Pareto Optimal Multirobot Motion Planning

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***Abstract*—This article studies a class of multirobot coordination problems where a team of robots aim to reach their goal regions with minimum time and avoid collisions with obstacles and other robots. A novel numerical algorithm is proposed to identify the Pareto optimal solutions where no robot can unilaterally reduce its traveling time without extending others'. The consistent approximation of the algorithm in the epigraphical profile sense is guaranteed using set-valued numerical analysis. Experiments on an indoor multirobot platform and computer simulations show the anytime property of the proposed algorithm, i.e., it is able to quickly return a feasible control policy that safely steers the robots to their goal regions and it keeps improving policy optimality if more time is given.**

***Index Terms*—Multirobot coordination, Pareto optimality, robotic motion planning.**

I. Introduction

ROBOTIC motion planning is a fundamental problem where a control sequence is found to steer a mobile robot from an initial state to a goal set while enforcing dynamic constraints and environmental rules. It is well-known that the problem is computationally challenging. For example, the piano-mover problem is shown to be PSPACE-hard in general [1]. Sampling-based algorithms are demonstrated to be efficient in addressing robotic motion planning in high-dimensional spaces. The Rapidly-exploring Random Tree (RRT) algorithm [2] and its variants are able to quickly find feasible paths. However, the optimality of returned paths is probably lost. In fact, computing optimal motion planners is much more computationally challenging than finding feasible motion planners [3]. It is shown that computing the shortest path in $\mathbb{R}^3$ populated with obstacles is NP-hard in the number of obstacles [3]. Recently, RRT* [4] and its variants are shown to be both computationally efficient and asymptotically optimal.

Multirobot optimal motion planning is even more computationally challenging, because the worst-case computational complexity exponentially grows as the robot number. Current multirobot motion planning mainly falls into three categories: centralized planning [5], [6], decoupled planning [7], [8], and priority planning [9], [10]. Noticeably, none of these multirobot motion planners are able to guarantee the optimality of returned solutions. Recent papers [11], [12] employ game theory to synthesize open-loop planners and closed-loop controllers to coordinate multiple robots, respectively. It is shown that the proposed algorithms converge to Nash equilibrium [13] where no robot can benefit from unilateral deviations. As RRTs, the algorithms in [11] and [12] leverage incremental sampling and steering functions, the latter of which require to solve two-point boundary value problems. There are only a very limited number of dynamic systems whose steering functions have known analytical solutions, including single integrators, double integrators, and Dubin's cars [14]. Heuristic methods are needed to compute steering functions when dynamic systems are complicated.

In the control community, distributed coordination of multirobot systems has been extensively studied in last decades [15]–[17]. A large number of algorithms have been proposed to accomplish a variety of missions, e.g., rendezvous [18], formation control [15], vehicle routing [19], and sensor deployment [20], [21]. This set of work is mainly focused on the design and analysis of algorithms, which are scalable with respect to network expansion. To achieve scalability, most algorithms adopt gradient descent methodologies, which are easy to implement. Their long-term behavior, e.g., asymptotic convergence, can be ensured but usually there is no guarantee on transient performance, e.g., aggregate costs, due to the myopic nature of the algorithms. Another set of more relevant papers is about (distributed) receding-horizon control or model predictive control (MPC) for multirobot coordination. Representative works include [22], [23] on formation stabilization, [24], [25] on vehicle platooning, and [26] on trajectory optimization. MPC bears the following benefits [27]–[29]. First, it has a unique ability to cope with hard constraints on controls and states. Second, it can deal with system uncertainties and control disturbances and its robust stability can be formally guaranteed. Third, it is suitable for control applications requiring rapid computations thanks to its online fashion of implementation. The infinite-horizon performance of N-horizon MPC policy exponentially converges to the optimal value function of the infinite-horizon optimal control problem as the computing horizon N extends to infinity [30]. In contrast, multirobot motion planning aims to find controllers that can optimize certain cost functionals over entire missions, e.g., finding collision-free paths with shortest distances or minimum fuel consumption.

Differential games extend optimal control from single players to multiple players. Linear-quadratic differential games are the most basic, and their solutions can be formulated as coupled Riccati equations [31]. For nonlinear systems with state and input constraints, there are a very limited number of differential games whose closed-form solutions are known, and some

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examples include the homicidal chauffeur and the lady-in-the-lake games [31], [32]. Otherwise, numerical algorithms are desired. Existing numerical algorithms are mainly based on partial differential equations [33]–[35] and viability theory [36]–[38]. Noticeably, this set of papers only considers zero-sum two-player scenarios.

Contribution statement: This article investigates a class of multirobot closed-loop motion planning problems where multiple robots aim to reach their respective goal regions as soon as possible. The robots are restricted to complex dynamic constraints and need to avoid the collisions with static obstacles and other robots. Pareto optimality is used as the solution notion where no robot can reduce its own travelling time without extending others'. A numerical algorithm is proposed to identify the Pareto optimal solutions. It is shown that under mild regularity conditions, the algorithm can consistently approximate the epigraph of the minimal arrival time function. The proofs are based on set-valued numerical analysis [36]–[38], which are the first to point out the promise in extending set-valued tools to multirobot motion planning problems. Experiments on an indoor multirobot platform and computer simulations on unicycle robots are conducted to demonstrate the anytime property of our algorithm, i.e., it is able to quickly return a feasible control policy that safely steers the robots to their goal regions and it keeps improving policy optimality if more time is given. Detailed proofs are provided in Section VII. Preliminary results are included in [39] where all the proofs and experimental results are removed due to space limitation.

II. Problem Formulation

Consider a team of mobile robots labeled by $\mathcal{V} \triangleq \{1, \ldots, N\}$. The dynamic of robot i is governed by

$$\dot{x}_i(s) = f_i(x_i(s), u_i(s)) \quad \forall i \in \mathcal{V} \tag{1}$$

where $x_i(s) \in X_i$ is the state of robot i and $u_i : [0, +\infty) \to U_i$ is the control of robot i. Here, the state space and the set of all possible control values for robot i are denoted by $X_i \subseteq \mathbb{R}^{d_i}$ and $U_i \subseteq \mathbb{R}^{m_i}$, respectively. The obstacle region and goal region for robot $i \in \mathcal{V}$ are denoted by $X_i^O \subseteq X_i$ and $X_i^G \subseteq X_i \setminus X_i^O$, respectively. Denote the minimum safety distance between any two robots as $\sigma > 0$. The free region for robot i is denoted by $X_i^F \triangleq \{x_i \in X_i \setminus X_i^O | \|x_i - x_j\| \geq \sigma, x_j \in X_j^G, i \neq j\}$. Let $\mathbf{X} \triangleq \prod_{i\in\mathcal{V}} X_i$, $\mathbf{X}^G \triangleq \prod_{i\in\mathcal{V}} X_i^G$, and $\mathbf{X}^F \triangleq \prod_{i\in\mathcal{V}} X_i^F$. Assume $\|x_i - x_j\| \geq \sigma \quad \forall x \in \mathbf{X}^G, i \neq j$. Define the safety region as $\mathbf{S} \triangleq \{x \in \mathbf{X}^F | \|x_i - x_j\| \geq \sigma, i \neq j\}$. Here, $\|\cdot\|$ denotes the two-norm.

The sets of state feedback control policies for robot i and the whole robot team are defined as $\varpi_i \triangleq \{\pi_i(\cdot) : \mathbf{X} \to U_i\}$ and $\varpi \triangleq \{\prod_{i\in\mathcal{V}} \pi_i(\cdot) | \pi_i(\cdot) \in \varpi_i\}$, respectively. Consider the scenario where the robot team starts from $x \in \mathbf{X}$ and executes policy $\pi \in \varpi$. The induced minimal arrival time vector is characterized as $\vartheta(x, \pi) \triangleq \inf\{t \in \bar{\mathbb{R}}^N_{\geq 0} | \quad \forall i \in \mathcal{V}, x_i(0) = x_i, \dot{x}_i(s) = f_i(x_i(s), \pi_i(x(s))), x(s) \in \mathbf{S}, x_i(t_i) \in X_i^G, 0 \leq s \leq \max_{i\in\mathcal{V}} t_i\}$, where the infimum uses the partial order in footnote 1. The ith element of $\vartheta(x, \pi)$ represents the first time robot i reaches its goal region without collisions when the robot team starts from initial state x and executes policy π. In our multirobot motion planning problem, the minimal arrival time function $\Theta^* : \mathbf{X} \rightrightarrows \bar{\mathbb{R}}^N_{\geq 0}$ is a set-valued map and is defined as $\Theta^*(x) \triangleq \mathcal{E}[\text{cl}(\{\vartheta(x, \pi) | \pi \in \varpi\})]$, where $\mathcal{E}$ is the Pareto minimization defined as $\mathcal{E}(\mathcal{T}) \triangleq \{\tau \in \mathcal{T} | \nexists \tau' \in \mathcal{T}$ s.t. $\tau' \neq \tau$ and $\tau' \preceq \tau\}$ for $\mathcal{T} \subseteq \mathbb{R}^N_{\geq 0}$ and $\text{cl}(\cdot)$ is the closure. The closure ensures the existence of $\Theta^*(x)$ per [40, Th. 4.1]. The vectors in $\Theta^*(x)$ indicate that no robot can unilaterally reach its goal region earlier without extending other robots' travelling times. The associated set of Pareto optimal solutions is defined as $\mathcal{U}^*(x) \triangleq \{\pi^* \in \varpi | \vartheta(x, \pi^*) \in \Theta^*(x)\}$. Note that the elements of $\vartheta(x, \pi^*)$ could be infinite, indicating that some robots cannot safely reach their goal regions. Infinite time may cause numerical issues. To tackle this, transformed minimal arrival time function is defined as $v^*(x) \triangleq \Psi(\Theta^*(x))$, where Kruzhkov transform $\Psi(t) \triangleq \begin{bmatrix} 1 - e^{-t_1} \\ 1 - e^{-t_2} \\ \vdots \\ 1 - e^{-t_N} \end{bmatrix}$ for $t \in \bar{\mathbb{R}}^N_{\geq 0}$ normalizes $[0, +\infty]$ to [0,1]. Notice that Kruzhkov transform is bijective and monotonically increasing.

The objective of this article is to identify optimal control policies in $\mathcal{U}^*(x)$ and the corresponding minimal arrival time function $\Theta^*(x)$ (or equivalently $v^*(x)$).

III. Assumptions and Notations

This section summarizes the assumptions, notions, and notations used throughout this article. Most notions and notations on sets and set-valued maps follow the presentation of [41].

The multirobot system (1) can be written in the differential inclusion form: $\dot{x}_i(s) \in F_i(x_i(s)) \quad \forall s \geq 0$, where the set-valued map $F_i : X_i \rightrightarrows \mathbb{R}^{d_i}$ is defined as $F_i(x_i) \triangleq \{f_i(x_i, u_i) | u_i \in U_i\}$. Let $F(x) \triangleq \prod_{i\in\mathcal{V}} F_i(x_i)$. The following assumptions are imposed.

Assumption III.1: The following properties hold for $i \in \mathcal{V}$.

A1) X_i and U_i are nonempty and compact.
A2) $f_i(x_i, u_i)$ is continuous over both variables.
A3) $f_i(x_i, u_i)$ is linear growth, i.e., $\exists c_i \geq 0$ s.t. $\forall x_i \in X_i$ and $\forall u_i \in U_i, \|f_i(x_i, u_i)\| \leq c_i(\|x_i\| + \|u_i\| + 1)$.
A4) For each $x_i \in X_i$, $F_i(x_i)$ is convex.
A5) $F_i(x_i)$ is Lipschitz with Lipschitz constant l_i.

Assumptions (A1) and (A2) imply $\|f_i(x_i, u_i)\|$ is bounded for each $i \in \mathcal{V}$. Define $M_i \triangleq \max_{x_i\in X_i, u_i\in U_i} \|f_i(x_i, u_i)\|$ and let $M^+ \triangleq \sqrt{\sum_{i\in\mathcal{V}} M_i^2}$ and $l^+ \triangleq \sqrt{\sum_{i\in\mathcal{V}} l_i^2}$. Then, F is bounded by M^+ and is l^+-Lipschitz.

Remark III.1: One sufficient condition of Assumption (A4) is that $f_i(x_i, u_i)$ is linear with respect to u_i and U_i is convex. One sufficient condition of Assumption (A5) is that $f_i(x_i, u_i)$ is Lipschitz continuous with respect to both variables on $X_i \times U_i$. □

Define the distance from a point $x \in \mathcal{X}$ to a set $A \subseteq \mathcal{X}$ as $d(x, A) \triangleq \inf\{\|x - a\| | a \in A\}$. A closed unit ball around $x \in \mathcal{X}$ in space $\mathcal{X}$ is denoted as $x + \mathcal{B}_{\mathcal{X}} \triangleq \{y \in \mathcal{X} | \|y - x\| \leq 1\}$. Similarly, δ expansion of a set $A \subseteq \mathcal{X}$ is defined as $A + \delta\mathcal{B}_{\mathcal{X}} \triangleq \{x \in \mathcal{X} | d(x, A) \leq \delta\}$ for some $\delta \geq 0$. Specifically, we denote $x + \mathcal{B}_N \triangleq \{y \in \mathbb{R}^N | \|y - x\| \leq 1\}$ if $x \in \mathbb{R}^N$. Similar notation applies to a set A. The subscript of

[1] Throughout this article, product order is imposed, i.e., two vectors $a, b \in \mathbb{R}^N$ are said "a is less than b in the Pareto sense," denoted by $a \preceq b$, if and only if $a_i \leq b_i \quad \forall i \in \{1, \ldots, N\}$. Similarly, strict inequality can be defined by $a \prec b \iff a_i < b_i \quad \forall i \in \{1, \ldots, N\}$.

closed unit ball may be omitted when there is no ambiguity. The Hausdorff distance that measures the distance of two sets A and B is defined by $d_H(A,B) \triangleq \inf\{\delta \geq 0 | A \subseteq B + \delta\mathcal{B}, B \subseteq A + \delta\mathcal{B}\}$. Kuratowski lower limit and Kuratowski upper limit of sets $\{A_n\} \subseteq \mathcal{X}$ are denoted by $\mathrm{Liminf}_{n\to+\infty} A_n = \{x \in \mathcal{X} | \lim_{n\to+\infty} d(x, A_n) = 0\}$ and $\mathrm{Limsup}_{n\to+\infty} A_n = \{x \in \mathcal{X} | \liminf_{n\to+\infty} d(x, A_n) = 0\}$, respectively. If $\mathrm{Liminf}_{n\to+\infty} A_n = \mathrm{Limsup}_{n\to+\infty} A_n$, the common limit is defined as Kuratowski limit $\mathrm{Lim}_{n\to+\infty} A_n$.

The Pareto frontier of a nonempty set $A \subseteq \mathcal{X}$ is denoted as $\mathcal{E}(A) \triangleq \{t \in A | \nexists t' \in A \text{ s.t. } t' \neq t, t' \preceq t\}$. Let $A + B \triangleq \{a + b | a \in A, b \in B\}$ be the sum of two sets A and B. Denote the n-fold Cartesian product of a set A by A^n. Specifically, when A is an interval, e.g., $A = [a,b]$, its n-fold product is denoted by $[a,b]^n$. When A is a singleton, e.g., $A = \{a\}$, its n-fold product is written as $\{a\}^n$. Let $A \times \{b\} \triangleq \{(a,b) | a \in A\}$ be the Cartesian product of a set A and a point b. Define Hadamard product for two vectors $a, b \in \mathbb{R}^N$ as $a \circ b \triangleq \begin{bmatrix} a_1 b_1 & \cdots & a_N b_N \end{bmatrix}^T$. Define $a \circ B \triangleq \{a \circ b | b \in B\}$. Denote N-dimensional zero vector and all-ones vector by $\mathbf{0}_N$ and $\mathbf{1}_N$, respectively. The subscript may be omitted when there is no ambiguity. The cardinality of a set is denoted as $|\cdot|$.

Define the distance between two set-valued maps $g, \bar{g} : \mathbf{X} \rightrightarrows [0,1]^N$ by $d_{\mathbf{X}}(g, \bar{g}) \triangleq \sup_{x\in\mathbf{X}} d_H(g(x), \bar{g}(x))$.

Definition III.1 (Epigraph): The epigraph of Θ is defined by $\mathcal{E}pi(\Theta) \triangleq \{(x,t) \in \mathcal{X} \times \mathbb{R}^N | \exists t' \in \Theta(x) \text{ s.t } t \succeq t'\}$.

Definition III.2 (Epigraphical Profile): The epigraphical profile of Θ is defined by $E_\Theta(x) \triangleq \Theta(x) + \mathbb{R}^N_{\geq 0}$.

Remark III.2: For a Kruzhkov transformed function v, we define its epigraphical profile by $E_v(x) \triangleq (v(x) + \mathbb{R}^N_{\geq 0}) \cap [0,1]^N$. □

IV. Algorithm Statement and Performance Guarantee

In this section, we present our algorithmic solution and summarize its convergence in Theorem IV.1.

A. Algorithm Statement

The proposed algorithms, Algorithms 1–3, are informally stated as follows. The state space of each robot is discretized by a sequence of finite grids $\{X_i^p\} \subseteq X_i$ s.t. $X_i^p \subseteq X_i^{p+1} \quad \forall p \geq 1$, where p is the grid index and by convention $X_i^0 = \emptyset$. The state space for the robot team is discretized by $\{\mathbf{X}^p\} \subseteq \mathbf{X}$ with monotonic spatial resolutions $h_p \to 0$, where $\mathbf{X}^p \triangleq \prod_{i\in\mathcal{V}} X_i^p$. The safety region $\mathbf{S}$ is discretized as $\mathbf{S}^p \triangleq (\mathbf{S} + h_p\mathcal{B}_{\mathbf{X}}) \cap \mathbf{X}^p$. On each grid $\mathbf{X}^p$, our algorithm chooses temporal resolution $\epsilon_p > 2h_p$. Denote $\mathbb{R}^p_{\geq 0}$ as an integer lattice on $\mathbb{R}_{\geq 0}$ consisting of segments of length h_p, and $(\mathbb{R}^N_{\geq 0})^p$ as a lattice on $\mathbb{R}^N_{\geq 0}$.

With these spatial and temporal discretizations, Algorithm 1 leverages the idea of multigrid methods to search for the minimal arrival time function. Specifically, Algorithm 1 iteratively executes the following two phases: initializing the solution on $\mathbf{X}^p$ by utilizing the results from $\mathbf{X}^{p-1}$ and partially solving a multirobot optimal control problem on grid $\mathbf{X}^p$. We start with the second phase, which consists of two steps: construction of set-valued

Algorithm 1: Pareto-Based Anytime Algorithm.

