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Author for correspondence:
Igor Rumanov
e-mail: igor.rumanov@colorado.edu

Whitham equations and phase shifts for the Korteweg–de Vries equation

Mark J. Ablowitz, Justin T. Cole and Igor Rumanov

Department of Applied Mathematics, University of Colorado, Boulder, CO 80309, USA

IR, 0000-0001-9799-1274

The semi-classical Korteweg–de Vries equation for step-like data is considered with a small parameter in front of the highest derivative. Using perturbation analysis, Whitham theory is constructed to the higher order. This allows the order one phase and the complete leading-order solution to be obtained; the results are confirmed by extensive numerical calculations.

1. Introduction

Whitham modulation theory has been widely used since it was first developed in 1965 [1]; see also the classic book [2]. An important application of the theory is the Korteweg–de Vries (KdV) equation with small dispersion. Averaging over the fast dynamics that occur over scales on the order of the small dispersion parameter ϵ, Whitham constructed PDEs governing the slowly varying parameters that change over order one space and time scales. These Whitham equations are hyperbolic first-order PDEs in space and time. Another breakthrough came several years later when Gurevich & Pitaevskii [3] found an important special self-similar solution of Whitham–KdV equations for the step initial condition (IC). The solution is a rarefaction wave solution of Whitham's equations; physically, it describes a *collisionless* shock wave, also called *dispersive shock wave* (DSW), which is a consequence of the small dispersion; there is no dissipation. At leading order in ϵ, the theory describes a modulated travelling wave KdV solution where the slowly modulated parameters obey the Whitham equations. The travelling wave fast phase of order $1/\epsilon$ is determined from these parameters. However, the finite phase shift of order $O(1)$ was not computed, so the description of the leading-order solution has remained incomplete until now. After this seminal work,

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Whitham theory developed in many different directions and its applications keep growing. Yet, for many years finding the finite phase shifts from Whitham theory was an open problem. In the Whitham approach, this determination requires the computation of higher orders, in particular, the next-to-leading order of Whitham theory which can be viewed as a nonlinear WKB-type expansion.

For *integrable* nonlinear PDEs like KdV, there is another way to find solutions—the inverse scattering transform (IST); see, e.g. [4,5]. This is equivalent to constructing and solving a Riemann–Hilbert type problem (RHP). It was originally developed for initial value problems with rapidly decaying ICs in both directions in space. Within the RHP framework, the small dispersion limit of KdV with fast decaying ICs was studied in [6] extending earlier work, see [7,8] and references therein, and using the steepest descent approach developed in [9,10]. In [6], a trivial, i.e. constant, finite phase shift was established and supporting numerical results were obtained in [11]. For step-like ICs the long time asymptotics have been considered by IST/RHP methods in [12,13], with space and time-dependent $O(1)$ phase shift as a result. Long time and small dispersion are, in general, different limits. However, the dispersion parameter ϵ can be removed from the KdV equation by rescaling space x and time t variables. Then, step IC is seen as very special since it remains intact by this rescaling. Thus, the Cauchy problem and solution for KdV with step IC depends only on x/ϵ and t/ϵ. Therefore, the long-time asymptotics for this IC should apparently be equivalent to the small dispersion limit. In appendix B, we express the long-time result of [13] for pure step IC in a simpler form. This facilitates comparison with the result of Whitham theory which we remark upon.

The IST/RHP approach, however, is only applicable to PDEs with known integrable structure. For non-integrable PDEs one has to resort to other methods, and here the nonlinear WKB/Whitham approach has been indispensable. Nevertheless, it is important to analyse the well-known PDEs such as KdV and develop key ideas in order to pave the way for understanding more complicated models. Moreover, even for the relatively simple situation of KdV with step IC, to our knowledge, the question of finding the $O(1)$ phase shift via Whitham theory remains unsettled. A large number of papers use the leading-order theory, associated simulations and experiments involving DSWs; see reviews [14,15]. However, very few deal with higher-order corrections. An early discussion of higher-order effects can be found in [16]. Finding the phase shift remains a vital part of the leading order modulated periodic solution.

In this paper, we derive the higher-order Whitham theory for the KdV equation, the leading order of which was established in [2,3]. The key ideas and main results of our approach to higher orders in ϵ are explained in §2. Using the (implicit) assumption that the phase shift is included in the total fast phase $\theta \sim 1/\epsilon$, we systematically compute the higher-order corrections in ϵ via singular perturbation theory, cf. e.g. [16,17]. This approach leads to an expansion in powers of ϵ^2 rather than ϵ *for the slow (Whitham) variables*. Then eventually only a constant finite (i.e. $O(1)$, at next-to-leading order in ϵ) phase shift appears in this theory. In other words, every $O(1)$ spacetime varying shift can be absorbed into a redefinition of the (other) basic slow variables and the fast phase θ determined by them. These results are presented in detail in §§3 and 4. In §5, we compare the numerical solution of KdV with step IC to the leading-order Whitham–GP solution of $O(1/\epsilon)$. The result indicates that the residual phase shift, apart from a constant, is $O(\epsilon)$. Then, the value of the constant can be found from the condition at the leading edge of DSW that the solution vanishes; see, e.g. [3,18] and below in §5.

In Whitham theory, the phase shift arises as an integration constant when integrating the leading-order ODE equation (4.4) in its fast oscillation phase variable. Any such 'constant' can, in general, be an arbitrary *slow variable*, i.e. a function of space and time which does not change significantly over a period of fast oscillations. Motivated by this observation, in §6 we also explore a modified Whitham theory approach by explicitly introducing a spacetime-dependent phase shift $\theta_*(x,t)$ into the Whitham–KdV theory from the beginning, i.e. represent the total phase as $\theta = \theta_0/\epsilon + \theta_*$. This changes the look of the higher-order Whitham perturbation theory and leads to the apparent possibility of non-trivial θ_* dynamics. Such a consideration was initiated back in 1988 by Haberman [19] who derived equations governing the phase shift θ_* but did not present any solution. We consider this approach in more detail here but do not find a non-singular/non-trivial phase shift consistent with our numerical results.

As an instructive comparison, we treat in §7 the linearized KdV equation in the Whitham framework; see also [2]. The linearized KdV equation with, for example, the same step IC has an exact Fourier solution (the analogue of IST solution for nonlinear case). Using Whitham theory, we also derive the corresponding approximate (WKB) solution in the region of fast oscillations. The Whitham approach here yields the exact leading-order amplitude and fast phase of the oscillations and allows one to conclude unambiguously that the $O(1)$ phase shift is constant. Only its value remains undetermined; it can be found from the exact solution. The simpler linear problem gives additional insight for the nonlinear case. Our conclusions are presented in §8.

2. Approach to higher-order Whitham theory; main results

We look for an asymptotic solution $u(x,t;\epsilon)$ of the KdV equation,

$$\partial_t u + 6u\partial_x u + \epsilon^2 \partial_{xxx} u = 0, \tag{2.1}$$

with fast and slow scales,

$$u = u(\theta, x, t; \epsilon), \quad 0 < \epsilon \ll 1, \tag{2.2}$$

where the single fast phase θ is $O(1/\epsilon)$. The fast phase θ satisfies equations

$$\partial_x \theta = \frac{k}{\epsilon} \quad \text{and} \quad \partial_t \theta = -\frac{kV}{\epsilon}, \tag{2.3}$$

with slowly varying quantities k and V. Then,

$$\partial_t k + \partial_x (kV) = 0. \tag{2.4}$$

This 'kinematic' equation remains intact at all higher orders in ϵ since it is just a consequence of definition of θ. The other Whitham equations are derived as secularity conditions ensuring that the solution u is periodic rather than growing in θ. In the next section, by the separation of fast and slow scales method, we find the Whitham–KdV equations to all orders—see equations (3.15)–(3.17).

The solution u is expanded in ϵ, $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$, where $u_0(\theta; x, t; \epsilon)$ is the leading-order solution given in equation (4.4), and further corrections satisfy linear equations of the form

$$\mathcal{L} u_n = F_n[u_0, u_1, \ldots, u_{n-1}], \quad n = 1, 2, \ldots \quad \mathcal{L} = k^2 \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + 6u_0(\theta) - V.$$

The forcing terms F_n at each order depend only on the solution at previous orders of ϵ-perturbation theory. The wavenumber k is determined from equation (4.3) ensuring the constant period of fast oscillations in θ. The solutions to the homogeneous equation $\mathcal{L}w = 0$ here are known, the first being $w_1 = u_0'(\theta)$, and the second, $w_2(\theta)$, explicitly given by equations (4.18), (4.19), is of the form

$$w_2 = K_2 u_0' \theta + \phi[u_0],$$

where K_2 is a slow variable and ϕ is a periodic function of θ determined by u_0. Therefore, using variation of parameters, the particular solutions u_n are given by the integral formulas

$$u_n = \frac{u_0'(\theta) \int_0^\theta w_2(z) F_n(z)\,\mathrm{d}z - w_2(\theta) \int_0^\theta u_0'(z) F_n(z)\,\mathrm{d}z}{k^2 W}, \tag{2.5}$$

where $W = u_0' w_2' - u_0'' w_2$ is the Wronskian which is independent of θ. Then, we can calculate $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$ at any order in terms of integrals. Moreover, u_1 depends on u_0; u_2 depends on u_0 and u_1 hence it depends only on u_0; similarly $u_j, j = 3, 4, \ldots$, depends only on u_0. The well-known equation (4.4) for u_0 is periodic in θ, so it can be written as a Fourier series in the variable θ. Hence all successive terms $u_j, j = 1, 2\ldots$, from equation (2.5) can be written in terms of a Fourier series in θ. We further show in §4 and appendix A how the initial/boundary conditions are satisfied. Thus, we have provided a method to solve for u_n and satisfy the initial/boundary conditions at all orders of ϵ.

Then, we express the Whitham equations in terms of three Riemann invariants order by order and find the higher-order corrections to them given the corrections u_n described above. The non-trivial corrections at $O(\epsilon^2)$ are given by equation (4.22) and more explicitly in the following equations, which is one of the main results of the paper.

