}$$

and by equation (4.16), have

$$\|\widetilde{\mathbf{X}}^{l+1} - \widetilde{\mathbf{X}}^l\|_1^2 \leq Ce^{Ct}\int_0^t D_{l-1}(s)ds. \tag{4.38}$$

Thus, we conclude that

$$D_l \leq Ce^{Ct}\int_0^t D_{l-1}(s)ds. \tag{4.39}$$

Moreover $D_l(0) = 0$. Hence, we deduce for $l \geq 2$ that $D_l \leq e^{Ct}\frac{(Ct)^{l-1}}{(n-1)!}$. This implies that $(\mathbf{X}^l, \kappa^l, E^l)$ is a Cauchy sequence. □

4.5. Well-posedness with surface tension

In this subsection we state and prove the main theorem of Section 4. We take s large enough. Let c_0, c_1, d be positive constants. We define an open subset $\mathcal{O} \subset H^{s+2}$, such that for every $\mathbf{X} \in \mathcal{O}$, the following conditions hold:

$$\|\mathbf{X}\|_{s+2} < d, \tag{4.40}$$

$$E(\alpha,\beta) > c_0, \tag{4.41}$$

$$\frac{\left|\mathbf{X}(\alpha,\beta) - \mathbf{X}(\alpha',\beta')\right|^2}{(\alpha-\alpha')^2 + (\beta-\beta')^2} > c_1, \quad \text{for all } \alpha \neq \beta. \tag{4.42}$$

Theorem 4.4. *Let the surface $\mathbf{X}_0 \in \mathcal{O}$ be globally parameterized by harmonic coordinates (namely (2.3) holds). Then, there exists a time $T > 0$ (T may depend on τ) and a unique solution $\mathbf{X} \in C^0([0,T), \overline{\mathcal{O}})$ for the Cauchy problem*

$$\begin{cases} \mathbf{X}_t = U\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2, \\ \mu_\alpha = \tau\kappa_\alpha - 2A_\nu\sqrt{E}\mathbf{W}\cdot\hat{\mathbf{t}}^1 - Rz_\alpha, \\ \mu_\beta = \tau\kappa_\beta - 2A_\nu\sqrt{E}\mathbf{W}\cdot\hat{\mathbf{t}}^2 - Rz_\beta, \\ \mathbf{X}(t=0) = \mathbf{X}_0. \end{cases} \tag{4.43}$$

Remark 6. When we say $\mathbf{X} \in H^s$, this means that $\mathbf{X}(\alpha,\beta) - (\alpha,\beta,0)$ is actually in H^s, since the surface $\mathbf{X}$ is asymptotic to the plane at infinity.

Proof. First, it remains to show that the limit of the iterates is a solution of the original system. We have proved that $(\mathbf{X}^l, \kappa^l, E^l)$ is a Cauchy sequence. Moreover $\widetilde{\mathbf{X}}^l$ and $\widehat{\mathbf{X}}^l$ are also Cauchy sequences. We denote the limit of $(\mathbf{X}^l, \kappa^l, E^l, \widetilde{\mathbf{X}}^l, \widehat{\mathbf{X}}^l)$ as $(\mathbf{X}, \kappa, E, \widetilde{\mathbf{X}}, \widehat{\mathbf{X}})$. Then the limit satisfies the system (4.14)-(4.19) without index l and $l+1$. In particular, we need to verify the system

$$\widetilde{\mathbf{X}}_t = U\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2,$$

$$\Delta\widehat{\mathbf{X}} - \widehat{\mathbf{X}} = 2\kappa\widetilde{\mathbf{X}}_\alpha \times \widetilde{\mathbf{X}}_\beta - \widetilde{\mathbf{X}}, \tag{4.44}$$

$$\Delta\mathbf{X} - \mathbf{X} = 2\kappa\widehat{\mathbf{X}}_\alpha \times \widehat{\mathbf{X}}_\beta - \widetilde{\mathbf{X}}, \tag{4.45}$$

$$\Delta E - E = 2(\mathbf{X}_{\alpha\beta}\cdot\mathbf{X}_{\alpha\beta} - \mathbf{X}_{\alpha\alpha}\cdot\mathbf{X}_{\beta\beta}) - \frac{1}{2}(\mathbf{X}_\alpha\cdot\mathbf{X}_\alpha + \mathbf{X}_\beta\cdot\mathbf{X}_\beta). \tag{4.46}$$

For the existence of a solution it is essential to prove the following relations:

$$\widetilde{\mathbf{X}} = \widehat{\mathbf{X}} = \mathbf{X}, \quad E = \mathbf{X}_\alpha\cdot\mathbf{X}_\alpha = \mathbf{X}_\beta\cdot\mathbf{X}_\beta, \quad \mathbf{X}_\alpha\cdot\mathbf{X}_\beta = 0, \quad \kappa = \frac{L+N}{2E}. \tag{4.47}$$

These relations imply then that $\Delta\mathbf{X} = 2E\kappa\hat{\mathbf{n}}$. The above relations all hold but we omit proof here and refer to [28] which gives all the details.

Now we demonstrate the highest regularity of $\mathbf{X}$. The solutions $(\mathbf{X}^l, E^l)$ of the iterated equations are in $H^{s+1}\times H^s$, uniformly bounded with respect to l, and thus the limit $(\mathbf{X}, E)$ is in this space with such a bound. Then $\mathbf{X}$ can be bounded in H^{s+2} since $\mathbf{X}$ satisfies (2.3). Considering $(\mathbf{X}^l, \kappa^l)$, at each time there is a subsequence which converges weakly in $H^{s+2}\times H^s$ and the limit must be $(\mathbf{X}, \kappa)$. Therefore at each time t, $(\mathbf{X}(\cdot,t), \kappa(\cdot,t)) \in H^{s+2}\times H^s$. It remains to show that $\mathbf{X} \in C^0([0,T]; H^{s+2})$. We do not include all the details, but this can be done by adapting the corresponding argument for regularity of solutions for the Navier-Stokes equations in Chapter 3 of [19]. We sketch this argument now.

First, we prove that the solution is strongly right-continuous in time at $t=0$. We will need to prove that $(\mathbf{X}, \kappa) \in H^{s+1}\times H^s$ is strongly right-continuous in time at $t=0$. The steps are to first show that $(\mathbf{X}, \kappa)$ is weakly continuous in time with values in $(H^{s+1}\times H^s)$; this follows easily from the uniform bound and the strong continuity in $(H^{s'+1}\times H^{s'})$, for $0 < s' < s$. (This continuity follows from the continuity in a low norm established when we proved the iterates form a Cauchy sequence, the uniform bound in the high norm, and interpolation.) Then, it is shown that the solution is strongly right-continuous in time at $t=0$ in the highest norm; this follows from the energy estimate and Fatou's Lemma.