Input: System dynamics f, state space $\mathbf{X}$, discretization grids $\{\mathbf{X}^p\}_{p=1}^P$, the associated resolutions h_p, ϵ_p and the number of value iterations to be executed n_p.
for $1 \leq p \leq P$ **do**
Grid refinement
  $\alpha_p = 2h_p + \epsilon_p h_p l^+ + \epsilon_p^2 l^+ M^+$
  $\mathbf{S}^p = (\mathbf{S} + h_p\mathcal{B}_{\mathbf{X}}) \cap \mathbf{X}^p$
Value function interpolation
  for $x \in \mathbf{X}^{p-1}$ **do**
    $\tilde{v}^{p-1}(x) = v^{p-1}_{\bar{n}_{p-1}}(x)$
  end for
  for $x \in \mathbf{S}^p \setminus \mathbf{X}^{p-1}$ **do**
    for $i \in \mathcal{V}$ **do**
      if $d(x_i, X_i^G) \leq M_i\epsilon_p + h_p$ **then**
        $\tilde{v}_i^{p-1}(x) = 0$
      else
        $\tilde{v}_i^{p-1}(x) = 1$
      end if
    end for
  end for
  for $x \in \mathbf{X}^p \setminus (\mathbf{S}^p \bigcup \mathbf{X}^{p-1})$ **do**
    $\tilde{v}^{p-1}(x) = \{\mathbf{1}_N\}$
  end for
Value function initialization
  for $x \in \mathbf{X}^{p-1}$ **do**
    $v_0^p(x) = \tilde{v}^{p-1}(x)$
  end for
  for $x \in \mathbf{X}^p \setminus \mathbf{X}^{p-1}$ **do**
    $v_0^p(x) = \bigcup_{\tilde{x}\in X_E^p(x)} \tilde{v}^{p-1}(\tilde{x})$
  end for
Value function update
  for $x \in \mathbf{S}^p \setminus (\mathbf{X}^G + (M^+\epsilon_p + h_p)\mathcal{B}_{\mathbf{X}})$ **do**
    $(\tilde{X}^p(x), \tilde{T}^p(x)) =$ `Set_Valued_Dynamic`$(x, \mathbf{S}^p)$
  end for
  $n = 0$
  while $n \leq n_p$ **and** $v_n^p \neq v_{n-1}^p$ **do**
    $n = n + 1$
    for $x \in \mathbf{S}^p \setminus (\mathbf{X}^G + (M^+\epsilon_p + h_p)\mathcal{B}_{\mathbf{X}})$ **do**
      $(v_n^p(x), \mathcal{U}^p(x)) =$ `Value_Iteration` $(x, \tilde{X}^p(x), \tilde{T}^p(x), v_{n-1}^p)$
    end for
  end while
  $\bar{n}_p = n$
  for $x \in (\mathbf{X}^p \cap (\mathbf{X}^G + (M^+\epsilon_p + h_p)\mathcal{B}_{\mathbf{X}})) \cup (\mathbf{X}^p \setminus \mathbf{S}^p)$ **do**
    $v^p_{\bar{n}_p}(x) = \tilde{v}^{p-1}(x)$
  end for
end for
Output: $v^p_{\bar{n}_p}, \mathcal{U}^p$

Algorithm 2: `Set_Valued_Dynamic` $(x, \mathbf{S}^p)$.

1: **Input:** $x, \mathbf{S}^p$
2: **for** $i \in \mathcal{V}$ **do**
3:   **if** $d(x_i, X_i^G) > M_i\epsilon_p + h_p$ **then**
4:     $\tilde{T}_i^p = \epsilon_p + 2h_p\mathcal{B}_1$;
    $\tilde{X}_i^p = x_i + \epsilon_p F_i(x_i) + \alpha_p\mathcal{B}_{X_i}$;
5:   **else**
6:     $\tilde{T}_i^p = \{0\}$; $\tilde{X}_i^p = \{x_i\}$;
7:   **end if**
8: **end for**
9: $\tilde{T}^p = (\prod_{i\in\mathcal{V}} \tilde{T}_i^p) \cap (\mathbb{R}^N_{\geq 0})^p$; $\tilde{X}^p = (\prod_{i\in\mathcal{V}} \tilde{X}_i^p) \cap \mathbf{S}^p$;
10: **Output:** $\tilde{X}^p, \tilde{T}^p$

Algorithm 3: `Value_Iteration` $(x, \tilde{X}^p, \tilde{T}^p, v_{n-1}^p)$.

1: **Input:** $x, \tilde{X}^p, \tilde{T}^p, v_{n-1}^p$
2: $v_n^p(x) = \mathcal{E}(\{\tau + \tilde{\tau} - \tau \circ \tilde{\tau} | \tilde{\tau} = \mathcal{E}(\Psi(\tilde{T}^p)), \tilde{x} \in \tilde{X}^p, \tau \in v_{n-1}^p(\tilde{x})\})$
3: $\mathcal{U}^p(x) = \{$the solutions to u in the above step$\}$
4: **Output:** $v_n^p, \mathcal{U}^p$

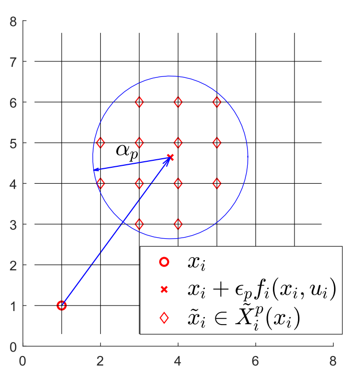

Fig. 1. Set-valued discretization of robot dynamics.

dynamics as Algorithm 2 and execution of value iteration as Algorithm 3.

Step 1: In lines 2–8 of Algorithm 2, the following set-valued dynamics are constructed to approximate system (1)

$$\tilde{X}_i^p(x_i) = \begin{cases} x_i + \epsilon_p F_i(x_i) + \alpha_p \mathcal{B}_{X_i} \\ \quad \text{if } d(x_i, X_i^G) > M_i\epsilon_p + h_p \\ x_i, \quad \text{otherwise} \end{cases} \tag{2}$$

and time dynamic $\dot{t} = 1$ is approximated by

$$\tilde{T}_i^p(x_i) = \begin{cases} \epsilon_p + 2h_p\mathcal{B}_1, & \text{if } d(x_i, X_i^G) > M_i\epsilon_p + h_p \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where $\alpha_p \triangleq 2h_p + \epsilon_p h_p l^+ + \epsilon_p^2 l^+ M^+$. Let $\tilde{X}^p(x) \triangleq \prod_{i\in\mathcal{V}} \tilde{X}_i^p(x_i) \cap \mathbf{S}^p$ and $\tilde{T}^p(x) \triangleq \prod_{i\in\mathcal{V}} \tilde{T}_i^p(x_i) \cap (\mathbb{R}^N_{\geq 0})^p$ as line 9 in Algorithm 2. The balls $\alpha_p\mathcal{B}_{X_i}$ in (2) and $2h_p\mathcal{B}_1$ in (3) represent perturbations on the dynamics. The perturbations ensure that the image set of any x is nonempty and the set-valued dynamic is well-defined. Fig. 1 illustrates the set-valued dynamics (2), where robot i at state x_i takes a constant control u_i for a time duration ϵ_p and transits to the red cross. The next state of robot i could be any red diamond, which lies in the intersection of the grid and the ball centered at $x_i + \epsilon_p f_i(x_i, u_i)$ with radius α_p. Let $\epsilon_p \to 0$ and $\frac{h_p}{\epsilon_p} \to 0$, i.e., the spatial resolution h_p diminishes faster than the temporal resolution ϵ_p. This ensures the validity of the approximation in three phases: when α_p is very small compared to ϵ_p and h_p, the set-valued dynamics transit on the grid $\mathbf{X}^p$; since h_p is diminishing faster than ϵ_p, the set-valued dynamics can well approximate the discrete-time system on $\mathbf{X}$ when p is sufficiently large; finally, as ϵ_p converges to 0, the discrete-time system further converges to the continuous-time system. When $d(x_i, X_i^G) \leq M_i\epsilon_p + h_p$, robot i is considered in the goal region, and hence, it could stay still and stop counting traveling time.

Step 2: Given the aforementioned set-valued dynamics, Algorithm 3 searches for Pareto optimal solutions of minimal arrival time vectors and stores values in Θ_n^p and the last controls in $\mathcal{U}^p$. The Bellman operator in the Pareto sense is defined by

$$(\mathbb{T}\Theta_n^p)(x) \triangleq \mathcal{E}(\{\tilde{t} + t | \tilde{t} \in \tilde{T}^p(x), \tilde{x} \in \tilde{X}^p(x), t \in \Theta_n^p(\tilde{x})\}) \tag{4}$$

where $\Theta_n^p : \mathbf{X}^p \rightrightarrows \bar{\mathbb{R}}^N_{\geq 0}$ is the estimate of Θ^* after n value iterations on grid $\mathbf{X}^p$. Since $\mathcal{E}(\tilde{T}^p(x))$ is a singleton, $\tilde{t} = \mathcal{E}(\tilde{T}^p(x))$. When no feasible control policy exists at x, $\Theta_n^p(x)$ is infinity. To remedy this numerical issue, we apply Kruzhkov transform on both sides of (4) and replace Θ_n^p with $\Psi^{-1}v_n^p$, which produces the transformed Bellman operator in the Pareto sense

$$(\mathbb{G}v_n^p)(x) = \mathcal{E}(\{\tilde{\tau} + \tau - \tilde{\tau} \circ \tau | \tilde{\tau} = \Psi(\mathcal{E}(\tilde{T}^p(x))) \quad \tilde{x} \in \tilde{X}^p(x), \tau \in v_n^p(\tilde{x})\}) \tag{5}$$

where $\mathbb{G} \triangleq \Psi\mathbb{T}\Psi^{-1}$ summarizes line 2 of Algorithm 3. Let $\mathcal{U}^p(x)$ be the set of controls that solve the last value iteration $v_n^p(x) = (\mathbb{G}v_{n-1}^p)(x)$ on grid $\mathbf{X}^p$. It corresponds to line 3 of Algorithm 3.

With the aforementioned two steps, Algorithm 1 iteratively calls Algorithms 2 and 3 to search for the minimal arrival time function. Denote the last estimate of minimal arrival time function on $\mathbf{X}^p$ by $v_{\bar{n}_p}^p$, where $\bar{n}_p$ denotes the total number of value iterations executed on $\mathbf{X}^p$. When proceeding to grid $\mathbf{X}^p$, Algorithm 1 first interpolates $v_{\bar{n}_{p-1}}^{p-1}$ to generate $\tilde{v}^p$ as lines 5–19 to reuse previous computational results, then initializes value function v_0^p as lines 20–25 to reduce coupling among robots. In particular, we maintain the estimates of minimal arrival time on the last grid $\mathbf{X}^{p-1}$, assuming the fixed points on two consecutive grids are close to each other. On new nodes $x \in \mathbf{X}^p \setminus \mathbf{X}^{p-1}$, $\tilde{v}^p(x)$ sets its ith element as 0 if robot i is considered in the goal region, indicating that robot i is not supposed to move and affect other robots' motions, and as 1 otherwise, meaning no feasible solution has been found for robot i yet. Define the set of equivalent nodes $X_E^p(x)$ of $x \in \mathbf{X}^p$ by

$$X_E^p(x) \triangleq \{x' \in \mathbf{X}^p | x_i = x_i' \quad \forall i \in \mathcal{V} \setminus \mathcal{V}_p^G(x) \quad d(x_i', X_i^G) \leq M_i\epsilon_p + h_p \quad \forall i \in \mathcal{V}_p^G(x)\} \tag{6}$$

where $\mathcal{V}_p^G(x) \triangleq \{i \in \mathcal{V} | d(x_i, X_i^G) \leq M_i\epsilon_p + h_p\}$ denotes the set of robots that are close to or already in the goal regions. Since robots in the goal regions never interfere with others and, thus, are excluded in value iterations, the values of equivalent nodes are the same. Then, the value function is initialized by $v_0^p(x) = \bigcup_{\tilde{x} \in X_E^p(x)} \tilde{v}^{p-1}(\tilde{x}) \quad \forall x \in \mathbf{X}^p \setminus \mathbf{X}^{p-1}$, i.e., line 24 in

Algorithm 1. With the initialized value function, Algorithm 1 in lines 26–28 first calls Algorithm 2 to construct set-valued dynamics and then in lines 30–35 calls Algorithm 3 to execute value iterations for n_p times or until a fixed point is reached. Notice that the total number of value iterations $\bar{n}_p$ may be less than n_p. After that, Algorithm 1 refines the grid and begins a new cycle of updates.

B. Performance Guarantee

Recall that n_p at line 30 of Algorithm 1 is the number of value iterations to be executed on grid $\mathbf{X}^p$. The choice of n_p needs to satisfy the following assumption to ensure the convergence of Algorithm 1.

Assumption IV.1: There is a subsequence $\{D_k\}$ of the grid index sequence $\{p\}$ with $D_0 = 0$ s.t. $D_k - D_{k-1} \leq \bar{D}$ for some constant $\bar{D}$ and all $k \geq 0$ and $\exp(-\sum_{p=D_{k-1}+1}^{D_k} n_p \kappa_p) \leq \gamma < 1$ for every $k \geq 0$, where $\kappa_p \triangleq (\lceil \frac{\epsilon_p}{h_p} \rceil - 2) h_p$ is the minimum running cost.

Assumption IV.1 implies that the distance between the estimate and the fixed point on the D_kth grid reduces at least by $\gamma \in [0, 1)$ over the update window length $\{D_{k-1}+1, \ldots, D_k\}$.

The choice of ϵ_p and h_p should satisfy the following technical assumptions.

Assumption IV.2: The following hold for the sequences of $\{\epsilon_p\}$ and $\{h_p\}$.

A6) $\epsilon_p > 2h_p \quad \forall p \geq 1$.

A7) $\epsilon_p \to 0$ and $\frac{h_p}{\epsilon_p} \to 0$ monotonically as $p \to +\infty$.

A8) $2h_p + \epsilon_p h_p l^+ + \epsilon_p^2 l^+ M^+ \geq h_{p-1} \quad \forall p \geq 1$.

A9) $[X_i^G + (\sigma + M_i \epsilon_1 + h_1)\mathcal{B}_{X_i}] \cap X_j^F = \emptyset \quad \forall i \neq j$.

The consistent approximation of v^* via Algorithm 1 in the epigraphical profile sense is summarized in Theorem IV.1.

Theorem IV.1: Suppose Assumptions III.1, IV.1, and IV.2 hold, then the sequence $\{v_{\bar{n}_p}^p\}$ in Algorithm 1 converges to v^* in the epigraphical profile sense, i.e., for any $x \in \mathbf{X}$

$$E_{v^*}(x) = \operatorname*{Lim}_{p\to+\infty} \bigcup_{\tilde{x}\in(x+h_p\mathcal{B}_{\mathbf{X}})\cap\mathbf{X}^p} E_{v_{\bar{n}_p}^p}(\tilde{x}).$$

C. Discussion

Our proposed algorithm extends [36] to multirobot scenario. For single robot scenario, i.e., $N = 1$, if we set $\bar{D} = 1$ and $\gamma = 0$ and only impose Assumptions III.1, (A6), and (A7), Algorithm 1 and Theorem IV.1 become [36, Algorithm 3.2.4, p. 211 and Cor. 3.7, p. 210], respectively.

However, from the analysis point of view, nonzero γ and nonuniform lengths for update windows in the multirobot scenario, i.e., $N \geq 2$, require a set of novel analysis, which is provided in Sections V and VII.

The progress toward v^* slows down or even stops as more value iterations are performed on a single grid. A γ close to one ensures that excessive value iterations are postponed to finer grids, and a longer update interval reduces each grid's efforts to reach the discount factor.

V. Analysis

In this section, we provide the major theoretic results that lead to the proof of Theorem IV.1, which consist of following four steps.

Step 1: We characterize the convergence of fixed points v_∞^p to the minimal arrival time function v^*, i.e., in Theorem V.1. The fixed point v_∞^p functions as a benchmark and we will show later that the last value function $v_{\bar{n}_p}^p$ on each grid $\mathbf{X}^p$ can closely follow v_∞^p to converge.

Step 2: We introduce an auxiliary Bellman operator $\hat{\mathbb{G}}$ defined in (9) to facilitate the analysis of the contraction property of the transformed Bellman operator $\mathbb{G}$ in the next step. Specifically, the contraction property requires to add perturbations around all nodes in value iteration, but $\mathbb{G}$ imposes zero perturbation when robots are close to their goal regions. Then, $\hat{\mathbb{G}}$ bridges this technical gap and is equivalent to $\mathbb{G}$ in terms of updating value functions, which is shown in Lemma V.5.

Step 3: We prove the contraction property of $\mathbb{G}$ via $\hat{\mathbb{G}}$ in Step 2 and it is summarized in Theorem V.2. The contraction property shows that the distance between the estimate of minimal arrival time function v_n^p and the fixed point v_∞^p is exponentially discounted as value iterations are executed.

Step 4: We integrate Step 3 with Step 1 and show that $v_{\bar{n}_p}^p$ can closely follow v_∞^p and, thus, converge to v^*. In particular, the approximation errors induced by grid refinement are shown to be suppressed by sufficient value iterations and thereby the distance between $v_{\bar{n}_p}^p$ and v_∞^p is decreasing to zero.

This section is organized as follows. Section V-A corresponds to Step 1 and introduces the convergence of fixed points, i.e., Theorem V.1. Section V-B corresponds to Step 2 and confirms the equivalence of $\mathbb{G}$ and $\hat{\mathbb{G}}$ in terms of updating value functions. Section V-C corresponds to Step 3 and proves the contraction property of $\mathbb{G}$. Step 4 is summarized in Section VII-D, which shows the proof of Theorem IV.1. We only keep theorem statements in this section and postpone all the proofs to Section VII.

A. Convergence of Fixed Points

The following theorem characterizes the convergence of fixed points v_∞^p to the optimal arrival time function v^*.

Theorem V.1: Suppose Assumption III.1 holds and let $\epsilon_p > 2h_p$, $h_p \to 0$, and $\frac{h_p}{\epsilon_p} \to 0$. Construct the sequence $\{v_n^p : \mathbf{X}^p \rightrightarrows [0,1]^N\}$ as follows:

$$\begin{cases} v_0^p(x) = \begin{cases} \{\mathbf{0}_N\}, & \text{if } x \in \mathbf{S}^p \\ \{\mathbf{1}_N\}, & \text{otherwise} \end{cases} \\ v_{n+1}^p(x) = \begin{cases} \mathbb{G}v_n^p(x), & \text{if } x \in \mathbf{S}^p \\ v_n^p(x), & \text{otherwise} \end{cases} \end{cases}$$

where $\mathbb{G}$ is defined in (5). Then, for each p, there exists v_∞^p s.t. $\mathbb{G}v_\infty^p = v_\infty^p$ and $v_\infty^p(x) = \operatorname{Lim}_{n\to+\infty} v_n^p(x) \quad \forall x \in \mathbf{X}^p$. Furthermore, the fixed points converge to v^* in the epigraphical sense, i.e., for any $\{\eta_p\}$ s.t. $\eta_p \geq h_p$ and $\lim_{p\to+\infty} \eta_p = 0$, the following holds:

$$\forall x \in \mathbf{X}, E_{v^*}(x) = \operatorname*{Lim}_{p\to+\infty} \bigcup_{\tilde{x}\in(x+h_p\mathcal{B}_{\mathbf{X}})\cap\mathbf{X}^p} E_{v^p_\infty}(\tilde{x}).$$

The proof of Theorem V.1 mainly follows those of [36, Lemma 3.6 and Cor. 3.7]. For the sake of completeness, we include the details of proofs in [42]. Please refer to [42, Th. IX.3 and Cor. IX.1].

