Rev. Mat. Iberoam. **37** (2021), no. 3, 821–910
DOI 10.4171/RMI/1208

The Dirichlet problem in domains with lower dimensional boundaries

Joseph Feneuil, Svitlana Mayboroda and Zihui Zhao

Abstract. The present paper pioneers the study of the Dirichlet problem with L^q boundary data for second order operators with complex coefficients in domains with lower dimensional boundaries, e.g., in $\Omega := \mathbb{R}^n \setminus \mathbb{R}^d$, with $d < n-1$. Following results of David, Feneuil and Mayboroda, we introduce an appropriate degenerate elliptic operator and show that the Dirichlet problem is solvable for all $q > 1$, provided that the coefficients satisfy the small Carleson norm condition.

Even in the context of the classical case $d = n-1$, (the analogues of) our results are new. The conditions on the coefficients are more relaxed than the previously known ones (most notably, we do not impose any restrictions whatsoever on the first $n-1$ rows of the matrix of coefficients) and the results are more general. We establish local rather than global estimates between the square function and the non-tangential maximal function and, perhaps even more importantly, we establish new Moser-type estimates at the boundary and improve the interior ones.

Contents

Mathematics Subject Classification (2010): Primary 35J25; Secondary 35J70.
Keywords: Degenerate elliptic operators, Dirichlet problem, complex coefficients, Moser-type estimates, non-tangential maximal function, square functionals, Carleson measures.

1. Introduction

1.1. State of the art

The main objective of the present article is the study of the well-posedness of the Dirichlet problem. Formally, given an open domain $\Omega \subset \mathbb{R}^n$, an elliptic operator L in divergence form, and $q \in (1, +\infty)$, we say that the Dirichlet problem (D_q) is well-posed if for any $g \in L^p(\partial\Omega, \sigma)$, we can find a unique function u such that

$$Lu = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

and

$$\|N(u)\|_{L^q(\partial\Omega,\sigma)} \leq C\|g\|_{L^q(\partial\Omega,\sigma)}, \tag{1.3}$$

where N is a non-tangential maximal function and the constant $C > 0$ is independent of g. Obviously, the statement above is not complete as we need to make precise the meaning of (1.1)–(1.3). We naturally expect (1.1) to be taken in the weak sense. But what is the meaning of (1.2) when u is defined on the open set Ω? If Ω is very irregular, a proper choice of the measure σ on $\partial\Omega$ and the definition of $N(u)$ is unclear as well. What (1.1)–(1.3) means in our context will be carefully explained later, but first, let us give a brief (and somewhat narrowly focused) presentation of the relevant history of the Dirichlet problem. Due to the huge literature on the topic, we will be unable to cite all the works, and we apologize in advance for the omissions.

We start the history on the topic with a work of Dahlberg (see [8]). Let Ω be a bounded Lipschitz domain. Dahlberg proved that the harmonic measure (for the Laplace operator) defined on the boundary $\partial\Omega$ is A^∞-absolutely continuous[1] with respect to the surface measure on $\partial\Omega$. This property is known to imply that the Dirichlet problem (D_q), associated with the domain Ω and the elliptic operator Δ, is well-posed for large enough q. It was proved just a little later that D_q is well-posed for the Laplacian on Lipschitz domains for any $2 - \varepsilon < q < \infty$, and that the range is sharp in the sense that for any $q < 2$, we can find a Lipschitz domain such that (D_q) is false [9], [21].

Consider the Laplacian in the domain Ω lying above the graph of a Lipschitz function $\varphi\colon x \in \mathbb{R}^{n-1} \to \mathbb{R}$. Let us keep in mind that a bi-Lipschitz change of variable $\rho\colon \Omega \to \rho(\Omega)$ preserves the well-posedness of the Dirichlet problem. In [20] Jerison and Kenig use the change of variable

$$\rho\colon (x,t) \in \mathbb{R}^n \mapsto (x, t - \varphi(x))$$

that flattens the domain Ω and that maps the Laplacian to another elliptic operator $\mathcal{L} = -\operatorname{div} A_0 \nabla$ with bounded, measurable, symmetric, and t-independent coefficient. By using a Rellich identity, they establish that these conditions on $\mathcal{L}$ are sufficient to ensure that (D_2) is well-posed, and hence, they extend the result

[1] A^∞ absolute continuity is a quantitative version of the mutual absolute continuity.

of Dahlberg to the case where the Laplacian is replaced by an elliptic operator $L = -\operatorname{div} A\nabla$, where A has real, bounded, symmetric, and t-independent coefficients. Analogous results for real non-symmetric operators have been proved much later in [23], [18] using square function/non-tangential maximal function estimates and elements of the solution of the Kato problem. The complete situation for operators with complex coefficients is still not clear, although the solution of the Kato problem [2] and later developments allowed to treat block matrices and some of their generalizations [3], [26].

The next breakthrough we shall talk about in the study of the Dirichlet problem is a result from Kenig and Pipher (see [24], which use methods developed in [23]). Consider again the Laplacian and $\Omega = \{t > \varphi(x)\}$, a domain that lies above the graph of the function $\varphi\colon x \in \mathbb{R}^{n-1} \to \mathbb{R}$. The change of variable

$$\rho\colon (x,t) \in \mathbb{R}^n \mapsto (x, ct - \varphi_t(x)),$$

where c is a large constant and φ_t is the convolution of φ by a smooth mollifier, also sends Ω to $\mathbb{R}^n_+$, but maps now the Laplacian $-\Delta$ to an elliptic operator $\mathcal{L} := -\operatorname{div} A_0\nabla$, where A_0 satisfies the conditions that $|\nabla A_0(x,t)| \leq C/t$ and $|t\nabla A_0(x,t)|^2\, dx\, dt/t$ is a Carleson measure. Kenig and Pipher showed that the two latter conditions are enough to ensure that (D_q) is well-posed if q is large enough, hence extending the result of Dahlberg to a new class of elliptic operators.

Dindoš, Petermichl, and Pipher studied in [12] the conditions needed for the well-posedness of (D_q) when $q > 1$ is small. They established that, for a given $q > 1$, the Dirichlet problem (D_q), associated to the Lipschitz domain Ω and the elliptic operator $L = -\operatorname{div} A\nabla$, is well-posed if both the Lipschitz constant of Ω and the Carleson norm of $|t\nabla A|^2\, dx\, dt/t$ are smaller than $\epsilon(q) \ll 1$.

One has to also mention a number of perturbation results, in L^∞ and in Carleson measure norm, which we shall not review here.

Our focus is on operators with coefficients whose gradient satisfies the Carleson measure condition, as above. All the previous results that we mentioned in this context were established in the case where $L = -\operatorname{div} A\nabla$ has real coefficients. In [14] Dindoš and Pipher introduced a notion of q-ellipticity based on a notion of L^q-dissipativity (see [6], [7]), and cleverly used this notion of q-ellipticity to obtain "q-Caccіopoli's inequalities" and "reverse Hölder inequalities" (see Subsection 1.4 for the precise statement), which can be seen as a weakened version of Moser's estimates. They used these partial estimates to get the well-posedness of the Dirichlet problem (D_q) whenever the q-ellipticity, in addition to appropriate Carleson measure estimates and some structural conditions on coefficients, holds.

In all the above works, the boundary of the domain $\Omega \subset \mathbb{R}^n$ has Hausdorff dimension $n-1$. In [10], Guy David and the first two authors of the present article launched an elliptic theory adapted to domains Ω that are the complement in $\mathbb{R}^n$ of sets Γ with dimensions $d < n-1$. Since the boundary Γ of the domain Ω is too thin to be "seen" by the Laplacian (or the general elliptic operators), the operators $L := -\operatorname{div} A\nabla$ used in this theory are degenerate and satisfy the ellipticity condition with a different homogeneity. Precise definition are given later, see (1.24), (1.25). We also mention that similar degenerate operators have been considered before, notably in [15], [16] but the well-posedness or any related boundary estimates have never been attacked.

Subsequently in [11], David, Feneuil, and Mayboroda established that if Γ is the graph of a function $\varphi\colon \mathbb{R}^d \to \mathbb{R}^{n-d}$ with small Lipschitz constant, we can find a particular degenerate elliptic operator $L := -\operatorname{div} A\nabla$ such that the harmonic measure on Γ (and associated to L) is A^∞-absolutely continuous with respect to the d-dimensional Hausdorff measure on Γ. While this operator, being the simplest one that we can treat, can be thought of as an analogue of the Laplacian in the domains with lower dimensional boundaries, it already carries most of the difficulties exhibited by the operators whose coefficients satisfy the aforementioned Carleson condition. Indeed, by necessity, it is not a constant coefficient operator, and it cannot be t-independent either; a rather delicate dependence of the coefficients on the distance to the boundary is exactly what makes the problem well-posed in the higher co-dimensional context. The proof in [11] uses some ideas from [13], [23], [22], and relies on a *new* change of variable ρ that sends Ω to $\mathbb{R}^n \setminus \mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^{n-d} \setminus \{0\})$ and that is almost, up to Carleson measure, an isometry in the last $n-d$ variables. As shown in [27], the A^∞-absolute continuity of the harmonic measure implies the well-posedness of the Dirichlet problem (D_q) when q is large.

The main aim of this article is to prove that, given $q > 1$ and $d < n-1$, the Dirichlet problem (D_q) is well-posed in the domain $\mathbb{R}^n \setminus \mathbb{R}^d$ for any degenerate elliptic operator (in the full generality of possibly complex coefficients) with conditions in the spirit of [24]. As a consequence, whenever Γ is the graph of a function with small Lipschitz constant and L is given as in [11], the Dirichlet problem (D_q) is well-posed.

While the article is written with $d < n-1$ in mind, all our computations can be adapted with very light changes to the case where $d = n-1$ and the domain is the upper half plane $\mathbb{R}^{d+1}_+$. In that context it can be viewed as an alternative to [14]. We build on their ideas and introduce new tools. As a result, even in the classical setting, our conditions on the operator are weaker and our results are somewhat stronger. Most notably, we do not impose any restrictions on the first $n-1$ rows of the coefficient matrix. We will make these statements more precise below.

1.2. Main result

Let $d < n-1$ and $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d = \{(x,t) \in \mathbb{R}^n,\ x \in \mathbb{R}^d \text{ and } t \in \mathbb{R}^{n-d} \setminus \{0\}\}$; the boundary $\partial\Omega$ is assimilated to $\mathbb{R}^d$. We write $X = (x,t)$ or $Y = (y,s)$ for the running points in Ω. The notation $B_l(x)$ is used for the ball in $\mathbb{R}^d$ with center x and radius l.

First, we introduce the square function and the averaged version of the non-tangential maximal function. Let a be a positive number. For $x \in \mathbb{R}^d$, we define the regular cones in $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,+\infty)$ as

$$\Gamma_a(x) := \{(z,r) \in \mathbb{R}^{d+1}_+,\ |z-x| < ar\},$$

and the higher co-dimensional cones as

$$\widehat{\Gamma}_a(x) := \{(y,s) \in \Omega,\ |y-x| < a|s|\}.$$

For $(z,r) \in \mathbb{R}^d \times (0,\infty)$, we write $W_a(z,r)$ for the Whitney box

$$W_a(z,r) = \{(y,s) \in \Omega,\ y \in B_{ar/2}(z),\ r/2 \le |s| \le 2r\}.$$

If, in addition, $q \in (1,+\infty)$, we define the q-adapted square function $S_{a,q}$ as

$$S_{a,q}(v)(x) := \Big(\iint_{\widehat{\Gamma}_a(x)} |\nabla v(y,s)|^2 \, |v(y,s)|^{q-2} \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{1/q}, \tag{1.4}$$

where v is a measurable function that satisfies $|v|^{q/2-1} v \in W^{1,2}_{\text{loc}}(\Omega)$. Note that the definition of $S_{a,q}$ makes sense even when $q < 2$, because, in particular, $\nabla v \equiv 0$ almost everywhere on $|v| = 0$. For any function $v \in L^q_{\text{loc}}(\Omega)$ and any $x \in \mathbb{R}^d$, the non-tangential maximal function (in the average sense) is

$$\widetilde{N}_{a,q}(v)(x) = \sup_{\Gamma_a(x)} v_{W,a,q}, \tag{1.5}$$

where $v_{W,a,q}$ is defined on $\mathbb{R}^{d+1}_+$ by

$$v_{W,a,q}(z,r) := \Big(\frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |v(y,s)|^q \, dy \, ds \Big)^{1/q}. \tag{1.6}$$

We say that $\mathcal{L} = -\operatorname{div} A\nabla$ is an elliptic operator *with weight* $|t|^{d+1-n}$ (we will always omit this weight for brevity) if there exists $C > 0$ such that the complex matrix A satisfies

$$\operatorname{Re}(A(X)\xi \cdot \bar{\xi}) \ge C^{-1} |t|^{d+1-n} |\xi|^2 \quad \text{for } X \in \mathbb{R}^n \setminus \mathbb{R}^d,\ \xi \in \mathbb{C}^n, \tag{1.7}$$

and

$$|A(X)\xi \cdot \bar{\zeta}| \le C |t|^{d+1-n} |\xi|\,|\zeta| \quad \text{for } X \in \mathbb{R}^n \setminus \mathbb{R}^d,\ \xi,\zeta \in \mathbb{C}^n. \tag{1.8}$$

Or alternatively, if the reduced matrix $\mathcal{A} := |t|^{n-d-1} A$ satisfies the classical elliptic and boundedness condition. We say that L is q-elliptic if the coefficient matrix $A(X)$ satisfies (1.7)–(1.8) and $\mathcal{A}(X)$ satisfies the condition (2.2) given in the next section. We refer the reader to the next section for the discussion of q-ellipticity, but let us cite one key property: $\mathcal{L}$ is q-elliptic for all $q \in (1,+\infty)$ if and only if the matrix $\mathcal{A}$ is real-valued. This means that q-ellipticity is a notion intrinsically linked to matrices with complex coefficients.

The next assumption we put on $\mathcal{L}$ will use Carleson measures. Let us first introduce the following definition.

Definition 1.1. We say that f satisfies the Carleson measure condition if

$$d\mu(x,r) := \sup_{W_a(x,r)} |f|^2 \, dx \, \frac{dr}{r}$$

is a Carleson measure, that is, if there exists $C_a > 0$ such that

$$\|f\|_{\mathrm{CM},a} := \sup_{x\in\mathbb{R}^d,\, r>0} \fint_{y\in B(x,r)} \int_0^r \sup_{W_a(y,s)} |f|^2 \, dy \, \frac{ds}{s} \leq C_a.$$

Let us make a few remarks. It is easy to check that the fact that f satisfies the Carleson measure condition does not depend on the choice of a, and that for any $a, b > 0$,

$$\|f\|_{\mathrm{CM},a} \leq C_{a,b} \|f\|_{\mathrm{CM},b}.$$

The quantity f in the above definition can be a measurable function, the gradient of a function, or a matrix-valued function. So $|f|$ is, respectively, the absolute value, the vector norm, or the matrix norm. Furthermore, the Carleson measure condition forces f to be locally Lipschitz.

Definition 1.2. An elliptic operator $\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ with complex coefficient satisfies the hypothesis $(\mathcal{H}^1)$ if the matrix $\mathcal{A}$ can be decomposed as

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 & bI \end{pmatrix} + \mathcal{C}, \tag{1.9}$$

where

(i) I is an $(n-d)$-identity matrix, b is a real scalar function, and $\mathcal{B}_3$ is a real matrix in $M_{(n-d)\times d}$,

(ii) b is uniformly bounded from above and below,

(iii) the quantities $|t|\nabla_x\mathcal{B}_3$, $|t|^{n-d}\operatorname{div}_t[|t|^{d+1-n}\mathcal{B}_3]$, $|t|\nabla b$ and $\mathcal{C}$ satisfy the Carleson measure condition, that is, there exists $C > 0$ such that

$$\sup_{\substack{z\in\mathbb{R}^d\\ r>0}} \fint_{y\in B(z,r)} \int_0^\infty \sup_{(x,t)\in W_1(y,s)} \left[|t|^2|\nabla_x\mathcal{B}_3|^2 + |t|^2|\nabla b|^2 + |\mathcal{C}|^2\right] \frac{ds}{s}\, dy \leq C, \tag{1.10}$$

and for any $j \leq d$,

$$\sup_{\substack{z\in\mathbb{R}^d\\ r>0}} \fint_{y\in B(z,r)} \int_0^\infty \sup_{(x,t)\in W_1(y,s)} |t|^{2(n-d)} \Big|\sum_{\ell>d} \partial_{t_\ell}[(\mathcal{B}_3)_{\ell j}|t|^{n-d-1}]\Big|^2 \frac{ds}{s}\, dy \leq C \tag{1.11}$$

if we write $\mathcal{B}_3$ as $(\mathcal{B}_3)_{1\leq j\leq d<i\leq n}$, and ∂_{t_ℓ} corresponds to the partial derivative with respect to the ℓ-th coordinate in $\mathbb{R}^n$.

In addition, we say that $\mathcal{L}$ satisfies the small Carleson hypothesis $(\mathcal{H}^1_\kappa)$ if $\mathcal{L}$ satisfies $(\mathcal{H}^1)$ and the constant C in (1.10)–(1.11) can be chosen to be smaller than κ.

Remark 1.3. If the block matrix $\mathcal{B}_3$ is not a real matrix, we could include its imaginary part in $\mathcal{C}$ as long as $\operatorname{Im}\mathcal{B}_3$ satisfies the appropriate Carleson measure condition. The assumptions that b, $\mathcal{B}_3$ are real-valued are used in the proof of Lemma 5.1, which is the key estimate in proving $S < N$ and $S < \kappa N + \mathrm{Tr}$. Observe that $|t|^{n-d}\operatorname{div}_t[|t|^{d+1-n}\mathcal{B}_3]$ equals 0, and so satisfies the Carleson measure condition, if for instance $\mathcal{B}_3 =$ has the form $(\mathcal{B}_3)_{ij} = b_i(x)t_j/|t|$.

We say that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{R}^d)$ is a weak solution to $\mathcal{L}u = 0$ if we have

$$\iint_{(x,t)\in\mathbb{R}^n} \mathcal{A}\nabla u \cdot \nabla \bar{\Psi}\, dx\, \frac{dt}{|t|^{n-d-1}} = 0 \quad \text{for any } \Psi \in C_0^\infty(\mathbb{R}^n \setminus \mathbb{R}^d).$$

Our main theorem is the following result.

Theorem 1.4. *Let $a > 0$, $M > 0$, and $q \in (1, +\infty)$. There exists a constant $\kappa_0 := \kappa_0(n, a, M, q) > 0$ such that if $\mathcal{L} := -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ is a q-elliptic operator with the following properties*:

(i) *it satisfies the hypothesis $(\mathcal{H}^1_{\kappa_0})$,*

(ii) *the q-ellipticity constant λ_q that appears in (2.2) is bigger that M^{-1},*

(iii) *$\|\mathcal{A}\|_\infty + \|b\|_\infty + \|b^{-1}\|_\infty \leq M$, where b is the one that is defined via (1.9),*

then the Dirichlet problem (D_q) is well-posed. That is, for any $g \in L^q(\mathbb{R}^d)$, there exists a unique weak solution $u := u_g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{R}^q)$ to $\mathcal{L}u = 0$ such that

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{W_a(z,r)} \iint_{W_a(z,r)} u(y,s)\, dy\, ds = g(x) \quad \textit{for a.a. } x \in \mathbb{R}^d, \tag{1.12}$$

and

$$\|\widetilde{N}_{a,2}(u)\|_q < +\infty.$$

Furthermore, we have

$$\|\widetilde{N}_{a,q}(u)\|_q \leq C\|g\|_q,$$

and

$$\|\widetilde{N}_{a,q}(u)\|_q \approx \|S_{a,q}(u)\|_q, \quad \|\widetilde{N}_{a,2}(u)\|_q \approx \|\widetilde{N}_{a,q}(u)\|_q,$$

with constants that depend only on the dimension n, a, q and M. One also has the stronger convergence

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{W_a(z,r)} \iint_{W_a(z,r)} |u(y,s) - g(x)|^q\, dy\, ds = 0 \quad \textit{for a.a. } x \in \mathbb{R}^d. \tag{1.13}$$

Remark 1.5. In particular (1.13) implies the pointwise non-tangential convergence of the solution u to g:

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \Big(\frac{1}{W_a(z,r)} \iint_{W_a(z,r)} |u(y,s)|^q\, dy\, ds\Big)^{1/q} = |g(x)| \quad \text{for a.a. } x \in \mathbb{R}^d.$$

Except for the main result in [11], the result above is the first treatment of the well-posedness of the Dirichlet problem for domains with higher co-dimensional boundary. In particular, in the case where the domain is $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$, the well-posedness of (D_q) for q small has never been considered before, even in the case where the operator $L = -\operatorname{div} A\nabla$ is such that A has real coefficients. At some point in the proof, we need to use the saw-tooth domains $\{|t| > h(x)\}$, where $h\colon \mathbb{R}^d \to \mathbb{R}_+$ is some non-negative Lipschitz function. The strategy in codimension 1 is to use a bi-Lipschitz change of variable that sends $\{t > h(x)\}$ to the upper half plane, and to deduce the result on saw-tooth domains from the one in the upper half plane. This method is not available anymore in higher co-dimension (notice that even the boundary of the saw-tooth domain becomes an object of mixed dimensions in this context), and one of the main difficulties here is to establish desired estimates directly on saw-tooth domains. Fortunately, emerging new ideas brought considerably stronger results in the "classical" co-dimension 1 setting of $\mathbb{R}^n_+ = \mathbb{R}^{d+1}_+$ as well.

Indeed, the methods used to prove Theorem 1.4 (and its result) can be easily adapted to the classical case where the boundary of the domain is of codimension 1, that is, $\Omega = \mathbb{R}^n_+$, with an obvious reformulation. Here are some remarks that are important even in codimension 1:

- As pointed out above, contrary to [14], we do not impose any conditions on the first d rows of the matrix of coefficients.
- We allow Carleson measure perturbations (addition of a matrix $\mathcal{C}$). This is vital in our method and, formally speaking, new even in the classical scenario. Indeed, known results about the Carleson measure perturbation are tied up to real coefficients [17] or to perturbing from the t-independence matrix [1], [19], which is not the setting of the present paper.
- This is the second time to the best of our knowledge (after [14]) that the Dirichlet problem for elliptic operators with complex coefficients whose gradients are Carleson measures is attacked. We keep the remarkable idea of Dindoš and Pipher to use a notion of q-ellipticity for elliptic operators, and we improve it in several ways (see the first two points above and the last two points below).
- We offer a new proof of the existence of solutions to the Dirichlet problem in unbounded domains. A general difficulty is to prove that the quantity $\|\widetilde{N}_{a,q}u\|_q$ is actually finite for a large class of solutions. Indeed, the formal proof of well-posedness consists of showing that $\|\widetilde{N}_{a,q}u\|_q \leq C\|S_{a,q}u\|_q$ and then that $\|S_{a,q}u\|_q \leq \eta\|\widetilde{N}_{a,q}u\|_q + \|\operatorname{Tr} u\|_q$. If η is small, the conclusion $\|\widetilde{N}_{a,q}u\|_q \leq C\|\operatorname{Tr} u\|_q$ holds, provided that $\|\widetilde{N}_{a,q}u\|_q$ is *a priori* finite. In the case of co-dimension 1 one can rely on a plethora of known results for smooth coefficients, on layer potential techniques, and other methods. The proof in this paper is self-contained (which partially explains the length).
- We prove local rather than global $S < N$ and $N < S$ estimates. Even in co-dimension one, such results in full generality are only available for real coefficients, typically by localization from the global estimates [12], [18], [24]. Unfortunately, localizing the global $S < N$ and $N < S$ estimates requires sufficient decay of solutions and a maximum principle, generally failing for complex coefficients. However, the local bounds are important both in the present argument and for the future use (for instance, in the extrapolation techniques on uniformly rectifiable domains) – a more detailed discussion will be presented in the next paragraph.

- We prove that, when the trace is smooth, the solutions u obtained by the Lax–Milgram theorem match the solution(s) of the Dirichlet problem (see [4] and [5] for a discussion of importance and possible failure of this property under various circumstances). In order to do it, we prove, in particular, the finiteness of the quantity $\int_\Omega |\nabla u|^2 |u|^{q-2}\, dm$ (where dm is the Lebesgue measure in the co-dimension 1 case, and $dm(x) = \operatorname{dist}(x,\Omega)^{d+1-n}\, dx$ when the boundary has higher codimension) for $q \geq 2$. This finiteness is new even in the codimension 1 case, and holds under the only condition that the operator is q-elliptic – in particular we do not require $\mathcal{L}$ to satisfy $(\mathcal{H}^1)$ – and that the domain is the complement of an Ahlfors regular set. Our control on (the energy of) solutions will allow us to derive the global $S < N$ and $N < S$ estimates from the local ones (contrary to the classical approach). A more detailed discussion can be found in Subsection 1.4.
- We improve the reverse Hölder estimates (that can be seen as weak Moser estimates) proven in Lemmas 2.6 and 2.7 of [14], and we also give a boundary version. These results hold under the sole assumption of q-ellipticity and will hopefully be useful in a wide variety of problems. They will be presented in Subsection 1.4 as well.

If an operator $L = -\operatorname{div} A\nabla$ has real coefficients, slight modifications of the argument of Theorem 1.4 gives the result below. Before stating it, let us introduce some additional notation. Thanks to the classical Moser estimate, we can directly work with the classical non-tangential maximal function N_a (rather than in the average sense, see (1.5)–(1.6)): For any function $v \in L^\infty_{\mathrm{loc}}(\Omega)$ and any $x \in \mathbb{R}^d$, we define

$$N_a(v)(x) = \sup_{\widehat{\Gamma}_a(x)} v. \tag{1.14}$$

We say that f satisfies the *real* Carleson measure condition if

$$d\mu(x,t) := |f(x,t)|^2\, dx\, \frac{dt}{|t|^{n-d}}$$

is a Carleson measure. Note that if the elliptic operator L has real coefficients, the assumption $(\mathcal{H}^1_\kappa)$ is effectively weakened by requiring only the real Carleson measure condition in (iii). Clearly, the assumption that b and $\mathcal{B}_3$ are real becomes void.

Theorem 1.6 (Analogue of Theorem 1.4 for real coefficients)**.** *Let $a > 0$, $M > 0$, and $q \in (1, +\infty)$. There exists a constant $\kappa_0 := \kappa_0(n,a,M,q) > 0$ such that if $\mathcal{L} := -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ is an elliptic operator with real coefficients that satisfies*

(i) *the hypothesis $(\mathcal{H}^1_{\kappa_0})$,*

(ii) *the ellipticity constant is bounded from below by M^{-1}, i.e.,*

$$\inf_{\substack{x\in\Omega,\, \xi\in\mathbb{R}^n \\ |\xi|=1}} \mathcal{A}(X)\xi\cdot\xi \geq M^{-1},$$

(iii) $\|\mathcal{A}\|_\infty + \|b\|_\infty + \|b^{-1}\|_\infty \leq M$, *where b is defined as in* (1.9),

then the Dirichlet problem (D_q) *is well-posed. That is, for any $g \in L^q(\mathbb{R}^d)$, there exists a unique weak solution $u := u_g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{R}^q)$ to $\mathcal{L}u = 0$ such that*

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{W_a(z,r)} \iint_{W_a(z,r)} u(y,s)\,dy\,ds = g(x) \quad \textit{for a.a. } x \in \mathbb{R}^d, \tag{1.15}$$

and

$$\|N_a(u)\|_q < +\infty.$$

Furthermore,

$$\|S_{a,q}(u)\|_q \approx \|N_a(u)\|_q \leq C\|g\|_q,$$

where the constants depend only on the dimension n, a, q and M.

A corollary of Theorem 1.6 is the well-posedness of the Dirichlet problem (D_q) when the domain Ω is the complement of the graph of a Lipschitz function with small Lipschitz constant. Let us be more precise. Let $\Gamma = \{(x,\varphi(x)),\ x \in \mathbb{R}^d\}$ be the graph of a Lipschitz function $\varphi\colon \mathbb{R}^d \to \mathbb{R}^{n-d}$, and let $\Omega = \mathbb{R}^n \setminus \Gamma$. The non-tangentially maximal function $N_{a,\Gamma}$ on Γ is the one defined as

$$N_{a,\Gamma}(u)(x') = N_a(v)(x) \quad \text{for } x' = (x,\varphi(x)) \in \Gamma,$$

where $v(y,s) = u(y, s-\varphi(x))$. Observe that the definition of $N_{a,\Gamma}$ makes sense if either a or the Lipschitz constant of φ is small. Indeed, in either case, the set $\widehat{\Gamma}_a(x)$ stays inside the domain of definition of v.

Corollary 1.7. *Let $\Gamma := \{(x,\varphi(x)),\ x \in \mathbb{R}^d\}$, where $\varphi\colon \mathbb{R}^d \to \mathbb{R}^{n-d}$ is a Lipschitz function, and $\Omega = \mathbb{R}^n \setminus \Gamma$. Choose $\alpha > 0$ and define*

$$D_\alpha(X) := \Big(\int_\Gamma |X-y|^{-d-\alpha}\,d\sigma(y)\Big)^{-1/\alpha},$$

with σ being the d-dimensional Hausdorff measure on Γ. Consider the elliptic operator

$$L := -\operatorname{div} D_\alpha(X)^{d+1-n}\nabla.$$

Let $a > 0$ and $q \in (1,+\infty)$. There exists a constant $\kappa_0 := \kappa_0(n,\alpha,a,q) > 0$ such that if $\|\nabla\varphi\|_\infty \leq \kappa_0$, then the Dirichlet problem (D_q) *is well-posed. That is, for any $g \in L^q(\Gamma)$, there exists a unique weak solution $u := u_g \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to $Lu = 0$ such that*

$$\lim_{X\in\Gamma_a(x')} \frac{1}{|B(X,\delta(X)/2)|} \iint_{B(X,\delta(X)/2)} u(Y)\,dY = g(x') \quad \textit{for a.a. } x' \in \Gamma,$$

and

$$\|N_a(u)\|_{L^q(\Gamma)} < +\infty.$$

Furthermore,

$$\|N_a(u)\|_{L^q(\Gamma)} \leq C\|g\|_{L^q(\Gamma)},$$

where the constant $C > 0$ depends only on the dimension n, α, a, and q.

Remark 1.8. The operator above is the one for which the A^∞ property of the elliptic measure, and hence the solvability of the Dirichlet problem for *some large* $q < \infty$ was proved in [11]. Clearly, the corollary could be extended to a more general class of elliptic operators which are tied up to the change of variables defined below in the proof. However, it does not seem to be possible to work, e.g., with the Euclidean distance, and so the precise choice of the coefficients is rather delicate.

Proof. Assume that $\|\nabla\varphi\| \ll 1$. We use the same bi-Lipschitz change of variable ρ as the one used in equation (3.3) of [11], that is,

$$\rho(x,t) = (x, \eta_{|t|} * \varphi(x)) + h(x,|t|)\, R_{x,|t|}(0,t) \quad \text{for } (x,t) \in \mathbb{R}^n, \tag{1.16}$$

where η_r is a mollifier, $R_{x,r}$ is a linear isometry of $\mathbb{R}^n$, and $h(x,r) > 0$ is a dilation factor. We construct $R_{x,r}$ (with a convolution formula and projections) so that it maps $\mathbb{R}^d$ to the d-plane $P(x,r)$ tangent to $\Gamma_r := \{(x, \eta_r * \varphi(x)),\, x \in \mathbb{R}^d\}$ at the point $\Phi_r(x) := (x, \eta_r * \varphi(x))$, and hence also $R_{x,r}$ maps $\mathbb{R}^{n-d} = (\mathbb{R}^d)^\perp$ to the orthogonal plane to $P(x,r)$ at $\Phi_r(x)$. The map ρ^{-1} sends Ω to $\mathbb{R}^n \setminus \mathbb{R}^d$ and L to a (real-coefficient) elliptic operator $\mathcal{L} = -\operatorname{div} A\nabla$. Lemma 3.40 in [11] and the fact that $D_\alpha(X) \simeq \operatorname{dist}(X,\Gamma)$ prove that A satisfies (1.7)–(1.8). Moreover, if $\|\nabla\varphi\|_\infty$ is small enough, Lemma 6.22 in [11] establishes that $\mathcal{A} := |t|^{n-d-1}A$ satisfies assumptions (i)–(iii) of Theorem 1.6.[2] Theorem 1.6 implies now that the Dirichlet problem (D_q) is well-posed when the domain is $\mathbb{R}^n \setminus \mathbb{R}^d$ and the operator is $\mathcal{L}$, and therefore, by using again the change of variable ρ, we conclude that the Dirichlet problem (D_q) is well-posed when the domain is $\Omega \setminus \Gamma$ and the elliptic operator is L. □

1.3. Local bounds

In this paragraph, we stay with $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$ and $\Gamma = \mathbb{R}^d$. We write (x,t) or (y,s) for a running point of $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$.

Let $a > 0$ be fixed. We need local versions of the square function and the non-tangential maximal function. The following definitions are in parallel to the previous definitions (1.4), (1.5) and (1.6). If Ψ is a cut-off function, that is, we ask Ψ to be at least a locally Lipschitz function and to satisfy $0 \leq \Psi \leq 1$ everywhere, then we define $S_{a,q}(\cdot|\Psi)$ and $\widetilde{N}_{a,q}(\cdot|\Psi)$ as

$$S_{a,q}(v|\Psi)(x) := \Big(\iint_{\widehat{\Gamma}_a(x)} |\nabla v(y,s)|^2\, |v(y,s)|^{q-2}\, \Psi(y,s)\, dy\, \frac{ds}{|s|^{n-2}} \Big)^{1/q} \tag{1.17}$$

and

$$\widetilde{N}_{a,q}(v|\Psi)(x) = \sup_{\Gamma_a(x)} (v|\Psi)_{W,a,q}, \tag{1.18}$$

[2] The smallness of the Carleson measure is not explicitly written in [11], but the smallness holds as long as the Ahlfors measure σ is close enough to a flat measure, which is the case if φ has a small Lipschitz constant and σ is the d-dimensional Hausdorff measure on Γ.

where v_W is defined on $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,+\infty)$ by

$$(1.19) \qquad (v|\Psi)_{W,a,q}(z,r) := \Big(\frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |v(y,s)|^q \, \Psi(y,s)\, dy\, ds \Big)^{1/q}.$$

Particularly we are interested in the following cut-off functions. Choose a function $\phi \in C_0^\infty(\mathbb{R}_+)$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $(0,1)$ and $\phi \equiv 0$ on $(2,+\infty)$, ϕ is non-increasing and $|\phi'| \leq 2$. If $e(x)$ is a positive a^{-1}-Lipschitz function, we define Ψ_e as

$$\Psi_e(x,t) = \phi\Big(\frac{e(x)}{|t|}\Big).$$

If $B \subset \mathbb{R}^d$ is a ball with radius bigger or equal to l, then

$$\Psi_{B,l}(x,t) = \phi\Big(\frac{a|t|}{l}\Big) \phi\Big(1 + \frac{\operatorname{dist}(x,B)}{l}\Big).$$

We keep in mind that the functions Ψ_e and $\Psi_{B,l}$ depend on a, but we do not write the dependence to lighten the notation. Whatever choice we make for $e > 0$, l, and B, observe that $\Psi_e \Psi_{B,l}$ is a smooth cut-off function which is compactly supported in $\mathbb{R}^n \setminus \mathbb{R}^d$. The following results hold.

Theorem 1.9. *Let $a > 0$ and $q \in (1,+\infty)$. Let $\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ be a q-elliptic operator that satisfies $(\mathcal{H}^1)$. For any a^{-1}-Lipschitz function e, any $l > 0$, any ball B whose radius is bigger than l, and any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{R}^d)$ to $\mathcal{L}u = 0$, we have*

(1) *if $k > 2$ and $1 < p < \infty$,*

$$\|S_{a,q}(u|\Psi_e^k \Psi_{B,l}^k)\|_p \leq C \|\widetilde{N}_{a,q}(u|\Psi_e^{k-2} \Psi_{B,l}^{k-2})\|_p,$$

(2) *if $k > 12$,*

$$\|\widetilde{N}_{a,q}(u|\Psi_e^k \Psi_{B,l}^k)\|_q^q \leq C \|S_{a,q}(u|\Psi_e^{k-12} \Psi_{B,l}^{k-12})\|_q^q + C \iint_{(y,s)\in\mathbb{R}^n} |u|^q \, \Psi_{B,l}^{k-3} \, \partial_r[\Psi_e^{k-3}] \, \frac{ds}{|s|^{n-d-1}} \, dy,$$

(3) *if $k > 12$,*

$$\|\widetilde{N}_{a,q}(u|\Psi_e^k \Psi_{B,l}^k)\|_q^q \leq C \|S_{a,q}(u|\Psi_e^{k-12} \Psi_{B,l}^{k-12})\|_q^q + C \iint_{(y,s)\in\mathbb{R}^n} |u|^q \, \Psi_e^{k-3} \, \partial_r[-\Psi_{B,l}^{k-3}] \, \frac{ds}{|s|^{n-d-1}} \, dy,$$

where ∂_r represents the derivative in the radial direction of t. The constants in the above inequalities depend on a, q, n, a lower bound for the value λ_q in (2.2), $\|\mathcal{A}\|_\infty$, an upper bound for the value C in (1.10), $\|b\|_\infty + \|b^{-1}\|_\infty$, and k. In (1), the constant depends also on p.

The proofs of the three parts of the theorem are in Lemmas 5.5, 6.9 and 6.11, respectively.

As we will see in the next paragraph, the fact that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{R}^d)$ is enough to ensure that the terms in Theorem 1.9 are finite. On the other hand, nothing guarantees, for instance, the finiteness of the quantities $\|S_{a,q}(u|\Psi^k_{B,l})\|_p$ or $\|\widetilde{N}_{a,q}(u|\Psi^k_{B,l})\|_p$ that are obtained by taking $e \to 0$, i.e., the boundary behavior.

The above definitions using the functions Ψ_e and $\Psi_{B,l}$ are smooth versions of the standard local square function and non-tangential maximal function. They are more powerful since they will allow us to hide terms from the right-hand side of an estimate. In order to build the reader's intuition, let us show some examples of the use of the theorem above.

If $\epsilon > 0$ and $l > 0$, we define the q-adapted square function $S^{\epsilon,l}_{a,q}$ and the non-tangential maximal function $\widetilde{N}^{\epsilon,l}_{a,q}$ in a similar manner to $S_{a,q}$ and $\widetilde{N}_{a,q}$, but with the truncated cones

$$\widehat{\Gamma}^{\epsilon,l}_a(x) := \{(y,s) \in \widehat{\Gamma}_a(x),\, \epsilon < |s| < l/a\}$$

and

$$\Gamma^{\epsilon,l}_a(x) := \{(z,r) \in \Gamma_a(x),\, \epsilon < r < l/a\},$$

respectively. Take a ball $B \subset \mathbb{R}^d$ with radius l and then $\epsilon > 0$. We construct the Lipschitz function e as $e \equiv \epsilon$ on $\mathbb{R}^d$. We choose then $B' = 2B$ and we write Ψ for $\Psi_e \Psi_{B',l}$. Due to (1) of Theorem 1.9, we have

$$\|S_{a,q}(u|\Psi^3)\|_p \leq C \|\widetilde{N}_{a,q}(u|\Psi)\|_p.$$

However, our choice of Ψ is such that

$$S_{a,q}(u|\Psi^3)(x) \geq S^{\epsilon,l}_{a,q}(u)(x) \quad \text{for } x \in B,$$

and, in addition, one can check that

$$\|\widetilde{N}_{a,q}(u|\Psi)\|_p \leq C_p \|\widetilde{N}^{\epsilon/2,2l}_{a,q}(u)\|_{L^p(4B)}.$$

So Theorem 1.9 implies that

$$\|S^{\epsilon,l}_{a,q}(u)\|_{L^p(B)} \leq C_p \|\widetilde{N}^{\epsilon/2,2l}_{a,q}(u)\|_{L^p(4B)}, \tag{1.20}$$

which is a more customary statement of the local $S < N$ bound.

