PRIMUS
Problems, Resources, and Issues in Mathematics Undergraduate Studies

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/upri20

Designing Activities to Support Prospective High School Teachers' Proofs of Congruence from a Transformation Perspective

Julia St. Goar & Yvonne Lai

PRIMUS
https://doi.org/10.1080/10511970.2021.1940403

Designing Activities to Support Prospective High School Teachers' Proofs of Congruence from a Transformation Perspective

Julia St. Goar and Yvonne Lai

ABSTRACT
Undergraduate mathematics instructors are called by many current standards to promote prospective teachers' learning of geometry from a transformation perspective, marking a change from previous standards. The novelty of this situation means it is unclear what is involved in undergraduate learning and teaching of geometry from a transformation perspective. To approach this problem, we illustrate how specific in-class activities and design principles might help prospective teachers make conceptual links between congruence proofs and a transformation approach to geometry. Additionally, to illustrate these activities for instructors, we provide examples of prospective teachers' work on some of these problems.

KEYWORDS
Geometry; transformations; secondary teacher education; classroom applications

1. INTRODUCTION

Instructors of secondary teacher preparation programs face a transition in geometry instruction. In the past several decades, geometry has been taught primarily from a perspective based on Euclid's *Elements* [12]. More recently, a *transformation perspective* has come to the fore [9, 8]. Thus, many K–12 teachers may have to teach geometry from a different perspective from the one they learned. Consequently, college geometry instructors will need to support teachers' transition to a new perspective. Unfortunately, at any level, there has been "limited research explicitly on the topics of congruency and similarity, and little on transformation geometry" ([6], p. 139).

To illustrate the pedagogical impact of the perspective, consider the well-known triangle congruence criterion "Angle-Side-Angle" (ASA): *For all* ΔABC *and* ΔDEF *such that* $\overline{AB} \cong \overline{ED}, \measuredangle BAC \cong \measuredangle EDF$, *and* $\measuredangle ABC \cong \measuredangle DEF$, *we have* $\Delta ABC \cong \Delta DEF$. In secondary and college geometry texts using an *Elements* approach, this criterion is often taken as a postulate: a mathematical truth without proof. Typically, instructors help future teachers establish conviction in such postulates through empirical exploration, such as constructing pairs of triangles satisfying the congruence criterion and measuring the remaining side lengths and

CONTACT Julia St. Goar stgoarj@merrimack.edu Department of Mathematics, Merrimack College, 315 Turnpike Street, North Andover, MA 01845, USA.

angles. In contrast, ASA is often a theorem in the transformation approach (e.g., [2,16]). To show that ΔABC and ΔDEF are congruent, one must show that no matter the triangles' locations, there exists a sequence of rigid motions that map the triangles to each other (see Wu [16] for a schematic for such a proof). Even if empirical exploration is beneficial, it is insufficient for this triangle congruence theorem in a transformation context. Moreover, strictly empirical exploration can undermine the development of deductive schema (e.g., [4]). An instructor must thus help future teachers move toward deductive proof. In an *Elements* approach when taking ASA to be a postulate, a proof would be mathematically impossible.

We note that ASA *does not have to be a postulate*; it is just simply taken as one in various sources (e.g., [1,3,7,11]). As Venema [15] notes, taking at least one triangle congruence criterion (e.g., SSS, ASA, or SAS) as a postulate is necessary in an *Elements* approach. Whether it is ASA or another triangle congruence criterion, one of them must be taken as a postulate.

In the current transition from an *Elements* approach to a transformation approach, some prospective teachers (as well as practicing teachers) may be unfamiliar with what can (or cannot) be proven and how proofs operate. This situation informs our agenda: *How can we better understand prospective teachers' thinking and work on transformation congruence proofs so that we can become more reflective and adaptable geometry instructors? What design principles for in-class college geometry activities could support prospective teachers' understanding of congruence proofs from a transformation perspective? On what basis do we (continue to) make improvements to these activities?*

We focus on congruence because it is a fundamental and relatively unexamined area where differences between *Elements* and transformation approaches are salient [6,15]. To our knowledge, even in the existing studies (e.g., [5]), there are few results on teachers' understandings of congruence *proofs*, particularly of figures beyond basic triangle congruence proofs.

In this paper, we first discuss the basic structure of transformation congruence proofs. We then describe design principles, based on our previous work [13], for in-class activities and for possible strategies spanning multiple lesson plans. Finally, we discuss the potential impact of these strategies as well as a deeper dive into prospective teachers' understanding of congruence proofs.

2. WHAT IS A TRANSFORMATION APPROACH TO GEOMETRY?

Following Usiskin and Coxford [14], we take a *transformation approach* to (planar) geometry as one that includes:

- Postulating preservation properties of transformations: in particular, that reflections, rotations, and translations preserve length and angle measure;

- Defining congruence via transformations: e.g., two subsets X and Y of the plane (e.g., two triangles or circles) are congruent if there exists a reflection, rotation, translation, or sequence of these transformations[1], that map X to Y;

Details may differ across texts, for instance, in postulates of transformations taken, such as whether length and angle, or only length are assumed to be preserved by transformations. Nonetheless, they have in common that the postulates are about transformations, rather than about congruence criteria for particular objects such as triangles. Hence, from a transformation perspective:

- [**Transformations-to-Congruence**] To establish a proof of congruence of two objects in the plane, one constructs a sequence of assertions that show that there exists a single one of or a sequence of reflections, rotations, or translations that maps one object to the other; and
- [**Congruence-to-Transformations**] When two objects are congruent, the transformation perspective provides that there then exists a single one of or a sequence of reflections, rotations, or translations that maps the first object to the other.