B. Auxiliary Bellman Operator $\hat{\mathbb{G}}$: Lemma V.5

In this section, an auxiliary Bellman operator $\hat{\mathbb{G}}$ is introduced as a stepping stone toward the contraction property of $\mathbb{G}$ in Section V-C. This section consists of following three phases.

1) First, $\hat{\mathbb{G}}$ is formally defined as (9). The auxiliary Bellman operator $\hat{\mathbb{G}}$ differs from $\mathbb{G}$ in the perturbations around nodes within one hop of the goal regions.
2) Second, the properties of $\hat{\mathbb{G}}$ are analyzed and it is shown that $\hat{\mathbb{G}}v^p_n$ is no less than $\mathbb{G}v^p_n$, as Lemmas V.3 and V.4.
3) Finally, $\hat{\mathbb{G}}v^p_n$ is no larger than $\mathbb{G}v^p_n$, either, and thereby the equivalence of $\hat{\mathbb{G}}$ and $\mathbb{G}$ is established in Lemma V.5.

We proceed to the first phase and derive the Bellman operator in terms of epigraphical profiles and its Kruzhkov transformed version. We start with (4) by adding $\mathbb{R}^N_{\geq 0}$ to both sides

$$\begin{aligned} E_{\mathbb{T}\Theta^p_n}(x) &= (\mathbb{T}\Theta^p_n)(x) + \mathbb{R}^N_{\geq 0} \\ &= \{\tilde{t}+t|\tilde{t} = \mathcal{E}(\tilde{T}^p(x)), \tilde{x}\in\tilde{X}^p(x), t\in\Theta^p_n(\tilde{x})+\mathbb{R}^N_{\geq 0}\} \\ &= \{\mathcal{E}(\tilde{T}^p(x))+t|\tilde{x}\in\tilde{X}^p(x), t\in E_{\Theta^p_n}(\tilde{x})\} \\ &= \mathcal{E}(\tilde{T}^p(x)) + \bigcup_{\tilde{x}\in\tilde{X}^p(x)} E_{\Theta^p_n}(\tilde{x}). \end{aligned}$$

Recall that $v^p_n(x) = (\Psi\Theta^p_n)(x)$. Denote $\Delta\tau(x) \triangleq \Psi(\mathcal{E}(\tilde{T}^p(x)))$. Applying Kruzhkov transform to both sides yields

$$E_{\mathbb{G}v^p_n}(x) = \Delta\tau(x) + (\mathbf{1}-\Delta\tau(x))\circ \bigcup_{\tilde{x}\in\tilde{X}^p(x)} E_{v^p_n}(\tilde{x}). \tag{7}$$

The ith element of $\Delta\tau$ can be written as

$$\Delta\tau_i(x) = \begin{cases} 0, & \text{if } i\in\mathcal{V}^G_p(x) \\ 1-e^{-\kappa_p}, & \text{otherwise} \end{cases} \tag{8}$$

where κ_p follows the definition in Assumption IV.1.

Now, we define the auxiliary Bellman operator $\hat{\mathbb{G}}$ by

$$E_{\hat{\mathbb{G}}v}(x) \triangleq \Delta\tau(x) + (\mathbf{1}-\Delta\tau(x))\circ \bigcup_{\hat{x}\in\hat{X}^p(x)} E_v(\hat{x}) \tag{9}$$

where $\hat{X}^p(x) \triangleq (\prod_{i\in\mathcal{V}} \hat{X}^p_i(x_i)) \cap \mathbf{S}^p$ and

$$\hat{X}^p_i(x_i) \triangleq \begin{cases} x_i + \epsilon_p F_i(x_i) + \alpha_p\mathcal{B}, & \text{if } d(x_i, X^G_i) > M_i\epsilon_p + h_p \\ x_i + \alpha_p\mathcal{B}, & \text{otherwise.} \end{cases}$$

If $d(x_i, X^G_i) \leq M_i\epsilon_p + h_p$, then $\tilde{X}^p_i(x_x) = x_i$ in $\mathbb{G}$ and $\hat{X}^p_i(x_i) = x_i + \alpha_p\mathcal{B}$ in $\hat{\mathbb{G}}$. This is the only difference between $\mathbb{G}$ and $\hat{\mathbb{G}}$.

Before we move on to the second phase, intermediate results are required to facilitate our analysis. The next lemma shows that the equivalent nodes of $x \in \mathbf{S}^p$ are also in the safety region.

Lemma V.1: Suppose Assumptions (A7) and (A9) are satisfied. Then, for any $p \geq 1$ and $x \in \mathbf{S}^p$, it holds that $X^p_E(x) \subseteq \mathbf{S}^p$.

The next lemma shows that for any robot $i \in \mathcal{V}^G_p(x)$, its estimate of travelling time is always 0.

Lemma V.2: For any $p \geq 1$, the following hold.

1) $\mathcal{V}^G_p(x) \subseteq \mathcal{V}^G_p(\tilde{x})$ for any $x \in \mathbf{X}^p$ and $\tilde{x} \in \tilde{X}^p(x)$.
2) $\mathcal{V}^G_p(x) \supseteq \mathcal{V}^G_{p+1}(x)$ for any $x \in \mathbf{X}^p$.
3) $\tau_i = 0$ for any $x \in \mathbf{S}^p$ $\tau \in v^p_n(x)$, $0 \leq n \leq \bar{n}_p$ and $i \in \mathcal{V}^G_p(x)$.

Remark V.1: Notice that $v^p_n = \mathbb{G}^n v^p_0$. Fix $p \geq 1$. It follows from the proof of the third property of Lemma V.2 that $\tau_i = 0$ for any $\tau \in \mathbb{G}^m v^p_0(x)$, $m \geq 0$ and $i \in \mathcal{V}^G_p(x)$. Specifically, by Theorem V.1, we have $v^p_\infty = \operatorname{Lim}_{n\to+\infty} v^p_n(x) = \operatorname{Lim}_{n\to+\infty}\mathbb{G}^n v^p_0(x)$. Then, the third property of Lemma V.2 also applies to v^p_∞. □

Remark V.2: Fix $x \in \mathbf{S}^p$ and $m \geq 0$ and let $\mathcal{V}^G_p(x) = \{1,\ldots,N_p\}$, where $0 \leq N_p \leq N$. By the third property of Lemma V.2, we have $\forall\tilde{\tau} \in [0,1]^{N_p} \times \{1\}^{N-N_p}$, $\exists\tau \in \mathbb{G}^m v^p_0(x)$ s.t. $\tilde{\tau} \succeq \tau$. This implies $[0,1]^{N_p} \times \{1\}^{N-N_p} \subseteq E_{\mathbb{G}^m v^p_0}(x) = (\mathbb{G}^m v^p_0(x) + \mathbb{R}^N_{\geq 0}) \cap [0,1]^N$. □

Define the set of partially perturbed state nodes $x' \in X^p_P(x)$ of $x \in \mathbf{X}^p$ by

$$X^p_P(x) \triangleq \{x' \in \mathbf{S}^p | x'_i = x_i \quad \forall i \in \mathcal{V}\setminus\mathcal{V}^G_p(x)\}.$$

The term "partially perturbed state node" means that x' differs from x only at the perturbations added to the positions of robots $i \in \mathcal{V}^G_p(x)$. It is a superset of $X^p_E(x)$ in (6).

The following lemma shows that on a fixed grid, the partially perturbed nodes cannot have less value.

Lemma V.3: Fix $p \geq 1$ s.t. Assumptions (A7) and (A9) are satisfied. Consider $v^p_n : \mathbf{X}^p \rightrightarrows [0,1]^N$. If $E_{v^p_n}(x') \subseteq E_{v^p_n}(x)$ for any pair of $x \in \mathbf{S}^p$ and $x' \in X^p_P(x)$, then $E_{\mathbb{G}^m v^p_n}(x') \subseteq E_{\mathbb{G}^m v^p_n}(x)$ holds for all $m \geq 1$ and any pair of $x \in \mathbf{S}^p$ and $x' \in X^p_P(x)$.

The next lemma extends Lemma V.3 to all the iterations of Algorithm 1.

Lemma V.4: For any pair of $x \in \mathbf{S}^p$ and $x' \in X^p_P(x)$, if Assumptions (A7) and (A9) are satisfied, it holds that $E_{v^p_n}(x') \subseteq E_{v^p_n}(x)$ for any $p \geq 1$ and $0 \leq n \leq \bar{n}_p$.

The next corollary shows that the values of all equivalent nodes are the same.

Corollary V.1: If all conditions in Lemma V.4 are satisfied, for any $p \geq 1$, $0 \leq n \leq \bar{n}_p$ and any pair of $x \in \mathbf{S}^p$ and $x' \in X^p_E(x)$, $E_{v^p_n}(x) = E_{v^p_n}(x')$. In addition, $E_{v^p_\infty}(x) = E_{v^p_\infty}(x')$.

Finally, we arrive at the last phase and the next lemma is the main result of this section that reveals the equivalence of $\mathbb{G}$ and $\hat{\mathbb{G}}$.

Lemma V.5: If Assumptions (A7) and (A9) are satisfied, for any $p \geq 1$, $0 \leq n \leq \bar{n}_p$ and $x \in \mathbf{S}^p$, it holds that $\bigcup_{\tilde{x}\in\tilde{X}^p(x)} E_{v^p_n}(\tilde{x}) = \bigcup_{\hat{x}\in\hat{X}^p(x)} E_{v^p_n}(\hat{x})$ and $E_{\mathbb{G}v^p_n}(x) = E_{\hat{\mathbb{G}}v^p_n}(x)$. In addition, $\bigcup_{\tilde{x}\in\tilde{X}^p(x)} E_{v^p_\infty}(\tilde{x}) = \bigcup_{\hat{x}\in\hat{X}^p(x)} E_{v^p_\infty}(\hat{x})$ and $E_{\mathbb{G}v^p_\infty}(x) = E_{\hat{\mathbb{G}}v^p_\infty}(x)$.

C. Contraction Property of $\mathbb{G}$: Theorem V.2

In this section, Theorem V.2 shows that the transformed Bellman operator $\mathbb{G}$ in (5) is contractive with factor $e^{-\kappa_p}$.

Before we proceed to the final conclusion, the following notations are defined to facilitate our analysis. Given a set-valued map $v : \mathbf{X}^p \rightrightarrows [0,1]^N$, define the interpolation operation $\mathbb{I}^p$ by

$$(\mathbb{I}^p v)(x) \triangleq \begin{cases} v(x), & \text{if } x \in \mathbf{X}^p \\ \{V^p(x)\}, & \text{if } x \in \mathbf{S}^{p+1} \setminus \mathbf{X}^p \\ \{\mathbf{1}_N\}, & \text{if } x \in \mathbf{X}^{p+1} \setminus (\mathbf{S}^{p+1} \cup \mathbf{X}^p) \end{cases}$$

where interpolation function $V^p : \mathbf{X}^p \to \{0,1\}^N$ is defined as

$$V_i^p(x) \triangleq \begin{cases} 0, & \text{if } d(x_i, X_i^G) \leq M_i \epsilon_{p+1} + h_{p+1} \\ 1, & \text{otherwise.} \end{cases} \quad (10)$$

Then, the interpolated value function $\tilde{v}^p : \mathbf{X}^{p+1} \rightrightarrows [0,1]^N$ in Algorithm 1 can be represented by $\tilde{v}^p \triangleq \mathbb{I}^p v^p_{\bar{n}_p}$. The interpolated fixed point $\tilde{v}^p_\infty : \mathbf{X}^{p+1} \rightrightarrows [0,1]^N$ is written as $\tilde{v}^p_\infty \triangleq \mathbb{I}^p v^p_\infty$. Correspondingly, define the initialization operator $\mathbb{P}$ by

$$E_{\mathbb{P}v}(x) \triangleq \begin{cases} E_v(x), & \text{if } x \in \mathbf{X}^{p-1} \\ \bigcup_{\tilde{x} \in X^p_E(x)} E_v(\tilde{x}), & \text{if } x \in \mathbf{X}^p \setminus \mathbf{X}^{p-1}. \end{cases}$$

Define the distance between two consecutive fixed points at $x \in \mathbf{X}$ by $b_p(x) \triangleq d_H(\bigcup_{\tilde{x} \in (x+\alpha_p \mathcal{B}) \cap \mathbf{X}^p} E_{\mathbb{P}\tilde{v}^{p-1}_\infty}(\tilde{x}), \bigcup_{\tilde{x} \in (x+\alpha_p \mathcal{B}) \cap \mathbf{X}^p} E_{v^p_\infty}(\tilde{x}))$. Define $b_p \triangleq \sup_{x \in \mathbf{X}} b_p(x)$. The next lemma shows the distance diminishes.

Lemma V.6: If Assumptions (A7) and (A8) are satisfied, it holds that $\lim_{p \to +\infty} b_p = 0$.

The following lemma shows that under $\mathbb{G}$, the distance of v^p_n and v^p_∞ at any node $x \in \mathbf{X}^p$ is discounted by $e^{-\kappa_p}$.

Lemma V.7: If Assumptions (A7) and (A9) are satisfied, then the following holds for any $p \geq 1$, $n \geq 0$ and $x \in \mathbf{S}^p$:

$$d_H((\mathbf{1} - \Delta\tau(x)) \circ A, (\mathbf{1} - \Delta\tau(x)) \circ B) \leq e^{-\kappa_p} d_H(A, B)$$

where $A \triangleq \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{v^p_n}(\tilde{x})$ and $B \triangleq \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{v^p_\infty}(\tilde{x})$.

Finally, we come to the contraction property of $\mathbb{G}$.

Theorem V.2: If Assumptions (A7) and (A9) are satisfied, the following holds for any $p \geq 1$, $n \geq 0$ and $x \in \mathbf{S}^p$:

$$d_{\mathbf{S}^p}(E_{\mathbb{G}v^p_n}, E_{\mathbb{G}v^p_\infty}) \leq e^{-\kappa_p} d_{\mathbf{X}}\Big(\bigcup_{\tilde{x} \in (x+\alpha_p \mathcal{B}) \cap \mathbf{X}^p} E_{v^p_n}(\tilde{x}), \bigcup_{\tilde{x} \in (x+\alpha_p \mathcal{B}) \cap \mathbf{X}^p} E_{v^p_\infty}(\tilde{x})\Big). \quad (11)$$

In addition, the following is also true:

$$d_{\mathbf{S}^p}(E_{\mathbb{G}v^p_n}, E_{\mathbb{G}v^p_\infty}) \leq e^{-\kappa_p} d_{\mathbf{S}^p}(E_{v^p_n}, E_{v^p_\infty}). \quad (12)$$

The next lemma derives a recursive relation of $d_{\mathbf{X}^p}(E_{v^p_{\bar{n}_p}}, E_{v^p_\infty})$.

Lemma V.8: If Assumptions III.1 and IV.2 are satisfied, the following inequality holds for each grid $\mathbf{X}^p$:

$$d_{\mathbf{X}^p}(E_{v^p_{\bar{n}_p}}, E_{v^p_\infty}) \leq \gamma_p d_{\mathbf{X}^{p-1}}(E_{v^{p-1}_{\bar{n}_{p-1}}}, E_{v^{p-1}_\infty}) + b_p \quad (13)$$

where $\gamma_p \triangleq e^{-n_p \kappa_p}$ and b_p is defined in Lemma V.6.

VI. Experiments and Simulations

This section presents the experiments on an indoor multi-robot platform and computer simulations conducted to assess the performance of Algorithm 1. The experiment environment, shown in Fig. 2, is a four-way intersection with no signs or signals. Each road is 420 mm wide and consists of two lanes of same width with opposite directions. Three Khepera III robots of diameters 170 mm can neither sense the environment nor communicate with each other. A centralized computer can measure robots' locations and heading angles via Vicon system, a motion capture system, and remotely command each robot's motion via Bluetooth.

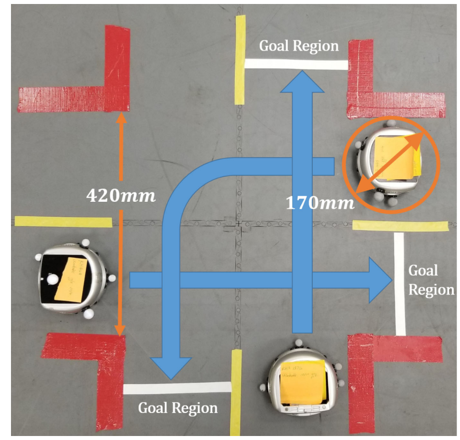

Fig. 2. Three Khepera III robots arrive at the intersection at the same time.

Each robot is modeled as a unicycle and its dynamics are given by $\dot{p}^x_i = v_i \cos\theta_i$ and $\dot{p}^y_i = v_i \sin\theta_i$, where $x_i = (p^x_i, p^y_i)$ denotes the ith robot's position and $u_i = (\theta_i, v_i) \in U_i = U^\theta_i \times U^v_i$ is its control, including heading angle θ_i and linear speed v_i. The goal for each robot is to pass the crossroads and arrive at its goal region without colliding with curbs or any other robot. The robots stop as long as they pass their respective white goal lines in Fig. 2.

In practice, the allowable computational times for the robots are varying and uncertain. Therefore, it is desired to compute control policies, which can safely steer the robots to their goal regions within a short time and keep improving the control policies if more time is given. This property is referred to the anytime property, which is widely adopted in robotic motion planning literature [43]–[46]. In the following, we demonstrate that our algorithm is an anytime algorithm, i.e., it is quickly feasible and increasingly optimal. In addition, the simulations are also used to analyze the computational complexity of our algorithm.