The next step is to use parabolic smoothing; From the estimate (4.11), we see that κ^l is uniformly bounded in the space $L^2([0, T]; H^{s+3/2})$. Since this is a Hilbert space, we see that our sequence κ^l has a subsequence with a weak limit in this space, and this weak limit must be κ. The existence theory can then be repeated in higher regularity spaces starting from almost any positive time, t, with initial data $\mathbf{X}(\cdot, t)$. Using the uniqueness of solutions, the solution starting from time t and the solution starting from time zero must be the same. It can then be concluded that the solution starting from time t is continuous in H^{s+2} (since H^{s+2} would no longer be the highest regularity), and we are able to do this for any arbitrarily small value of t. Together with the right-continuity at time zero, this argument implies $\mathbf{X} \in C^0([0, T]; H^{s+2})$.

Finally, to obtain uniqueness of solutions, we argue as in the proof that $(\mathbf{X}^l, \kappa^l, E^l)$ is a Cauchy sequence, making an estimate for the difference of two solutions. □

5. The zero surface tension limit

We now consider the behavior of solutions of the system (4.43) as $\tau \to 0$. We will demonstrate that as τ vanishes, the sequence of solutions forms a Cauchy sequence. First, we will find solutions exist on a uniform time interval, and then we are able to take the limit as $\tau \to 0$. To get the uniform time of existence, we will revisit the energy estimate in the case that a stability condition is satisfied. For the new energy estimate, our first step is to make some decompositions, in order to make clear the effect of surface tension.

5.1. Decompositions

As we continue to rewrite the equations of motion, we will begin to isolate the contribution from surface tension. That is, for quantities which are related to the velocity, such as μ and $\mathbf{W}$, we want to decompose them into two parts, one of which is proportional to τ, and one of which is not. We decompose the equation (2.16) as

$$\tilde{\mu} + A_\nu \mathfrak{D}\tilde{\mu} = Rz, \tag{5.1}$$

$$\mu^{\text{s.t.}} + A_\nu \mathfrak{D}\mu^{\text{s.t.}} = \kappa. \tag{5.2}$$

(Recall that the operator $\mathfrak{D}$ was introduced in (2.17).) Equation (5.1) and (5.2) can be solved for $\tilde{\mu}$ and $\mu^{\text{s.t.}}$ respectively since $I + A_\nu \mathfrak{D}$ is an invertible operator for all $|A_\nu| \le 1$ [14]. Continuing, we write $\mu_\alpha = \tau\mu_\alpha^{\text{s.t.}} + \widetilde{\mu}_\alpha$ and $\mu_\beta = \tau\mu_\beta^{\text{s.t.}} + \widetilde{\mu}_\beta$, where

$$\mu_\alpha^{\text{s.t.}} = \kappa_\alpha - 2A_\nu\sqrt{E}\mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^1, \qquad \mu_\beta^{\text{s.t.}} = \kappa_\beta - 2A_\nu\sqrt{E}\mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^2, \tag{5.3}$$

$$\widetilde{\mu}_\alpha = -2A_\nu\sqrt{E}\widetilde{\mathbf{W}} \cdot \hat{\mathbf{t}}^1 - Rz_\alpha, \qquad \widetilde{\mu}_\beta = -2A_\nu\sqrt{E}\widetilde{\mathbf{W}} \cdot \hat{\mathbf{t}}^2 - Rz_\beta. \tag{5.4}$$

Similarly, we define $\mathbf{W}^{\text{s.t.}}$ and $\widetilde{\mathbf{W}}$, so that $\mathbf{W} = \tau\mathbf{W}^{\text{s.t.}} + \widetilde{\mathbf{W}}$:

$$\mathbf{W}^{\text{s.t.}}(\vec{\alpha}) = -\frac{1}{4\pi}\text{PV}\iint_{\mathbb{R}^2} (\mu_\alpha^{\text{s.t.}\prime}\mathbf{X}'_\beta - \mu_\beta^{\text{s.t.}\prime}\mathbf{X}'_\alpha) \times \frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3}\, d\vec{\alpha}', \tag{5.5}$$

$$\widetilde{\mathbf{W}}(\vec{\alpha}) = -\frac{1}{4\pi}\text{PV}\iint_{\mathbb{R}^2} (\widetilde{\mu}'_\alpha\mathbf{X}'_\beta - \widetilde{\mu}'_\beta\mathbf{X}'_\alpha) \times \frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3}\, d\vec{\alpha}'. \tag{5.6}$$

As before, in (5.5) and (5.6), quantities followed by a prime are evaluated at $\vec{\alpha}'$, while quantities without a prime are evaluated at $\vec{\alpha}$.

Given these decompositions, we then decompose the normal velocity as $U = \tau U^{\text{s.t.}} + \widetilde{U}$:

$$U^{\text{s.t.}} = \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{n}}, \qquad \widetilde{U} = \widetilde{\mathbf{W}} \cdot \hat{\mathbf{n}}.$$

We can also decompose V_i as $V_i = \tau V_i^{\text{s.t.}} + \widetilde{V}_i$, for $i \in \{1, 2\}$, where the pieces should satisfy the following equations:

$$\left(\frac{V_1^{\text{s.t.}}}{\sqrt{E}}\right)_\alpha - \left(\frac{V_2^{\text{s.t.}}}{\sqrt{E}}\right)_\beta = \frac{U^{\text{s.t.}}(L-N)}{E}, \tag{5.7}$$

$$\left(\frac{V_1^{\text{s.t.}}}{\sqrt{E}}\right)_\beta + \left(\frac{V_2^{\text{s.t.}}}{\sqrt{E}}\right)_\alpha = \frac{2U^{\text{s.t.}}M}{E}, \tag{5.8}$$

$$\left(\frac{\widetilde{V}_1}{\sqrt{E}}\right)_\alpha - \left(\frac{\widetilde{V}_2}{\sqrt{E}}\right)_\beta = \frac{\widetilde{U}(L-N)}{E}, \tag{5.9}$$

$$\left(\frac{\widetilde{V}_1}{\sqrt{E}}\right)_\beta + \left(\frac{\widetilde{V}_2}{\sqrt{E}}\right)_\alpha = \frac{2\widetilde{U}M}{E}. \tag{5.10}$$

We also make the decompositions $g = \tau g^{\text{s.t.}} + \widetilde{g}$, $\Xi_1 = \tau \Xi_1^{\text{s.t.}} + \widetilde{\Xi}_1$ and $\Xi_2 = \tau \Xi_2^{\text{s.t.}} + \widetilde{\Xi}_2$. We only give the definition of $\Xi_1^{\text{s.t.}}$, but omit the details of $\widetilde{\Xi}_1$, $\Xi_2^{\text{s.t.}}$ and $\widetilde{\Xi}_2$ since they are almost the same:

$$g^{\text{s.t.}} = \mu_\beta^{\text{s.t.}}\mathbf{X}_\alpha - \mu_\alpha^{\text{s.t.}}\mathbf{X}_\beta, \qquad \widetilde{g} = \widetilde{\mu}_\beta \mathbf{X}_\alpha - \widetilde{\mu}_\alpha \mathbf{X}_\beta. \tag{5.11}$$

