Error Correction for Correlated Quantum Systems

Mark Byrd, Alvin Gonzales, Daniel Dilley, and Purva Thakre

Abstract Modeling open quantum systems is a difficult task for many experiments. A standard method for modeling open system evolution uses an environment that is initially uncorrelated with the system in question, evolves the two unitarily, and then traces over the bath degrees of freedom to find an effective evolution of the system. This model can be insufficient for physical systems that have initial correlations. Specifically, there are evolutions $\rho_S = \text{tr}_E(\rho_{SE}) \rightarrow \rho'_S = \text{tr}_E(U\rho_{SE}U^\dagger)$ which cannot be modeled as $\rho_S = \text{tr}_E(\rho_{SE}) \rightarrow \rho'_S = \text{tr}_E(U\rho_S \otimes \rho_E U^\dagger)$. An example of this is $\rho_{SE} = \left|\Phi^+\right\rangle\left\langle\Phi^+\right|$ and $U_{SE} = \text{CNOT}$ with control on the environment. Unfortunately, there is no known method of modeling an open quantum system which is completely general. We first present some restrictions on the availability of completely positive (CP) maps via the standard prescription. We then discuss some implications a more general treatment would have for quantum control methods. In particular, we provide a theorem that restricts the reversibility of a map that is not completely positive (NCP). Let Φ be NCP and $\tilde{\Phi}$ be the corresponding CP map given by taking the absolute value of the coefficients in Φ. The theorem shows that the CP reversibility conditions for $\tilde{\Phi}$ do not provide reversibility conditions for Φ unless Φ is positive on the domain of the code space.

Keywords Quantum error correction · Quantum control · Open quantum systems

1 Introduction

Precise modeling and control of quantum systems will be required for quantum technologies. This includes quantum computers, quantum cryptography, and quantum simulation of quantum systems. Unfortunately, even though great progress has been

M. Byrd (✉) · A. Gonzales · D. Dilley · P. Thakre
Southern Illinois University Carbondale, 1263 Lincoln Drive, Carbondale, IL 62901-6899, USA
e-mail: mbyrd@siu.edu

A. Gonzales
e-mail: agonza@siu.edu

D. M. Kilgour et al. (eds.), *Recent Developments in Mathematical, Statistical and Computational Sciences*, Springer Proceedings in Mathematics & Statistics 343,
https://doi.org/10.1007/978-3-030-63591-6_34

made in this area, there are still questions concerning the types of possible evolutions and how to describe them. In the case that the system and environment have no prior correlations, there is a standard prescription for describing the evolution. Let the system (environment) state be ρ_S (ρ_E). Supposing that the two evolve under a joint unitary transformation U_{SE}, then a map can be defined by

$$\Phi(\rho_S) = \mathrm{tr}_E(U_{SE}\rho_S \otimes \rho_E U_{SE}^\dagger). \tag{1}$$

This map is not only positive (It maps all positive operators to positive operators.), it is also completely positive (The extended map $\mathbb{I}_n \otimes \Phi(\rho)$ is positive for all n and any positive input ρ.). This also provides a way to model a quantum system.

However, not all evolutions of a quantum system are able to be described this way. In particular, the assumption of an initial product state may not be satisfied. In this case, some discussions have arisen in the literature about what one should do if the standard assumption of an initially uncorrelated state no longer applies [1–13]. This is very relevant given that such examples are not difficult to find [14].

In general, the evolution of a system can be defined by a dynamical map, A, where we first vectorize the system density matrix [15]. The vectorization is done by writing all the elements as a column vector. For a single qubit state, this is given by

$$\mathrm{vec}(\rho) = \begin{bmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{bmatrix}. \tag{2}$$

The transformation is then done on this vectorized form and is given by

$$\rho' = A\rho. \tag{3}$$

Using the restrictions for a valid density matrix, it being Hermitian, positive semi-definite and having trace one, the restrictions on the A matrix are given by

$$A_{rs,r's'} = (A_{sr,s'r'})^*, \tag{4}$$

$$\sum_{rsr's'} x_r^* x_s A_{rs,r's'} y_{r'} y_{s'}^* \geq 0, \tag{5}$$

and

$$\sum_r A_{rr,r's'} = \delta_{r's'}, \tag{6}$$

respectively.