By reasoning similar to the above, (3) of Theorem 1.9 gives the bound

$$\begin{aligned} \|\widetilde{N}^{\epsilon,l}_{a,q}(u)\|_{L^q(B)} \leq{}& C \|S^{\epsilon/2,2l}_{a,q}(u)\|_{L^p(10B)} \\ &+ C \iint_{(y,s)\in\mathbb{R}^n} |u|^q\, \Psi_e^{k-3}\, \partial_r[-\Psi^{k-3}_{B',l}]\, \frac{ds}{|s|^{n-d-1}}\, dy. \end{aligned} \tag{1.21}$$

Since ϕ is non-increasing and non-negative, the term $\Psi_e^{k-3}\, \partial_r[-\Psi^{k-3}_{B',l}]$ is non-negative. Moreover, by the construction of Ψ_e and $\Psi_{B',l}$, the term $\Psi_e^{k-3}\, \partial_r[-\Psi^{k-3}_{B',l}]$

is bounded by C/l and supported in $\{(y,s) \in \mathbb{R}^n,\ y \in 10B,\ l \leq a|s| \leq 2l\}$. So the last term in (1.21) is bounded by Cl^d times the average of the function $|u|^q$ over a Whitney box W_B associated to the ball B. If we assume that $\iint_{W_B} u = 0$, the Poincaré inequality implies that the average of $|u|^q$ over W_B can be bounded by $\|S^{\epsilon/2,2l}_{a,q}(u)\|_{L^q(10B)}$. We obtain then, if the average of u over the Whitney box W_B is 0, that

$$\|\widetilde{N}^{\epsilon,l}_{a,q}(u)\|_{L^q(B)} \leq C\|S^{\epsilon/2,2l}_{a,q}(u)\|_{L^q(10B)}. \tag{1.22}$$

The latter is a customary statement of the local $N < S$ bound.

1.4. Reverse Hölder estimates (weak Moser) and energy solutions

The results in the present paragraph require a lot fewer assumptions than before, either on the domain or on the elliptic operator L. The ball in $\mathbb{R}^n$ with center $X \in \mathbb{R}^n$ and radius r is denoted by $B(X,r)$. Let $d < n-1$ be a positive number (not necessarily integer), and let Γ be a d-dimensional Ahlfors regular set, that is, there exists $C > 0$ and a measure σ on Γ such that

$$C^{-1}r^d \leq \sigma(B(X,r)) \leq Cr^d \quad \text{for any } X \in \Gamma,\ r > 0. \tag{1.23}$$

It is well known that if Γ is Ahlfors regular, then (1.23) also holds for $\sigma = \mathcal{H}^d_{|\Gamma}$, the d-dimensional Hausdorff measure on Γ (and a different constant C), see Theorem 6.9 of [25]. Set now $\Omega = \mathbb{R}^n \setminus \Gamma$, $\delta(x) = \operatorname{dist}(x,\Gamma)$, and a measure m defined as

$$m(E) = \int_E \delta(X)^{d+1-n}\, dx,$$

that is, $dm = \delta^{d+1-n}\, dX$. Note that when $\Gamma = \mathbb{R}^d$ and $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$, for any $X = (x,t) \in \mathbb{R}^n$, we have $\delta(X) = |t|$ and $dm(X) = |t|^{d+1-n}\, dt\, dx$, and we recognize the weight used in the previous subsections.

We say an operator $L = -\operatorname{div} A\nabla$ on Ω is elliptic *with weight* $\delta(X)^{d+1-n}$ if there exists $C > 0$ such that the complex matrix A satisfies

$$\operatorname{Re}(A(X)\xi \cdot \bar{\xi}) \geq C^{-1}\delta(X)^{d+1-n}|\xi|^2 \quad \text{for } X \in \Omega,\ \xi \in \mathbb{C}^n, \tag{1.24}$$

and

$$|A(X)\xi \cdot \bar{\zeta}| \leq C\delta(X)^{d+1-n}|\xi||\zeta| \quad \text{for } X \in \Omega,\ \xi,\zeta \in \mathbb{C}^n. \tag{1.25}$$

These two conditions are the generalization of (1.7)–(1.8) in the case where Γ is Ahlfors regular. The operator L is said to be *q-elliptic* if L is elliptic and $\mathcal{A}(X) := \delta^{n-d-1}(X)A(X)$ satisfies the condition (2.2) given in the next section. In addition, we say that u is a weak solution to $Lu = 0$ if

$$\int_\Omega A\nabla u \cdot \nabla\bar{\Psi}\, dx = \int_\Omega \mathcal{A}\nabla u \cdot \nabla\bar{\Psi}\, dm = 0 \quad \text{for any } \Psi \in C_0^\infty(\Omega).$$

In the complex case, the classical Moser's estimates (i.e., L^∞-local bounds) do not necessarily hold but we have the following weaker version.

Proposition 1.10. *Let $L = -\operatorname{div} A\nabla$ be a q-elliptic operator. For any ball $B \subset \mathbb{R}^n$ of radius r that satisfies $3B \subset \Omega$ and any weak solution $u \in W^{1,2}_{\rm loc}(\Omega)$ to $Lu = 0$, we have*

$$\int_B |\nabla u|^2 |u|^{q-2}\, dm \leq \frac{C}{r^2} \int_{2B} |u|^q\, dm.$$

Moreover, the following reverse Hölder estimates hold:

(i) *If $q > 2$,*

$$\Big(\frac{1}{m(B)} \int_B |u|^q\, dm\Big)^{1/q} \leq C\Big(\frac{1}{m(2B)} \int_{2B} |u|^2\, dm\Big)^{1/2}.$$

(ii) *If $q < 2$,*

$$\Big(\frac{1}{m(B)} \int_B |u|^2\, dm\Big)^{1/2} \leq C\Big(\frac{1}{m(2B)} \int_{2B} |u|^q\, dm\Big)^{1/q}.$$

In all three inequalities, the constant $C > 0$ depends only on n, q, a lower bound on constant λ_q in (2.2), and the ellipticity constant in (1.24)–(1.25).

The proposition validates the fact that the quantities invoked in Theorem 1.9 are indeed finite. The analogue of this result in codimension 1 is written in the next subsection, and we will see that the bounds (in co-dimension 1) when $q > 2$ were already stated in Lemma 2.6 of [14], but when $q < 2$, our proposition is an improvement of Lemma 2.7 in [14].

Before we state our next result, let us introduce a bit of the theory given in [10]. We denote by W the weighted Sobolev space of functions $u \in L^1_{\rm loc}(\Omega)$ whose distribution gradient in Ω lies in $L^2(\Omega, dm)$:

$$W = \Big\{u \in L^1_{\rm loc}(\Omega) : \|u\|_W := \Big(\int_\Omega |\nabla u|^2\, dm\Big)^{1/2} < +\infty\Big\}.$$

Clearly W is contained in $W^{1,2}_{\rm loc}(\Omega)$. Lemma 3.2 and Lemma 4.2 in [10] establish that

$$W = \{f \in L^2_{\rm loc}(\mathbb{R}^n, dm),\ \nabla u \in L^2(\mathbb{R}^n, dm)\}.$$

This observation is useful to see that we will not have any problems integrating $u \in W$ across the boundary of Ω. We denote by $\mathcal{M}(\Gamma)$ the set of measurable functions on Γ, and we set

$$H := \Big\{g \in \mathcal{M}(\Gamma),\ \int_\Gamma \int_\Gamma \frac{|g(x) - g(y)|^2}{|x-y|^{d+1}}\, d\sigma(x)\, d\sigma(y)\Big\}.$$

Now, by Theorem 3.4 in [10], the trace operator $\operatorname{Tr}\colon W \to H$, defined by

$$\operatorname{Tr} u(x) = \lim_{\epsilon \to 0} \fint_{y \in B_\epsilon(x)} \fint_{|s| \leq \epsilon} u(y,s)\, ds\, dy, \quad x \in \Gamma,$$

is linear and bounded. We are ready to state a version of Proposition 1.10 at the boundary.

Proposition 1.11. *Let $L := -\operatorname{div} A\nabla$ be an q-elliptic operator. For any weak solution $u \in W$ to $Lu = 0$ and for any ball B of radius r centered on Γ that satisfies* $\operatorname{Tr} u = 0$ *on $3B$, we have*

$$\int_B |u|^{q-2} |\nabla u|^2 \, dm \leq \frac{C}{r^2} \int_{2B \setminus B} |u|^q \, dm.$$

Furthermore, if $q > 2$,

$$\Big(\frac{1}{m(B)} \int_B |u|^q \, dm\Big)^{1/q} \leq C \Big(\frac{1}{m(2B)} \int_{2B} |u|^2 \, dm\Big)^{1/2},$$

and if $q < 2$,

$$\Big(\frac{1}{m(B)} \int_B |u|^2 \, dm\Big)^{1/2} \leq C \Big(\frac{1}{m(2B)} \int_{2B} |u|^q \, dm\Big)^{1/q}.$$

The constant $C > 0$ depends only on n, q, a lower bound on constant λ_q in (2.2), and $\|A\|_\infty$.

The following result (proved as Lemma 9.1 in [10] in the case where $\mathcal{A}$ has real coefficients but valid with the same proof in the present setting) gives the existence of weak solutions in W.

Lemma 1.12. *Let $L := -\operatorname{div} A\nabla$ be an elliptic operator. For any $g \in H$, there exists a unique $u_g \in W$ such that*

$$\int_\Omega A\nabla u_g \cdot \nabla \bar{\varphi} \, dx = 0 \quad \textit{for any } \varphi \in C_0^\infty(\Omega)$$

and $\operatorname{Tr} u_g = g$ *σ-a.e. on Γ. Moreover, $\|u\|_W \leq C\|g\|_H$.*

A weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is called an *energy solution* to $Lu = 0$ if $u \in W$ and $\operatorname{Tr} u \in C_0^\infty(\Gamma)$. Since Γ is a closed set in $\mathbb{R}^n$, by an extension theorem of Whitney type (see Chapter VI, Section 2.2 of [29]), the assumption $\operatorname{Tr} u \in C_0^\infty(\Gamma)$ implies that there exists $g \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{Tr} u = \operatorname{Tr} g = g$ for a.e. $x \in \Gamma$. Since $\operatorname{Tr} g \in H$, Lemma 1.12 shows that there exists a unique energy solution $u := u_g$ to $Lu = 0$ that satisfies $\operatorname{Tr} u = \operatorname{Tr} g$.

Theorem 1.13. *Let L be a q-elliptic operator. For any energy solution $u \in W$ to $Lu = 0$, we have*

(i) *if $q \geq 2$,*

$$\int_\Omega |\nabla u|^2 \, |u|^{q-2} \, dm < +\infty,$$

(ii) *if $q \in (1, 2)$, there exists a ball B such that*

$$\int_{\Omega \setminus B} |\nabla u|^2 \, |u|^{q-2} \, dm < +\infty.$$

As we have previously mentioned, we prove the existence in the Dirichlet problem by proving that the energy solutions satisfy $\|N_{a,q}(u)\|_q < +\infty$ whenever L is q-elliptic (the range of such q is, as discussed in the next section, an open subset of $(1, +\infty)$ which is symmetric around 2). Thanks to the a-priori finiteness proved in Theorem 1.13, we will be able to pass the local estimates in Theorem 1.9 to global estimates by taking $e \to 0$, $B \nearrow \mathbb{R}^d$, and $l \to +\infty$, whenever u is an energy solution (see Section 7).

1.5. The analogues of the results from the previous section in domains with co-dimension 1 boundaries

As mentioned above, all our results have analogues in the upper half space or, respectively, a domain above an $(n-1)$-dimensional Lipschitz graph in $\mathbb{R}^n$. This statement is also valid for results of the previous section, but the geometric conditions become slightly more involved, and for that reason, we choose to restate the results carefully.

In this subsection, we say that $L = -\operatorname{div} A\nabla$ is a q-elliptic operator if the matrix A lies in $L^\infty(\Omega)$ and A satisfies (2.2).

Proposition 1.14. *Let Ω be a domain in $\mathbb{R}^n$ and let $L = -\operatorname{div} A\nabla$ be a q-elliptic operator. For any ball $B \subset \mathbb{R}^n$ of radius r that satisfies $2B \subset \Omega$ and any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to $Lu = 0$, we have*

$$\int_B |\nabla u|^2 |u|^{q-2}\, dX \le \frac{C}{r^2} \int_{2B} |u|^q\, dX.$$

Moreover, the following reverse Hölder estimates hold:

(i) *If $q > 2$,*

$$\Big(\frac{1}{|B|}\int_B |u|^q\, dX\Big)^{1/q} \le C\Big(\frac{1}{|2B|}\int_{2B} |u|^2\, dX\Big)^{1/2}.$$

(ii) *If $q < 2$,*

$$\Big(\frac{1}{|B|}\int_B |u|^2\, dX\Big)^{1/2} \le C\Big(\frac{1}{|2B|}\int_{2B} |u|^q\, dX\Big)^{1/q}.$$

In all three inequalities, the constant $C > 0$ depends only on n, q, a lower bound on constant λ_q in (2.2), and $\|A\|_\infty$.

Observe that when $q \ge 2$, the above result is the same as Lemma 2.6 of [14]. However, when $q < 2$ and under the assumptions of Proposition 1.14, Lemma 2.7 of [14] states that for any $\epsilon > 0$, we can find $C_\epsilon > 0$ such that

$$\int_B |\nabla u|^2 |u|^{q-2}\, dX \le \frac{C_\epsilon}{r^2} \int_{2B} |u|^q\, dX + \frac{\epsilon}{r^2}\Big(\int_{2B} |u|^2\, dX\Big)^{q/2}$$

and

$$\Big(\frac{1}{|B|}\int_B |u|^2\, dX\Big)^{1/2} \le C_\epsilon\Big(\frac{1}{|2B|}\int_{2B} |u|^q\, dX\Big)^{1/q} + \epsilon\Big(\frac{1}{|2B|}\int_{2B} |u|^2\, dX\Big)^{1/2}.$$

Our result is stronger than the one in [14] because we can remove the second terms of the two right-hand sides above.

For the sequel, let us introduce three topological conditions on Ω. We say that Ω satisfies the interior Corkscrew point condition when there exists a constant $C_1 > 0$ such that for $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$,

$$\text{one can find a point } A_{x,r} \in \Omega \cap B(x,r) \text{ such that } B(A_{x,r}, C_1^{-1} r) \subset \Omega. \tag{1.26}$$

Similarly, we say that Ω satisfies the exterior Corkscrew point condition if the complement Ω^c satisfies the interior Corkscrew condition. We also say that Ω satisfies the Harnack chain condition if there is a constant $C_2 \geq 1$ and, for each $\Lambda \geq 1$, an integer $N \geq 1$ such that, whenever $X, Y \in \Omega$ and $r \in (0, \operatorname{diam}(\partial\Omega))$ are such that

$$\min\{\operatorname{dist}(X,\partial\Omega), \operatorname{dist}(Y,\partial\Omega)\} \geq r \quad \text{and} \quad |X-Y| \leq \Lambda r, \tag{1.27}$$

we can find a chain of $N+1$ points $Z_0 = X, Z_1, \dots, Z_N = Y$ in Ω such that

$$C_2^{-1} r \leq \operatorname{dist}(Z_i, \partial\Omega) \leq C_2 \Lambda r \quad \text{and} \quad |Z_{i+1} - Z_i| \leq \operatorname{dist}(Z_i,\partial\Omega)/2 \tag{1.28}$$

for $1 \leq i \leq N$.

Similar to the higher codimension case, we define the space

$$W := \Big\{ u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_W := \Big(\int_\Omega |\nabla u|^2 \, dX \Big)^{1/2} < +\infty \Big\},$$

which is clearly contained in $W^{1,2}_{\mathrm{loc}}(\Omega)$. If Ω satisfies the interior Corkscrew point condition, the exterior Corkscrew point condition, and the Harnack chain condition, and if the boundary $\partial\Omega$ is Ahlfors regular, i.e., verifies (1.23) with $d = n-1$, then we can define notion of trace on W, that is, there exists a bounded operator Tr from W to $L^2_{\mathrm{loc}}(\Gamma, \sigma)$ such that $\operatorname{Tr} u = u$ if $u \in W \cap C^0(\overline{\Omega})$.

We are now ready for the analogue of Proposition 1.14 at the boundary, which is completely new.

Proposition 1.15. *Let Ω satisfy the Corkscrew point condition and the Harnack chain condition, and assume that its boundary $\partial\Omega$ is Ahlfors regular of dimension $n-1$. Let $L := -\operatorname{div} A\nabla$ be a q-elliptic operator. Let $u \in W$ be a weak solution to $Lu = 0$ in Ω and let B be a ball of radius r, centered on $\partial\Omega$, such that* $\operatorname{Tr} u = 0$ *on $2B \cap \partial\Omega$. We have*

$$\int_{B\cap\Omega} |u|^{q-2} |\nabla u|^2 \, dx \leq \frac{C}{r^2} \int_{(2B \cap \setminus B)\cap \Omega} |u|^q \, dx.$$

Furthermore, if $q > 2$,

$$\Big(\frac{1}{|B\cap\Omega|} \int_{B\cap\Omega} |u|^q \, dx \Big)^{1/q} \leq C \Big(\frac{1}{|2B\cap\Omega|} \int_{2B\cap\Omega} |u|^2 \, dx \Big)^{1/2},$$

and if $q < 2$,

$$\Big(\frac{1}{|B\cap\Omega|} \int_{B\cap\Omega} |u|^2 \, dx \Big)^{1/2} \leq C \Big(\frac{1}{|2B\cap\Omega|} \int_{2B\cap\Omega} |u|^q \, dx \Big)^{1/q}.$$

The constant $C > 0$ depends only on n, q, a smaller bound on constant λ_q in (2.2), and $\|A\|_\infty$.

This result, along with a few others that we did not recall here, can be used to establish the finiteness of the following integrals.

Theorem 1.16. *Retain the assumptions of Proposition 1.11. For any solution $u \in W$ to $Lu = 0$ whose trace $\operatorname{Tr} u$ is a restriction to $\partial\Omega$ of a function in $C_0^\infty(\mathbb{R}^n)$, we have*

(i) *if $q \geq 2$,*

$$\int_\Omega |\nabla u|^2 \, |u|^{q-2} \, dx < +\infty,$$

(ii) *if $q \in (1, 2)$, there exists a ball B centered on $\partial\Omega$ such that*

$$\int_{\Omega \setminus B} |\nabla u|^2 \, |u|^{q-2} \, dx < +\infty.$$

1.6. Plan of the article

Section 2 is devoted to the presentation of the q-ellipticity. In Section 3, we prove the results stated in Subsection 1.4, which hold in the general context when Ω is the complement in $\mathbb{R}^n$ of an Ahlfors regular set. Section 4 serves as an introduction to the work with the square function and the non-tangential maximal function, for instance, we establish there the equivalence $\|\widetilde{N}_{a,q}(u)\|_p \approx \|\widetilde{N}_{1,2}(u)\|_p$ whenever u is a weak solution to $\mathcal{L}u = 0$, $a > 0$, and q is in the range of ellipticity of $\mathcal{L}$. Sections 5 and 6 are devoted to, respectively, the local $S < N$ and $N < S$ bounds, so altogether, these sections contain the proof of Theorem 1.9. Finally, in Sections 7 and 8, we prove, respectively, the existence and uniqueness of the solutions to the Dirichlet problem (D_q), and the combination of Lemma 7.9 (existence), Lemma 8.1 (equivalence between S and N), Lemma 8.2 (improvement (1.13)), and Lemma 8.7 (uniqueness) gives Theorem 1.4.

2. The q-ellipticity and its consequences

Throughout this section, we assume that $\Gamma \subset \mathbb{R}^n$ is an Ahlfors–David regular set of dimension $d < n-1$ and $\Omega := \mathbb{R}^n \setminus \Gamma$. For $x \in \Omega$, define $\delta(x) = \operatorname{dist}(x, \Gamma)$, and we write $dm(x)$ for $\delta^{d+1-n}(x) \, dx$.

Consider a matrix $\mathcal{A}(X)$ with complex coefficients, the usual ellipticity assumption is that there exist constants $\lambda = \lambda_{\mathcal{A}} > 0$ and $\Lambda = \|\mathcal{A}\|_\infty < \infty$ such that for almost every $x \in \Omega$ and every $\xi, \zeta \in \mathbb{C}^n$,

$$\lambda|\xi|^2 \leq \operatorname{Re}(\mathcal{A}(X)\xi \cdot \overline{\xi}) \quad \text{and} \quad |\mathcal{A}(X)\xi \cdot \overline{\eta}| \leq \Lambda|\xi|\,|\eta|. \tag{2.1}$$

A stronger form of ellipticity was introduced in [6] and [14], and see also [7] where you can find the older, but related, notion of L^p dissipativity. For $q > 1$, we say that the matrix $\mathcal{A}$ is q-elliptic if there exists $\lambda_q(\mathcal{A}) > 0$ such that for almost every $X \in \Omega$ and every $\xi, \zeta \in \mathbb{C}^n$,

$$\lambda_q |\xi|^2 \leq \operatorname{Re}(\mathcal{A}(X)\xi \cdot \overline{\mathcal{J}_q \xi}) \quad \text{and} \quad |\mathcal{A}(X)\xi \cdot \overline{\eta}| \leq \Lambda |\xi|\,|\eta|, \tag{2.2}$$

where $\mathcal{J}_q\colon \mathbb{C}^n \to \mathbb{C}^n$ is defined as

$$\mathcal{J}_q(\alpha + i\beta) = \frac{1}{q}\,\alpha + \frac{i}{q'}\,\beta \quad \text{and} \quad \frac{1}{q} + \frac{1}{q'} = 1. \tag{2.3}$$

Let us make a few simple remarks on q-ellipticity. First, a matrix $\mathcal{A}$ is elliptic in the usual sense (i.e., it satisfies (2.1)) if and only if $\mathcal{A}$ is 2-elliptic. Second, a matrix $\mathcal{A}$ can be q-elliptic only if $q \in (1, +\infty)$, and, moreover, $\mathcal{A}$ is q-elliptic if and only if $\mathcal{A}$ is q' elliptic. In Proposition 5.17 of [6] (see also the discussions in [14]) the following nice result can be found.

Proposition 2.1. *Let $\mathcal{A} \in L^\infty(\Omega, \mathbb{C})$ be an elliptic matrix, i.e., a matrix satisfying (2.1). Then $\mathcal{A}$ is q-elliptic if and only if*

$$\mu(\mathcal{A}) = \operatorname*{ess\,inf}_{X \in \Omega} \min_{\xi \in \mathbb{C}^n \setminus \{0\}} \operatorname{Re} \frac{\mathcal{A}(X)\xi \cdot \overline{\xi}}{|\mathcal{A}(X)\xi \cdot \xi|} > \left|1 - \frac{2}{q}\right|. \tag{2.4}$$

In other words, $\mathcal{A}$ is q-elliptic if and only if $q \in (q_0, q_0')$, where $q_0 = 2/(1 + \mu(\mathcal{A}))$. Moreover, we can find λ_q satisfying (2.2) such that

$$C^{-1}\lambda_q \leq \mu(\mathcal{A}) - |1 - 2/q| \leq C\lambda_q,$$

where C depends only on $\mu(\mathcal{A})$, $\lambda_{\mathcal{A}}$ and $\|\mathcal{A}\|_\infty$.

Again, let us make a few comments. The minimum in ξ shall be taken over the ξ such that $|\mathcal{A}(X)\xi \cdot \xi| \neq 0$. Writing $|\mathcal{A}(X)\xi \cdot \xi|$ is not a mistake for $|\mathcal{A}(X)\xi \cdot \overline{\xi}|$. By taking $\xi \in \mathbb{R}^n$, it is easy to check that $\mu(\mathcal{A}) \leq 1$, and thus $q_0 \in [1, 2]$; and since the ellipticity condition on $\mathcal{A}$ implies $\mu(\mathcal{A}) \geq \lambda / \|\mathcal{A}\|_\infty$, we have $q_0 < 2$. If $\mathcal{A} \in L^\infty(\Omega, \mathbb{C})$ is elliptic, we also have the following (not completely immediate) equivalence: $\mathcal{A}$ is real valued if and only if $q_0 = 1$. This means that the notion of q-ellipticity is not relevant when $\mathcal{A}$ has real coefficients, and that being complex-valued prevents $\mathcal{A}$ to be q-elliptic on the full range of $q \in (1, +\infty)$.

The notion of q-ellipticity will be used via the following result (whose proof is completely identical to the one of Theorem 2.4 in [14]).

Proposition 2.2. *Assume that $\mathcal{A} \in L^\infty(\Omega)$ is a q-elliptic matrix. Then there exists $\lambda_q' := \lambda_q'(\lambda, \|\mathcal{A}\|_\infty, \lambda_q) = \lambda_q'(\lambda_{\mathcal{A}}, \|\mathcal{A}\|_\infty, \mu(\mathcal{A}), q) > 0$ such that for any nonnegative function $\chi \in L^\infty(\Omega)$, and any function u such that $|u|^{q-2}|\nabla u|^2 \chi \in L^2(\Omega, dm)$, one has*

$$\operatorname{Re} \int_\Omega \mathcal{A}\nabla u \cdot \nabla[|u|^{q-2}\,\overline{u}]\,\chi\, dm \geq \lambda_q' \int_\Omega |u|^{q-2}\,|\nabla u|^2 \chi\, dm.$$

In particular, the right-hand side above is finite.

We finish the section with a last observation. We will use repeatedly the following fact (see Lemma 2.5 in [14]). For any $q > 1$, any u such that $v := |u|^{q/2-1}u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$, and any X for which $u(X) \neq 0$, we have $\nabla|u| = |u|^{-1}\operatorname{Re}(\overline{u}\nabla u)$, and thus

$$C^{-1}|u(X)|^{q-2}|\nabla u(X)|^2 \leq |\nabla v(X)|^2 \leq C|u(X)|^{q-2}|\nabla u(X)|^2, \tag{2.5}$$

where $C > 0$ depends only on q.

3. Moser and energy estimates

The goal of this section is to prove Moser's estimates and the energy estimates, i.e., the estimates of the gradient of solutions. We will start with interior estimates, and then prove boundary estimates for solutions with vanishing or non-vanishing traces. Meanwhile, we will use these estimates to show the a priori finiteness of the square function, that is, we will prove Theorem 1.16 for energy solutions. We keep the same assumption on Γ as the ones given in Subsection 1.4 and Section 2.

3.1. Interior estimates

We aim to prove the following result, which easily implies Proposition 1.10. The notation $\fint_E f\,dm$ is used to denote $m(E)^{-1}\int_E f\,dm$.

Lemma 3.1. *Let $L = -\operatorname{div} A\nabla$ be an elliptic operator, that is, assume that L satisfies (1.24)–(1.25). Set $\mathcal{A}(X) := \delta(X)^{n-d-1}A(X)$ and let $q_0 \in [1,2)$ be given by (2.4). Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $Lu = 0$ and B is a ball of radius r that satisfies $3B \subset \Omega$.*

(i) *Let Ψ be a smooth function satisfying $0 \leq \Psi \leq 1$ and $|\nabla\Psi(X)| \leq 100/\delta(X)$, and let $k > 2$. For $q \in (q_0, q_0')$, we have*

$$\int_B |u|^{q-2}|\nabla u|^2\Psi^k\,dm \leq \frac{C}{r^2}\int_{2B}|u|^q\,\Psi^{k-2}\,dm. \tag{3.1}$$

(ii) *For $q \in (2, q_0'n/(n-2))$, we have*

$$\Big(\fint_B |u|^q\,dm\Big)^{1/q} \leq C\Big(\fint_{2B}|u|^2\,dm\Big)^{1/2}. \tag{3.2}$$

(iii) *For $q \in (q_0, 2)$, we have*

$$\Big(\fint_B |u|^2\,dm\Big)^{1/2} \leq C\Big(\fint_{2B}|u|^q\,dm\Big)^{1/q}. \tag{3.3}$$

Each of the above constants $C > 0$ depends on n, q, $\lambda_{\mathcal{A}}$, $\|\mathcal{A}\|_\infty$, $\mu(\mathcal{A})$, and in case (i), it depends also on k.

Remark 3.2. The above lemma and its proof are inspired by Lemmas 2.6 and 2.7 of [14]. However, our Lemma is stronger than Lemmas 2.6 and 2.7 of [14] in the case $q < 2$. In addition, our proof is direct, that is, contrary to the proof in [14], we do not approximate L by some elliptic operators L_j with smooth coefficients.

Proof. We set $\Phi = \Psi\eta_B$, where $\eta_B \in C_0^\infty(2B)$ is a smooth function satisfying $0 \leq \eta_B \leq 1$, $\eta_B \equiv 1$ on B, and $|\nabla\eta_B| \leq C/r$.

Step 1: estimate of the gradient. For the purpose of a priori finiteness which will be used later, we also define $u_N = \min\{|u|, N\}$ if $q \geq 2$, and $u_N = \max\{|u|, 1/N\}$ if $q < 2$. Note that u_N is a real-valued and non-negative function, and for all $X \in \Omega$, u_N converges monotonically to $|u|$ as $N \to \infty$. It is also easy to see that $u_N^{q/2-1}|u| \in W^{1,2}_{\mathrm{loc}}(\Omega)$, and this guarantees a priori boundedness of the following integrals.

For the case $q \geq 2$, let $E_1 = \{X \in \Omega : |u| \leq N\}$ and $E_2 = \{X \in \Omega : |u| > N\}$. Then

$$\int_\Omega u_N^{q-2}\,|\nabla u|^2\,\Phi^k\,dm = \int_{E_1} |u|^{q-2}\,|\nabla u|^2\,\Phi^k\,dm + \int_{E_2} N^{q-2}\,|\nabla u|^2\,\Phi^k\,dm.$$

By the q-ellipticity, we can apply Proposition 2.2 to $\chi = \Phi^k \mathbb{1}_{E_1}$ and get

$$\mathrm{Re}\int_{E_1} \mathcal{A}\nabla u \cdot \nabla[|u|^{q-2}\,\overline{u}]\,\Phi^k\,dm \geq \lambda_q' \int_{E_1} |u|^{q-2}\,|\nabla u|^2\,\Phi^k\,dm.$$

Similarly, by the 2-ellipticity, we have

$$\mathrm{Re}\int_{E_2} \mathcal{A}\nabla u \cdot \nabla\overline{u}\,\Phi^k\,dm \geq \lambda_2' \int_{E_2} |\nabla u|^2\,\Phi^k\,dm.$$

Therefore,

$$\begin{aligned}
\int_\Omega u_N^{q-2}\,|\nabla u|^2\,\Phi^k\,dm &\lesssim \mathrm{Re}\int_\Omega \mathcal{A}\nabla u \cdot \nabla[u_N^{q-2}\,\bar{u}]\Phi^k\,dm \\
&= \mathrm{Re}\int_\Omega \mathcal{A}\nabla u \cdot \nabla[u_N^{q-2}\,\bar{u}\,\Phi^k]\,dm \\
&\quad - \mathrm{Re}\int_\Omega \mathcal{A}\nabla u \cdot \nabla[\Phi^k]\,u_N^{q-2}\,\bar{u}\,dm \\
&=: T_1 + T_2.
\end{aligned} \tag{3.4}$$

A similar argument gives (3.4) in the case $q < 2$.

Observe that $|\nabla u_N| \leq |\nabla u|$, so if $q \geq 2$,

$$|\nabla[u_N^{q-2}\,\bar{u}]| \leq (q-1)u_N^{q-2}\,|\nabla u| \leq (q-1)N^{q-2}\,|\nabla u|, \tag{3.5}$$

and if $q < 2$,

$$|\nabla[u_N^{q-2}\,\bar{u}]| \leq (3-q)u_N^{q-2}\,|\nabla u| \leq (3-q)N^{2-q}|\nabla u|. \tag{3.6}$$

In any case, $u_N^{q-2}\bar u \in W^{1,2}_{\rm loc}$, hence, since Φ is compactly supported in Ω, we have that $u_N^{q-2}\bar u\Phi^k$ lies in W_0 and is compactly supported in Ω. Lemma 8.3 in [10] shows that $u_N^{q-2}\bar u\Phi^k$ is a valid test function for $u \in W^{1,2}_{\rm loc}(\Omega)$, and hence $T_1 = 0$. As for T_2, by Hölder's inequality,

$$
\begin{aligned}
(3.7)\qquad |T_2| &\lesssim \int_\Omega |\nabla u|\,\Phi^{k-1}|\nabla\Phi|\,u_N^{q-2}|u|\,dm \\
&\le \Big(\int_\Omega |\nabla u|^2\,\Phi^k\,u_N^{q-2}\,dm\Big)^{1/2}\Big(\int_\Omega u_N^{q-2}|u|^2\,\Phi^{k-2}|\nabla\Phi|^2\,dm\Big)^{1/2}.
\end{aligned}
$$

Combining (3.4), $T_1 = 0$ and (3.7), we conclude

$$
\begin{aligned}
(3.8)\qquad &\int_\Omega u_N^{q-2}|\nabla u|^2\,\Phi^k\,dm \\
&\lesssim \Big(\int_\Omega u_N^{q-2}|\nabla u|^2\,\Phi^k\,dm\Big)^{1/2}\Big(\int_\Omega u_N^{q-2}|u|^2\,\Phi^{k-2}|\nabla\Phi|^2\,dm\Big)^{1/2}.
\end{aligned}
$$

Note that by the definition of u_N,

$$
\int_\Omega u_N^{q-2}|\nabla u|^2\,\Phi^k\,dm \le N^{|q-2|}\int_\Omega |\nabla u|^2\,\Phi^k\,dm < \infty,
$$

hence we may divide the same term on both sides of (3.8) and obtain

$$
(3.9)\qquad \int_\Omega u_N^{q-2}|\nabla u|^2\,\Phi^k\,dm \lesssim \int_\Omega u_N^{q-2}|u|^2\,\Phi^{k-2}|\nabla\Phi|^2\,dm,
$$

with a constant depending on q.

Since $u_N \to |u|$ pointwise, by Fatou's lemma and the observation that $u_N^{q-2} \le |u|^{q-2}$ for both $q \le 2$ and $q \ge 2$, we have

$$
\begin{aligned}
(3.10)\qquad \int_\Omega |u|^{q-2}|\nabla u|^2\,\Phi^k\,dm &\le \liminf_{N\to\infty}\int_\Omega u_N^{q-2}|\nabla u|^2\,\Phi^k\,dm \\
&\lesssim \liminf_{N\to\infty}\int_\Omega u_N^{q-2}|u|^2\,\Phi^{k-2}|\nabla\Phi|^2\,dm \\
&\le \int_\Omega |u|^q\,\Phi^{k-2}|\nabla\Phi|^2\,dm.
\end{aligned}
$$

Now, observe that $\Phi^k \ge \Psi^k\mathbb{1}_B$ and $\Phi^{k-2} \le \Psi^{k-2}\mathbb{1}_{2B}$. In addition, on $2B$, we have

$$
(3.11)\qquad |\nabla\Phi| \le |\nabla\Psi| + |\nabla\eta_B| \lesssim \delta^{-1} + r^{-1} \lesssim \frac{1}{r},
$$

since $\delta(X) \ge r$ on $2B$. The bound (3.1) follows. However, we remark that the estimate (3.1) could be an empty statement unless we prove that its right-hand side is finite.

Step 2: Moser estimate. Case $q \geq 2$. Let $2 < k' \in \mathbb{R}$. Since $u_N = \min\{|u|, N\} \leq |u|$, for any p such that $q \leq p \leq qn/(n-2)$, we have

$$\begin{aligned}(3.12)\qquad \Big(\int_{2B} u_N^{p-2}|u|^2\Phi^{k'}\,dm\Big)^{1/p} &\leq \Big(\int_{2B} u_N^{p-2p/q}|u|^{2p/q}\Phi^{k'}\,dm\Big)^{1/p}\\ &= \Big(\int_{2B}\big(u_N^{q/2-1}|u|\Phi^{k'q/2p}\big)^{2p/q}\,dm\Big)^{1/p}.\end{aligned}$$

We remark that the first inequality is not true for the case $q < 2$. By the Sobolev–Poincaré inequality, see Lemma 4.2 and the following Remark 4.3 of [10], there is a constant C, independent of q, p, such that

$$\begin{aligned}(3.13)\qquad &\Big(\fint_{2B}\big(u_N^{q/2-1}|u|\Phi^{k'q/2p}\big)^{2p/q}\,dm\Big)^{q/2p}\\ &\qquad\leq Cr\Big(\fint_{2B}\big|\nabla\big(u_N^{q/2-1}|u|\Phi^{k'q/2p}\big)\big|^2\,dm\Big)^{1/2}.\end{aligned}$$

Here we used that the power $2p/q$ is less or equal to the Sobolev exponent $2n/(n-2)$ (and ∞ if $n=2$). Therefore, by combining (3.12) and (3.13), we get

$$\begin{aligned}(3.14)\qquad \Big(\fint_{2B} u_N^{p-2}|u|^2\Phi^{k'}\,dm\Big)^{1/p} &\lesssim r^{2/q}\Big(\fint_{2B} u_N^{q-2}|\nabla u|^2\Phi^{k'q/p}\,dm\Big)^{1/q}\\ &\quad+\Big(\fint_{2B} u_N^{q-2}|u|^2\Phi^{k'q/p-2}|\nabla\Phi|^2\,dm\Big)^{1/q}.\end{aligned}$$

Suppose $k'q/p \geq 2$. Then we can apply the gradient estimate (3.9) and get

$$\int_{2B} u_N^{q-2}|\nabla u|^2\Phi^{k'q/p}\,dm \lesssim \int_{2B} u_N^{q-2}|u|^2\Phi^{k'q/p-2}|\nabla\Phi|^2\,dm.$$

Combining this with (3.14), we obtain

$$(3.15)\qquad \Big(\fint_{2B} u_N^{p-2}|u|^2\Phi^{k'}\,dm\Big)^{1/p} \lesssim \Big(r^2\fint_{2B} u_N^{q-2}|u|^2\Phi^{k'q/p-2}|\nabla\Phi|^2\,dm\Big)^{1/q}.$$

Recall the definition $\Phi = \Psi\eta_B$ and (3.11). We have

$$\begin{aligned}(3.16)\qquad \Big(\fint_{B} u_N^{p-2}|u|^2\Psi^{k'}\,dm\Big)^{1/p} &\lesssim \Big(\fint_{2B} u_N^{p-2}|u|^2\Phi^{k'}\,dm\Big)^{1/p}\\ &\lesssim \Big(\fint_{2B} u_N^{q-2}|u|^2\Psi^{k'q/p-2}\,dm\Big)^{1/q}.\end{aligned}$$

In particular, if we choose $\Psi \equiv 1$ and k' big enough (depending only on n), the estimate above becomes

$$(3.17)\qquad \Big(\fint_{B} u_N^{p-2}|u|^2\,dm\Big)^{1/p} \lesssim \Big(\fint_{2B} u_N^{q-2}|u|^2\,dm\Big)^{1/q} < \infty,$$

whenever $q \in [2, q_0')$ and $q \leq p \leq qn/(n-2)$.

