Commun. Math. Sci.
Vol. 18, No. 6, pp. 1743–1754

GLOBAL EXISTENCE FOR NERNST-PLANCK-NAVIER-STOKES SYSTEM IN $\mathbb{R}^N$*

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Abstract. In this note, we study the Nernst-Planck-Navier-Stokes system for the transport and diffusion of ions in electrolyte solutions. The key feature is to establish three energy-dissipation equalities. As their direct consequence, we obtain global existence for two-ionic species case in $\mathbb{R}^n$, $n \geq 2$, and multi-ionic species case in $\mathbb{R}^n$, $n=2,3$.

Keywords. Electrolyte; electro-osmosis; electrochemical transport and diffusion; global weak solution; entropy method.

AMS subject classifications. 35Q30; 35Q35; 35Q92.

1. Introduction

The Nernst-Planck-Navier-Stokes (NPNS) system, describing the transport and diffusion of ions in electrolyte solutions, plays an important role in many physical and biological system [1, 5], such as ion particles in the electrokinetic fluids [7, 10], and ion channels in cell membranes [2, 8]. An introduction to some of the basic physical, biological and mathematical issues can be found in [11].

The NPNS system [6] reads

$$\partial_t u + u \cdot \nabla u + \nabla p = \Delta u - \Big(\sum_{i=1}^{N} z_i c_i\Big)\nabla\phi, \tag{1.1}$$

$$\partial_t c_i + \nabla\cdot(c_i u) = \Delta c_i + \nabla\cdot(z_i c_i \nabla\phi), \quad i=1,\dots,N, \tag{1.2}$$

$$-\Delta\phi = \sum_{i=1}^{N} z_i c_i, \quad i=1,\dots,N, \tag{1.3}$$

$$\nabla\cdot u = 0, \tag{1.4}$$

where $x \in \mathbb{R}^n$, $t>0$. We impose the following initial conditions

$$c_i(x,0) = c_i^0(x), \quad i=1,\dots,N, \qquad x \in \mathbb{R}^n, \tag{1.5}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n. \tag{1.6}$$

Here $u=u(x,t)$ and $p=p(x,t)$ are the velocity and pressure of electrolyte solutions, respectively, $c_i = c_i(x,t)$ are the i-th ionic species concentrations, $z_i \in \mathbb{R}$ is valence of the i-th ion, $i=1,\dots,N$, and ϕ is the electric potential. In the above system, we choose all physical parameters to be 1 for simplicity in representation. When initial data c_i^0 are non-negative functions, then c_i are still non-negative, $i=1,\dots,N$.

*Received: December 09, 2018; Accepted (in revised form): April 17, 2020. Communicated by Chun Liu.

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She is partially supported by National Natural Science Foundation of China (Grant No. 11926338) and Key Project of Education Department of Liaoning Province (Grant No. LZD201701).

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The work of J-G Liu was partially supported by KI-Net NSF RNMS (Grant No. 1107444) and NSF DMS (Grant No. 1812573).

The system (1.1)-(1.4) has the following two free energy-dissipation relations

$$\frac{d}{dt}\mathcal{F}_1(t)+\mathcal{D}_1(t)=0, \tag{1.7}$$

$$\frac{d}{dt}\mathcal{F}_2(t)+\mathcal{D}_2(t)=0, \tag{1.8}$$

where

$$\mathcal{F}_1(t)=\frac{1}{2}\int_{\mathbb{R}^n}|u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla\phi|^2\,dx, \tag{1.9}$$

$$\mathcal{D}_1(t)=\int_{\mathbb{R}^n}|\nabla u|^2\,dx+\int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)^2\,dx+\int_{\mathbb{R}^n}(\sum_{i=1}^N z_i^2c_i)|\nabla\phi|^2\,dx, \tag{1.10}$$

$$\mathcal{F}_2(t)=\sum_{i=1}^N\int_{\mathbb{R}^n}c_i\log c_i\,dx, \tag{1.11}$$

$$\mathcal{D}_2(t)=\sum_{i=1}^N\int_{\mathbb{R}^n}\frac{|\nabla c_i|^2}{c_i}\,dx+\int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)^2\,dx. \tag{1.12}$$