$$\begin{aligned}\Xi_1^{\text{s.t.}} = {}& [H_1, \hat{\mathbf{n}}]\left(\frac{\mu_{\alpha\alpha}^{\text{s.t.}}}{2E^{1/2}}\right) + [H_2, \hat{\mathbf{n}}]\left(\frac{\mu_{\alpha\beta}^{\text{s.t.}}}{2E^{1/2}}\right) - [H_1, \hat{\mathbf{t}}^1]\left(\frac{\mu_\alpha^{\text{s.t.}}L}{2E}\right) - [H_1, \hat{\mathbf{t}}^2]\left(\frac{\mu_\alpha^{\text{s.t.}}M}{2E}\right) \\ & - [H_2, \hat{\mathbf{t}}^1]\left(\frac{\mu_\beta^{\text{s.t.}}M}{2E}\right) - [H_2, \hat{\mathbf{t}}^2]\left(\frac{\mu_\beta^{\text{s.t.}}N}{2E}\right) - [H_1, \hat{\mathbf{n}}]\left(\frac{\mu_\alpha^{\text{s.t.}}E_\alpha}{4E^{3/2}}\right) - [H_2, \hat{\mathbf{n}}]\left(\frac{\mu_\beta^{\text{s.t.}}E_\alpha}{4E^{3/2}}\right).\end{aligned}$$

Now, we are ready to give the decomposition of $\mathbf{X}_t$:

$$\mathbf{X}_t = \tau U^{\text{s.t}}\hat{\mathbf{n}} + \tau V_1^{\text{s.t}}\hat{\mathbf{t}}^1 + \tau V_2^{\text{s.t}}\hat{\mathbf{t}}^2 + \widetilde{U}\hat{\mathbf{n}} + \widetilde{V}_1\hat{\mathbf{t}}^1 + \widetilde{V}_2\hat{\mathbf{t}}^2. \tag{5.12}$$

We need to decompose the evolution equation for κ, (2.58), more carefully. Firstly, we decompose Q_i as $Q_i = \tau Q_i^{\text{s.t.}} + \widetilde{Q}_i$. We give some of the formulas but omit others as they are similar:

$$Q_1^{\text{s.t.}} = -\left(\frac{2A_\nu U^{\text{s.t.}}}{2\sqrt{E}}\right)\Lambda\kappa + \left(\frac{V_1^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu \mu_\alpha^{\text{s.t.}}}{2E}\right)\kappa_\alpha + \left(\frac{V_2^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu \mu_\beta^{\text{s.t.}}}{2E}\right)\kappa_\beta,$$

$$\widetilde{Q}_1 = -\left(\frac{2A_\nu \widetilde{U} + Rn_3}{2\sqrt{E}}\right)\Lambda\kappa + \left(\frac{\widetilde{V}_1 - \widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E}\right)\kappa_\alpha + \left(\frac{\widetilde{V}_2 - \widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\beta}{2E}\right)\kappa_\beta,$$

$$
\begin{aligned}
Q_2^{\text{s.t.}} &= \frac{R_3^{\text{s.t.}}}{2E} + \frac{V_1^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^1}{2E^{3/2}}(\kappa E_\alpha) + \frac{V_2^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^2}{2E^{3/2}}(\kappa E_\beta) + + \frac{U^{\text{s.t.}} M^2}{E} + \frac{MH_1\left(\kappa \mu_\alpha^{\text{s.t.}}\right)}{2E} \\
&+ \frac{MH_2\left(\kappa\mu_\beta^{\text{s.t.}}\right)}{2E} - \frac{2U^{\text{s.t.}}M^2}{E^2} - \frac{\Xi_1^{\text{s.t.}} \cdot \hat{\mathbf{t}}^{\mathbf{1}} + D_\alpha \mathcal{J}[\mathbf{X}]g^{\text{s.t.}} \cdot \hat{\mathbf{t}}^{\mathbf{1}} + \Xi_2^{\text{s.t.}} \cdot \hat{\mathbf{t}}^{\mathbf{2}} + D_\beta \mathcal{J}[\mathbf{X}]g^{\text{s.t.}} \cdot \hat{\mathbf{t}}^{\mathbf{2}}}{2E^{3/2}} M \\
&+ \frac{L}{2E}\left(\frac{V_1^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^1}{\sqrt{E}}\right)_\alpha + \frac{N}{2E}\left(\frac{V_2^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^2}{\sqrt{E}}\right)_\beta + \kappa\left(\frac{V_{1,\alpha}^{\text{s.t.}}}{\sqrt{E}} - \frac{U^{\text{s.t.}}L}{E} + \frac{V_2^{\text{s.t.}}E_\beta}{2E^{3/2}}\right).
\end{aligned}
$$

We now conclude that the evolution of κ satisfies

$$
\begin{aligned}
\kappa_t &= \tau\frac{\Lambda}{2E}\left(\frac{\Delta\kappa}{2E^{1/2}}\right) - \tau A_\nu \frac{\Lambda}{2E}\left([H_1, \hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right) \cdot \hat{\mathbf{t}}^{\mathbf{1}} + [H_2, \hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right) \cdot \hat{\mathbf{t}}^{\mathbf{1}}\right) \\
&- \tau A_\nu \frac{\Lambda}{2E}\left([H_1, \hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right) \cdot \hat{\mathbf{t}}^{\mathbf{2}} + [H_2, \hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right) \cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) - \frac{E_\alpha}{8E^{5/2}}\tau H_1 \Delta\kappa - \frac{E_\beta}{8E^{5/2}}\tau H_2\Delta\kappa \\
&- \tau A_\nu \frac{\Lambda}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha - \kappa_{\alpha\alpha}\mathbf{X}_\beta\right) \cdot \hat{\mathbf{t}}^{\mathbf{1}} + \mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha - \kappa_{\alpha\beta}\mathbf{X}_\beta\right) \cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) \\
&+ \frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha - \kappa_{\alpha\alpha}\mathbf{X}_\beta\right) \cdot \hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha + \frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha - \kappa_{\alpha\beta}\mathbf{X}_\beta\right) \cdot \hat{\mathbf{t}}^{\mathbf{2}}\right) \\
&+ \tau Q_1^{\text{s.t.}} + \tau Q_2^{\text{s.t.}} + \widetilde{Q}_1 + \widetilde{Q}_2. \qquad (5.13)
\end{aligned}
$$

We note that the evolution equation without surface tension, i.e. in the case $\tau = 0$, is the following:

$$
\begin{aligned}
\kappa_t = \widetilde{Q}_1 + \widetilde{Q}_2 &= -\frac{2A_\nu\widetilde{U} + Rn_3}{2\sqrt{E}}\Lambda\kappa + \left(\frac{\widetilde{V}_1 - \widetilde{\mathbf{W}} \cdot \hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E}\right)\kappa_\alpha \\
&+ \left(\frac{\widetilde{V}_2 - \widetilde{\mathbf{W}} \cdot \hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\beta}{2E}\right)\kappa_\beta + \widetilde{Q}_2. \qquad (5.14)
\end{aligned}
$$

As shown in [2], the Cauchy problem for (5.14) is well-posed under the assumption

$$
\frac{2A_\nu\widetilde{U} + Rn_3}{2\sqrt{E}}(\vec{\alpha}, 0) > \bar{k} > 0, \qquad (5.15)
$$

for some constant $\bar{k}$. In this section, we also make the above assumption. Note that this is an assumption only on the data.