Iteration. By the same argument, if we replace $2B$ by λB, with $\lambda \in (1,2]$, then (3.17) also holds, namely,

$$\Big(\fint_B u_N^{p-2}|u|^2\,dm\Big)^{1/p} \lesssim \Big(\fint_{\lambda B} u_N^{q-2}|u|^2\,dm\Big)^{1/q}, \tag{3.18}$$

with a constant depending on the value of λ (as well as q, $\|\mathcal{A}\|_\infty$ and the q-ellipticity of $\mathcal{A}$). In particular, (3.18) holds for any $q \in [2, q_0')$ and $p = qn/(n-2)$ (for $n > 2$; any $p < \infty$ for $n = 2$). Now let $p \in (2, q_0' n/(n-2))$ be fixed. Let $\ell \in \mathbb{N}$ be the first integer so that $p((n-2)/n)^\ell \le 2$, and $\lambda = 2^{1/\ell} \in (1,2)$. We can iterate (3.18) ℓ times and obtain

$$\Big(\fint_B u_N^{p-2}|u|^2\,dm\Big)^{1/p} \lesssim \Big(\fint_{2B} |u|^2\,dm\Big)^{1/2}.$$

Note that the right-hand side is finite if $u \in W^{1,2}_{\rm loc}(\Omega)$. Therefore, by passing $N \to \infty$, we conclude

$$\Big(\fint_B |u|^p\,dm\Big)^{1/p} \le C_p\Big(\fint_{2B} |u|^2\,dm\Big)^{1/2}, \tag{3.19}$$

for any $p \in (2, q_0' n/(n-2))$. Note that in the iterative process, we get estimates with ℓ constants depending on the powers $p(n-2)/n, p((n-2)/n)^2, \dots, ((n-2)/n)^{\ell-1}$, and we combine them into one constant C_p depending on p and n. This is Moser's estimate for $p > 2$. It also justifies the right-hand side of (3.1) is finite when $q \ge 2$.

Step 3: Moser estimate. Case $q \le 2$. By Hölder's inequality, if $u \in W^{1,2}_{\rm loc}(\Omega)$, we have $u \in L^q(2B, dm)$ for all $q \le 2$. Hence we do not need to use u_N to approximate $|u|$ in this case. Similar to the previous case, for any $q \in (q_0, 2]$ and $q \le p \le qn/(n-2)$, we have

$$\begin{aligned}\Big(\fint_B |u|^p\,\Psi^{k'}\,dm\Big)^{1/p} &= \Big(\fint_B (|u|^{q/2}\,\Psi^{k'q/2p})^{2p/q}\,dm\Big)^{1/p}\\ &\le C\Big(r^2\fint_B |\nabla(|u|^{q/2}\,\Psi^{k'q/2p})|^2\,dm\Big)^{1/q}\\ &\lesssim \Big(r^2\fint_B |u|^{q-2}\,|\nabla u|^2\,\Psi^{k'q/p}\,dm\Big)^{1/q}\\ &\quad + \Big(r^2\fint_B |u|^q\,\Psi^{k'q/p-2}\,|\nabla\Psi|^2\,dm\Big)^{1/q}.\end{aligned}$$

We have $|\nabla\Psi| \lesssim 1/r$ on B. Furthermore, since $u \in L^q(2B, dm)$ for $q \le 2$, the right-hand side of (3.1) is finite, thus we can plug (3.1) in the above estimate. Therefore, we get

$$\Big(\fint_B |u|^p\,\Psi^{k'}\,dm\Big)^{1/p} \lesssim \Big(\fint_{2B} |u|^q\,\Psi^{k'q/p-2}\,dm\Big)^{1/q}, \tag{3.20}$$

whenever $q \in (q_0, 2]$ and $q \leq p \leq qn/(n-2)$. Again, we take $\Psi \equiv 1$ and k' large and we obtain

$$\left(\fint_B |u|^p \, dm\right)^{1/p} \lesssim \left(\fint_{2B} |u|^q \, dm\right)^{1/q}, \tag{3.21}$$

with $q \in (q_0, 2]$ and $q \leq p \leq qn/(n-2)$.

Iteration. Applying a similar iterative process as in the previous case, we conclude that

$$\left(\fint_B |u|^2 \, dm\right)^{1/2} \leq C_q \left(\fint_{2B} |u|^q \, dm\right)^{1/q}, \tag{3.22}$$

for any $q \in (q_0, 2)$. □

Here is a side product of Lemma 3.1, that will be useful later on.

Lemma 3.3. *Let $L = -\operatorname{div} A\nabla$ be an elliptic operator, that is, assume that L satisfies (1.24)–(1.25). Set $\mathcal{A}(X) := \delta(X)^{n-d-1} A(X)$ and let $q_0 \in [1, 2)$ be given by (2.4). Take $k > 3$.*

For any $q \in (q_0, q_0')$, there exists $\epsilon := \epsilon(n, k, q) > 0$ such that if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $Lu = 0$, if B is a ball of radius r that satisfies $3B \subset \Omega$, and if Ψ is a smooth function satisfying $0 \leq \Psi \leq 1$ and $|\nabla \Psi(X)| \leq 100/\delta(X)$, then we have

$$\left(\fint_B |u|^q \, \Psi^k \, dm\right)^{1/q} \leq C \left(\fint_{2B} |u|^{q-\epsilon} \, \Psi^{k-3} \, dm\right)^{1/q-\epsilon}, \tag{3.23}$$

where the constant $C > 0$ depends on n, q, $\lambda_{\mathcal{A}}$, $\|\mathcal{A}\|_\infty$, $\mu(\mathcal{A})$, and k.

Proof. Actually, this lemma is almost already proven. We just need to find the constant ϵ.

Indeed, the estimate (3.16) gives that (here we switch the roles of p and q) for $2 \leq p < q_0'$, $p \leq q \leq pn/(n-2)$, and $k > 2q/p$,

$$\left(\fint_B u_N^{q-2} |u|^2 \, \Psi^k \, dm\right)^{1/q} \lesssim \left(\fint_{2B} u_N^{p-2} |u|^2 \, \Psi^{kp/q-2} \, dm\right)^{1/p}.$$

Thanks to Lemma 3.1, the right-hand side above is bounded uniformly in N, and by taking $N \to +\infty$, we get

$$\left(\fint_B |u|^q \, \Psi^k \, dm\right)^{1/q} \lesssim \left(\fint_{2B} |u|^p \, \Psi^{kp/q-2} \, dm\right)^{1/p}.$$

Given $q \in (2, q_0')$, we choose $p = q - \epsilon$, where $\epsilon = \frac{1}{2}\min\{q-2, 2q/n, q(1-2/k)\} > 0$, and we obtain (3.23) in the case $q > 2$.

As for the case $q \leq 2$, we use (3.20), and we have

$$\left(\fint_B |u|^q \, \Psi^k \, dm\right)^{1/q} \lesssim \left(\fint_{2B} |u|^p \, \Psi^{kp/q-2} \, dm\right)^{1/p},$$

whenever $q \in (q_0, 2]$, $p \leq q \leq pn/(n-2)$ and $k > 2q/p$. We choose again $p = q - \epsilon$, but where ϵ is now $\frac{1}{2}\min\{q - q_0, 2q/n, q(1-2/k)\} > 0$, and we obtain (3.23) in the case $q \leq 2$. □

3.2. Boundary estimates

We want now to prove Theorem 1.13. In Theorem 1.13, we assume a stronger assumption, that is, we assume that $u \in W$ is an energy solution. Taking $u \in W$ instead of $u \in W^{1,2}_{\rm loc}(\Omega)$ allows us to define a trace on the boundary, and then to get estimates similar to the ones in Lemma 3.1 at the boundary.

We introduce the space W_0 defined as

$$W_0 := \{v \in W,\ \mathrm{Tr}\, v = 0\}.$$

Recall that the space W_0 is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_W$, see Lemma 5.5 of [10].

Lemma 3.4 (Boundary estimates with vanishing trace)**.** *Let $L = -\operatorname{div} A\nabla$ be an elliptic operator, that is, assume that L satisfies (1.24)–(1.25). Set $\mathcal{A}(X) := \delta(X)^{n-d-1}A(X)$ and let $q_0 \in [1,2)$ be given by (2.4). Let $u \in W$ be a weak solution to $Lu = 0$ such that for a ball B of radius r centered on Γ, we have $\mathrm{Tr}\, u = 0$ on $3B$.*

(i) *For $q \in (q_0, q_0')$,*

$$\int_B |u|^{q-2}|\nabla u|^2\, dm \le \frac{C}{r^2}\int_{2B\setminus B} |u|^q\, dm. \tag{3.24}$$

(ii) *For $q \in (2, q_0' n/(n-2))$,*

$$\Big(\fint_B |u|^q\, dm\Big)^{1/q} \le C\Big(\fint_{2B} |u|^2\, dm\Big)^{1/2}. \tag{3.25}$$

(iii) *For $q \in (q_0, 2)$,*

$$\Big(\fint_B |u|^2\, dm\Big)^{1/2} \le C\Big(\fint_{2B} |u|^q\, dm\Big)^{1/q}. \tag{3.26}$$

Each of the above constant $C > 0$ depends on n, q, $\lambda_{\mathcal{A}}$, $\|\mathcal{A}\|_\infty$, and $\mu(\mathcal{A})$.

Proof. The proof of this lemma is similar to the one of Lemma 3.1 and we will only talk about the differences.

Step 1: *estimate of the gradient.* Here, we take $\Psi \equiv 1$ and thus $\Phi = \eta_B$, where $\eta_B \in C_0^\infty(2B)$, $0 \le \eta_B \le 1$, $\eta_B \equiv 1$ on B, and $|\nabla \eta_B| \le C/r$. We take $k = 3$ (k has no importance as long as it is bigger than 2). The function u_N is defined as in step 1 of the proof of Lemma 3.1.

The proof of (3.24) is then done as the one of (3.1). The only delicate point is to verify

$$T_1 := \mathrm{Re} \int_\Omega \mathcal{A}\nabla u \cdot \nabla[u_N^{q-2}\,\bar u\,\Phi^k]\, dm = 0. \tag{3.27}$$

Since $u \in W$ is a solution, according to Lemma 8.3 in [10], it suffices to show that $u_N^{q-2}\,\bar u\,\Phi^k \in W_0$. The bounds (3.5)–(3.6) prove that $u_N^{q-2}\,\bar u \in W$. The fact that $u_N^{q-2}\,\bar u\,\Phi^k$ lies in W_0 is then a consequence of $\mathrm{Tr}\, u \equiv 0$ on $3B$, $\operatorname{supp} \Phi \subset 2B$, and Lemmas 5.4 and 6.1 in [10].

Step 2: *Moser estimate.* The proof of both (3.25) and (3.26) is identical of the ones of respectively (3.2) and (3.3), and is based on (3.24) and the boundary Poincaré's inequality given in Lemma 4.2 of [10]. $\square$

Now we set out to prove Theorem 1.13.

Lemma 3.5. *Let $L := -\operatorname{div} A\nabla$ be an elliptic operator. Set $\mathcal{A}(X)$ to be the quantity $\delta(X)^{n-d-1}A(X)$, let $q_0 \in [1,2)$ be given by (2.4) and $q \in (q_0, q_0')$. For any energy solution $u \in W$ to $Lu = 0$, there exists a ball B centered on Γ such that*

$$\int_{\Omega\setminus B} |\nabla u|^2 |u|^{q-2}\, dm < +\infty.$$

Proof. If $u \in W$ is an energy solution, there exists $g \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{Tr} u = g$ on Γ. So we can find a ball B' centered on Γ such that $\operatorname{supp} g \subset B'$, that is, the support of $\operatorname{Tr} u$ is in $B' \cap \Gamma$. We choose $B = 10B'$, and let r be the radius of B.

We take $k = 3$ and then $\Phi \in C^\infty(\mathbb{R}^n)$ such that $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ outside B, $\Phi \equiv 0$ in $\frac{1}{2}B$, and $|\nabla\Phi| \leq C/r$. We can apply the argument in step 1 of Lemma 3.1 to Φ and get an estimate similar to (3.10):

$$\int_\Omega |u|^{q-2} |\nabla u|^2 \Phi^k\, dm \lesssim \int_\Omega |u|^q \Phi^{k-2} |\nabla\Phi|^2\, dm. \tag{3.28}$$

The only delicate point in the proof of (3.28) is the proof of the fact that

$$\operatorname{Re} \int_\Omega \mathcal{A}\nabla u \cdot \nabla[u_N^{q-2}\, \bar{u}\, \Phi^k]\, dm = 0, \tag{3.29}$$

that can be established with the same reasoning used to prove (3.27). By using the properties of Φ, the bound (3.28) becomes

$$\int_{\Omega\setminus B} |u|^{q-2} |\nabla u|^2\, dm \lesssim \frac{1}{r^2} \int_{B\setminus\frac{1}{2}B} |u|^q\, dm. \tag{3.30}$$

The annulus $B \setminus \frac{1}{2}B$ can be covered by a finite number of balls $(D_i)_{i\in I}$ of radius $r/100$ that does not intersect $2B'$. Due to the Moser estimates (3.2) and (3.25) (if $q > 2$) or simply by Hölder's inequality (if $q \leq 2$), we have

$$\Big(\fint_{D_i} |u|^q\, dm\Big)^{1/q} \lesssim \Big(\fint_{D_i} |u|^2\, dm\Big)^{1/2}.$$

Therefore, we deduce from (3.30) that

$$\int_{\Omega\setminus B} |u|^{q-2} |\nabla u|^2\, dm \lesssim r^{(1+d)(1-q/2)-2} \Big(\int_{2B\setminus 2B'} |u|^2\, dm\Big)^{q/2},$$

where C_B depends on the ball B (and, in particular, its radius r). But we omit the dependence since B is fixed. The Poincaré inequality implies now, since $\operatorname{Tr} u = 0$ on $2B \setminus 2B'$,

$$\int_{2B\setminus 2B'} |u|^2\, dm \lesssim r^2 \int_{2B\setminus 2B'} |\nabla u|^2\, dm.$$

Therefore,

$$\int_{\Omega\setminus B} |u|^{q-2} |\nabla u|^2 \, dm \lesssim r^{(d-1)(1-q/2)} \Big(\int_{2B\setminus 2B'} |\nabla u|^2 \, dm \Big)^{q/2} < +\infty.$$

The lemma follows. □

When the trace does not vanish, we can still apply similar argument in Lemma 3.1 to $u - g \in W_0$. As long as $q \geq 2$, we can morally speaking bound the integral of $|u|^{q-2} |\nabla u|^2$ by that of $u - g$ and g. We make it rigorous in the following lemma.

Lemma 3.6 (Boundary estimates with non-vanishing trace: case $q \geq 2$)**.** *Let $L := -\operatorname{div} A\nabla$ be an elliptic operator. Set $\mathcal{A}(X) := \delta(X)^{n-d-1} A(X)$, let $q_0 \in [1,2)$ given by (2.4), and take $q \in [2, q_0')$, $p \in [2, q_0' n/(n-2))$.*

Choose $g \in C_0^\infty(\mathbb{R}^n)$ and set $u \in W$ to be the (unique) energy solution to $Lu = 0$ satisfying $\operatorname{Tr} u = g$. For any ball B of radius r centered on Γ, we have

$$\int_B |\nabla u|^2 |u|^{q-2} \, dm \lesssim \frac{1}{r^2} \int_{2B} |u|^q \, dm + r^{q-2} \int_{2B} |\nabla g|^q + r^{d-1} \|g\|^q_{L^\infty(2B)} \tag{3.31}$$

and

$$\Big(\fint_B |u|^p \, dm \Big)^{1/p} \lesssim \Big(\fint_{2B} |u|^2 \, dm \Big)^{1/2} + \Big(\fint_{2B} |g|^p \, dm \Big)^{1/p} + r \Big(\fint_{2B} |\nabla g|^{p_*} \, dm \Big)^{1/p_*}, \tag{3.32}$$

where $p_* = \max\{2, p(n-2)/n\}$, and where the constant depends only on n, $\lambda_{\mathcal{A}}$, $\mu(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, and, respectively, q and p.

As a consequence, for any energy solution $u \in W$ to $Lu = 0$ and any ball $B \subset \mathbb{R}^n$ centered on Γ, one has

$$\int_B |\nabla u|^2 |u|^{q-2} \, dm < +\infty. \tag{3.33}$$

Proof. We define $v = u - g \in W_0$. The proof follows the same arguments as the ones given in Lemma 3.1, step 1 and 2 or Lemma 3.4, the only difference being that here we have $Lv = -Lg$ instead.

Step 1: estimate of the gradient. We set k large, say $k = 100$, and then $\Phi \in C_0^\infty(2B)$, $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on B, and $|\nabla\Phi| \leq C/r$. We also define $v_N = \min\{|v|, N\}$ to ensure the a priori finiteness of the integrals we work with.

Using q-ellipticity and 2-ellipticity, we obtain similarly to (3.4) that

$$\begin{aligned}
\int_\Omega v_N^{q-2} |\nabla v|^2 \Phi^k \, dm &\lesssim \operatorname{Re} \int_\Omega \mathcal{A}\nabla v \cdot \nabla[v_N^{q-2} \bar v] \Phi^k \, dm \\
&= \operatorname{Re} \int_\Omega \mathcal{A}\nabla v \cdot \nabla[v_N^{q-2} \bar v \Phi^k] \, dm - \operatorname{Re} \int_\Omega \mathcal{A}\nabla v \cdot \nabla[\Phi^k] v_N^{q-2} \bar v \, dm \\
&=: T_1 + T_2. \qquad (3.34)
\end{aligned}$$

Since $q \geq 2$, we have

$$|\nabla[v_N^{q-2}\bar{v}]| \leq (q-1)v_N^{q-2}|\nabla v| \leq (q-1)N^{q-2}|\nabla v|,$$

which ensures that $v_N^{q-2}\bar{v} \in W$. Moreover, since $\operatorname{Tr} v = 0$, Lemma 6.1 in [10] gives that $\operatorname{Tr}[v_N^{q-2}\bar{v}] = 0$ and then Lemma 5.24 gives that $\varphi := v_N^{q-2}\bar{v}\Phi^k \in W_0$. So the term T_1 is v tested against the function $\varphi \in W_0$. Since $v = u - g$ and u is a solution to $Lu = 0$, we deduce

$$\begin{aligned}
|T_1| &= \Big| \int_\Omega \mathcal{A}\nabla g \cdot \nabla[v_N^{q-2}\,\bar{v}\,\Phi^k]\,dm \Big| \lesssim \int_\Omega |\nabla g|\,|\nabla[v_N^{q-2}\bar{v}\Phi^k]|\,dm \\
&\lesssim \int_\Omega |\nabla g|\,|\nabla v|\,v_N^{q-2}\,\Phi^k\,dm + \int_\Omega |\nabla g|\,|\nabla\Phi|\,v_N^{q-2}\,|v|\,\Phi^{k-1}\,dm \\
&\lesssim \Big[\Big(\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm\Big)^{1/2} + \Big(\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,|\nabla\Phi|^2\,dm\Big)^{1/2}\Big] \\
&\quad \times \Big(\int_\Omega |\nabla g|^2\,v_N^{q-2}\,\Phi^k\,dm\Big)^{1/2}.
\end{aligned}$$

The last term in the last inequality can be treated as follows: Using the fact that $a^\theta b^{1-\theta} \lesssim a + b$, where in our case $\theta = 1 - 2/q$, $a = r^{-2}\,v_N^q$ and $b = r^{q-2}\,|\nabla g|^q$, we have

$$\begin{aligned}
\int_\Omega v_N^{q-2}\,|\nabla g|^2\,\Phi^k\,dm &\lesssim \frac{1}{r^2}\int_\Omega v_N^q\,\Phi^k\,dm + r^{q-2}\int_\Omega |\nabla g|^q\,\Phi^k\,dm \\
&\lesssim \frac{1}{r^2}\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^k\,dm + r^{q-2}\int_\Omega |\nabla g|^q\,\Phi^k\,dm.
\end{aligned} \tag{3.35}$$

Therefore, together with the fact that $|\nabla\Phi| \leq C/r$, the bound on T_1 becomes

$$\begin{aligned}
|T_1| &\lesssim \Big(\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm\Big)^{1/2} \\
&\quad \times \Big[\frac{1}{r^2}\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,dm + r^{q-2}\int_\Omega |\nabla g|^q\,\Phi^k\,dm\Big]^{1/2} \\
&\quad + \frac{1}{r^2}\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,dm + r^{q-2}\int_\Omega |\nabla g|^q\,\Phi^k\,dm.
\end{aligned} \tag{3.36}$$

We turn to the estimate of T_2. One has, by Cauchy–Schwarz's inequality,

$$\begin{aligned}
|T_2| &\lesssim \int_\Omega |\nabla v|\,v_N^{q-2}\,|v|\,|\nabla\Phi|\,\Phi^{k-1} \\
&\lesssim \Big(\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm\Big)^{1/2}\Big(\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,|\nabla\Phi|^2\,dm\Big)^{1/2} \\
&\lesssim \Big(\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm\Big)^{1/2}\Big(\frac{1}{r^2}\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,dm\Big)^{1/2},
\end{aligned} \tag{3.37}$$

where we use the fact that $|\nabla\Phi| \lesssim r^{-1}$ in the last line. The combination of (3.34), (3.36), and (3.37) implies that

$$\begin{aligned}\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm \lesssim \Big(\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm\Big)^{1/2}&\\ \times\Big[\frac{1}{r^2}\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,dm + r^{q-2}\int_\Omega |\nabla g|^q\,\Phi^k\,dm\Big]^{1/2}&\\ +\frac{1}{r^2}\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,dm + r^{q-2}\int_\Omega |\nabla g|^q\,\Phi^k\,dm,&\end{aligned}$$

which self-improves, since $\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm$ is finite, to

$$\int_\Omega v_N^{q-2}\,|\nabla v|^2\,\Phi^k\,dm \lesssim \frac{1}{r^2}\int_\Omega v_N^{q-2}\,|v|^2\,\Phi^{k-2}\,dm + r^{q-2}\int_\Omega |\nabla g|^q\,\Phi^k\,dm. \tag{3.38}$$

Recall that $\Phi \equiv 1$ on B and $\Phi \equiv 0$ outside $2B$, hence

$$\int_B v_N^{q-2}\,|\nabla v|^2\,dm \lesssim \frac{1}{r^2}\int_{2B} v_N^{q-2}\,|v|^2\,dm + r^{q-2}\int_{2B} |\nabla g|^q\,dm. \tag{3.39}$$

Step 2: Moser estimate. Using similar arguments as the ones invoked in step 2 of the proof of Lemma 3.1 (mainly based on the Poincaré inequality), we obtain an analogue of (3.14), that is,

$$\begin{aligned}&\Big(\fint_{2B} v_N^{p-2}\,|v|^2\,\Phi^{k'}\,dm\Big)^{1/p}\\ &\quad\lesssim r^{2/q}\Big(\fint_{2B} v_N^{q-2}\,|\nabla v|^2\,\Phi^{k'q/p}\,dm + \fint_{2B} v_N^{q-2}\,|v|^2\,\Phi^{k'q/p-2}\,|\nabla\Phi|^2\,dm\Big)^{1/q},\end{aligned}$$

whenever $2 \le q \le p \le qn/(n-2)$. Assuming that k' is such that $k'q/p - 2 > 0$ and using (3.38) with $k = k'q/p$ for the first term in the right-hand side, we obtain that

$$\begin{aligned}\Big(\fint_{2B} v_N^{p-2}|v|^2\,\Phi^{k'}\,dm\Big)^{1/p} \lesssim r^{2/q}\Big(&\frac{1}{r^2}\fint_{2B} v_N^{q-2}\,|v|^2\,\Phi^{k'q/p-2}\,dm\\ &+ r^{q-2}\fint_{2B}|\nabla g|^q\,\Phi^{k'q/p}\,dm\\ &+\fint_{2B} v_N^{q-2}\,|v|^2\,\Phi^{k'q/p-2}|\nabla\Phi|^2\,dm\Big)^{1/q},\end{aligned}$$

whenever $2 \le q < q_0'$. Now, use the fact that $|\nabla\Phi| \le C/r$, $\Psi \equiv 1$ on B, and $\Psi \le 1$, to get

$$\Big(\fint_B v_N^{p-2}|v|^2\,dm\Big)^{1/p} \lesssim r^{2/q}\Big(\frac{1}{r^2}\fint_{2B} v_N^{q-2}\,|v|^2\,dm + r^{q-2}\fint_{2B}|\nabla g|^q\,dm\Big)^{1/q}$$

or

$$\Big(\fint_B v_N^{p-2}|v|^2\,dm\Big)^{1/p} \lesssim \Big(\fint_{2B} v_N^{q-2}\,|v|^2\,dm\Big)^{1/q} + r\Big(\fint_{2B}|\nabla g|^q\,dm\Big)^{1/q}.$$

The arguments can be slightly modified to get

$$\Big(\fint_B v_N^{p-2}|v|^2\,dm\Big)^{1/p} \leq C_\lambda\Big(\fint_{\lambda B} v_N^{q-2}|v|^2\,dm\Big)^{1/q} + C_\lambda r\Big(\fint_{\lambda B}|\nabla g|^q\,dm\Big)^{1/q},$$

where $\lambda \in (1,2)$ and C_λ depends on λ. We use then an iterative argument as the one given page 845 and we get, if $p \in [2, q_0' n/(n-2))$ and $p_* = \max\{2, p(n-2)/n\}$,

$$\Big(\fint_B v_N^{p-2}|v|^2\,dm\Big)^{1/p} \lesssim \Big(\fint_{2B}|v|^2\,dm\Big)^{1/2} + r\Big(\fint_{2B}|\nabla g|^{p_*}\,dm\Big)^{1/p_*}. \tag{3.40}$$

Step 3: Conclusion. The estimate (3.40) gives a uniform (and finite) bound on

$$\Big(\fint_B v_N^{p-2}|v|^2\,dm\Big)^{1/p},$$

so, taking $N \to +\infty$, we obtain, if $p \in [2, q_0' n/(n-2))$ and $p_* = \max\{2, p(n-2)/n\}$,

$$\Big(\fint_B |v|^p\,dm\Big)^{1/p} \lesssim \Big(\fint_{2B}|v|^2\,dm\Big)^{1/2} + r\Big(\fint_{2B}|\nabla g|^{p_*}\,dm\Big)^{1/p_*}. \tag{3.41}$$

The inequality above is not (3.32), because the estimate is on $v = u - g$ and not on u. But by the triangle inequality

$$\Big(\fint_B |u|^p\,dm\Big)^{1/p} \leq \Big(\fint_B |v|^p\,dm\Big)^{1/p} + \Big(\fint_B |g|^p\,dm\Big)^{1/p},$$

and Hölder's inequality

$$\Big(\fint_{2B} |v|^2\,dm\Big)^{1/2} \leq \Big(\fint_{2B} |u|^2\,dm\Big)^{1/2} + \Big(\fint_{2B} |g|^p\,dm\Big)^{1/p},$$

we easily obtain the desired estimate (3.32).

Thanks to (3.41), the right-hand side of (3.39) is uniformly bounded in N. Thus we may take $N \to +\infty$ in (3.39) and get

$$\int_B |v|^{q-2}|\nabla v|^2\,dm \lesssim \frac{1}{r^2}\int_{2B}|v|^q\,dm + r^{q-2}\int_{2B}|\nabla g|^q\,dm. \tag{3.42}$$

Again, we want to turn (3.42) into an estimate on $u = v + g$. Yet, observe that

$$\frac{1}{r^2}\int_{2B}|v|^q\,dm \lesssim \frac{1}{r^2}\int_{2B}|u|^q\,dm + r^{d-1}\|g\|^q_{L^\infty(2B)}. \tag{3.43}$$

Besides, we have

$$\begin{aligned}\int_B |u|^{q-2}|\nabla u|^2\,dm &\lesssim \int_B |v|^{q-2}|\nabla v|^2\,dm + \int_B |v|^{q-2}|\nabla g|^2\,dm \\ &\qquad + \int_B |g|^{q-2}|\nabla v|^2\,dm + \int_B |g|^{q-2}|\nabla g|^2\,dm \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We do not change I_1. We use the fact that $a^{q-2}b^2 \lesssim r^{d-1}a^q + r^{(d-1)(1-q/2)}b^q$ and we get

$$\begin{aligned}
I_3 &\le \|g\|^{q-2}_{L^\infty(B)} \int_B |\nabla v|^2\, dm \\
&\lesssim r^{d-1}\|g\|^q_{L^\infty(B)} + r^{(d-1)(1-q/2)}\Big(\int_B |\nabla v|^2\, dm\Big)^{q/2} \\
&\lesssim r^{d-1}\|g\|^q_{L^\infty(B)} + r^{(d-1)(1-q/2)}\Big(\frac{1}{r^2}\int_{2B} |v|^2\, dm + \int_{2B} |\nabla g|^2\, dm\Big)^{q/2} \\
&\lesssim r^{d-1}\|g\|^q_{L^\infty(B)} + \frac{1}{r^2}\int_{2B} |v|^q\, dm + r^{q-2}\int_{2B} |\nabla g|^q\, dm.
\end{aligned}$$

Similarly, since $a^{q-2}b^2 \lesssim r^{-2}a^q + r^{q-2}b^q$, we have

$$I_2 \lesssim \frac{1}{r^2}\int_B |v|^q\, dm + r^{q-2}\int_B |\nabla g|^q\, dm$$

and

$$I_4 \lesssim \frac{1}{r^2}\int_B |g|^q\, dm + r^{q-2}\int_B |\nabla g|^q\, dm \lesssim r^{d-1}\|g\|^q_{L^\infty(B)} + r^{q-2}\int_B |\nabla g|^q\, dm.$$

Altogether, we deduce

$$\begin{aligned}
\int_B |u|^{q-2}|\nabla u|^2\, dm \lesssim \int_B |v|^{q-2}|\nabla v|^2\, dm + \frac{1}{r^2}\int_B |v|^q\, dm + r^{d-1}\|g\|^q_{L^\infty(B)} \\
+ r^{q-2}\int_B |\nabla g|^q\, dm.
\end{aligned} \tag{3.44}$$

The combination of (3.42), (3.43), and (3.44) gives (3.31).

Eventually, by combining (3.31) and (3.32), we obtain

$$\begin{aligned}
&\int_B |\nabla u|^2 |u|^{q-2}\, dm \\
&\quad\lesssim \frac{1}{r^2}\int_{2B} |u|^q\, dm + r^{q-2}\int_{2B} |\nabla g|^q + r^{d-1}\|g\|^q_{L^\infty(2B)} \\
&\quad\lesssim r^{(d+1)(1-q/2)-2}\Big(\int_{4B} |u|^2\, dm\Big)^{q/2} + r^{q-2}\int_{4B} |\nabla g|^q + r^{d-1}\|g\|^q_{L^\infty(2B)} < +\infty
\end{aligned}$$

because $g \in C_0^\infty(\mathbb{R}^n)$ and $u \in W$ (recall that the latter forces $u \in L^2_{\text{loc}}(\mathbb{R}^n, dm)$, see Lemma 3.1 in [10]). The lemma follows. □

Combining the Lemmas 3.5 and 3.6, we easily prove Theorem 1.13.

4. Preliminaries to the local estimates

In this section, and for the rest of the article, we take $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d = \{(x,t) \in \mathbb{R}^n,\ x \in \mathbb{R}^d \text{ and } t \in \mathbb{R}^{n-d} \setminus \{0\}\}$; the boundary Γ is assimilated to $\mathbb{R}^d$. We also write $X = (x, t)$ or $Y = (y, s)$ for the running points in Ω, and we write $B_l(x)$ for the ball in $\mathbb{R}^d$ with center x and radius l. In this particular case, the measure m satisfies

$$dm(X) = \frac{dt}{|t|^{n-d-1}}\, dx.$$

4.1. Properties of the non-tangential maximal function N

Lemma 4.1. *Let $a, b > 0$, $p \in [1, +\infty]$, and let $\mathcal{L}$ be an elliptic operator. Set $q_0 \in (1, 2)$ given by Proposition 2.1 and choose $q_1, q_2 \in (q_0, q_0')$. Then for any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to $\mathcal{L}u = 0$ that satisfies $\|\widetilde{N}_{a,q_1}(u)\|_p < +\infty$, we have*

$$\|\widetilde{N}_{b,q_2}(u)\|_p \leq C\|\widetilde{N}_{a,q_1}(u)\|_p < +\infty,$$

where $C > 0$ depends only on n, a, b, q_1, q_2, $\lambda_{\mathcal{A}}$, $\|\mathcal{A}\|_\infty$ and $\mu(A)$.

Proof. We use the notation introduced in Subsection 1.2. The proof of Lemma 3.1 can be easily adapted to get (3.2)–(3.3), where the ball B is replaced by $W_b(z, r)$ and $2B$ is replaced by

$$\{(y, s) \in \Omega,\, y \in B_{br}(z),\, r/4 \leq |s| \leq 4r\} \subset W_{4b}(z, r/2) \cup W_{4b}(z, 2r).$$

So we obtain, for any $(z, r) \in \mathbb{R}^{d+1}_+$, that

$$\left(\fint_{W_b(z,r)} |u|^{q_2}\, dm\right)^{1/q_2} \lesssim \left(\fint_{W_{4b}(z,r/2)\cup W_{4b}(z,2r)} |u|^{q_1}\, dm\right)^{1/q_1}. \tag{4.1}$$

In addition, we have

$$|W_{4b}(z, r/2)| \simeq b^d r^n \simeq |W_{4b}(z, 2r)|$$

and

$$dm(X) \simeq r^{d+1-n}\, dx\, dt \quad \text{for any } X = (x, t) \in W_{4b}(z, r/2) \cup W_{4b}(z, 2r).$$

We deduce that (4.1) becomes

$$u_{W,b,q_2}(z, r) \lesssim u_{W,4b,q_1}(z, r/2) + u_{W,4b,q_1}(z, 2r).$$

By taking the supremum on $(z, r) \in \Gamma_b(x)$, where $x \in \mathbb{R}^d$, we get the pointwise bound

$$\widetilde{N}_{b,q'}(u)(x) \leq C\widetilde{N}_{4b,q}(u)(x),$$

with a constant C independent of x. We take the L^p-norm of the inequality above to obtain

$$\|\widetilde{N}_{b,q_2}(u)\|_p \lesssim \|\widetilde{N}_{4b,q_1}(u)\|_p \lesssim \|\widetilde{N}_{a,q_1}(u)\|_p,$$

where the last inequality is obtained by a classical real variable method argument (see Chapter II, equation (25) in [28]). □

We also have the following corollary of Lemma 3.3, that will be useful in Section 6.

Lemma 4.2. *Let $a > 0$ and $\mathcal{L} := -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator. Set $q_0 \in (1,2)$ given by Proposition 2.1 and choose $q \in (q_0, q_0')$. Take also $k > 3$.*

There exists $\epsilon := \epsilon(n,k,q,q_0)$ such that for any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to $\mathcal{L}u = 0$, and for any smooth function $\Psi \in C_0^\infty(\mathbb{R}^n)$ that satisfies $0 \le \Psi \le 1$ and $|\nabla\Psi| \le 100/|t|$, we have

$$\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q \le C\|\widetilde{N}_{a,q-\epsilon}(u|\Psi^{k-3})\|_q,$$

where $C > 0$ depends only on n, a, q, $\lambda_{\mathcal{A}}$, $\|\mathcal{A}\|_\infty$, $\mu(A)$ and k. In particular, if the right-hand side above is finite, so is the left-hand side.

Proof. As in the previous lemma, the use of Lemma 3.3 with the sets $W_a(z,r)$ instead of balls gives that

$$(u|\Psi^k)_{W,a,q}(z,r) \lesssim (u|\Psi^{k-3})_{W,4a,q-\epsilon}(z,r/2) + (u|\Psi^{k-3})_{W,4a,q-\epsilon}(z,2r).$$

By taking the supremum on the cones $\Gamma_a(x)$, and then the L^q norm on $x \in \mathbb{R}^d$, we get

$$\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q \lesssim \|\widetilde{N}_{4a,q-\epsilon}(u|\Psi^{k-3})\|_q.$$

We can upgrade $\widetilde{N}_{4a,q-\epsilon}$ into $\widetilde{N}_{a,q-\epsilon}$ by using, as in the proof of Lemma 4.1, a real variable argument. The lemma follows. □

4.2. The Carleson inequality and good cut-off functions

Proposition 4.3 (Carleson inequality)**.** *Let $a > 0$ and $q \in (1, +\infty)$. Let f be a measurable function (scalar, vector-valued, or matrix-valued) that satisfies the Carleson measure condition (see Definition 1.1). For any function $u \in L^q_{\mathrm{loc}}(\Omega)$ and any non-negative function $\Psi \in C_0^\infty(\Omega)$, we have*

$$\int_{(x,t)\in\Omega} |f|^2\,|u|^q\,\Psi\,\frac{dt}{|t|^{n-d}}\,dx \le C\|f\|_{\mathrm{CM},a}\|\widetilde{N}_{a,q}(u|\Psi)\|_q^q,$$

where the constant C depends only on the dimensions n and d.

Proof. Observe that

$$\begin{aligned}
&\iint_{(x,t)\in\Omega} |f|^2\,|u|^q\,\Psi\,\frac{dt}{|t|^{n-d}}\,dx \\
&\quad\lesssim \iint_{(x,r)\in\mathbb{R}^{d+1}_+} \frac{1}{|W_a(x,r)|}\iint_{(y,s)\in W_a(x,r)} |f|^2\,|u|^q\,\Psi\,ds\,dy\,\frac{dr}{r}\,dx \\
&\quad\lesssim \iint_{(x,r)\in\mathbb{R}^{d+1}_+} (u|\Psi)^q_{W,a,q}\Big(\sup_{(y,s)\in W_a(x,r)}|f|^2\Big)\,\frac{dr}{r}\,dx \\
&\quad\lesssim \|f\|_{\mathrm{CM},a}\|\widetilde{N}_{a,q}(u|\Psi)\|_q^q,
\end{aligned}$$

where we used the classical Carleson inequality, see, for instance, Chapter II, Section 2.2 of [28]. □

Let us define 'good' cut-off functions.

Definition 4.4. Let $a > 0$. A function Ψ satisfies $(\mathcal{H}^2_a)$ if Ψ is locally Lipschitz on Ω and there exists $M > 0$ such that

(i) $0 \leq \Psi \leq 1$ on Ω,

(ii) $|\nabla \Psi(x,t)| \leq C/|t|$ for any $(x,t) \in \Omega$,

(iii) for any $x \in \mathbb{R}^d$,

$$\iint_{(z,r)\in\Gamma_b(x)} \Big(\sup_{W_b(z,r)} |\nabla \Psi| \Big) \frac{dr}{r^d}\, dz \leq M, \tag{4.2}$$

where $b = a/100$.

We say that Ψ satisfies $(\mathcal{H}^2_{a,M})$ if Ψ satisfies (iii) with the constant M.

On first read it appears that the condition $(\mathcal{H}^2_{a,M})$ depends on the constant C in (ii); however, it can be shown that condition (iii) implies (ii), with a constant C depending on a, M, d, and condition (ii) is only here to simplify the reading. Indeed, a point $(x,t) \in \Omega$ belongs to $W_b(z,r)$ for any $z \in \mathbb{R}^d$, $r > 0$, which satisfies $|t|/2 \leq r \leq 2|t|$ and $z \in B_{b|t|/4}(x)$, so

$$\begin{aligned} |\nabla \Psi(x,t)| &\lesssim \frac{1}{|t|} \int_{B_{b|t|/4}} \int_{|t|/2 \leq r \leq 2|t|} \sup_{W_b(z,r)} |\nabla \Psi| \frac{dr}{r^d}\, dz \\ &\leq \frac{1}{|t|} \iint_{(z,r)\in\Gamma_b(x)} \Big(\sup_{W_b(z,r)} |\nabla \Psi| \Big) \frac{dr}{r^d}\, dz \lesssim \frac{M}{|t|} \end{aligned}$$

if (iii) is satisfied.

The following observations are crucial.

Lemma 4.5. *Let $\phi \in C_0^\infty(\mathbb{R}_+)$ be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $[0,1]$, $\phi \equiv 0$ on $[2,+\infty)$, and $|\phi'| \leq 2$.*

(1) *If $e(x)$ is a positive a^{-1}-Lipschitz function and*

$$\Psi_e(x,t) = \phi\Big(\frac{e(x)}{|t|}\Big),$$

then Ψ_e satisfies $(\mathcal{H}^2_{a,M})$ for an M that depends only on a and the dimensions n, d.