In the two-ionic species case including one species of cations ($z_1>0$) and one species of anions ($z_2<0$), denote $c_+=z_1c_1$ and $c_-=-z_2c_2$. Then the NPNS system is reduced to

$$\partial_t u+u\cdot\nabla u+\nabla p=\Delta u-(c_+-c_-)\nabla\phi, \tag{1.13}$$

$$\partial_t c_++\nabla\cdot(c_+u)=\Delta c_++|z_1|\nabla\cdot(c_+\nabla\phi), \tag{1.14}$$

$$\partial_t c_-+\nabla\cdot(c_-u)=\Delta c_--|z_2|\nabla\cdot(c_-\nabla\phi), \tag{1.15}$$

$$-\Delta\phi=c_+-c_-, \tag{1.16}$$

$$\nabla\cdot u=0 \tag{1.17}$$

with the initial conditions

$$u(x,0)=u_0(x),\quad c_+(x,0)=c_+^0(x),\quad c_-(x,0)=c_-^0(x). \tag{1.18}$$

Local existence for the NPNS system coupled from the Navier-Stokes equations in the whole space was obtained in [9]. In this note, we will prove global existence for this model in the whole space. There is a family of additional free energy-dissipation relations for (1.13)-(1.17), i.e., for any $p\geq 1$

$$\frac{d}{dt}(|z_2|\|c_+\|_{L^p}^p+|z_1|\|c_-\|_{L^p}^p)+\mathcal{D}_3(t)=0, \tag{1.19}$$

where

$$\mathcal{D}_3(t):=\frac{4(p-1)}{p}\left(|z_2|\|\nabla c_+^{\frac{p}{2}}\|_{L^2}^2+|z_1|\|\nabla c_-^{\frac{p}{2}}\|_{L^2}^2\right)$$
$$+(p-1)|z_1||z_2|\int_{\mathbb{R}^n}(c_+^p-c_-^p)(c_+-c_-)\,dx\geq 0. \tag{1.20}$$

For the narrative convenience, we specially take $z_1=1, z_2=-1$ below. Using the free energy-dissipation relations (1.7) and (1.19), which will be proved in Section 2, together with some standard analysis, we have the following theorem on global existence of bounded solutions for models (1.13)-(1.17).

THEOREM 1.1 (Existence for two-ionic species case). *Assume that* $n\geq 2$, $u_0\in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_+^0,c_-^0\in L_+^1\cap L^2(\mathbb{R}^n)$ *and* $\mathcal{F}_1(0)<\infty$. *Then for any* $T>0$, *there is a global weak solution* (u,c_+,c_-) *satisfying regularities*

$$u\in L^\infty(0,\infty;L^2(\mathbb{R}^n;\mathbb{R}^n))\cap L^2(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n)), \tag{1.21}$$

$$c_+,c_-\in L^\infty(0,\infty;L_+^1\cap L^2(\mathbb{R}^n))\cap L^2(0,T;H^1(\mathbb{R}^n)), \tag{1.22}$$

$$\nabla\phi\in L^\infty(0,\infty;L^2(\mathbb{R}^n))\cap L^2(0,T;H^1(\mathbb{R}^n)), \tag{1.23}$$

$$\partial_t u\in L^2(0,T;W^{-1,\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^n)), \tag{1.24}$$

$$\partial_t c_+,\partial_t c_-\in L^2(0,T;W^{-1,\frac{n}{n-1}}(\mathbb{R}^n)). \tag{1.25}$$

Moreover, if $c_+^0,c_-^0\in L_+^1\cap L^\infty(\mathbb{R}^n)$, *then the weak solutions have the uniform* L^∞*-bound, i.e., there exists a constant* C *such that*

$$\|c_+\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^n))}+\|c_-\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^n))}\leq C. \tag{1.26}$$

For the multi-ionic species system (1.1)-(1.4), recently, Constantin and Ignatova [6], using the relative entropy method, obtained global existence and stability results in two dimensional bounded domain with blocking or selective boundary conditions for the ionic concentrations. In this paper, we will prove global existence for the model (1.1)-(1.4) in the whole space $\mathbb{R}^n$, $n=2,3$. As usual, we can use the first moment $m_1(t)$ to show the tightness of c_i, $i=1,\dots,N$. Let

$$m_1(t):=\sum_{i=1}^N m_1^i(t)=\sum_{i=1}^N\int_{\mathbb{R}^n}|x|c_i\,dx,\quad m_0(t):=\sum_{i=1}^N\int_{\mathbb{R}^n}c_i\,dx\equiv m_0. \tag{1.27}$$

We have the following existence theorem.