A. Demonstration of Quick Feasibility

In this section, an experiment on three physical robots is conducted to examine the quick feasibility of our algorithm for multiple robots. In our MATLAB codes, we normalize the road width to 1 and scale robot radii to 0.2. We choose $\epsilon_p = \sqrt{h_p}$. The constraint sets of controls are given as: $U^v_i = [0, 0.25]$, $U^\theta_1 = [-\pi, -\pi/2]$, $U^\theta_2 = [-\pi/2, \pi/2]$, and $U^\theta_3 = [0, \pi]$. The dimension of state space is 6. For the purpose of collision avoidance, we set the inter-robot safety distance as 0.6 and ignore perturbations added to $\mathbf{S}$ in line 4 of Algorithm 1, i.e., we choose $\mathbf{S}^p = \mathbf{S} \cap \mathbf{X}^p$. In order to efficiently address the failure of arrival caused by coarse resolutions of discrete grids, we use finer grids near goal regions. Specifically, in the one-hop expansion of each robot's goal region $\{x \in \mathbf{X} | d(x_i, X^G_i) \leq M_i \epsilon_p + h_p, i \in \mathcal{V}\}$, we refine the grids, perform Algorithm 1 on the new nodes, and replace coarse controller with the refined one. Since Algorithm 1 only returns control policies on discrete grids, we

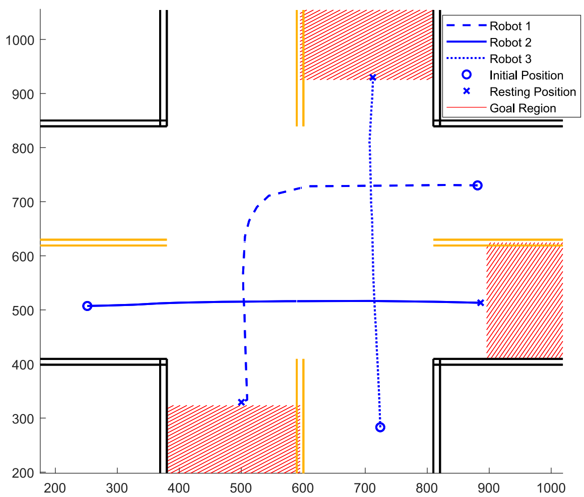

Fig. 3. Trajectories of centers of three robots when the computation time is 1.05 s.

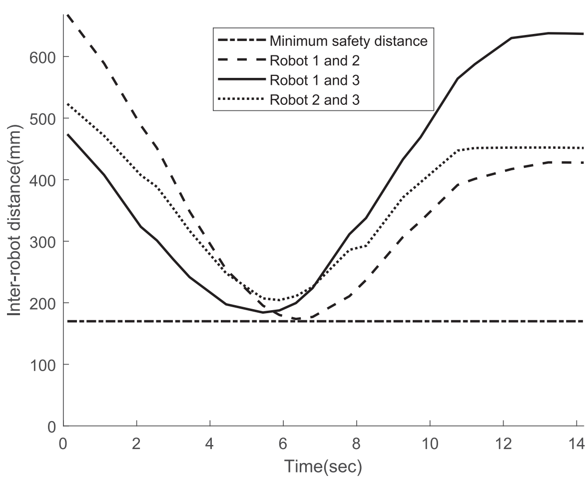

Fig. 4. Inter-robot distances over time.

need to interpolate the control policies into the continuous state space. In particular, Unif$(\cdot)$ is used to uniformly select one control from $\mathcal{U}^p(x)$ for $x \in \mathbf{S}^p$. For state $x \in \mathbf{X} \setminus \mathbf{S}^p$, the control is interpolated by nearest neighbor method, i.e., we take $u = \text{Unif}(\mathcal{U}^p(\arg\min_{\hat{x}\in\mathbf{S}^p} \|\hat{x} - x\|))$. Algorithm 1 is executed in MATLAB on a 3.40-Ghz Intel Core i7 computer.

Each physical robot has inertia in changing its heading angle θ_i and is subject to $\dot{\theta}_i = \omega_i$, where ω_i is the angular velocity that robot i can directly command. To address this difference in dynamics, a PID controller is leveraged to modulate robots' heading angles, i.e., $\omega_i = \text{PID}(u_{i,1} - \theta_i)$, where $u_{i,1}$ is the returned heading angle of robot i.

Fig. 3 shows the trajectories of the robots when they apply the interpolated control policies computed in 1.05 s. Fig. 4 shows the inter-robot distances over time corresponding to Fig. 3, indicating that no collision is caused throughout the movement of the robots. Fig. 5 displays the linear speeds of each robot over time. At around 2 s, robot 2 and robot 3 slow down so that robot 1 can first pass the intersection. At 8 s, robot 3 is no more than one hop away from its goal region and stops owing to the coarse resolution of the grid. After this moment, the robots switch to the refined controller, hence robot 3 continues to move until it rests at its goal region. The results show that given short computational time, i.e., 1.05 s, our algorithm can already generate a feasible policy that accomplishes the planning task without violating any hard constraint. Therefore, the quick feasibility is verified.

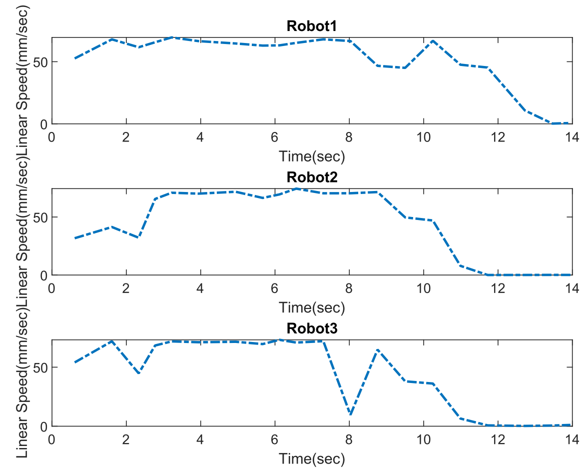

Fig. 5. Robot linear speeds over time.

B. Demonstration of Increasing Optimality

A set of computer simulations is performed to examine the increasing optimality of Algorithm 1. The parameters are identical to the previous experiment with the differences that robot 3 is excluded and safety distance is 0.4. The operating region of the robot team is discretized by the sequence of uniform square grids $\{\mathbf{X}^p\}$ for $p \in \{1,\ldots,4\}$ with resolutions $h_p \in \{0.2, 0.1, 0.05, 0.025\}$, each of which contains $145, 3403, 34\,344, 416\,689$ nodes, respectively. All the grids are within the same update window. We choose $\epsilon_p = \sqrt{h_p/M^+}$. In computations, we only update values of nodes in the safety region $\mathbf{S}^p$ as nodes in $\mathbf{X}^p \setminus \mathbf{S}^p$ indicate collisions and, therefore, are irrelevant. In addition, we ignore the perturbation added to $\mathbf{S}^p$ to avoid excessive computations. In line 24 of Algorithm 1, we choose any single node $x_E(x) \in X_E^p(x) \cap \mathbf{X}^1$ to represent the whole equivalent set $X_E^p(x)$ as it is the minimizer of $\bigcup_{\tilde{x}\in X_E^p(x)} \tilde{v}^{p-1}(\tilde{x})$. Our algorithm refines grids if the relative difference between two consecutive value functions v_n^p and v_{n-1}^p is less than 10% of the total difference between v_n^p and v_0^p, i.e., $\mathcal{D}_{n-1,n}^p/\mathcal{D}_{0,n}^p \leq 10\%$, where $\mathcal{D}_{n_1,n_2}^p \triangleq \sqrt{\sum_{x\in\mathbf{S}^p} d_H^2(v_{n_1}^p(x), v_{n_2}^p(x))}$ is the two-norm difference between $v_{n_1}^p$ and $v_{n_2}^p$. The benchmark $v^\star$ is the estimate of minimal arrival time function computed on the finest grid $\mathbf{S}^4$ with resolution $h_p = 0.025$. To measure approximation errors, we use nearest neighbor method to interpolate each estimate of minimal arrival time function v_n^p into $\hat{v}_n^p$ so that both $\hat{v}_n^p$ and $v^\star$ share the finest grid as their domains. Note that $\hat{v}_n^p(x) \triangleq v_n^p(\arg\min_{\hat{x}\in\mathbf{S}^p} \|\hat{x} - x\|)$ for every $x \in \mathbf{S}^4$. Then, approximation error of $\hat{v}_n^p$ is measured by $\sqrt{\sum_{x\in\mathbf{S}^4} d_H^2(\hat{v}_n^p(x), v^\star(x))}$. Fig. 6 shows the approximation errors over time. The nth dot from the left in Fig. 6 represents the total computational time after n value iterations and the associated approximation error. The peak at 2 s is caused by the nonlinearity of Kruzhkov transform, where the initial value $\mathbf{1}_2$ is closer to the benchmark values.

TABLE I
COMPUTATIONAL TIMES ON EACH GRID

Grid index	Grid size	Total time/sec	Construction of set-valued dynamics		Execution of value iteration	
			Computational time/sec	Percentage in total time	Computational time/sec	Percentage in total time
1	145	2.21	2.14	96.8%	0.07	3.2%
2	3403	54.35	50.08	92.1%	4.27	7.9%
3	34344	624.56	494.03	79.1%	130.53	20.9%
4	416689	21586.20	6206.59	28.7%	15379.61	71.3%

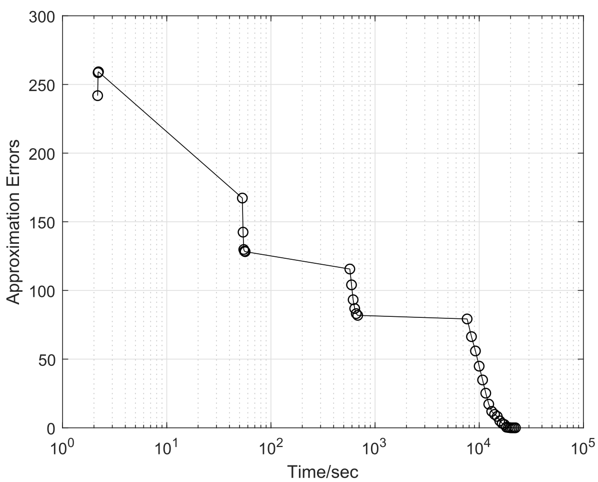

Fig. 6. Approximation errors over time.

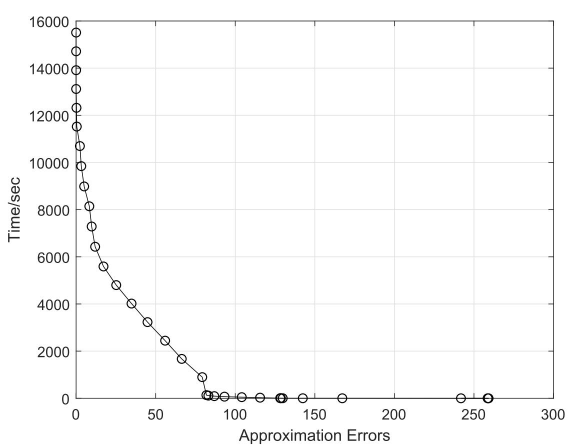

Fig. 7. Value iteration time over approximation errors.

Other than this, the approximation errors are monotonically decreasing over time.

C. Computational Complexity

Algorithms 2 and 3 correspond to two steps: construction of set-valued dynamics and execution of value iteration. In Fig. 7, the nth dot from the right represents the time to execute n value iterations and the resulting approximation error except the rightmost ones around 250. Fig. 7 shows the time to perform value iteration exponentially increases as approximation errors decrease.

Table I summarizes the total time to compute the last estimate $v^p_{\bar{n}_p}$ on each grid $\mathbf{S}^p$ and its size. The total computational time grows polynomially with respect to the grid size. Specifically, the time to construct set-valued dynamics is linear with respect to the grid size while the time to execute value iteration grows polynomially. As a result, most of the total computational time is spent on constructing set-valued dynamics on the coarse grids while the time to execute value iteration dominates on the fine grids.

VII. PROOFS

In this section, detailed proofs of theoretic results in Section V are provided.

A. Preliminary

In this section, some preliminary properties of Hausdorff distance are introduced. All the proofs are removed in this section due to space limitation. Please refer to [42] for details.

The following lemma shows the union of two expanded sets is the expansion of their unions.

Lemma VII.1: Given two sets $A, B \subseteq \mathcal{X}$ and $\eta > 0$, the following holds $(A + \eta\mathcal{B}) \cup (B + \eta\mathcal{B}) = (A \cup B) + \eta\mathcal{B}$.

The following lemma compares set distances given their set inclusion relationships.

Lemma VII.2: Given four nonempty compact sets $A \subseteq B$ and $C \subseteq D$, the following relationships hold:

$$d_H(A \cup D, B) \leq d_H(D, B) \leq \max\{d_H(A, D), d_H(B, C)\}. \tag{14}$$

The next lemma shows the triangle inequality holds for d_H.

Lemma VII.3: Given three set-valued maps $g^l : \mathcal{X} \rightrightarrows [0,1]^N$, $g^l(x)$ is compact for all $x \in \mathcal{X}$, $l \in \{1,2,3\}$. It holds that $d_{\mathcal{X}}(g^1, g^2) \leq d_{\mathcal{X}}(g^1, g^3) + d_{\mathcal{X}}(g^3, g^2)$.

Lemma VII.4 reveals that for two perturbed set-valued maps, the union of images of fewer nodes contributes to larger distance.

Lemma VII.4: Given two subsets $\mathcal{X}^1, \mathcal{X}^2 \subseteq \mathcal{X}$, consider two set-valued maps $g_1, g_2 : \mathcal{X} \rightrightarrows [0,1]^N$ and perturbation radii $\eta_l > 0$ s.t. $(x + \eta_l\mathcal{B}) \cap \mathcal{X}^l \neq \emptyset \quad \forall x \in \mathcal{X}, l \in \{1,2\}$. The following holds for any set-valued map $Y : \mathcal{X} \rightrightarrows \mathcal{X}$ s.t. $Y(x) \neq \emptyset \quad \forall x \in \mathcal{X}$:

$$\begin{aligned} &d_{\mathcal{X}}\Big(\bigcup_{\tilde{x} \in (Y(x)+\eta_1\mathcal{B})\cap\mathcal{X}^1} g_1(\tilde{x}), \bigcup_{\tilde{x} \in (Y(x)+\eta_2\mathcal{B})\cap\mathcal{X}^2} g_2(\tilde{x})\Big) \\ &\leq d_{\mathcal{X}}\Big(\bigcup_{\tilde{x} \in (x+\eta_1\mathcal{B})\cap\mathcal{X}^1} g_1(\tilde{x}), \bigcup_{\tilde{x} \in (x+\eta_2\mathcal{B})\cap\mathcal{X}^2} g_2(\tilde{x})\Big). \end{aligned} \tag{15}$$

If $\mathcal{X}^1 = \mathcal{X}^2 \triangleq \bar{\mathcal{X}}$ and $\eta_1 = \eta_2 \triangleq \bar{\eta}$, we have

$$d_{\mathcal{X}}\Big(\bigcup_{\tilde{x} \in (x+\eta\mathcal{B})\cap\bar{\mathcal{X}}} g_1(\tilde{x}), \bigcup_{\tilde{x} \in (x+\eta\mathcal{B})\cap\bar{\mathcal{X}}} g_2(\tilde{x})\Big) \leq d_{\bar{\mathcal{X}}}(g_1, g_2). \tag{16}$$

Lemma VII.5 shows that an exponentially diminishing sequence subject to diminishing perturbations remains diminishing.

Lemma VII.5: A sequence $\{a_p\} \subseteq \mathbb{R}_{\geq 0}$ satisfies $a_{p+1} \leq \gamma(a_p + c_p)$, where $\gamma \in [0,1)$, $c_p \geq 0\ \ \forall p \geq 1$ and $\lim_{p\to+\infty} c_p = 0$. Then, $\lim_{p\to+\infty} a_p = 0$.

B. Auxiliary Bellman Operator $\hat{\mathbb{G}}$

In this section, an auxiliary Bellman operator $\hat{\mathbb{G}}$ is introduced to facilitate the analysis of $\mathbb{G}$ in Section V-C.

Proof of Lemma V.1: Fix $p \geq 1$, $x \in \mathbf{S}^p$ and $\tilde{x} \in X_E^p(x) \subseteq \mathbf{X}^p$. Without loss of generality, we denote $\mathcal{V}_G^p(x) = \{1,\ldots,N_p\}$. It follows from the definition of X_E^p that $\mathcal{V}_G^p(\tilde{x}) = \{1,\ldots,N_p\}$. It follows from the definition of $\mathbf{S}^p$ that $\exists x' \in \mathbf{S}$ s.t. $\|x - x'\| \leq h_p$. Construct $\tilde{x}'$ s.t. $\tilde{x}'_i \triangleq \begin{cases} \tilde{x}_i, & \text{if } i \in \{1,\ldots,N_p\} \\ \tilde{x}_i + x'_i - x_i, & \text{otherwise.} \end{cases}$

Clearly, $\|\tilde{x} - \tilde{x}'\| \leq \|x - x'\| \leq h_p$.

Now, we proceed to show that $\tilde{x}' \in \mathbf{S}$. It again follows from the definition of X_E^p that $\tilde{x}_i = x_i\ \ \forall i \in \{N_p+1,\ldots,N\}$. Therefore, we may rewrite $\tilde{x}'$ as

$$\tilde{x}'_i = \begin{cases} \tilde{x}_i, & \text{if } i \in \{1,\ldots,N_p\} \\ x'_i, & \text{otherwise.} \end{cases}$$

By Assumptions (A7) and (A9), we have $\forall i \in \{1,\ldots,N_p\}$ and $j \in \{1,\ldots,N\}$, $\|\tilde{x}_i - \tilde{x}_j\| \geq \sigma$. Since $x' \in \mathbf{S}$, it follows from the definition of $\mathbf{S}$ that $\|x'_i - x'_j\| \geq \sigma\ \ \forall i \neq j$ and $i,j \in \{N_p+1,\ldots,N\}$. This indicates that $\|\tilde{x}'_i - \tilde{x}_j\| \geq \sigma\ \ \forall i \neq j$ and $i,j \in \{N_p+1,\ldots,N\}$. In summary, we have $\|\tilde{x}'_i - \tilde{x}_j\| \geq \sigma$ holds for every $i \neq j$, which implies $\tilde{x}' \in \mathbf{S}$.

Since $\|\tilde{x} - \tilde{x}'\| \leq h_p$ and $\tilde{x} \in \mathbf{X}^p$, we arrive at $\tilde{x} \in (\mathbf{S} + h_p\mathcal{B}) \cap \mathbf{X}^p = \mathbf{S}^p$. □

Proof of Lemma V.2: The first property follows from the definition of $\tilde{X}^p$. For any pair of $i \in \mathcal{V}_p^G(x)$ and $\tilde{x} \in \tilde{X}^p(x)$, it holds that $\tilde{x}_i = x_i$, then $i \in \mathcal{V}_p^G(\tilde{x})$.

Now, we proceed to show the second property. Since both ϵ_p and h_p are monotonically decreasing, $\forall i \in \mathcal{V}_{p+1}^G(x)$, $d(x_i, X_i^G) \leq M_i\epsilon_{p+1} + h_{p+1} < M_i\epsilon_p + h_p$. It follows from the definition of $\mathcal{V}_p^G$ that $i \in \mathcal{V}_p^G(x)$. Then, the second property is proven.

We are now in a position to prove the third property. Throughout the rest of the proof, given any $p \geq 1$, $n \geq 0$, and $x \in \mathbf{S}^p$, define a value in $v_n^p(x)$ by $\tau^{p,n} \in v_n^p(x)$. The ith element of $\tau^{p,n}$ is denoted by $\tau_i^{p,n}$. The grid index p in $\tau^{p,n}$ may be omitted when omission causes no ambiguity. The proof is based on induction on p. Denote the induction hypothesis for p by $H(p)$ as $\tau_i^{p,n} = 0$ for any $x \in \mathbf{S}^p, i \in \mathcal{V}_p^G(x), 0 \leq n \leq \bar{n}_p$ and $\tau^{p,n} \in v_n^p(x)$.