We emphasize and name these statements, "Transformation-to-Congruence" (a sequence of transformations is used to establish congruence of two figures) and "Congruence-to-Transformations" (a congruence implies the existence of a sequence of transformations mapping one figure to the other), for mathematical and pedagogical reasons. First, mathematically, they unpack the two directions of the definition of congruence from a transformation approach, when the definition is taken as an if-and-only-if statement. Second, our experiences teaching geometry suggest that prospective teachers understand these directions differently. Sometimes teachers can use one direction in their proofs, but not the other, and vice versa [13].

Finally, in a transformation approach, a sequence of transformations used to establish congruence of two figures must map one *entire* figure to the other *entire* figure. For instance, if we want to show that the union of a pair of intersecting triangles $\Delta ABC \cup \Delta AOB$ is congruent to a second union of two triangles $\Delta DEF \cup \Delta DPE$, then it would not be enough to show that, say, $\Delta ABC \cong \Delta DEF$ and $\Delta AOB \cong \Delta DPE$, by using one sequence of transformations to map ΔABC to ΔDEF and a second, non-equivalent sequence of transformations to map ΔAOB to ΔDPE. We would need to show that there is a single sequence of transformations that carry $\Delta ABC \cup \Delta AOB$ to $\Delta DEF \cup \Delta DPE$. We point this out for mathematical and pedagogical reasons. Mathematically, it points to an advantage of geometry from a transformation approach, that we can now work with more complex figures than two single triangles or circles. Pedagogically, we have found that we need to reinforce this idea over time when we are teaching. The idea of focusing on a figure

[1] Note that glide reflections can be expressed as compositions of reflections and translations.

holistically, rather than focusing on parts of figures in isolation, is something that is often new to teachers in our courses.

3. DESIGN PRINCIPLES FOR GEOMETRY PROOF ACTIVITIES

In this section, we share the design principles that we have begun using in our classes after several iterations of teaching this material. These design principles come from our understanding of the transformation approach combined with our experience teaching geometry courses. The first author has taught this material three times, and the second author has taught this material four times. We then discuss how these principles shaped our within-lesson and across-lesson plans. Then, in Section 4, we illustrate, by example, some patterns in teachers' understandings based on the design principles. Finally, we conclude in Section 5 how we believe our materials and teaching may have impacted teachers' understandings and takeaways for other instructors.

3.1. Design Principles and Rubric

Our design principles for teaching geometry from a transformation perspective are to focus teachers' attention on:

- Map the ENTIRE figure: Mapping the *entire* first figure to the *entire* second figure
- Say WHY: Saying *why* their work maps the *entire* first figure to the *entire* second figure
- Say HOW: Saying *how* they used the definition of congruence.

Moreover:

- During class discussion and feedback, we also help teachers pay attention to how they used the Transformation-to-Congruence or Congruence-to-Transformation directions of the definition of congruence.

One of the most useful things we have done for ourselves as instructors is to turn these principles into a rubric. Figure 1 shows the relationship between this rubric and our design principles. We have found that this rubric helps us communicate with future teachers about the mathematics, improves our grading, and clarifies for ourselves what to emphasize in feedback, whether as a part of grading or during class discussions.

Both authors used a rubric similar to the one shown in Figure 1, although the rubric is most similar to the one used by the second author. We find it helpful for more accurately and quickly assessing future teachers' work. The second author shared the rubric with their prospective teachers as yet another tool to make the structure of transformation congruence proofs more explicit. An added benefit of sharing the rubric with prospective teachers is the possibility of having them grade each other's work as an in-class activity.

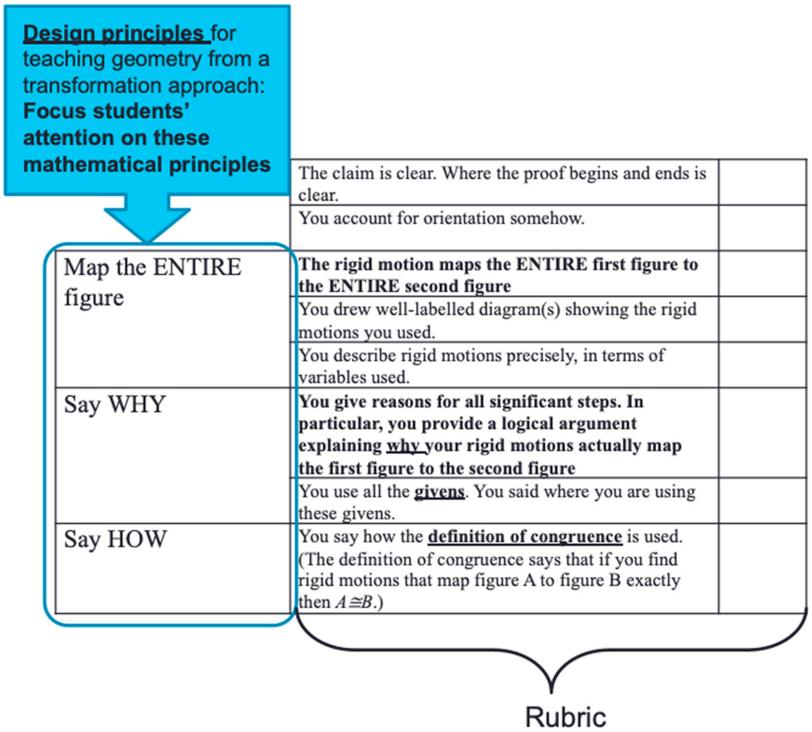


Figure 1. The column on the right-hand side is a rubric similar to one the second author used in their class. The column on the left-hand side indicates the rubric's correspondence to our design principles.