THEOREM 1.2 (Existence for the multi-ionic species case). *Assume that* $n=2,3$, $u_0\in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_i^0\in L_+^1\cap L\log L(\mathbb{R}^n)$, $m_1(0)<\infty$, $\mathcal{F}_1(0)<\infty$ *and* $\mathcal{F}_2(0)<\infty$. *Then for any* $T>0$, *there is a global weak solution* $(u,c_1,\dots,c_N)$ *satisfying regularities*

$$u\in L^\infty(0,\infty;L^2(\mathbb{R}^n;\mathbb{R}^n))\cap L^2(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n)), \tag{1.28}$$

$$c_i\in L^\infty(0,\infty;L_+^1\cap L\log L(\mathbb{R}^n)),\quad c_i\in L^{\frac{8}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{R}^n)),\quad i=1,\dots,N, \tag{1.29}$$

$$\nabla\phi\in L^\infty(0,\infty;L^2(\mathbb{R}^n))\cap L^2(0,T;H^1(\mathbb{R}^n)), \tag{1.30}$$

$$\partial_t u\in L^2(0,T;W^{-1,\frac{4}{3}}(\mathbb{R}^n;\mathbb{R}^n)),\quad \partial_t(\nabla\phi)\in L^{\frac{4}{3}}(0,T;W^{-2,\frac{4}{3}}(\mathbb{R}^n)). \tag{1.31}$$

REMARK 1.1. We remark that the family of additional free energy-dissipation relations (1.19) may not hold for the case with more than two ionic species. The last term in (1.20) may be negative for the multi-ionic species. For example, taking $N=3, z_1=1, z_2=-\frac{1}{2}, z_3=-\frac{1}{2}$, we can easily construct three real numbers (a,b,c) such that

$$\Big(a^p-\frac{b^p+c^p}{2}\Big)\Big(a-\frac{b+c}{2}\Big)<0.$$

REMARK 1.2. Theorem 1.1 and Theorem 1.2 show global existence for two-ionic species case in $\mathbb{R}^n$, $n\geq 2$, and multi-ionic species case in $\mathbb{R}^n$, $n=2,3$, respectively. Research on the uniqueness of solutions will be our further work.

2. Free energy-dissipation equalities

In this section, we derive mass conservation, two energy-dissipation equalities and a L^p-energy-dissipation equality.

PROPOSITION 2.1 (Mass conservation). *Let $c_i(x,t)$ be non-negative solutions to (1.1)-(1.4). Then $c_i(x,t)$ has the following conservation of mass*

$$\int_{\mathbb{R}^n} c_i(x,t)\,dx \equiv \int_{\mathbb{R}^n} c_i^0(x)\,dx =: m_0^i, \quad i=1,\ldots,N. \tag{2.1}$$

The proof of Proposition 2.1 is standard, refer to [3]. The second property below gives the two free energy-dissipation equalities to the model (1.1)-(1.4).

PROPOSITION 2.2 (Two free energy-dissipation equalities). *Let $(u,c_1,\ldots,c_N)$ be solutions to (1.1)-(1.4). Then the two energy-dissipation relations in (1.7)-(1.8) hold.*

Proof. Multiplying u and $z_i\phi$ to (1.1) and (1.2) respectively, integrating them in $\mathbb{R}^n$ and using $\nabla\cdot u=0$, we obtain that

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}|u|^2\,dx = -\int_{\mathbb{R}^n}|\nabla u|^2\,dx - \int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)u\cdot\nabla\phi\,dx, \tag{2.2}$$

$$\int_{\mathbb{R}^n} z_i\phi\partial_t c_i\,dx + \int_{\mathbb{R}^n} z_i\phi\nabla\cdot(c_iu)\,dx = \int_{\mathbb{R}^n} z_i\phi\Delta c_i\,dx + \int_{\mathbb{R}^n} z_i\phi\nabla\cdot(z_ic_i\nabla\phi)\,dx, \tag{2.3}$$

$i=1,\ldots,N$. Summing (2.3) from 1 to N, we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}|\nabla\phi|^2\,dx = &\int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)u\cdot\nabla\phi\,dx - \int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)^2\,dx\\ &-\int_{\mathbb{R}^n}(\sum_{i=1}^N z_i^2c_i)|\nabla\phi|^2\,dx.\end{aligned} \tag{2.4}$$