As mentioned above, we first find the Whitham equations to all orders in the original 'physical' variables: the conservation of waves equation (2.4) and the secularity equations (3.15) and (3.16). Equations (3.15) and (3.16) are another important result of the paper. This is the first time such equations have appeared in Whitham–KdV theory; for a simpler system, the nonlinear Klein–Gordon equation, a similar result was obtained long ago in [16]. For KdV, there are only three equations for five dependent variables so they are not closed. However, these 'non-perturbative' Whitham equations can be expanded in ϵ^2 order by order. Their leading order, equations (4.6), (4.7) and (2.4), is classical [2]. In this case, the Whitham system is closed by using two additional relations, equations (4.3) and (4.5). Equation (4.3) must hold to all orders since it enforces the constant period condition. Equation (4.5) is replaced at higher orders by corrections u_n integrated over a period in θ. This allows one to find corrections to all slow Whitham variables order by order. Equations (2.4), (3.15) and (3.16) are a convenient means to derive higher orders of Whitham theory.

The fact that these equations contain only ϵ^2, and not ϵ, and the oddness of u_1 as a function of θ lead eventually to the triviality (constancy) of the $O(1)$ phase shift to the order $1/\epsilon$ fast phase of the single DSW solution. This is confirmed by extensive numerics for the special important case of pure step IC; the figures and comments in §5b are also key results in this paper.

3. Whitham equations to all orders

After introducing fast and slow scales, the KdV equation takes the form

$$\left(-\frac{kV}{\epsilon}\partial_\theta + \partial_t\right)u + 6u\left(\frac{k}{\epsilon}\partial_\theta + \partial_x\right)u + \epsilon^2\left(\frac{k}{\epsilon}\partial_\theta + \partial_x\right)^3 u = 0. \tag{3.1}$$

We denote $\partial_\theta f = f'$ and $\bar{f}$ the average of f over a period in θ. Equation (3.1) can be written as

$$k(k^2u'' + 3u^2 - Vu)' + \epsilon kF' = 0, \tag{3.2}$$

$$kF' = \partial_t u + 6u\partial_x u + 3k^2\partial_x u'' + \frac{3}{2}\partial_x(k^2)u'' + \epsilon\left(3k\partial_{xx}u' + 3\partial_x k\partial_x u' + \partial_{xx}ku'\right) + \epsilon^2\partial_{xxx}u. \tag{3.3}$$

Imposing periodicity of the solution u and integrating equation (3.2) over a period in θ yields an exact (non-perturbative and asymptotic to all orders) secularity condition $\overline{F'} = 0$, or, explicitly,

$$\partial_t\overline{u} + 3\partial_x(\overline{u^2}) + \epsilon^2\partial_{xxx}\overline{u} = 0. \tag{3.4}$$

The other needed secularity condition is readily derived when one notices that $u(k^2u'' + 3u^2 - Vu)' = (k^2(uu'' - (u')^2/2) + 2u^3 - Vu^2/2)'$ is a total derivative in θ. Thus, multiplying equation (3.2) by u and integrating over the period, one finds the second exact secularity condition $\overline{uF'} = 0$, or, explicitly,

$$\partial_t\overline{u^2} + 4\partial_x(\overline{u^3}) - 3\partial_x(k^2\overline{(u')^2}) + 3\epsilon\partial_x[k\overline{(u\partial_x u' - u'\partial_x u)}] + \epsilon^2\partial_x[\partial_{xx}\overline{u^2} - 3\overline{(\partial_x u)^2}] = 0. \tag{3.5}$$

To derive equation (3.5) we used the identities $\overline{u(3k\partial_{xx} + 3\partial_x k\partial_x + \partial_{xx}k)u'} = \frac{3}{2}\overline{(u(k\partial_{xx} + \partial_x k\partial_x)u' - u'(k\partial_{xx} + \partial_x k\partial_x)u)}$, $\quad k(u\partial_{xx}u' - u'\partial_{xx}u) + \partial_x k(u\partial_x u' - u'\partial_x u) = \partial_x[k(u\partial_x u' - u'\partial_x u)]$, $\quad \overline{u\partial_{xxx}u} = \partial_x[\partial_{xx}\overline{u^2} - 3\overline{(\partial_x u)^2}]/2$. Using the notation $Q = \overline{u}$, $Q_n = \overline{u^n}, n > 1$, $G = k^2\overline{(u')^2}$, the two derived secularity equations read:

$$\partial_t Q + 3\partial_x Q_2 + \epsilon^2 \partial_{xxx} Q = 0, \tag{3.6}$$

$$\partial_t Q_2 + 4\partial_x Q_3 - 3\partial_x G + 3\epsilon\partial_x \left(\overline{k(u\partial_x u' - u'\partial_x u)}\right) + \epsilon^2 \partial_x \left(\partial_{xx} Q_2 - 3\overline{(\partial_x u)^2}\right) = 0. \tag{3.7}$$

Next, we transform the obtained equations to a more convenient form with fewer dependent variables, guided by the well-known leading-order KdV manipulations. We integrate equation (3.2) to obtain

$$k^2 u'' + 3u^2 - Vu + \epsilon F = C_1, \tag{3.8}$$

where $C_1 = C_1(x,t)$ is an arbitrary integration constant, i.e. slow variable in our case, and F is defined as a certain antiderivative of equation (3.3),

$$kF = \partial_t J_1 + 3\partial_x J_2 + 3k\partial_x(ku') + \epsilon\,(3k\partial_{xx}u + 3\partial_x k\partial_x u + \partial_{xx}ku) + \epsilon^2 \partial_{xxx} J_1. \tag{3.9}$$

Functions J_n such that $J_n' = u^n$ contain secular (non-periodic) terms proportional to θ which we explicitly separate writing

$$J_n = \overline{u^n}\theta + \hat{J}_n \quad \text{and} \quad \hat{J}_n' = u^n - \overline{u^n}, \tag{3.10}$$

so that $\hat{J}_n$ are periodic. We define all $\hat{J}_n$ so that

$$\overline{\hat{J}_n} = 0,\ n \in \mathbb{N}. \tag{3.11}$$

Next, we multiply equation (3.8) by $2u'$ and integrate it again over θ obtaining

$$k^2(u')^2 + 2u^3 - Vu^2 + 2\epsilon\left(uF - \int^\theta uF'\right) = 2C_1 u + C_2, \tag{3.12}$$

with the arbitrary integration slow variable C_2 and $\int^\theta uF'$, as follows from equation (3.3), of the form

$$k\int^\theta uF' = \frac{\partial_t J_2}{2} + \partial_x\left(2J_3 - \frac{3}{2}G_1\right) + 3k^2 u\partial_x u' + \frac{3}{2}\partial_x(k^2)uu'$$
$$+\,\epsilon\left(\frac{3}{2}\partial_x\left(k\int^\theta (u\partial_x u' - u'\partial_x u)\right) + \frac{u}{2}(3k\partial_{xx} + 3\partial_x k\partial_x + \partial_{xx}k)u\right) + \frac{\epsilon^2}{2}\left(\partial_{xxx}J_2 - 3\partial_x\int^\theta (\partial_x u)^2\right). \tag{3.13}$$

Here, we defined $G_1 = k^2\int^\theta (u')^2 = G\theta + \hat{G}_1$, $\overline{\hat{G}_1} = 0$. Equations (3.10), (3.11) imply that $\overline{J_1} = Q/2$, $\overline{J_2} = Q_2/2$, $\overline{J_3} = Q_3/2$, $\overline{G_1} = G/2$. We similarly fix the other antiderivatives in equation (3.13) as secular θ-term plus periodic 'hat'-term with zero average so that

$$\overline{\int^\theta (u\partial_x u' - u'\partial_x u)} = \frac{\overline{u\partial_x u' - u'\partial_x u}}{2} \quad \text{and} \quad \overline{\int^\theta (\partial_x u)^2} = \frac{\overline{(\partial_x u)^2}}{2}. \tag{3.14}$$

Proposition 3.1. *Secularity conditions equations (3.6) and (3.7) for KdV are equivalent to*

$$\partial_t Q + \partial_x(VQ + C_1) - \epsilon^2\partial_x\left(2\partial_{xx}Q + \frac{3\partial_x k}{k}\partial_x Q + \frac{\partial_{xx}k}{k}Q\right) = 0, \tag{3.15}$$

$$\partial_t P + \partial_x H - \epsilon^2\left[\partial_t Q_d + \partial_x(VQ_d) + \partial_x\left(\frac{\partial_{xx}P}{2} + \frac{3\partial_x k}{2k}\partial_x P + \frac{\partial_{xx}k}{k}P\right)\right]$$
$$+\,\epsilon^4\partial_x\left(\frac{\partial_{xx}Q_d}{2} + \frac{3\partial_x k}{2k}\partial_x Q_d + \frac{\partial_{xx}k}{k}Q_d\right) = 0, \tag{3.16}$$

where we denoted

$$P = VQ + C_1, \quad H = VP - 3C_2, \quad Q_d = \frac{3\partial_x(k\partial_x Q) + \partial_{xx}kQ}{k}. \tag{3.17}$$

Proof. Integration of equations (3.8) and (3.12) over a period gives, respectively,

$$3Q_2 - VQ + \epsilon\overline{F} = C_1 \tag{3.18}$$

and

$$G + 2Q_3 - VQ_2 + 2\epsilon\left(\overline{uF} - \overline{\int^\theta uF'}\right) = 2C_1Q + C_2, \tag{3.19}$$

while multiplying equation (3.8) by u and integrating over a period, one gets

$$-G + 3Q_3 - VQ_2 + \epsilon\overline{uF} = C_1Q. \tag{3.20}$$

Taking into account the secularity condition equation (3.4) in equation (3.9) brings the 'forcing' F to the explicitly periodic form

$$kF = \partial_t\hat{J}_1 + 3\partial_x\hat{J}_2 + 3k\partial_x(ku') + \epsilon\,(3k\partial_{xx}u + 3\partial_x k\partial_x u + \partial_{xx}ku) + \epsilon^2\partial_{xxx}\hat{J}_1. \tag{3.21}$$

The last equation integrated over a period becomes very simple

$$k\overline{F} = \epsilon(3k\partial_{xx} + 3\partial_x k\partial_x + \partial_{xx}k)Q. \tag{3.22}$$