For $p = 1$, fix $x \in \mathbf{S}^p$ and $i \in \mathcal{V}_p^G(x)$ and take $n = 0$. Since $\mathbf{X}^0 = \emptyset$, $v_0^1(x) = V^0(x)$. It follows from (10) that $\tau_i^0 = 0$ for every $\tau^0 \in v_0^1(x)$. Moreover, $\tilde{T}_i(x_i) = 0$ and $\tilde{X}_i(x_i) = x_i$. Now, we adopt induction on n to prove that $\tau_i^{1,n} = 0$ for all $x \in \mathbf{S}^1 \subseteq \mathbf{X}^1$, $i \in \mathcal{V}_1^G(x)$ and $0 \leq n \leq \bar{n}_p$. For $n = 0$, it has been proven. Assume it holds up to $0 \leq n \leq \bar{n}_p$. Then, $\tau_i^{1,n} + \tilde{T}_i(x_i) - \tau_i^{1,n}\tilde{T}_i(x_i) = \tau_i^{1,n} = 0$ holds for any $x \in \mathbf{S}^1$, $i \in \mathcal{V}_1^G(x)$, and $\tau^{1,n} \in v_n^p(x)$. Therefore, it follows from (5) that $\tau_i^{1,n+1} = 0$.

Assume $H(p)$ holds and let us consider $p+1$. Fix $x \in \mathbf{S}^{p+1}$ and $i \in \mathcal{V}_{p+1}^G(x)$. By the second property of this lemma, $i \in \mathcal{V}_p^G(x)$. Take $\tau^0 \in v_0^{p+1}(x)$. If $x \in \mathbf{X}^p$, that is, $x \in \mathbf{S}^{p+1} \cap \mathbf{X}^p = (\mathbf{S} + h_{p+1}\mathcal{B}) \cap \mathbf{X}^{p+1} \cap \mathbf{X}^p \subseteq \mathbf{S}^p$, we have $v_0^{p+1}(x) = \tilde{v}^p(x) = v_{\bar{n}_p}^p(x)$. Therefore, $\forall \tau^0 \in v_0^{p+1}(x)$, it follows from $H(p)$ that $\tau_i^0 = 0$. If $x \in \mathbf{S}^{p+1} \setminus \mathbf{X}^p$, $v_0^{p+1}(x) = \bigcup_{\tilde{x} \in X_E^{p+1}(x)} \tilde{v}^p(\tilde{x})$. Notice that when $x \in \mathbf{S}^{p+1}$, it follows from Lemma V.1 that $\tilde{x} \in \mathbf{S}^{p+1}$. Then, if $\tilde{x} \in \mathbf{S}^{p+1} \setminus \mathbf{X}^p$, $\tilde{v}^p(\tilde{x}) = V^p(\tilde{x})$; hence, it follows from the definition of V^p in (10) that we have $V_i^p(x) = 0$. If $\tilde{x} \in \mathbf{S}^{p+1} \cap \mathbf{X}^p \subseteq \mathbf{S}^p$, $\tilde{v}^p(\tilde{x}) = v_{\bar{n}_p}^p(\tilde{x})$; hence, it follows from $H(p)$ that $\forall \tilde{\tau} \in \tilde{v}^p(\tilde{x})$, $\tilde{\tau}_i = 0$. Therefore, $\forall \tilde{x} \in X_E^{p+1}(x)$ and $\tilde{\tau} \in \tilde{v}^p(\tilde{x})$, $\tilde{\tau}_i = 0$. That is to say, we have $\tau_i^0 = 0$ for $x \in \mathbf{S}^{p+1} \setminus \mathbf{X}^p$ and $\tau^0 \in v_0^{p+1}(x)$. In summary, $\tau_i^0 = 0$ for every $x \in \mathbf{S}^{p+1}$, $i \in \mathcal{V}_{p+1}^G(x)$, and $\tau^0 \in v_0^{p-1}(x)$. For $1 \leq n \leq \bar{n}_p$, we follow the arguments for $p = 1$ and it holds that $\tau_i^n = 0\ \ \forall \tau^n \in v_n^{p+1}(x)$. Then, $H(p+1)$ is proven and the proof of the third property is finished. □

Proof of Lemma V.3: Throughout the proof, we adopt the shorthand notation $v \triangleq v_n^p$. Without loss of generality, let $\mathcal{V}_p^G(x) = \{1,\ldots,N_p\}$ for some $0 \leq N_p \leq N$ and $\mathcal{V}_p^G(x') = \{1,\ldots,N'_p\}$ for some $0 \leq N'_p \leq N_p$. Specifically, when $N_p = 0$ (resp. $N'_p = 0$), $\mathcal{V}_p^G(x) = \emptyset$ (resp. $\mathcal{V}_p^G(x') = \emptyset$).

Notice that when $N_p = N$, i.e., all robots are in their goal regions at state x, it follows from the third property of Lemma V.2 that $E_{\mathbb{G}^m v}(x) = E_v(x) = [0,1]^N \supseteq E_{\mathbb{G}^m v}(x')$ for any $x' \in X_P^p(x)$, hence the lemma trivially holds. When $N_p = 0$, it holds that $N'_p = 0$, $x = x'$, and the lemma also trivially holds. In the following proof, we restrict $1 \leq N_p \leq N-1$.

The lemma is proven by induction on m. Denote the induction hypothesis for m by $H(m)$. Then, $H(0)$ trivially holds. Assume $H(m)$ holds and let us consider $m+1$. It follows from (7) that

$$E_{\mathbb{G}^{m+1}v}(x) = \Delta\tau(x) + (\mathbf{1} - \Delta\tau(x)) \circ \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{\mathbb{G}^m v}(\tilde{x})$$

$$E_{\mathbb{G}^{m+1}v}(x') = \Delta\tau(x') + (\mathbf{1} - \Delta\tau(x')) \circ \bigcup_{\tilde{x}' \in \tilde{X}^p(x')} E_{\mathbb{G}^m v}(\tilde{x}').$$

First, we focus on the unions on the right-hand side, especially the one-hop neighbors $\tilde{X}^p$.

Claim VII.1: For all $\tilde{x}' \in \tilde{X}^p(x')$, $\exists \tilde{x} \in \tilde{X}^p(x)$ s.t. $\tilde{x}' \in X_P^p(\tilde{x})$.

Proof: Fix $\tilde{x}' \in \tilde{X}^p(x')$ and define $\tilde{x} \in \mathbf{X}^p$ s.t.

$$\tilde{x}_i = \begin{cases} x_i, & \text{if } i \in \{1,\ldots,N_p\} \\ \tilde{x}'_i, & \text{otherwise.} \end{cases}$$

We proceed to show $\tilde{x} \in \tilde{X}^p(x) = (\prod_{i=1}^N \tilde{X}_i^p(x_i)) \cap \mathbf{S}^p$.

For $i \in \{1,\ldots,N_p\}$, we have $\tilde{X}_i^p(x_i) = \{x_i\}$; therefore, $\tilde{x}_i \in \tilde{X}_i^p(x_i)$. For $i \in \{N_p+1,\ldots,N\}$, it follows from $x' \in X_P^p(x)$ that $x_i = x'_i$. Then, $\tilde{x}_i = \tilde{x}'_i \in \tilde{X}_i^p(x'_i) = \tilde{X}_i^p(x_i)$. Therefore, $\tilde{x} \in \prod_{i=1}^N \tilde{X}_i^p(x_i)$.

Notice that $\tilde{x}' \in \tilde{X}^p(x') \subseteq \mathbf{S}^p$, thus $\exists \tilde{y}' \in \mathbf{S}$ s.t. $\|\tilde{x}' - \tilde{y}'\| \leq h_p$. Define $\tilde{y}$ s.t.

$$\tilde{y}_i = \begin{cases} x_i, & \text{if } i \in \{1,\ldots,N_p\} \\ \tilde{y}'_i, & \text{otherwise.} \end{cases}$$

Clearly, $\|\tilde{x} - \tilde{y}\| \leq \|\tilde{x}' - \tilde{y}'\| \leq h_p$. Since $\tilde{y}' \in \mathbf{S}$, it holds that $\|\tilde{y}_i - \tilde{y}_j\| = \|\tilde{y}'_i - \tilde{y}'_j\| \geq \sigma\ \ \forall i,j \in \{N_p+1,\ldots,N\}, i \neq j$. For $i \in \{1,\ldots,N_p\}$, we have $\tilde{x}_i = x_i \in X_i^G + (M_i\epsilon_p + h_p)\mathcal{B}$. Then, by Assumptions (A7) and (A9), we have $\|\tilde{y}_i - \tilde{y}_j\| = \|x_i - \tilde{y}_j\| \geq \sigma\ \ \forall i \in \{1,\ldots,N_p\}$ and

$j \in \{1, \ldots, N\}$. Therefore, $\tilde{y} \in \mathbf{S}$ and $\tilde{x} \in (\tilde{y} + h_p\mathcal{B}) \cap \mathbf{X}^p \subseteq \mathbf{S}^p$. Thus, $\tilde{x} \in \tilde{X}^p(x)$ is proven.

Now, we proceed to show $\tilde{x}' \in X_P^p(\tilde{x})$. By the first property of Lemma V.2, $\mathcal{V}_p^G(\tilde{x}) \supseteq \mathcal{V}_p^G(x)$. Then, for $i \in \mathcal{V} \setminus \mathcal{V}_p^G(\tilde{x}) \subseteq \mathcal{V} \setminus \mathcal{V}_p^G(x) = \{N_p+1, \ldots, N\}$, $\tilde{x}'_i = \tilde{x}_i$. By the definition of X_P^p, $\tilde{x}' \in X_P^p(\tilde{x})$ and, therefore, the claim is proven. □

It follows from Claim VII.1 and $H(m)$ that $\forall \tilde{x}' \in \tilde{X}^p(x')$, $\exists \tilde{x} \in \tilde{X}^p(x)$ s.t. $E_{\mathbb{G}^m v}(\tilde{x}') \subseteq E_{\mathbb{G}^m v}(\tilde{x})$. Hence, $\bigcup_{\tilde{x}' \in \tilde{X}^p(x')} E_{\mathbb{G}^m v}(\tilde{x}') \subseteq \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{\mathbb{G}^m v}(\tilde{x})$ and we have

$$E_{\mathbb{G}^{m+1} v}(x') \subseteq \Delta\tau(x') + (\mathbf{1} - \Delta\tau(x')) \circ \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{\mathbb{G}^m v}(\tilde{x}). \tag{17}$$

Next, we prove that the right-hand side of (17) is a subset of $E_{\mathbb{G}^{m+1} v}(x)$.

Claim VII.2: The following relationship holds:

$$\Delta\tau(x') + (\mathbf{1} - \Delta\tau(x')) \circ \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{\mathbb{G}^m v}(\tilde{x})$$
$$\subseteq \Delta\tau(x) + (\mathbf{1} - \Delta\tau(x)) \circ \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{\mathbb{G}^m v}(\tilde{x}).$$

Proof: For any $\tilde{x} \in \tilde{X}^p(x)$ and $\tau \in E_{\mathbb{G}^m v}(\tilde{x})$, construct $\hat{\tau}$ s.t. $\hat{\tau}_i = \begin{cases} (1 - e^{-\kappa_p}) + e^{-\kappa_p}\tau_i, & \text{if } i \in \{N'_p + 1, \ldots, N_p\} \\ \tau_i, & \text{otherwise} \end{cases}$.
Since $\hat{\tau} \succeq \tau$, $\hat{\tau} \in E_{\mathbb{G}^m v}(\tilde{x})$. Recall $\Delta\tau(x') = \Psi(\mathcal{E}(T^p(x')))$ and $\Delta\tau_i(x') = \begin{cases} 0, & \text{if } i \in \{1, \ldots, N'_p\} \\ 1 - e^{-\kappa_p}, & \text{otherwise.} \end{cases}$

Then, the following holds:

$$\Delta\tau(x') + (\mathbf{1} - \Delta\tau(x')) \circ \tau$$
$$= \begin{bmatrix} \mathbf{0}_{N'_p} \\ (1 - e^{-\kappa_p})\mathbf{1}_{N_p - N'_p} \\ (1 - e^{-\kappa_p})\mathbf{1}_{N - N_p} \end{bmatrix} + \begin{bmatrix} \mathbf{1}_{N'_p} \\ e^{-\kappa_p}\mathbf{1}_{N_p - N'_p} \\ e^{-\kappa_p}\mathbf{1}_{N - N_p} \end{bmatrix} \circ \tau$$
$$= \begin{bmatrix} \mathbf{0}_{N'_p} \\ \mathbf{0}_{N_p - N'_p} \\ (1 - e^{-\kappa_p})\mathbf{1}_{N - N_p} \end{bmatrix} + \begin{bmatrix} \mathbf{1}_{N'_p} \\ \mathbf{1}_{N_p - N'_p} \\ e^{-\kappa_p}\mathbf{1}_{N - N_p} \end{bmatrix} \circ \hat{\tau}$$
$$= \Delta\tau(x) + (\mathbf{1} - \Delta\tau(x)) \circ \hat{\tau}.$$

In summary, for every $\tilde{x} \in \tilde{X}^p(x)$ and $\tau \in E_{\mathbb{G}^m v}(\tilde{x})$, there is $\hat{\tau} \in E_{\mathbb{G}^m v}(\tilde{x})$ s.t. $\Delta\tau(x') + (\mathbf{1} - \Delta\tau(x')) \circ \tau = \Delta\tau(x) + (\mathbf{1} - \Delta\tau(x)) \circ \hat{\tau}$. Hence, the proof of the claim is finished. □

Together with (17), Claim VII.2 indicates $E_{\mathbb{G}^{m+1} v}(x') \subseteq \Delta\tau(x) + (\mathbf{1} - \Delta\tau(x)) \circ \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{\mathbb{G}^m v}(\tilde{x}) = E_{\mathbb{G}^{m+1} v}(x)$. Then, $H(m+1)$ holds and the lemma is proven. □

Proof of Lemma V.4: Fix a pair of $x \in \mathbf{S}^p$ and $x' \in X_P^p(x)$. Without loss of generality, let $\mathcal{V}_p^G(x) = \{1, \ldots, N_p\}$ and $\mathcal{V}_p^G(x') = \{1, \ldots, N'_p\}$ for some $0 \le N'_p \le N_p \le N$. Specifically, when $N_p = 0$ (resp. $N'_p = 0$), $\mathcal{V}_p^G(x) = \emptyset$ (resp. $\mathcal{V}_p^G(x') = \emptyset$).

Notice that when $N_p = N$, by the third property of Lemma V.2, $E_{v_n^p}(x') \subseteq [0,1]^N = E_{v_n^p}(x)$ for any $x' \in X_P^p(x)$ and $0 \le n \le \bar{n}_p$, and the lemma trivially holds. When $N_p = 0$, it holds that $N'_p = 0$ and $x = x'$, and the lemma also trivially holds. In the following proof, we restrict $1 \le N_p \le N - 1$.

The lemma is proven by induction on p. Denote the induction hypothesis for p by $H(p)$ as $E_{v_n^p}(x') \subseteq E_{v_n^p}(x)$ holds for all $x \in \mathbf{S}^p$, $x' \in X_P^p(x)$, and $0 \le n \le \bar{n}_p$.

For $p = 1$, it follows from the definition of V^0 that $E_{v_0^1}(x) = E_{V^0}(x) = [0,1]^{N_1} \times \{1\}^{N - N_1} \supseteq [0,1]^{N'_1} \times \{1\}^{N - N'_1} = E_{v_0^1}(x')$, where $E_{V^1}(x) \triangleq (\{V^1(x)\} + \mathbb{R}^N_{\ge 0}) \cap [0,1]^N$ is the epigraphical profile of interpolation function $V^1(x)$. Since this holds for every $x \in \mathbf{S}^1$ and $x' \in X_P^1(x)$, it follows from Lemma V.3 that $E_{v_n^1}(x) \supseteq E_{v_n^1}(x')$ holds for all $0 \le n \le \bar{n}_p$. Hence, $H(1)$ holds.

Assume $H(p)$ holds for $p \ge 1$. For $p+1$, pick a pair of $x \in \mathbf{S}^{p+1}$ and $x' \in X_P^{p+1}(x)$. There are four cases, which are as follows:

Case 1: $x, x' \in \mathbf{S}^p$;
Case 2: $x \in \mathbf{S}^{p+1} \setminus \mathbf{S}^p$ and $x' \in \mathbf{S}^p$;
Case 3: $x \in \mathbf{S}^p$ and $x' \in \mathbf{S}^{p+1} \setminus \mathbf{S}^p$;
Case 4: $x, x' \in \mathbf{S}^{p+1} \setminus \mathbf{S}^p$.

Claim VII.3: $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 1.

Proof: It follows from the definitions of v_0^{p+1} and $\tilde{v}^p$ that $E_{v_0^{p+1}}(x) = E_{\tilde{v}^p}(x) = E_{v_{\bar{n}_p}^p}(x)$ and $E_{v_0^{p+1}}(x') = E_{\tilde{v}^p}(x') = E_{v_{\bar{n}_p}^p}(x')$. By $H(p)$, we have $E_{v_0^{p+1}}(x') = E_{v_{\bar{n}_p}^p}(x') \subseteq E_{v_{\bar{n}_p}^p}(x) = E_{v_0^{p+1}}(x)$. Then, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 1. □

Claim VII.4: $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 2.

Proof: Notice that $E_{v_0^{p+1}}(x) = \bigcup_{\tilde{x} \in X_E^{p+1}(x)} E_{\tilde{v}^p}(\tilde{x})$ and $E_{v_0^{p+1}}(x') = E_{v_{\bar{n}_p}^p}(x')$. Now, we are going to construct $\tilde{x} \in X_E^{p+1}(x)$ s.t. $\tilde{x} \in \mathbf{S}^p$ and prove that $E_{v_{\bar{n}_p}^p}(x') \subseteq E_{\tilde{v}^p}(\tilde{x})$. For $i \in \{N_{p+1}+1, \ldots, N\}$, let $\tilde{x}_i = x_i$; for $i \in \{1, \ldots, N_{p+1}\}$, pick $\tilde{x}_i \in X_i^G \cap X_i^p$. Since $x' \in \mathbf{S}^p$ and $x' \in X_P^{p+1}(x)$, we have $\tilde{x}_i = x_i = x'_i \in X_i^p \quad \forall i \in \{N_{p+1}+1, \ldots, N\}$. Therefore, we have $\tilde{x} \in X_E^{p+1}(x)$ and $\tilde{x} \in \mathbf{S}^p$.