Hence by (2.2) and (2.4), we deduce

$$\frac{d\mathcal{F}_1}{dt} + \int_{\mathbb{R}^n}|\nabla u|^2\,dx + \int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)^2\,dx + \int_{\mathbb{R}^n}(\sum_{i=1}^N z_i^2c_i)|\nabla\phi|^2\,dx = 0. \tag{2.5}$$

Now we prove the second free energy-dissipation relation (1.8). Taking $1+\log c_i$ as a test function on both sides of (1.2), summing them and using $\nabla\cdot u=0$, we have

$$\frac{d\mathcal{F}_2}{dt} + 4\sum_{i=1}^N\int_{\mathbb{R}^n}|\nabla\sqrt{c_i}|^2\,dx + \int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)^2\,dx = 0. \tag{2.6}$$

This completes the proof of Proposition 2.2. ∎

Moreover, for two-ionic species case we also have the L^p-energy-dissipation relation (1.19).

PROPOSITION 2.3. *Let (u,c_+,c_-) be solutions to the model (1.13)-(1.17). Then the L^p-energy-dissipation relation (1.19) holds.*

Proof. Multiplying pc_+^{p-1} and pc_-^{p-1} $(p\geq 1)$ to equations (1.14) and (1.15) respectively, integrating them in $\mathbb{R}^n$, and using $\nabla\cdot u=0$, we get

$$\frac{d}{dt}\|c_+\|_{L^p}^p + \frac{4(p-1)}{p}\int_{\mathbb{R}^n}|\nabla c_+^{p/2}|^2\,dx = \int_{\mathbb{R}^n} pc_+^{p-1}\nabla\cdot(c_+\nabla\phi)\,dx, \tag{2.7}$$

$$\frac{d}{dt}\|c_-\|_{L^p}^p+\frac{4(p-1)}{p}\int_{\mathbb{R}^n}|\nabla c_-^{p/2}|^2\,dx=-\int_{\mathbb{R}^n}pc_-^{p-1}\nabla\cdot(c_-\nabla\phi)\,dx. \tag{2.8}$$

A simple computation gives that

$$\int_{\mathbb{R}^n}pc_+^{p-1}\nabla\cdot(c_+\nabla\phi)\,dx=-(p-1)\int_{\mathbb{R}^n}c_+^{p+1}\,dx+(p-1)\int_{\mathbb{R}^n}c_+^pc_-\,dx, \tag{2.9}$$

$$\int_{\mathbb{R}^n}pc_-^{p-1}\nabla\cdot(c_-\nabla\phi)\,dx=-(p-1)\int_{\mathbb{R}^n}c_-^pc_+\,dx+(p-1)\int_{\mathbb{R}^n}c_-^{p+1}\,dx. \tag{2.10}$$

Hence summing (2.7) and (2.8), and using (2.9) and (2.10), we have

$$\frac{d}{dt}(\|c_+\|_{L^p}^p+\|c_-\|_{L^p}^p)+\frac{4(p-1)}{p}\int_{\mathbb{R}^n}|\nabla c_+^{p/2}|^2\,dx+\frac{4(p-1)}{p}\int_{\mathbb{R}^n}|\nabla c_-^{p/2}|^2\,dx$$
$$+(p-1)\int_{\mathbb{R}^n}(c_+^{p+1}+c_-^{p+1}-c_+^pc_--c_-^pc_+)\,dx=0. \tag{2.11}$$

Due to

$$c_+^{p+1}+c_-^{p+1}-c_+^pc_--c_-^pc_+=(c_+^p-c_-^p)(c_+-c_-)\geq 0,$$

hence (1.19) holds. □

3. Global existence for the two-ionic species case

In this section, we show global existence of bounded solutions for the model (1.13)-(1.17) by using the energy-dissipation equalities (1.7) and (1.19), in order to prove Theorem 1.1. The process is standard. For completeness, we outline a proof below.