5.2. *Estimates for the decomposed quantities*

First, we establish the higher regularity of $\widetilde{\mu}$ and $\mu^{\text{s.t.}}$.

Lemma 5.1. *If* $\mathbf{X} \in H^{s+2}$ *and* $\kappa \in H^s$, *then there exists a nondecreasing function* $C(\cdot)$ *such that*

$$
\|\widetilde{\mu}\|_{s+j} \leq C(\|\mathbf{X}\|_{s+j}), \quad \text{for } j = 1, 2, \qquad (5.16)
$$

$$
\|\mu^{\text{s.t.}}\|_s \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}). \qquad (5.17)
$$

Remark 7. We do not prove this lemma here. It is similar to the proof of Lemma 7 of [2].

We immediately conclude the estimates for $g^{\mathrm{s.t}}$ and $\widetilde{g}$ from (5.11):

$$\|g^{\mathrm{s.t.}}\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}), \qquad \|\widetilde{g}\|_{s+1} \leq C(\|\mathbf{X}\|_{s+2}). \tag{5.18}$$

Now we consider $\widetilde{\mathbf{W}}$, $\mathbf{W}^{\mathrm{s.t.}}$. In this subsection, we assume that $\mathbf{X} \in H^{s+2}$ and $\kappa \in H^s$.

Lemma 5.2. *We have the following estimates for the velocities:*

$$\|\widetilde{U}\|_s \leq C(\|\mathbf{X}\|_{s+1}), \tag{5.19}$$

$$\|\widetilde{U}\|_{s+1} \leq C(\|\mathbf{X}\|_{s+2}), \tag{5.20}$$

$$\|U^{\mathrm{s.t.}}\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}), \tag{5.21}$$

$$\|\mathbf{W}^{\mathrm{s.t.}} \cdot \hat{\mathbf{t}}^i\|_s \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}), \quad i = 1, 2, \tag{5.22}$$

$$\|\widetilde{\mathbf{W}} \cdot \hat{\mathbf{t}}^i\|_s \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}), \quad i = 1, 2. \tag{5.23}$$

Proof. We will give the proof of the estimate of $U^{\mathrm{s.t}}$ and $\mathbf{W}^{\mathrm{s.t}} \cdot \hat{\mathbf{t}}^1$ and omit the remaining details.

First recall the equation (5.5),

$$\mathbf{W}^{\mathrm{s.t.}} = H_1\left(\frac{\mu_\alpha^{\mathrm{s.t.}}}{2E^{1/2}}\hat{\mathbf{n}}\right) + H_2\left(\frac{\mu_\beta^{\mathrm{s.t.}}}{2E^{1/2}}\hat{\mathbf{n}}\right) + \mathcal{J}[\mathbf{X}]g^{\mathrm{s.t}}, \tag{5.24}$$

where

$$\begin{aligned}\mathcal{J}[\mathbf{X}]g^{\mathrm{s.t}} = \mathcal{K}[\mathbf{X}]g^{\mathrm{s.t.}} &+ G_{11}\left(\frac{g^{\mathrm{s.t}} \times \mathbf{X}_{\alpha\alpha}}{2E^{3/2}} - \frac{3(g^{\mathrm{s.t.}} \times \mathbf{X}_\alpha)E_\alpha}{4E^{5/2}}\right)\\ &+ G_{12}\left(\frac{g^{\mathrm{s.t.}} \times \mathbf{X}_{\alpha\beta}}{2E^{3/2}} - \frac{3(g^{\mathrm{s.t.}} \times \mathbf{X}_\alpha)E_\beta + 3(g^{\mathrm{s.t.}} \times \mathbf{X}_\beta)E_\alpha}{4E^{5/2}}\right)\\ &+ G_{22}\left(\frac{g^{\mathrm{s.t.}} \times \mathbf{X}_{\beta\beta}}{2E^{3/2}} - \frac{3(g^{\mathrm{s.t.}} \times \mathbf{X}_\beta)E_\beta}{4E^{5/2}}\right).\end{aligned}$$

By Lemma 3.1 and the estimate (5.18) we have

$$\|\mathcal{K}[\mathbf{X}]g^{\mathrm{s.t.}}\|_s \leq C(1 + \|\mathbf{X}\|_{s+1})^2\|g^{\mathrm{s.t.}}\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}).$$

Using the fact that the operator G_{ij} is of order -1, we estimate the last three terms on the right-hand side of $\mathcal{J}[\mathbf{X}]g^{\mathrm{s.t}}$. Now we are ready to conclude that

$$\|\mathcal{J}[\mathbf{X}]g^{\mathrm{s.t.}}\|_s \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}).$$

It is obvious that $\|H_1(\frac{\mu_\alpha^{\mathrm{s.t}}}{2E^{1/2}}\hat{\mathbf{n}})\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1})$ and $\|H_2(\frac{\mu_\beta^{\mathrm{s.t}}}{2E^{1/2}}\hat{\mathbf{n}})\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1})$. This completes

$$\|U^{\text{s.t.}}\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}).$$

Rewriting $\mathbf{W}^{\text{s.t}}$, and then taking the inner product with $\hat{\mathbf{t}}^i$, we have

$$\mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^i = \mathcal{J}[\mathbf{X}]g^{\text{s.t.}} \cdot \hat{\mathbf{t}}^i + \left[H_1, \hat{\mathbf{n}}\right]\left(\frac{\mu_\alpha^{\text{s.t.}}}{2E^{1/2}}\right) \cdot \hat{\mathbf{t}}^i + \left[H_2, \hat{\mathbf{n}}\right]\left(\frac{\mu_\beta^{\text{s.t.}}}{2E^{1/2}}\right) \cdot \hat{\mathbf{t}}^i. \tag{5.25}$$

So, applying Lemma 3.4 and the estimate for $\mu^{\text{s.t}}$ will complete the estimate $\|\mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^i\|_s \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1})$. □

Now we are ready to estimate $\widetilde{V}_i$, $V_i^{\text{s.t}}$, $i = 1, 2$, using the equations (5.7), (5.8), (5.9), and (5.10).

Lemma 5.3. *We have the following estimates:*

$$\|\widetilde{V}_i\|_{s+1} \leq C(\|\mathbf{X}\|_{s+1}), \tag{5.26}$$

$$\|\widetilde{V}_i\|_{s+2} \leq C(\|\mathbf{X}\|_{s+2}), \tag{5.27}$$

$$\|V_i^{\text{s.t}}\|_s \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}). \tag{5.28}$$

We omit the details of the proof. Moving on, to estimate the terms $R_i^{\text{s.t}}$, we need estimates for $\Xi_i^{\text{s.t}}$ for $i = 1, 2$ first.

Lemma 5.4. *We have the following estimates:*

$$\|\Xi_i^{\text{s.t}}\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}), \tag{5.29}$$

$$\|\Xi_i^{\text{s.t}} \cdot \hat{\mathbf{n}}\|_s \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}), \tag{5.30}$$

for $i = 1, 2$.