3.2. Within-Lesson Plan Design

The first author designed new worksheets and homework assignments to support the design principles. We share one example of these worksheets in Figure 2. Solutions are in the appendix. The first author used this worksheet to help teachers understand the Map-the-ENTIRE figure principle, the Say-WHY principle, and the Say-HOW principle, including both Transformation-to-Congruence and Congruence-to-Transformation.

In the worksheet section Applications of Congruence, Problems 1 and 2 contrast the use of Congruence-to-Transformation (Problem 2) with Transformations-to-Congruence (Problem 1). Problem 3 and 5a further explore the use of Transformations-to-Congruence, as well as begin to approach the Map-the-ENTIRE-figure principle. The tasks are designed to position teachers to realize they must apply their sequence of rigid motions to the entire union and not just to the parts. Placing these concepts side-by-side brings to life the multiple uses and subtleties of the definition of congruence.

The worksheet then transitions to the section Applying Congruence Definition in Proofs for Non-Triangles. This section features three non-standard proofs, the first two of which were originally created by the second author, that offer teachers the opportunity not only to practice applications of the definition of congruence but also encounter all of the design principles. Such proofs with non-standard

Page 1
Applications of congruence:
Use the definition of congruence to answer the following.

1.) Say r is a sequence of rigid motions and that $r(\Delta ABC) = \Delta DEF$. Then we can conclude____
2.) Say quadrilateral $ABCD$ is congruent to quadrilateral $EFGH$. Then we can construct ____
3.) I want to show that triangle $\Delta ABC \cup D$ is congruent to $\Delta EFG \cup H$ where D is a point on the interior of ΔABC and H is a point on the interior of ΔEFG. In order to show that these figures are congruent, what do I need to show? I.e. what is my goal?
4.) Suppose triangle ΔABC is congruent to triangle ΔDEF.
   a. Why do I know then that $\overline{AB}$ is congruent to $\overline{DE}$?
   b. Why do I know that $\measuredangle ABC$ is congruent to $\measuredangle DEF$?
5.) Trickier: Suppose I am trying to show that $\Delta ABC \cup \Delta ABO \cong \Delta DEF \cup \Delta DEM$.
   a. What do I need to show to finish the proof? What is my goal?
   b. Suppose I am in the middle of the proof and I have already succeeded in showing that $f(\Delta ABC) = \Delta DEF$ for some sequence of rigid motions f, and I know that ΔABO is congruent to ΔDEM. Is it true that $f(\Delta ABO) = \Delta DEM$? If yes, why? If no, why and state what I do know instead.

Page 2
Applying congruence definition in proofs for non-triangles:
Note: You may assume that any two lines and any two rays are congruent.

1.) Let ℓ, m be lines. Among all the points that are a unit distance from ℓ, choose one point P. Among all the points that are a unit distance from m, choose one point Q. Prove that no matter what points P and Q you chose, it is always true that $\ell \cup P \cong m \cup Q$.
2.) Suppose $ABCD$ and $EFGH$ are rectangles with the same dimensions. That is suppose assume all angles in each rectangle are 90 degrees, and assume $\overline{AB} \cong \overline{CD} \cong \overline{EF} \cong \overline{GH}$ and $\overline{BC} \cong \overline{DA} \cong \overline{FG} \cong \overline{HE}$. Show $ABCD \cong EFGH$.
3.) Consider figures $ABCD \cup \Delta ABO$ and $EFGH \cup \Delta EFM$, where $ABCD$ and $EFGH$ are rectangles, O is a point in the interior of $ABCD$ and M is a point in the interior of $EFGH$. Suppose further that $ABCD$ and $EFGH$ have the same dimensions, $\overline{AB} \cong \overline{EF}$, $\measuredangle ABO \cong \measuredangle EFM$, and $\overline{BO} \cong \overline{FM}$. Show $ABCD \cup \Delta ABO \cong EFGH \cup \Delta EFM$.

Figure 2. The above is a sample worksheet that supports the design principles.

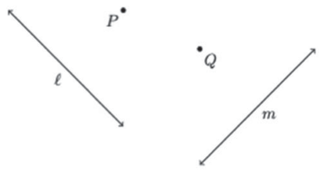
and non-triangular figures offer an opportunity for teachers to grasp the definition of congruence more broadly and offer further opportunities to repeatedly revisit contexts that require all of the design principles.

3.3. Across-Lesson Plan Design: Sequencing Figures

In our experience, the key aspects in the construction of a transformation congruence proof require multiple experiences over time for future teachers to grasp them. Therefore, it is advantageous to engage teachers in many examples over time, both in class and in homework assignments. In order to have ample fodder for such examples, it may be necessary for instructors to go beyond the stereotypical triangle congruence proofs (e.g., Angle-Side-Angle or Side-Angle-Side) and have teachers work on proofs with non-standard figures or other shapes such as unions of a line and a point (Figure 3), unions of triangles (Figure 4), the union of a rectangle and a triangle, or rectangles (Figure 2).

One possible way to approach transformation congruence proofs is to begin with a basic figure that may be used as a scaffold for additional proofs. For instance, the second author asked whether the claim that "line segments of equal length are

Line-Point Task:



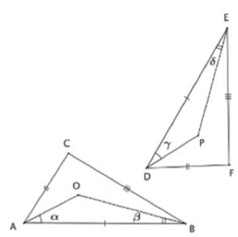
Say that P is one unit from line ℓ, and point Q is one unit from line m. Show that $\overline{PR}\cup\ell \cong \overline{QS}\cup m$.