These conditions can be translated to an equivalent B matrix by just relabeling. Let

$$B_{rr',ss'} \equiv A_{rs,r's'}. \tag{7}$$

Then, the conditions are hermiticity

$$B_{rr',ss'} = (B_{ss',rr'})^*, \tag{8}$$

positivity

$$\sum_{rsr's'} z^*_{rr'} B_{rr',ss'} z_{ss'} \geq 0, \tag{9}$$

and trace preserving

$$\sum_r B_{rr',rs'} = \delta_{r's'}. \tag{10}$$

Since B is Hermitian, it has an eigenvector/eigenvalue decomposition

$$B_{r'r,s's}\rho_{rs} = \sum_\alpha \gamma(\alpha) C^\alpha_{r'r} \rho_{rs} (C^\alpha_{s's})^*,$$

where the C are the eigenvectors and γ the eigenvalues.

One may also write this as

$$\Phi(\rho) = B\rho = \sum_\alpha \eta_\alpha A_\alpha \rho A^\dagger_\alpha \quad \left(= \sum_\alpha A_\alpha \rho A^\dagger_\alpha, \forall \eta_\alpha = 1 \right), \tag{11}$$

where $A_\alpha \equiv \sqrt{|\gamma|} C^\alpha$ so that $\eta_\alpha = \pm 1$. It is known that the map is completely positive (CP) if and only if all $\eta_\alpha = 1$.

This form is often called the "Operator-Sum representation", or "Kraus decomposition" and is often used to describe open-system quantum dynamics.

2 Freedom in the Operator-Sum Representation

It is important to realize that the operator-sum decomposition, Eq. (11), is not unique and this non-uniqueness can be useful for finding different operator bases. This freedom is often called the "unitary freedom" [16].

Unitary Theorem: The form of a completely positive Hermiticity-preserving map, $\Phi(\rho) = \sum_\alpha A_\alpha \rho A^\dagger_\alpha$, defined by operators $\{A_\alpha\}$ is not unique, but the operators $\{F_\beta\}$

give the same map, if and only if there is a unitary matrix with elements $u_{\alpha\beta}$ such that $F_\beta = \sum_\alpha u_{\beta\alpha} A_\alpha, \ \forall\beta$.

This theorem can be used to prove the error-correcting code conditions below.

The more general map, the map that may be not completely positive (NCP), has a freedom in it as well. This is called the "pseudo-unitary freedom" [17].

Pseudo-Unitary Theorem: The form of a Hermiticity-preserving map,

$$\Phi(\rho) = \sum_\alpha \eta_\alpha A_\alpha \rho A_\alpha^\dagger,$$

defined by $\{A_\alpha\}$ and $\{\eta_\alpha\}$ is not unique, but the operators $\{F_\beta\}$ give the same map, if and only if there is a pseudo-unitary matrix with elements $u_{\alpha\beta}$ such that $F_\alpha = \sum_\beta u_{\alpha\beta} A_\beta, \ \forall\alpha$. The signature of the matrix $(u_{\alpha\beta}) \in U(p, q)$ is determined by the number of input and output elements in the sets $\{A_\alpha\}$ and $\{F_\beta\}$.

Note that a unitary matrix, V, is defined by the equation $VIV^\dagger = I$, whereas a pseudounitary matrix, U, is defined by the equation $U\eta U^\dagger = \eta$. In general, there are many choices for η. However, in our case, $\eta = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1)$ where there are p ones and q minus ones.