The estimates (5.29) and (5.30) follow from the estimates (3.6) and (3.7).

Lemma 5.5. *We have the following estimates:*

$$\|Q_1^{\text{s.t}}\|_{s-1} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}), \tag{5.31}$$

$$\|Q_2^{\text{s.t}}\|_{s-3/2} \leq C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}). \tag{5.32}$$

Proof. First, we have the expression

$$Q_1^{\text{s.t.}} = -\frac{2A_\nu U^{\text{s.t.}}}{2\sqrt{E}}\Lambda(\kappa) + \left(\frac{V_1^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu \mu_\alpha^{\text{s.t.}}}{2E}\right)\kappa_\alpha + \left(\frac{V_2^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu \mu_\beta^{\text{s.t.}}}{2E}\right)\kappa_\beta.$$

To estimate the first term on the right-hand side, by Lemma 3.9 we have

$$\left\| -\frac{2A_\nu U^{\text{s.t.}}}{2\sqrt{E}}\Lambda(\kappa)\right\|_{s-1} \le c\left\|\frac{1}{\sqrt{E}}\right\|_s \|U^{\text{s.t.}}\Lambda(\kappa)\|_{s-1}$$
$$\le c\|E\|_{s-1}(\|U^{\text{s.t.}}\|_d\|\Lambda(\kappa)\|_{s-1} + \|U^{\text{s.t.}}\|_{s-1}\|\Lambda(\kappa)\|_d).$$

So by Lemma 5.2, we have $\| -\frac{2A_\nu U^{\text{s.t.}}}{2\sqrt{E}}\Lambda(\kappa)\|_{s-1} \le C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1})$. The remaining terms are similar and we omit further details. □

When considering the estimate of $\widetilde{Q}_2$, it is similar to the case of $\tau = 0$ (see [2]). We will state a conclusion here without further proof.

Lemma 5.6. *We have the following estimate:*

$$\|\widetilde{Q}_2\|_s \le C(\|\mathbf{X}\|_{s+2}) \le C(\|\kappa\|_s, \|\mathbf{X}\|_{s+1}).$$

5.3. Uniform time of existence

Recalling Remark 5, we know that, thus far, the time of existence of a solution is dependent of τ. Therefore before we may take a limit as τ vanishes we must revisit the energy estimate to get a uniform time of existence. We will introduce an open subset of $\mathcal{O}$. Letting $\bar{k} > 0$, $\mathcal{O}_k \subset \mathcal{O}$ is defined as

$$\mathcal{O}_k = \left\{ \mathbf{X} \in \mathcal{O} : \forall\vec{\alpha}, \frac{2A_\nu\widetilde{U} + Rn_3}{2\sqrt{E}}(\vec{\alpha}) > \bar{k} > 0 \right\}. \tag{5.33}$$

We have the following theorem:

Theorem 5.7. *Let the surface $\mathbf{X}_0 \in \mathcal{O}_k$ be globally parameterized by harmonic coordinates (namely (2.3) holds). There exists $T > 0$ such that for all $\tau \in (0, 1)$, the solution of Cauchy problem (4.43) with initial $\mathbf{X}^\tau(\cdot, 0) = \mathbf{X}_0$ exists on [0,T], and $\mathbf{X}^\tau \in C([0, T]; \overline{\mathcal{O}}_k)$.*

Proof. We do energy estimates in the same way but use the new expressions for the evolution with respect to the decompositions. We define the following:

$$\mathcal{E}_0 = \frac{1}{2}\iint \kappa^2 d\alpha d\beta, \tag{5.34}$$

$$\mathcal{E}_1 = \frac{1}{2}\iint \left(J^{s+1}\mathbf{X}\right)^2 d\alpha d\beta, \tag{5.35}$$

$$\mathcal{E}_2 = \frac{1}{2}\iint \left(\Lambda^s\kappa\right)^2 d\alpha d\beta, \tag{5.36}$$

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2. \tag{5.37}$$

To begin with, we take the time derivative of E_0:

$$\frac{d\mathcal{E}_0}{dt} = \iint \kappa\kappa_t d\alpha d\beta.$$

Since s is sufficiently large, using the equation for κ_t and the preliminary estimates, it immediately follows that

$$\frac{d\mathcal{E}_0}{dt} \le C_1 \exp\{C_2\mathcal{E}\}.$$

Here, the constants C_1 and C_2 may be taken to be uniform with respect to $\tau \in (0, 1)$.

Recalling equation (5.12), we next take the time derivative of $\mathcal{E}_1$:

$$\begin{aligned}
\frac{d\mathcal{E}_1}{dt} &= \iint J^{s+2}\mathbf{X} \cdot J^s\mathbf{X}_t d\alpha d\beta \le \|\mathbf{X}\|_{s+2}\|\mathbf{X}_t\|_s \\
&\le \|\mathbf{X}\|_{s+2}\|\tau U^{\text{s.t}}\hat{\mathbf{n}} + \tau V_1^{\text{s.t}}\hat{\mathbf{t}}^1 + \tau V_2^{\text{s.t}}\hat{\mathbf{t}}^2 + \widetilde{U}\hat{\mathbf{n}} + \widetilde{V}_1\hat{\mathbf{t}}^1 + \widetilde{V}_2\hat{\mathbf{t}}^2\|_s \\
&\le \tau\|\mathbf{X}\|_{s+2}\|U^{\text{s.t}}\|_s\|\hat{\mathbf{n}}\|_s + \|\mathbf{X}\|_{s+2}\|\tau V_1^{\text{s.t}}\hat{\mathbf{t}}^1 + \tau V_2^{\text{s.t}}\hat{\mathbf{t}}^2 + \widetilde{U}\hat{\mathbf{n}} + \widetilde{V}_1\hat{\mathbf{t}}^1 + \widetilde{V}_2\hat{\mathbf{t}}^2\|_s \\
&\le \tau\|\kappa\|_{s+1}C(\mathcal{E}) + C(\mathcal{E}).
\end{aligned}$$

By Young's inequality, $\tau\|\kappa\|_{s+1}C(\mathcal{E}) \le \frac{\tau}{n}\|\kappa\|_{s+1}^2 + n\tau C(\mathcal{E})^2$, with parameter n to be chosen. So we conclude that

$$\frac{d\mathcal{E}_1}{dt} \le \frac{\tau}{n}\|\kappa\|_{s+1}^2 + C(\mathcal{E}). \tag{5.38}$$

Finally we take the time derivative of $\mathcal{E}_2$. Recalling the estimate (4.10), we have

$$\begin{aligned}
\frac{d\mathcal{E}_2}{dt} &= \iint \left(\Lambda^s\kappa\right)\Lambda^s\kappa_t d\alpha d\beta \\
&\le -\tau\iint \frac{1}{4E^{3/2}}\left(\Lambda^{s+3/2}\kappa\right)\Lambda^{s+3/2}\kappa d\alpha d\beta + \frac{3\tau\|\Lambda^{s+3/2}\kappa\|_0^2}{n} + C\tau\mathcal{E}_2 \\
&+ \tau\iint\left(\Lambda^{s+3/2}\kappa\right)\Lambda^{s-3/2}\left(Q_1^{\text{s.t.}} + Q_2^{\text{s.t.}}\right)d\alpha d\beta + \iint\left(\Lambda^s\kappa\right)\Lambda^s\left(\widetilde{Q}_1 + \widetilde{Q}_2\right)d\alpha d\beta.
\end{aligned} \tag{5.39}$$