(2) *If $B \subset \mathbb{R}^d$ is a (boundary) ball of radius bigger than or equal to l, and*

$$\Psi_{B,l}(x,t) = \phi\Big(\frac{a|t|}{l}\Big)\phi\Big(1 + \frac{\mathrm{dist}(x,B)}{l}\Big),$$

then $\Psi_{B,l}$ satisfies $(\mathcal{H}^2_{a,M})$ for an M that depends only on a, n.

(3) *If Ψ_1 and Ψ_2 are two functions satisfying $(\mathcal{H}^2_{a,M_1})$ and $(\mathcal{H}^2_{a,M_2})$, respectively, then $\Psi_1\Psi_2$ satisfies $(\mathcal{H}^2_{a,M_1+M_2})$.*

Proof. Part (3) follows easily from Definition 4.4. We shall only prove (1), and the proof for (2) is similar. The assertion (i) of Definition 4.4 is trivial, and as we said, (ii) follows from (iii), so we only need to verify (iii).

Let $b = a/100$ and take $x \in \mathbb{R}^d$. We want to show that

$$\iint_{(z,r)\in\Gamma_b(x)} \Big(\sup_{W_b(z,r)} |\nabla\Psi_e| \Big) \frac{dr}{r^d}\, dz \le M.$$

By the properties of ϕ, we first observe that $|\nabla\Psi_e(y,s)| \le C/|s|$, and thus $\sup_{W_b(z,r)} |\nabla\Psi_e| \le C'/r$. Another simple observation is that

$$\{(y,s) \in \mathbb{R}^n,\, |\nabla\Psi_e(y,s)| \neq 0\} \subset \{(y,s) \in \mathbb{R}^n, e(y)/2 \le |s| \le e(y)\}.$$

If $(y,s) \in W_b(z,r)$ for some $(z,r) \in \Gamma_b(x)$, then necessarily $(y,s) \in \widehat{\Gamma}_{3b}(x)$. So we want to find for which values of $(y,s) \in \widehat{\Gamma}_{3b}(x)$, we have $|\nabla\Psi_e(y,s)| \neq 0$. If $y = x$, then we have

$$S_x := \{s \in \mathbb{R}^{n-d},\, |\nabla\Psi_e(x,s)| \neq 0\} \subset \{s \in \mathbb{R}^{n-d},\, e(x)/2 \le |s| \le e(x)\}.$$

Besides, if $(y,s) \in \widehat{\Gamma}_{3b}(x)$ is such that $|\nabla\Psi_e(y,s)| \neq 0$, since e is a^{-1}-Lipschitz, we can find $s_x \in S_x$ such that

$$|s - s_x| \le \frac{1}{a}\,|y-x| \le \frac{3b}{a}\,|s| \le \frac{|s|}{2}.$$

We deduce that

$$\frac{e(x)}{3} \le \frac{2}{3}\,|s_x| \le |s| \le 2s_x \le 2e(x),$$

and then the values $(z,r) \in \Gamma_b(x)$ for which $W_b(z,r)$ contains such (y,s) satisfy

$$\frac{e(x)}{6} \le \frac{|s|}{2} \le r \le 2|s| \le 4e(x).$$

Therefore,

$$\iint_{(z,r)\in\Gamma_b(x)} \Big(\sup_{W_b(z,r)} |\nabla\Psi_e| \Big) \frac{dr}{r^d}\, dz \lesssim \int_{e(x)/6\le r\le 4e(x)} \int_{B_{br}(x)} dz\, \frac{dr}{r^{d+1}} \lesssim 1.$$

Part (1) of the lemma follows. $\square$

Now, let us state another important property of functions which satisfy $(\mathcal{H}^2_a)$.

Lemma 4.6. *Let $a > 0$ and $q \in (1, +\infty)$. Let Ψ satisfy $(\mathcal{H}^2_{a,M})$. If $v \in L^q_{\mathrm{loc}}(\Omega)$ and $0 \le \Phi \in C_0^\infty(\Omega)$, then*

$$\iint_\Omega |v|^q\,\Phi\,|\nabla\Psi|^2\,\frac{dt}{|t|^{n-d-2}}\,dx + \iint_\Omega |v|^q\,\Phi\,|\nabla\Psi|\,\frac{dt}{|t|^{n-d-1}}\,dx \le CM\|\widetilde{N}_{a,q}(v|\Phi)\|_q^q, \tag{4.3}$$

with a constant $C > 0$ that depends only on n, a.

Proof. The proof is almost straightforward. Let $b = a/100$. First, due to (ii) in Definition 4.4, we have

$$\iint_\Omega |v|^q \, \Phi \, |\nabla \Psi|^2 \, \frac{dt}{|t|^{n-d-2}} \, dx \leq C \iint_\Omega |v|^q \, \Phi \, |\nabla \Psi| \, \frac{dt}{|t|^{n-d-1}} \, dx,$$

so we only need to bound the second term in the left-hand side of (4.3). By Fubini's lemma, for any non-negative function f,

$$\begin{aligned}
&\iint_{(z,r)\in\mathbb{R}^d\times(0,\infty)} \Big(\frac{1}{|W_b(z,r)|} \iint_{W_b(z,r)} f \, dt \, dx \Big) dz \, dr \\
&\qquad \approx \iint_{(x,t)\in\Omega} f \Big(\iint_{\substack{(z,r)\in\mathbb{R}^d\times(0,\infty) \\ \text{s.t. } (x,t)\in W_b(z,r)}} dz \, \frac{dr}{r^n} \Big) \, dt \, dx \\
&\qquad \geq \iint_\Omega f \Big(\int_{|t|/2}^{2|t|} \int_{z\in B_{b|t|/2}(x)} dz \, \frac{dr}{r^n} \Big) \, dt \, dx \approx \iint_\Omega f \, \frac{dt}{|t|^{n-d-1}} \, dx,
\end{aligned}$$

where the constants depend on b and the dimension d. Thus

$$\begin{aligned}
&\iint_\Omega |v|^q \, \Phi \, |\nabla \Psi| \, \frac{dt}{|t|^{n-d-1}} \, dx \\
&\qquad \lesssim \iint_{(z,r)\in\mathbb{R}^d\times(0,\infty)} \Big(\frac{1}{|W_b(z,r)|} \iint_{W_b(z,r)} |v|^q \, \Phi \, |\nabla \Psi| \Big) \, dz \, dr \\
&\qquad \lesssim \iint_{(z,r)\in\mathbb{R}^d\times(0,\infty)} |(v|\Phi)_{W,b,q}(z,r)|^q \Big(\sup_{W_b(z,r)} |\nabla \Psi| \Big) \, dz \, dr \\
&\qquad \lesssim \int_{x\in\mathbb{R}^d} \iint_{(z,r)\in\Gamma_a(x)} |(v|\Phi)_{W,b,q}(z,r)|^q \Big(\sup_{W_b(z,r)} |\nabla \Psi| \Big) dz \, \frac{dr}{r^d} \, dx \\
&\qquad \lesssim \int_{x\in\mathbb{R}^d} |\widetilde{N}_{b,q}(v|\Phi)(x)|^q \iint_{(z,r)\in\Gamma_b(x)} \Big(\sup_{W_b(z,r)} |\nabla \Psi| \Big) \, dz \, \frac{dr}{r^d} \, dx \\
&\qquad \lesssim M \int_{\mathbb{R}^d} |\widetilde{N}_{a,q}(v|\Phi)|^q \, dx,
\end{aligned}$$

by (iii) in Definition 4.4. □

5. The local estimate $S < N$

In this section, we estimate the q-adapted functional $S_{a,q}$ by the functional $\widetilde{N}_{a,q}$, by using methods similar to the ones in [12] and Section 7 of [11]. In fact, the general method was pioneered in [24]. To get boundary estimates, the derivative(s) in the transversal t-direction clearly plays an essential role. To estimate the t-derivative(s), it is convenient to tweak the elliptic matrix $\mathcal{A}$ so that its lower left corner becomes zero:

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 & bI \end{pmatrix} + \mathcal{C} \;\to\; \mathcal{A}' := \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 + (\mathcal{B}_3)^T \\ 0 & bI \end{pmatrix} + \mathcal{C}. \tag{5.1}$$

By simple algebra, such an $\mathcal{A}'$ satisfies the following:

- For $X = (0,t)$ and $Y = (y,s) \in \mathbb{R}^n$, we have

$$\mathcal{A}'Y \cdot X = bs \cdot t + \mathcal{C}Y \cdot X. \tag{5.2}$$

If Y denotes the full derivative, then roughly speaking $\mathcal{A}'$ highlights the t-derivative(s).

- The difference between the two elliptic operators $-\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ and $-\operatorname{div}|t|^{d+1-n}\mathcal{A}'\nabla$ is of first-order; to be more precise,

$$\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla = -\operatorname{div}|t|^{d+1-n}\mathcal{A}'\nabla + |t|^{d+1-n}\mathcal{D}' \cdot \nabla, \tag{5.3}$$

where, for any $1 \leq j \leq d < i \leq n$,

$$(\mathcal{D}')_j = -\sum_{\ell > d} |t|^{n-d-1} \partial_{t_\ell}[(\mathcal{B}_3)_{\ell j}|t|^{d+1-n}], \quad (\mathcal{D}')_i = \sum_{\ell \leq d} \partial_{x_\ell}(\mathcal{B}_3)_{i\ell}. \tag{5.4}$$

Observe that the condition $(\mathcal{H}^1)$ imposes, in particular, that $|t|\mathcal{D}'$ satisfies the Carleson measure condition.

- Moreover, if we assume that the matrix $\mathcal{B}_3$ is real-valued, then $\mathcal{A}'\xi \cdot \overline{\xi}$ has the same real part as $\mathcal{A}\xi \cdot \overline{\xi}$ (and we have $\mathcal{A}'\xi \cdot \xi = \mathcal{A}\xi \cdot \xi$ without even assuming that $\mathcal{B}_3$ is real). Hence, by Proposition 2.4, $\mathcal{A}'$ is q-elliptic if and only if $\mathcal{A}$ is q-elliptic; more precisely, $\lambda_q(\mathcal{A}') = \lambda_q(\mathcal{A})$.

Lemma 5.1 (Local $S < \kappa N$)**.** *Let $\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator that satisfies $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$ (see Definition 1.2). We define $\mathcal{A}'$ as in (5.1). Let $a > 0$ and $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Take $k > 2$ and a function $\Psi \in C_0^\infty(\Omega)$ which verifies $0 \leq \Psi \leq 1$ and $|\nabla \Psi| \leq 100/|t|$ everywhere. Then for any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to $\mathcal{L}u = 0$, we have*

$$\begin{aligned} c\|S_{a,q}(u|\Psi^k)\|_q^q &\leq \frac{1}{2}\operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u \cdot \nabla[|u|^{q-2}\,\bar{u}]\,\frac{\Psi^k}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &\leq C\kappa\|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_q^q - \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u \cdot \nabla_x[\Psi^k]\,\frac{|u|^{q-2}\,\bar{u}}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &\quad + \frac{1}{q}\iint_\Omega \Big(\frac{|u|^q}{|t|}\nabla|t| - \nabla_t[|u|^q]\Big) \cdot \nabla_t[\Psi^k]\,\frac{dt}{|t|^{n-d-2}}\,dx, \end{aligned} \tag{5.5}$$

where the constants c, C depend on a, q, n, k, $\|\mathcal{A}\|_\infty$, $\lambda_q(\mathcal{A})$ and $\|b\|_\infty + \|b^{-1}\|_\infty$.

Remark 5.2. (i) In (5.5) the differential operators ∇_x and ∇_t denote an n-dimensional vector, where the missing derivatives are taken to be zero.

(ii) This lemma is a key step in proving the $S < N$ estimates. The corollaries of this lemma are Lemma 5.3, and with some additional a priori estimate, Lemma 7.6. In fact, the second and last terms on the right-hand side can be roughly bounded by $\|\widetilde{N}_{a,q}(u|\Psi^{k-1})\|_q^q + \|S_{a,q}(u|\Psi^k)\|_q^{q/2}\|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_q^{q/2}$ (see Lemma 5.3); with an appropriate choice of cut-off function they can be bounded more precisely by $\|\operatorname{Tr} u\|_q^q$ (see Lemmas 7.5 and 7.8).

Proof. First of all, Proposition 1.10 guarantees that $|u|^q$ and $|\nabla u|^2|u|^{q-2}$ lie in $L^1_{\mathrm{loc}}(\Omega)$. Therefore, since $\mathcal{A}'$ and b^{-1} are bounded and

$$|\nabla u|^2|u|^{q-2} \simeq |\nabla u||\nabla[u^{q-2}\bar{u}]|,$$

by (2.5), all the quantities invoked in (5.5) are well defined and finite.

Let T be the quantity

$$T := \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u \cdot \nabla[|u|^{q-2}\bar{u}]\,\frac{\Psi^k}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx.$$

Since $\mathcal{A}$ is q-elliptic and $\mathcal{B}_3$ is real-valued, we know $\mathcal{A}'$ is also q-elliptic, and, more precisely, that $\lambda_q(\mathcal{A}') = \lambda_q(\mathcal{A})$. Together with Proposition 2.2 and the fact that b is uniformly bounded from above, we obtain

$$\|S_{a,q}(u|\Psi^k)\|_q^q \simeq \iint_{(x,t)\in\mathbb{R}^n} |\nabla u|^2|u|^{q-2}\,\Psi^k\,\frac{dt}{|t|^{n-d-2}}\,dx \lesssim T,$$

which is exactly the first estimate in (5.5). Here we use the assumption that b is a real-valued scalar.

We claim that

$$\begin{aligned} T \leq{}& C\kappa^{1/2}T^{1/2}\|\widetilde{N}_{q,a}(u|\Psi^{k-2})\|^{q/2}_{L^q(B)} \\ &- \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla_x[\Psi^k]\,\frac{|u|^{q-2}\bar{u}}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &+ \frac{1}{q}\iint_\Omega\Big(\frac{|u|^q}{|t|}\nabla|t| - \nabla_t[|u|^q]\Big)\cdot\nabla_t[\Psi^k]\,\frac{dt}{|t|^{n-d-2}}\,dx, \end{aligned} \tag{5.6}$$

which easily implies the second inequality in (5.5) since T is finite. So let us prove the claim.

First, check that we can write

$$\begin{aligned} T &= \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla[|u|^{q-2}\bar{u}]\,\frac{|t|\Psi^k}{b}\,\frac{dt}{|t|^{n-d-1}}\,dx \\ &= \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla\Big[\frac{|t||u|^{q-2}\bar{u}\,\Psi^k}{b}\Big]\,\frac{dt}{|t|^{n-d-1}}\,dx \\ &\quad - \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla[\Psi^k]\,\frac{|u|^{q-2}\bar{u}}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &\quad + \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla b\,\frac{|u|^{q-2}\bar{u}\,\Psi^k}{b^2}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &\quad - \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla|t|\,\frac{|u|^{q-2}\bar{u}\,\Psi^k}{b}\,\frac{dt}{|t|^{n-d-1}}\,dx \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Let us start with T_2. The term T_2 is similar to the second term in the right-hand side of (5.6), except that $\nabla[\Psi^k]$ is replaced by $\nabla_x[\Psi^k]$. We write

$$\begin{aligned} T_2 &= -\operatorname{Re} \iint_\Omega \mathcal{A}' \nabla u \cdot \nabla_x[\Psi^k] \, \frac{|u|^{q-2}\,\bar{u}}{b} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &\quad - \operatorname{Re} \iint_\Omega \mathcal{A}' \nabla u \cdot \nabla_t[\Psi^k] \, \frac{|u|^{q-2}\,\bar{u}}{b} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &=: T_{21} + T_{22}. \end{aligned}$$

The term T_{21} is the second term in the right-hand side of (5.6). As for T_{22}, by (5.2) we know that

$$\begin{aligned} T_{22} &\leq -\operatorname{Re} \iint_\Omega \nabla_t u \cdot \nabla_t[\Psi^k] \, |u|^{q-2}\,\bar{u} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &\quad + C \iint_\Omega |\mathcal{C}| \, |\nabla u| \, |\nabla_t[\Psi^k]| \, |u|^{q-1} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &=: T_{221} + T_{222}, \end{aligned}$$

where T_{222} was obtained by using the fact that b^{-1} is uniformly bounded. Notice that $\nabla|u|^q = q\operatorname{Re}(\nabla u\,|u|^{q-2}\,\bar{u})$, which gives that T_{221} is part of the last term in the right-hand side of (5.6). Observe now that by the assumption on Ψ, we have

$$|\nabla_t[\Psi^k]| \leq \frac{C}{|t|} \, \Psi^{k-1},$$

hence

$$\begin{aligned} T_{222} &\leq C \int_\Omega |\mathcal{C}| \, |\nabla u| \, \Psi^{k-1} \, |u|^{q-1} \, \frac{dt}{|t|^{n-d-1}} \, dx \\ &\leq C \Big(\int_\Omega |\nabla u|^2 \, |u|^{q-2} \, \Psi^k \, \frac{dt}{|t|^{n-d-2}} \, dx \Big)^{1/2} \Big(\int_\Omega |\mathcal{C}|^2 \, |u|^q \, \Psi^{k-2} \, \frac{dt}{|t|^{n-d}} \, dx \Big)^{1/2}. \end{aligned}$$

Since the matrix $\mathcal{A}'$ is q-elliptic and $b^{-1} \leq C$, the first integral in the right-hand side above is thus bounded by

$$\operatorname{Re} \int_\Omega \mathcal{A}' \nabla u \cdot \nabla[|u|^{q-2}\,\bar{u}] \, \frac{\Psi^k}{b} \, \frac{dt}{|t|^{n-d-2}} \, dx = T,$$

while we bound the second integral by using Proposition 4.3 (the Carleson inequality) and the fact that $\mathcal{C}$ satisfies the Carleson measure condition. We obtain

$$T_{222} \lesssim T^{1/2} \kappa^{1/2} \, \|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_q^{q/2}.$$

The bound of T_2 follows.

We turn to the estimate of T_3. We use the boundedness of $\mathcal{A}'$ and b^{-1}, and then Cauchy–Schwarz's inequality to obtain

$$\begin{aligned} |T_3| &\lesssim \iint_\Omega |\nabla u| \, |\nabla b| \, \Psi^k \, |u|^{q-1} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &\lesssim \Big(\int_\Omega |\nabla u|^2 \, |u|^{q-2} \, \Psi^k \, \frac{dt}{|t|^{n-d-2}} \, dx \Big)^{1/2} \Big(\iint_\Omega |\nabla b|^2 \, \Psi^k \, |u|^q \, \frac{dt}{|t|^{n-d-2}} \, dx \Big)^{1/2}. \end{aligned}$$

As we did for T_{222}, the q-ellipticity of $\mathcal{A}'$ and the fact that $|t||\nabla b|$ satisfies the Carleson measure condition give that

$$|T_3| \lesssim T^{1/2}\kappa^{1/2}\,\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2} \leq T^{1/2}\kappa^{1/2}\,\|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_q^{q/2},$$

where the last inequality stands because $\Psi \leq 1$.

The term T_4 is a bit more complicated. By (5.1), the bottom left corner of $\mathcal{A}'$ is a zero matrix. Thus, similar to (5.2), we have

$$\begin{aligned}
T_4 = &- \operatorname{Re} \iint_\Omega b\nabla_t u \cdot \nabla_t|t|\, \frac{|u|^{q-2}\,\bar{u}\,\Psi^k}{b}\, \frac{dt}{|t|^{n-d-1}}\, dx \\
&- \operatorname{Re} \iint_\Omega \mathcal{C}\nabla u \cdot \nabla_t|t|\, \frac{|u|^{q-2}\,\bar{u}\,\Psi^k}{b}\, \frac{dt}{|t|^{n-d-1}}\, dx \\
= &- \operatorname{Re} \iint_\Omega \mathcal{C}\nabla u \cdot \nabla_t|t|\, \frac{|u|^{q-2}\,\bar{u}\,\Psi^k}{b}\, \frac{dt}{|t|^{n-d-1}}\, dx \\
&- \operatorname{Re} \iint_\Omega \nabla_t u \cdot \nabla_t|t|\, |u|^{q-2}\,\bar{u}\,\Psi^k\, \frac{dt}{|t|^{n-d-1}}\, dx \\
=: &\, T_{41} + T_{42}.
\end{aligned}$$

The term T_{41} can be dealt as T_{222} or T_3. By using Cauchy–Schwarz's inequality and then Carleson inequality (Proposition 4.3), we have

$$|T_{41}| \lesssim T^{1/2}\,\kappa^{1/2}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2} \leq T^{1/2}\,\kappa^{1/2}\|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_q^{q/2}.$$

As for T_{42}, recall that $\nabla_t|u|^q = q|u|^{q-2}\operatorname{Re}(\bar{u}\nabla_t u)$, hence

$$\begin{aligned}
T_{42} = &-\frac{1}{q} \iint_\Omega \nabla_t[|u|^q\,\Psi^k] \cdot \nabla_t|t|\, \frac{dt}{|t|^{n-d-1}}\, dx \\
&+ \frac{1}{q} \iint_\Omega \nabla_t[\Psi^k] \cdot \nabla_t|t|\,|u|^q\, \frac{dt}{|t|^{n-d-1}}\, dx \\
=: &\, T_{421} + T_{422}.
\end{aligned}$$

The first term T_{421} is 0. Indeed, for any function v, $\nabla_t v \cdot \nabla_t|t|$ equals $\partial_r v$, the derivative in the radial direction. We switch then to polar coordinates, and T_{421} is the integral of a derivative. The last term T_{422} is part of the last term in the right-hand side of (5.6).

It remains to treat T_1. As stated in (5.3), we know that if u is a weak solution of $\mathcal{L}u = 0$, then it is also a weak solution to $\mathcal{L}'u = 0$, $\mathcal{L}' = -\operatorname{div}|t|^{d+1-n}\mathcal{A}'\nabla + \mathcal{D}' \cdot \nabla$, where the matrix $\mathcal{A}'$ is defined in (5.1), and $\mathcal{D}'$ is as in (5.4). Note that $|t||\mathcal{D}'| \lesssim |t||\nabla \mathcal{B}_3|$, so $|t|\mathcal{D}'$ lies in $L^\infty_{\mathrm{loc}}(\Omega)$ and satisfies the Carleson measure condition.

Now, set $v := \frac{|t|}{b}\,|u|^{q-2}\,\bar{u}\,\Psi^k$. We want to find the value of

$$\iint_{(x,t)\in\mathbb{R}^n} \mathcal{A}'\nabla u \cdot \nabla v\, \frac{dt}{|t|^{n-d-1}}\, dx + \iint_{(x,t)\in\mathbb{R}^n} \mathcal{D}' \cdot \nabla u\, v\, \frac{dt}{|t|^{n-d-1}}\, dx, \tag{5.7}$$

which is formally 0, because (5.7) is only $\mathcal{L}'u$ tested against a test function v. The problem is that the test function v is not smooth, and not even necessarily in $W^{1,2}_{\rm loc}(\Omega)$, so the fact that (5.7) equals 0 needs to be proven.

The function Ψ is compactly supported in Ω, hence is v; besides, since the functions $\Psi, b^{-1}, \nabla\Psi, \nabla b$ are all in $L^\infty_{\rm loc}(\Omega)$, we have $|v| \lesssim |u|^{q-1}$ and $|\nabla v| \lesssim |\nabla u||u|^{q-2} + |u|^{q-1}$ uniformly on the support of Ψ. Due to Proposition 1.10, the function $\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v$ is now in $L^1(\Omega, dt/|t|^{n-d-1}\,dx)$ and we will see that an argument similar to the Lebesgue domination convergence proves that (5.7) is 0. We separate two cases: if $q \geq 2$, then we set $u_N = \min\{N, |u|\}$ and

$$v_N := \frac{|t|}{b}\, u_N^{q-2}\,\bar{u}\,\Psi^k.$$

The function v_N is in $W^{1,2}_{\rm loc}(\Omega)$, because $|t|\Psi^{2q}$ is Lipschitz, $u_N^{q-2}\bar{u} \in W^{1,2}_{\rm loc}(\Omega)$, since $|\nabla[u_N^{q-2}\bar{u}]| \leq (q-1)u_N^{q-2}|\nabla\bar{u}| \in L^2_{\rm loc}(\Omega)$, and $|b| \geq C^{-1}$, $b \in W^{1,\infty}_{\rm loc}(\Omega)$, by assumption. Moreover, v_N is compactly supported in Ω, so by Lemma 8.3 of [10], for any $N \in \mathbb{N}$,

$$\iint_{(x,t)\in\mathbb{R}^n} \mathcal{A}'\nabla u\cdot\nabla v_N\,\frac{dt}{|t|^{n-d-1}}\,dx + \iint_{(x,t)\in\mathbb{R}^n} \mathcal{D}'\cdot\nabla u\, v_N\,\frac{dt}{|t|^{n-d-1}}\,dx = 0.$$

In addition, we have

$$\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v = \mathcal{A}'\nabla u\cdot\nabla v_N + \mathcal{D}'\cdot\nabla u\, v_N \quad \text{in } \{|u| < N\}.$$

It follows that

$$\Big|\iint_{(x,t)\in\mathbb{R}^n} [\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v]\,\frac{dt}{|t|^{n-d-1}}\,dx\Big|$$
$$\leq \int_{\{|u|\geq N\}} \big(|\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v| + |\mathcal{A}'\nabla u\cdot\nabla v_N + \mathcal{D}'\cdot\nabla u\, v_N|\big)\,\frac{dt}{|t|^{n-d-1}}\,dx \to 0$$

as $N \to +\infty$, because $\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v$ and $\mathcal{A}'\nabla u\cdot\nabla v_N + \mathcal{D}'\cdot\nabla u\, v_N$ are integrable.

We turn now to the case where $q < 2$. We define $u_\epsilon := \max\{|u|, \epsilon\}$, $v_\epsilon := \frac{|t|}{b} u_\epsilon^{q-2}\,\bar{u}\,\Psi^k$. Check, similarly, as before, that $v_\epsilon \in W^{1,2}_{\rm loc}(\Omega)$, which gives

$$\iint_{(x,t)\in\mathbb{R}^n} \mathcal{A}'\nabla u\cdot\nabla v_\epsilon\,\frac{dt}{|t|^{n-d-1}}\,dx + \iint_{(x,t)\in\mathbb{R}^n} \mathcal{D}'\cdot\nabla u\, v_\epsilon\,\frac{dt}{|t|^{n-d-1}}\,dx = 0$$

and, by the Lebesgue domination theorem, since $\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v$ and $\mathcal{A}'\nabla u\cdot\nabla v_\epsilon + \mathcal{D}'\cdot\nabla u\, v_\epsilon$ are integrable,

$$\Big|\iint_{(x,t)\in\mathbb{R}^n} [\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v]\,\frac{dt}{|t|^{n-d-1}}\,dx\Big|$$
$$\leq \int_{\{|u|\leq \epsilon\}} \big(|\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v| + |\mathcal{A}'\nabla u\cdot\nabla v_\epsilon + \mathcal{D}'\cdot\nabla u\, v_\epsilon|\big)\,\frac{dt}{|t|^{n-d-1}}\,dx$$
$$\to \int_{\{|u|=0\}} \big(|\mathcal{A}'\nabla u\cdot\nabla v + \mathcal{D}'\cdot\nabla u\, v| + |\mathcal{A}'\nabla u\cdot\nabla v_\epsilon + \mathcal{D}'\cdot\nabla u\, v_\epsilon|\big)\,\frac{dt}{|t|^{n-d-1}}\,dx$$

as $\epsilon \to 0$. However, $\nabla u = 0$ almost everywhere on $\{|u| = 0\}$. We deduce

$$\iint_{(x,t)\in\mathbb{R}^n} [\mathcal{A}'\nabla u \cdot \nabla v + \mathcal{D}' \cdot \nabla u\, v] \frac{dt}{|t|^{n-d-1}}\, dx = 0.$$

We just have shown that the term in (5.7) is zero, thus our term T_1 becomes

$$\begin{aligned} |T_1| &\lesssim \left| \iint_{(x,t)\in\mathbb{R}^n} \mathcal{D}' \cdot \nabla u\, v \frac{dt}{|t|^{n-d-1}}\, dx \right| \\ &\lesssim \iint_{(x,t)\in\mathbb{R}^n} |t|\,|\mathcal{D}'|\,|\nabla u|\,|u|^{q-1}\,\Psi^k \frac{dt}{|t|^{n-d-1}}\, dx. \end{aligned}$$

Recall that $|t|\mathcal{D}'$ satisfies the Carleson measure condition. So, using Cauchy–Schwarz's inequality and then Carleson's inequality (Proposition 4.3), similar to T_{222}, T_3 or T_{41},

$$|T_1| \lesssim T^{1/2}\kappa^{1/2} \|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|^{q/2}_{L^q(B)}.$$

The estimate (5.6) and then the lemma follows. □

From Lemma 5.1, we can easily deduce the local bounds given in the following lemma.

Lemma 5.3. *Let $\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator that satisfies $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$. Let $a > 0$ and $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose a function $\Psi \in C^\infty_0(\Omega)$ that satisfies $(\mathcal{H}^2_{a,M})$ and $k > 2$. Then, for any weak solution $u \in W^{1,2}_{\rm loc}(\Omega)$, we have*

$$\|S_{a,q}(u|\Psi^k)\|_q^q \leq C(\kappa + M)\|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_q^q, \tag{5.8}$$

where the constant C depends on a, q, n, $\|\mathcal{A}\|_\infty$, $\lambda_q(\mathcal{A})$, $\|b\|_\infty + \|b^{-1}\|_\infty$, and k.

Proof. First, recall that

$$\|S^l_{a,q,e}(u|\Psi^k)\|_q^q \simeq T := \iint_{\mathbb{R}^n} |\nabla u|^2\,|u|^{q-2}\,\Psi^k \frac{ds}{|s|^{n-d-2}}\, dy. \tag{5.9}$$

By Lemma 5.1, we have

$$\begin{aligned} \|S_{a,q}(u|\Psi^k)\|_q^q &\leq C\kappa\|\widetilde{N}_{q,a}(u|\Psi^{k-2})\|^q_{L^q(4B)} \\ &\quad - \operatorname{Re}\iint_\Omega \mathcal{A}'\nabla u \cdot \nabla_x[\Psi^k] \frac{|u|^{q-2}\,\bar{u}}{b} \frac{dt}{|t|^{n-d-2}}\, dx \\ &\quad + \frac{1}{q}\operatorname{Re}\iint_\Omega \Big(\frac{|u|^q}{|t|}\nabla|t| - \nabla_t[|u|^q]\Big) \cdot \nabla_t[\Psi^k] \frac{dt}{|t|^{n-d-2}}\, dx \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

We claim

$$|T_2 + T_3| \lesssim M\|\widetilde{N}_{q,a}(u|\Psi^{k-1})\|_q^q + T^{1/2}M^{1/2}\|\widetilde{N}_{q,a}(u|\Psi^{k-2})\|_q^{q/2}. \tag{5.10}$$

From the claim, we deduce that $T \lesssim (\kappa + M)\|\widetilde{N}_{q,a}(u|\Psi^{k-2})\|^q_{L^q(4B)}$. From this and (5.9), the lemma follows.

So it remains to prove (5.10). By using the fact that $\nabla_t |u|^q = q|u|^{q-2}\operatorname{Re}(\bar u \nabla_t u)$ and that $\mathcal{A}'/b$ is bounded, we get

$$\begin{aligned}|T_2 + T_3| &\lesssim \iint_\Omega |\nabla u|\,|\nabla[\Psi^k]|\,|u|^{q-1}\,\frac{dt}{|t|^{n-d-2}}\,dx + \iint_\Omega |u|^q\,|\nabla[\Psi^k]|\,\frac{dt}{|t|^{n-d-1}}\,dx\\ &\lesssim \iint_\Omega |\nabla u|\Psi^{k-1}|\nabla\Psi||u|^{q-1}\,\frac{dt}{|t|^{n-d-2}}\,dx + \iint_\Omega |u|^q\,\Psi^{k-1}|\nabla\Psi|\,\frac{dt}{|t|^{n-d-1}}\,dx\\ &=: T_4 + T_5.\end{aligned}$$

Applying Lemma 4.6 to $\Phi = \Psi^{k-1}$, we get

$$T_5 \lesssim M\|\widetilde{N}_{q,a}(u|\Psi^{k-1})\|_q^q.$$

As for T_4, we use Cauchy–Schwarz's inequality to get

$$\begin{aligned}\iint_\Omega |\nabla u|\Psi^{k-1}|\nabla\Psi||u|^{q-1}\,\frac{dt}{|t|^{n-d-2}}\,dx &\le T^{1/2}\Big(\iint_\Omega |u|^q\,\Psi^{k-2}\,|\nabla\Psi|^2\,\frac{dt}{|t|^{n-d-2}}\,dx\Big)\\ &\lesssim T^{1/2}M^{1/2}\|\widetilde{N}_{q,a}(u|\Psi^{k-2})\|_q^q,\end{aligned}$$

by applying Lemma 4.6 again to $\Phi = \Psi^{k-2}$. The claim (5.10) and then the lemma follows. □

We denote by $\mathcal{M}$ the Hardy–Littlewood maximal function on $\mathbb{R}^d$, that is, if f is a locally integrable function on $\mathbb{R}^d$,

$$\mathcal{M}f(x) := \sup_{\text{balls } B\ni x} \fint_B |f|.$$

As it will be useful later on, we also introduce here the maximal operator $\mathcal{M}_q$ defined for any $q \in (1, +\infty)$ on $L^q_{\text{loc}}(\mathbb{R}^d)$ as

$$\mathcal{M}_q[f](x) = [\mathcal{M}[f^q](x)]^{1/q}.$$

It is well known that the operator $\mathcal{M}_q$ is weak type (q, q) and strong type (p, p) for $p > q$.

A key result of our paper is Lemma 5.5, which states that the L^q bounds for the q-adapted square function given in Lemma 5.3 self-improve to L^p bounds for all $p > 0$. We start by proving a preliminary good-λ inequality.

Lemma 5.4. *Let $\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator that satisfies $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \ge 0$. Let $a > 0$ and $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose a function $\Psi \in C_0^\infty(\Omega)$ that satisfies $(\mathcal{H}^2_{a,M})$ and $k > 2$. Then there exists $\eta \in (0, 1)$ that depends only on d and q such that for any weak solution $u \in W^{1,2}_{\text{loc}}(\Omega)$, any $\nu > 0$ and any $\gamma \in (0, 1)$,*

$$\begin{aligned}&\big|\{x \in \mathbb{R}^d,\ S_{a,q}(u|\Psi^k)(x) > \nu,\ \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^{k-2})](x) \le \gamma\nu\}\big|\\ &\qquad\le C\gamma^q\,|\{x \in \mathbb{R}^d,\ \mathcal{M}[S_{a,q}(u|\Psi^k)](x) > \eta\nu\}|,\end{aligned}\tag{5.11}$$

where the constant C depends on a, q, n, $\|\mathcal{A}\|_\infty$, $\lambda_q(\mathcal{A})$, $\|b\|_\infty + \|b^{-1}\|_\infty$, κ, k and M.

Proof. Let η to be chosen later on. Take $\nu > 0$. We define the set

$$\mathcal{S} := \{x \in \mathbb{R}^d,\ \mathcal{M}[S_{a,q}(u|\Psi^k)](x) > \eta\nu\},$$

which is open (Ψ is compactly supported in Ω, which makes $S_{a,q}(u|\Psi^k)$ continuous) and bounded. Indeed, $\mathcal{S}$ is bounded because Ψ^k is compactly supported, so we have $S_{a,q}(u|\Psi^k) \equiv 0$ outside of a big ball.

We construct a Whitney decomposition of $\mathcal{S}$ in the following manner. For any $x \in \mathcal{S}$, we set B_x as the ball of center x and radius $\mathrm{dist}(x, \mathcal{S}^c)/10$. The balls $(B_x)_{x\in\mathcal{S}}$ clearly cover $\mathcal{S}$. Moreover, the radii of the balls are uniformly bounded, since $\mathcal{S}$ is bounded. So, by Vitali's lemma, there exists a non-overlapping subcollection of balls $(B_{x_i})_{i\in I}$ such that $\bigcup_{i\in I} 5B_{x_i} \supset \mathcal{S}$. We write $B_i = 10B_{x_i}$ and l_i for its radius. By construction, for every $i \in I$,

$$\text{there exists } y_i \in \mathbb{R}^d \text{ such that } |x_i - y_i| = l_i \text{ and } \mathcal{M}[S_{a,q}(u|\Psi^k)](y_i) \leq \eta\nu. \tag{5.12}$$

The balls $B_{x_i} = B_i/10$ are mutually disjoint sets contained in $\mathcal{S}$, so we deduce

$$\sum_{i\in I}|B_i| = 10^d \sum_{i\in I}|B_{x_i}| \lesssim 10^d |\mathcal{S}|. \tag{5.13}$$

Thanks to (5.13), the estimate (5.11) will be obtained if we can prove that

$$F^i_\gamma := \{x \in B_i,\ S_{a,q}(u|\Psi^k)(x) > \nu,\ \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^{k-2})](x) \leq \gamma\nu\} \leq C\gamma^q |B_i|, \tag{5.14}$$

where C is independent of $\gamma \in (0,1)$. If $F^i_\gamma = \emptyset$, there is nothing to prove, so we can assume $F^i_\gamma \neq \emptyset$. Set the function Φ_i as

$$\Phi_i(x,t) = \phi\Big(\frac{a|t|}{l_i}\Big)\phi\Big(\frac{|x-x_i|}{2l_i}\Big),$$

where $\phi \in C_0^\infty(\mathbb{R})$ is such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $[-1,1]$, $\phi \equiv 0$ outside $[-2,2]$, and $|\phi'| \leq 2$. We claim that we can find η small such that

$$S_{a,q}(u|\Psi^k\Phi_i^k)(z) \geq \frac{\nu}{2} \quad \text{for } z \in F^i_\gamma. \tag{5.15}$$

Indeed, for any $z \in \mathbb{R}^d$, one has, by definition,

$$S_{a,q}(u|\Psi^k[1-\Phi_i^k])(z) = \Big(\iint_{(y,s)\in\widehat{\Gamma}_a(z)} |\nabla u|^2 |u|^{q-2}\,\Psi^k[1-\Phi_i^k]\,\frac{ds}{|s|^{n-2}}\,dy\Big)^{1/q}. \tag{5.16}$$

By definition, $1 - \Phi_i^k \equiv 0$ if $|s| \leq l_i/a$, thus in the above integral (5.16) we only need to consider the integration region $\widehat{\Gamma}_a(z)$ above the level of l_i/a. Note that we can find N points $(z_j)_{j\leq N} \in B_{l_i}(z)$, with N depending only on d (for example, by taking each z_j to be at least $l_i/3a$-distance away from all the other z_j's), such that

$$\widehat{\Gamma}_a(z) \cap \{|s| > l_i/a\} \subset \bigcup_{i=1}^{N} \widehat{\Gamma}_a(z_j, l_i/2a).$$

Here we define

$$\widehat{\Gamma}_a(x,\rho) := \{(y,s) \in \Omega,\ |y-x| < a(|s|-\rho)\}, \tag{5.17}$$

which are cones raised to the level ρ. Hence

$$S_{a,q}(u|\Psi^k[1-\Phi_i^k])(z) \le \sum_{j=1}^N \Big(\iint_{(y,s)\in\widehat{\Gamma}_a(z_j,l_i/2a)} |\nabla u|^2\,|u|^{q-2}\,\Psi^k\,[1-\Phi_i^k]\frac{ds}{|s|^{n-2}}\,dy\Big)^{1/q}.$$

By simple geometry, we observe that

$$\widehat{\Gamma}_a(z_j,l_i/2a) \subset \widehat{\Gamma}_a(z') \quad \text{for any } z' \in B_{l_i/2}(z_j);$$

if the point z belongs to $F_\gamma^i \subset B_i = B(x_i,l_i)$ and $|x_i - y_i| = l_i$ (see (5.12)), we have $B_{l_i/2}(z_j) \subset B_{4l_i}(y_i)$. We conclude that

$$\begin{aligned}
&S_{a,q}(u|\Psi^k[1-\Phi_i^k])(z)\\
&\quad\le \sum_{j=1}^N \fint_{z'\in B_{l_i/2}(z_j)} \Big(\iint_{(y,s)\in\widehat{\Gamma}_a(z')} |\nabla u|^2\,|u|^{q-2}\,\Psi^k\,[1-\Phi_i^k]\frac{ds}{|s|^{n-2}}\,dy\Big)^{1/q}\,dz'\\
&\quad\lesssim \sum_{j=1}^N \fint_{z'\in B_{4l_i}(y_i)} |S_{a,q}(u|\Psi^k)(z')|\,dz'\\
&\quad\le C_d\mathcal{M}[S_{a,q}(u|\Psi^k)](y_i) \le C_d\,\eta\nu,
\end{aligned}$$

where we used (5.12) in the last inequality. We choose $\eta := \eta(d)$ such that $(C_d\eta)^q \le (1-2^{-q})$. Consequently, $|S_{a,q}(u|\Psi^k[1-\Phi_i^k])(z)|^q \le \nu(1-2^{-q})$ and then

$$|S_{a,q}(u|\Psi^k\,\Phi_i^k)(z)|^q = |S_{a,q}(u|\Psi^k)(z)|^q - |S_{a,q}(u|\Psi^k[1-\Phi_i^k])(z)|^q \ge \frac{\nu^q}{2^q},$$

which is the desired claim (5.15).