3 Modeling Open Quantum Systems

The standard prescription, Eq. (1), is used to justify completely positive maps, and often suffices for modeling quantum systems. However, it is clear that it is not the most general possible evolution. Many people, including the recent work of Pechukas, which spurred much discussion [1], have pointed out that a more general evolution may be derived from a potentially correlated system and environment:

$$\Phi(\rho_S) = \mathrm{tr}_E(U_{SE}\rho_{SE}U_{SE}^\dagger). \tag{12}$$

Finding examples which do not obey the assumption of an uncorrelated system and environment is not difficult. Consider the following two qubit example. Suppose that initial and final states of the system are known to be, respectively,

$$\rho_S = (1/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho_S' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Further assume that it is known that they evolve according to a system-environment coupling

$$U_{SE} = (1/\sqrt{2}) \begin{pmatrix} -i & 0 & 0 & -i \\ 0 & -i & -i & 0 \\ -i & 0 & 0 & i \\ 0 & -i & i & 0 \end{pmatrix}.$$

Then it is easy to show that there is no state ρ_E such that Eq. (1) is satisfied. This example can be shown to be robust to initial condition variations, as well as variations in the unitary transformation. This makes it experimentally verifiable.

Furthermore, finding such examples is not difficult. Consider a transformation from

$$\rho_S = \mathrm{Tr}_E(\rho_{SE})$$

to

$$\rho'_S = \mathrm{Tr}_B(U_{SE}\rho_{SE}U^\dagger_{SE}).$$

We say this is *U-generated* by U_{SE}. The set of local unitary transformations will be denoted LU, the set of unitaries that are equivalent via local unitaries to the swap unitary will be denoted as SWAP, and the set of unitary transformations that are equivalent via local unitaries to a controlled unitary will be denoted U_{C2}. Then we have the following theorem [14].

Theorem: Suppose that the system and environment consist of two qubits. Every U-generated physical transformation $\rho_S \to \rho'_S$ can be U-generated by a product state iff U belongs to LU $\cup$ SWAP. If U belongs to U_{C2}, the transformation can be U-generated by a quantum-classical state. On the other hand, if U does not belong to LU $\cup$ SWAP $\cup$ U_{C2}, then there exist physical transformations that cannot be U-generated by any initial separable state.

Therefore, there are plenty of examples where the standard prescription fails. This is our motivation for studying evolutions that do not necessarily correspond to a completely positive map.

4 Reversing a Quantum Operation Corresponding to a Completely Positive Map

The reversibility of a quantum operation depends on the operation elements satisfying certain conditions. These conditions are known as the *quantum error correcting code conditions*. There are several ways to state these conditions, one is to consider a map of the form of Eq. (11) when the map is completely positive with operation elements A_α, and some logical (encoded states) $|i_L\rangle$, $|j_L\rangle$ [18]

$$\langle i_L|A^\dagger_\alpha A_\beta|j_L\rangle = m_{\alpha\beta}\delta_{ij}.$$

This has an intuitive interpretation as a “disjointness condition.” It states that one state $|i_L\rangle$ acted on by one operator A_α cannot have any overlap with another state $|j_L\rangle$ acted on by another error A_β.

One can show that this condition is equivalent to the following condition [19]

$$PA_\alpha^\dagger A_\beta P = c_{\alpha\beta} P,$$

where P is a projector onto the code space and c is a Hermitian matrix.

5 Reversing a Quantum Operation that Is Not a Completely Positive Map

We first show that it is possible to reverse a map that is NCP.

Example 1 Let the NCP map be the three qubit map

$$\Phi(\rho) = c_0\rho + c_1 \sum_i X_i \rho X_i - c_2 \left|010\right\rangle \left\langle 010\right| \rho \left|010\right\rangle \left\langle 010\right|, \tag{13}$$

where X_i is the Pauli matrix σ_x acting on qubit i and $c_0 + 3c_1 = 1$ and $0 < c_2 < 1$. Thus, Φ is a trace decreasing map. Suppose we know that Φ has occurred. Then, the projector onto the code space is

$$P = \left|000\right\rangle \left\langle 000\right| + \left|111\right\rangle \left\langle 111\right| \tag{14}$$

and the recovery map is

$$R(\rho) = P\rho P + \sum_i PX_i\rho X_i P. \tag{15}$$

It is easy to check that any state $P\rho P$ in the code space is recovered.