Possible Proof:

Draw segments $\overline{PR}$ and $\overline{QS}$ such that R is on ℓ, S is on m, $m(\overline{PR}) = 1$, and $m(\overline{QS}) = 1$. Draw $\overline{RS}$, and reflect over the perpendicular bisector of $\overline{RS}$ to create $\overline{R'P'}\cup\ell'$. Because we know that when lines are reflected over a perpendicular bisector they land on themselves, $\overleftrightarrow{RS} = \overleftrightarrow{S'R'}$. By line-segment-on-ray, $R' = S$. Create the angle bisector $\overrightarrow{SV}$ of $\angle QSP'$, and reflect over that bisector. Because $\overrightarrow{SV}$ is fixed in the second reflection above, $\angle QSV = \angle VSP''$, and by ray-on-directed-angle, it holds that $(\overrightarrow{SP''}) = \overrightarrow{SQ}$. Because $\overline{SP''} \cong \overline{SQ}$ and by line-segment-on-ray, $P'' = Q$. Draw points G on ℓ and K on m so that G'' and K both lie on the same side of $\overleftrightarrow{QS}$. Because $(\angle PRG)'' \cong \angle QSK$ and by ray-on-directed-angle, $(\overrightarrow{R''G''}) = \overrightarrow{SK}$. Therefore $\ell = m$. As a result $(\overline{PR}\cup\ell)'' = \overline{QS}\cup m$. Therefore by the definition of congruence, $\overline{PR}\cup\ell \cong \overline{QS}\cup m$.

Figure 3. The above illustrates the idea of Transformations-to-Congruence in a possible proof of the line-point task, which was created by the second author.

Boomerang Task:



Show that the two figures above are congruent as marked.

Possible Proof:

By SSS, there exists a sequence of rigid motions r such that $r(\triangle ABC) = \triangle DEF$. By ray-on-directed-angle, $r\left(\overrightarrow{AO}\right) = \overrightarrow{DP}$ and $r\left(\overrightarrow{BO}\right) = \overrightarrow{EP}$. Note that $\overrightarrow{DP}$ and $\overrightarrow{EP}$ have both P and $r(O)$ in common. Because two distinct rays can intersect at only one point $r(O) = P$. So $r(\triangle AOB) = \triangle DPE$. So $r(\triangle ABC \cup \triangle AOB) = \triangle DEF \cup \triangle DPE$. Therefore $\triangle ABC\cup\triangle AOB \cong \triangle DEF \cup \triangle DPE$.

Figure 4. The above illustrates both Transformations-to-Congruence and Congruence-to-Transformation in a possible proof of the boomerang task, which was created by the second author.

congruent" requires proof from a transformation perspective (it does).[2] Then as a class, teachers constructed a proof via transformations. Similarly, the first author began with a proof of the fact that vertical angles are congruent.

[2] This prompt is from the Park City Mathematics Institute [10] geometry materials.

Then, using the equal line segment activity as a scaffold, teachers in the second author's class discussed how this proof might be a foothold for a transformation-based proof for the Leg-Leg (LL) right triangle congruence criterion (If two legs of one right triangle are congruent to two legs of a second right triangle, then the two triangles are congruent). The equal length segment proof helped make visible the structure of a congruence proof, which then organized a discussion of approaches to a proof of LL.

LL proofs then scaffolded proofs of the "side-angle-side" (SAS) and "angle-side-angle" (ASA) triangle congruence criteria, and supported the Say-WHY principle. Class discussion focused on how teachers know where the images of vertices must be, and why the image of the entire triangle is the same as the other entire triangle. These proofs then scaffolded even more complex problems, such as those involving unions of a line and a point or unions of triangles, shown in Figures 3 and 4.

3.4. Across-Lesson Plan Design: Sequencing Proof Approaches

The above sequence of proofs helped teachers understand two types of proof approaches to geometry from a transformation approach. We call these approaches:

- "from scratch" and
- "building on a known congruence".

These approaches correspond to the ideas of Transformations-to-Congruence and Congruence-to-Transformations, respectively.

In the "from scratch" approach, one explicitly constructs rigid motions that map one figure to the other. This approach is shown in Figure 3. In the "building on a known congruence" approach, one uses a known congruence to suppose the existence of a sequence of transformations. For example, the equal length segment claim implies that there exists a rigid motion that maps one segment to a corresponding segment. From there, one can show how this rigid motion can be combined with others to map one figure exactly to the other. This approach can be seen in Figure 4. Once teachers understand the full potential of Congruence-to-Transformations, they gain further insight into the definition of congruence and this understanding opens up a wider array of possible congruence proofs. This wider scope of examples enriches class discussions and bolsters understanding of congruence proofs overall.

The authors note however that though they saw advantages in their course to the use of Congruence-to-Transformation in constructing rigid motions, it may be useful to require teachers to use explicit constructions earlier in the course. This claim is backed by the last iteration of the second author's teaching, which spent more time early on focused on explicit constructions than the first author's course or any previous iteration of the second author's geometry course. In this last iteration, unlike any previous one, and unlike the first author's course, almost all future teachers showed understanding of the Map-the-ENTIRE-figure principle on homework and exams. This is remarkable because these same future teachers had not shown this understanding in the first class's activities; it was an understanding that was built over time. Meanwhile, in the second iteration of the first author's course, where teachers abstractly constructed rigid motions using Congruence-to-Transformation earlier in the course but who also worked on activities such as those described in Section 3.2, many teachers showed evidence of the Say-WHY principle but several teachers struggled significantly with the Map-the-ENTIRE principle. The authors conclude that a combination of both approaches, requiring explicit constructions early on but diving into Congruence-to-Transformation and activities such as that in Section 3.2, may be beneficial.

4. FUTURE TEACHERS' UNDERSTANDINGS

Knowledge of the design principles can help us to better understand future teachers' work and can offer constructive insight into teachers' brilliance and struggles. We offer examples of teachers' work here that can be illuminated in the light of the design principles.