Then, we show that $x' \in X_P^p(\tilde{x})$. Consider $j \in \{1, \ldots, N\}$ s.t. $\tilde{x}_j \notin X_j^G + (M_j\epsilon_p + h_p)\mathcal{B}$. By Assumption (A7), we have $\tilde{x}_j \notin X_j^G + (M_j\epsilon_{p+1} + h_{p+1})\mathcal{B}$. The fact that $\tilde{x} \in X_E^{p+1}(x)$ implies $\tilde{x}_i \in X_i^G + (M_i\epsilon_{p+1} + h_{p+1})\mathcal{B} \quad \forall i \in \{1, \ldots, N_{p+1}\}$. Therefore, $j \in \{N_{p+1}+1, \ldots, N\}$ and, hence, $\tilde{x}_j = x_j$. Moreover, since $x' \in X_P^{p+1}(x)$, it follows from $j \in \{N_{p+1}+1, \ldots, N\}$ that $x'_j = x_j = \tilde{x}_j$. This holds for every $j \in \mathcal{V}$ s.t. $\tilde{x}_j \notin X_j^G + (M_j\epsilon_p + h_p)\mathcal{B}$. By the definition of X_P^p, we conclude that $x' \in X_P^p(\tilde{x})$.

By utilizing $H(p)$, it follows from $x' \in X_P^p(\tilde{x})$ that $E_{v_0^{p+1}}(x') = E_{v_{\bar{n}_p}^p}(x') \subseteq E_{v_{\bar{n}_p}^p}(\tilde{x})$. Since $\tilde{x} \in \mathbf{S}^p$, $E_{v_{\bar{n}_p}^p}(\tilde{x}) = E_{\tilde{v}^p}(\tilde{x})$. Moreover, it follows from $\tilde{x} \in X_E^{p+1}(x)$ that $E_{\tilde{v}^p}(\tilde{x}) \subseteq \bigcup_{\tilde{x} \in X_E^{p+1}(x)} E_{\tilde{v}^p}(\tilde{x}) = E_{v_0^{p+1}}(x)$. In summary, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$. Then, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 2. □

Claim VII.5: $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 3.

Proof: Notice $E_{v_0^{p+1}}(x) = E_{\tilde{v}^p}(x) = E_{v_{\bar{n}_p}^p}(x)$ and $E_{v_0^{p+1}}(x') = \bigcup_{\tilde{x}' \in X_E^{p+1}(x')} E_{\tilde{v}^p}(\tilde{x}')$. For each $\tilde{x}' \in X_E^{p+1}(x')$, following two cases arise.

Case 3.1, $\tilde{x}' \in \mathbf{S}^{p+1} \setminus \mathbf{S}^p$: Then, $\tilde{v}^p(\tilde{x}') = \{V^p(\tilde{x}')\}$. Since $\tilde{x}' \in X_E^{p+1}(x')$, $E_{\tilde{v}^p}(\tilde{x}') = E_{V^p}(\tilde{x}') = [0,1]^{N'_{p+1}} \times \{1\}^{N-N'_{p+1}}$. It follows from $N'_{p+1} \leq N_{p+1}$ that $[0,1]^{N'_{p+1}} \times \{1\}^{N-N'_{p+1}} \subseteq [0,1]^{N_{p+1}} \times \{1\}^{N-N_{p+1}}$. By the third property of Lemma V.2, $[0,1]^{N_p} \times \{1\}^{N-N_p} \subseteq E_{v^p_{\bar{n}_p}}(x)$. Therefore, we have $E_{\tilde{v}^p}(\tilde{x}') \subseteq E_{v_0^{p+1}}(x)$.

Case 3.2, $\tilde{x}' \in \mathbf{S}^p$: Then, $E_{\tilde{v}^p}(\tilde{x}') = E_{v^p_{\bar{n}_p}}(\tilde{x}')$. Since $\tilde{x}' \in X_E^{p+1}(x')$, thus for any $i \in \{N_{p+1}+1, \ldots, N\}$, we have $\tilde{x}'_i = x'_i = x_i \notin X_i^G + (M_i\epsilon_{p+1} + h_{p+1})\mathcal{B}$. Using the second property of Lemma V.2, we have $\tilde{x}'_i = x_i \quad \forall i \in \mathcal{V} \setminus \mathcal{V}_p^G(x) \subseteq \mathcal{V} \setminus \mathcal{V}_{p+1}^G(x) = \{N_{p+1}+1, \ldots, N\}$. Therefore, $\tilde{x}' \in X_P^p(x)$. By $H(p)$, $E_{v^p_{\bar{n}_p}}(\tilde{x}') \subseteq E_{v^p_{\bar{n}_p}}(x) = E_{v_0^{p+1}}(x)$. That is, $E_{\tilde{v}^p}(\tilde{x}') \subseteq E_{v_0^{p+1}}(x)$.

In summary, $\forall \tilde{x}' \in X_E^{p+1}(x')$, $E_{\tilde{v}^p}(\tilde{x}') \subseteq E_{v_0^{p+1}}(x)$. Then, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 3. □

Claim VII.6: $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 4.

Proof: Then, $E_{v_0^{p+1}}(x) = \bigcup_{\tilde{x} \in X_E^{p+1}(x)} E_{\tilde{v}^p}(\tilde{x})$ and $E_{v_0^{p+1}}(x') = \bigcup_{\tilde{x}' \in X_E^{p+1}(x')} E_{\tilde{v}^p}(\tilde{x}')$. Consider x', and there are two scenarios, which are as follows.

Case 4.1: $\exists j \in \{N'_{p+1}+1, \ldots, N\}$ s.t. $x'_j \in X_j^{p+1} \setminus X_j^p$. Then, $\forall \tilde{x}' \in X_E^{p+1}(x')$, we have $\tilde{x}'_j = x'_j \in X_j^{p+1} \setminus X_j^p$. This indicates $\tilde{x}' \in \mathbf{S}^{p+1} \setminus \mathbf{S}^p$. Following Case 3.1, we have $E_{v_0^{p+1}}(x') = \bigcup_{\tilde{x}' \in X_E^{p+1}(x')} E_{\tilde{v}^p}(\tilde{x}') = \bigcup_{\tilde{x}' \in X_E^{p+1}(x')} ([0,1]^{N'_{p+1}} \times \{1\}^{N-N'_{p+1}}) = [0,1]^{N'_{p+1}} \times \{1\}^{N-N'_{p+1}}$ and $[0,1]^{N_{p+1}} \times \{1\}^{N-N_{p+1}} \subseteq E_{v_0^{p+1}}(x)$. Notice that $N'_{p+1} \leq N_{p+1}$. Then, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$. Hence, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 4.1.

Case 4.2: $\forall j \in \{N'_{p+1}+1, \ldots, N\}, x'_j \in X_j^p$ while $\exists j \in \{1, \ldots, N'_{p+1}\}$ s.t. $x'_j \in X_j^{p+1} \setminus X_j^p$. We show that $\exists \tilde{x} \in X_E^{p+1}(x)$ s.t. $\tilde{x} \in \mathbf{S}^p$. Since $x' \in X_P^{p+1}(x)$, $x_i = x'_i \in X_i^p \subseteq X_i^{p+1} \quad \forall i \in \{N_{p+1}+1, \ldots, N\} \subseteq \{N'_{p+1}+1, \ldots, N\}$. By picking $\tilde{x}_i \in X_i^G \cap X_i^p$ for $i \in \{1, \ldots, N_{p+1}\}$ and $\tilde{x}_i = x_i \quad \forall i \in \{N_{p+1}+1, \ldots, 1\}$, we have $\tilde{x} \in X_E^{p+1}(x)$. In addition, since $x \in \mathbf{S}^{p+1}$, we have $\exists y \in \mathbf{S}$ s.t. $\|x - y\| \leq h_{p+1} \leq h_p$. Define $\tilde{y}$ s.t. $\tilde{y}_i = \tilde{x}_i \quad \forall i \in \{1, \ldots, N_{p+1}\}$ and $\tilde{y}_i = y_i \quad \forall i \in \{N_{p+1}+1, \ldots, N\}$. Then, $\|\tilde{y} - \tilde{x}\| = \sqrt{\sum_{i=N_{p+1}+1}^{N} (y_i - x_i)^2} \leq \|y - x\| \leq h_{p+1}$. Then, $\forall i, j \in \{N_{p+1}+1, \ldots, N\}$, $\|\tilde{y}_i - \tilde{y}_j\| = \|y_i - y_j\| \geq \sigma$. Moreover, it follows from Assumptions (A7) and (A9) that $\forall i \in \{1, \ldots, N_{p+1}\}$ and $j \in \{1, \ldots, N\}$, $\|\tilde{y}_i - \tilde{y}_j\| = \|\tilde{x}_i - \tilde{x}_j\| \geq \sigma$. This indicates that $\tilde{y} \in \mathbf{S}$ and, hence, $\tilde{x} \in \mathbf{S}^p$. By the definition of X_E^{p+1}, $X_E^{p+1}(x) = X_E^{p+1}(\tilde{x})$. This means we can replace $X_E^{p+1}(x)$ with $X_E^{p+1}(\tilde{x})$ and degenerate the current case to Case 3. Then, by Claim VII.5, $H(p+1)$ holds for $x' \in \mathbf{S}^{p+1} \setminus \mathbf{S}^p$ and $\tilde{x} \in \mathbf{S}^p$. Thus, $E_{v_0^{p+1}}(x') = \bigcup_{\tilde{x}' \in X_E^{p+1}(x')} E_{\tilde{v}^p}(\tilde{x}') \subseteq E_{v^p_{\bar{n}_p}}(\tilde{x}) \subseteq \bigcup_{\tilde{x} \in X_E^{p+1}(x)} E_{\tilde{v}^p}(\tilde{x}) = E_{v_0^{p+1}}(x)$.

By the two cases discussed, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for Case 4. □

By the four cases above, $E_{v_0^{p+1}}(x') \subseteq E_{v_0^{p+1}}(x)$ holds for all $x \in \mathbf{S}^{p+1}$ and $x' \in X_P^{p+1}(x)$. By Lemma V.3, $H(p+1)$ is proven. Then, the lemma is established. □

Proof of Corollary V.1: Fix $p \geq 1$, $0 \leq n \leq \bar{n}_p$, $x \in \mathbf{S}^p$ and $x' \in X_E^p(x)$. On one hand, by Lemma V.4, $\forall x' \in X_E^p(x) \subseteq X_P^p(x)$, $E_{v_n^p}(x') \subseteq E_{v_n^p}(x)$. On the other hand, $x \in X_E^p(x')$; thus again by Lemma V.4, we also have $E_{v_n^p}(x) \subseteq E_{v_n^p}(x')$, which indicates that $E_{v_n^p}(x') = E_{v_n^p}(x)$.

Since Lemma V.3 holds for every $m \geq 0$, the aforementioned proof can be directly extended to $\mathbb{G}^m v_0^p$ for any $m > \bar{n}_p$. By Theorem V.1, v_∞^p exists and $v_\infty^p(x) = \mathrm{Lim}_{m \to +\infty} \mathbb{G}^m v_0^p(x)$ for any $x \in \mathbf{S}^p$. Hence, the equivalence $E_{\mathbb{G}^m v_0^p}(x) = E_{\mathbb{G}^m v_0^p}(x')$ can be further extended to $E_{v_\infty^p}(x) = E_{v_\infty^p}(x)$ by taking $m \to +\infty$. □

Proof of Lemma V.5: We fix $p \geq 1$, $0 \leq n \leq \bar{n}_p$ and $x \in \mathbf{S}^p$. Recall that

$$E_{\mathbb{G} v_n^p}(x) = \Delta\tau(x) + (\mathbf{1} - \Delta\tau(x)) \circ \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{v_n^p}(\tilde{x})$$

$$E_{\hat{\mathbb{G}} v_n^p}(x) = \Delta\tau(x) + (\mathbf{1} - \Delta\tau(x)) \circ \bigcup_{\hat{x} \in \hat{X}^p(x)} E_{v_n^p}(\hat{x}).$$

The difference of $\mathbb{G}$ and $\hat{\mathbb{G}}$ solely depends on $\tilde{X}^p$ and $\hat{X}^p$. The proof of either of the two equivalences automatically proves the other.

By the definitions of $\tilde{X}^p(x)$ and $\hat{X}^p(x)$, $\hat{X}_i^p(x_i) = \tilde{X}_i^p(x_i) \quad \forall i \in \mathcal{V} \setminus \mathcal{V}_p^G(x)$. Therefore, $\forall \hat{x} \in \hat{X}^p(x)$, $\exists \tilde{x} \in \prod_{i \in \mathcal{V}} \tilde{X}_i^p(x)$ s.t. $\hat{x}_i = \tilde{x}_i \quad \forall i \in \mathcal{V} \setminus \mathcal{V}_p^G(x)$. It follows from the definitions of $\tilde{X}_i^p$ and $\mathcal{V}_p^G$ that $\tilde{x}_i = x_i \in X_i^G + (M_i\epsilon_p + h_p)\mathcal{B} \quad \forall i \in \mathcal{V}_p^G(x)$. By Assumptions (A7) and (A9), we see that $\|\tilde{x}_i - \tilde{x}_j\| \geq \sigma \quad \forall i \in \mathcal{V}_p^G(x)$ and $j \in \mathcal{V}$ s.t. $i \neq j$. It follows from $\tilde{x}_i = \hat{x}_i \quad \forall i \in \mathcal{V} \setminus \mathcal{V}_p^G(x)$ and $\hat{x} \in \hat{X}^p(x) \subseteq \mathbf{S}^p$ that $\|\tilde{x}_i - \tilde{x}_j\| \geq \sigma \quad \forall i, j \in \mathcal{V} \setminus \mathcal{V}_p^G(x)$ s.t. $i \neq j$. This indicates $\tilde{x} \in \mathbf{S}^p$. Therefore, it follows from the definition of $\tilde{X}^p$ that $\tilde{x} \in \tilde{X}^p(x)$. Since $\mathcal{V}_p^G(\tilde{x}) = \mathcal{V}_p^G(x)$, then $\hat{x} \in X_P^p(\tilde{x})$. By Lemma V.4, $E_{v_n^p}(\hat{x}) \subseteq E_{v_n^p}(\tilde{x})$. We see that this holds for all $\hat{x} \in \hat{X}^p(x)$, then $\bigcup_{\hat{x} \in \hat{X}^p(x)} E_{v_n^p}(\hat{x}) \subseteq \bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{v_n^p}(\tilde{x})$.

In addition, $\tilde{X}^p(x) \subseteq \hat{X}^p(x)$, we have $\bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{v_n^p}(\tilde{x}) \subseteq \bigcup_{\hat{x} \in \hat{X}^p(x)} E_{v_n^p}(\hat{x})$. It is concluded that $\bigcup_{\tilde{x} \in \tilde{X}^p(x)} E_{v_n^p}(\tilde{x}) = \bigcup_{\hat{x} \in \hat{X}^p(x)} E_{v_n^p}(\hat{x})$ and $E_{\mathbb{G} v_n^p}(x) = E_{\hat{\mathbb{G}} v_n^p}(x)$. Since this holds for every $p \geq 0$, $0 \leq n \leq n_p$ and $x \in \mathbf{S}^p$, the first part of the lemma is proven.

Since Lemma V.3 holds for every $m \geq 0$, the aforementioned proof can be directly extended to $\mathbb{G}^m v_0^p$ for any $m > \bar{n}_p$. By Theorem V.1, the fixed point v_∞^p exists and $v_\infty^p(x) = \mathrm{Lim}_{m \to +\infty} \mathbb{G}^m v_0^p(x)$ for any $x \in \mathbf{S}^p$. By the equivalence of $\mathbb{G}$ and $\hat{\mathbb{G}}$, we have $v_\infty^p(x) = \mathrm{Lim}_{m \to +\infty} \hat{\mathbb{G}}^m v_0^p(x)$ and $\hat{\mathbb{G}} v_\infty^p = v_\infty^p = \mathbb{G} v_\infty^p$. Therefore, $E_{\mathbb{G} v_\infty^p}(x) = E_{\hat{\mathbb{G}} v_\infty^p}(x)$ and the second conclusion is proven. □

C. Contraction Property of $\mathbb{G}$

In this section, Theorem V.2 shows that the transformed Bellman operator $\mathbb{G}$ in (5) is contractive with factor $e^{-\kappa_p}$.

Proof of Lemma V.6: We first consider $x \in \mathbf{X} \setminus \mathbf{S}$. Since $\mathbf{S}$ is closed and α_p is monotonically decreasing, then there exists $q > 0$ s.t. $\forall p \geq q$, $(x + \alpha_p\mathcal{B}) \cap \mathbf{S}^{p-1} = (x + \alpha_p\mathcal{B}) \cap (\mathbf{S} + h_{p-1}\mathcal{B}) \cap \mathbf{X}^{p-1} = \emptyset$. This renders at $\bigcup_{\tilde{x} \in (x+\alpha_p\mathcal{B}) \cap \mathbf{X}^p} E_{\mathbb{P}\tilde{v}_\infty^{p-1}}(\tilde{x}) = \{\mathbf{1}_N\}$. In addition, it also indicates that $(x + \alpha_p\mathcal{B}) \cap \mathbf{S}^p = \emptyset$. This renders at $\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_\infty}(\tilde{x}) = \{\mathbf{1}_N\}$. Therefore, we have $b_p(x) = 0$. This holds for all $x \in \mathbf{X}\setminus\mathbf{S}$.

Then, we fix $x \in \mathbf{S}$. The following shorthand notations are used throughout the proof:

$$A^p_{11}(x) \triangleq \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})}\ \bigcup_{\tilde{x}'\in X^p_E(\tilde{x})\setminus\mathbf{X}^{p-1}} E_{\tilde{v}^{p-1}_\infty}(\tilde{x}')$$

$$A^p_{12}(x) \triangleq \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})}\ \bigcup_{\tilde{x}'\in X^p_E(\tilde{x})\cap\mathbf{X}^{p-1}} E_{\tilde{v}^{p-1}_\infty}(\tilde{x}')$$

$$A^p_{13}(x) \triangleq \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap(\mathbf{X}^p\setminus(\mathbf{S}^p\cup\mathbf{X}^{p-1}))} E_{\tilde{v}^{p-1}_\infty}(\tilde{x}')$$

$$A^p_{2}(x) \triangleq \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^{p-1}} E_{\mathbb{P}\tilde{v}^{p-1}_\infty}(\tilde{x})$$

$$B^p(x) \triangleq \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^{p}} E_{v^{p}_\infty}(\tilde{x}).$$

We drop the dependence of the aforementioned notations on x for notational simplicity. Now, we are going to simplify A^p_{11}, A^p_{12}, A^p_{13}, and A^p_2. From the definitions of V^p and $\tilde{v}^{p-1}_\infty$, the following holds:

$$A^p_{11} = \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})}\ \bigcup_{\tilde{x}'\in X^p_E(\tilde{x})\setminus\mathbf{X}^{p-1}} E_{V^{p-1}}(\tilde{x}')$$

$$= \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})} E_{V^{p-1}}(\tilde{x})$$

$$A^p_{12} = \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})}\ \bigcup_{\tilde{x}'\in X^p_E(\tilde{x})\cap\mathbf{X}^{p-1}} E_{v^{p-1}_\infty}(\tilde{x}')$$

$$A^p_{13} = \{\mathbf{1}_N\}.$$

By the definitions of $\mathbb{P}$ and $\tilde{v}^{p-1}_\infty$

$$A^p_2 = \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^{p-1}} E_{\tilde{v}^{p-1}_\infty}(\tilde{x}) = \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^{p-1}} E_{v^{p-1}_\infty}(\tilde{x}).$$

By Assumption (A8), we have $(x+\alpha_p\mathcal{B})\cap\mathbf{X}^{p-1} \neq \emptyset$ and $A^p_2 \neq \emptyset$. It follows from the third property of Lemma V.2 that $\forall\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p$, $E_{V^{p-1}}(\tilde{x}) \subseteq E_{v^p_\infty}(\tilde{x})$. This indicates that $A^p_{11} \subseteq B^p$. In addition, it trivially holds that $A^p_{13} \subseteq B^p$. By the first inequality of (14) in Lemma VII.2, $d_H(A^p_{11}\cup A^p_{12}\cup A^p_{13}\cup A^p_2, B^p) \leq d_H(A^p_{12}\cup A^p_2, B^p)$.