At first, a regularized problem for (1.13)-(1.17) is constructed as follows

$$\partial_t u_\varepsilon+u_\varepsilon\cdot\nabla u_\varepsilon+\nabla p_\varepsilon=\Delta u_\varepsilon-(c_+^\varepsilon-c_-^\varepsilon)\nabla\phi_\varepsilon, \tag{3.1}$$
$$\partial_t c_+^\varepsilon+\nabla\cdot(c_+^\varepsilon u_\varepsilon)=\Delta c_+^\varepsilon+J_\varepsilon*(\nabla\cdot((J_\varepsilon*c_+^\varepsilon)J_\varepsilon*\nabla\phi_\varepsilon)), \tag{3.2}$$
$$\partial_t c_-^\varepsilon+\nabla\cdot(c_-^\varepsilon u_\varepsilon)=\Delta c_-^\varepsilon-J_\varepsilon*(\nabla\cdot((J_\varepsilon*c_-^\varepsilon)J_\varepsilon*\nabla\phi_\varepsilon)), \tag{3.3}$$
$$-\Delta\phi_\varepsilon=c_+^\varepsilon-c_-^\varepsilon, \tag{3.4}$$
$$\nabla\cdot u_\varepsilon=0, \tag{3.5}$$
$$c_+^\varepsilon(x,0)=c_+^{\varepsilon 0}(x):=c_+^0(x)*J_\varepsilon,\quad c_-^\varepsilon(x,0)=c_-^{\varepsilon 0}(x):=c_-^0(x)*J_\varepsilon, \tag{3.6}$$
$$u_\varepsilon(x,0)=u_{\varepsilon 0}(x):=u_0(x)*J_\varepsilon, \tag{3.7}$$

where $J_\varepsilon(x)$ is defined by the standard mollifier $J(x)$ satisfying $\int_{\mathbb{R}^n}J(x)\,dx=1$.

Since $u_0\in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_+^0,c_-^0\in L_+^1\cap L^\infty(\mathbb{R}^n)$ and $\mathcal{F}_1(0)<\infty$, we have

$$\|u_{\varepsilon 0}\|_{L^2(\mathbb{R}^n)}\leq\|u_0\|_{L^2(\mathbb{R}^n)},\quad \mathcal{F}_1^\varepsilon(0)\leq\mathcal{F}_1(0), \tag{3.8}$$
$$\|c_+^{\varepsilon 0}\|_{L^1\cap L^\infty(\mathbb{R}^n)}\leq\|c_+^0\|_{L^1\cap L^\infty(\mathbb{R}^n)},\quad \|c_-^{\varepsilon 0}\|_{L^1\cap L^\infty(\mathbb{R}^n)}\leq\|c_-^0\|_{L^1\cap L^\infty(\mathbb{R}^n)}, \tag{3.9}$$

where $\mathcal{F}_1^\varepsilon(t)$ is defined by

$$\mathcal{F}_1^\varepsilon(t)=\frac{1}{2}\int_{\mathbb{R}^n}|u_\varepsilon|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla\phi_\varepsilon|^2\,dx.$$

Next, we give some uniform estimates of solutions to the model (3.1)-(3.7). The process is similar to that of obtaining the energy-dissipation equalities (1.7) and (1.19).

Proposition 3.1. *Assume that* $u_0 \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_+^0, c_-^0 \in L_+^1 \cap L^2(\mathbb{R}^n)$ *and* $\mathcal{F}_1(0) < \infty$. *Let* $(u_\varepsilon, c_+^\varepsilon, c_-^\varepsilon)$ *be solutions to (3.1)-(3.7). Then* u_ε *and* ϕ_ε *satisfy the following uniform estimates*

$$\|u_\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^n))} + \|\nabla u_\varepsilon\|_{L^2(0,\infty;L^2(\mathbb{R}^n))} \leq C, \tag{3.10}$$

$$\|\nabla \phi_\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^n))} \leq C. \tag{3.11}$$

Proof. Multiplying u_ε, ϕ_ε and $-\phi_\varepsilon$ to (3.1), (3.2) and (3.3) respectively, integrating them in $\mathbb{R}^n$ and using $\nabla \cdot u_\varepsilon = 0$, we obtain that

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n} |u_\varepsilon|^2\,dx = -\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\,dx - \int_{\mathbb{R}^n} (c_+^\varepsilon - c_-^\varepsilon) u_\varepsilon \cdot \nabla \phi_\varepsilon\,dx, \tag{3.12}$$