The tasks we use to illustrate these are the Line-Point task (Figure 3) and the Boomerang task (Figure 4). The solutions to each represent a possible correct solution that a teacher could produce. As discussed earlier, in the Line-Point task, the solution used the "from scratch" approach, with explicitly described rigid motions. Meanwhile in the Boomerang task, the solution was a "building on known congruence" and made use of Congruence-to-Transformation. For the purposes of clarity, portions of each proof that is underlined with a solid line indicate where the Map-the-ENTIRE-figure principle is being used. Meanwhile, portions of the proofs underlined with a dotted line indicate where the Say-WHY principle is being used.

We note that in each of our respective courses in question in this article, we had taught and prospective teachers had made use of these properties, which we asked teachers to take as postulates[3]:

- Ray-on-directed angle (RODA): Given $\vec{r}$, if $\vec{r}, \vec{s}, \vec{t}$ are all in the same plane and if $\vec{s}$ and $\vec{t}$ both form an angle of the same directed angle measure with $\vec{r}$, then $\vec{s} = \vec{t}$.
- Line-segment-on-ray (LSOR): Given $\vec{r}$ with endpoint P, if Q and R are both on $\vec{r}$ and $\overline{PQ} \cong \overline{PR}$, then $Q = R$. (Note that this property could also be stated by saying $\overline{PQ}$ and $\overline{PR}$ have equal length.)

Certainly, the list above does not constitute a comprehensive list of the postulates, properties and assumptions prospective teachers used in our courses, but they are particularly salient here given that teachers frequently made use of these in the examples shown throughout this paper. For consistency, we also incorporated these properties into our proof illustrations in the line-point and boomerang tasks. However, we are by no means claiming that instructors could not successfully make use

[3] Although it may be possible to prove these properties in some axiomatic systems for geometry from a transformation approach, we elected not to do so, in order to focus teachers' attention on the problem-solving aspects of working with complex figures such as in the Line-Point and Boomerang Tasks.

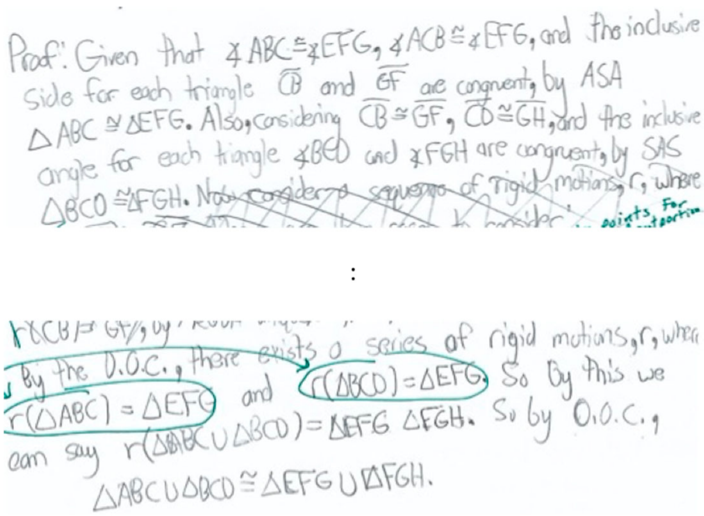


Figure 5. The above shows a future teacher's full attempt (with crossed out portions omitted) at the boomerang problem. Marks in green are a portion of the comments written by the instructor. The abbreviation "D.O.C." appears to refer to the definition of congruence.

of different postulates and properties in their own courses to serve as foundations for proof writing.

Figure 5 shows an interesting attempt at the boomerang problem that indicates that the future teacher does not show evidence of the Map-the-ENTIRE-figure principle. The teacher attempted to use two different statements of congruence to conclude the existence of a single sequence of rigid motions. As we know, it is possible that Congruence-to-Transformation may lead to a *different* rigid motion for each given sequence of rigid motions. As a result, the teacher shows a misunderstanding of the applications of Congruence-to-Transformation and does not show evidence of the Map-the-ENTIRE-figure principle because the teacher attempted to treat multiple potentially different sequences of rigid motions as if they were the same sequence.

Future teachers' work like that shown in Figure 5 convinced both authors in future iterations of their courses to insist that future teachers complete early congruence proofs in the course using explicit constructions of rigid motions (that is spelling out exactly which translations, rotations, and reflections they are using), prior to allowing them to use Congruence-to-Transformation to conclude the existence of rigid motions. For example, the authors plan to have future teachers prove side-angle-side and angle-side-angle using only explicit rigid motions.

Meanwhile in Figure 6, we see a future teachers' attempt at the line point problem. Note that this teacher succeeds in explicitly constructing a sequence of rigid motions. However, the teacher ends the proof shortly thereafter without a deductive argument explaining *why* the image of one figure under the sequence of rigid motions is exactly the other. As a result, this teacher does not demonstrate evidence of the Say-WHY principle. Similarly, other teachers included a statement that rigid motions preserve distance as justification for ending the proof, and anecdotally in

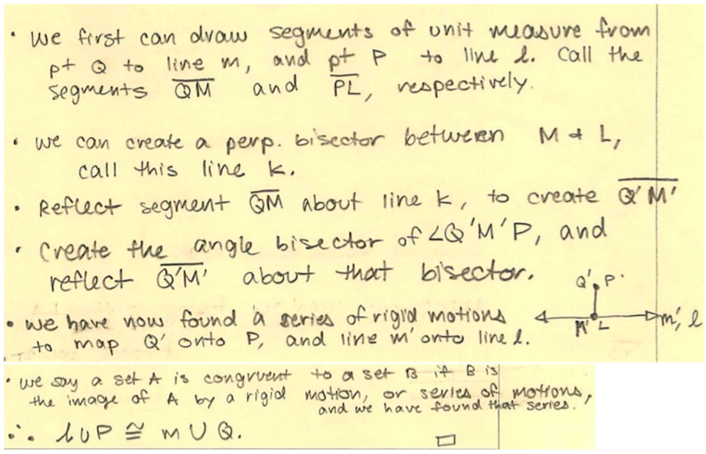


Figure 6. The above shows a future teacher's full attempt at the line-point problem.

class, both authors noticed prospective teachers making this statement in place of a deductive argument during group work.