In turn, taking into account the secularity condition equation (3.5) in equation (3.13) lets one bring the quantity $\int^\theta uF'$ to explicitly periodic form and its average over a period reads

$$k\overline{\int^\theta uF'} = \frac{3k^2}{2}\overline{(u\partial_x u' - u'\partial_x u)} + \epsilon\left[\frac{3k}{2}\left(\frac{\partial_{xx}Q_2}{2} - \overline{(\partial_x u)^2}\right) + \frac{3}{2}\partial_x k\frac{\partial_x Q_2}{2} + \partial_{xx}k\frac{Q_2}{2}\right]. \tag{3.23}$$

Taking the combination of averaged equations 2 · (3.20) − (3.19) yields

$$4Q_3 - 3G - VQ_2 + 2\epsilon\overline{\int^\theta uF'} = -C_2. \tag{3.24}$$

Upon using equation (3.23), equation (3.24) acquires the form containing exactly the combination entering the secularity equation (3.5)

$$4Q_3 - 3G + 3\epsilon k\overline{(u\partial_x u' - u'\partial_x u)} - 3\epsilon^2\overline{(\partial_x u)^2} = VQ_2 - C_2 - \frac{\epsilon^2}{k}\left(\frac{3}{2}\partial_x(k\partial_x Q_2) + \partial_{xx}kQ_2\right). \tag{3.25}$$

We substitute the right-hand side of equation (3.25) into equation (3.5) and the last becomes

$$\partial_t Q_2 + \partial_x(VQ_2 - C_2) - \epsilon^2\partial_x\left(\frac{\partial_{xx}Q_2}{2} + \frac{3\partial_x k}{2k}\partial_x Q_2 + \frac{\partial_{xx}k}{k}Q_2\right) = 0. \tag{3.26}$$

Finally, we substitute into equations (3.4) and (3.26) Q_2 expressed from equations (3.18) and (3.22) as

$$Q_2 = \frac{VQ + C_1}{3} - \epsilon^2\left(\frac{\partial_x(k\partial_x Q)}{k} + \frac{\partial_{xx}k}{3k}Q\right). \tag{3.27}$$

Thus, we obtain the secularity conditions in their final form of equations (3.15) and (3.16), as claimed. ∎

The secularity equations (3.15), (3.16) and the kinematic equation (2.4) comprise exact non-perturbative Whitham–KdV equations in physical variables V, C_1, C_2, k and Q. The system of Whitham PDEs is not closed as it stands. Still it is a very convenient starting point to get the Whitham equations to any needed higher order in ϵ. We see that the system (3.15), (3.16) is perturbed only by ϵ^2. This suggests that under a perturbation expansion of u in powers of ϵ the secularity conditions will have all non-trivial higher-order corrections expanded in ϵ^2. We demonstrate in some detail that this is indeed the case in the next section. This, in particular, implies that the first non-trivial correction to the fast phase θ is going to be of order ϵ. As for an order $O(1)$ phase shift, it can then only be a pure constant, the value of which only affects the initial/boundary conditions but not the equations. As we will see in §5b, this picture is consistent with the numerical results.

4. Perturbations of Whitham variables

Let $u = u_0 + \epsilon\tilde{u}$, where u_0 is *defined* as the solution of first-order ODE

$$k^2(u_0')^2 = -2u_0^3 + Vu_0^2 + 2C_1u_0 + C_2, \tag{4.1}$$

compare with equation (3.12). Equation (4.1) includes all leading-order terms of equation (3.12); therefore, $u - u_0$ starts at order ϵ indeed. Let $\lambda_1 \le \lambda_2 \le \lambda_3$ be the roots of the cubic in the right-hand side of equation (4.1). They are related to V, C_1 and C_2 as

$$\frac{V}{2} = e_1 \equiv \lambda_1 + \lambda_2 + \lambda_3, \quad C_1 = -e_2 \equiv -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \quad \frac{C_2}{2} = e_3 \equiv \lambda_1\lambda_2\lambda_3. \tag{4.2}$$

The normalization of the elliptic cnoidal solution to equation (4.1) with unit period in θ implies that

$$k^2 = \frac{\lambda_3 - \lambda_1}{8K^2(m)}, \tag{4.3}$$

where $K(m)$ is the first complete elliptic integral; hence the solution to equation (4.1) is

$$u_0 = \lambda_2 + (\lambda_3 - \lambda_2)\mathrm{cn}^2(2K(m)\theta; m) \quad \text{and} \quad m = \frac{\lambda_3 - \lambda_2}{\lambda_2 - \lambda_1}. \tag{4.4}$$

Also we have $Q = Q_0 + \epsilon\tilde{Q}$, where $Q_0 = \overline{u_0}$ so that

$$Q_0 = \lambda_1 + (\lambda_3 - \lambda_1)\frac{E(m)}{K(m)}, \tag{4.5}$$

in terms of the first and second complete elliptic integrals $K(m)$ and $E(m)$. Then, keeping only terms starting at leading order in equations (3.15), (3.16) and adding the relations (4.3), (4.5) yields the closed familiar Whitham–KdV system of equations (2.4), (4.3), (4.5), and equations

$$\partial_t Q_0 + \partial_x(VQ_0 + C_1) = 0, \tag{4.6}$$

$$\partial_t(VQ_0 + C_1) + \partial_x\left(V(VQ_0 + C_1) - 3C_2\right) = 0. \tag{4.7}$$

At this point the Whitham–KdV system can be viewed as three equations (2.4), (4.6) and (4.7) for three unknowns $\lambda_j, j = 1, 2, 3$, where the constraints equations (4.3) and (4.5) were substituted. We introduce the KdV Riemann invariants $r_1 \le r_2 \le r_3$ [2,3] in terms of which this system diagonalizes with respect to space and time derivatives and takes the form

$$\partial_t r_j + v_j(r_1, r_2, r_3)\partial_x r_j = 0.$$

Explicitly r_j-variables are linearly related with the cubic roots λ_i

$$\lambda_1 = r_1 + r_2 - r_3, \quad \lambda_2 = r_1 - r_2 + r_3, \quad \lambda_3 = -r_1 + r_2 + r_3, \tag{4.8}$$

and the other slow variables are the following functions of the three r-s: $m = (r_2 - r_1)/(r_3 - r_1)$,

$$\left.\begin{aligned}
&Q_0 = r_2 - r_3 + r_1 + 2(r_3 - r_1)\frac{E(m)}{K(m)}, \quad \frac{V}{2} = r_1 + r_2 + r_3,\\
&\frac{C_1}{2} = \frac{r_1^2 + r_2^2 + r_3^2}{2} - r_1r_2 - r_2r_3 - r_3r_1,\\
&C_2 = 2(-r_1^3 - r_2^3 - r_3^3 + r_1^2(r_2 + r_3) + r_2^2(r_3 + r_1) + r_3^2(r_1 + r_2) - 2r_1r_2r_3), \quad k^2 = \frac{r_3 - r_1}{4K^2(m)}.
\end{aligned}\right\} \tag{4.9}$$

On the other hand, the all orders Whitham system derived before is equation (2.4) and

$$\partial_t Q + \partial_x(VQ + C_1) - \epsilon^2\Phi_1 = 0, \tag{4.10}$$

$$\partial_t(VQ + C_1) + \partial_x[V(VQ + C_1) - 3C_2] - \epsilon^2\Phi_2 = 0, \tag{4.11}$$

$$P \equiv VQ + C_1, \quad Q_d \equiv \frac{3\partial_x(k\partial_x Q) + \partial_{xx}kQ}{k},$$

$$\Phi_1 = \partial_x\left(2\partial_{xx}Q + \frac{3\partial_x k}{k}\partial_x Q + \frac{\partial_{xx}k}{k}Q\right), \tag{4.12}$$

$$\Phi_2 = \partial_t Q_d + \partial_x\left(VQ_d + \frac{\partial_{xx}P}{2} + \frac{3\partial_x k}{2k}\partial_x P + \frac{\partial_{xx}k}{k}P\right)$$
$$- \epsilon^2\partial_x\left(\frac{(k\partial_{xx} + 3\partial_x k\partial_x + 2\partial_{xx}k)Q_d}{2k}\right). \tag{4.13}$$

It is remarkable that only the mean of the KdV solution $Q \equiv \bar{u}$ has to be found using the higher-order corrections to KdV itself, no other information from them is needed. These corrections u_n to $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$ are found from

$$k^2u_1'' + (6u_0 - V)u_1 + F_1 = 0, \quad k^2u_2'' + (6u_0 - V)u_2 + F_2 = 0, \ldots, \tag{4.14}$$

the forcing terms F_1 and F_2 are, with $(\hat{J}_1^n)' = u_n - \bar{u}_n$, $(\hat{J}_2^n)' = u_n^2 - \overline{u_n^2}$, $\overline{\hat{J}_m^n} = 0$,

$$kF_1 = \partial_t\hat{J}_1^0 + 3\partial_x\hat{J}_2^0 + 3k^2\partial_x u_0' + 3k\partial_x k u_0' \tag{4.15}$$

and

$$kF_2 = 3u_1^2 + \partial_t\hat{J}_1^1 + 3\partial_x\hat{J}_2^1 + 3k^2\partial_x u_1' + 3k\partial_x k u_1' + 3k\partial_{xx}u_0 + 3\partial_x k\partial_x u_0 + \partial_{xx}ku_0. \tag{4.16}$$

Using variation of parameters, the solution to equation (4.14) is given by the integral formulas

$$u_n = \frac{u_0'(\theta)\int_0^\theta w_2(z)F_n(z)\,\mathrm{d}z - w_2(\theta)\int_0^\theta u_0'(z)F_n(z)\,\mathrm{d}z}{k^2W}, \quad n = 1, 2, \ldots. \tag{4.17}$$

Here, u_1 depends on u_0 only, u_2 depends on u_0 and u_1 only and every subsequent u_n depends only on the previous ones, u_0 through u_{n-1}. In equation (4.17), $w_2(\theta)$ is the second homogeneous solution of the linear operator $\mathcal{L} = k^2\mathrm{d}^2/\mathrm{d}\theta^2 + 6u_0 - V$, the first being $u_0'(\theta)$. Explicitly, w_2 reads

$$w_2 = \alpha(u_0'\hat{J}_1^0 - 2u_0^2) + (\alpha Q_0 + \beta)u_0'\theta + \delta u_0 + \chi, \tag{4.18}$$

it can be normalized so that the constants in equation (4.18) are

$$\alpha = \frac{V^2}{3} + 4C_1, \quad \beta = \frac{VC_1}{3} + 3C_2, \quad \delta = 2\beta + V\alpha, \quad \chi = \frac{4C_1\alpha - V\beta}{3}. \tag{4.19}$$

This w_2 can be obtained, for example, by looking for a solution of a form like equation (4.18), which gives δ, χ in terms of α, β as in equation (4.19) and fixes the ratio α/β as $(VC_1 + 9C_2)\alpha = (V^2 + 12C_1)\beta$. Then, the Wronskian $W = u_0'w_2' - u_0''w_2$, a slow variable independent of θ, is given by

$$k^2W = \left(VC_2 - \frac{4C_1^2}{3}\right)\alpha + \left(\frac{VC_1}{3} + 3C_2\right)\beta. \tag{4.20}$$

The lower limits of integration in equation (4.17) ensure that $u_n(x, 0) = 0$ for $n \geq 1$, i.e. the IC has to be satisfied by the leading order solution u_0. This is true when $\theta(x, 0) = 0$, see next section and equation (5.4) in particular, when the fast phase is 'born' after wave breaking at time $t = 0$, which is the situation we study here. More general, IC consideration is given in appendix A. There it is also shown that u_n in equation (4.17) are indeed periodic hence non-secular.