Claim VII.7: There is $q \geq 1$ s.t. $\forall p \geq q$, if $x_i \in X^G_i$, $x_i + \alpha_p\mathcal{B} \subseteq X^G_i + (M_i\epsilon_p + h_p)\mathcal{B}$; if $x_i \notin X^G_i$, $(x_i+\alpha_p\mathcal{B}) \cap (X^G_i + (M_i\epsilon_p+h_p)\mathcal{B}) = \emptyset$.

Proof: By Assumption (A7), we have $\exists q_i \geq 0$ s.t. $\forall p \geq q_i$, $\alpha_p \leq M_i\epsilon_p + h_p$. Then, for any $p \geq q_i$, if $x_i \in X^G_i$, $x_i + \alpha_p\mathcal{B} \subseteq X^G_i + (M_i\epsilon_p+h_p)\mathcal{B}$. It again follows from Assumption (A7) that for each $i \in \mathcal{V}$ s.t. $x_i \notin X^G_i$, there exists $q_i \geq 1$ s.t. $\forall p \geq q_i$, $(x_i+\alpha_p\mathcal{B})\cap(X^G_i + (M_i\epsilon_p+h_p)\mathcal{B}) = \emptyset$. Then, the desired q is defined as $q \triangleq \max_{i\in\mathcal{V}} q_i$. □

Claim VII.8: For $p \geq q$ and any pair of $\tilde{x} \in x + \alpha_p\mathcal{B}$ and $i \in \mathcal{V}^G_p(\tilde{x})$, $x_i \in X^G_i$.

Proof: For every $i \in \mathcal{V}^G_p(\tilde{x})$, $\tilde{x}_i \in X^G_i + (M_i\epsilon_p + h_p)\mathcal{B}$. Assume $x_i \notin X^G_i$. It follows from Claim VII.7 that $(x_i + \alpha_p\mathcal{B}) \cap (X^G_i + (M_i\epsilon_p+h_p)\mathcal{B}) = \emptyset$. This contradicts the fact that $\tilde{x}_i \in X^G_i + (M_i\epsilon_p+h_p)\mathcal{B}$. Then, $x_i \in X^G_i$. □

Fix $p \geq q$ and $\tilde{x} \in (x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})$ s.t. $X^p_E(\tilde{x})\cap\mathbf{X}^{p-1}\neq\emptyset$. Define $\hat{x}$ s.t. $\hat{x}_i = \begin{cases}\tilde{x}_i, & \text{if } \tilde{x}_i \in X^{p-1}_i\\ \arg\min_{\hat{x}_i\in X^{p-1}_i}\|\hat{x}_i - x_i\|, & \text{otherwise.}\end{cases}$

Notice that $\hat{x}\in\mathbf{X}^{p-1}$. It follows from the definition of $\mathbf{X}^{p-1}$ that $\|\hat{x}_i - x_i\| \leq h_{p-1} \leq \alpha_p \quad \forall i \text{ s.t. } \tilde{x}_i \notin X^{p-1}_i$. Since $\tilde{x} \in x + \alpha_p\mathcal{B}$, we have $\|\hat{x}_i - x_i\| = \|\tilde{x}_i - x_i\| \leq \alpha_p \quad \forall i \text{ s.t. } \tilde{x}_i \in X^{p-1}_i$. Then, it holds that $\hat{x} \in x + \alpha_p\mathcal{B}$.

Claim VII.9: For $p \geq q$, $\hat{x} \in X^{p-1}_E(\tilde{x})$.

Proof: Since $\exists\tilde{x}' \in X^p_E(\tilde{x})\cap\mathbf{X}^{p-1}$, it follows from the definition of X^p_E that $\tilde{x}_i = \tilde{x}'_i \in X^{p-1}_i \quad \forall i \in \mathcal{V}\setminus\mathcal{V}^G_p(\tilde{x})$. Then, the following two properties hold for $\tilde{x}$: (a) $\forall i\in\mathcal{V}\setminus\mathcal{V}^G_p(\tilde{x}), \tilde{x}_i \in X^{p-1}_i$; (b) $\exists i \in \mathcal{V}^G_p(\tilde{x})$ s.t. $\tilde{x}_i \in X^p_i \setminus X^{p-1}_i$. Property (b) is a result of $\tilde{x} \in \mathbf{S}^p\setminus\mathbf{X}^{p-1}$.

Fix $j \in \mathcal{V}$ s.t. $\hat{x}_j \neq \tilde{x}_j$. Now, we are to show $\hat{x}_j \in X^G_j + (M_j\epsilon_{p-1} + h_{p-1})\mathcal{B}$. By properties (a) and (b), $j \in \mathcal{V}^G_p(\tilde{x})$. It follows from Claim VII.8 that $x_j \in X^G_j$. It follows from Claim VII.7 that $x_j + \alpha_p\mathcal{B} \subseteq X^G_j + (M_j\epsilon_p + h_p)\mathcal{B}$. Therefore, $\hat{x}_j \in x_j + \alpha_p\mathcal{B} \subseteq X^G_j + (M_j\epsilon_p+h_p)\mathcal{B}$. By Assumption (A7), it renders at $\hat{x}_j \in X^G_j + (M_j\epsilon_{p-1} + h_{p-1})\mathcal{B}$.

This holds for all $j\in\mathcal{V}$ s.t. $\hat{x}_j \neq \tilde{x}_j$. By the definition of X^{p-1}_E, we have $\hat{x} \in X^{p-1}_E(\tilde{x})$. □

Claim VII.10: There is $q \geq 1$ s.t. $A^p_{12} \subseteq A^p_2$ holds for all $p \geq q$.

Proof: If $A^p_{12} = \emptyset$, the claim trivially holds. Throughout the proof, assume that $\exists\tilde{x} \in (x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})$ s.t. $X^p_E(\tilde{x})\cap\mathbf{X}^{p-1}\neq\emptyset$.

Pick any $\tilde{x}' \in X^p_E(\tilde{x})\cap\mathbf{X}^{p-1}$. It follows from Assumption (A7) that $X^p_E(\tilde{x}) \subseteq X^{p-1}_E(\tilde{x})$; then $\tilde{x}' \in X^{p-1}_E(\tilde{x})$. It follows from Claim VII.9 that $\exists\hat{x}\in X^{p-1}_E(\tilde{x})$. Since $\tilde{x}' \in X^{p-1}_E(\tilde{x})$, by the definition of X^{p-1}_E, we have $\tilde{x}' \in X^{p-1}_E(\hat{x})$. Then, by Corollary V.1, $E_{v^{p-1}_\infty}(\tilde{x}') = E_{v^{p-1}_\infty}(\hat{x})$. Since $\hat{x} \in (x+\alpha_p\mathcal{B})\cap\mathbf{X}^{p-1}$, then $E_{v^{p-1}_\infty}(\tilde{x}') \subseteq A^p_2$. This holds for every pair of $\tilde{x} \in (x+\alpha_p\mathcal{B})\cap(\mathbf{S}^p\setminus\mathbf{X}^{p-1})$ and $\tilde{x}' \in X^p_E(\tilde{x})\cap\mathbf{X}^{p-1}$. Then, $A^p_{12} \subseteq A^p_2$. □

It follows from Lemma VII.2 and Claim VII.10 that $b_p(x) \leq d_H(A^p_{12}(x)\cup A^p_2(x), B^p(x)) = d_H(A^p_2(x), B^p(x))$ holds for $p \geq q(x)$. Recall $\alpha_p \geq h_p$ and Assumption (A8). It follows from Theorem V.1 that $\lim_{p\to+\infty} b_p(x) \leq \lim_{p\to+\infty} d_H(A^p_2(x), B^p(x)) = 0$. Since this holds for all $x \in \mathbf{X}$, the lemma is proven. □

Proof of Lemma V.7: Take $\delta' > \delta \triangleq d_H(A,B)$. Then, $A \subseteq B + \delta'\mathcal{B}_N$, $B \subseteq A + \delta'\mathcal{B}_N$. Focus on the first relationship and we want to show

$$(\mathbf{1} - \Delta\tau(x))\circ A \subseteq (\mathbf{1}-\Delta\tau(x))\circ B + e^{-\kappa_p}\delta'\mathcal{B}_N. \quad (18)$$

This is equivalent to show that $\forall a \in A$, $\exists b \in B$ s.t. $\|(\mathbf{1}-\Delta\tau(x))\circ a - (\mathbf{1}-\Delta\tau(x))\circ b\| \leq e^{-\kappa_p}\delta'$.

We start with $A \subseteq B + \delta'\mathcal{B}_N$, which implies $\forall a\in A, \exists b' \in B$ s.t. $\|a - b'\| \leq \delta'$. Fix a and b'. Denote the one-hop neighbor of x that attains b' by $\tilde{x}$, i.e., $\exists\tilde{x} \in \tilde{X}^p(x)$ s.t. $b' \in E_{v^p_\infty}(\tilde{x})$. Construct $b \in [0,1]^N$ s.t. $b_i = a_i$, if $i \in \mathcal{V}^G_p(\tilde{x})$; $b_i = b'_i$, otherwise. Since $b' \in E_{v^p_\infty}(\tilde{x})$, $\exists\tau \in v^p_\infty(\tilde{x})$ s.t. $b' \succeq \tau$, that is, $b'_i \geq \tau_i$ for all $i \in \mathcal{V}$. Specifically, by the third property of Lemma V.2, for $i \in \mathcal{V}^G_p(\tilde{x})$, $b'_i \geq \tau_i = 0$. Since $b_i = a_i \geq 0 = \tau_i \quad \forall i \in \mathcal{V}^G_p(\tilde{x})$

and $b_i = b'_i \geq \tau_i \quad \forall i \in \mathcal{V} \setminus \mathcal{V}_p^G(\tilde{x})$, we have $b \succeq \tau$ and, thus, $b \in E_{v_\infty^p}(\tilde{x})$.

Now, we have $\|(\mathbf{1}-\Delta\tau(x)) \circ a - (\mathbf{1}-\Delta\tau(x)) \circ b\|^2 = \sum_{i\in\mathcal{V}\setminus\mathcal{V}_p^G(\tilde{x})}(1-\Delta\tau_i(x))^2(a_i - b'_i)^2$. By the first property of Lemma V.2, $\mathcal{V} \setminus \mathcal{V}_p^G(x) \supseteq \mathcal{V} \setminus \mathcal{V}_p^G(\tilde{x})$. Then, it follows from (8) that $1 - \Delta\tau_i(x) = e^{-\kappa_p} \quad \forall i \in \mathcal{V} \setminus \mathcal{V}_p^G(\tilde{x})$. Therefore

$$\|(\mathbf{1}-\Delta\tau(x)) \circ a - (\mathbf{1}-\Delta\tau(x)) \circ b\|^2$$
$$= (e^{-\kappa_p})^2 \sum_{i\in\mathcal{V}\setminus\mathcal{V}_p^G(\tilde{x})} (a_i - b'_i)^2$$
$$\leq (e^{-\kappa_p})^2 \|a - b'\|^2 \leq (e^{-\kappa_p}\delta')^2.$$

Since this holds $\forall a \in A$ and $b \in B$, then (18) is proven. A similar relationship for $B \subseteq A + \delta'\mathcal{B}_N$ can be obtained by swapping A and B

$$(\mathbf{1}-\Delta\tau(x)) \circ B \subseteq (\mathbf{1}-\Delta\tau(x)) \circ A + e^{-\kappa_p}\delta'\mathcal{B}_N. \quad (19)$$

Combining (18) and (19), we arrive at $d_H((\mathbf{1}-\Delta\tau(x)) \circ A, (\mathbf{1}-\Delta\tau(x)) \circ B) \leq \delta' e^{-\kappa_p}$. Since these two relationships hold for all $\delta' > \delta$, the lemma is then proven. □

Proof of Theorem V.2: Fix $x \in \mathbf{S}^p$. For simplicity, shorthand notations listed below are used in the rest of the proof

$$\tilde{A}(x) = \bigcup_{\tilde{x}\in\tilde{X}^p(x)} E_{v_n^p}(\tilde{x}), \quad \tilde{B}(x) = \bigcup_{\tilde{x}\in\tilde{X}^p(x)} E_{v_\infty^p}(\tilde{x})$$
$$\hat{A}(x) = \bigcup_{\hat{x}\in\hat{X}^p(x)} E_{v_n^p}(\hat{x}), \quad \hat{B}(x) = \bigcup_{\hat{x}\in\hat{X}^p(x)} E_{v_\infty^p}(\hat{x}).$$

Since translating each term in the Hausdorff distance with a common vector $\Delta\tau(x)$ does not change the distance, we focus on the discounted terms in (7). The following holds:

$$d_H(E_{\mathbb{G}v_n^p}(x), E_{\mathbb{G}v_\infty^p}(x))$$
$$= d_H((\mathbf{1}-\Delta\tau(x)) \circ \tilde{A}(x), (\mathbf{1}-\Delta\tau(x)) \circ \tilde{B}(x))$$
$$\leq e^{-\kappa_p} d_H(\tilde{A}(x), \tilde{B}(x))$$

where the last inequality follows from Lemma V.7. By Lemma V.5, the right-hand side of the above may be rewritten as $e^{-\kappa_p} d_H(\hat{A}(x), \hat{B}(x))$. Taking supremum over all $x \in \mathbf{S}^p$ on both sides makes the left-hand side yield to $d_{\mathbf{S}^p}(E_{\mathbb{G}v_n^p}, E_{\mathbb{G}v_\infty^p})$. Then, the following holds:

$$d_{\mathbf{S}^p}(E_{\mathbb{G}v_n^p}, E_{\mathbb{G}v_\infty^p}) \leq e^{-\kappa_p} d_{\mathbf{S}^p}(\hat{A}(x), \hat{B}(x)). \quad (20)$$

It follows from (15) in Lemma VII.4 that

$$d_{\mathbf{S}^p}(\hat{A}(x), \hat{B}(x))$$
$$\leq d_{\mathbf{S}^p}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_n^p}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_\infty^p}(\tilde{x})\Big).$$

Notice that $\forall \tilde{x} \in \mathbf{X}^p \setminus \mathbf{S}^p$ and $\tilde{x}' \in \mathbf{X}^p$, it holds that $E_{v_n^p}(\tilde{x}) = \{\mathbf{1}_N\} \subseteq E_{v_n^p}(\tilde{x}')$. Then, the aforementioned inequality can be extended to the following one:

$$d_{\mathbf{S}^p}(\hat{A}(x), \hat{B}(x))$$
$$\leq d_{\mathbf{S}^p}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_n^p}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_\infty^p}(\tilde{x})\Big) \quad (21)$$
$$\leq d_{\mathbf{X}}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_n^p}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_\infty^p}(\tilde{x})\Big).$$

Combine (20) and (21), then (11) is proven. Inequality (12) is a direct result of (16) in Lemma VII.4. □

Proof of Lemma V.8: For each grid $\mathbf{X}^p$, from line 30 of Algorithm 1, one can see that the value iterations on grid $\mathbf{X}^p$ terminate when (1) $n > n_p$; or (2) the fixed point v_∞^p is reached. Two cases arise, which are as follows.

Case 1: Value iterations terminate before the fixed point is attained, i.e., $v_{\bar{n}_p}^p = v_{n_p}^p$. Notice that $\forall x \in \mathbf{X}^p \setminus \mathbf{S}^p$, $E_{v_n^p}(x) = E_{v_\infty^p}(x) = \{\mathbf{1}_N\}$. Then, the following holds:

$$d_{\mathbf{X}^p}(E_{v_{n_p}^p}, E_{v_\infty^p})$$
$$= \max\{d_{\mathbf{S}^p}(E_{\mathbb{G}v_{n_p-1}^p}, E_{\mathbb{G}v_\infty^p}), d_{\mathbf{X}^p\setminus\mathbf{S}^p}(E_{v_{n_p}^p}, E_{v_\infty^p})\}$$
$$= d_{\mathbf{S}^p}(E_{\mathbb{G}v_{n_p-1}^p}, E_{\mathbb{G}v_\infty^p}).$$

We apply inequality (12) in Theorem V.2 for $n_p - 1$ times to $d_{\mathbf{X}^p}(E_{v_{n_p}^p}, E_{v_\infty^p})$, then the following inequalities are obtained:

$$d_{\mathbf{X}^p}(E_{v_{n_p}^p}, E_{v_\infty^p}) = d_{\mathbf{S}^p}(E_{\mathbb{G}v_{n_p-1}^p}, E_{\mathbb{G}v_\infty^p})$$
$$\leq e^{-\kappa_p} d_{\mathbf{S}^p}(E_{v_{n_p-1}^p}, E_{v_\infty^p}) = e^{-\kappa_p} d_{\mathbf{X}^p}(E_{\mathbb{G}v_{n_p-2}^p}, E_{\mathbb{G}v_\infty^p})$$
$$\leq \cdots \leq e^{-(n_p-1)\kappa_p} d_{\mathbf{S}^p}(E_{\mathbb{G}v_0^p}, E_{\mathbb{G}v_\infty^p})$$
$$\leq e^{-n_p\kappa_p} d_{\mathbf{X}}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_0^p}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_\infty^p}(\tilde{x})\Big)$$

where the last equality is a result of (11) in Theorem V.2.