We define now the cut-off function χ_i as

$$\chi_i(x,t) = \phi\Big(\frac{\operatorname{dist}(x,F_\gamma^i)}{a|t|}\Big).$$

It is easy to check that for any $z \in F_\gamma^i$, we have $\widehat{\Gamma}_a(z) \subset \{\operatorname{dist}(x,F_\gamma^i) \le a|t|\}$, and thus $\chi_i \equiv 1$ on $\widehat{\Gamma}_a(z)$. In fact χ_i describes a smooth version of the classic sawtooth domain on top of F_γ^i. Therefore,

$$S_{a,q}(u|\Psi^k\,\Phi_i^k\chi_i^k)(z) = S_{a,q}(u|\Psi^k\,\Phi_i^k) \ge \frac{\nu}{2} \quad \text{for } z \in F_\gamma^i$$

and

$$|F_\gamma^i| \le \frac{2^q}{\nu^q}\int_{\mathbb{R}^d} |S_{a,q}(u|\Psi^k\,\Phi_i^k\,\chi_i^k)(z)|^q\,dz.$$

Since Ψ satisfies $(\mathcal{H}^2_{a,M})$, by Lemma 4.5, $\Psi\Phi_i\chi_i$ satisfies $(\mathcal{H}^2_{M+M'})$, with M' depending only on a and n. Lemma 5.3 entails that

$$|F^i_\gamma| \lesssim \nu^{-q} \int_{\mathbb{R}^d} |\widetilde{N}_{a,q}(u|\Psi^{k-2}\Phi_i^{k-2}\chi_i^{k-2})(z)|^q\,dz. \tag{5.18}$$

The function Φ_i is supported in $\{(x,t) \in \Omega,\ x \in 4B_i,\ |t| \leq 2l_i/a\}$. It forces $\widetilde{N}_{a,q}(u|\Psi^{k-2}\Phi_i^{k-2}\chi_i^{k-2})$ to be supported in, say, $20B_i$. So (5.18) becomes

$$|F^i_\gamma| \lesssim \nu^{-q} \int_{20B_i} |\widetilde{N}_{a,q}(u|\Psi^{k-2}\chi_i^{k-2})(x)|^q\,dx. \tag{5.19}$$

For $(z,r) \in \mathbb{R}^d \times (0,\infty)$, we want to compute $(u|\Psi^{k-2}\chi_i^{k-2})_{W,a,q}(z,r)$. On the one hand, for any $z'' \in B_{ar}(z)$, we have $(z,r) \in \Gamma_a(z'')$, and thus

$$(u|\Psi^{k-2}\chi_i^{k-2})_{W,a,q}(z,r) \leq \widetilde{N}_{a,q}(u|\Psi^{k-2}\chi_i^{k-2})(z'') \leq \widetilde{N}_{a,q}(u|\Psi^{k-2})(z'').$$

Hence

$$(u|\Psi^{k-2}\chi_i^{k-2})_{W,a,q}(z,r) \leq \fint_{B_{ar}(z)} \widetilde{N}_{a,q}(u|\Psi^{k-2})(z'')dz''. \tag{5.20}$$

On the other hand, if $(u|\Psi^{k-2}\chi_i^{k-2})_{W,a,q}(z,r)$ is not 0, it means that $W_a(z,r) \cap \operatorname{supp}\chi_i \neq \emptyset$, and so that there exists $z' \in F^i_\gamma$ such that $|z-z'| \leq 6ar$. Combining this with (5.20), we get

$$\begin{aligned}(u|\Psi^{k-2}\chi_i^{k-2})_{W,a,q}(z,r) &\lesssim \fint_{B_{7ar}(z')} \widetilde{N}_{a,q}(u|\Psi^{k-2})(z'')\,dz'' \\ &\leq \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^{k-2})](z') \leq \gamma\nu.\end{aligned} \tag{5.21}$$

The last inequality is by the definition of F^i_γ. Since the estimate (5.21) holds for all $(z,r) \in \mathbb{R}^d \times (0,\infty)$, we deduce that $\widetilde{N}_{a,q}(u|\Psi^{k-2}\chi_i^{k-2})(x) \lesssim \gamma\nu$ for all $x \in \mathbb{R}^d$. Therefore, one can rewrite (5.19) as

$$|F^i_\gamma| \lesssim \nu^{-q}|20B_i|(\gamma\nu)^q \lesssim \gamma^q|B_i|. \tag{5.22}$$

The claim (5.14) and then the lemma follows. □

Lemma 5.5. *Let $\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator that satisfies $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$. Let $a > 0$, $1 < p < \infty$, and $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose a function $\Psi \in C_0^\infty(\Omega)$ that satisfies $(\mathcal{H}^2_{a,M})$ and $k > 2$. Then, for any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$,*

$$\|S_{a,q}(u|\Psi^k)\|_p \leq C\|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_p, \tag{5.23}$$

where the constant C depends on a, p, q, n, $\|\mathcal{A}\|_\infty$, $\lambda_q(\mathcal{A})$, $\|b\|_\infty + \|b^{-1}\|_\infty$, κ, k and M.

Remark 5.6. The result is proven for $p > 1$, but can be easily extended to $p > 0$ if, in Lemma 5.4, we replace the Hardy–Littlewood maximal function $\mathcal{M}$ by $\mathcal{M}_r$ with $0 < r < 1$.

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |S_{a,q}(u|\Psi^k)|^p \, dx \\
&\quad = c_p \int_0^\infty \nu^{p-1} |\{x \in \mathbb{R}^d,\ S_{a,q}(u|\Psi^k)(x) > \nu\}| \, d\nu \\
&\quad \le C \int_0^\infty \nu^{p-1} \big|\{x \in \mathbb{R}^d,\ S_{a,q}(u|\Psi^k)(x) > \nu,\ \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^{k-2})](x) \le \gamma\nu\}\big| \, d\nu \\
&\qquad + C \int_0^\infty \nu^{p-1} |\{x \in \mathbb{R}^d,\ \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^{k-2})](x) > \gamma\nu\}| \, d\nu.
\end{aligned}$$

Lemma 5.4 implies the existence of $\eta := \eta(d)$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |S_{a,q}(u|\Psi^k)|^p \, dx \\
&\quad \le C\gamma^q \int_0^\infty \nu^{p-1} |\{x \in \mathbb{R}^d,\ \mathcal{M}[S_{a,q}(u|\Psi^k)](x) > \eta\nu\}| \, d\nu \\
&\qquad + C\gamma^{1-p} \int_0^\infty (\gamma\nu)^{p-1} |\{x \in \mathbb{R}^d,\ \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^{k-2})](x) > \gamma\nu\}| \, d\nu \\
&\quad \le C\gamma^q \int_{\mathbb{R}^d} |\mathcal{M}[S_{a,q}(u|\Psi^k)]|^p \, dx + C\gamma^{1-p} \int_{\mathbb{R}^d} |\mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^{k-2})]|^p \, dx \\
&\quad \le C\gamma^q \int_{\mathbb{R}^d} |S_{a,q}(u|\Psi^k)|^p \, dx + C\gamma^{1-p} \int_{\mathbb{R}^d} |\widetilde{N}_{a,q}(u|\Psi^{k-2})|^p \, dx,
\end{aligned}$$

where the last inequality is due to the Hardy–Littlewood maximal inequality. We choose γ such that $C\gamma^q = 1/2$. On the other hand, since Ψ is compactly supported in Ω, we know $\|S_{a,q}(u|\Psi^k)\|_p < +\infty$. Therefore,

$$\int_{\mathbb{R}^d} |S_{a,q}(u|\Psi^k)|^p \, dx \le C \int_{\mathbb{R}^d} |\widetilde{N}_{a,q}(u|\Psi^{k-2})|^p \, dx.$$

The lemma follows. □

6. The local estimate $N < S$

First, for $\rho > 0$, we introduce similarly to (5.17) the new set defined as

$$\Gamma_a(x,\rho) := \{(z,r) \in \mathbb{R}^{d+1}_+,\ |z-x| < a(r-\rho)\},$$

which is a $(d+1)$-dimensional cone raised to the level ρ. Let $a > 0$. For $\nu > 0$ and v, a continuous and compactly supported function, we define the map $h_{\nu,a}(v)\colon \mathbb{R}^d \to \mathbb{R}$ as

$$h_\nu(x) := h_{\nu,a}(v)(x) = \inf\big\{r > 0 : \sup_{Y \in \Gamma_a(x,r)} v(Y) < \nu\big\}.$$

Since v is compactly supported, it is clear that $\{r > 0 : \sup_{Y\in\Gamma_a(x,r)} v(Y) < \nu\}$ is non-empty and then that $h_\nu(x)$ is well defined for all $x \in \mathbb{R}^d$.

The function h_ν will play a very important role in the future, so it is important to understand its purpose. The original idea (see [23], [14]) is the following: we want to get rid of the supremum in the quantity $\widetilde{N}_{a,q}(v)(x)$, so we would like to replace $\widetilde{N}_{a,q}(v)(x)$ by $v_{W,a,q}(x, h_\nu(x))$ for some 'good' function h_ν. The function h_ν is defined so that it captures, roughly, the level sets of $\widetilde{N}_{a,q}(v)$ (see Lemma 6.2). In Lemma 6.4, we prove local L^q estimates on $u_{W,a,q}(\cdot, h_\nu(\cdot))$ when $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $\mathcal{L}u = 0$. Lemma 6.6 gathers Lemmas 6.2 and 6.4 to prove a good λ-inequality for the bound $\widetilde{N}_{a,q} \lesssim S_{a,q}$; Lemma 6.7 transforms this weak estimate into one in L^p, $p > q$; Lemma 6.9 proves that Lemma 6.7 can self-improve, which allows us to get an estimate in L^q. Lemma 6.11 is the aim of the section, and is an easy consequence of Lemma 6.7 and Lemma 6.4.

We start by proving some basic properties of the function h_ν.

Lemma 6.1. *Let $a > 0$. Let v be a continuous and compactly supported function. Choose a positive number ν. Then the following properties hold*:

(i) *The function $h_\nu = h_{\nu,a}(v)$ is Lipschitz with Lipschitz constant a^{-1}, that is, for $x, y \in \mathbb{R}^d$,*

$$|h_\nu(x) - h_\nu(y)| \leq \frac{|x-y|}{a}.$$

(ii) *For an arbitrary $x \in \{y \in \mathbb{R}^d : \sup_{\Gamma_a(y)} v > \nu\}$, we set $r_x := h_\nu(x) > 0$. Then there exists a point $(z, r_z) \in \partial\Gamma_a(x, r_x)$ such that $v(z, r_z) = \nu$ and $h_\nu(z) = r_z$.*

Proof. Let us prove (i). Pick a pair of points $x, y \in \mathbb{R}^d$ and set $r_x = h_\nu(x)$ and $r_y = h_\nu(y)$. Without loss of generality, we can assume that $r_x < r_y$. We want to argue by contradiction, hence we also assume

$$|x - y| < a(r_y - r_x), \tag{6.1}$$

which can be rewritten $(y, r_y) \in \Gamma_a(x, r_x)$. As a consequence, $\overline{\Gamma_a(y, r_y)}$ is a subset of $\Gamma_a(x, r_x)$. Now, it is easy to improve the inclusion $\overline{\Gamma_a(y, r_y)} \subset \Gamma_a(x, r_x)$ to

$$\overline{\Gamma_a(y, r_y - \eta)} \subset \Gamma_a(x, r_x + \eta),$$

with $\eta > 0$ small enough. We deduce that

$$\sup_{\Gamma_a(y,r_y-\eta)} v \leq \sup_{\Gamma_a(x,r_x+\eta)} v < \nu,$$

the last inequality being true by the definition of r_x. It follows that

$$r_y - \eta \geq h_\nu(y) = r_y,$$

which is a contradiction. We deduce that (6.1) is false and thus part (i) of the lemma follows.

We turn now to the proof of part (ii). For any $x \in \mathbb{R}^d$, the function $r \mapsto \sup_{\Gamma_a(x,r)} v$ is continuous (and non-increasing). By the definition of $r_x := h_\nu(x)$, for any $m \in \mathbb{N}$ big enough, the quantity $\sup_{\Gamma_a(x,r_x-1/m)} v$ is bigger than or equal to ν, and the aformentioned continuity implies then the existence of $Z_m \in \Gamma_a(x, r_x - 1/m)$ such that $v(Z_m) = \nu$. Besides, again by definition of r_x, none of the Z_m lies in $\Gamma_a(x, r_x)$. Obviously, the sequence $(Z_m)_m$ lies in the compact set $\operatorname{supp} v_W$, so Z_m has at least one accumulation point Z. Such accumulation point Z has to lie in $\bigcap_m \Gamma_a(x, r_x - 1/m) \setminus \Gamma_a(x, r_x) = \partial\Gamma_a(x, r_x)$, and has to satisfy $v(Z) = \nu$ by the continuity of v. Hence, $r_z = h_{\nu,a}(z)$. $\square$

The function h_ν has the following interesting property.

Lemma 6.2. *Let $a > 0$ and $q \in (1, +\infty)$. Choose a function $v \in L^q_{\text{loc}}(\Omega)$, a smooth function Ψ which satisfies $0 \leq \Psi \leq 1$, and $k > 2$. For any $\nu > 0$ and any point x satisfying $\widetilde{N}_{a,q}(v|\Psi^k)(x) > \nu$, one has*

$$\mathcal{M}\left[\left(\fint_{y \in B_{ah_\nu(\cdot)/2}(\cdot)} \int_{s \in \mathbb{R}^{n-d}} |v|^q \, \Psi^k \, \partial_r[-\chi_\nu^k] \frac{ds}{|s|^{n-d-1}} \, dy\right)^{1/q}\right](x) \geq c\nu, \tag{6.2}$$

where c depends only on k and n, $h_\nu := h_{\nu,a}((v|\Psi^k)_{W,a,q})$, and χ_ν is a cut-off function defined as

$$\chi_\nu(y,s) = \phi\Big(\frac{|s|}{h_\nu(y)}\Big), \quad \textit{where } \phi(r) = \begin{cases} 1 & \textit{if } 0 \leq r < 1/5, \\ \frac{25}{24} - \frac{5}{24} r & \textit{if } 1/5 \leq r \leq 5, \\ 0 & \textit{if } r > 5, \end{cases} \tag{6.3}$$

and $\chi_\nu(y, \cdot) \equiv 0$ if $h_\nu(y) = 0$.

Remark 6.3. (i) Since $\partial_r[-\chi_\nu^k] \approx 1/h_\nu(y)$ is supported in $\{(y,s), h_\nu(y)/5 \leq |s| \leq 5h_\nu(y)\}$, and for $y \in B_{ah_\nu(x)/2}(x)$, we have $h_\nu(y) \approx h_\nu(x)$, we have that (roughly speaking)

$$\left(\fint_{y \in B_{ah_\nu(x)/2}(x)} \int_{s \in \mathbb{R}^{n-d}} |v|^q \, \Psi^k \, \partial_r[-\chi_\nu^k] \, \frac{ds}{|s|^{n-d-1}} \, dy\right)^{1/q} \approx (v|\Psi^k)_{W,a,q}(x, h_\nu(x)).$$

(ii) Since we define the non-tangential maximal function by the average of $|v|^q \Psi^k$, instead of the pointwise values, the assumption $\widetilde{N}_{a,q}(v|\Psi^k)(x) > \nu$ can only give us information about the average value of $|v|^q \Psi^k$ on the level of $h_\nu(y)$. Therefore, we use a function χ_ν such that $\partial_r[-\chi_\nu^k]$ is supported in a band of width $\approx h_\nu(y)$.

Proof. First, observe that $v_{W,a,q}$ is the L^q average of a locally L^q function, so $(v|\Psi^k)_{W,a,q}$ is continuous and the function $h_\nu := h_{\nu,a}((v|\Psi^k)_{W,a,q})$ is well defined.

Fix $x \in \mathbb{R}^d$ such that $\widetilde{N}_{a,q}(v|\Psi^k)(x) > \nu$. Set $r_x := h_\nu(x) > 0$. From part (ii) of Lemma 6.1, there exists $Z = (z, r_z) \in \partial\Gamma_a(x, r_x) \subset \Gamma_a(x)$ such that $(v|\Psi^k)_{W,a,q}(z, r_z) = \nu$ and $h_\nu(z) = r_z$. Define now B as the ball $B_{ar_z/2}(z) \subset \mathbb{R}^d$. Since Z belongs to the boundary of $\Gamma_a(x, r_x)$, we deduce $|x - z| = a(r_z - r_x) < ar_z$, and thus $x \in 2B$. Besides, due to Lemma 6.1, the function h_ν is a^{-1}-Lipschitz and we obtain that

$$0 < \frac{r_z}{2} \leq r_y := h_\nu(y) \leq \frac{3r_z}{2} \quad \text{for } y \in B, \tag{6.4}$$

and, in particular,

$$\frac{r_z}{2} \geq \frac{r_y}{3} \quad \text{and} \quad 2r_z \leq 4r_y \quad \text{for } y \in B.$$

The inequalities above imply that $W_a(z, r_z) \subset \{(y, s) \in \Omega,\, y \in B,\, h_\nu(y)/3 \leq |s| \leq 4h_\nu(y)\}$. Besides, notice that, by the definition of χ_ν,

$$\partial_r[-\chi_\nu^k] \gtrsim \frac{1}{r_z} \quad \text{on } \{(y, s) \in \Omega,\, y \in R,\, h_\nu(y)/3 \leq |s| \leq 4h_\nu(y)\} \supset W(z, r_z).$$

This yields that

$$\fint_{y\in B} \int_{s\in\mathbb{R}^{n-d}} |v|^q\, \Psi^k\, \partial_r[-\chi_\nu^k]\, \frac{ds}{|s|^{n-d-1}}\, dy \gtrsim |(v|\Psi^k)_{W,a,q}(z, r_z)|^q = \nu^q.$$

(It is also easy to see that the left-hand side is finite). For any $y \in B$, we define $B_y = B_{ar_y/100}(y)$. By Vitali's covering lemma, we can find N points $(y_i)_{i\leq N}$ such that the balls B_{y_i} are non-overlapping and $\bigcup 5B_{y_i} \supset B$. We deduce first that, since $r_y \simeq r_z$ for any $y \in B$, the value N depends only on the dimension d, and second that there exists at least one $i \leq N$ such that

$$\begin{aligned}\fint_{y\in 5B_{y_i}} \int_{s\in\mathbb{R}^{n-d}} &|v|^q\, \Psi^k\, \partial_r[-\chi_\nu^k]\, \frac{ds}{|s|^{n-d-1}}\, dy \\ &\gtrsim \fint_{y\in B} \int_{s\in\mathbb{R}^{n-d}} |v|^q\, \Psi^k\, \partial_r[-\chi_\nu^k]\, \frac{ds}{|s|^{n-d-1}}\, dy \gtrsim \nu^q.\end{aligned}$$

For any $y' \in B_{ar_z/10}(y_i) \subset 2B$, we let $r_{y'} = h_\nu(y')$. By Lemma 6.1 and (6.4), we get

$$\frac{2}{5} r_z \leq r_{y'} \leq \frac{8}{5} r_z,$$

and thus $r_{y'} \simeq r_z \simeq r_{y_i}$. And, moreover, simple computations show that $5B_{y_i} = B_{ar_{y_i}/20}(y_i) \subset B_{ar_{y'}/2}(y')$. Therefore,

$$\begin{aligned}\nu &\lesssim \Big(\fint_{y\in 5B_{y_i}} \int_{s\in\mathbb{R}^{n-d}} |v|^q\, \Psi^k\, \partial_r[-\chi_\nu^k]\, \frac{ds}{|s|^{n-d-1}}\, dy\Big)^{1/q} \\ &\lesssim \fint_{y'\in B_{ar_z/10}(y_i)} \Big(\fint_{y\in B_{ar_{y'}/2}(y')} \int_{s\in\mathbb{R}^{n-d}} |v|^q\, \Psi^k\, \partial_r[-\chi_\nu^k]\, \frac{ds}{|s|^{n-d-1}}\, dy\Big)^{1/q} dy' \\ &\lesssim \fint_{y'\in 2B} \Big(\fint_{y\in B_{ar_{y'}/2}(y')} \int_{s\in\mathbb{R}^{n-d}} |v|^q\, \Psi^k\, \partial_r[-\chi_\nu^k]\, \frac{ds}{|s|^{n-d-1}}\, dy\Big)^{1/q} dy'.\end{aligned}$$

Since $x \in 2B$, we conclude that

$$\mathcal{M}\Big[\Big(\fint_{y\in B_{ah_\nu(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |v|^q\, \Psi^k\, \partial_r[-\chi_\nu^k]\, \frac{ds}{|s|^{n-d-1}}\, dy\Big)^{1/q}\Big](x) \gtrsim \nu.$$

The lemma follows. $\square$

Lemma 6.4. *Let $\mathcal{L} = -\operatorname{div}|t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator that satisfies $(\mathcal{H}^1_\kappa)$. Let $a > 0$ and $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose $k > 2$ and a function Ψ compactly supported in Ω and that satisfies $(\mathcal{H}^2_{a,M})$. Then for any weak solution $u \in W^{1,2}_{\rm loc}(\Omega)$ to $\mathcal{L}u = 0$, we have*

$$\tag{6.5} \begin{aligned} &\Big| \iint_\Omega |u|^q\, \partial_r[\Psi^k]\, \frac{dt}{|t|^{n-d-1}}\, dx \Big| \\ &\qquad \lesssim \|S_{a,q}(u|\Psi^k)\|_q^q + \|S_{a,q}(u|\Psi^{k-2})\|_q^{q/2}\, \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2}, \end{aligned}$$

where the constant depends only on a, q and n, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty$, κ and M.

Remark 6.5. A careful reader will notice that the constants do not depend on the ellipticity constants, or on an upper bound on $|b|$. In addition, we do not need for the proof of this lemma to assume that b and $\mathcal{B}_3$ are real-valued.

Observe also that if u is a constant function (and since Ψ has to be compactly supported in Ω), both the right-hand term and the left-hand term in (6.5) are zero.

Proof. First, Proposition 1.10 ensures that $|u|^q$ and $|\nabla u|^2\,|u|^{q-2}$ both lie in $L^1_{\rm loc}(\Omega)$, and so all the quantities in (6.5) are well defined and finite.

We also use the same trick as in Lemma 5.1, which says that u is also a weak solution to $\mathcal{L}'u = 0$, where $\mathcal{L}' = -\operatorname{div}\mathcal{A}'\nabla + \mathcal{D}'\cdot\nabla$,

$$\tag{6.6} \mathcal{A}' = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 + (\mathcal{B}_3)^T \\ 0 & bI \end{pmatrix} + \mathcal{C},$$

and for any $1 \leq j \leq d < i \leq n$,

$$\tag{6.7} (\mathcal{D}')_j = -\sum_{k>d} |t|^{n-d-1}\,\partial_{t_k}[(\mathcal{B}_3)_{kj}|t|^{d+1-n}] \quad \text{and} \quad (\mathcal{D}')_i = \sum_{k\leq d} \partial_{x_k}(\mathcal{B}_3)_{ik}.$$

Note in particular that $|t|\mathcal{D}'$ satisfies the Carleson measure condition. Unless we assume $\mathcal{B}_3$ is real-valued, nothing guarantees here that the matrix $\mathcal{A}'$ is elliptic, but we shall not use this assumption.

Let us denote

$$T := \iint_\Omega |u|^q \partial_r[\Psi^k]\, \frac{dt}{|t|^{n-d-1}}\, dx.$$

By using the product rule and the fact that $\partial_r[|u|^q] = \frac{q}{2}|u|^{q-2}[\bar u\,\partial_r u + u\,\partial_r \bar u]$, one obtains

$$\begin{aligned} T &= \iint_\Omega \partial_r[|u|^q\Psi^k]\, \frac{dt}{|t|^{n-d-1}}\, dx - \frac{q}{2}\iint_\Omega \partial_r[u]\,|u|^{q-2}\,\bar u\,\Psi^k\, \frac{dt}{|t|^{n-d-1}}\, dx \\ &\quad - \frac{q}{2}\iint_\Omega \partial_r[\bar u]\,|u|^{q-2}\,u\,\Psi^k\, \frac{dt}{|t|^{n-d-1}}\, dx \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

The term T_1 is 0. Indeed, we switch to cylindrical coordinates, and we have

$$T_1 = \int_{x\in\mathbb{R}^d}\int_{\theta\in\mathbb{S}_{n-d-1}}\int_0^\infty \partial_r[|u|^q\Psi^k]\, dr\, d\theta\, dx = 0,$$

since Ψ is compactly supported in Ω.

We need to bound T_2 and T_3. Since, $T_3 = \overline{T_2}$, we only need to treat T_2. We remark that the estimate of T_2 is in some sense the reverse of the estimate of the term T_4 in Lemma 5.1. Observe that

$$T_2 = -\frac{q}{2} \iint_{(x,t)\in\Omega} b\nabla_t u \cdot \nabla_t |t| \, \frac{|u|^{q-2}\, \bar u\, \Psi^k}{b} \, \frac{dt}{|t|^{n-d-1}} \, dx,$$

and so with (6.6), we have

$$\begin{aligned} T_2 &= -\frac{q}{2} \iint_{(x,t)\in\Omega} \mathcal{A}'\nabla u \cdot \nabla |t| \, \frac{|u|^{q-2}\, \bar u\, \Psi^k}{b} \, \frac{dt}{|t|^{n-d-1}} \, dx \\ &\quad + \frac{q}{2} \iint_{(x,t)\in\Omega} \mathcal{C}\nabla u \cdot \nabla |t| \, \frac{|u|^{q-2}\, \bar u\, \Psi^k}{b} \, \frac{dt}{|t|^{n-d-1}} \, dx \\ &=: T_{21} + T_{22}. \end{aligned}$$

We use the fact that $|b| \gtrsim 1$, and then Cauchy–Schwarz's inequality to bound T_{22} as follows:

$$\begin{aligned} |T_{22}| &\lesssim \iint_{(x,t)\in\Omega} |\mathcal{C}|\, |\nabla u| \, \frac{|u|^{q-1} \Psi^k}{|b|} \, \frac{dt}{|t|^{n-d-1}} \, dx \\ &\lesssim \iint_{(x,t)\in\Omega} |\mathcal{C}|\, |\nabla u|\, |u|^{q-1}\, \Psi^k \, \frac{dt}{|t|^{n-d-1}} \, dx \\ &\lesssim \Big(\iint_{(x,t)\in\Omega} |\nabla u|^2\, |u|^{q-2}\, \Psi^k \, \frac{dt}{|t|^{n-d-2}} \, dx \Big)^{1/2} \Big(\iint_{(x,t)\in\Omega} |\mathcal{C}|^2\, |u|^q\, \Psi^k \, \frac{dt}{|t|^{n-d}} \, dx \Big)^{1/2}. \end{aligned}$$

The first integral in the right-hand side above can be bounded by $\|S_{a,q}(u|\Psi^k)\|_q^q \leq \|S_{a,q}(u|\Psi^{k-2})\|_q^q$. Since $\mathcal{C}$ satisfies the Carleson measure condition, we use the Proposition 4.3 (Carleson inequality) to bound the second integral as follows:

$$\iint_{(x,t)\in\Omega} |\mathcal{C}|^2\, |u|^q\, \Psi^k \, \frac{dt}{|t|^{n-d}} \, dx \leq \kappa \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q.$$

As a consequence,

$$|T_{22}| \lesssim \|S_{a,q}(u|\Psi^{k-2})\|_q^{q/2} \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2}.$$

Now, we deal with T_{21}. We write

$$\begin{aligned} T_{21} &= -\frac{q}{2} \iint_{(x,t)\in\Omega} \mathcal{A}'\nabla u \cdot \nabla \Big[\frac{|t|\, |u|^{q-2}\, \bar u\, \Psi^k}{b} \Big] \, \frac{dt}{|t|^{n-d-1}} \, dx \\ &\quad + \frac{q}{2} \iint_{(x,t)\in\Omega} \mathcal{A}'\nabla u \cdot \nabla [|u|^{q-2}\, \bar u] \, \frac{\Psi^k}{b} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &\quad + \frac{q}{2} \iint_{(x,t)\in\Omega} \mathcal{A}'\nabla u \cdot \nabla [\Psi^k] \, \frac{|u|^{q-2}\, \bar u}{b} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &\quad - \frac{q}{2} \iint_{(x,t)\in\Omega} \mathcal{A}'\nabla u \cdot \nabla b \, \frac{|u|^{q-2}\, \bar u\, \Psi^k}{b^2} \, \frac{dt}{|t|^{n-d-2}} \, dx \\ &=: T_{211} + T_{212} + T_{213} + T_{214}. \end{aligned}$$

By using the boundedness of $\mathcal{A}'/b$, the term T_{212} can be bounded as follows:

$$|T_{212}| \lesssim \iint_{(x,t)\in\Omega} |\nabla u|\,|\nabla[|u|^{q-2}\,\bar{u}]|\,\Psi^k \,\frac{dt}{|t|^{n-d-2}}\,dx.$$

But using the product rule and the fact that $|\nabla|u|| \leq |\nabla u|$, one has $|\nabla[|u|^{q-2}\,\bar{u}]| \leq (q-1)|u|^{q-2}|\nabla u|$ if $q \leq 2$ and $|\nabla[|u|^{q-2}\,\bar{u}]| \leq (3-q)|u|^{q-2}|\nabla u|$ if $q \geq 2$. Hence one can bound

$$|T_{212}| \lesssim \iint_{(x,t)\in\Omega} |\nabla u|^2\,|u|^{q-2}\,\Psi^k \,\frac{dt}{|t|^{n-d-2}}\,dx \lesssim \|S_{a,q}(u|\Psi^k)\|_q^q.$$

The terms T_{213} and T_{214} are bounded in a similar manner as T_{22}. Let us start with T_{213}. Since $\mathcal{A}'/b$ is uniformly bounded,

$$\begin{aligned}
|T_{213}| &\lesssim \iint_{(x,t)\in\Omega} |\nabla u|\,|\nabla[\Psi^k]|\,|u|^{q-1}\,\frac{dt}{|t|^{n-d-2}}\,dx \\
&\lesssim \iint_{(x,t)\in\Omega} |\nabla u|\,|\nabla\Psi|\,\Psi^{k-1}\,|u|^{q-1}\,\frac{dt}{|t|^{n-d-2}}\,dx \\
&\lesssim \Big(\iint_{(x,t)\in\Omega} |\nabla u|^2\,|u|^{q-2}\,\Psi^{k-2}\,\frac{dt}{|t|^{n-d-2}}\,dx\Big)^{1/2} \\
&\qquad \times \Big(\iint_{(x,t)\in\Omega} |u|^q\,\Psi^k\,|\nabla\Psi|^2\,\frac{dt}{|t|^{n-d-2}}\,dx\Big)^{1/2},
\end{aligned}$$

by Cauchy–Schwarz's inequality. The first integral in the right-hand side above is bounded by $\|S_{a,q}(u|\Psi^{k-2})\|_q^q$. As for the second one, Lemma 4.6 gives that

$$\iint_{(x,t)\in\Omega} |u|^q\,\Psi^k\,|\nabla\Psi|^2\,\frac{dt}{|t|^{n-d-2}}\,dx \lesssim M\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q.$$

We conclude that

$$|T_{213}| \lesssim \|S_{a,q}(u|\Psi^{k-2})\|_q^{q/2}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2}.$$

Then we deal with T_{214}. Using the fact that $\mathcal{A}'/b$ is uniformly bounded, Cauchy–Schwarz's inequality and the fact that $|t|\nabla b$ satisfies the Carleson measure condition, similar to the bound on T_{22} or T_{213}, we obtain

$$\begin{aligned}
|T_{214}| &\lesssim \iint_{(x,t)\in\Omega} |\nabla u|\,|\nabla b|\,|u|^{q-1}\,\Psi^k\,\frac{dt}{|t|^{n-d-2}}\,dx \\
&\lesssim \Big(\iint_{(x,t)\in\Omega} |\nabla u|^2\,|u|^{q-2}\,\Psi^k\,\frac{dt}{|t|^{n-d-2}}\,dx\Big)^{1/2} \\
&\qquad \times \Big(\iint_{(x,t)\in\Omega} (|t|\,|\nabla b|)^2\,|u|^q\,\Psi^k\,\frac{dt}{|t|^{n-d}}\,dx\Big)^{1/2} \\
&\lesssim \kappa^{1/2}\|S_{a,q}(u|\Psi^k)\|_q^{q/2}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2} \\
&\lesssim \|S_{a,q}(u|\Psi^{k-2})\|_q^{q/2}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2}.
\end{aligned} \tag{6.8}$$

It remains to bound T_{211}. However, T_{211} can be treated as the term T_1 in Lemma 5.1, by using the fact that u is a weak solution to $\mathcal{L}'u = 0$, and we will eventually obtain that

$$|T_{211}| \lesssim \Big| \iint_{(x,t)\in\mathbb{R}^n} \mathcal{D}' \cdot \nabla u \, \frac{|t|\,|u|^{q-2}\,\bar{u}\,\Psi^k}{b} \, \frac{dt}{|t|^{n-d-1}} \, dx \Big|$$
$$\lesssim \iint_{(x,t)\in\mathbb{R}^n} |t|\,|\mathcal{D}'|\,|\nabla u|\,|u|^{q-1}\,\Psi^k \, \frac{dt}{|t|^{n-d-1}} \, dx.$$

However, recall that $|\mathcal{D}'| \lesssim |\nabla \mathcal{B}_3|$, so $|t|\mathcal{D}'$ satisfies the Carleson measure condition. By using Cauchy–Schwarz's inequality and then Carleson's inequality (Proposition 4.3), similar to T_{22},

$$|T_{111}| \lesssim \kappa^{1/2} \|S_{a,q}(u|\Psi^{k-2})\|_q^{q/2} \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2}.$$

The lemma follows. □

Lemma 6.6. *Let $\mathcal{L}$ be an elliptic operator satisfying (H^1_κ) for some constant $\kappa \geq 0$. Let $a, l > 0$ and $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose $k > 2$, a positive a^{-1}-Lipschitz function e and a ball $B := B_{l'}(x_B) \subset \mathbb{R}^d$ of radius $l' \geq l$. Then there exists $\eta \in (0,1)$ that depends only on d such that for any weak solution $u \in W^{1,2}_{\rm loc}(\Omega)$, any $\nu > 0$ and any $\gamma \in (0,1)$,*

$$\tag{6.9} \begin{aligned} &\big|\{x \in \mathbb{R}^d,\ \widetilde{N}_{a,q}(u|\Psi_e^k \Psi_{B,l}^k)(x) > \nu\} \cap E_{\nu,\gamma}\big| \\ &\qquad \leq C\gamma^q \big|\{x \in \mathbb{R}^d,\ \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi_e^k \Psi_{B,l}^k)](x) > \eta\nu\}\big|, \end{aligned}$$

where $\Psi_e, \Psi_{B,l}$ are defined as in Lemma 4.5,

$$E_{\nu,\gamma} := \Big\{ x \in \mathbb{R}^d,\ \mathcal{M}_q\Big[\Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |u|^q \, \Psi_{B,l}^k \, \partial_r[\Psi_e^k] \, \frac{ds}{|s|^{n-d-1}}\Big)^{1/q}\Big](x) \leq \gamma\nu$$
$$\text{and } \mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi_e^k \Psi_{B,l}^k)](x)\, \mathcal{M}_q[S_{a,q}(u|\Psi_e^{k-2}\Psi_{B,l}^{k-2})](x) \leq \gamma^2\nu^2 \Big\},$$

and the constant $C > 0$ depends on a, q, n, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty$, κ and k.

Proof. Take some η to be fixed later. Choose $\nu > 0$. To lighten the notation, we write Ψ for $\Psi_e \Psi_{B,l}$. We define

$$\mathcal{S} := \{x \in \mathbb{R}^d,\ \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^k)](x) > \eta\nu\},$$

which is open and bounded. Indeed, $\mathcal{S}$ is bounded because Ψ is compactly supported, and then $\widetilde{N}_{a,q}(u|\Psi^k)] \equiv 0$ outside a big ball, and $\mathcal{S}$ is open because $(u|\Psi^k)_W$ is continuous.

We construct a Whitney decomposition as follows (the construction is classical, and we only aim to prove that (6.10) is possible). For any $x \in \mathcal{S}$, we set $B_x \subset \mathbb{R}^d$ the ball of center x and radius $\mathrm{dist}(x, \mathcal{S}^c)/10$. The balls B_x have uniformly bounded radius (because $|B_x| \leq |\mathcal{S}| < +\infty$) and therefore Vitali's covering lemma entails the existence of a non-overlapping collection of balls $(B_{x_i})_{i\in I}$ such that $\bigcup_{i\in I} 5B_{x_i} = \mathcal{S}$. We write B_i for $10B_{x_i}$ and l_i for its radius, so by construction,

$$(6.10)\quad \bigcup_{i\in I} B_i = \mathcal{S}, \quad \sum_{i\in I} |B_i| \leq 10^d |\mathcal{S}| \quad \text{and for } i \in I, \text{ there exists } y_i \text{ in } \overline{B}_i \cap \mathcal{S}^c.$$

Then the point y_i satisfies

$$(6.11)\qquad |x_i - y_i| = l_i \quad \text{and} \quad \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^k)](y_i) \leq \eta\nu.$$

Since $\sum_{i\in I} |B_i| \lesssim |\mathcal{S}|$, the estimate (6.9) will be proven if we establish that

$$(6.12)\qquad F^i_\gamma := \{x \in \mathbb{R}^d,\ \widetilde{N}_{a,q}(u|\Psi^k)(x) > \nu\} \cap E_{\nu,\gamma} \cap B_i \leq C\gamma^q |B_i|,$$

where C is independent of $\gamma \in (0, 1)$. If $F^i_\gamma = \emptyset$, there is nothing to prove, so we can assume that F^i_γ contains some point z_i.