Since u_1 is odd in θ because F_1 is, we have $\bar{u}_1 = 0$. This is also true of all odd order corrections u_{2n+1}. Therefore, the total mean Q can be written as

$$Q = \bar{u}_0 + \epsilon^2 q = Q_0 + \epsilon^2(q_2 + \epsilon^2 q_4 + O(\epsilon^4)), \quad q_2 = \bar{u}_2,\ q_4 = \bar{u}_4, \ldots, \tag{4.21}$$

and q is determined order by order by averages of even u-corrections u_{2n}.

The Whitham equations can be represented order by order in terms of the three Riemann variables r_j, $j = 1, 2, 3$, defined exactly as in the leading-order equations above. We transform equations (2.4), (4.10) and (4.11) using equations (4.8) and (4.9) and obtain the Whitham equations diagonalized to leading order in Riemann variables:

$$\partial_t r_j + v_j \partial_x r_j + \frac{\epsilon^2 (X_2 + 6(r_j - r_l - r_m) X_1)}{24(r_j - r_l)(r_j - r_m)\partial_j k/k} = 0, \quad j \neq l \neq m \neq j,\ j = 1,2,3, \tag{4.22}$$

$$X_1 \equiv \partial_t q + \partial_x (Vq) - \Phi_1, \quad X_2 \equiv \partial_t (Vq) + \partial_x (V^2 q) - \Phi_2, \quad \frac{\partial_j k}{k} \equiv \frac{\partial k}{k \partial r_j}.$$

The quantity q defined in equation (4.21) is also expressed order by order in terms of the r_j-variables by solving equation (4.14), etc. Asymptotically, the Whitham equations, e.g. in Riemann form equation (4.22), can be solved order by order in the r-variables. The first corrections come at order ϵ^2 which implies a non-trivial (non-constant) correction to the fast phase θ of order ϵ only. Thus, the most important $O(1)$ phase shift has to be a pure constant rather than a slow variable. The value of the constant is then fixed by initial/boundary conditions. For example, in the case of pure step IC of [3] or §5, the constant is fixed by the condition at the leading (solitonic) edge of the DSW. We also note that when solving order by order, u_n do not have any singularities for $0 < m < 1$. For example, Whitham equations to order $O(\epsilon^2)$ take the form of equation (4.22) where

$$X_1 = \partial_t q_2 + \partial_x (V q_2) - \partial_x \left(2\partial_{xx} Q_0 + \frac{3\partial_x k}{k} \partial_x Q_0 + \frac{\partial_{xx} k}{k} Q_0 \right), \quad q_2 = \overline{u_2}, \quad P_0 = V Q_0 + C_1$$

$$X_2 = \partial_t (V q_2) + \partial_x (V^2 q_2) - \partial_t Q_d^0 + \partial_x \left(V Q_d^0 + \left[\frac{\partial_{xx}}{2} + \frac{3\partial_x k}{2k} \partial_x + \frac{\partial_{xx} k}{k} \right] P_0 \right),$$

$$Q_d^0 = \frac{3\partial_x (k \partial_x Q_0) + \partial_{xx} k Q_0}{k},$$

and k, Q_0, V, C_1 are given in terms of $r_j, j = 1,2,3$, by equation (4.9). One could obtain the Fourier series for u_2 and take the average in θ to get q_2, leaving only slow variables.

The system (4.22) can, in principle, also be solved numerically. In this regard, we consider an iteration where the terms without ϵ^2 are iterates at level $n+1$ and the terms with ϵ^2 are iterates at level n. At $n = 0$, we take the perturbing iterate with $\epsilon = 0$; i.e. we have our unperturbed solution. The $n = 1$ term is solved by calculating the perturbed terms with equation (4.21), and ODEs and definitions given above for equation (4.22), u_1, u_2, etc.

5. Step initial value problem: numerical results

Consider the step IC for the KdV equation

$$u(x, 0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{5.1}$$

Then, Whitham theory gives the famous Gurevich–Pitaevskii (GP) DSW solution involving the leading-order cnoidal function with modulated parameters. In the GP solution, $r_1 = 0$, $r_3 = 1$ and $r_2 = m = m(x/t)$ is implicitly given by

$$\frac{x}{t} = v_2(m) = 2(1 + m) - \frac{4m(1 - m)K(m)}{E(m) - (1 - m)K(m)}. \tag{5.2}$$

Only the leading-order phase $\theta_0(x, t)/\epsilon$ is known in the GP solution. It is determined by formula

$$\frac{\theta_0(x, t)}{\epsilon} = \int_0^x \frac{k(\eta, t)}{\epsilon} \, \mathrm{d}\eta - \int_0^t \frac{k(0, \tau) V(0, \tau)}{\epsilon} \, \mathrm{d}\tau. \tag{5.3}$$

The lower limits of integration here reflect the fact that the fast phase is 'born' at time $t = 0$ at the jump point $x = 0$. Explicitly,

$$\frac{\theta_0(x, t)}{\epsilon} = -(r_3 - r_1)^{3/2} \frac{t}{\epsilon K'(m)} = -\frac{t}{\epsilon} \cdot \frac{2m(1 - m)}{E(m) - (1 - m)K(m)}. \tag{5.4}$$

Hence $\theta(x,0)=0$ and the lower limits of integration in equation (4.17) vanish. The leading-order modulated travelling wave solution u_0 in equation (4.4) which now takes the form

$$u_0 = 1 - m + 2m\,\mathrm{cn}^2\,(2K(m)\theta; m)\,, \tag{5.5}$$

satisfies the step IC behind the step for $x<0$ where the amplitude of the cnoidal function $2m=0$. In order for u_0 to satisfy also the IC $u_0=0$ in front of the step for $x>0$, where $m\to 1$, one needs to have $\lim_{m\to 1}\mathrm{cn}(2K(m)\theta; m)=0$ there which implies $\theta(x,0)=\pm 1/2$ for $x>0$. Thus, for consistency with equation (5.4), the total phase θ must contain a constant phase shift $\pm 1/2$, i.e. $\theta(x,t)=\theta_0(x,t)/\epsilon \pm 1/2+\cdots$, where dots stand for any possible further corrections of order zero or higher in ϵ which must vanish identically at $t=0$. Then, all higher-order corrections to u_0 must also vanish identically at $t=0$. The numerics presented below for the step IC clearly demonstrate that the additional phase shift described by dots in the last equation is of order ϵ. We numerically determine the positions of the maxima of the computed solution $u(x,t)$ and compare them with those of the approximate theoretical leading order solution u_0 with $\theta=\theta_0/\epsilon+1/2$.

(a) Numerical methods

This section describes the numerical scheme used to solve the KdV equation (2.1) with step initial data equation (5.1). The idea is to introduce a source function that 'cancels out' the solutions of equation (2.1) at $x\to\pm\infty$. This 'cancellation' function reformulates the problem into one that has zero boundary conditions (BCs) on both sides. From there, we utilize fast Fourier methods to approximate spatial derivatives and implement an exponential time-differencing Runge–Kutta (ETDRK) scheme to integrate. The ETD class of methods are ideal for problems like KdV since they solve the rapidly oscillating part of the equation exactly.

To begin, consider decomposing the solution $u(x,t)$ as

$$u(x,t)=v(x,t)+w(x), \tag{5.6}$$

with the accompanying boundary conditions: $v\to 0$ as $|x|\to\infty$, and $w(x)$ satisfies the step BC to match equation (5.1). The localized function $v(x,t)$ is unknown and must be solved for. The 'cancellation' function $w(x)$ is chosen with the appropriate BCs. We typically take something simple that is easy to differentiate exactly, such as

$$w(x)=\frac{1-\tanh(x)}{2}. \tag{5.7}$$

For solutions of the form in equation (5.6), the governing equation (2.1) is expressed as

$$v_t+3(v^2)_x+6(vw)_x+\epsilon^2 v_{xxx}=-3(w^2)_x-\epsilon^2 w_{xxx}, \tag{5.8}$$

with IC: $v(x,0)=u(x,0)-w(x)$. The initial step $u(x,0)$ is numerically approximated by a sharp (relative to ϵ) hyperbolic tangent function of

$$u(x,0)=\frac{1-\tanh(x/\delta)}{2}, \tag{5.9}$$

where $\delta=\epsilon/10$. Note that this equation has zero BCs at infinity. We approximate all spatial derivatives of $v(x,t)$ by Fourier methods. A wide computational domain is used to ensure waves from the linear edge of the DSW do not propagate through the periodic BCs and back into the DSW region. We integrate (5.8) by the ETDRK4 scheme described in [20].

After numerically solving for $u(x,t)$, the maxima values in the DSW region are computed. These points are all the local maxima located within the interval $-6t\le x\le 4t$. A diagram illustrating this is shown in figure 1. A maximum value $x_{\max}$ at time t is converted to

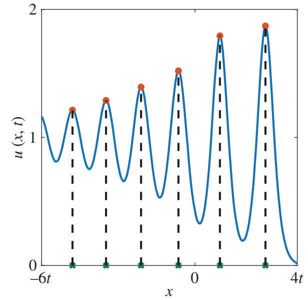

Figure 1. Locations of DSW maxima, $x_{\max}$. The maxima in terms of the modulus are computed via equation (5.10). (Online version in colour.)