Using the Hölder inequality and the estimate of $Q_1^{\text{s.t.}}$, $Q_2^{\text{s.t.}}$ from Lemma 5.5, it is immediate that

$$\tau\iint\left(\Lambda^{s+3/2}\kappa\right)\Lambda^{s-3/2}\left(Q_1^{\text{s.t.}} + Q_2^{\text{s.t.}}\right)d\alpha d\beta \le \frac{\tau\|\Lambda^{s+3/2}\kappa\|_0^2}{n} + C(\mathcal{E}).$$

Thus by Lemma 5.6 and the Hölder inequality, we conclude that

$$\iint\left(\Lambda^s\kappa\right)\Lambda^s\widetilde{Q}_2 d\alpha d\beta \le C(\mathcal{E}).$$

Next we consider

$$\iint \left(\Lambda^s\kappa\right)\Lambda^s \widetilde{Q}_1 d\alpha d\beta =$$
$$\iint \Bigg[\left(\Lambda^s\kappa\right)\Lambda^s \left(-\frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}}\Lambda\kappa\right) + \left(\Lambda^s\kappa\right)\Lambda^s \Bigg(\left(\frac{\widetilde{V}_1-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E}\right)\kappa_\alpha$$
$$+ \left(\frac{\widetilde{V}_2-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\beta}{2E}\right)\kappa_\beta\Bigg)\Bigg] d\alpha d\beta. \quad (5.40)$$

Notice that we have the following estimate:

$$\iint \left(\Lambda^s\kappa\right)\Lambda^s \left(\left(\frac{\widetilde{V}_1-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E}\right)\kappa_\alpha + \left(\frac{\widetilde{V}_2-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\beta}{2E}\right)\kappa_\beta\right) d\alpha d\beta \le C(\mathcal{E}),$$

since

$$\left\| \frac{\widetilde{V}_1-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E} \right\|_s \le C(\mathcal{E})$$

and

$$\left\| \frac{\widetilde{V}_2-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\beta}{2E} \right\|_s \le C(\mathcal{E}).$$

It remains to deal with one term,

$$\iint \left(\Lambda^s\kappa\right)\Lambda^s \left(-\frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}}\Lambda\kappa\right) d\alpha d\beta = -\iint \frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}} (\Lambda^{s+1/2}\kappa)^2 d\alpha d\beta$$
$$- \iint \left(\Lambda^{s+1/2}\kappa\right)\left[\Lambda^{s-1/2}, \frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}}\right]\Lambda\kappa d\alpha d\beta.$$

By Lemma 3.8, we have

$$\left\| \left[\Lambda^{s-1/2}, \frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}}\right]\Lambda\kappa \right\| \le C(\mathcal{E})\|\kappa\|_{s+1/2} \le C(\mathcal{E}) + \frac{\bar{k}}{2}\|\Lambda^{s+1/2}\kappa\|^2.$$

As long as the solution remains in the set $\mathcal{O}_k$, we have

$$\iint \left(\Lambda^s\kappa\right)\Lambda^s \widetilde{Q}_1 d\alpha d\beta \le -\frac{\bar{k}}{2}\|\Lambda^{s+1/2}\kappa\|^2 + C(\mathcal{E}).$$

We now conclude that

$$\frac{d\mathcal{E}}{dt} \le -m\tau\|\Lambda^{s+3/2}\kappa\|^2/5 - \frac{\bar{k}}{2}\|\Lambda^{s+1/2}\kappa\|^2 + C(\mathcal{E}), \quad (5.41)$$

where m and n are as given previously. □

Remark 8. From the above energy estimate, it is also proved that the solutions are uniformly bounded, that is, there exists $d > 0$, for all $\tau \in (0,1)$, $\|\mathbf{X}\|_{s+2} \le d$. This implies that solutions cannot leave the set $\mathcal{O}_k$ arbitrarily fast and thus that solutions exist in the set $\mathcal{O}_k$ for an interval of time which is uniform for $\tau \in (0,1)$.

5.4. *Cauchy sequence as* $\tau \to 0^+$

We denote for any fixed $\tau > 0$, $\tau' > 0$, the corresponding solutions $\mathbf{X} \in C^0([0,T], \overline{\mathcal{O}}_k)$, $\mathbf{X}' \in C^0([0,T], \overline{\mathcal{O}}_k)$, respectively. Both surfaces $\mathbf{X}$ and $\mathbf{X}'$ are globally isothermally parameterized and uniformly bounded in H^{s+2}. So for the curvatures κ, κ' are uniformly bounded in H^s. Now we will prove that they are Cauchy sequences; the main result is the following theorem (recall again throughout that we take s to be large enough).

Theorem 5.8. *For any $\eta > 0$, there exists $\delta > 0$ such that if $|\tau - \tau'| < \delta$, then*

$$\sup_{t\in[0,T]} \|\mathbf{X} - \mathbf{X}'\|_3 + \|\kappa - \kappa'\|_2 < \eta. \tag{5.42}$$

Proof. We denote $\delta\mathbf{X} = \mathbf{X} - \mathbf{X}'$ and $\delta\kappa = \mathbf{X} - \mathbf{X}'$. We define

$$\mathcal{D} = \frac{1}{2}\|\delta\mathbf{X}\|_3^2 + \frac{1}{2}\|\delta\kappa\|_0^2 + \frac{1}{2}\|\Lambda^2\delta\kappa\|_0^2. \tag{5.43}$$

First, we notice that

$$\Delta\delta\mathbf{X} = 2E\kappa\hat{\mathbf{n}} - 2E'\kappa'\hat{\mathbf{n}}'.$$

Therefore we have

$$\|\delta\mathbf{X}\|_4 \le C\|\delta\kappa\|_2 + C\|\delta\mathbf{X}\|_3 \le C\mathcal{D}^{1/2}.$$

Furthermore, we see also that

$$\|\delta E\|_4 \le C\|\delta\mathbf{X}\|_4 \le C\|\delta\kappa\|_2 + C\|\delta\mathbf{X}\|_3 \le C\mathcal{D}^{1/2},$$

because of the following:

$$\Delta\delta E = 2(\mathbf{X}_{\alpha\beta}\cdot\mathbf{X}_{\alpha\beta} - \mathbf{X}'_{\alpha\beta}\cdot\mathbf{X}'_{\alpha\beta}) - 2(\mathbf{X}_{\alpha\alpha}\cdot\mathbf{X}_{\beta\beta} - \mathbf{X}'_{\alpha\alpha}\cdot\mathbf{X}'_{\beta\beta}).$$