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_\varepsilon \partial_t c_+^\varepsilon\,dx = &\int_{\mathbb{R}^n} \nabla \phi_\varepsilon \cdot (c_+^\varepsilon u)\,dx + \int_{\mathbb{R}^n} \phi_\varepsilon \Delta c_+^\varepsilon\,dx \\ &- \int_{\mathbb{R}^n} \nabla(J_\varepsilon * \phi_\varepsilon) \cdot (J_\varepsilon * c_+^\varepsilon \nabla(J_\varepsilon * \phi_\varepsilon))\,dx,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_\varepsilon \partial_t c_-^\varepsilon\,dx = &\int_{\mathbb{R}^n} \nabla \phi_\varepsilon \cdot (c_-^\varepsilon u)\,dx + \int_{\mathbb{R}^n} \phi_\varepsilon \Delta c_-^\varepsilon\,dx \\ &+ \int_{\mathbb{R}^n} \nabla(J_\varepsilon * \phi_\varepsilon) \cdot (J_\varepsilon * c_-^\varepsilon \nabla(J_\varepsilon * \phi_\varepsilon))\,dx.\end{aligned} \tag{3.14}$$

Subtracting (3.14) from (3.13), we have

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n} |\nabla \phi_\varepsilon|^2\,dx - \int_{\mathbb{R}^n} (c_+^\varepsilon - c_-^\varepsilon) u_\varepsilon \cdot \nabla \phi_\varepsilon\,dx \\ &= -\int_{\mathbb{R}^n} (c_+^\varepsilon - c_-^\varepsilon)^2\,dx - \int_{\mathbb{R}^n} J_\varepsilon * (c_+^\varepsilon + c_-^\varepsilon)|\nabla(J_\varepsilon * \phi_\varepsilon)|^2\,dx.\end{aligned} \tag{3.15}$$

Hence by (3.12) and (3.15), we deduce

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n} |u_\varepsilon|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \phi_\varepsilon|^2\,dx\right) \\ &= -\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\,dx - \int_{\mathbb{R}^n} (c_+^\varepsilon - c_-^\varepsilon)^2\,dx - \int_{\mathbb{R}^n} J_\varepsilon * (c_+^\varepsilon + c_-^\varepsilon)|\nabla(J_\varepsilon * \phi_\varepsilon)|^2\,dx,\end{aligned} \tag{3.16}$$

which implies the estimates (3.10) and (3.11). ∎

Proposition 3.2. *Assume that* $u_0 \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_+^0, c_-^0 \in L_+^1 \cap L^2(\mathbb{R}^n)$ *and* $\mathcal{F}_1(0) < \infty$. *Let* $(u_\varepsilon, c_+^\varepsilon, c_-^\varepsilon)$ *be solutions to (3.1)-(3.7). Then* c_+^ε *and* c_-^ε *satisfy the following uniform estimates*

$$\|c_+^\varepsilon\|_{L^\infty(0,\infty;L^1\cap L^2(\mathbb{R}^n))} + \|c_-^\varepsilon\|_{L^\infty(0,\infty;L^1\cap L^2(\mathbb{R}^n))} \leq C, \tag{3.17}$$

$$\|\nabla c_+^\varepsilon\|_{L^2(0,\infty;L^2(\mathbb{R}^n))} + \|\nabla c_-^\varepsilon\|_{L^2(0,\infty;L^2(\mathbb{R}^n))} \leq C, \tag{3.18}$$

$$\|\nabla \phi_\varepsilon\|_{L^\infty(0,\infty;L^r(\mathbb{R}^n))} \leq C, \quad \frac{n}{n-1} < r \leq \frac{2n}{n-2}, \tag{3.19}$$

where C *is a constant independent of* ε.

Proof. Multiplying $2c_+^\varepsilon$ and $2c_-^\varepsilon$ to equations (3.2) and (3.3) respectively, integrating them in $\mathbb{R}^n$, and using $\nabla \cdot u_\varepsilon = 0$, we get

$$\frac{d}{dt}\|c_+^\varepsilon\|_{L^2}^2 + 2\int_{\mathbb{R}^n} |\nabla c_+^\varepsilon|^2\,dx = 2\int_{\mathbb{R}^n} (J_\varepsilon * c_+^\varepsilon)\nabla \cdot (J_\varepsilon * c_+^\varepsilon \nabla(J_\varepsilon * \phi_\varepsilon))\,dx, \tag{3.20}$$

$$\frac{d}{dt}\|c_-^\varepsilon\|_{L^2}^2 + 2\int_{\mathbb{R}^n} |\nabla c_-^\varepsilon|^2\,dx = -2\int_{\mathbb{R}^n} (J_\varepsilon * c_-^\varepsilon)\nabla\cdot(J_\varepsilon * c_-^\varepsilon \nabla(J_\varepsilon * \phi_\varepsilon))\,dx. \tag{3.21}$$