Readers should note that Figure 6 represents a more extreme example of a teacher not showing evidence of the Say-WHY principle. We also saw instances of teachers who attempted to continue the proof after constructing the rigid motion but who stopped short of fully demonstrating evidence of the Say-WHY principle. In such cases, teachers may inappropriately imitate older proofs or make a series of statements that do not constitute deductive logic. We hypothesize that such prospective teachers do not truly grasp the purpose of this portion of the proof.

This issue has been the source of an on-going discussion among the authors about how best to communicate the need for proof. One tactic used by both authors is creating sketches where the figures are not superimposed. Of course, to create these sketches one must distort the figures, and the authors hypothesize that not all teachers may initially understand the purpose of these warped diagrams, since they appear to contradict what teachers may view as "obvious." Activities like those shown in Figure 2 and proofs involving figures more complex than triangles already appear to help teachers with the Say-WHY principle in the authors' experience. Additionally, it seems clear that the Say-WHY principle needs repeated reinforcement among the teachers over time. As a result, the authors have been working to prompt teachers frequently and consistently over time about why teachers know their figures are truly superimposed or why they know the diagrams they drew really represent the situation in all cases. The activities and proofs described in this paper are all opportunities for this repeated talking point.

5. CONCLUSION

Our impression based on homework and exam performances, as well as in-class contributions, is that after incorporating the design principles (including the Map-the-ENTIRE-figure, Say-WHY, and Say-HOW principles), as well as the activities and lessons described in this paper into our courses, prospective teachers demonstrated more willingness to use rigid motions in their proofs of congruence and to connect rigid motions to congruence explicitly. We saw more explicit use during in-class discussions of teachers citing congruence results to provide the existence of a rigid motion to begin a proof, as well as more teachers concluding their proofs establishing congruence citing the definition of congruence. We interpret this as the potential success of these materials in supporting teachers' development of the Congruence-to-Transformation and Transformations-to-Congruence actions. For the second author, the scaffolding using the equal length segment result seemed particularly crucial to teachers' understanding, and well after this proof was constructed, teachers would refer to it productively and without prompting.

The first author found that the handout (see Figure 2) helped teachers apply Congruence-to-Transformation and Transformations-to-Congruence in congruence proofs. The handout helped teachers by providing the first and final steps of congruence proofs. The first author emphasized to teachers throughout the semester that they should write these items down early in the proof-writing process. Problem 5b in particular generated a lot of discussion and debate among the groups in the class. The problem asks if a sequence of rigid motions, f, that carries ΔABC to ΔDEF would also apply to congruent triangles ΔABO and ΔDEM. Several teachers initially answered "yes" to this question. However, the class refuted this in two different ways. Some generated examples where $\Delta ABC \cup \Delta ABO$ and $\Delta DEF \cup \Delta DEM$ may not be congruent. Others discussed the fact that while there does exist some sequence of transformations carrying ΔABO to ΔDEM, this sequence may not be the same as the sequence f and would need to be assigned a different name. Teachers who grasped these arguments gained additional understanding of Transformations-to-Congruence and progress towards the Map-the-ENTIRE-figure principle.

The three problems on the second page of the handout in Figure 2 helped with all aspects of the congruence proof-writing process through non-standard problems. On the rectangle proof (#2), many teachers in the room took the approach of beginning the proof by defining a transformation r taking one segment of the rectangle to another segment and then reflecting over this segment as-needed. This strategy resulted in a large portion of the proof being dedicated to progressively extending the transformation to the three other sides of the rectangle. The teachers found this repetitive process enlightening, with several professing "light-bulb moments" on understanding the process of showing deductively that a transformation extends to an entire figure, thus contributing toward the Say-WHY principle.

In the second author's experience, sequencing figures and sequencing proof approaches, in combination with explicit discussion of Transformations-to-Congruence and Congruence-to-Transformations and careful work with the notation to distinguish the image from preimages, was critical to teachers' understanding and construction of congruence proofs from a transformation approach. The first time she taught this course, she did not emphasize Transformations-to-Congruence and Congruence-to-Transformations, nor did she pose any questions of whether a statement needed proof. Teachers in this course were confused as to how proofs from a transformation perspective differed from their previous experiences with an *Elements* approach, and frequently answered homework questions using an *Elements* approach. Then, after problematizing the need for proof (or not, and from which approach), teachers engaged more willingly with proofs with transformations and often generated multiple and beautiful proof approaches that the second author had not even anticipated. Class discussions were lively, but proofs in class, in homework, and on exams sometimes showed a lack of attention to the Map-the-ENTIRE-figure principle. Finally, bringing in sequencing figures, sequencing proof approaches, and notational attention, almost all teachers in the final exam showed understanding of the Map-the-ENTIRE-figure principle as well as flexibility with different proof approaches.

Both authors found that having prospective teachers work on proofs of the congruence of non-standard figures offered teachers the opportunity not only to practice applications of the definition of congruence but also frequently encounter the design principles. Such proofs with non-standard and non-triangular figures offer an opportunity for teachers to grasp the definition of congruence more broadly and offer further opportunities to repeatedly revisit contexts that require all of the design principles.