Similar to Lemma 4.5 (2), we denote

$$(6.13)\qquad \Phi_i = \Psi_{3B_i,2l_i}(x, t) = \phi\Big(\frac{a|t|}{2l_i}\Big)\phi\Big(1 + \frac{\operatorname{dist}(x, 3B_i)}{2l_i}\Big).$$

We claim that we can find η small enough such that

$$(6.14)\qquad \widetilde{N}_{a,q}(u|\Psi^k\,\Phi^k_i)(z) > \eta \quad \text{for } z \in F^i_\gamma.$$

Indeed, letting $z \in B_i$, for any $(z', r') \in \Gamma_a(z)$ such that $r' \geq l_i/a$,

$$(u|\Psi^k)_{W,a,q}(z', r') \leq \widetilde{N}_{a,q}(u|\Psi^k)(z'') \quad \text{for } z'' \in B_{ar'}(z').$$

Now, since

$$|z' - y_i| \leq |z' - z| + |z - y_i| < ar' + 2l_i \leq 3ar',$$

we deduce

$$\begin{aligned}(u|\Psi^k)_{W,a,q}(z', r') &\leq \fint_{B_{ar'}(z')} \widetilde{N}_{a,q}(u|\Psi^k)(z'')\, dz'' \\ &\leq C_d \fint_{B_{3ar'}(z')} \widetilde{N}_{a,q}(u|\Psi^k)(z'')\, dz'' \\ &\leq C_d \mathcal{M}[\widetilde{N}_{a,q}(u|\Psi^k)](y_i) \leq C_d\eta\nu,\end{aligned}$$

by (6.11). We choose η such that $C_d\eta \leq 1$ and we obtain that

$$(6.15)\qquad (u|\Psi^k)_{W,a,q}(z', r') \leq \nu \quad \text{for } (z', r') \in \Gamma_a(z),\ r' \geq l_i/a.$$

We observe that $\Phi_i \equiv 1$ on $W_a(z'', r'')$ if $(z'', r'') \in \Gamma_a(z)$ and $r'' \leq l_i/a$, and thus

$$(u|\Psi^k)_{W,a,q}(z'', r'') = (u|\Psi^k\,\Phi^k_i)_{W,a,q}(z'', r'').$$

Recall that $\widetilde{N}_{a,q}(u|\Psi^k)(z) > \nu$ for $z \in F^i_\gamma$ and (6.15), so we necessarily have

$$\widetilde{N}_{a,q}(u|\Psi^k\,\Phi^k_i)(z) > \nu.$$

The claim (6.14) follows.

We are now ready to use Lemma 6.2. Set $h_\nu := h_{\nu,a}((u|\Psi^k\,\Phi^k_i)_{W,a,q})$ and χ_ν as in (6.3). Lemma 6.2 gives that, for any $z \in F^i_\gamma$,

$$\mathcal{M}\Big[\Big(\fint_{y\in B_{ah_\nu(\cdot)/2}(\cdot)}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\Psi^k\,\Phi^k_i\,\partial_r[-\chi^k_\nu]\,\frac{ds}{|s|^{n-d-1}}\,dy\Big)^{1/q}\Big](z) \gtrsim \nu,$$

and so, if we integrate with respect to $z \in \mathbb{R}^d$,

$$|F^i_\gamma| \lesssim \frac{1}{\nu^q}\int_{z\in\mathbb{R}^d}\Big|\mathcal{M}\Big[\Big(\fint_{y\in B_{ah_\nu(\cdot)/2}(\cdot)}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\Psi^k\,\Phi^k_i\,\partial_r[-\chi^k_\nu]\frac{ds}{|s|^{n-d-1}}\,dy\Big)^{1/q}\Big](z)\Big|^q dz.$$

The Hardy–Littlewood maximal inequality entails that

$$|F^i_\gamma| \lesssim \frac{1}{\nu^q}\int_{z\in\mathbb{R}^d}\fint_{y\in B_{ah_\nu(z)/2}(z)}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\Psi^k\,\Phi^k_i\,\partial_r[-\chi^k_\nu]\,\frac{ds}{|s|^{n-d-1}}\,dy\,dz. \tag{6.16}$$

If $y \in B_{ah_\nu(z)/2}(z)$, then $|y-z| \leq ah_\nu(z)/2$, and since h_ν is a^{-1}-Lipschitz,

$$\frac{1}{2}\,h_\nu(z) \leq h_\nu(y) \leq \frac{3}{2}\,h_\nu(z), \tag{6.17}$$

in other words $h_\nu(z) \leq 2h_\nu(y)$. It follows that $z \in B_{ah_\nu(y)}(y)$, and so, by Fubini's theorem and (6.17),

$$\begin{aligned}|F^i_\gamma| &\lesssim \frac{1}{\nu^q}\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\Psi^k\,\Phi^k_i\,\partial_r[-\chi^k_\nu]\,\frac{ds}{|s|^{n-d-1}}\\ &\qquad\times\Big(\int_{z\in B_{ah_\nu(y)}(y)}(ah_\nu(z))^{-d}\,dz\Big)\,dy\\ &\lesssim \frac{1}{\nu^q}\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\Psi^k\,\Phi^k_i\,\partial_r[-\chi^k_\nu]\,\frac{ds}{|s|^{n-d-1}}\,dy.\end{aligned} \tag{6.18}$$

Now we estimate the right-hand side of (6.18). By the product rule,

$$\begin{aligned}\Psi^k\,\Phi^k_i\,\partial_r[-\chi^k_\nu] &= -\partial_r[\Psi^k\,\Phi^k_i\chi^k_\nu] + \chi^k_\nu\,\partial_r[\Psi^k\,\Phi^k_i]\\ &= -\partial_r[\Psi^k\,\Phi^k_i\,\chi^k_\nu] + \chi^k_\nu(\Psi^k_e\,\partial_r[\Psi^k_{B,l}\,\Phi^k_i] + \Psi^k_{B,l}\,\Phi^k_i\,\partial_r[\Psi^k_e])\\ &\leq -\partial_r[\Psi^k\,\Phi^k_i\,\chi^k_\nu] + \chi^k_\nu\Psi^k_{B,l}\,\Phi^k_i\,\partial_r[\Psi^k_e],\end{aligned}$$

where the last line holds because $\Psi_{B,l}\Phi_i$ is decreasing in r (because we build $\Psi_{B,l}$ and Φ_i with the help of ϕ which is decreasing). Set

$$T_1 := -\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}}|u|^q\partial_r[\Psi^k\,\Phi^k_i\,\chi^k_\nu]\,\frac{ds}{|s|^{n-d-1}}\,dy$$

and

$$T_2 := \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_\nu^k \Phi_i^k \Psi_{B,l}^k \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\, dy.$$

We have

$$|F_\gamma^i| \lesssim \nu^{-q}(T_1+T_2), \tag{6.19}$$

and we want to bound T_1 and T_2. Let us start with T_2. Let $y\in\mathbb{R}^d$ and $z\in B_{ae(y)/4}(y)$. Since e is a^{-1}-Lipschitz, similar to (6.17), one has

$$\frac{3}{4}\, e(y) \le e(z) \le \frac{5}{4}\, e(y). \tag{6.20}$$

In particular, $y\in B_{ae(z)/3}(z)$ and for any $y\in\mathbb{R}^d$,

$$1 \lesssim \int_{z\in B_{ae(y)/4}} (ae(z))^{-d}\, dz.$$

The bound on T_2 then becomes

$$\begin{aligned} T_2 &\lesssim \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_\nu^k \Phi_i^k \Psi_{B,l}^k \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}} \int_{z\in B_{ae(y)/4}} (ae(z))^{-d}\, dz\, dy \\ &\lesssim \int_{z\in\mathbb{R}^d} \fint_{y\in B_{ae(z)/3}(z)} \int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_\nu^k \Phi_i^k \Psi_{B,l}^k \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\, dy\, dz, \end{aligned}$$

by Fubini's lemma. We want to see for which $z\in\mathbb{R}^d$, the quantity

$$\fint_{y\in B_{ae(z)/3}(z)} \int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_\nu^k \Phi_i^k \Psi_{B,l}^k \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\, dy$$

is non-zero. First, by the definition (6.13), we know

$$\operatorname{supp}\Phi_i \subset \{(y,s)\in\mathbb{R}^n,\ |s|\le 4l_i/a,\ y\in 5B_i\}.$$

Thus we need $e(z)\le 10l_i/a$, because otherwise $\Phi_i^k\,\partial_r[\Psi_e^k]\equiv 0$ (we also use (6.20) here). We also need $B_{ae(z)/3}(z)\cap 5B_i\neq\emptyset$ to guarantee $\Phi_i\neq 0$. Altogether, z needs to lie in, say, $10B_i$. Recall that there exists some $z_i\in F_\gamma^i\subset 10B_i$, so we conclude

$$\begin{aligned} T_2 &\lesssim \int_{z\in 10B_i} \fint_{y\in B_{ae(z)/3}(z)} \int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_\nu^k \Phi_i^k \Psi_{B,l}^k \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\, dy\, dz \\ &\lesssim \int_{z\in 10B_i} \fint_{y\in B_{ae(z)/3}(z)} \int_{s\in\mathbb{R}^{n-d}} |u|^q \Psi_{B,l}^k \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\, dy\, dz \\ &\lesssim |10B_i| \Big|\mathcal{M}_q\Big[\Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |u|^q \Psi_{B,l}^k \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\Big)^{1/q}\Big](z_i)\Big|^q \\ &\lesssim |B_i|\gamma^q\nu^q. \end{aligned} \tag{6.21}$$

We turn now to the treatment of T_1. Observe that, by invoking the same argument as the one used to prove Lemma 4.5 (1), we can prove that χ_ν satisfies $(\mathcal{H}^2_{a,M})$ with M that depends only on a and n,d. (Note, in particular, that the Lipschitz constant for h_ν is a^{-1}, independent of ν.) Therefore, Lemma 6.4 entails that

$$\begin{aligned}|T_1| &\lesssim \|S_{a,q}(u|\Psi^k\,\Phi_i^k\chi_\nu^k)\|_q^q + \|S_{a,q}(u|\Psi^{k-2}\,\Phi_i^{k-2}\,\chi_\nu^{k-2})\|_q^{q/2}\|\widetilde{N}_{a,q}(u|\Psi^k\,\Phi_i^k\chi_i^k)\|_q^{q/2}\\ &\lesssim \|S_{a,q}(u|\Psi^k\,\Phi_i^k)\|_q^q + \|S_{a,q}(u|\Psi^{k-2}\,\Phi_i^{k-2})\|_q^{q/2}\|\widetilde{N}_{a,q}(u|\Psi^k\,\Phi_i^k)\|_q^{q/2}.\end{aligned}$$

Since Φ_i is supported in $\{(x,t)\in\Omega,\ x\in 5B_i,\ |t|\le 4l_i/a\}$, we have that the functions $S_{a,q}(u|\Psi^{k-2}\Phi_i^{k-2})$ and $\widetilde{N}_{a,q}(u|\Psi^k\,\Phi_i^k)$ are supported in, say, $10B_i$. The bound on T_1 then becomes

$$\begin{aligned}|T_1| &\lesssim \|S_{a,q}(u|\Psi^k)\|^q_{L^q(10B_i)} + \|S_{a,q}(u|\Psi^{k-2})\|^{q/2}_{L^q(10B_i)}\|\widetilde{N}_{a,q}(u|\Psi^k)\|^{q/2}_{L^q(10B_i)}\\ &\lesssim |B_i|\cdot\big(|\mathcal{M}_q[S_{a,q}(u|\Psi^k)](z_i)|^q\\ &\quad + |\mathcal{M}_q[S_{a,q}(u|\Psi^k)](z_i)|^{q/2}\,|\mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi^k)](z_i)|^{q/2}\big).\end{aligned}$$

Since $z_i\in F^i_\gamma$, we have

$$|\mathcal{M}_q[S_{a,q}(u|\Psi^k)](z_i)|^{q/2}\,|\mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi^k)](z_i)|^{q/2}\le\gamma^q\nu^q.$$

In addition,

$$\begin{aligned}|\mathcal{M}_q[S_{a,q}(u|\Psi^k)](z_i)|^q &= \Big(\frac{|\mathcal{M}_q[S_{a,q}(u|\Psi^k)](z_i)|\,|\mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi^k)](z_i)|}{|\mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi^k)](z_i)|}\Big)^q\\ &\le\frac{\gamma^{2q}\nu^{2q}}{\nu^q}\le\gamma^{2q}\nu^q\le\gamma^q\nu^q.\end{aligned}$$

For the first inequality we also used the fact that $\widetilde{N}_{a,q}(u|\Psi^k)(z_i)\ge\nu$ and that $\widetilde{N}_{a,q}(u|\Psi^k)$ is continuous. We deduce

$$|T_1|\lesssim|B_i|\gamma^q\nu^q. \tag{6.22}$$

The combination of (6.19), (6.21) and (6.22) proves (6.12). The lemma follows. □

Lemma 6.7. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa\ge 0$. Let $a,l>0$, $q\in(q_0,q_0')$, where q_0 is given by Proposition 2.1, and $p>q$. Choose $k>2$, a positive a^{-1}-Lipschitz function e and a ball $B\subset\mathbb{R}^d$ of radius $l'\ge l$. Then for any weak solution $u\in W^{1,2}_{\rm loc}(\Omega)$,*

$$\begin{aligned}&\|\widetilde{N}_{a,q}(u|\Psi_e^k\Psi_{B,l}^k)\|_p^p \\ &\lesssim\|S_{a,q}(u|\Psi_e^{k-2}\Psi_{B,l}^{k-2})\|_p^p+\Big\|\Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\Psi_{B,l}^k\,\partial_r[\Psi_e^k]\frac{ds}{|s|^{n-d-1}}\,dy\Big)^{1/q}\Big\|_p^p,\end{aligned}\tag{6.23}$$

where the constant depends on a, q, n, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty$, κ, k.

Remark 6.8. The reader may think of the last term as the average of u in the local region in consideration (determined by B and e). It appears on the right side because, roughly speaking, we are estimating u by its gradient.

The aforementioned region is close to boundary, and the reader may be used to see it lying in a Whitney region of the boundary ball B. However, we find it easier to get the self-improvement given by Lemma 6.9 with the estimate (6.23), and Lemma 6.4 will be used (again) in Lemma 6.11 to recover a "classical" right-hand term.

Proof. As in the previous proof, we write Ψ for $\Psi_e \Psi_{B,l}$. Besides, $E_{\nu,\gamma}$ denotes the set

$$E_{\nu,\gamma} := \Big\{ x \in \mathbb{R}^d,\ \mathcal{M}_q\Big[\Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_{B,l}^k\, \partial_r[\Psi_e^k]\, \frac{ds}{|s|^{n-d-1}}\Big)^{1/q}\Big](x) \leq \gamma\nu \\ \text{and } \mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi_e^k\,\Psi_{B,l}^k)](x)\mathcal{M}_q[S_{a,q}(u|\Psi_e^{k-2}\,\Psi_{B,l}^{k-2})](x) \leq \gamma^2\nu^2 \Big\}$$

By Lemma 6.6, there exists $\eta = \eta(d)$ such that for any $\gamma \in (0,1)$,

$$\begin{aligned}
\|\widetilde{N}_{a,q}(u|\Psi^k)\|_p^p &= c_p \int_0^\infty \nu^{p-1} |\{x \in \mathbb{R}^d,\ \widetilde{N}_{a,q}(u|\Psi^k)(x) > \nu\}|\, d\nu \\
&\leq C \int_0^\infty \nu^{p-1} \Bigg(|\{x \in \mathbb{R}^d,\ \widetilde{N}_{a,q}(u|\Psi^k)(x) > \nu\} \cap E_{\nu,\gamma}| \\
&\quad + \Big|\Big\{ x\in\mathbb{R}^d,\ \mathcal{M}_q\Big[\Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_{B,l}^k\, \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\, dy\Big)^{1/q}\Big] > \gamma\nu \Big\}\Big| \\
&\quad + \Big|\Big\{ x\in\mathbb{R}^d,\ \mathcal{M}_q^{1/2}[\widetilde{N}_{a,q}(u|\Psi^k)](x)\mathcal{M}_q^{1/2}[S_{a,q}(u|\Psi^{k-2})](x) > \gamma\nu \Big\}\Big| \Bigg)\, d\nu \\
&\leq C\gamma^q \int_0^\infty \nu^{p-1} |\{x\in\mathbb{R}^d,\ \mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi^k)](x) > \eta\nu\}|\, d\nu \\
&\quad + C\gamma^{1-p} \Big\| \mathcal{M}_q\Big[\Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_{B,l}^k\, \partial_r[\Psi_e^k]\, \frac{ds}{|s|^{n-d-1}}\Big)^{1/q}\Big]\Big\|_p^p \\
&\quad + C\gamma^{1-p} \big\| \mathcal{M}_q^{1/2}[\widetilde{N}_{a,q}(u|\Psi^k)]\ \mathcal{M}_q^{1/2}[S_{a,q}(u|\Psi^{k-2})]\big\|_p^p.
\end{aligned}$$

Now applying the Hardy–Littlewood maximal theorem with power $p/q > 1$ to each term, and using also Cauchy–Schwarz's inequality for the last term, we get

$$\begin{aligned}
\|\widetilde{N}_{a,q}(u|\Psi^k)\|_p^p &\leq C\gamma^q \|\mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi^k)]\|_p^p \\
&\quad + C\gamma^{1-p} \Big\| \mathcal{M}_q\Big[\Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_{B,l}^k\, \partial_r[\Psi_e^k]\, \frac{ds}{|s|^{n-d-1}}\Big)^{1/q}\Big]\Big\|_p^p \\
&\quad + C\gamma^{1-p} \|\mathcal{M}_q[\widetilde{N}_{a,q}(u|\Psi^k)]\|_p^{1/2} \|\mathcal{M}_q^{p/2}[S_{a,q}(u|\Psi^{k-2})]\|_p^{p/2} \\
&\leq C\gamma^q \|\widetilde{N}_{a,q}(u|\Psi^k)\|_p^p \\
&\quad + C\gamma^{1-p} \Big\| \Big(\fint_{y\in B_{ae(\cdot)/2}(\cdot)} \int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_{B,l}^k\, \partial_r[\Psi_e^k] \frac{ds}{|s|^{n-d-1}}\Big)^{1/q} \Big\|_p^p \\
&\quad + C\gamma^{1-p} \|\widetilde{N}_{a,q}(u|\Psi^k)\|_p^{p/2} \|S_{a,q}(u|\Psi^{k-2})\|_p^{p/2}.
\end{aligned}$$

The last term in the above inequality can be bounded by

$$\frac{1}{4}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_p^p + C\|S_{a,q}(u|\Psi^{k-2})\|_p^p.$$

We choose γ such that $C\gamma^q = 1/4$. So all the term $\|\widetilde{N}_{a,q}(u|\Psi^k)\|_p^p$ in the right-hand side can be hidden in the left-hand side. The estimate (6.23) and then the lemma follows. □

Combined with Moser's estimate and Lemma 5.5, we can improve Lemma 6.7.

Lemma 6.9 (Self-improvement). *Let $\mathcal{L}$ be an elliptic operator satisfying (H^1_κ) for some constant $\kappa \geq 0$. Let $a > 0$, $l > 0$, and $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose $k > 12$, a positive a^{-1}-Lipschitz function e and a ball $B \subset \mathbb{R}^d$ of radius $l' \geq l$. Then for any weak solution $u \in W^{1,2}_{\rm loc}(\Omega)$,*

$$\begin{aligned}\|\widetilde{N}_{a,q}(u|\Psi_e^k\Psi_{B,l}^k)\|_q^q &\lesssim \|S_{a,q}(u|\Psi_e^{k-12}\Psi_{B,l}^{k-12})\|_q^q \\ &\quad + \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q\,\Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy,\end{aligned} \tag{6.24}$$

where the constant depends on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty + \|b\|_\infty$, κ and k.

Remark 6.10. The last term is bounded by the L^q average on a local region compactly contained in Ω, so it is finite. Thus this integral has no singularity at the boundary $\mathbb{R}^d \times \{0\}$, and in fact it is the same as

$$\iint_\Omega |u|^q\,\Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy.$$

Proof. As before, we write Ψ for $\Psi_e\Psi_{B,l}$. Lemma 4.2 entails that there exists ϵ sufficiently small (depending on n, k, q, q_0 and, without loss of generality, we can assume $q - 2\epsilon > q_0$) such that

$$\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q \lesssim \|\widetilde{N}_{a,q-\epsilon}(u|\Psi^{k-3})\|_q^q.$$

Now, since $q - \epsilon < q$, we can use Lemma 6.7 to get

$$\begin{aligned}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q &\lesssim \|\widetilde{N}_{a,q-\epsilon}(u|\Psi^{k-3})\|_q^q \\ &\lesssim \|S_{a,q-\epsilon}(u|\Psi^{k-5})\|_q^q \\ &\quad + \left\|\left(\fint_{y\in B_{ae(\cdot)/2}(\cdot)}\int_{s\in\mathbb{R}^{n-d}} |u|^{q-\epsilon}\Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy\right)^{1/(q-\epsilon)}\right\|_q^q \\ &=: T_1 + T_2.\end{aligned} \tag{6.25}$$

We first estimate T_2. By Hölder's inequality, for any $x \in \mathbb{R}^d$,

$$\begin{aligned}&\left(\fint_{y\in B_{ae(x)/2}(x)}\int_{s\in\mathbb{R}^{n-d}} |u|^{q-\epsilon}\,\Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy\right)^{1/(q-\epsilon)} \\ &\leq \left(\fint_{y\in B_{ae(x)/2}(x)}\int_{s\in\mathbb{R}^{n-d}} |u|^q\,\Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy\right)^{1/q} \\ &\quad \times \left(\fint_{y\in B_{ae(x)/2}(x)}\int_{s\in\mathbb{R}^{n-d}} \Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy\right)^{\epsilon/[q(q-\epsilon)]}.\end{aligned} \tag{6.26}$$

Note that $0 \leq \Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}] \leq (k-3)\partial_r\Psi_e$, $\partial_r\Psi_e \leq 2e(x)/|s|^2$, and it is nonzero only if $e(y)/2 \leq |s| \leq e(y)$. For any $y \in B_{ae(x)/2}(x)$, since $|e(y) - e(x)| \leq |y-x|/a < e(x)/2$, it follows that

$$\frac{e(x)}{2} < e(y) < \frac{3e(x)}{2}.$$

Hence

$$\tag{6.27} \begin{aligned} &\fint_{y\in B_{ae(x)/2}(x)} \int_{s\in\mathbb{R}^{n-d}} \Psi_{B,l}^{k-3}\,\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy \\ &\qquad \lesssim \fint_{y\in B_{ae(x)/2}(x)} \int_{e(x)/4\leq|s|\leq 3e(x)/2} \frac{e(x)}{|s|^2}\,\frac{ds}{|s|^{n-d-1}}\,dy \lesssim 1. \end{aligned}$$

Combining (6.27) with (6.26), we deduce that

$$\tag{6.28} \begin{aligned} T_2 &\lesssim \int_{x\in\mathbb{R}^d} \fint_{y\in B_{ae(x)/2}(x)} \int_{s\in\mathbb{R}^{n-d}} |u|^q\,\Psi_{B,l}^{k-3}\,\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy\,dx \\ &\lesssim \int_{y\in\mathbb{R}^d} \int_{s\in\mathbb{R}^{n-d}} |u|^q\,\Psi_{B,l}^{k-3}\,\partial_r[\Psi_e^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy, \end{aligned}$$

where, in the last line, we use the same argument as the one used to go from (6.16) to (6.18).

It remains to bound T_1. For any $x \in \mathbb{R}^d$,

$$\begin{aligned} |S_{a,q-\epsilon}(u|\Psi^{k-5})(x)|^q &= \iint_{(y,s)\in\widehat{\Gamma}_a(x)} |\nabla u|^2\,|u|^{q-2-\epsilon}\,\Psi^{k-5}\,\frac{ds}{|s|^{n-2}}\,dy \\ &\leq \Big(\iint_{(y,s)\in\widehat{\Gamma}_a(x)} |\nabla u|^2\,|u|^{q-2-2\epsilon}\,\Psi^{k+2}\,\frac{ds}{|s|^{n-2}}\,dy\Big)^{1/2} \\ &\quad\times \Big(\iint_{(y,s)\in\widehat{\Gamma}_a(x)} |\nabla u|^2\,|u|^{q-2}\,\Psi^{k-12}\,\frac{ds}{|s|^{n-2}}\,dy\Big)^{1/2} \\ &\leq |S_{a,q-2\epsilon}(u|\Psi^{k+2})(x)|^{q/2}|S_{a,q}(u|\Psi^{k-12})(x)|^{q/2}. \end{aligned}$$

So, using Cauchy–Schwarz's inequality, we can bound T_1 as follows:

$$T_1 \leq \|S_{a,q-2\epsilon}(u|\Psi^{k+2})\|_q^{q/2}\|S_{a,q}(u|\Psi^{k-12})\|_q^{q/2}.$$

The use of Lemma 5.5 and then Hölder's inequality gives that

$$\tag{6.29} \begin{aligned} T_1 &\lesssim \|\widetilde{N}_{a,q-2\epsilon}(u|\Psi^k)\|_q^{q/2}\|S_{a,q}(u|\Psi^{k-12})\|_q^{q/2} \\ &\lesssim \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2}\|S_{a,q}(u|\Psi^{k-12})\|_q^{q/2}. \end{aligned}$$

The combination of (6.25), (6.28), and (6.29) proves that

$$\begin{aligned} \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q &\leq C\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2}\|S_{a,q}(u|\Psi^{k-12})\|_q^{q/2} \\ &\quad + C\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q\Psi_{B,l}^{k-3}\partial_r[\Psi_e^{k-3}]\frac{ds}{|s|^{n-d-1}}\,dy, \end{aligned}$$

which can be easily improved into (6.24). The lemma follows. $\square$

Lemma 6.11. *Let $\mathcal{L}$ be an elliptic operator satisfying (H^1_κ) for some constant $\kappa \geq 0$. Let $a, l > 0$, $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose $k > 12$, a positive a^{-1}-Lipschitz function e and a ball $B := B_{l'}(x_B) \subset \mathbb{R}^d$ of radius $l' \geq l$. Then for any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$,*

$$\|\widetilde{N}_{a,q}(u|\Psi_e^k \Psi_{B,l}^k)\|_q^q \lesssim \|S_{a,q}(u|\Psi_e^{k-12}\Psi_{B,l}^{k-12})\|_q^q + \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_e^{k-3} \partial_r[-\Psi_{B,l}^{k-3}]\, \frac{ds}{|s|^{n-d-1}}\, dy, \tag{6.30}$$

where the constant depends on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty + \|b\|_\infty$, κ and k.

Proof. Let Ψ be the product $\Psi_e \Psi_{B,l}$. By Lemma 6.9,

$$\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q \lesssim \|S_{a,q}(u|\Psi^{k-12})\|_q^q + \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_{B,l}^{k-3}\, \partial_r[\Psi_e^{k-3}]\, \frac{ds}{|s|^{n-d-1}}\, dy.$$

It follows by the product rule that

$$\begin{aligned}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q &\lesssim \|S_{a,q}(u|\Psi^{k-12})\|_q^q \\ &\quad + \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q\, \Psi_e^{k-3} \partial_r[-\Psi_{B,l}^{k-3}]\, \frac{ds}{|s|^{n-d-1}}\, dy \\ &\quad + \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q\, \partial_r[\Psi^{k-3}]\, \frac{ds}{|s|^{n-d-1}}\, dy.\end{aligned} \tag{6.31}$$

So it remains to bound the last term in the right-hand side of (6.31). By simply using Hölder's inequality with different powers, the proof of Lemma 6.4 can be easily adapted to obtain

$$\begin{aligned}\left|\iint_\Omega |u|^q \partial_r[\Psi^{k-3}]\, \frac{dt}{|t|^{n-d-1}}\, dx\right| &\lesssim \|S_{a,q}(u|\Psi^{k-3})\|_q^q \\ &\quad + \|S_{a,q}(u|\Psi^{k-8})\|_q^{q/2} \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^{q/2} \\ &\leq C_\eta \|S_{a,q}(u|\Psi^{k-8})\|_q^q + \eta \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q \\ &\leq C_\eta \|S_{a,q}(u|\Psi^{k-12})\|_q^q + \eta \|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q\end{aligned} \tag{6.32}$$

for all $\eta > 0$. By choosing η small enough, the combination of (6.31) and (6.32) gives (6.30). The lemma follows. □

7. From local estimates to global ones and existence of solutions to the Dirichlet problem

By Lemma 1.12, for any $g \in C_0^\infty(\mathbb{R}^d) \subset H$, there is a (unique) energy solution $u \in W$ to $\mathcal{L}u = 0$ such that $\mathrm{Tr}\, u = g$. The idea of this section is to first prove that if $\mathcal{L}$ satisfies (H^1_κ) with κ sufficiently small, then any energy solution satisfies

$$\|\widetilde{N}_{a,q}(u)\|_q \leq C \|\mathrm{Tr}\, u\|_q, \tag{7.1}$$

with a universal constant $C > 0$. Then, for any $g \in L^q(\mathbb{R}^d)$, the existence of a weak solution u to $\mathcal{L}u = 0$, whose non-tangential limit on $\mathbb{R}^d$ is given by g, can be obtained by a density argument; moreover, we can show that this solution u satisfies $\|\widetilde{N}_{a,q}(u)\|_q \lesssim \|g\|_q$.

The local inequalities proven in Lemmas 5.1 and 6.11 increase the integral regions from left to right. Actually, we use some cut-off functions and we lose power on these cut-off functions, but the idea is the same: by combining Lemma 6.11 and Lemma 5.1, we do not loop back to the same local non-tangential function, so even if κ is small, we cannot hide the term on the right-hand side to the left-hand side at the local level, where we are sure that everything is finite. The idea to prove (7.1) is then to pass the local estimates to infinity. The main obstacle however is that we do not know *a priori* that the energy solutions satisfy that $\|\widetilde{N}_{a,q}(u)\|_q$, or even $\|S_{a,q}(u)\|_q$, is finite.

For the sequel we use the following notations for cut-off functions. Choose the same function $\phi \in C_0^\infty([0,\infty))$ such that $0 \le \phi \le 1$, $\phi \equiv 1$ on $[0,1]$, $\phi \equiv 0$ outside $[0,2]$, ϕ decreasing, and $|\phi'| \le 2$. For $\epsilon > 0$, we define Ψ_ϵ as

$$\Psi_\epsilon(x,t) = \Psi_\epsilon(t) = \phi\Big(\frac{\epsilon}{|t|}\Big).$$

For $l > 0$, we define χ_l as

$$\chi_l(x,t) = \chi_l(t) = \phi\Big(\frac{a|t|}{l}\Big),$$

and if $B \subset \mathbb{R}^d$ is a ball, we define

$$\Phi_{B,l}(x,t) = \Phi_{B,l}(x) = \phi\Big(1 + \frac{\mathrm{dist}(x,B)}{l}\Big).$$

The reader may recall Lemma 4.5 and recognize that Ψ_ϵ is the function Ψ_e there with $e(x) \equiv \epsilon$, and the product $\chi_l \Phi_{B,l}$ is the function $\Psi_{B,l}$ there. These cut-off functions correspond to smooth cut-off away from the boundary, at infinity in the t and x variables, respectively. Also recall that, when the radius of B is bigger than l, the function $\Psi_\epsilon \chi_l \Phi_{B,l}$ satisfies $(\mathcal{H}^2_{a,M_0})$ for some M_0 depending only on a and the dimensions d, n (see Lemma 4.5).

Lemma 7.1 ($N < S + \mathrm{Tr}$ when $q \ge 2$)**.** *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \ge 0$. Let $a, l > 0$, $2 \le q < q_0'$, where q_0' is the conjugate of q_0 given by Proposition 2.1. Choose $k > 12$. Then for any energy solution $u \in W$ to $\mathcal{L}u = 0$,*

$$\|\widetilde{N}_{a,q}(u|\chi_l^k)\|_q^q \lesssim \|S_{a,q}(u|\chi_l^{k-12})\|_q^q + \|\mathrm{Tr}\, u\|_q^q < +\infty, \tag{7.2}$$

where the constant of the first inequality depends on a, q, n, d, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty + \|b\|_\infty$, κ and k.

Remark 7.2. An immediate consequence of the lemma is that if $u \in W$ is an energy solution to $\mathcal{L}u = 0$, then $\|\widetilde{N}_{a,q}(u|\chi_l^k)\|_q$ is finite for any $k > 0$ (not only for $k > 12$). Indeed, since $\chi_l^k \le \chi_{2l}^{20}$ for all $k > 0$, $l > 0$, we have

$$\|\widetilde{N}_{a,q}(u|\chi_l^k)\|_q \le \|\widetilde{N}_{a,q}(u|\chi_{2l}^{20})\|_q < +\infty.$$

The same remark holds for Lemma 7.4

Proof. We start by proving finiteness. First, if u is an energy solution and $q \ge 2$, we have that $\int_\Omega |\nabla u|^2\,|u|^{q-2}\,|t|^{d+1-n}\,dt\,dx < +\infty$, by Theorem 1.13 (i), and also that $\operatorname{Tr} u \in C_0^\infty(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$. Then, since $\chi_l \le 1$ is supported in $\{(x,t) \in \Omega,\ |t| \le 2l/a\}$, we have

$$\begin{aligned}\|S_{a,q}(u|\chi_l^{k-12})\|_q^q &\simeq \iint_{(x,t)\in\Omega} |\nabla u|^2\,|u|^{q-2}\chi_l^{k-12}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &\le \frac{2l}{a}\iint_{(x,t)\in\Omega} |\nabla u|^2\,|u|^{q-2}\,\frac{dt}{|t|^{n-d-1}}\,dx < +\infty.\end{aligned} \tag{7.3}$$

So, indeed, we have

$$\|S_{a,q}(u|\chi_l^{k-12})\|_q^q + \|\operatorname{Tr} u\|_q^q < +\infty.$$

The proof of the first inequality in (7.2), in simple words, is by passing to the limit the estimate in Lemma 6.9.

Step 1. We claim that for any ball B with radius $l' \ge l$ and for any $k > 12$,

$$\|\widetilde{N}_{a,q}(u|\chi_l^k\Phi_{B,l}^k)\|_q^q < +\infty. \tag{7.4}$$

We take $\epsilon > 0$ sufficiently small ($\epsilon < l/a$). We write Ψ for $\chi_l\,\Phi_{B,l}\,\Psi_\epsilon$. According to Lemma 6.11,

$$\begin{aligned}\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q &\lesssim \|S_{a,q}(u|\Psi^{k-12})\|_q^q \\ &\quad + \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q\,\Psi_\epsilon^{k-3}\,\Phi_{B,l}^{k-3}\,\partial_r[-\chi_l^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy.\end{aligned}$$

Note that $\Phi_{B,l}$ is independent of t, so we can pull it out of the r-derivative. Since $\Psi \le \chi_l$, $\Psi_\epsilon \equiv 1$ on $\operatorname{supp}\partial_r\chi_l \subset \{l/a \le |t| \le 2l/a\}$, and $|\partial_r\chi_l(s)| \lesssim 1/|s|$, we deduce

$$\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q \lesssim \|S_{a,q}(u|\chi_l^{k-12})\|_q^q + \iint_{(y,s)\in\operatorname{supp}\Phi_{B,l}\partial_r\chi_l} |u|^q\,\frac{ds}{|s|^{n-d}}\,dy.$$

Notice that the right-hand side is independent of ϵ, and it is finite. Indeed, by (7.3), $\|S_{a,q}(u|\chi_l^{k-12})\|_q^q < +\infty$; moreover, $\Phi_{B,l}\partial_r\chi_l$ is compactly supported in Ω and by Lemma 3.1 $u \in L^q_{\text{loc}}(\Omega)$. We deduce that

$$\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q = \|\widetilde{N}_{a,q}(u|\chi_l^k\Phi_{B,l}^k\Psi_\epsilon^k)\|_q^q$$

is uniformly bounded in ϵ, and so the claim (7.4) follows by passing $\epsilon \to 0$.

Step 2: We want to prove that for any $k \geq 1$ and any ball B with radius $l' \geq l$, we have

$$\limsup_{\epsilon\to 0} \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_l^k\, \Phi_{B,l}^k\, \partial_r[\Psi_\epsilon^k]\, \frac{ds}{|s|^{n-d-1}}\, dy \lesssim \int_{x\in 4B} |\mathrm{Tr}\, u|^q\, dx. \tag{7.5}$$

Note that $\operatorname{supp} \partial_r \Psi_\epsilon \subset \{(x,t)\in\Omega,\ \epsilon/2 \leq |t| \leq \epsilon\}$, $0 \leq \partial_r\Psi_\epsilon \lesssim 1/\epsilon$. Hence

$$\begin{aligned}\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} &|u|^q \chi_l^k\, \Phi_{B,l}^k\, \partial_r[\Psi_\epsilon^k]\, \frac{ds}{|s|^{n-d-1}}\, dy \\ &\lesssim \int_{y\in 2B} \fint_{\epsilon/2\leq|s|\leq\epsilon} |u|^q \chi_l^k\, \Phi_{B,l}^k\, ds\, dy \\ &\lesssim \int_{x\in 3B}\fint_{y\in B_{a\epsilon/2}(x)} \fint_{\epsilon/2\leq|s|\leq\epsilon} |u|^q \chi_l^k\, \Phi_{B,l}^k\, ds\, dy\, dx.\end{aligned} \tag{7.6}$$

For the last inequality, we use Fubini's lemma. By (7.4) of step 1,

$$\begin{aligned}\fint_{y\in B_{a\epsilon/2}(x)} \fint_{\epsilon/2\leq|s|\leq\epsilon} |u|^q \chi_l^k\, \Phi_{B,l}^k\, ds\, dy &= \left|(u|\chi_l^k\, \Phi_{B,l}^k)_{W,a,q}(x,\epsilon)\right|^k \\ &\leq |\widetilde{N}_{a,q}(u|\chi_l^k\, \Phi_{B,l}^k)(x)|^q\end{aligned} \tag{7.7}$$

is integrable uniformly in ϵ. When $q \geq 2$, Moser's estimate (Lemma 3.1 (ii)) gives that

$$\Big(\fint_{y\in B_{a\epsilon}(x)} \fint_{\epsilon/2\leq|s|\leq\epsilon} |u|^q ds\, dy\Big)^{1/q} \lesssim \Big(\fint_{y\in B_{2a\epsilon}(x)} \fint_{\epsilon/4\leq|s|\leq 2\epsilon} |u|^2\, ds\, dy\Big)^{1/2}. \tag{7.8}$$

(We remark that when $q < 2$, the above estimate also holds by Hölder's inequality.) We claim that

$$\limsup_{\epsilon\to 0}\Big(\fint_{y\in B_{2a\epsilon}(x)} \fint_{\epsilon/4\leq|s|\leq 2\epsilon} |u|^2\, ds\, dy\Big)^{1/2} \lesssim |\mathrm{Tr}\, u(x)|, \tag{7.9}$$

for σ-almost every $x \in \Gamma$. Then, by the reverse Fatou lemma and the pointwise domination (7.7), we get

$$\limsup_{\epsilon\to 0}\int_{x\in 4B}\Big(\fint_{y\in B_{2a\epsilon}(x)} \fint_{\epsilon/4\leq|s|\leq 2\epsilon} |u|^2 ds\, dy\Big)^{q/2} dx \lesssim \int_{x\in 4B} |\mathrm{Tr}\, u|^q\, dx.$$

This estimate, combined with (7.6) and (7.8), proves (7.5).