A simple computation gives

$$2\int_{\mathbb{R}^n} (J_\varepsilon * c_+^\varepsilon)\nabla\cdot(J_\varepsilon * c_+^\varepsilon \nabla(J_\varepsilon * \phi_\varepsilon))\,dx = -\int_{\mathbb{R}^n} (J_\varepsilon * c_+^\varepsilon)^2 (J_\varepsilon * (c_+^\varepsilon - c_-^\varepsilon))\,dx, \tag{3.22}$$

$$2\int_{\mathbb{R}^n} (J_\varepsilon * c_-^\varepsilon)\nabla\cdot(J_\varepsilon * c_-^\varepsilon \nabla(J_\varepsilon * \phi_\varepsilon))\,dx = -\int_{\mathbb{R}^n} (J_\varepsilon * c_-^\varepsilon)^2 (J_\varepsilon * (c_+^\varepsilon - c_-^\varepsilon))\,dx. \tag{3.23}$$

Hence summing (3.20) and (3.21), and using (3.22) and (3.23), we have

$$\begin{aligned}&\frac{d}{dt}(\|c_+^\varepsilon\|_{L^2}^2 + \|c_-^\varepsilon\|_{L^2}^2) + 2\int_{\mathbb{R}^n} |\nabla c_+^\varepsilon|^2 + |\nabla c_-^\varepsilon|^2\,dx\\ &= -\int_{\mathbb{R}^n} \Big((J_\varepsilon * c_+^\varepsilon)^2 - (J_\varepsilon * c_-^\varepsilon)^2\Big)\Big(J_\varepsilon * c_+^\varepsilon - J_\varepsilon * c_-^\varepsilon\Big)\,dx \le 0,\end{aligned}$$

which implies (3.17) and (3.18).

Moreover, using the Equation (3.4) and the weak Young inequality, we obtain

$$\|\nabla\phi_\varepsilon\|_{L^r} \le C\Big\|\frac{1}{|x|^{n-1}}\Big\|_{L_w^{\frac{n}{n-1}}} \|c_+^\varepsilon - c_-^\varepsilon\|_{L^s}, \quad \frac{1}{r} + 1 = \frac{n-1}{n} + \frac{1}{s}, \quad 1 < s \le 2.$$

Together with (3.17), this implies (3.19). □

Furthermore, using the regularized equations (3.1)-(3.3), and the uniform estimates in Proposition 3.1 and Proposition 3.2, we can directly obtain the following proposition. Since the estimate is standard, we omit the details.

Proposition 3.3. *Assume that $u_0 \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_+^0, c_-^0 \in L_+^1 \cap L^2(\mathbb{R}^n)$ and $\mathcal{F}_1(0) < \infty$. Let $(u_\varepsilon, c_+^\varepsilon, c_-^\varepsilon)$ be solutions to (3.1)-(3.7). Then for any $T > 0$, there is a constant C such that $(u_\varepsilon, c_+^\varepsilon, c_-^\varepsilon)$ satisfy the following estimates uniformly in ε*

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;W^{-1,\frac{n}{n-1}}(\mathbb{R}^n))} \le C, \tag{3.24}$$

$$\|\partial_t c_+^\varepsilon\|_{L^2(0,T;W^{-1,\frac{n}{n-1}}(\mathbb{R}^n))} + \|\partial_t c_-^\varepsilon\|_{L^2(0,T;W^{-1,\frac{n}{n-1}}(\mathbb{R}^n))} \le C. \tag{3.25}$$

Finally, we use compactness argument to complete the proof of Theorem 1.1.