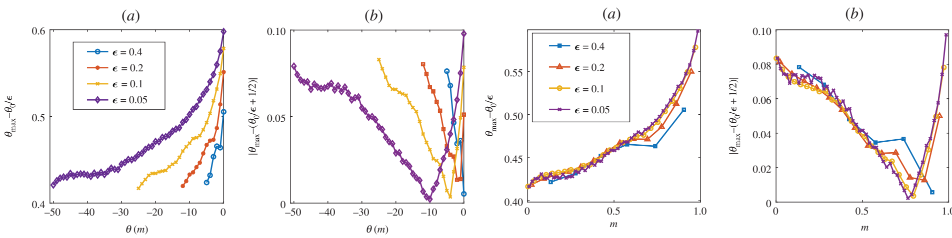

Figure 2. Left: The difference between the numerically computed DSW maxima (integers) and the asymptotic approximations for different values of ϵ at time $t = 1$: (*a*) θ_0/ϵ and (*b*) $\theta_0/\epsilon + 1/2$ versus the function $\theta_0(m)$ given by equation (5.4). This is evaluated at the numerically computed maxima locations shown in figure 1. Right: The difference between the numerically computed DSW maxima (integers) and the asymptotic approximations for different values of ϵ at time $t = 1$: (*a*) θ_0/ϵ and (*b*) $\theta_0/\epsilon + 1/2$ versus the modulus m. (Online version in colour.)

corresponding value of the elliptic modulus $m_{\max}$ through the GP formula equation (5.2)

$$\frac{x_{\max}}{t} = 2(1 + m_{\max}) + \frac{4m_{\max}(1 - m_{\max})}{1 - m_{\max} - E(m_{\max})/K(m_{\max})}, \tag{5.10}$$

using a root-finding method to invert equation (5.10). The maxima of the asymptotic solution equation (5.5) occur at integer values of θ, i.e. where

$$\theta_{\max} = \theta(m_{\max}) = \frac{\theta_0}{\epsilon} + \theta_* + \cdots \bigg|_{m=m_{\max}} = n \in \mathbb{Z}. \tag{5.11}$$

To approximate the phase shift θ_* we take the difference between a set of integers and θ_0/ϵ. It is arbitrary where to begin the integers; different starting values result in θ_* being shifted by an integer amount. We take $n = 0$ at the largest DSW peak, that is the maximum nearest to $m = 1$. The integers decrease as m decreases, resulting in the phase values $\theta_{\max} = \{0, -1, -2, \ldots\}$.

(b) Numerical results

The difference in the maximum phase values, $\theta_{\max}$ and θ_0/ϵ is shown in figure 2 left(*a*) at time $t = 1$. As ϵ decreases, the number of maxima points increases. For each value of ϵ, the value of θ_* is found to be approximately 1/2, i.e. θ_0/ϵ takes half-integer values at the maxima. A comparison between $\theta_{\max}$ and the approximation $\theta_0/\epsilon + 1/2$ is given in figure 2 left(*b*). Overall there is excellent agreement between the two curves, with error less than 0.1 (i.e. $\leq 2\epsilon$) for all cases considered. The largest disagreement comes near $m = 0$, where there is no improvement as ϵ decreases, and $m = 1$, where there is a slight growth in the error as ϵ decreases. Another

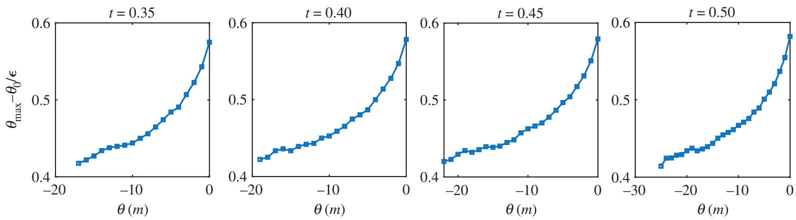

Figure 3. Difference between DSW maxima $\theta_{\max}$ and θ_0/ϵ at different times for $\epsilon = 0.05$. (Online version in colour.)

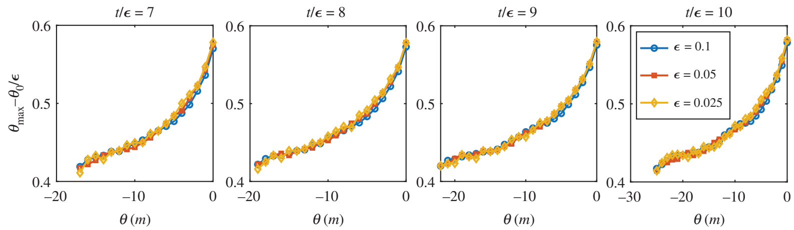

Figure 4. Difference between $\theta_{\max}$ and θ_0/ϵ for different values of t/ϵ. Each curve corresponds to a different value of ϵ, and t is chosen so that t/ϵ is constant. (Online version in colour.)

enlightening view of these results is presented in figure 2 right. Here, the abscissa is converted to values of m using the GP formula (5.4). We see that the results are tending to an asymptotic result as $\epsilon \to 0$. There might be intermediate/transition regions near $m = 1$ and perhaps $m = 0$ where the solution is governed by different scalings and the formulae in the DSW region may not be uniformly applicable.

Next, we seek to establish that, for large times, θ_* depends only on m, and not *both* m and t. The difference between $\theta_{\max}$ and θ_0/ϵ at several different times t is shown in figure 3. The snapshot series in figure 3 indicates that once t/ϵ is sufficiently large, the corrections $\theta_* + O(\epsilon)$ approach a steady state. The small deviations can be attributed to the sensitive nature of tracking the maximum point $x_{\max}$ and converting it into a modulus $m_{\max}$. Finally, numerics imply that the mode profile has time dependence $\theta_*(m(t/\epsilon))$. In figure 4, the phase difference is shown for the same value of t/ϵ, but different values of ϵ. These figures demonstrate that the phase does not depend on t and ϵ separately.

6. Whitham equations with additional phase shift

There is a variation of Whitham theory which explicitly introduces some phase shift that is independent of the leading order fast phase. This was studied, for example, by Haberman [19]. We employ this idea here. It is a natural generalization of the previous approach: when one integrates the leading order ODE equation (4.1), the 'integration constant' θ_* is in general a slow function of x, t, independent of θ. We now look for a solution u of equation (2.1) with fast and slow scales in the form

$$u = u(\theta, x, t; \epsilon), \quad \theta = \frac{\theta_0}{\epsilon} + \theta_*, \quad 0 < \epsilon \ll 1, \tag{6.1}$$

where we define the slowly varying quantities k and V by $\partial_x \theta_0 = k$, $\partial_t \theta_0 = -kV$. Then, the consideration parallels that of §3. Again we have equation (2.4) and the other Whitham equations are derived as secularity conditions ensuring that u is periodic in θ. The KdV equation, after introducing fast and slow scales, takes the form of equation (3.1) with substitutions $-(kV/\epsilon) \to -(kV/\epsilon) + \partial_t \theta_*$ and $k/\epsilon \to (k/\epsilon) + \partial_x \theta_*$. We again denote $\partial_\theta f = f'$ and also use the notation $\tilde{k} = k + \epsilon \partial_x \theta_*$. Multiplied by ϵ, equation (3.1) with the above replacements can be written as equation (3.2) with

$$kF' = \partial_t u + 6u\partial_x u + \frac{(\tilde{k}^3 - k^3)}{\epsilon}u''' + \partial_t\theta_* u' + 6\partial_x\theta_* uu' + 3\tilde{k}^2\partial_x u'' + \frac{3}{2}\partial_x(\tilde{k}^2)u''$$
$$+ \epsilon\left(3\tilde{k}\partial_{xx}u' + 3\partial_x\tilde{k}\partial_x u' + \partial_{xx}\tilde{k}u'\right) + \epsilon^2\partial_{xxx}u. \tag{6.2}$$

As in §3, two exact secularity conditions are obtained from equation (3.2) as $\overline{F'} = 0$ and $\overline{uF'} = 0$:

$$\partial_t Q + 3\partial_x Q_2 + \epsilon^2\partial_{xxx}Q = 0 \tag{6.3}$$

$$\partial_t Q_2 + 4\partial_x Q_3 - 3\partial_x\left(\frac{\tilde{k}^2}{k^2}G\right) + 3\epsilon\partial_x\left(\tilde{k}\overline{(u\partial_x u' - u'\partial_x u)}\right) + \epsilon^2\partial_x\left(\partial_{xx}Q_2 - 3\overline{(\partial_x u)^2}\right) = 0. \tag{6.4}$$

Next, we bring equations (6.3), (6.4) to the form with fewer dependent variables, exactly like in §3. We integrate equation (3.2) to get equation (3.8) again only with different F in it, now

$$kF = \partial_t J_1 + 3\partial_x J_2 + \frac{(\tilde{k}^3 - k^3)}{\epsilon}u'' + \partial_t\theta_* u + 3\partial_x\theta_* u^2 + 3\tilde{k}^2\partial_x u' + \frac{3}{2}\partial_x(\tilde{k}^2)u'$$
$$+ \epsilon\left(3\tilde{k}\partial_{xx}u + 3\partial_x\tilde{k}\partial_x u + \partial_{xx}\tilde{k}u\right) + \epsilon^2\partial_{xxx}J_1, \tag{6.5}$$

functions J_n are defined by equations (3.10) and (3.11). Multiplying equation (3.8) by $2u'$ and integrating over θ one obtains equation (3.12) only with different uF and $\int uF'$,

$$k\int uF' = \frac{\partial_t J_2}{2} + \partial_x(2J_3 - \frac{3}{2}G_1) + \frac{(\tilde{k}^3 - k^3)}{\epsilon}((uu')' - \frac{3}{2}(u')^2) + \partial_t\theta_*\frac{u^2}{2} + 2\partial_x\theta_* u^3$$
$$+ 3\tilde{k}u\partial_x(\tilde{k}u') + \epsilon\left(\frac{3}{2}\partial_x\left(\tilde{k}\int(u\partial_x u' - u'\partial_x u)\right) + \frac{u}{2}(3\tilde{k}\partial_{xx} + 3\partial_x\tilde{k}\partial_x + \partial_{xx}\tilde{k})u\right)$$
$$+ \frac{\epsilon^2}{2}\left(\partial_{xxx}J_2 - 3\partial_x\int(\partial_x u)^2\right). \tag{6.6}$$

Here, we defined $G_1 = \tilde{k}^2\int(u')^2 = (\tilde{k}^2/k^2)G\theta + \hat{G}_1$, $\overline{\hat{G}_1} = 0$, so that $\overline{G_1} = \tilde{k}^2/k^2 \cdot G/2$ and the other formulas before and in equation (3.14) of §3 still hold.