By the triangle inequality,

$$\begin{aligned}\fint_{y\in B_{2a\epsilon}(x)} &\fint_{\epsilon/4\leq|s|\leq 2\epsilon} |u|^2\, ds\, dy \\ &\lesssim \fint_{y\in B_{2a\epsilon}(x)} \fint_{\epsilon/4\leq|s|\leq 2\epsilon} |u - \mathrm{Tr}\, u(x)|^2\, ds\, dy + |\mathrm{Tr}\, u(x)|^2 \\ &\lesssim \fint_{y\in B_{2a\epsilon}(x)} \fint_{|s|\leq 2\epsilon} |u - \mathrm{Tr}\, u(x)|^2\, ds\, dy + |\mathrm{Tr}\, u(x)|^2.\end{aligned} \tag{7.10}$$

Recall the Lebesgue density property in Theorem 3.4 of [10] guarantees the first term above tends to 0 as $\epsilon \to 0$, if we replace the power two by one. We claim that a similar L^2 Lebesgue density property holds, and thus (7.9) follows immediately. In fact,

$$\begin{aligned}(7.11)\qquad &\fint_{y\in B_{2a\epsilon}(x)}\fint_{|s|\le 2\epsilon}|u-\operatorname{Tr}u(x)|^2\,ds\,dy\\ &\qquad\le 2|u_{x,\epsilon}-\operatorname{Tr}u(x)|^2+2\fint_{y\in B_{2a\epsilon}(x)}\fint_{|s|\le 2\epsilon}|u-u_{x,\epsilon}|^2\,ds\,dy.\end{aligned}$$

The first term goes to 0 as $\epsilon \to 0$, thanks to (3.23) in [10]. We use (2.13) and Lemma 4.2, both from [10], to bound the second term as follows:

$$\begin{aligned}(7.12)\qquad &\fint_{y\in B_{2a\epsilon}(x)}\fint_{|s|\le 2\epsilon}|u-u_{x,\epsilon}|^2\,ds\,dy\\ &\qquad\lesssim G_\epsilon(x):=\epsilon^{1-d}\int_{y\in B_{2a\epsilon}(x)}\int_{|s|\le 2\epsilon}|\nabla u|^2\,\frac{ds}{|s|^{n-d-1}}\,dy.\end{aligned}$$

But since $\nabla u \in L^2(\mathbb{R}^n\setminus\Gamma, w)$ implies

$$\int_\Gamma G_\epsilon(x)\,dx=\epsilon\cdot\int_{y\in\Gamma}\int_{|s|\le 2\epsilon}|\nabla u|^2\,\frac{ds}{|s|^{n-d-1}}\,dy\to 0,$$

we deduce that $G_\epsilon(x) \to 0$ for a.e. $x \in \Gamma$. The latter convergence, combined with (7.11) and (7.12), gives the L^2 Lebesgue density property, i.e., the left-hand side of (7.11) converges to 0 as $\epsilon \to 0$ for σ-almost every $x \in \Gamma$.

Step 3: Conclusion. Let $B = B_{l'}$ be a ball with center 0 and radius $l' \ge l$ and $\epsilon > 0$. Applying Lemma 6.9 to the function $\Psi = \chi_l\,\Phi_{B,l}\,\Psi_\epsilon$, we get

$$\begin{aligned}&\|\widetilde{N}_{a,q}(u|\Psi^k)\|_q^q\lesssim\|S_{a,q}(u|\Psi^{k-12})\|_q^q\\ &\qquad+\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}}|u|^q\chi_l^{k-3}\,\Phi_{B,l}^{k-3}\,\partial_r[\Psi_\epsilon^{k-3}]\,\frac{ds}{|s|^{n-d-1}}\,dy.\end{aligned}$$

By step 2, taking the limit as ϵ goes to 0 gives that

$$\|\widetilde{N}_{a,q}(u|\chi_l^k\,\Phi_{B,l}^k)\|_q^q\lesssim\|S_{a,q}(u|\chi_l^{k-12}\,\Phi_{B,l}^{k-12})\|_q^q+\int_{x\in 4B}|\operatorname{Tr}u|^q\,dx.$$

We take now the limit as the radius l' goes to $+\infty$ and we obtain (7.2). □

Remark 7.3. We remark that the above proof does not use $q \ge 2$ per se: We only need this assumption to guarantee that $\|S_{a,q}(u|\chi_l^{k-12})\|_q$ is finite (see step 1). That is to say, we can use the same argument for the case $q < 2$, if we know a priori the corresponding square function is integrable.

Lemma 7.4 ($N < S + \mathrm{Tr}$ when $q < 2$)**.** *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$. Let $a, l > 0$, $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose $k > 12$. Then for any energy solution $u \in W$ to $\mathcal{L}u = 0$,*

$$\|\widetilde{N}_{a,q}(u|\chi_l^k)\|_q^q \lesssim \|S_{a,q}(u|\chi_l^{k-12})\|_q^q + \|\mathrm{Tr}\, u\|_q^q < +\infty, \tag{7.13}$$

where the constant of the first inequality depends on a, q, n, d, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty + \|b\|_\infty$, κ and k.

Proof. We start by proving a priori finiteness.

Step 1. We claim that if l is sufficiently large (depending only on a and the support of $\mathrm{Tr}\, u$) and $B \subset \mathbb{R}^d$ is any ball centered at zero with radius greater than l, then

$$\begin{aligned} &\|\widetilde{N}_{a,q}(u|\chi_l^k(1-\Phi_{B,l})^k)\|_q^q \\ &\qquad \lesssim \|S_{a,q}(u|\chi_l^{k-12}(1-\Phi_{B,l})^{k-12})\|_q^q + \|\mathrm{Tr}\, u\|_q^q < +\infty. \end{aligned} \tag{7.14}$$

We choose l so that the ball in $\mathbb{R}^n$ centered at zero with radius l/a contains two times the ball $B_0 \subset \mathbb{R}^n$ given by Theorem 1.13 (ii). (Recall that B_0 depends on the support of $\mathrm{Tr}\, u$.) Recall that

$$\begin{aligned} &\|S_{a,q}(u|\chi_l^{k-12}(1-\Phi_{B,l})^{k-12})\|_q^q \\ &\quad \simeq \iint_{(x,t)\in\Omega} |\nabla u|^2\, |u|^{q-2} \chi_l^{k-12} (1-\Phi_{B,l})^{k-12}\, \frac{dt}{|t|^{n-d-2}}\, dx \\ &\quad \lesssim l \iint_{(x,t)\in\Omega} |\nabla u|^2\, |u|^{q-2} (1-\Phi_{B,l})^{k-12}\, \frac{dt}{|t|^{n-d-1}}\, dx, \end{aligned}$$

because $|t| \leq 2l$ in the support of χ_l. In addition, the support of $(1-\Phi_{B,l})^{k-12}$ is contained in the complement of a cylinder of radius $\sim l/a$, thus by the choice of l it is contained in $\mathbb{R}^n \setminus B_0$. Lemma 3.5 then allows us to conclude that the right-hand side above, and so the left-hand side, is finite. Again by Remark 7.3, once we know $\|S_{a,q}(u|\chi_l^{k-12}(1-\Phi_{B,l})^{k-2})\|_q < +\infty$, (7.14) follows by the same argument as in the proof of Lemma 7.1.

Step 2. We claim that for any $k > 2$,

$$\|S_{a,q}(u|\chi_l^k)\|_q \lesssim \|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q < +\infty. \tag{7.15}$$

On the one hand we observe that for any $l > 0$ and any ball B' of radius at least l, one has

$$\|\widetilde{N}_{a,q}(u|\chi_l^{k-2}\,\Phi_{B',l}^{k-2})\|_q < +\infty. \tag{7.16}$$

Indeed, the above finiteness is an immediate consequence of Hölder's inequality, since

$$\widetilde{N}_{a,q}(u|\chi_l^{k-2}\,\Phi_{B',l}^{k-2}) \leq \widetilde{N}_{a,2}(u|\chi_l^{k-2}),$$

where $\widetilde{N}_{a,q}(u|\chi_l^{k-2}\Phi_{B',l}^{k-2})$ has compact support, and $\|\widetilde{N}_{a,2}(u|\chi_l^{k-2})\|_2 < +\infty$ by Lemma 7.1. On the other hand, by step 1, $\|\widetilde{N}_{a,q}(u|\chi_l^{k-2}(1-\Phi_{B,l})^{k-2})\|_q < +\infty$ if l is sufficiently large. (A priori we only have finiteness when $k-2 > 12$, but since $0 \leq \chi_l \leq 1$, the finiteness clearly holds for any $k > 2$.) A simple computation shows that $1 \leq \Phi_{2B,l}^{k-2} + (1-\Phi_{B,l})^{k-2}$, and hence

$$\|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q \lesssim \|\widetilde{N}_{a,q}(u|\chi_l^{k-2}\Phi_{2B,l}^{k-2})\|_q + \|\widetilde{N}_{a,q}(u|\chi_l^{k-2}(1-\Phi_{B,l})^{k-2})\|_q < +\infty.$$

Now take an increasing sequence of balls $(B_i)_{i\geq 1}$ such that B_1 is of radius l, $B_i \subset B_{i+1}$ and $\bigcup B_i = \mathbb{R}^d$. We apply Lemma 5.3 to the functions $\Psi = \chi_l \Phi_{B_i,l}\Psi_{1/i}$, which gives that

$$\|S_{a,q}(u|\Psi^k)\|_q \lesssim \|\widetilde{N}_{a,q}(u|\Psi^{k-2})\|_q.$$

By the finiteness of $\|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q$, we may take the limit as $i \to +\infty$ and obtain (7.15).

Step 3: Conclusion. Having shown the finiteness of $\|S_{a,q}(u|\chi_l^{k-12})\|_q$, by Remark 7.3, the same argument in the proof of Lemma 7.1 gives (7.13). □

The following estimate is a byproduct of Lemma 7.1, in particular, of (7.5). It will be used later in the proof of Lemma 7.6.

Corollary 7.5. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}_\kappa^1)$ for some constant $\kappa \geq 0$. Let $a > 0$, $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose $k \geq 1$. Then for any energy solution $u \in W$ to $\mathcal{L}u = 0$,*

$$\limsup_{\epsilon\to 0} \iint_\Omega |u|^q \,\partial_r[\Psi_\epsilon^k] \,\frac{ds}{|s|^{n-d-1}}\, dy \lesssim \|\mathrm{Tr}\, u\|_q^q, \tag{7.17}$$

where the constant depends on a, q, n, d, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty + \|b\|_\infty$, κ and k.

Proof. Let $l > 0$ be fixed. Since $|\widetilde{N}_{a,q}(u|\chi_l^k)|^q$ is integrable, the same argument in step 2 of Lemma 7.1 allows us to conclude

$$\limsup_{\epsilon\to 0} \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_l^k \,\partial_r[\Psi_\epsilon^k]\, \frac{ds}{|s|^{n-d-1}}\, dy \lesssim \int_{x\in\mathbb{R}^d} |\mathrm{Tr}\, u|^q \, dx.$$

Since $\chi_l \equiv 1$ on $\operatorname{supp} \partial[\Psi_\epsilon^k]$ if ϵ is small enough ($\epsilon < l/a$), it follows that

$$\begin{aligned}\limsup_{\epsilon\to 0} &\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q \partial_r[\Psi_\epsilon^k] \frac{ds}{|s|^{n-d-1}}\, dy \\ &= \limsup_{\epsilon\to 0} \int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}} |u|^q \chi_l^k \,\partial_r[\Psi_\epsilon^k] \frac{ds}{|s|^{n-d-1}}\, dy \\ &\lesssim \int_{x\in\mathbb{R}^d} |\mathrm{Tr}\, u|^q \, dx. \end{aligned}$$ □

The next lemma can be seen as the analogue of Lemma 5.1 for energy solutions.

Lemma 7.6 ($S < \kappa N + \mathrm{Tr}$)**.** *Let $\mathcal{L}$ be an elliptic operator satisfying (H^1_κ) for some constant $\kappa \geq 0$. Let $a, l > 0$, $q \in (q_0, q_0')$ where q_0 is given by Proposition 2.1. Choose $k > 2$. Then for any energy solution $u \in W$ to $\mathcal{L}u = 0$,*

$$c\|S_{a,q}(u|\chi_l^k)\|_q^q \leq C\kappa \|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q^q + C\|\mathrm{Tr}\, u\|_q^q + \frac{1}{q}\iint_\Omega \Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q]\Big)\partial_r[\chi_l^k]\, \frac{dt}{|t|^{n-d-2}}\, dx, \tag{7.18}$$

where the constants $c, C > 0$ depend on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b\|_\infty + \|b^{-1}\|_\infty$, k and (the upper bound of) κ.

Proof. We take $\epsilon > 0$ and a sequence of balls $(B_i)_{i\geq 1}$ such that B_1 is of radius $100l$, $B_i \subset B_{i+1}$, $\bigcup B_i = \mathbb{R}^d$. We apply Lemma 5.1 with the function $\Psi_i = \chi_l \Phi_{B_i,l}\Psi_\epsilon$, which gives that

$$\begin{aligned} c\|S_{a,q}(u|\Psi_i^k)\|_q^q &\leq \frac{1}{2}\mathrm{Re}\iint_\Omega \mathcal{A}'\nabla u \cdot \nabla[|u|^{q-2}\bar{u}]\,\frac{\Psi_i^k}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &\leq C\kappa\|\widetilde{N}_{q,a}(u|\Psi_i^{k-2})\|_q^q - \mathrm{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla_x[\Psi_i^k]\,\frac{|u|^{q-2}\bar{u}}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx \\ &\quad + \frac{1}{q}\iint_\Omega\Big(\frac{|u|^q}{|t|}\nabla|t| - \nabla_t[|u|^q]\Big)\cdot\nabla_t[\Psi_i^k]\,\frac{dt}{|t|^{n-d-2}}\,dx. \end{aligned} \tag{7.19}$$

Passing $i \to +\infty$ and then $\epsilon \to 0$, the left-hand side converges to $c\|S_{a,q}(u|\chi_l^k)\|_q^q$, which is finite by Lemmas 7.1 and 7.4. Clearly we have that $\|\widetilde{N}_{q,a}(u|\Psi_i^{k-2})\|_q \leq \|\widetilde{N}_{q,a}(u|\chi_l^{k-2})\|_q$. So to prove (7.18), it suffices to establish

$$\lim_{i\to\infty}\mathrm{Re}\iint_\Omega \mathcal{A}'\nabla u\cdot\nabla_x[\Psi_i^k]\,\frac{|u|^{q-2}\bar{u}}{b}\,\frac{dt}{|t|^{n-d-2}}\,dx = 0 \tag{7.20}$$

and

$$\begin{aligned} \limsup_{\epsilon\to 0}\lim_{i\to\infty}\frac{1}{q}\iint_\Omega\Big(\frac{|u|^q}{|t|}\nabla|t| - \nabla_t[|u|^q]\Big)\cdot\nabla_t[\Psi_i^k]\,\frac{dt}{|t|^{n-d-2}}\,dx \\ \leq C\|\mathrm{Tr}\,u\|_q^{q/2}\|S_{a,q}(u|\chi_l^k)\|_q^{q/2} + C\|\mathrm{Tr}\,u\|_q^q \\ + \frac{1}{q}\iint_\Omega\Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q]\Big)\partial_r[\chi_l^k]\,\frac{dt}{|t|^{n-d-2}}\,dx. \end{aligned} \tag{7.21}$$

In the last inequality, we have the additional term $C\|\mathrm{Tr}\,u\|_q^{q/2}\|S_{a,q}(u|\chi_l^k)\|_q^{q/2}$, which does not appear in the right-hand side of (7.18), but this term can be bounded by $C_\eta\|\mathrm{Tr}\,u\|_q^q + \eta\|S_{a,q}(u|\chi_l^k)\|_q^q$, and then we can hide $\eta\|S_{a,q}(u|\chi_l^k)\|_q^q$ in the left-hand side by taking η small enough.

Let us start with (7.20). Since χ_l and Ψ_ϵ are x-independent, we have

$$\nabla_x[\Psi_i^k] = k\Psi_i^{k-1}\cdot\chi_l\Psi_\epsilon\nabla_x\Phi_{B_i,l}.$$

But $\nabla_x \Phi_{B_i,l}$ is supported outside the ball B_i, and $\Phi_{B_i/2,l} \equiv 0$ on $\operatorname{supp} \nabla_x \Phi_{B_i,l}$ if $l_i \geq 2l$. So $\nabla_x \Phi_{B_i,l} = (1-\Phi_{B_i/2,l})^k \nabla_x \Phi_{B_i,l}$ and

$$\begin{aligned}
(7.22) \qquad T_1 &:= \Big| \operatorname{Re} \iint_\Omega \mathcal{A}' \nabla u \cdot \nabla_x [\Psi_i^k] \, \frac{|u|^{q-2} \bar{u}}{b} \, \frac{dt}{|t|^{n-d-2}} \, dx \Big| \\
&\lesssim \iint_\Omega |\nabla u| (1-\Phi_{B_i/2,l})^k \, \Psi_i^{k-1} \chi_l \, \Psi_\epsilon |\nabla_x \Phi_{B_i,l}| \, |u|^{q-1} \, \frac{dt}{|t|^{n-d-2}} \, dx \\
&\lesssim \Big(\iint_\Omega |\nabla u|^2 \, |u|^{q-2} (1-\Phi_{B_i/2,l})^k \, \Psi_i^k \, \frac{dt}{|t|^{n-d-2}} \, dx \Big)^{1/2} \\
&\quad \times \Big(\iint_\Omega |u|^q (1-\Phi_{B_i/2,l})^k \, \Psi_i^{k-2} \, |\nabla_x [\Phi_{B_i,l} \chi_l]|^2 \, \frac{dt}{|t|^{n-d-2}} \, dx \Big)^{1/2} \\
&\lesssim \| S_{a,q}(u|\Psi_i^k (1-\Phi_{B_i/2,l})^k) \|_q^{q/2} \| \widetilde{N}_{a,q}(u|\Psi_i^{k-2}(1-\Phi_{B_i/2,l})^k) \|_q^{q/2},
\end{aligned}$$

by Lemma 4.6 and the fact that $\Phi_{B_i,l}\chi_l$ satisfies $(\mathcal{H}_a^2)$. It then follows from Lemma 5.3 that

$$(7.23) \qquad T_1 \lesssim \| \widetilde{N}_{a,q}(u|\Psi_i^{k-2}(1-\Phi_{B_i/2,l})^{k-2}) \|_q^q.$$

By the construction of Ψ_i, $\widetilde{N}_{a,q}(u|\Psi_i^{k-2}(1-\Phi_{B_i/2,l})^{k-2})$ is supported in $\mathbb{R}^d \setminus (B_i/4)$, so the right-hand side of (7.23) is bounded by $\| \widetilde{N}_{a,q}(u|\chi_l^{k-2}) \|^q_{L^q(\mathbb{R}^d \setminus B_i/4)}$. Lemmas 7.1 and 7.4 imply that $\widetilde{N}_{a,q}(u|\chi_l^{k-2})$ is integrable in $L^q(\mathbb{R}^d)$. Therefore, since the balls B_i increase to $\mathbb{R}^d$, we have

$$\lim_{i\to\infty} T_1 = 0.$$

The claim (7.20) follows.

We turn to the proof of (7.21). First, since Ψ_i depends only on $x, |t|$ and not on $t/|t|$, by simple algebra, we have

$$\begin{aligned}
\Big| \Big(\frac{|u|^q}{|t|} \nabla |t| - \nabla_t [|u|^q] \Big) \cdot \nabla_t [\Psi_i^k] \Big| &= \Big| \Big(\frac{|u|^q}{|t|} - \partial_r [|u|^q] \Big) \, \partial_r [\Psi_i^k] \Big| \\
&\leq \Big| \Big(\frac{|u|^q}{|t|} - \partial_r [|u|^q] \Big) \, \partial_r [\chi_l^k \Psi_\epsilon^k] \Big|.
\end{aligned}$$

That is, the integrand is bounded by a function independent of i. Moreover, we let the reader check that the latter is integrable:

$$\begin{aligned}
&\iint_\Omega \Big| \Big(\frac{|u|^q}{|t|} - \partial_r [|u|^q] \Big) \, \partial_r [\chi_l^k \Psi_\epsilon^k] \Big| \, \frac{dt}{|t|^{n-d-2}} \, dx \\
&\qquad \lesssim \| \widetilde{N}_{a,q}(u|\Psi_i^{k-1}) \|_q^q + \| \widetilde{N}_{a,q}(u|\Psi_i^{k-2}) \|_q^{q/2} \| S_{a,q}(u|\Psi_i^k) \|_q^{q/2} \\
&\qquad \lesssim \| \widetilde{N}_{a,q}(u|\Psi_i^{k-2}) \|_q^q \lesssim \| \widetilde{N}_{a,q}(u|\chi_l^{k-2}) \|_q^q < +\infty.
\end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}
&\lim_{i\to\infty} \frac{1}{q} \iint_\Omega \Big(\frac{|u|^q}{|t|} \nabla |t| - \nabla_t [|u|^q] \Big) \cdot \nabla_t [\Psi_i^k] \, \frac{dt}{|t|^{n-d-2}} \, dx \\
&\qquad = \frac{1}{q} \iint_\Omega \Big(\frac{|u|^q}{|t|} - \partial_r [|u|^q] \Big) \, \partial_r [\chi_l^k \Psi_\epsilon^k] \, \frac{dt}{|t|^{n-d-2}} \, dx < +\infty.
\end{aligned}$$

We denote the second integral as T_2. If ϵ is small ($\epsilon < l/a$), then $\Psi_\epsilon \equiv 1$ on $\operatorname{supp} \partial_r \chi_l$ and $\chi_l \equiv 1$ on $\operatorname{supp} \partial_r \Psi_\epsilon$. Hence

$$\partial_r[\chi_l^k \Psi_\epsilon^k] = \partial_r[\Psi_\epsilon^k] + \partial_r[\chi_l^k],$$

which implies that

$$\begin{aligned} T_2 &= \frac{1}{q} \iint_\Omega \Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q] \Big) \partial_r[\Psi_\epsilon^k] \frac{dt}{|t|^{n-d-2}} \, dx \\ &\quad + \frac{1}{q} \iint_\Omega \Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q] \Big) \partial_r[\chi_l^k] \frac{dt}{|t|^{n-d-2}} \, dx \\ &=: T_3 + T_4. \end{aligned}$$

Both T_3 and T_4 are finite when ϵ is small, because T_3 and T_4 are just the decomposition of the integral T_2 into two integrals on disjoint sets $\operatorname{supp} \partial_r \Psi_\epsilon$ and $\operatorname{supp} \partial_r \chi_l$ (and the function under the integral of T_2 is integrable). The term T_4 is independent of ϵ, so we do not need to touch it anymore. Now, we consider

$$U_\epsilon := \iint_\Omega |u|^q \partial_r[\Psi_\epsilon^k] \frac{dt}{|t|^{n-d-1}} \, dx \geq 0.$$

Since $|\partial_r[|u|^q]| \lesssim |u|^{q-1} |\nabla u|$ and $\partial_r[\Psi_\epsilon^k] \geq 0$ by construction, by Hölder's inequality, we have

$$T_3 \lesssim U_\epsilon + U_\epsilon^{1/2} \Big(\iint_\Omega |\nabla u|^2 \, |u|^{q-2} \, \partial_r[\Psi_\epsilon^k] \frac{dt}{|t|^{n-d-3}} \, dx \Big)^{1/2}.$$

Observe now that $\partial_r[\Psi_\epsilon] \lesssim 1/|t|$, and if ϵ is small, then $\chi_l \equiv 1$ on $\operatorname{supp} \partial_r[\Psi_\epsilon^k]$. We deduce

$$T_3 \lesssim U_\epsilon + U_\epsilon^{1/2} \Big(\iint_\Omega |\nabla u|^2 \, |u|^{q-2} \chi_l^k \frac{dt}{|t|^{n-d-2}} \, dx \Big)^{1/2} \lesssim U_\epsilon + U_\epsilon^{1/2} \| S_{a,q}(u|\chi_l^k) \|_q^{q/2}.$$

By taking the limit as $\epsilon \to 0$, and recalling that Lemma 7.5 shows $\limsup_{\epsilon \to 0} U_\epsilon \lesssim \|\operatorname{Tr} u\|_q^q$, it follows that

$$\limsup_{\epsilon \to 0} T_3 \lesssim \|\operatorname{Tr} u\|_q^q + \|\operatorname{Tr} u\|_q^{q/2} \| S_{a,q}(u|\chi_l^k) \|_q^{q/2}.$$

The claim (7.21) and then the lemma follows. □

The next step is to prove a bound on the growth of the energy solution u. It will be used to get a global estimate in the proof of Lemma 7.8.

Lemma 7.7. *Let $\mathcal{L}$ be an elliptic operator. Let $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Choose $k \geq 1$. Then for any energy solution $u \in W$ to $\mathcal{L}u = 0$,*

$$\lim_{l \to +\infty} \frac{1}{l} \iint_\Omega |u|^q \partial_r[-\chi_l^k] \frac{dt}{|t|^{n-d-1}} \, dx = 0. \tag{7.24}$$

Proof. First, by the definition of χ_l and ϕ, we have $0 \leq \partial_r[-\chi_l^k] \lesssim 1/l$ and $\operatorname{supp} \partial_r[-\chi_l^k] \subset \{(x,t) \in \Omega,\, l \leq a|t| \leq 2l\}$. Therefore,

$$\frac{1}{l} \iint_\Omega |u|^q \partial_r[-\chi_l^k] \frac{dt}{|t|^{n-d-1}}\, dx \lesssim \frac{1}{l^2} \int_{x\in\mathbb{R}^d} \int_{l\leq a|t|\leq 2l} |u|^q \frac{dt}{|t|^{n-d-1}}\, dx. \tag{7.25}$$

Let $(B_i)_{i\in I}$ be balls of radius l that form a finitely overlapping covering of $\mathbb{R}^d \setminus B_{4l}(0)$. Without loss of generality, we only consider balls that intersect $\mathbb{R}^d \setminus B_{4l}(0)$, thus

$$\mathbb{R}^d \setminus B_{4l}(0) \subset \bigcup_{i\in I} B_i \subset \mathbb{R}^d \setminus B_{2l}(0).$$

Let D_0 be a cylindrical annulus:

$$D_0 := \{(x,t) \in \mathbb{R}^n,\, x \in B_{4l} \setminus B_{2l},\, a|t| \leq 2l\} \cup \{(x,t) \in \mathbb{R}^n,\, x \in B_{2l},\, l \leq a|t| \leq 2l\}.$$

The bound (7.25) then becomes

$$\begin{aligned} &\frac{1}{l} \iint_\Omega |u|^q\, \partial_r[-\chi_l^k] \frac{dt}{|t|^{n-d-1}}\, dx \\ &\quad\lesssim \frac{1}{l^2} \iint_{(x,t)\in D_0} |u|^q \frac{dt}{|t|^{n-d-1}}\, dx + \frac{1}{l^2} \sum_{i\in I} \int_{x\in B_i} \int_{a|t|\leq 2l} |u|^q \frac{dt}{|t|^{n-d-1}}\, dx \\ &\quad= \frac{1}{l^2} \iint_{(x,t)\in D_0} ||u|^{q/2-1}u|^2 \frac{dt}{|t|^{n-d-1}}\, dx \\ &\qquad+ \frac{1}{l^2} \sum_{i\in I} \int_{x\in B_i} \int_{a|t|\leq 2l} ||u|^{q/2-1}u|^2 \frac{dt}{|t|^{n-d-1}}\, dx. \end{aligned} \tag{7.26}$$

Now, since u is an energy solution, there exists l_0 such that $\operatorname{supp} \operatorname{Tr} u \subset B_{2l_0}(0)$. If $l \geq l_0$, all the cubes $\{(x,t) \in \mathbb{R}^n,\, x \in B_i,\, a|t| \leq 2l\}$ and the domain D_0 intersect the boundary $\partial\Omega = \mathbb{R}^d$, where $\operatorname{Tr} u$ is 0. Thus $\operatorname{Tr} |u|^{q/2-1}u = 0$ by boundary Moser's estimate (when $q > 2$) or Holder's inequality (when $q < 2$). So by Poincaré's inequality (see Lemma 4.2 in [10] (the proof there is written when the domains are balls but it goes through for our domains), we have

$$\begin{aligned} \frac{1}{l} \iint_\Omega |u|^q\, \partial_r[-\chi_l^k] \frac{dt}{|t|^{n-d-1}}\, dx &\lesssim \iint_{(x,t)\in D_0} |\nabla[|u|^{q/2-1}u]|^2 \frac{dt}{|t|^{n-d-1}}\, dx \\ &\quad+ \sum_{i\in I} \int_{x\in B_i} \int_{a|t|\leq 2l} |\nabla[|u|^{q/2-1}u]|^2 \frac{dt}{|t|^{n-d-1}}\, dx \\ &\lesssim \iint_{(x,t)\in D_0} |\nabla u|^2\, |u|^{q-2} \frac{dt}{|t|^{n-d-1}}\, dx \\ &\quad+ \sum_{i\in I} \int_{x\in B_i} \int_{a|t|\leq 2l} |\nabla u|^2\, |u|^{q-2} \frac{dt}{|t|^{n-d-1}}\, dx, \end{aligned}$$

where the last line is due to (2.5). Due to the finite overlapping of the covering $(B_i)_{i\in I}$ and the fact that we always avoid $\{(x,t) \in \Omega,\, x \in B_{2l}(0),\, a|t| \leq l\}$ in the integrations, we have, when $l \geq l_0$,

$$\frac{1}{l}\iint_\Omega |u|^q\,\partial_r[-\chi_l^k]\,\frac{dt}{|t|^{n-d-1}}\,dx \lesssim \iint_{\Omega\setminus\{(x,t)\in\Omega,\, x\in B_{2l}(0),\, a|t|\leq l\}} |\nabla u|^2\,|u|^{q-2}\,\frac{dt}{|t|^{n-d-1}}\,dx.$$

Thanks to Lemma 3.5, the right-hand side above converges to 0 as l goes to $+\infty$. The lemma follows. □

The following lemma is the key point in proving the existence of solutions to the Dirichlet problem, when the boundary function (trace) is smooth.

Lemma 7.8 (Global estimate $N <$ Tr for energy solutions)**.** *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$. Let $a > 0$, $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. There exist two values $\kappa_0 > 0$ and $C > 0$, both depending only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b\|_\infty + \|b^{-1}\|_\infty$, such that if $\kappa \leq \kappa_0$, then for any energy solution $u \in W$ to $\mathcal{L}u = 0$,*

$$\|\widetilde{N}_{a,q}(u)\|_q \leq C\|\mathrm{Tr}\, u\|_q. \tag{7.27}$$

Proof. Let us fix for the proof some k large, say $k = 20$. In addition, we always consider κ_0 smaller than 1 and $\kappa \leq \kappa_0$. Assume that u is not trivially zero, otherwise there is nothing to prove.

The combination of Lemma 7.1 and Lemma 7.6 gives that for $l > 0$,

$$\|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q \leq C\kappa\|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q^q + C\|\mathrm{Tr}\, u\|_q^q + \frac{C}{q}\iint_\Omega\Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q]\Big)\,\partial_r[\chi_l^k]\,\frac{dt}{|t|^{n-d-2}}\,dx, \tag{7.28}$$

where the constant $C > 0$ depends only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b\|_\infty + \|b^{-1}\|_\infty$. In particular, note that $k = 20$ is fixed and $\kappa \leq \kappa_0 \leq 1$, the constant C does not depend κ and l.

In order to control $\|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q$ by $\|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q$, we make an additional assumption on ϕ, the smooth function from which χ_l is defined. Recall that $\phi \in C_0^\infty([0,\infty))$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $[0,1]$, $\phi \equiv 0$ outside $[0,2]$, ϕ is decreasing and $|\phi'| \leq 2$. We assume, in addition, that

$$\phi(x) = 2 - x \quad \text{when } x \in (9/8, 15/8). \tag{7.29}$$

With this additional assumption, it is not hard to verify that

$$\chi_l^{k-2} \lesssim \chi_l^{k+12} + l\,\partial_r[-\chi_{3l/2}^k],$$

where the constant is universal (recall $k = 20$ is fixed). Indeed, observe that the assumption (7.29), in particular, implies that $\partial_r[-\chi_{3l/2}^k]$ has a strictly positive lower bound on the interval $\frac{3l}{2a}(9/8, 15/8) \supset [15l/(8a), 2l/a]$, where the value of χ_l is very small. It then follows that

$$\begin{aligned} \|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q^q \\ \lesssim \|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q + l\int_{x\in\mathbb{R}^d} \Big|\sup_{\Gamma_a(x)} (u|\partial_r[-\chi_{3l/2}^k])_{W,a,q}\Big|^q\,dx. \end{aligned} \tag{7.30}$$

Since $\partial_r[-\chi_{3l/2}^k]$ is supported in a domain, where $|s|\approx l/a$, we deduce

$$\begin{aligned} \Big|\sup_{\Gamma_a(x)} (u|\partial_r[-\chi_{3l/2}^k])_{W,a,q}\Big|^q &\lesssim \big|(u|\partial_r[-\chi_{3l/2}^k])_{W,10a,q}(x, 3l/2a)\big|^q \\ &\lesssim l^{-n}\int_{y\in B_{8l}(x)}\int_{3l/4\le a|s|\le 3l} |u|^q\,\partial_r[-\chi_{3l/2}^k]\,ds\,dy \\ &\lesssim l^{-d-1}\int_{y\in B_{8l}(x)}\int_{s\in\mathbb{R}^{n-d}} |u|^q\,\partial_r[-\chi_{3l/2}^k]\,\frac{ds}{|s|^{n-d-1}}\,dy. \end{aligned}$$

And then, by Fubini's lemma,

$$\int_{x\in\mathbb{R}^d}\Big|\sup_{\Gamma_a(x)} (u|\partial_r[-\chi_{3l/2}^k])_{W,a,q}\Big|^q\,dx \lesssim l^{-1}\iint_{(y,s)\in\Omega} |u|^q\,\partial_r[-\chi_{3l/2}^k]\,\frac{ds}{|s|^{n-d-1}}\,dy.$$

The estimate (7.30) becomes

$$\|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q^q \lesssim \|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q + \iint_{(x,t)\in\Omega} |u|^q\,\partial_r[-\chi_{3l/2}^k]\,\frac{dt}{|t|^{n-d-1}}\,dx. \tag{7.31}$$

We eventually want to estimate the last term by $\|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q$. Notice that for any $l' \le (2/3)^2 l < l/2$, we have

$$\begin{aligned} &\iint_{(x,t)\in\Omega} |u|^q\,\partial_r[-\chi_{l'}^k]\,\frac{dt}{|t|^{n-d-1}}\,dx \\ &\quad\lesssim (l')^{d-n}\int_{x\in\mathbb{R}^d}\int_{l'/2\le a|t|\le 2l'} |u|^q\,dt\,dx \\ &\quad\lesssim \int_{x\in\mathbb{R}^d}\Big((l')^{-n}\int_{y\in B_{l'/2}(x)}\int_{l'/2\le a|t|\le 2l'} |u|^q\,dt\,dy\Big)\,dx \\ &\quad\lesssim \int_{x\in\mathbb{R}^d} |\widetilde{N}_{a,q}(u|\chi_l^{k+12})|^q\,dx = \|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q. \end{aligned} \tag{7.32}$$

The last inequality is because $\chi_l \equiv 1$ when $a|t| \le l$, so, in particular, when $a|t| \le 2l' < l$. Let $w\colon (0,\infty) \to \mathbb{R}$ denote the continuous function

$$w(l) = \iint_{(x,t)\in\Omega} |u|^q\,\partial_r[-\chi_l^k]\,\frac{dt}{|t|^{n-d-1}}\,dx.$$

Then the combination of (7.28), (7.31), and (7.32) gives that

$$\begin{aligned}\sup_{l'\le 4l/9} w(l') &\lesssim \|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q \\ &\le C\kappa\|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q + C\kappa\, w(3l/2) + C\|\mathrm{Tr}\, u\|_q^q \\ &\quad + \frac{C}{q}\iint_\Omega \Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q]\Big)\partial_r[\chi_l^k]\,\frac{dt}{|t|^{n-d-2}}\,dx,\end{aligned} \tag{7.33}$$

where the constant $C>0$ is independent of κ.

The last term of the above estimate (7.33) can also be written in terms of the function $w(l)$. In fact, we define

$$v(r) = \int_{x\in\mathbb{R}^d}\int_{\theta\in\mathbb{S}^{n-d-1}} |u(x,r\theta)|^q\, d\theta\, dx,$$

which is finite for almost every $r>0$, since $u\in L^q_{\mathrm{loc}}(\Omega)$. On the one hand, the last term of (7.33) can be rewritten by polar coordinates:

$$\iint_\Omega \Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q]\Big)\partial_r[\chi_l^k]\,\frac{dt}{|t|^{n-d-2}}\,dx = \int_0^\infty [rv'(r) - v(r)]\,\partial_r[-\chi_l^k]\,dr. \tag{7.34}$$

On the other hand, we have

$$w(l) = \iint_{(x,t)\in\Omega} |u|^q\partial_r[-\chi_l^k]\,\frac{dt}{|t|^{n-d-1}}\,dx = \int_0^\infty v(r)\,\partial_r[-\chi_l^k]\,dr. \tag{7.35}$$

By the construction of χ_l, we can write $\partial_r[-\chi_l^k]$ as $\frac{a}{l}\xi(\frac{ar}{l})$ with a non-negative function $\xi = -\phi^k\,\Phi' \in C_0^\infty([0,\infty))$, and hence

$$\begin{aligned} w'(l) &= \int_0^{+\infty} v(r)\,\partial_l\Big[\frac{a}{l}\,\xi\Big(\frac{ar}{l}\Big)\Big]\,dr \\ &= -\frac{1}{l}\int_0^{+\infty} v(r)\,\frac{a}{l}\,\xi\Big(\frac{ar}{l}\Big)\,dr - \frac{1}{l}\int_0^{+\infty} rv(r)\,\frac{a^2}{l^2}\,\xi'\Big(\frac{ar}{l}\Big)\,dr \\ &= -\frac{1}{l}\int_0^{+\infty} v(r)\,\frac{a}{l}\,\xi\Big(\frac{ar}{l}\Big)\,dr - \frac{1}{l}\int_0^{+\infty} rv(r)\,\frac{a}{l}\,\partial_r\Big[\xi\Big(\frac{ar}{l}\Big)\Big]\,dr \\ &= -\frac{1}{l}\int_0^{+\infty} v(r)\,\frac{a}{l}\,\xi\Big(\frac{ar}{l}\Big)\,dr + \frac{1}{l}\int_0^{+\infty} \partial_r[rv(r)]\,\frac{a}{l}\,\xi\Big(\frac{ar}{l}\Big)\,dr \\ &= \frac{1}{l}\int_0^{+\infty} rv'(r)\,\frac{a}{l}\,\xi\Big(\frac{ar}{l}\Big)\,dr \\ &= \frac{1}{l}\int_0^{+\infty} rv'(r)\,\partial_r[-\chi_l^k]\,dr, \end{aligned} \tag{7.36}$$

where we use integration by parts and the fact that ξ has compact support. By combining (7.34), (7.35) and (7.36), we can rewrite (7.33) as

$$\begin{aligned}(7.37)\qquad \sup_{l'\le 4l/9} w(l') &\lesssim \|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q\\ &\le C\kappa\|\widetilde{N}_{a,q}(u|\chi_l^{k+12})\|_q^q + C\kappa w(3l/2)\\ &\quad + \frac{C}{q}(lw'(l)-w(l)) + C\|\mathrm{Tr}\, u\|_q^q.\end{aligned}$$

We claim that for any $l_0>0$, there exists $l\ge l_0$ such that

$$(7.38)\qquad w\Big(\frac{3}{2}\,l\Big) \le \Big(\frac{3}{2}\Big)^5 \sup_{l'\le 4l/9} w(l') \quad \text{and} \quad lw'(l)-w(l)\le 0.$$

Assume the claim holds. Then there is a sequence $l_i\to\infty$ such that

$$C\kappa w\Big(\frac{3}{2}\,l_i\Big) \le C\kappa\Big(\frac{3}{2}\Big)^5 \sup_{l'\le 4l_i/9} w(l') \le \frac{1}{3}\|\widetilde{N}_{a,q}(u|\chi_{l_i}^{k+12})\|_q^q.$$

Thus, by choosing κ sufficiently small satisfying $C\kappa(3/2)^5\le 1/3$, we obtain (7.27) by combining (7.37) and (7.38).