In this paper, we have provided principles and examples, based on our experiences teaching as well as our previous research into teachers' understanding of congruence proofs [13], to help provide instructors further insight into the work and thinking of prospective teachers on transformation congruence proof. Additionally, we have illustrated how this insight has begun to impact our own teaching practice. Changes in our practice have included offering prospective teachers more opportunities to practice transformation congruence proof by leveraging congruent non-standard shapes, providing rubrics that help offer teachers a more explicit illustration of the structure of transformation congruence proofs, providing an example of an activity that allows teachers to practice various aspects of the structure of congruence proofs, and using congruence proofs of simpler figures to provide scaffolding for proofs involving more complex congruent figures. It is our hope that these examples of prospective teachers' work and of possible changes in teaching practice will provide immediate suggestions for concrete changes in classrooms. However, we also hope readers will be inspired to further the work of adapting instruction, of creating activities, and of re-designing lesson plans to better reflect and reinforce prospective teachers' understanding of transformation congruence proof.

ACKNOWLEDGEMENTS

The authors are grateful to Rachel Funk for insight into the data and for feedback on task design. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

DISCLOSURE STATEMENT

No potential conflict of interest was reported by the author(s).

FUNDING

This work is partially supported by Division of Undergraduate Education [10.13039/100000179NSF DUE-1937512 and NSF DUE-1726744].

APPENDIX

This appendix includes possible solutions to the handout questions posed in Figure 2. Note that other solutions may be possible; these are merely included as a convenient reference for the reader.

Page 1

Applications of congruence:

Use the *definition of congruence* to answer the following.

1.) Say r is a sequence of rigid motions and that $r(\Delta ABC) = \Delta DEF$. Then we can conclude $\underline{\Delta ABC \cong \Delta DEF.}$

2.) Say quadrilateral $ABCD$ is congruent to quadrilateral $EFGH$. Then we can construct *a sequence of rigid motions, r, such that* $r(ABCD) = EFGH$.

3.) I want to show that triangle $\Delta ABC \cup D$ is congruent to $\Delta EFG \cup H$ where D is a point on the interior of ΔABC and H is a point on the interior of ΔEFG. In order to show that these figures are congruent, what do I need to show? i.e. what is my goal?

Show that there is a sequence of rigid motions, f, such that $f(\Delta ABC \cup D) = \Delta EFG \cup H$.

4.) Suppose triangle ΔABC is congruent to triangle ΔDEF.

a. Why do I know then that $\overline{AB}$ is congruent to $\overline{DE}$?

Because $\Delta ABC \cong \Delta DEF$, *there exists a sequence of rigid motions, g, such that* $g(\Delta ABC) = \Delta DEF$. *Then* $g(\overline{AB}) = \overline{DE}$. *By the definition of congruence,* $\overline{AB} \cong \overline{DE}$.

b. Why do I know that $\angle ABC$ is congruent to $\angle DEF$?

Because $\Delta ABC \cong \Delta DEF$, *there exists a sequence of rigid motions, g, such that* $g(\Delta ABC) = \Delta DEF$. *Then,* $g(A) = D, g(B) = E$, *and* $g(C) = F$. *This implies that* $g(\overrightarrow{BA}) = \overrightarrow{ED}$ *and* $g(\overrightarrow{BC}) = \overrightarrow{EF}$. *Therefore,* $g(\angle ABC) = \angle DEF$. *By the definition of congruence,* $\angle ABC \cong \angle DEF$.

5.) Trickier: Suppose I am trying to show that $\Delta ABC \cup \Delta ABO \cong \Delta DEF \cup \Delta DEM$.

a. What do I need to show to finish the proof? What is my goal?

Show that there exists a sequence of rigid motions, r, such that $r(\Delta ABC \cup \Delta ABO) = \Delta DEF \cup \Delta DEM$.

b. Suppose I am in the middle of the proof and I have already succeeded in showing that $f(\Delta ABC) = \Delta DEF$ for some sequence of rigid motions f, and I know that ΔABO is congruent to ΔDEM. Is it true that $f(\Delta ABO) = \Delta DEM$? If yes, why? If no, why and state what I do know instead.

Two possible types of answers:

(1) *Because $\triangle ABO \cong \triangle DEM$ by the definition of congruence, there exists a sequence of rigid motions, r, such that $r(\triangle ABO) = \triangle DEM$. However, the definition of congruence does NOT necessarily imply that $r = f$.*

(2) *It is possible to draw $\triangle ABC \cup \triangle ABO$ and $\triangle DEF \cup \triangle DEM$ are not congruent. One way to do this is to place point O on the interior of $\triangle ABC$, and draw point M so that it is not on the interior of $\triangle DEF$ because M and $\triangle DEF$ lie on opposite sides of $\overline{DE}$. Then f could not apply to the entire union.*

Page 2

Applying congruence definition in proofs for non-triangles:

Note: You may assume that any two lines and any two rays are congruent.

1.) Let ℓ, m be lines. Among all the points that are a unit distance from ℓ, choose one point P. Among all the points that are a unit distance from m, choose one point Q. Prove that no matter what points P and Q you chose, it is always true that $\ell \cup P \cong m \cup Q$.

See Figure 3 for the solution to this problem.

2.) Suppose $ABCD$ and $EFGH$ are rectangles with the same dimensions. That is suppose assume all angles in each rectangle are 90 degrees, and assume $\overline{AB} \cong \overline{CD} \cong \overline{EF} \cong \overline{GH}$ and $\overline{BC} \cong \overline{DA} \cong \overline{FG} \cong \overline{HE}$. Show $ABCD \cong EFGH$.