The estimates for decompositions of differences are similar to the results of Section 5.2, when we take $s = 2$. For example:

$$\begin{aligned}
&\|\delta\mu^{\text{s.t.}}\|_2 \le C\mathcal{D}^{1/2},\\
&\|\delta\widetilde{\mu}\|_{2+j} \le c\|\delta\mathbf{X}\|_{2+j},\ j = 1, 2,\\
&\|\delta U^{\text{s.t}}\|_1 \le C\mathcal{D}^{1/2},\\
&\|\delta Q_1^{\text{s.t}}\|_1 \le C\mathcal{D}^{1/2},\\
&\|\delta Q_2^{\text{s.t}}\|_{1/2} \le C\mathcal{D}^{1/2}.
\end{aligned}$$

We begin estimating time derivatives:

$$\begin{aligned}
&\frac{d\|\delta\mathbf{X}\|_3^2}{2dt} \le \|\delta\mathbf{X}\|_4^2 + \|\delta\mathbf{X}\|_1^2 \\
&\le \|\delta\mathbf{X}\|_4^2 + \|\tau U^{\text{s.t}}\hat{\mathbf{n}} - \tau' U'^{\text{s.t}}\hat{\mathbf{n}}'\|_1 + \|\tau V_1^{\text{s.t}}\hat{\mathbf{t}}^1 + \tau V_2^{\text{s.t}}\hat{\mathbf{t}}^2 - \tau' V_1'^{\text{s.t}}\hat{\mathbf{t}}'^1 - \tau V_2'^{\text{s.t}}\hat{\mathbf{t}}'^2\|_1 \\
&+ \|\widetilde{U}\hat{\mathbf{n}} + \widetilde{V}_1\hat{\mathbf{t}}^1 + \widetilde{V}_2\hat{\mathbf{t}}^2 - \widetilde{U}'\hat{\mathbf{n}}' - \widetilde{V}_1'\hat{\mathbf{t}}'^1 + \widetilde{V}_2'\hat{\mathbf{t}}'^2\|_1 \\
&\le C\mathcal{D} + c|\tau - \tau'|.
\end{aligned}$$

We next consider the time derivative of $\delta\kappa$:

$$\begin{aligned}
\delta\kappa_t = {} & \tau\frac{\Lambda}{2E}\left(\frac{\Delta\delta\kappa}{2\sqrt{E}}\right) - \tau A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\delta\kappa_{\alpha\alpha}}{2\sqrt{E}}\right)\cdot\hat{\mathbf{t}}^{1,l} + [H_2,\hat{\mathbf{n}}]\left(\frac{\delta\kappa_{\alpha\beta}}{2\sqrt{E}}\right)\cdot\hat{\mathbf{t}}^1\right) \\
& - \tau A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\delta\kappa_{\alpha\beta}}{2\sqrt{E}}\right)\cdot\hat{\mathbf{t}}^2 + [H_2,\hat{\mathbf{n}}]\left(\frac{\delta\kappa_{\beta\beta}}{2\sqrt{E^l}}\right)\cdot\hat{\mathbf{t}}^2\right) \\
& - \frac{E_\alpha}{8(E)^{5/2}}\tau H_1\Delta\delta\kappa - \frac{E_\beta}{8(E)^{5/2}}\tau H_2\Delta\delta\kappa \\
& - \tau A_\nu\frac{\Lambda}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\delta\kappa_{\alpha\beta}\mathbf{X}_\alpha - \delta\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^1 + \mathcal{J}[\mathbf{X}]\left(\delta\kappa_{\beta\beta}\mathbf{X}_\alpha - \delta\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^2\right) \\
& + \frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\delta\kappa_{\alpha\beta}\mathbf{X}_\alpha - \delta\kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^1\right)_\alpha + \frac{\tau}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\delta\kappa_{\beta\beta}\mathbf{X}_\alpha - \delta\kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^2\right) \\
& + (\tau-\tau')F(\kappa',\mathbf{X}',E') + \tau(F(\kappa',\mathbf{X},E) - F(\kappa',\mathbf{X}',E')) + \tau(Q_1^{\text{s.t.}} + Q_2^{\text{s.t.}} - Q_1'^{\text{s.t.}} - Q_2'^{\text{s.t.}}) \\
& + (\tau-\tau')(Q_1'^{\text{s.t.}} + Q_2'^{\text{s.t.}}) + \widetilde{Q}_1 + \widetilde{Q}_2 - \widetilde{Q}_1' - \widetilde{Q}_2',
\end{aligned}$$

where $F(\kappa, \mathbf{X}, E)$ is a function depending on $\kappa, \mathbf{X}, E$:

$$\begin{aligned}
F(\kappa,\mathbf{X},E) = {} & \frac{\Lambda}{2E}\left(\frac{\Delta\kappa}{2E^{1/2}}\right) - A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\alpha}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}} + [H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right) \\
& - A_\nu\frac{\Lambda}{2E}\left([H_1,\hat{\mathbf{n}}]\left(\frac{\kappa_{\alpha\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}} + [H_2,\hat{\mathbf{n}}]\left(\frac{\kappa_{\beta\beta}}{2E^{1/2}}\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right) - \frac{E_\alpha}{8E^{5/2}}H_1\Delta\kappa - \frac{E_\beta}{8E^{5/2}}H_2\Delta\kappa \\
& - A_\nu\frac{\Lambda}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha - \kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}} + \mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha - \kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right) \\
& + \frac{1}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\alpha\beta}\mathbf{X}_\alpha - \kappa_{\alpha\alpha}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{1}}\right)_\alpha + \frac{1}{2E}\left(\mathcal{J}[\mathbf{X}]\left(\kappa_{\beta\beta}\mathbf{X}_\alpha - \kappa_{\alpha\beta}\mathbf{X}_\beta\right)\cdot\hat{\mathbf{t}}^{\mathbf{2}}\right). \quad (5.44)
\end{aligned}$$

It is easy to conclude that

$$\|F(\kappa',\mathbf{X},E) - F(\kappa',\mathbf{X}',E')\|_{1/2} \le C\|\delta\mathbf{X}\|_3 \quad (5.45)$$

and

$$\|\delta\widetilde{Q}_2\|_2 \le C\|\delta\mathbf{X}\|_4 \le C\mathcal{D}^{1/2}. \quad (5.46)$$

For $\delta\widetilde{Q}_1$, we need to take more care:

$$\delta\widetilde{Q}_1 = -\left(\frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}}\right)\Lambda\delta\kappa + \left(\frac{\widetilde{V}_1-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E}\right)\delta\kappa_\alpha + \left(\frac{\widetilde{V}_2-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\beta}{2E}\right)\delta\kappa_\beta + \mathcal{Y}, \tag{5.47}$$

where $\mathcal{Y}$ is given by

$$\mathcal{Y} = -\left(\frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}} - \frac{2A_\nu\widetilde{U}'+Rn_3'}{2\sqrt{E}'}\right)\Lambda\kappa' + \left(\frac{\widetilde{V}_1-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E} - \frac{\widetilde{V}_1'-\widetilde{\mathbf{W}}'\cdot\hat{\mathbf{t}}'^1}{\sqrt{E'}} + \frac{A_\nu\widetilde{\mu}_\alpha'}{2E'}\right)\kappa_\alpha' + \left(\frac{\widetilde{V}_2-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\beta}{2E} - \frac{\widetilde{V}_2'-\widetilde{\mathbf{W}}'\cdot\hat{\mathbf{t}}'^2}{\sqrt{E'}} + \frac{A_\nu\widetilde{\mu}_\beta'}{2E'}\right)\kappa_\beta'.$$