Proposition 6.1.

$$\partial_t Q + \partial_x(VQ + C_1) - \epsilon\partial_x\left(\frac{\partial_t\theta_* Q + \partial_x\theta_*(VQ + C_1)}{\tilde{k}}\right)$$
$$- \epsilon^2\partial_x\left(2\partial_{xx}Q + \frac{3\partial_x\tilde{k}}{\tilde{k}}\partial_x Q + \frac{\partial_{xx}\tilde{k}}{\tilde{k}}Q\right) = 0 \tag{6.7}$$

and

$$\partial_t P + \partial_x H - \epsilon\left[\partial_t Q_* + \partial_x(VQ_*) + \partial_x\left(\frac{\partial_t\theta_* P + \partial_x\theta_* H}{\tilde{k}}\right)\right]$$
$$- \epsilon^2\left[\partial_t Q_d + \partial_x(VQ_d) - \partial_x\left(\frac{(\partial_t\theta_* + V\partial_x\theta_*)Q_*}{\tilde{k}}\right) + \partial_x\left(\frac{\partial_{xx}P}{2} + \frac{3\partial_x\tilde{k}}{2\tilde{k}}\partial_x P + \frac{\partial_{xx}\tilde{k}}{\tilde{k}}P\right)\right]$$
$$+ \epsilon^3\partial_x\left[\frac{(\partial_t\theta_* + V\partial_x\theta_*)Q_d}{\tilde{k}} + \frac{(\tilde{k}\partial_{xx} + 3\partial_x\tilde{k}\partial_x + 2\partial_{xx}\tilde{k})Q_*}{2\tilde{k}}\right]$$
$$+ \epsilon^4\partial_x\left(\frac{\partial_{xx}Q_d}{2} + \frac{3\partial_x\tilde{k}}{2\tilde{k}}\partial_x Q_d + \frac{\partial_{xx}\tilde{k}}{\tilde{k}}Q_d\right) = 0, \tag{6.8}$$

where we denoted (recall that $\tilde{k} = k + \epsilon\partial_x\theta_*$*)* $P = VQ + C_1$, $H = VP - 3C_2$,

$$\tilde{k}Q_* = \partial_t\theta_* Q + \partial_x\theta_*(VQ + C_1), \quad \tilde{k}Q_d = 3\partial_x(\tilde{k}\partial_x Q) + \partial_{xx}\tilde{k}Q. \tag{6.9}$$

The proof is similar to that of proposition 3.1 and is omitted. Equations (6.7), (6.8) and (2.4) comprise the exact non-perturbative Whitham system in physical variables V, C_1, C_2, k, Q and θ_*; again it is not closed but is a starting point to get the Whitham equations to any order in ϵ.

Recall the consideration in the beginning of §4. We keep the same definition equation (4.1) here and obtain the same equations (4.4)–(4.7) leading to the KdV Riemann invariants $r_1 \le r_2 \le r_3$ as in equation (4.8) and to expressions in equation (4.9). To order ϵ, secularity equations (6.7) and (6.8) are

$$\partial_t Q + \partial_x(VQ + C_1 - \epsilon Q_*) = 0, \quad Q_* \equiv \frac{\partial_t\theta_* Q_0 + \partial_x\theta_*(VQ_0 + C_1)}{k} \tag{6.10}$$

$$\begin{aligned}&\partial_t(VQ + C_1) + \partial_x\left(V(VQ + C_1) - 3C_2\right)\\ &\quad - \epsilon\left[\partial_t Q_* + \partial_x\left(2VQ_* + \frac{C_1\partial_t\theta_* - 3C_2\partial_x\theta_*}{k}\right)\right] = 0,\end{aligned} \tag{6.11}$$

where now Q_* is the leading order of Q_* in the previous subsection. We subtract the leading order equation (4.6) from equation (6.10) and similarly equation (4.7) from equation (6.11). The result is a linear homogeneous closed PDE system for the phase shift θ_* and $O(\epsilon)$ correction to the mean Q_1, $\epsilon Q_1 = Q - Q_0$. We will work with the first of these, equation (6.12), and their combination, the second minus V times the first, as the PDE system to study:

$$\partial_t Q_1 + \partial_x(VQ_1 - Q_*) = 0 \tag{6.12}$$

$$(\partial_t V + V\partial_x V)Q_1 - \partial_t Q_* - Q_*\partial_x V - \partial_x\left(VQ_* + \frac{C_1\partial_t\theta_* - 3C_2\partial_x\theta_*}{k}\right) = 0. \tag{6.13}$$

(a) Solution for the step IC

Consider the step IC equation (5.1) and recall the GP solution given by equations (5.2) and (5.4). We try to solve equations (6.12) and (6.13) for θ_* and Q_1 in this case. From step IC, we expect the problem to be self-similar; this leads us to assume that $\theta_* = \theta_*(m)$. Then $\partial_x\theta_* = \theta_*'(m)/tv_2'(m)$, $\partial_t\theta_* = -(v_2\theta_*'(m)/tv_2'(m))$, by the GP formula. It is now convenient to change independent variables from x, t to m, t; for the derivatives we obtain

$$\partial_x = \frac{1}{tv_2'(m)}\partial_m, \quad \partial_t \to \partial_t - \frac{v_2(m)}{tv_2'(m)}\partial_m, \tag{6.14}$$

where in the last formula the t-derivative at constant x is expressed in terms of t-derivative at constant m and m-derivative. For the step IC, we have, according to equations (4.8) and (4.9),

$$\begin{aligned}&r_1 = 0, \quad r_2 = m, \quad r_3 = 1, \quad k^2 = \frac{1}{4K^2(m)}, \quad Q_0 = 2\frac{E(m)}{K(m)} - 1 + m,\\ &V = 2(1+m), \quad C_1 = (1-m)^2, \quad C_2 = -2(1-m)(1-m^2),\end{aligned} \tag{6.15}$$

i.e. all of these quantities are functions of m only. We obtain

$$Q_* = \frac{[C_1 - (v_2 - V)Q_0]\theta_*'(m)}{tkv_2'(m)} = \frac{q_*(m)}{t}, \quad q_*(m) = \frac{[C_1 - (v_2 - V)Q_0]\theta_*'(m)}{kv_2'(m)}. \tag{6.16}$$

Thus, equations (6.12) and (6.13) become (from now on 'prime' means $f' \equiv df(m)/dm$)

$$\partial_t Q_1 - \frac{(v_2 - V)}{tv_2'}\partial_m Q_1 + \frac{V'Q_1}{tv_2'} - \frac{q_*'}{t^2 v_2'} = 0 \tag{6.17}$$

$$-\frac{(v_2 - V)V'}{tv_2'}Q_1 + \frac{q_*}{t^2} + \frac{(v_2 - V)}{t^2 v_2'}q_*' - \frac{2V'}{t^2 v_2'}q_* + \frac{1}{t^2 v_2'}\left(\frac{(C_1 v_2 + 3C_2)\theta_*'}{kv_2'}\right)' = 0. \tag{6.18}$$

One is thus led to take Q_1 of the form $Q_1 = q_1(m)/t$, consistent with the implication of similarity, i.e. the ϵ-expansion is in fact an expansion in powers of ϵ/t. Then, the PDEs become ODEs in m

$$[(v_2 - V)q_1 + q_*]' = 0, \Longrightarrow (v_2 - V)q_1 + q_* = 3s_1 = \text{const.} \tag{6.19}$$

$$-V'[(v_2 - V)q_1 + q_*] + \left((v_2 - V)q_* + \frac{(C_1 v_2 + 3C_2)\theta_*'}{kv_2'}\right)' = 0. \tag{6.20}$$

After using equation (6.19) in equation (6.20), the last becomes a total derivative and integrates to

$$(v_2 - V)q_* + \frac{(C_1 v_2 + 3C_2)\theta_*'}{kv_2'} - 3s_1 V = s_2 = \text{const.} \tag{6.21}$$

Now $\theta_*'(m)$ is determined from equation (6.21) and then $q_1(m)$ is found from equation (6.19). There are two unknown constants s_1 and s_2 here. We find that $\theta_*'(m)$ has a singularity as $m \to 1$. In order to make the singularity the mildest possible, one has to choose s_2 such that $3s_1 V + s_2 = -6s_1 (1 - m)$. Then equations (6.15), (5.2) and expressions $K'(m) = (E(m) - (1 - m)K(m))/(2m(1 - m))$, $E'(m) = (K(m) - E(m))/2m$, when used in equation (6.21), give the final formula for θ_*', with still undetermined constant s_1,

$$\theta_*'(m) = \frac{3s_1[2(2m-1)E(m) + (1-m)(2-3m)K(m)]}{2(1-m)[(1+m)E^2(m) + 2(4m^2+m-1)K(m)E(m) + (1-m^2)(1-3m)K^2(m)]}. \tag{6.22}$$

Then one can verify that

$$\theta_*'(m) \approx \frac{3s_1}{4(1-m)\ln(16/(1-m))} \quad \text{as } m \to 1,$$

but even this mildest possible singularity is not integrable, therefore we do not get non-singular solution for θ_*. Moreover, function $q_1(m)$ can be then found from equation (6.19) in the form

$$q_1(m) = \frac{3s_1 K'}{4K} \cdot \frac{[(1-m)(5m-1)(K')^2 - 8mKK' - 2K^2]}{[(1-m^2)(K')^2 + (1+3m)KK' + K^2]}. \tag{6.23}$$

As $m \to 1$, $K(m) \approx \ln(16/(1-m))/2$ and $K'(m) \approx 1/(2(1-m))$, so one finds

$$q_1(m) \approx -\frac{3s_1}{2(1-m)\ln(16/1-m)} \quad \text{as } m \to 1,$$

the same non-integrable singularity. Recalling that $Q_1 = q_1/t$ is the first correction to the mean $\overline{u}$, its singularity looks particularly unrealistic. To remove it we need to take $s_1 = 0$ which gives $\theta_* = \text{const}$ and returns us to the situation of §3 and 4. One could ascribe the singularities at the solitonic edge of the DSW region to the necessity of a transition layer there with different scaling not captured by the current theory. Still then the only way to have s_1 consistent with numerics in §5 is to make it small, not larger than of order ϵ. This again leads back to the setting of §3 and 4. We also note another difficulty for the current approach: equations (2.4), (6.7), and (6.8) may not suffice to solve for all the variables V, C_1, C_2, Q and θ_*.