Because $\overline{AB} \cong \overline{EF}$, by the definition of congruence there exists a sequence of rigid motions, f, such that $f(\overline{AB}) = \overline{EF}$. Because $\measuredangle DAB \cong \measuredangle HEF$, $\measuredangle ABC \cong \measuredangle EFG$, and by ray-on-directed-angle, we can conclude that $f(\overrightarrow{AD}) = \overrightarrow{EH}$ and $f(\overrightarrow{BC}) = \overrightarrow{FG}$. Because $\overline{AD} \cong \overline{EH}$, $\overline{BC} \cong \overline{FG}$, and by line-segment-on-ray, we can conclude that $f(D) = H$ and $f(C) = G$. Then $f(ABCD) = EFGH$. Therefore by the definition of congruence, $ABCD \cong EFGH$.

3.) Consider figures $ABCD \cup \triangle ABO$ and $EFGH \cup \triangle EFM$, where $ABCD$ and $EFGH$ are rectangles, O is a point in the interior of $ABCD$ and M is a point in the interior of $EFGH$. Suppose further that $ABCD$ and $EFGH$ have the same dimensions, $\overline{AB} \cong \overline{EF}$, $\measuredangle ABO \cong \measuredangle EFM$, and $\overline{BO} \cong \overline{FM}$. Show $ABCD \cup \triangle ABO \cong EFGH \cup \triangle EFM$.

By #2, $ABCD \cong EFGH$. Then by the definition of congruence, there exists a sequence of rigid motions, g, such that $g(ABCD) = EFGH$. Because $\measuredangle ABO \cong \measuredangle EFM$ and by ray-on-directed angle, we can conclude that $g(\overrightarrow{BO}) = \overrightarrow{FM}$. Because $\overline{BO} \cong \overline{FM}$ and by line-segment-on-ray, we can conclude that $g(O) = M$. Then $g(ABCD \cup \triangle ABO) = EFGH \cup \triangle EFM$, so by the definition of congruence, $ABCD \cup \triangle ABO \cong EFGH \cup \triangle EFM$.

ORCID

Julia St. Goar http://orcid.org/0000-0001-6980-4611
Yvonne Lai http://orcid.org/0000-0002-2547-9914

REFERENCES

[1] Boyd, C. J., J. Cummins, C. Malloy, J. Carter, and A. Flores. 2005. *Glencoe Mathematics: Geometry: Teacher Wraparound Edition*. New York, NY, Columbus, OH, Chicago, IL, Peoria, IL, Woodland Hills, CA: Glencoe/McGraw-Hill.

[2] Douglas, L. and H. Picciotto. August 2017. *Transformational Proof in High School Geometry: A Guide for Teachers and Curriculum Developers*. https://www.mathedpage.org/transformations/proof/transformational-proof.pdf.
[3] Education Development Center. 2009. *CME Project: Geometry*. Boston, MA: Pearson.
[4] Harel, G. and L. Sowder. 2007. Toward comprehensive perspectives on the learning and teaching of proof. *Second Handbook of Research on Mathematics Teaching and Learning*. 2: 805–842.
[5] Hegg, M., D. Papadopoulos, B. Katz, and T. Fukawa-Connelly. 2018. Preservice teacher proficiency with transformations-based congruence proofs after a college proof-based geometry class. *The Journal of Mathematical Behavior*. 51: 56–70.
[6] Jones, K. and M. Tzekaki. 2016. Research on the teaching and learning of geometry. In A. Gutiérrez, G. Leder, and P. Boero (Eds), *The Second Handbook of Research on the Psychology of Mathematics Education: The Journey Continues*, pp. 109–149. Rotterdam, The Netherlands: Sense.
[7] Musser, G. L., L. E. Trimpe, and V. R. Maurer. 2008. *College Geometry: A Problem-Solving Approach with Applications*. Boston, MA: Pearson.
[8] National Council of Teachers of Mathematics. 2018. *Catalyzing Change in High School Mathematics: Initiating Critical Conversations*. Reston, VA: Author.
[9] National Governors Association Center for Best Practices, Council of Chief State School Officers. 2010. *Common Core State Standards for Mathematics*. Washington, DC: Author.
[10] Park City Mathematics Institute. 2016. Geometry transformed. http://projects.ias.edu/pcmi/hstp/sum2016/wg/geometry/abstract.geometry.html.
[11] Serra, M. 2008. *Discovering Geometry: An Investigative Approach*. Emeryville, CA: Key Curriculum Press.
[12] Sinclair, N. 2008. *The History of the Geometry Curriculum in the United States*. Charlotte, NC: IAP.
[13] St. Goar, J., Y. Lai, and R. Funk. 2019. Prospective High School Teachers' understanding and application of the connection between congruence and transformation in congruence proofs. In A. Weinberg, D. Moore-Russo, H. Soto, and M. Wawro (Eds), *Proceedings of the 22nd Annual Conference on Research on Undergraduate Mathematics Education*, pp. 247–254. Oklahoma City, OK: Special Interest Group of the Mathematical Association of America on Research on Undergraduate Mathematics Education.
[14] Usiskin, Z. P. and A. F. Coxford. 1972. A transformation approach to tenth-grade geometry. *The Mathematics Teacher*. 65(1): 21–30.
[15] Venema, G. A. 2006. *Foundations of Geometry*. Upper Saddle River, NJ: Pearson.
[16] Wu, H. 2013. *Teaching Geometry According to the Common Core Standards*. https://math.berkeley.edu/ ~ wu/Progressions_Geometry.pdf.

BIOGRAPHICAL SKETCHES

Julia St. Goar is an assistant professor of mathematics at Merrimack College. She received her Ph.D. in mathematics from the University of Nebraska-Lincoln and her B.A. from Agnes Scott College. Her research interests focus on how pre-service teachers learn mathematical content and apply content knowledge to their future teaching.

Yvonne Lai is an associate professor at the University of Nebraska-Lincoln. Her research focuses on mathematical knowledge for teaching, particularly at the secondary level. She is particularly interested in how the mathematical experiences of prospective and practicing teachers influence their teaching practice.