We estimate $\mathcal{Y}$ as follows:

$$\|\mathcal{Y}\|_2 \le c\|\delta\widetilde{U}\|_2 + c\|\delta\widetilde{\mathbf{W}}\|_2 + c\|\delta E\|_2 + c\|\delta\widetilde{V}_1\|_2 + c\|\delta\widetilde{V}_2\|_2 + c\|\delta\widetilde{\mu}\|_3 \le C\|\delta\mathbf{X}\|_3.$$

We use the above calculations and estimates to estimate the time derivative of part of the energy:

$$\begin{aligned}
\frac{d(\Lambda^2\delta\kappa)^2}{2dt} &= \iint \left(\Lambda^2\delta\kappa\right)\Lambda^2\delta\kappa_t\, d\alpha d\beta \\
&\le -\tau\iint \frac{1}{4E^{3/2}}\left(\Lambda^{2+3/2}\delta\kappa\right)\Lambda^{2+3/2}\delta\kappa\, d\alpha d\beta + \frac{3\tau\|\Lambda^{2+3/2}\delta\kappa\|_0^2}{n} + C\tau\mathcal{D} \\
&\quad + \tau\iint\left(\Lambda^{2+3/2}\delta\kappa\right)\Lambda^{1/2}\Big(F(\kappa',\mathbf{X},E) - F(\kappa',\mathbf{X}',E') + Q_1^{\text{s.t.}} + Q_2^{\text{s.t.}} - Q_1'^{\text{s.t.}} - Q_2'^{\text{s.t.}}\Big)d\alpha d\beta \\
&\quad + \iint\left(\Lambda^2\delta\kappa\right)\Lambda^2\Big((\tau-\tau')F(\kappa',\mathbf{X}',E') + (\tau-\tau')(Q_1'^{\text{s.t.}} + Q_2'^{\text{s.t.}})\Big)d\alpha d\beta \\
&\quad + \iint\left(\Lambda^2\delta\kappa\right)\Lambda^2(\widetilde{Q}_1 + \widetilde{Q}_2 - \widetilde{Q}_1' - \widetilde{Q}_2')\,d\alpha d\beta \\
&\le -\tau\iint \frac{1}{4E^{3/2}}\left(\Lambda^{2+3/2}\delta\kappa\right)\Lambda^{2+3/2}\delta\kappa\, d\alpha d\beta + \frac{4\tau\|\Lambda^{2+3/2}\delta\kappa\|_0^2}{n} + C\mathcal{D} + c|\tau-\tau'|\mathcal{D}^{1/2} \\
&\quad + \iint\left(\Lambda^2\delta\kappa\right)\Lambda^2\Bigg(-\left(\frac{2A_\nu\widetilde{U}+Rn_3}{2\sqrt{E}}\right)\Lambda\delta\kappa + \left(\frac{\widetilde{V}_1-\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^1}{\sqrt{E}} - \frac{A_\nu\widetilde{\mu}_\alpha}{2E}\right)\delta\kappa_\alpha
\end{aligned}$$

$$+\left(\frac{\widetilde{V}_2 - \widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}^2}{\sqrt{E}} - \frac{A_\nu \widetilde{\mu}_\beta}{2E}\right)\delta\kappa_\beta\Bigg)d\alpha d\beta.$$

Using the fact there exists n such that $-\frac{1}{4E^{3/2}} + \frac{4}{n} \le -k_0 < 0$ and $-\frac{2A_\nu \widetilde{U} + Rn_3}{2\sqrt{E}} \le -\bar{k} < 0$, we conclude that

$$\frac{d\mathcal{D}}{dt} \le d_1\mathcal{D} + d_2|\tau - \tau'|\mathcal{D}^{1/2}. \tag{5.48}$$

Solving the differential inequality, we conclude

$$\mathcal{D} \le \mathcal{D}(0)e^{d_1 t} + d_2|\tau - \tau'|(e^{d_1 t} - 1)/d_1.$$

We know that the two surfaces start with the same initial condition, i.e. $\mathcal{D}(0) = 0$, and the proof is thus complete. □

From the paper [2] (see also [14]), we know that 3D Darcy flow without surface tension is well-posed in the presence of the stability condition. That is, for the system (4.43) when $\tau = 0$, there exists a bounded solution $\mathbf{X} \in C^0([0, T']; \overline{\mathcal{O}}_k)$. Now we will prove that the limit as surface tension vanishes for Darcy flow with surface tension is the Darcy flow without surface tension, when the stability condition holds. This is the content of our final theorem.

Theorem 5.9. *Let $\mathbf{X}_0 \in \overline{\mathcal{O}}_k$ be globally parameterized by harmonic coordinates. Let T be the minimum of T in the Theorem 5.7 and T'. For all $\tau \in (0, 1)$, let $\mathbf{X}^\tau$ be the solution for the Cauchy problem* (4.43)*. Letting s' be given such that $0 \le s' < s + 2$, we have*

$$\lim_{\tau\to 0}\sup_{t\in[0,T]} \|\mathbf{X}^\tau - \mathbf{X}\|_{s'} = 0.$$

Proof. From Theorem 5.8, we see that $\mathbf{X}^\tau$ is a Cauchy sequence in H^3. Since the solutions $\mathbf{X}^\tau$ are uniformly bounded in H^{s+2} with respect to τ, by Sobolev interpolation, we have that the sequence $\mathbf{X}^\tau$ is a Cauchy sequence in $H^{s'}$. Therefore, there exists a limit $\mathbf{X} \in C^0([0, T]; \overline{\mathcal{O}}_k)$, such that $\mathbf{X}^\tau \to \mathbf{X}$ as $\tau \to 0$.

Now we will prove that the limit $\mathbf{X}$ is exactly the solution of Cauchy problem without surface tension, i.e. when $\tau = 0$. We call the right-hand side of (5.12) by name $\mathcal{B}^\tau$. Since s is sufficiently large and $\mathbf{X}^\tau \to \mathbf{X} \in C^0([0, T], H^{s'})$, we see that μ^τ converges uniformly to $\mu^{\tau=0}$, and furthermore $\mathcal{B}^\tau$ converges uniformly to $B^{\tau=0}$. We integrate (5.12) in time,

$$\mathbf{X}^\tau(\cdot, t) = \mathbf{X}_0 + \int_0^t \mathcal{B}^\tau(\cdot, s)ds.$$

We pass to the limit as $\tau \to 0^+$ since $\mathcal{B}^\tau$ uniformly bounded, finding

$$\mathbf{X}(\cdot, t) = \mathbf{X}_0 + \int_0^t \mathcal{B}^{\tau=0}(\cdot, s)ds.$$

This implies that $\mathbf{X}$ solves the Cauchy problem without surface tension. □

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