By Lemma VII.3, the right-hand side of the above becomes

$$d_{\mathbf{X}}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_\infty^p}(\tilde{x})\Big)$$
$$\leq d_{\mathbf{X}}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{\mathbb{P}\tilde{v}_\infty^{p-1}}(\tilde{x})\Big)$$
$$+ d_{\mathbf{X}}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{\mathbb{P}\tilde{v}_\infty^{p-1}}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{v_\infty^p}(\tilde{x})\Big)$$

where the second term is b_p in Lemma V.6. As for the first term, it follows from (16) in Lemma VII.4 that

$$d_{\mathbf{X}}\Big(\bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\alpha_p\mathcal{B})\cap\mathbf{X}^p} E_{\mathbb{P}\tilde{v}_\infty^{p-1}}(\tilde{x})\Big)$$
$$\leq d_{\mathbf{X}^p}(E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}, E_{\mathbb{P}\tilde{v}_\infty^{p-1}}).$$

We focus on the right-hand side of the aforementioned inequality and proceed to show that

$$d_{\mathbf{X}^p}(E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}, E_{\mathbb{P}\tilde{v}_\infty^{p-1}}) \leq d_{\mathbf{X}^{p-1}}(E_{v_{n_{p-1}}^{p-1}}, E_{v_\infty^{p-1}}). \quad (22)$$

For each $x \in \mathbf{X}^p$, if $x \in \mathbf{X}^{p-1}$, it follows from the definition of $\mathbb{P}$ that $E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}(x) = E_{\tilde{v}_{n_{p-1}}^{p-1}}(x)$ and $E_{\mathbb{P}\tilde{v}_\infty^{p-1}}(x) = E_{\tilde{v}_\infty^{p-1}}(x)$. By the definition of $\tilde{v}$, $E_{\tilde{v}_{n_{p-1}}^{p-1}}(x) = E_{v_{n_{p-1}}^{p-1}}(x)$ and $E_{\tilde{v}_\infty^{p-1}}(x) = E_{v_\infty^{p-1}}(x)$. Therefore

$$d_{\mathbf{X}^{p-1}}(E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}, E_{\mathbb{P}\tilde{v}_\infty^{p-1}}) = d_{\mathbf{X}^{p-1}}(E_{v_{n_{p-1}}^{p-1}}, E_{v_\infty^{p-1}}). \quad (23)$$

If $x \in \mathbf{X}^p \setminus \mathbf{X}^{p-1}$, it follows the definition of $\mathbb{P}$ that $E_{\mathbb{P}\tilde{v}_{n_{p-1}}^{p-1}}(x) = \bigcup_{x'\in X_E^p(x)} E_{\tilde{v}_{n_{p-1}}^{p-1}}(x')$ and $E_{\mathbb{P}\tilde{v}_\infty^{p-1}}(x) = \bigcup_{x'\in X_E^p(x)} E_{\tilde{v}_\infty^{p-1}}(x')$. For each $x' \in X_E^p(x)$, if $x' \in \mathbf{X}^p \setminus \mathbf{X}^{p-1}$, we have $E_{\tilde{v}_{n_{p-1}}^{p-1}}(x') = E_{V^{p-1}}(x')$ and $E_{\tilde{v}_\infty^{p-1}}(x') = E_{V^{p-1}}(x')$. Otherwise, i.e., $x' \in \mathbf{X}^p$, it follows from the definition of $\tilde{v}$ that $E_{\tilde{v}_{n_{p-1}}^{p-1}}(x') = E_{v_{n_{p-1}}^{p-1}}(x')$ and $E_{\tilde{v}_\infty^{p-1}}(x') = E_{v_\infty^{p-1}}(x')$. By the third properties of Lemma V.2, $E_{V^{p-1}}(x') \subseteq E_{v^{p-1}_{\tilde{n}_{p-1}}}(x')$ and $E_{V^{p-1}}(x') \subseteq E_{v^{p-1}_{\infty}}(x')$. Then, we have

$$d_{\mathbf{X}^p\setminus\mathbf{X}^{p-1}}(E_{\mathbb{P}\tilde{v}^{p-1}_{\tilde{n}_{p-1}}}, E_{\mathbb{P}\tilde{v}^{p-1}_{\infty}})$$

$$= d_{\mathbf{X}^p\setminus\mathbf{X}^{p-1}}(\bigcup_{x'\in X^p_E(x)\cap\mathbf{X}^p} E_{v^{p-1}_{\tilde{n}_{p-1}}}(x'), \bigcup_{x'\in X^p_E(x)\cap\mathbf{X}^p} E_{v^{p-1}_{\infty}}(x'))$$

$$= d_{\mathbf{X}^{p-1}}(E_{v^{p-1}_{\tilde{n}_{p-1}}}, E_{v^{p-1}_{\infty}}).$$

Then, (22) is a result of (23) and the aforementioned inequality. Therefore, inequality (13) is obtained for this case.

Case 2: The fixed point is reached, i.e., $v^p_{\tilde{n}_p} = v^p_\infty$. The left-hand side of (13) is zero and it is trivially true.

In summary, the lemma is proven. □

D. Proof of Theorem IV.1

We set out to finish the proof of Theorem IV.1. For each grid $\mathbf{X}^p$, we distinguish the following two cases.

Case 1: $p = D_{k+1}$ for some $k \geq 0$. We look back to D_kth grid and apply Lemma V.8 for $D_{k+1} - D_k$ times

$$d_{\mathbf{X}^p}(E_{v^p_{\tilde{n}_p}}, E_{v^p_\infty}) \leq \gamma_p d_{\mathbf{X}^{p-1}}(E_{v^{p-1}_{\tilde{n}_{p-1}}}, E_{v^{p-1}_\infty}) + b_p$$

$$\leq \gamma_p\gamma_{p-1} d_{\mathbf{X}^{p-2}}(E_{v^{p-2}_{\tilde{n}_{p-2}}}, E_{v^{p-2}_\infty}) + \gamma_p b_{p-1} + b_p$$

$$\leq (\prod_{q=D_k+1}^{D_{k+1}} \gamma_q) d_{\mathbf{X}^{D_k}}(E_{v^{D_k}_{\tilde{n}_{D_k-1}}}, E_{v^{D_k}_\infty}) + \sum_{q=D_k+1}^{D_{k+1}} (\prod_{r=q+1}^{D_{k+1}} \gamma_r) b_q$$

where b_q is defined in Lemma V.6. By Assumption IV.1, $\prod_{q=D_k+1}^{D_{k+1}} \gamma_q = \exp(-\sum_{q=D_k+1}^{D_{k+1}} n_q\kappa_q) \leq \gamma$. Since $D_{k+1} - D_k \leq \bar{D}$ and $\gamma_r \leq 1$

$$d_{\mathbf{X}^{D_{k+1}}}(E_{v^{D_{k+1}}_{\tilde{n}_p}}, E_{v^{D_{k+1}}_\infty}) \leq \gamma d_{\mathbf{X}^{D_k}}(E_{v^{D_k}_{\tilde{n}_{D_k}}}, E_{v^{D_k}_\infty}) + \sum_{q=D_k+1}^{D_k+\bar{D}} b_q.$$

By Lemma V.6, $b_q \to 0$ as $q \to +\infty$; hence $\lim_{k\to+\infty} \sum_{q=D_k+1}^{D_k+\bar{D}} b_q = 0$. Therefore, by Lemma VII.5, $\lim_{k\to+\infty} d_{\mathbf{X}^{D_k}}(E_{v^{D_k}_{\tilde{n}_{D_k}}}, E_{v^{D_k}_\infty}) = 0$.

Case 2: $p \neq D_{k+1}$ for any $k \geq 0$. Then, $\exists k \geq 0$ s.t. $D_k + 1 \leq p < D_{k+1}$. We apply Lemma V.8 for $p - D_k$ times

$$d_{\mathbf{X}^p}(E_{v^p_{\tilde{n}_p}}, E_{v^p_\infty}) \leq (\prod_{q=D_k+1}^{p} \gamma_q) d_{\mathbf{X}^{D_k}}(E_{v^{D_k}_{\tilde{n}_{D_k}}}, E_{v^{D_k}_\infty})$$

$$+ \sum_{q=D_k}^{p} (\prod_{r=q+1}^{D_{k+1}} \gamma_r) b_q \leq d_{\mathbf{X}^{D_k}}(E_{v^{D_k}_{\tilde{n}_{D_k}}}, E_{v^{D_k}_\infty}) + \bar{D}\bar{B}_{D_k}$$

where $\bar{B}_p \triangleq \sup_{q\geq p+1} b_q$. It follows from Lemma V.6 that $\lim_{p\to+\infty} \bar{B}_p = 0$. Hence, by Lemma VII.5, $\lim_{p\to+\infty} d_{\mathbf{X}^p}(E_{v^p_{\tilde{n}_p}}, E_{v^p_\infty}) = 0$.

Combining the aforementioned two cases, we may rewrite the result as $\lim_{p\to+\infty} d_{\mathbf{X}^p}(E_{v^p_{\tilde{n}_p}}, E_{v^p_\infty}) = 0$. Pick $x \in \mathbf{X}$. By (16) in Lemma VII.4, the following holds:

$$d_H(\bigcup_{\tilde{x}\in(x+\eta_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_{\tilde{n}_p}}(\tilde{x}), \bigcup_{\tilde{x}\in(x+\eta_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_\infty}(\tilde{x}))$$

$$\leq d_{\mathbf{X}^p}(E_{v^p_{\tilde{n}_p}}, E_{v^p_\infty}).$$

Take the limit $p \to +\infty$ on both sides, then the aforementioned relationship yields

$$\lim_{p\to+\infty} d_H(\bigcup_{\tilde{x}\in(x+h_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_{\tilde{n}_p}}(\tilde{x}), \bigcup_{\tilde{x}\in(x+h_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_\infty}(\tilde{x})) = 0.$$

Since this holds for all $x \in \mathbf{X}$

$$\lim_{p\to+\infty} d_{\mathbf{X}}(\bigcup_{\tilde{x}\in(x+h_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_{\tilde{n}_p}}(\tilde{x}), \bigcup_{\tilde{x}\in(x+h_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_\infty}(\tilde{x})) = 0.$$

By Theorem V.1, $\operatorname{Lim}_{p\to+\infty} \bigcup_{\tilde{x}\in(x+h_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_\infty}(\tilde{x})$ exists for any $x \in \mathbf{X}$ and equals to $E_{v^*}(x)$. Hence, it holds that $\forall x \in \mathbf{X}$, $\operatorname{Lim}_{p\to+\infty} \bigcup_{\tilde{x}\in(x+h_p\mathcal{B})\cap\mathbf{X}^p} E_{v^p_{\tilde{n}_p}}(\tilde{x}) = E_{v^*}(x)$. Then, the theorem is proven.

VIII. Conclusion

In this article, a numerical algorithm is proposed to find the Pareto optimal solution of a class of multirobot motion planning problems. The consistent approximation of the algorithm is guaranteed using set-valued analysis. A set of experiments on an indoor multirobot platform and computer simulations are conducted to assess the anytime property. There are a couple of interesting problems to solve in the future. First, the proposed algorithm is centralized. It is of interest to study distributed implementation. Second, it is interesting to find more efficient ways to construct set-valued dynamics and perform value iteration.

References

[1] J. H. Reif, “Complexity of the mover’s problem and generalizations,” in *Proc. 20th Annu. Symp. Found. Comput. Sci.*, Oct. 1979, pp. 421–427.

[2] S. M. LaValle and J. J. Kuffner, “Randomized kinodynamic planning,” *Int. J. Robot. Res.*, vol. 20, no. 5, pp. 378–400, May 2001.

[3] J. Canny, *The Complexity of Robot Motion Planning*. Cambridge, MA, USA: MIT Press, 1988.

[4] S. Karaman and E. Frazzoli, “Sampling-based algorithms for optimal motion planning,” *Int. J. Robot. Res.*, vol. 30, no. 7, pp. 846–894, 2011.

[5] G. Sánchez and J.-C. Latombe, “On delaying collision checking in PRM planning: Application to multi-robot coordination,” *Int. J. Robot. Res.*, vol. 21, no. 1, pp. 5–26, 2002.

[6] E. K. Xidias and N. A. Aspragathos, “Motion planning for multiple non-holonomic robots: A geometric approach,” *Robotica*, vol. 26, no. 4, pp. 525–536, 2008.

[7] K. Kant and S. W. Zucker, “Toward efficient trajectory planning: The path-velocity decomposition,” *Int. J. Robot. Res.*, vol. 5, no. 3, pp. 72–89, 1986.

[8] T. Siméon, S. Leroy, and J.-P. Lauumond, “Path coordination for multiple mobile robots: A resolution-complete algorithm,” *IEEE Trans. Robot. Autom.*, vol. 18, no. 1, pp. 42–49, Feb. 2002.

[9] S. J. Buckley, “Fast motion planning for multiple moving robots,” in *Proc. IEEE Int. Conf. Robot. Autom.*, May 1989, pp. 322–3261.

[10] M. Erdmann and T. Lozano-Perez, “On multiple moving objects,” *Algorithmica*, vol. 2, no. 1–4, pp. 477–521, 1987.

[11] M. Zhu, M. Otte, P. Chaudhari, and E. Frazzoli, “Game theoretic controller synthesis for multi-robot motion planning Part I: Trajectory based algorithms,” in *Proc. IEEE Int. Conf. Robot. Autom.*, May 2014, pp. 1646–1651.

[12] D. K. Jha, M. Zhu, and A. Ray, “Game theoretic controller synthesis for multi-robot motion planning Part II: Policy-based algorithms,” in *Proc. 5th IFAC Workshop Distrib. Estimation Control Netw. Syst.*, vol. 48, pp. 168–173, 2015.

[13] J. Nash, “Non-cooperative games,” *Ann. Math.*, vol. 54, no. 2, pp. 286–295, 1951.

[14] S. Karaman and E. Frazzoli, “Sampling-based optimal motion planning for non-holonomic dynamical systems,” in *Proc. IEEE Int. Conf. Robot. Autom.*, 2013, pp. 5041–5047.

[15] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*. London, U.K.: Springer, 2008.

[16] F. Bullo, J. Cortés, and S. Martìnez, *Distributed Control of Robotic Networks: A Mathematical Approach to Motion Coordination Algorithms*. Princeton, NJ, USA: Princeton Univ. Press, 2009.

[17] M. Zhu and S. Martínez, *Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments*. Cham, Switzerland: Springer, 2015.

[18] M. Cao, A. S. Morse, and B. D. Anderson, "Reaching a consensus in a dynamically changing environment: A graphical approach," *SIAM J. Control Optim.*, vol. 47, no. 2, pp. 575–600, 2008.

[19] E. Frazzoli and F. Bullo, "Decentralized algorithms for vehicle routing in a stochastic time-varying environment," in *Proc. 43rd IEEE Conf. Decis. Control*, 2004, pp. 3357–3363.

[20] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Trans. Robot. Autom.*, vol. 20, no. 2, pp. 243–255, Apr. 2004.

[21] M. Schwager, D. Rus, and J.-J. Slotine, "Decentralized, adaptive coverage control for networked robots," *Int. J. Robot. Res.*, vol. 28, no. 3, pp. 357–375, 2009.

[22] W. B. Dunbar and R. M. Murray, "Distributed receding horizon control for multi-vehicle formation stabilization," *Automatica*, vol. 42, no. 4, pp. 549–558, 2006.

[23] M. Zhu and S. Martínez, "On distributed constrained formation control in operator–vehicle adversarial networks," *Automatica*, vol. 49, no. 12, pp. 3571–3582, 2013.

[24] W. B. Dunbar and D. S. Caveney, "Distributed receding horizon control of vehicle platoons: Stability and string stability," *IEEE Trans. Autom. Control*, vol. 57, no. 3, pp. 620–633, Mar. 2012.

[25] H. Li, Y. Shi, and W. Yan, "Distributed receding horizon control of constrained nonlinear vehicle formations with guaranteed γ-gain stability," *Automatica*, vol. 68, pp. 148–154, 2016.

[26] Y. Kuwata, A. Richards, T. Schouwenaars, and J. P. How, "Distributed robust receding horizon control for multivehicle guidance," *IEEE Trans. Control Syst. Technol.*, vol. 15, no. 4, pp. 627–641, Jul. 2007.

[27] A. Bemporad and M. Morari, "Robust model predictive control: A survey," in *Robustness in Identification and Control*, T. A. Garulli, Ed. London, U.K.: Springer, 1999, pp. 207–226.

[28] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[29] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[30] L. Grune and A. Rantzer, "On the infinite horizon performance of receding horizon controllers," *IEEE Trans. Autom. Control*, vol. 53, no. 9, pp. 2100–2111, Oct. 2008.

[31] T. Baar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed. Philadelphia, PA, USA: SIAM, 1998.

[32] R. Isaacs, *Differential Games: A Mathematical Theory With Applications to Warfare and Pursuit, Control and Optimization*. Mineola, NY, USA: Dover, 1999.

[33] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Boston, MA, USA: Birkhaüser, 2008.

[34] M. Bardi, M. Falcone, and P. Soravia, "Numerical methods for pursuit-evasion games via viscosity solutions," in *Stochastic and Differential Games*, M. Bardi, T. Raghavan, and T. Parthasarathy, Eds. New York, NY, USA: Springer, 1999, pp. 105–175.

[35] P. E. Souganidis, "Two-player, zero-sum differential games and viscosity solutions," in *Stochastic and Differential Games*, M. Bardi, T. Raghavan, and T. Parthasarathy, Eds. New York, NY, USA: Springer, 1999, pp. 69–104.

[36] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre, "Set-valued numerical analysis for optimal control and differential games," in *Stochastic and Differential Games*, M. Bardi, T. Raghavan, and T. Parthasarathy, Eds. New York, NY, USA: Springer, 1999, pp. 177–247.

[37] J.-P. Aubin, *Viability Theory*. Boston, MA, USA: Birkhaüser, 2009.

[38] J.-P. Aubin, A. M. Bayen, and P. Saint-Pierre, *Viability Theory: New Directions*, 2nd ed. Berlin, Germany: Springer-Verlag, 2011.

[39] G. Zhao and M. Zhu, "Pareto optimal multi-robot motion planning," in *Proc. Amer. Control Conf.*, Milwaukee, WI, USA, 2018, pp. 4020–4025.

[40] R. Hartley, "On cone-efficiency, cone-convexity and cone-compactness," *SIAM J. Appl. Math.*, vol. 34, no. 2, pp. 211–222, 1978.

[41] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*. Basel, Switzerland: Birkhaüser, 2009.

[42] G. Zhao and M. Zhu, "Pareto optimal multi-robot motion planning," Sep. 2020. [Online]. Available: http://arxiv.org/abs/1802.09099

[43] J. Pineau, G. Gordon, and S. Thrun, "Point-based value iteration: An anytime algorithm for POMDPs," in *Proc. 18th Int. Joint Conf. Artif. Intell.*, 2003, pp. 1025–1032.

[44] M. Likhachev, D. Ferguson, G. Gordon, A. Stentz, and S. Thrun, "Anytime dynamic A*: An anytime, replanning algorithm," in *Proc. 15th Int. Conf. Autom. Plan. Scheduling*, 2005, pp. 262–271.

[45] D. Ferguson and A. Stentz, "Anytime RRTs," in *Proc. IEEE/RSJ Int. Conf. Intell. Robots Syst.*, 2006, pp. 5369–5375.

[46] S. Karaman, M. R. Walter, A. Perez, E. Frazzoli, and S. Teller, "Anytime motion planning using the RRT," in *Proc. IEEE Int. Conf. Robot. Autom.*, 2011, pp. 1478–1483.

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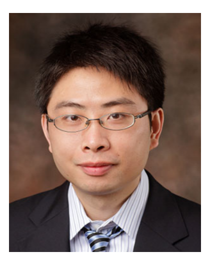

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