7. Whitham theory: linear case

To compare the above with what Whitham theory for linearized KdV equation

$$\partial_t u + \epsilon^2 \partial_{xxx} u = 0, \tag{7.1}$$

yields, we consider its leading order $k^2 u_0''' - V u_0' = 0$ solved by

$$u_0 = c + A\cos\theta = c + A\cos\left(\frac{\theta_0}{\epsilon} + \theta_*\right), \quad \partial_x\theta_0 = k, \quad \partial_t\theta_0 = -kV, \tag{7.2}$$

c, A, V and θ_* being independent parameters. The linear dispersion relation (or fixed constant period condition which is equivalent to it here) implies

$$k = (-V)^{1/2}, \tag{7.3}$$

where we have in effect chosen a normalization for θ_0 and the period. Then conservation of waves is a Hopf equation for V,

$$\partial_t V + 3V\partial_x V = 0. \tag{7.4}$$

The first usual secularity condition for the next order equation yields $\partial_t \overline{u}_0 = 0$ which means that c is a constant (not slowly varying). The second secularity condition from §3 (at leading order, with terms only from linear part of KdV) fixes the evolution equation for the amplitude A as

$$\partial_t A^2 + 3\partial_x(VA^2) = 0. \tag{7.5}$$

The correction u_1 has even and odd parts. Its even part u_{1e} is again tied with θ_* as

$$[k^3(u''_{1e} + u_{1e}) + u_0 D\theta_* - 3cV\partial_x\theta_*]' = 0, \tag{7.6}$$

where $D \equiv \partial_t + 3V\partial_x$. The general even solution of equation (7.6) reads

$$k^3 u_{1e} = -\frac{AD\theta_*}{2}\theta\sin\theta - C_e + \gamma_0 A\cos\theta, \tag{7.7}$$

with arbitrary slow C_e and γ_0. Enforcing periodicity yields the first-order PDE for θ_*

$$\partial_t\theta_* + 3V\partial_x\theta_* = 0. \tag{7.8}$$

Thus, here, in contrast to the nonlinear case, the dynamics of θ_* is fixed by a usual secularity condition controlling periodicity rather than growth of the solution u as function of fast variable θ_0/ϵ. Then, the IC $u_{1e}(x,0) = 0$ is achieved if one takes $C_e(x,0) = \gamma_0(x,0) = 0$. Both equations (7.5) and (7.8) depend on the solution of Hopf equation (7.4) implicitly given by

$$x = 3Vt + x_0(V) \quad \text{and} \quad x_0(V(x,0)) = x, \tag{7.9}$$

for a given IC $V(x,0)$. For example, if the IC for θ_* is written as $\theta_*(x,0) = \phi(V(x,0))$, then

$$x = 3\phi^{-1}(\theta_*)t + x_0(\phi^{-1}(\theta_*)) \tag{7.10}$$

is the implicit solution for $\theta_*(x,t)$. These formulae can describe the evolution of the total phase given an IC for slow variables. For example, if $\theta_*(x,0) = \text{const.}$ initially, it remains constant for all times.

On the other hand, e.g. for the step IC given by equation (5.1), the exact solution of linearized KdV is obtained by Fourier transform (the analogue of IST here); it is

$$u(x,t) = \int_{\xi}^{\infty} \mathrm{Ai}(\zeta)\,\mathrm{d}\zeta \quad \text{and} \quad \xi = \frac{x}{(3\epsilon^2 t)^{1/3}}. \tag{7.11}$$

Only its asymptotics for $\xi \ll -1$ can be described by periodic Whitham theory; it has indeed the form of leading-order solution (7.2)

$$\xi \ll -1: \quad u(x,t) \approx 1 + \frac{1}{\sqrt{\pi}|\xi|^{3/4}}\cos\left(\frac{2}{3}|\xi|^{3/2} - \frac{3\pi}{4}\right). \tag{7.12}$$

This corresponds to the following solution for the Whitham variables: constant $c = 1$ and

$$V = \frac{x}{3t}, \quad A^2 = \frac{1}{\pi|\xi|^{3/2}} = \frac{\epsilon(3t)^{1/2}}{\pi|x|^{3/2}}, \quad \frac{\theta_0}{\epsilon} = -\frac{2}{3}|\xi|^{3/2} = -\frac{2|x|^{3/2}}{3\epsilon(3t)^{1/2}}, \quad \theta_* = \frac{3\pi}{4}. \tag{7.13}$$

With this assignment all equations (7.4), (7.5), (7.2) with (7.3) and (7.8) are satisfied. However, V and θ_0 are singular as $t \to 0$; these solutions result from the asymptotics of the exact solution given in equation (7.11). Any $\theta_* = \text{const}$ is a solution of equation (7.8) but one needs to enforce ICs to get the correct value $3\pi/4$ from Whitham theory. It is unclear how to do this only within the context of Whitham theory due to the singular nature of V and θ_0. The numerics below clearly show that the phase shift is constant; there is only small 'numerical noise' around $\theta_* = 3\pi/4$ in this linear case.

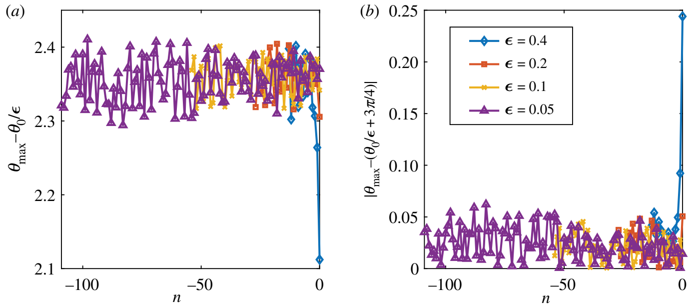

Figure 5. Difference between $\theta_{\max}$ (see text) and (*a*) θ_0/ϵ; (*b*) $\theta_0/\epsilon + \theta_*$. Each curve corresponds to a different value of ϵ. (Online version in colour.)

(a) Linear KdV numerics

For $\xi \ll -1$, the asymptotic solution of the linearized KdV equation is equation (7.12) implying equation (7.13). The maxima of $\cos(\theta)$ occur when $\theta_{\max} = 2\pi n$, $n \in \mathbb{Z}$. A comparison of the exact phase maxima and the asymptotic approximations of equation (7.12) is shown in figure 5. The difference between the exact maxima and the $O(1/\epsilon)$ approximation yields a nearly constant value for all n. In figure 5*b*, the constant is found to indeed be $\theta_* = 3\pi/4$ to within $O(\epsilon)$ error bounds. As ϵ decreases, ξ decreases, resulting in a better approximation. This improvement is different from the nonlinear case shown in figure 2 where there is more structure and indications of possible intermediate/transition regions in the neighbourhoods of $m = 0$, $m = 1$ as $\epsilon \to 0$. In figure 5, we observe only small random fluctuations around the constant phase shift which do not imply any functional dependence. This difference with the nonlinear case is expected and lends additional support to our theoretical findings.

8. Conclusion

We developed higher-order Whitham theory with a single fast phase for the KdV equation. This allowed us to determine the slow phase shift in the leading order solution and show that it is asymptotically constant for a wide range of ICs including step or steplike ICs. Other analytical possibilities which could be consistent with existing analytical results were ruled out. The role of nonlinearity and existence of a non-trivial $O(\epsilon)$ phase shift due to it predicted by our analysis is also clearly seen from comparison of our numerics for KdV. We also studied the linearized KdV equation and found that its phase shift is a constant.

The conclusion that the phase shift as a function of space and time is asymptotically constant illuminates why for so many years since the seminal work [3] the leading order solution with constant phase shift is so widely and effectively used in applications; see, e.g. recent reviews [14,15] and their rather comprehensive lists of references. We note that considerable work has been devoted to the leading order solutions for multiple phases; see, e.g. [14,18] and references therein.

Certain interesting and important issues remain for future work, in particular, the clarification of the relationship between small dispersion and long time limits originating from the space and time scaling properties of KdV equation. Numerical results also indicate the existence of intermediate/transition regions around the DSW edges. Their analytical description also presents an important problem for future work. To our knowledge, these transition regions from leading order DSW to constant solutions for steplike initial/boundary conditions have not been analytically described except for some partial results at leading order by IST in [21] and by matched asymptotic expansions assuming linear initial approximation and various matching conditions in several regions for the pure step problem in [22], both for the long-time regime.