CP and NCP maps are closely related and in the paper by Shabani and Lidar [20], they state:

Corollary 1 *Consider a Hermitian noise map*

$$\Phi_H(\rho) = \sum_{i=1}^N \eta_i A_i \rho A_i^\dagger$$

and associate to it a CP map

$$\tilde{\Phi}_{CP}(\rho) = \sum_{i=1}^N |\eta_i| A_i \rho A_i^\dagger.$$

Then any quantum error correcting code and corresponding CP recovery map for $\tilde{\Phi}_{CP}(\rho)$ are also a quantum error correcting code and CP recovery map for $\Phi_H(\rho)$.

Their corollary gives a result which is proportional to the original density operator on average. However, the standard procedure for a quantum error correction, which reverses a quantum operation, proceeds in two steps. The first is to measure an error syndrome which identifies the error. The second step is the recovery operation. Since the first projects out one of the terms in the sum, the terms in the sum should all be positive if they are independent. Otherwise, they can give a negative result for the measurement, which corresponds to a negative probability for the result to occur. We deem this nonphysical.

Theorem 1 *Suppose, using the pseudo-unitary (PU) degree of freedom, that*

$$PF_i^\dagger F_j P = d_{ij} P$$

and

$$\Phi(\rho) = \Phi_1(\rho) - \Phi_2(\rho),$$

where $F_i = u_{ij} A_j$, $\{u_{ij}\} \in PU$, $\Phi_2(P\rho P) \neq 0$, and $\{d_{ij}\}$ is diagonal. Then $\Phi(P\rho P)$ is not positive, i.e., the code space is not in the domain of the error map.

Sketch of Proof: Let our input density matrix be $P\rho P$, i.e., in the code space. The proof relies on the orthogonality of the rotated code space. The code space projector P, when acted on by the individual operators F_i are rotated to a set of orthogonal projectors due to the error correcting condition. From the polar decomposition, we have

$$F_i P = U_i \sqrt{PF_i^\dagger F_i P} = \sqrt{d_{ii}} U_i P \qquad (16)$$

This is actually a rotation on P. Thus, we can define

$$P_i \equiv U_i P U_i^\dagger \qquad (17)$$

and when $i \neq j$ we get

$$P_j P_i = 0. \qquad (18)$$

This means that we can pick out individual terms in the map.

Any NCP map can be written as the difference of two completely maps because we can group the negative terms and factor out the minus sign. Since the map $\Phi(\rho) = \Phi_1(\rho) - \Phi_2(\rho)$ and $\Phi_2(P\rho P) \neq 0$, we can get a measurement result P_i which corresponds to an outcome $P_i U_i \rho U_i^\dagger P_i$ by measuring the output density matrix in the $\{P_k\}$ basis. For $P_i U_i \rho U_i^\dagger P_i \in \Phi_2(P\rho P)$, this measurement probability is negative because the probability is given by

$$\mathrm{tr}(-\,|d_{ii}|\,P_i U_i \rho U_i^{\dagger} P_i) = -\,|d_{ii}|\,. \tag{19}$$

Since valid density matrices are positive semi-definite, the code space is not in the domain of $\Phi(\rho)$.

6 Discussion/Conclusions

The general problem of reversing the open-system evolution of a quantum system is an important open problem. Here we have provided a restriction on the ability to perform such an operation. In particular, we have shown that it is possible to arrive at a nonphysical result when attempting to use the same recovery operation for a map that is not completely positive as for the corresponding positive one. Furthermore, our theorem shows that there is a general restriction on the type of encoding that one may hope to use for reversing the quantum operations.

The general problem of how to reverse a quantum operation is still unsolved. However, we hope to present results elsewhere that can, in particular instances, enable the reversibility. Since the control of quantum systems is required for reliable quantum devices, we hope the results presented here, and in our future work, will help with the development of strategies for quantum control.

Acknowledgements Funding for this research was provided by the NSF, MPS under award number PHY-1820870.

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