Recall in Lemma 7.7 we proved (7.24), which says $w(l)/l\to 0$ as $l\to+\infty$. Thus, it is not difficult to show each estimate of the claim (7.38) holds for infinitely many l individually. But we need to find a sequence $l_i\to\infty$ such that both estimates hold at the same time. We prove the claim by contradiction. Assume not, then there exists $l_0>0$ such that for all $l\ge l_0$, either

$$(7.39)\qquad w\Big(\frac{3}{2}l\Big) > \Big(\frac{3}{2}\Big)^5 \sup_{l'\le 4l/9} w(l')$$

or

$$(7.40)\qquad lw'(l)-w(l)>0.$$

We define M as the positive quantity

$$M := \inf_{[(2/3)^4 l_0, l_0]} w > 0.$$

Indeed, note that $w(l)$ is continuous, and if M is zero, then there exists $l\in[16l_0/81, l_0]$ such that $w(l)=0$. Hence $u(x,t)\equiv 0$ almost everywhere whenever t lies in $\operatorname{supp}\partial_r(-\chi_l^k)$, which is a non-trivial band contained in $\{l\le a|t|\le 2l\}$. Extending u by zero outside of this band, we get another weak solution in W to $\mathcal{L}u=0$ with the same boundary value. By the uniqueness of weak solutions (see Lemma 1.12), we conclude that $u(x,t)\equiv 0$ almost everywhere beyond the band, in particular, whenever $|t|\ge 2l_0/a$. Then (7.39) and (7.40) clearly fail, and thus M is strictly positive.

Let I_k denote the interval $[(3/2)^{k-4}l_0, (3/2)^{k-3}l_0)$. We will show by induction that the following property holds for all $k\in\mathbb{N}$:

$$\mathcal{P}(k):\ \text{there exists } l\in I_k \text{ such that } w(l)\ge \frac{l}{l_0}M.$$

The base steps when $k = 0, 1, 2, 3$ are immediate by the definition of M. Assume that $\mathcal{P}(k)$ and $\mathcal{P}(k+3)$ hold, that is, there exists $l_1 \in I_k$ such that $w(l_1) \geq l_1 M/l_0$, and there exists $l_2 \in I_{k+3}$ such that $w(l_2) \geq l_2 M/l_0$. We want to prove that $\mathcal{P}(k+4)$ holds. Let $l_3 = (3/2)^k l_0$ denote the upper endpoint for the interval I_{k+3}. We distinguish two cases. Either there exists $l \in [l_2, l_3) \subset I_{k+3}$ such that (7.39) is satisfied, that is,

$$w\Big(\frac{3}{2}\,l\Big) > \Big(\frac{3}{2}\Big)^5 \sup_{l' \leq 4l/9} w(l').$$

Since $4l/9 \geq \sup I_k \geq l_1$, $3l/2 \in I_{k+4}$ and $(3/2)^5 l_1 \geq \sup I_{k+4} > 3l/2$, it follows that

$$w\Big(\frac{3}{2}\,l\Big) > \Big(\frac{3}{2}\Big)^5 w(l_1) \geq \Big(\frac{3}{2}\Big)^5 \frac{l_1}{l_0}\,M \geq \frac{3l/2}{l_0}\,M.$$

Therefore, $\mathcal{P}(k+4)$ holds for $3l/2 \in I_{k+4}$. Alternatively, in the second case, (7.39) is never satisfied on the interval $[l_2, l_3)$ and so by assumption (7.40) is always satisfied on the same interval, that is,

$$l w'(l) - w(l) > 0 \quad \text{for all } l \in [l_2, l_3).$$

By Grönwall's inequality and the continuity of w, this implies

$$w(l_3) \geq \frac{l_3}{l_2}\,w(l_2) \geq \frac{l_3}{l_0}\,M,$$

i.e., $\mathcal{P}(k+4)$ holds for $l_3 \in I_{k+4}$. The induction step follows, and we conclude that $\mathcal{P}(k)$ holds for all $k \in \mathbb{N}$. Thus, in particular, we have

$$\limsup_{l \to \infty} \frac{w(l)}{l} \geq \frac{M}{l_0} > 0,$$

contradicting Lemma 7.7. Therefore, the claim (7.38) holds. □

The next result proves the existence of a solution to the Dirichlet problem with boundary value in $L^q(\mathbb{R}^d)$, by approximating using energy solutions.

Lemma 7.9. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$. Let $a > 0$, $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. There exists two values $\kappa_0 > 0$ and $C > 0$, both depending only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b\|_\infty + \|b^{-1}\|_\infty$, such that if $\kappa \leq \kappa_0$, then for any $g \in L^q(\mathbb{R}^d)$, there exists a weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to $\mathcal{L}u = 0$ such that*

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u(y,s) - g(x)|^q \, dy\, ds = 0 \tag{7.41}$$

for almost every $x \in \mathbb{R}^d$, and

$$\|\widetilde{N}_{a,q}(u)\|_q \leq C\|g\|_q, \tag{7.42}$$

where the constant depends only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, and $\|b\|_\infty + \|b^{-1}\|_\infty$.

Remark 7.10. (i) The trace operator is defined for functions in W. In general, for functions in $W^{1,2}_{\text{loc}}(\Omega)$, we use the equality (7.41) as a weaker version or an adaptation of $\operatorname{Tr} u = g$ almost everywhere on $\mathbb{R}^d$.

(ii) By Moser's estimate or Hölder's inequality, the Lebesgue density property, i.e. (7.41), with power q is equivalent to that of power 2.

Proof. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^q(\mathbb{R}^d)$, we can find a collection $(g_i)_i$ of functions in $C_0^\infty(\mathbb{R}^d)$ such that $g_i \to g$ in $L^q(\mathbb{R}^d)$. In particular, $(g_i)_i$ is a Cauchy sequence in $L^q(\mathbb{R}^d)$. By passing to a subsequence if necessary, we may assume that $\|g_i - g_{i+1}\|_q \leq 2^{-2i}$ for all i. Since $C_0^\infty(\mathbb{R}^d) \subset H$, Lemma 1.12 guarantees the existence of an energy solution $u_i \in W$ such that $\operatorname{Tr} u_i = g_i$.

Lemma 1.12 also guarantees that for boundary value $g_i - g_j \in H$, the corresponding solution in W is unique, so by the linearity of the operator $\mathcal{L}$ the solution is $u_i - u_j$. Thus, by Lemma 7.8, we have, for all i, j,

$$\|\widetilde{N}_{a,q}(u_i)\|_q \lesssim \|g_i\|_q \tag{7.43}$$

and

$$\|\widetilde{N}_{a,q}(u_i - u_j)\|_q \lesssim \|g_i - g_j\|_q. \tag{7.44}$$

We claim that $\widetilde{N}_{a,q}(u_i - u_{i+1})(x) \to 0$ as $i \to \infty$, for almost every $x \in \mathbb{R}^d$. In fact, for any $\lambda > 0$, denote

$$E_\lambda = \{x \in \mathbb{R}^d, \widetilde{N}_{a,q}(u_i - u_{i+1})(x) < \lambda/2^i \text{ for all } i\}.$$

Then, by (7.44) and the assumption on $(g_i)_i$, we have

$$\begin{aligned} |E_\lambda^c| &\leq \sum_i \left|\{x \in \mathbb{R}^d, \widetilde{N}_{a,q}(u_i - u_{i+1})(x) \geq \lambda/2^i\}\right| \\ &\leq \sum_i \left(\frac{2^i}{\lambda}\right)^q \int_{x\in\mathbb{R}^d} |\widetilde{N}_{a,q}(u_i - u_{i+1})(x)|^q \, dx \\ &\leq \sum_i \left(\frac{2^i}{\lambda} 2^{-2i}\right)^q \leq \frac{C}{\lambda^q}. \end{aligned}$$

Consider an arbitrary sequence $\lambda_j \to \infty$, the above estimate implies $\bigcap_{\lambda_j} E_{\lambda_j}^c$ has measure zero, and thus its complement $\bigcup_{\lambda_j} E_{\lambda_j}$ has full measure in $\mathbb{R}^d$. Therefore, for almost every $x \in \mathbb{R}^d$, there exists some $\lambda > 0$ (depending on x) such that

$$\widetilde{N}_{a,q}(u_i - u_{i+1})(x) < \frac{\lambda}{2^i} \quad \text{for all } i.$$

Thus, by the triangle inequality for the L^q norm, we have that for any $j > i$,

$$\widetilde{N}_{a,q}(u_i - u_j)(x) \leq \frac{2\lambda}{2^i},$$

and by the definition of $\widetilde{N}_{a,q}$, this means that for any $(z, r) \in \Gamma_a(x)$,

$$\|u_i - u_j\|_{L^q(W_a(z,r))} = \left(\iint_{W_a(z,r)} |u_i - u_j|^q \, dy \, ds\right)^{1/q} \leq \frac{2\lambda}{2^i} |W_a(z,r)|^{1/q}. \tag{7.45}$$

That is to say $(u_i)_i$ is a Cauchy sequence in $L^q(W_a(z,r))$, and thus it converges to a function u. Since this holds for almost every $x \in \mathbb{R}^d$ and any $(z,r) \in \Gamma_a(x)$, this function u is well defined on all of Ω and $u_i \to u$ in $L^q_{\rm loc}$. Moreover, by passing $j \to \infty$ in (7.45), we have

$$(7.46)\quad \widetilde{N}_{a,q}(u_i - u)(x) = \sup_{(z,r)\in\Gamma_a(x)} \Big(\frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u_i - u|^q \, dy \, ds \Big)^{1/q} \leq \frac{2\lambda}{2^i},$$

which converges to zero as $i \to \infty$. In particular, for any $(z,r) \in \Gamma_a(x)$,

$$\big|(u_i)_{W,a,q}(z,r) - (u)_{W,a,q}(z,r)\big| \leq \widetilde{N}_{a,q}(u_i - u)(x) \leq \frac{2\lambda}{2^i}.$$

We deduce that

$$(7.47)\qquad \lim_{i\to\infty} \widetilde{N}_{a,q}(u_i)(x) = \widetilde{N}_{a,q}(u)(x)$$

for almost every $x \in \mathbb{R}^d$. The estimate (7.42) follows by (7.47), Fatou's lemma and (7.43).

We recall the interior Cacciopoli inequality (see Lemma 8.6 in [10]): Let B be a fixed ball with radius $r > 0$ such that the distance from $4B$ to the boundary is roughly r. We know

$$\iint_B |\nabla(u_i - u_j)|^2 \, dm \lesssim \frac{1}{r^2} \iint_{2B} |u_i - u_j|^2 \, dm < +\infty.$$

By assumption, the distances from B and $2B$ to the boundary are roughly r, so the above estimate is equivalent to

$$(7.48)\qquad \iint_B |\nabla u_i - \nabla u_j|^2 \, dx \, dt \lesssim \frac{1}{r^2} \iint_{2B} |u_i - u_j|^2 \, dx \, dt.$$

We have shown that $(u_i)_i$ is a Cauchy sequence in $L^q(4B)$. Either by Hölder's inequality (in the case of $q \geq 2$) or by Moser's estimate (in the case of $q \leq 2$), see (3.26), it follows that $(u_i)_i$ is a Cauchy sequence in $L^2(2B)$. Therefore, (7.48) implies that $(\nabla u_i)_i$ is also a Cauchy sequence in $L^2(B)$, and thus it converges in $L^2(B)$. By the uniqueness of the limit, $\nabla u_i \to \nabla u$ in $L^2_{\rm loc}(\Omega)$. The convergence forces u to be, like the u_i's, a weak solution to $\mathcal{L}u = 0$.

We now turn to the proof of (7.41). Since $u_i \in W$, by Theorem 3.4 of [10] (see also step 2 of Lemma 7.1), we have

$$(7.49)\qquad \sup_{\substack{(z,r)\in\Gamma_a(x) \\ r<\delta}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u_i(y,s) - g_i(x)|^2 \, dy \, ds \to 0 \quad \text{as } \delta \to 0,$$

for almost every $x \in \mathbb{R}^d$. The set such that (7.49) holds depends *a priori* on i, but since a countable union of sets of zero measure is still a set of measure zero, we have (7.49) for every $i \in \mathbb{N}$ and almost every $x \in \mathbb{R}^d$. Either by Hölder's inequality (when $q \leq 2$) or by Moser's estimate (when $q \geq 2$, see (3.2)) applied to the weak solution $u_i - g_i(x)$, it follows that

$$\sup_{\substack{(z,r)\in\Gamma_a(x)\\ r<\delta}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u_i(y,s) - g_i(x)|^q \, dy\, ds \to 0 \quad \text{as } \delta \to 0, \tag{7.50}$$

for any $i \in \mathbb{N}$ and almost every $x \in \mathbb{R}^d$. Since $g_i \to g$ in $L^q(\mathbb{R}^d)$, by passing to a subsequence, $g_i(x) \to g(x)$ for almost every $x \in \mathbb{R}^d$. Recall also that (7.47) holds for almost every $x \in \mathbb{R}^d$. As a consequence, (7.41) is obtained by taking the limit as $i \to \infty$ in (7.50). □

8. Uniqueness in the Dirichlet problem

In this section, we assume that $u \in W^{1,2}_{\rm loc}(\Omega)$ is a weak solution to $\mathcal{L}u = 0$, where $\mathcal{L}$ satisfies $(\mathcal{H}^1_\kappa)$, and that u satisfies $\|\widetilde{N}_{a,q}(u)\|_q < +\infty$. We want to prove that if κ is sufficiently small and that

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} u(y,s)\, dy\, ds = 0 \quad \text{for a.e. } x \in \mathbb{R}^d, \tag{8.1}$$

then u has to be 0. This, in turn, proves the uniqueness of solution.

Lemma 8.1. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$. Let $a > 0$, $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. For any weak solution $u \in W^{1,2}_{\rm loc}(\Omega)$ to $\mathcal{L}u = 0$ that satisfies $\|\widetilde{N}_{a,q}(u)\|_q < +\infty$, we have*

$$\|\widetilde{N}_{a,q}(u)\|_q \approx \|S_{a,q}(u)\|_q, \tag{8.2}$$

where the constants depend only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b\|_\infty + \|b^{-1}\|_\infty$ and κ.

Proof. Since $\|\widetilde{N}_{a,q}(u)\|_q$ is finite, the proof of

$$\|S_{a,q}(u)\|_q \leq C\|\widetilde{N}_{a,q}(u)\|_q < +\infty$$

can be achieved by using Lemma 5.5 with an increasing sequence of cut-off functions $\Psi_i(x)$ such that $\Psi_i(x) \to 1$ for all $x \in \Omega$.

With the notation of Section 6, for a fixed $k = 15$, for any $\epsilon > 0$, $l > 0$, and any ball B of radius $l' \geq l$, Lemma 6.11 shows that

$$\begin{aligned} \|\widetilde{N}_{a,q}(u|\Psi^k_\epsilon \Psi^k_{B,l})\|^q_q &\lesssim \|S_{a,q}(u|\Psi^{k-12}_\epsilon \Psi^{k-12}_{B,l})\|^q_q \\ &\quad + \int_{y\in\mathbb{R}^d} \int_{s\in\mathbb{R}^{n-d}} |u|^q \,\Psi^{k-3}_\epsilon \partial_r[-\Psi^{k-3}_{B,l}]\, \frac{ds}{|s|^{n-d-1}}\, dy. \end{aligned} \tag{8.3}$$

By Fubini's lemma and the fact that B has radius $l' \geq l$, the second term in the right-hand side is bounded by

$$\int_{y\in\mathbb{R}^d} \fint_{l\leq a|s|\leq 2l} |u|^q \, ds\, dy \lesssim \int_{x\in\mathbb{R}^d} \Big(\frac{1}{|W_a(x,l/a)|} \iint_{(y,s)\in W_a(x,l/a)} |u|^q \, ds\, dy \Big) \, dx$$
$$\leq \|N_{a,q}(u)\|^q_{L^q(\mathbb{R}^d\setminus B/2)} \to 0,$$

which tends to zero as the ball $B \nearrow \mathbb{R}^d$. By taking the limit as $l \to \infty$ and $\epsilon \to 0$, we obtain then

$$\|\widetilde{N}_{a,q}(u)\|_q \leq C \|S_{a,q}(u)\|_q.$$

The lemma follows. □

Lemma 8.2. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa \geq 0$. Let $a > 0$, $q \in (q_0, q_0')$, where q_0 is given by Proposition 2.1. Suppose that g lies in $L^q(\mathbb{R}^d)$ and $u \in W^{1,2}_{\rm loc}(\Omega)$ is weak solution to $\mathcal{L}u = 0$ that satisfy both $\|\widetilde{N}_{a,q}(u)\|_q < \infty$ and*

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} u(y,s)\, dy\, ds = g(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{8.4}$$

Then (8.4) self-improves itself into the q-Lebesgue property

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u(y,s) - g(x)|^q\, dy\, ds = 0 \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{8.5}$$

Proof. Let us write $\bar u(z,r)$ for $\frac{1}{|W_a(z,r)|}\iint_{W_a(z,r)} u(y,s)\,dy\,ds$. Observe that

$$\Big(\frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u(y,s) - g(x)|^q\, dy\, ds\Big)^{1/q}$$
$$\leq \Big(\frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u(y,s) - \bar u(z,r)|^q\, dy\, ds\Big)^{1/q} + |\bar u(z,r) - g(x)| =: T_1 + T_2.$$

Due to (8.4), the quantity T_2 tends to 0 as $(z,r) \in \Gamma_a(x)$, $r \to 0$ for almost every $x \in \mathbb{R}^d$. It remains to prove that

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \Big(\frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |u(y,s) - \bar u(z,r)|^q\, dy\, ds\Big)^{1/q} = 0 \tag{8.6}$$

for a.e. $x \in \mathbb{R}^d$. We split the proof of the claim into two cases: $q \geq 2$ and $q \leq 2$.

First, let us treat the case $q \geq 2$. Thanks to Proposition 1.10 and then the Poincaré inequality, we obtain that

$$T_1 \lesssim \Big(\frac{1}{|W_{4a}(z,r/2)\cup W_{4a}(z,2r)|} \iint_{W_{4a}(z,r/2)\cup W_{4a}(z,2r)} |u(y,s) - \bar u(z,r)|^2\, dy\, ds\Big)^{1/2}$$
$$\lesssim r\Big(\frac{1}{|W_{4a}(z,r/2)\cup W_{4a}(z,2r)|} \iint_{W_{4a}(z,r/2)\cup W_{4a}(z,2r)} |\nabla u(y,s)|^2\, dy\, ds\Big)^{1/2}.$$

Notice that $W_{4a}(z,r/2) \simeq a^d r^n \simeq W_{4a}(z,2r)$, from which we deduce that

$$\begin{aligned} T_1 &\lesssim \Big(\iint_{W_{4a}(z,r/2)} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-2}} + \iint_{W_{4a}(z,2r)} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{1/2} \\ &\lesssim \Big(\iint_{W_{8a}(x,r/2)} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{1/2} + \Big(\iint_{W_{8a}(x,2r)} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{1/2}. \end{aligned}$$

Therefore, the convergence of T_1 to 0 will be established if we can show that

$$\lim_{r\to 0} \iint_{W_{16a}(x,r)} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-2}} = 0 \quad \text{for a.e. } x \in \mathbb{R}^d,$$

which is a consequence of the fact that for any ball $B \subset \mathbb{R}^d$,

$$\lim_{r\to 0} \int_{x\in B} \iint_{W_{16a}(x,r)} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-2}} \, dx = 0. \tag{8.7}$$

Since, for r small enough,

$$\int_{x\in B} \iint_{W_{16a}(x,r)} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-2}} \, dx \leq \int_{2B} \int_{|t|\leq r} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-d-2}} \, dx,$$

the convergence (8.7) is an immediate byproduct of the fact that

$$\int_{2B} \int_{|t|\leq r} |\nabla u(y,s)|^2 \, dy \, \frac{ds}{|s|^{n-d-2}} \, dx \leq \int_{2B} |S_{a,2}(u)|^2 \, dx$$

is finite. Thanks to Lemma 5.5 and an increasing sequence of compactly supported cut-off functions $\Psi_i \uparrow 1$, it is easy to obtain that

$$\|S_{a,2}(u)\|_{L^2(2B)} \leq C_B \|S_{a,2}(u)\|_q \lesssim \|\widetilde{N}_{a,2}(u)\|_q \leq \|\widetilde{N}_{a,q}(u)\|_q < +\infty. \tag{8.8}$$

The claim (8.6), in the case $q \geq 2$, follows.

Let us turn to the proof of the claim in the case $q \leq 2$. By Poincaré's inequality,

$$\begin{aligned} T_1 &\lesssim r \Big(\frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |\nabla u(y,s)|^q \, dy \, ds \Big)^{1/q} \\ &\lesssim \Big(\iint_{W_{2a}(x,r)} |\nabla u(y,s)|^q \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{1/q}. \end{aligned}$$

Then, the use of Hölder's inequality with $\alpha = 2/q \geq 1$ gives that

$$\begin{aligned} T_1 &\lesssim \Big(\iint_{W_{2a}(x,r)} |\nabla u|^2 |u|^{q-2} \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{1/2} \Big(\iint_{W_{2a}(x,r)} |u|^q \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{(1-q/2)/q} \\ &\leq \Big(\iint_{W_{2a}(x,r)} |\nabla u|^2 |u|^{q-2} \, dy \, \frac{ds}{|s|^{n-2}} \Big)^{1/2} |\widetilde{N}_{2a,q}(u)|^{1-q/2}. \end{aligned}$$

In order to prove the claim (8.6), we will show that T_1 converges to 0 in $L^q(\mathbb{R}^d)$. We have, by Hölder's inequality again,

$$\begin{aligned}\int_{x\in\mathbb{R}^d} T_1\,dx &\lesssim \int_{x\in\mathbb{R}^d}\Big(\iint_{W_{2a}(x,r)}|\nabla u|^2\,|u|^{q-2}\,dy\,\frac{ds}{|s|^{n-2}}\Big)^{q/2}|\widetilde{N}_{2a,q}(u)|^{q(1-q/2)}\,dx\\ &\lesssim \Big\|\Big(\iint_{W_{2a}(\cdot,r)}|\nabla u|^2\,|u|^{q-2}\,dy\,\frac{ds}{|s|^{n-2}}\Big)^{1/q}\Big\|_{L^q}^{q/2}\|\widetilde{N}_{2a,q}(u)\|_q^{1-q/2}\\ &\lesssim \Big(\int_{x\in\mathbb{R}^d}\int_{|t|\le 2r}|\nabla u|^2\,|u|^{q-2}\,dx\,\frac{dt}{|t|^{n-d-2}}\Big)^{1/2}\|\widetilde{N}_{2a,q}(u)\|_q^{1-q/2}.\end{aligned}$$

The right-hand term above converges to 0 as $r\to 0$. Indeed, due to Lemma 4.1,

$$\|\widetilde{N}_{2a,q}(u)\|_q \lesssim \|\widetilde{N}_{a,q}(u)\|_q < +\infty$$

and

$$\int_{x\in\mathbb{R}^d}\int_{|t|\le 2r}|\nabla u|^2\,|u|^{q-2}\,dx\,\frac{dt}{|t|^{n-d-2}} \to 0 \quad \text{as } r\to 0$$

is an easy consequence of the fact that, thanks to Lemma 5.5, one has $\|S_{a,q}(u)\|_q \lesssim \|\widetilde{N}_{a,q}(u)\|_q < +\infty$. The claim (8.6) when $q\le 2$ follows. $\square$

Lemma 8.3. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa\ge 0$. Let $a>0$, $q\in(q_0,q_0')$, where q_0 is given by Proposition 2.1. Choose $k\ge 1$. Let $g\in L^q$ and $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution to $\mathcal{L}u=0$ that satisfies both $\|\widetilde{N}_{a,q}(u)\|_q<\infty$ and*

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|}\iint_{W_a(z,r)} u(y,s)\,dy\,ds = g(x) \quad \textit{for a.e. } x\in\mathbb{R}^d. \tag{8.9}$$

We have

$$\limsup_{\epsilon\to 0}\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\partial_r[\Psi_\epsilon^k]\,\frac{ds}{|s|^{n-d-1}}\,dy \lesssim \|g\|_q^q, \tag{8.10}$$

where the constant depends on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty+\|b\|_\infty$, κ and k.

Remark 8.4. (i) The existence of such a u is guaranteed by Lemma 7.9.

(ii) Here the solution u is only assumed to lie in $W^{1,2}_{\mathrm{loc}}(\Omega)$, so its trace may not be defined. Instead we use the assumption (8.9) to describe "$u=g$ on the boundary". In particular, if $u\in W$ is a weak solution to $\mathcal{L}u=0$ such that $\operatorname{Tr} u = g$, then u satisfies the assumption (8.9).

(iii) This lemma is an analogue and a generalization of Lemma 7.5, which only holds for energy solutions in W.

Proof. Observe first that we have the uniform bound

$$\begin{aligned}(8.11)\qquad &\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\partial_r[\Psi_\epsilon^k]\,\frac{ds}{|s|^{n-d-1}}\,dy\\ &\quad\lesssim\int_{y\in\mathbb{R}^d}\frac{1}{|W_a(y,\epsilon)|}\iint_{W_a(y,\epsilon)}|u(x,t)|^q\,dx\,dt\,dy\\ &\quad\lesssim\int_{y\in\mathbb{R}^d}|\widetilde{N}_{a,q}(u)(y)|^q\,dy<+\infty.\end{aligned}$$

By (8.9) and Lemma 8.2, we have

$$\frac{1}{|W_a(y,\epsilon)|}\iint_{W_a(y,\epsilon)}|u(x,t)|^q\,dx\,dt\to|g(y)|^q\quad\text{as }\epsilon\to0,$$

for almost every $y\in\mathbb{R}^d$. Therefore, the reverse Fatou lemma and the pointwise domination (8.11) imply that

$$\begin{aligned}&\limsup_{\epsilon\to0}\int_{y\in\mathbb{R}^d}\int_{s\in\mathbb{R}^{n-d}}|u|^q\,\partial_r[\Psi_\epsilon^k]\,\frac{ds}{|s|^{n-d-1}}\,dy\\ &\quad\lesssim\limsup_{\epsilon\to0}\int_{y\in\mathbb{R}^d}\frac{1}{W_a(y,\epsilon)}\iint_{W_a(y,\epsilon)}|u(x,t)|^q\,dx\,dt\,dy\lesssim\int_{y\in\mathbb{R}^d}|g(y)|^q\,dy.\end{aligned}$$

The lemma follows. □

Lemma 8.5. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}_\kappa^1)$ for some constant $\kappa\geq0$. Let $a>0$, $q\in(q_0,q_0')$, where q_0 is given by Proposition 2.1. For any $g\in L^q$, let $u\in W^{1,2}_{\rm loc}(\Omega)$ be a weak solution to $\mathcal{L}u=0$ that satisfies both $\|\widetilde{N}_{a,q}(u)\|_q<\infty$ and*

$$(8.12)\qquad \lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to0}}\frac{1}{|W_a(z,r)|}\iint_{W_a(z,r)}u(y,s)\,dy\,ds=g(x)\quad\textit{for a.e. }x\in\mathbb{R}^d.$$

Then

$$(8.13)\qquad \|S_{a,q}(u)\|_q^q\leq C\kappa\|\widetilde{N}_{a,q}(u)\|_q^q+C\|g\|_q^q,$$

where the constant $C>0$ depends only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$, $\|b^{-1}\|_\infty+\|b\|_\infty$.

Proof. Let χ_l be the function defined in the beginning of Section 7. By Lemma 5.3,

$$(8.14)\qquad \|S_{a,q}(u|\chi_l^k)\|_q\lesssim\|\widetilde{N}_{a,q}(u|\chi_l^{k-2})\|_q\leq\|\widetilde{N}_{a,q}(u)\|_q<+\infty.$$

We only proved Lemma 7.6 for energy solutions in W such that the trace operator is defined. But by the a priori estimate (8.14), in a similar manner, we can get

$$\begin{aligned}(8.15)\qquad c\|S_{a,q}(u|\chi_l^k)\|_q^q&\leq C\kappa\|\widetilde{N}_{q,a}(u|\chi_l^{k-2})\|_q^q+C\|g\|_q^q\\ &\quad+\frac{1}{q}\iint_\Omega\Big(\frac{|u|^q}{|t|}-\partial_r[|u|^q]\Big)\partial_r[\chi_l^k]\,\frac{dt}{|t|^{n-d-2}}\,dx.\end{aligned}$$

Note that we use Lemma 8.3 for the second term on the right. The last term can be bounded as follows:

$$\begin{aligned}
&\Big|\frac{1}{q}\iint_\Omega \Big(\frac{|u|^q}{|t|} - \partial_r[|u|^q]\Big)\partial_r[\chi_l^k]\,\frac{dt}{|t|^{n-d-2}}\,dx\Big| \\
&\quad \lesssim \int_{x\in\mathbb{R}^d} \fint_{l\leq a|t|\leq 2l} |u|^q\,dt\,dx + \int_{x\in\mathbb{R}^d}\int_{l\leq a|t|\leq 2l} |u|^{q-1}|\nabla u|\,\frac{dt}{|t|^{n-d-1}}\,dx \\
&\quad \lesssim \int_{x\in\mathbb{R}^d} \fint_{l\leq a|t|\leq 2l} |u|^q\,dt\,dx \\
&\qquad + \Big(\int_{x\in\mathbb{R}^d}\fint_{l\leq a|t|\leq 2l} |u|^q\,dt\,dx\Big)^{1/2}\Big(\iint_{(x,t)\in\Omega} |\nabla u|^2\,|u|^{q-2}\,\frac{dt}{|t|^{n-d-2}}\,dx\Big)^{1/2} \\
&\quad \lesssim \int_{x\in\mathbb{R}^d\setminus B_{l/4}(0)} |\widetilde{N}_{a,q}(u)|^q\,dx + \Big(\int_{x\in\mathbb{R}^d\setminus B_{l/4}(0)} |\widetilde{N}_{a,q}(u)|^q\,dx\Big)^{1/2}\|S_{a,q}(u)\|_q^{q/2} \to 0
\end{aligned}$$

as $l\to+\infty$. Hence the lemma follows by taking $l\to+\infty$ in (8.15). □

Lemma 8.6. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa\geq 0$. Let $a>0$, $q\in(q_0,q_0')$, where q_0 is given by Proposition 2.1. There exist two values $\kappa_0>0$ and $C>0$, both depending only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$ and $\|b^{-1}\|_\infty+\|b\|_\infty$, such that if $\kappa\leq\kappa_0$, then for any $g\in L^q$ and any weak solution $u\in W^{1,2}_{\rm loc}(\Omega)$ to $\mathcal{L}u=0$ that satisfy both $\|\widetilde{N}_{a,q}(u)\|_q<\infty$ and*

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|}\iint_{W_a(z,r)} u(y,s)\,dy\,ds = g(x) \quad \text{for a.e. } x\in\mathbb{R}^d, \tag{8.16}$$

we have $\|\widetilde{N}_{a,q}(u)\|_q\leq C\|g\|_q$.

Proof. Let $\kappa_0\leq 1$ to be fixed later and $\kappa\leq\kappa_0$. Lemmas 8.5 and 8.1 give that

$$\|\widetilde{N}_{a,q}(u)\|_q^q \leq C\kappa\|\widetilde{N}_{a,q}(u)\|_q^q + C\|g\|_q^q,$$

where the constant C does not depend on κ anymore (since $\kappa\leq 1$). We choose κ_0 such that $C\kappa_0\leq 1/2$ and the lemma follows. □

Lemma 8.7. *Let $\mathcal{L}$ be an elliptic operator satisfying $(\mathcal{H}^1_\kappa)$ for some constant $\kappa\geq 0$. Let $a>0$, $q\in(q_0,q_0')$, where q_0 is given by Proposition 2.1. There exists $\kappa_0>0$ depending only on a, q, n, $\lambda_q(\mathcal{A})$, $\|\mathcal{A}\|_\infty$ and $\|b^{-1}\|_\infty+\|b\|_\infty$ such that if $\kappa\leq\kappa_0$, then for any couple of weak solution $u_1,u_2\in W^{1,2}_{\rm loc}(\Omega)$ to $\mathcal{L}u_1=\mathcal{L}u_2=0$ that satisfy*

$$\|\widetilde{N}_{a,q}(u_1)\|_q + \|\widetilde{N}_{a,q}(u_2)\|_q < \infty$$

and

$$\begin{aligned}
&\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|}\iint_{W_a(z,r)} u_1(y,s)\,dy\,ds \\
&\qquad = \lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|}\iint_{W_a(z,r)} u_2(y,s)\,dy\,ds
\end{aligned} \tag{8.17}$$

for almost every $x\in\mathbb{R}^d$, we have $u_1=u_2$ almost everywhere in $\mathbb{R}^n$.

Proof. Set the weak solution $v = u_1 - u_2 \in W^{1,2}_{\rm loc}(\Omega)$. The function v satisfies

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} |v(y,s)|^q \, dy = 0 \quad \text{for a.e. } x \in \mathbb{R}^d$$

and

$$\|\widetilde{N}_{a,q}(v)\|_q < +\infty.$$

So if κ_0 is chosen as in Lemma 8.6, we have $\|\widetilde{N}_{a,q}(v)\|_q = 0$ and so $v \equiv 0$. The lemma follows. □

Remark 8.8. It is easy to see that the same conclusion holds if we assume, in place of (8.17), that

$$\lim_{\substack{(z,r)\in\Gamma_a(x)\\ r\to 0}} \frac{1}{|W_a(z,r)|} \iint_{W_a(z,r)} u_i(y,s) \, dy \, ds = g(x) \quad \text{for a.e. } x \in \mathbb{R}^d,$$

with $i = 1, 2$. Moreover, by Proposition 4.1, $\|\widetilde{N}_{a,q}(u_i)\|_q < +\infty$ if and only if $\|\widetilde{N}_{a,2}(u_i)\|_q < +\infty$. Therefore, we finish the proof of the uniqueness of solutions to Dirichlet problem with given boundary value g.

References

[1] Auscher, P. and Axelsson, A.: Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I. *Invent. Math.* **184** (2011), no. 1, 47–115.

[2] Auscher, P., Hofmann, S., Lacey, M., McIntosh, A. and Tchamitchian P.: The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$. *Ann. of Math. (2)* **156** (2002), no. 2, 633–654.

[3] Auscher, P., McIntosh, A. and Mourgoglou, M.: On L^2 solvability of BVPs for elliptic systems. *J. Fourier Anal. Appl.* **19** (2013), no. 3, 478–494.

[4] Axelsson, A.: Non-unique solutions to boundary value problems for non-symmetric divergence form equations. *Trans. Amer. Math. Soc.* **362** (2010), no. 2, 661–672.

[5] Barton, A. and Mayboroda, S.: Layer potentials and boundary-value problems for second order elliptic operators with data in Besov spaces. *Mem. Amer. Math. Soc.* **243** (2016), no. 1149, v+110 pp.

[6] Carbonaro, A. and Dragičević, O.: Convexity of power functions and bilinear embedding for divergence-form operators with complex coefficients. To appear in *J. Eur. Math. Soc.*, doi: `https://doi.org/10.4171/JEMS/984`.

[7] Cialdea, A. and Maz'ya, V.: Criterion for the L^p-dissipativity of second order differential operators with complex coefficients. *J. Math. Pures Appl. (9)* **84** (2005), no. 8, 1067–1100.

[8] Dahlberg, B. E. J.: Estimates of harmonic measure. *Arch. Rational Mech. Anal.* **65** (1977), no. 3, 275–288.

[9] Dahlberg, B. E. J.: On the Poisson integral for Lipschitz and C^1-domains. *Studia Math.* **66** (1979), no. 1, 13–24.

[10] David, G., Feneuil, J. and Mayboroda, S.: Elliptic theory for sets with higher co-dimensional boundaries. To appear in *Mem. Amer. Math. Soc.*

[11] David, G., Feneuil, J. and Mayboroda, S.: Dahlberg's theorem in higher co-dimension. *J. Funct. Anal.* **276** (2019), no. 9, 2731–2820.

[12] Dindoš, M., Petermichl, S. and Pipher, J.: The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function. *J. Funct. Anal.* **249** (2007), no. 2, 372–392.

[13] Dindoš, M., Petermichl, S. and Pipher, J.: BMO solvability and the A_∞ condition for second order parabolic operators. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017), no. 5, 1155–1180.

[14] Dindoš, M. and Pipher, J.: Regularity theory for solutions to second order elliptic operators with complex coefficients and the L^p Dirichlet problem. *Adv. Math.* **341** (2019), 255–298.

[15] Fabes, E., Jerison, D. and Kenig, C.: The Wiener test for degenerate elliptic equations. *Ann. Inst. Fourier (Grenoble)* **32** (1982), no. 3, 151–182.

[16] Fabes, E. B., Kenig, C. E. and Serapioni, R. P.: The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations* **7** (1982), no. 1, 77–116.

[17] Fefferman, R. A., Kenig, C. E. and Pipher, J.: The theory of weights and the Dirichlet problem for elliptic equations. *Ann. of Math. (2)* **134** (1991), no. 1, 65–124.

[18] Hofmann, S., Kenig, C., Mayboroda, S. and Pipher, J.: Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators. *J. Amer. Math. Soc.* **28** (2015), no. 2, 483–529.

[19] Hofmann, S., Mayboroda, S. and Mourgoglou, M.: Layer potentials and boundary value problems for elliptic equations with complex L^∞ coefficients satisfying the small Carleson measure norm condition. *Adv. Math.* **270** (2015), 480–564.

[20] Jerison, D. S. and Kenig, C. E.: The Dirichlet problem in nonsmooth domains. *Ann. of Math. (2)* **113** (1981), no. 2, 367–382.

[21] Kenig, C. E.: *Harmonic analysis techniques for second order elliptic boundary value problems.* CBMS Regional Conference Series in Mathematics 83, American Mathematical Society, Providence, RI, 1994.

[22] Kenig, C., Kirchheim, B., Pipher, J. and Toro, T.: Square functions and the A_∞ property of elliptic measures. *J. Geom. Anal.* **26** (2016), no. 3, 2383–2410.

[23] Kenig, C., Koch, H., Pipher, J. and Toro, T.: A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations. *Adv. Math.* **153** (2000), no. 2, 231–298.

[24] Kenig, C. E. and Pipher, J.: The Dirichlet problem for elliptic equations with drift terms. *Publ. Mat.* **45** (2001), no. 1, 199–217.

[25] Mattila, P.: *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 1995.

[26] Mayboroda, S.: The connections between Dirichlet, regularity and Neumann problems for second order elliptic operators with complex bounded measurable coefficients. *Adv. Math.* **225** (2010), no. 4, 1786–1819.

[27] Mayboroda, S. and Zhao, Z.: Square function estimates, the BMO Dirichlet problem, and absolute continuity of harmonic measure on lower-dimensional sets. *Anal. PDE* **12** (2019), no. 7, 1843–1890.

[28] Stein, E. M.: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton Mathematical Series 43, Princeton University Press, Princeton, NJ, 1993.

[29] Stein, E. M.: *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series 30, Princeton University Press, Princeton, NJ, 1970.

Received February 4, 2019. Published online October 8, 2020.

Joseph Feneuil: Department of Mathematics, Temple University, Wachman Hall, 1805 North Broad Street, Philadelphia, PA 19122, USA.
E-mail: joseph.feneuil@temple.edu

Svitlana Mayboroda: School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455, USA.
E-mail: svitlana@math.umn.edu

Zihui Zhao: Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA.
E-mail: zhaozh@uchicago.edu

The second author was supported by the Alfred P. Sloan Fellowship, the NSF INSPIRE Award DMS 1344235, NSF CAREER Award DMS 1220089, the NSF RAISE-TAQ grant DMS 1839077, and the Simons Foundation grant 563916, SM. The third author was partially supported by NSF grant numbers DMS-1361823, DMS-1500098, DMS-1664867, DMS-1902756 and the Institute for Advanced Study.