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Appendix A. Non-secular corrections to the solution

In general, formulas (4.17) should be written as

$$u_n = \frac{u_0'(\theta)\int_{\theta(x,0)}^{\theta} w_2(z)F_n(z)\,\mathrm{d}z - w_2(\theta)\int_{\theta(x,0)}^{\theta} u_0'(z)F_n(z)\,\mathrm{d}z}{k^2W} + A_{n1}u_0'(\theta) + A_{n2}w_2(\theta), \tag{A 1}$$

where $n = 1, 2, \ldots$. Requiring periodicity of u_1 in equation (A 1) leads one to consider the difference $u_1(\theta+1) - u_1(\theta)$ (recall that we normalize θ to have period 1). We have

$$w_2(\theta+1) - w_2(\theta) = K_2u_0'(\theta) \quad \text{and} \quad K_2 = \alpha Q + \beta, \tag{A 2}$$

see end of §4. Let $\theta_i = \theta(x, 0)$. Consider the numerator of the first term in equation (A 1). We have, using periodicity of u_0' and F_1 and equation (A 2),

$$\begin{aligned}
\Delta_1 &\equiv \int_{\theta_i}^{\theta+1} (u_0'(\theta+1)w_2(z) - w_2(\theta+1)u_0'(z))kF_1(z)\,\mathrm{d}z \\
&\quad - \left(\int_{\theta_i}^{\theta} (u_0'(\theta)w_2(z) - w_2(\theta)u_0'(z))kF_1(z)\,\mathrm{d}z\right) \\
&= u_0'(\theta)\int_{\theta}^{\theta+1} w_2(z)kF_1(z)\,\mathrm{d}z - K_2u_0'(\theta)\int_{\theta_i}^{\theta+1} u_0'(z)kF_1(z)\,\mathrm{d}z \\
&\quad - w_2(\theta)\int_{\theta}^{\theta+1} u_0'(z)kF_1(z)\,\mathrm{d}z.
\end{aligned} \tag{A 3}$$

The antiderivative $\int u_0'(z)F_1(z)\,\mathrm{d}z$ is an explicit odd periodic function found by direct integration

$$\begin{aligned}
\Lambda_1(\theta) &\equiv k\int^{\theta} u_0'(z)F_1(z)\,\mathrm{d}z \\
&= (D + 2\partial_x V)\left(\frac{\hat{u}_0\hat{J}_1}{2} + \left(Q - \frac{V}{6}\right)\hat{J}_1 + \frac{k^2u_0'}{6}\right) + \frac{\hat{u}_0D\hat{J}_1 - \hat{J}_1D\hat{u}_0}{2} \\
&\quad + \frac{k}{5}\partial_x\left(\frac{\alpha}{k}\hat{J}_1 + 2k\left(u_0 - \frac{V}{6}\right)u_0'\right),
\end{aligned} \tag{A 4}$$

where we denoted $D = \partial_t + V\partial_x$, $\hat{u}_0 = u_0 - Q$, $\hat{J}_1 \equiv \hat{J}_1^0$. Thus, the averages over the period

$$\overline{\Lambda_1(\theta)} = 0, \quad \int_{\theta}^{\theta+1} u_0'(z)F_1(z)\,\mathrm{d}z = \int_0^1 u_0'(z)F_1(z)\,\mathrm{d}z = 0. \tag{A 5}$$

Next, writing $w_2(\theta) = K_2u_0'(\theta)\theta + \tilde{w}_2(\theta)$, where $\tilde{w}_2(\theta)$ is an even periodic function, and taking into account that $F_1(\theta)$ is odd periodic, we obtain

$$\int_{\theta}^{\theta+1} \tilde{w}_2(z)F_1(z)\,\mathrm{d}z = 0. \tag{A 6}$$

Thus, since $\int_\theta^{\theta+1} \Lambda_1(z)\,\mathrm{d}z = 0$ due to $\Lambda_1(z)$ being odd periodic by equation (A 4)

$$\begin{aligned} k\int_\theta^{\theta+1} w_2(z)F_1(z)\,\mathrm{d}z &= K_2\int_\theta^{\theta+1} zu_0'(z)kF_1(z)\,\mathrm{d}z \\ &= K_2\, z\Lambda_1(z)|_\theta^{\theta+1} - K_2\int_\theta^{\theta+1}\Lambda_1(z)\,\mathrm{d}z \\ &= K_2\Lambda_1(\theta) - K_2\int_\theta^{\theta+1}\Lambda_1(z)\,\mathrm{d}z = K_2\Lambda_1(\theta). \end{aligned} \tag{A 7}$$

Finally, use of equations (A 5) and (A 7) in equation (A 3) yields

$$\Delta_1 = K_2u_0'(\theta)\Lambda_1(\theta) - K_2u_0'(\theta)\int_{\theta_i}^{\theta} u_0'(z)kF_1(z)\,\mathrm{d}z = K_2u_0'(\theta)\Lambda_1(\theta_i). \tag{A 8}$$

This depends on the ICs. When $\theta_i \equiv \theta(x,0) = 0$, also $\Delta_1 = 0$ and the first term in equation (A 1) is periodic in θ. Since it is zero at $t = 0$, we obtain equation (4.17) for $n = 1$ indeed (i.e. slow variables $A_{11} = A_{12} = 0$ in this case). In general, periodicity of u_1 in equation (A 1) is achieved by taking, as is seen from equations (A 8) and (A 2),

$$A_{12} = -\frac{\Lambda_1(\theta_i)}{k^3W}. \tag{A 9}$$

Then, the IC $u_1(x,0) = 0$ is ensured by taking A_{11} such that

$$A_{11}u_0'(x,0) + A_{12}w_2(x,0) = 0. \tag{A 10}$$

Quite similar considerations apply to u_2 and higher order corrections to the solution.

Appendix B. KdV with step IC phases from IST/RHP

For the KdV equation with steplike ICs, the total phase including the phase shift was computed in [12,13] by solving a vector RHP via the steepest descent approach of Deift and Zhou [9,10]. The solution is constructed for long-time asymptotics rather than for the small dispersion limit. However, for the pure step, the former appear to be equivalent to the latter since, by rescaling x and t, the KdV equation is seen to depend only on x/ϵ and t/ϵ while the IC (5.1) does not depend on scaling. Reintroducing ϵ into the formulas of [12,13] (where $\epsilon = 1$) and taking the initial jump $c^2 = 1$ there, their result for the total phase in the DSW region reads

$$\theta = \frac{tB(\xi)}{2\pi\epsilon} + \frac{\Delta(\xi)}{2\pi} \pm \frac{1}{2}, \quad \xi \equiv \frac{x}{12t}, \tag{B 1}$$

where the leading order phase function is

$$B(\xi) = 24\int_{\sqrt{m(\xi)}}^{1}\left(\xi + \frac{1-m(\xi)}{2} - s^2\right)\sqrt{\frac{s^2 - m(\xi)}{1-s^2}}\,\mathrm{d}s, \tag{B 2}$$

and the phase shift is determined by

$$\Delta(\xi) = \frac{1}{K(m)}\int_{\sqrt{m(\xi)}}^{1}\frac{\log(4s\sqrt{1-s^2})}{\sqrt{(1-s^2)(s^2-m(\xi))}}\,\mathrm{d}s. \tag{B 3}$$

The elliptic modulus parameter $m(\xi)$ is implicitly given by an integral equation which is an equivalent form of GP equation (5.2). Note that in [12,13] the solution is expressed in terms of second log-derivative of elliptic theta-function; to match its total phase with that of the cnoidal function one has to add $\pm 1/2$ to their phase $tB(\xi)/2\pi\epsilon + \Delta(\xi)/2\pi$; see, e.g. [23]. Equations (B 2) and

(B 3) can be expressed in terms of complete elliptic integrals. Using the identities; see, e.g. [23],

$$\left.\begin{aligned} I_0 &= \int_{\sqrt{m}}^{1} \frac{\mathrm{d}s}{\sqrt{(1-s^2)(s^2-m)}} = K(1-m),\\ I_2 &= \int_{\sqrt{m}}^{1} \frac{s^2\mathrm{d}s}{\sqrt{(1-s^2)(s^2-m)}} = E(1-m),\\ \text{and}\quad I_4 &= \int_{\sqrt{m}}^{1} \frac{s^4\mathrm{d}s}{\sqrt{(1-s^2)(s^2-m)}} = \frac{2(1+m)E(1-m)-mK(1-m)}{3}, \end{aligned}\right\} \tag{B 4}$$

one gets the expression for $B(\xi)$ in the form $B(\xi)/24 = (\xi + (1+m)/2)I_2 - m(\xi + (1-m)/2)I_0 - I_4$. This is further simplified using the Legendre relation $K(m)E(1-m) + K(1-m)E(m) - K(m)K(1-m) = \pi/2$ to get the exact match of the fast phase equation (5.4) from Whitham theory

$$\frac{tB(\xi)}{2\pi\epsilon} = -\frac{t}{\epsilon}\cdot\frac{2m(1-m)}{E(m)-(1-m)K(m)}.$$

Also, using the formulas [23, p. 288],

$$\left.\begin{aligned} &\int_{\sqrt{m}}^{1} \frac{\log(s)ds}{\sqrt{(1-s^2)(s^2-m)}} = \frac{K(1-m)\ln m}{4},\\ \text{and}\quad &\int_{0}^{\sqrt{1-m}} \frac{\log(s)\,\mathrm{d}s}{\sqrt{(1-s^2)(1-m-s^2)}} = \frac{K(1-m)\ln(1-m)}{4} - \frac{\pi K(m)}{4}, \end{aligned}\right\} \tag{B 5}$$

one brings the slow phase shift to the simple form

$$\theta_* = \frac{\Delta(m(\xi))}{2\pi} \pm \frac{1}{2} = \frac{K(1-m)}{2\pi K(m)}\log[4(m(1-m))^{1/4}] - \frac{1}{8} \pm \frac{1}{2}. \tag{B 6}$$

As $m \to 1$, the phase shift approaches a constant

$$m \to 1: \quad \frac{\Delta}{2\pi} \pm \frac{1}{2} \to -\frac{1}{4} \pm \frac{1}{2}.$$

Note this is not just the $\pm\frac{1}{2}$ as it should be at this edge. In the other limit $m \to 0$, Δ diverges as

$$m \to 0: \quad \frac{\Delta}{2\pi} \approx -\frac{(\ln m)^2}{8\pi^2}.$$

The last formulas imply that proper matching of the solutions in the DSW region and, respectively, the regions ahead and behind it should remove the discrepancy and the singularity.

We observe complete agreement for the *leading order* fast phase. The non-trivial result of [13] for the next order phase shift contrasts sharply with our analytical and numerical findings. The reason is a matter for future investigation. A possible source of the discrepancy could be short times $t \sim \epsilon$ where the apparent equivalence of small dispersion and long time results for scale-invariant ICs could be broken by subtle effects related to interchange of the two limits. Also the above limits as $m \to 1$ and $m \to 0$ of the phase shift of [13] at the very least imply the necessity of intermediate/transition regions at the edges of DSW. Their existence at both edges could be consistent with our numerical results in §5. For decaying ICs, such regions have been described analytically in [24,25] in the small dispersion limit.

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