Proof. (**Proof of Theorem 1.1.**) Since the solutions $(u_\varepsilon, c_+^\varepsilon, c_-^\varepsilon)$ of the regularized problem (3.1)-(3.7) satisfy all the estimates in Proposition 3.1, Proposition 3.2 and Proposition 3.3, using the Aubin-Lions lemma, we can obtain that there is a subsequence still denoted as u_ε, c_+^ε and c_-^ε and limit functions u, c_+ and c_- satisfying the regularities (1.21)-(1.25) such that

$$u_\varepsilon \to u \quad \text{in } L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^n)), \tag{3.26}$$

$$c_+^\varepsilon \to c_+ \quad \text{in } L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^n)), \tag{3.27}$$

$$c_-^\varepsilon \to c_- \quad \text{in } L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^n)). \tag{3.28}$$

Hence the standard compactness argument implies that there is a global weak solution (u, c_+, c_-) for the model (1.13)-(1.17), and they satisfy all the regularities (1.21)-(1.25).

Now we prove that weak solutions have the uniform L^∞-bound. Using (2.11), we can get for any $t > 0$ and $p \ge 1$, it holds that

$$\|c_+\|_{L^p}^p + \|c_-\|_{L^p}^p + \int_0^t \mathcal{D}_3(s)\,ds \le \|c_0^+\|_{L^p}^p + \|c_0^-\|_{L^p}^p.$$

Due to $c_+^0, c_-^0 \in L^1 \cap L^\infty(\mathbb{R}^n)$, we know that there is a constant C independent of p such that

$$\|c_+(\cdot,t)\|_{L^p(\mathbb{R}^n)} + \|c_-(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C, \quad t \in [0,\infty), \tag{3.29}$$

which means (1.26). □

4. Global existence for the multi-ionic species case in $\mathbb{R}^n$, $n=2,3$

As usual, we first show the bound of the first moment $m_1(t)$ and use it to determine the boundedness of the Fisher information, in order to give weak convergence of c_i^ε in the Sobolev space $L^2(0,T;L^{\frac{4}{3}}(\mathbb{R}^n))$. Moreover, we estimate the time derivative of $\nabla\phi_\varepsilon$ for proving its strong convergence in $L^2(0,T;L^4_{\text{loc}}(\mathbb{R}^n))$. Let us begin from estimating the first moment $m_1(t)$.

Proposition 4.1 (Estimate of the first moment). *Let $(u,c_1,\dots,c_N)$ be solutions to (1.1)-(1.4). Then for any $\sigma>0$, there is a constant $C(\sigma)$ such that the first moment $m_1(t)$ satisfies the following relation*

$$\begin{aligned} 2\frac{d}{dt}m_1(t) \le C(\sigma) + \sigma\sum_{i=1}^N \|\nabla\sqrt{c_i}\|^2_{L^2(\mathbb{R}^n)} + \sigma\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \\ + \sigma\sum_{i=1}^N \int_{\mathbb{R}^n} z_i^2 c_i |\nabla\phi|^2\,dx, \quad i=1,\dots,N. \end{aligned} \tag{4.1}$$

Proof. Multiplying $|x|$ to the Equation (1.2), integrating in $\mathbb{R}^n$, we have

$$\frac{d}{dt}m_1^i(t) = \int_{\mathbb{R}^n} \frac{x\cdot u}{|x|} c_i\,dx - \int_{\mathbb{R}^n} \frac{x\cdot\nabla c_i}{|x|}\,dx - \int_{\mathbb{R}^n} z_i c_i \frac{x}{|x|}\cdot\nabla\phi\,dx. \tag{4.2}$$

Notice that for any $0<t<\infty$, it holds that

$$\Big|\int_{\mathbb{R}^n} \frac{x}{|x|} c_i u\,dx\Big| \le \|c_i\|_{L^{\frac{4}{3}}(\mathbb{R}^n)} \|u\|_{L^4(\mathbb{R}^n)}, \tag{4.3}$$

$$\Big|-\int_{\mathbb{R}^n} \frac{x}{|x|}\nabla c_i\,dx\Big| \le \int_{\mathbb{R}^n} |\nabla c_i|\,dx \le 2\|c_i\|^{\frac{1}{2}}_{L^1(\mathbb{R}^n)} \|\nabla\sqrt{c_i}\|_{L^2(\mathbb{R}^n)}, \tag{4.4}$$

$$\Big|-\int_{\mathbb{R}^n} z_i c_i \frac{x}{|x|}\cdot\nabla\phi\,dx\Big| \le \Big(\int_{\mathbb{R}^n} z_i^2 c_i|\nabla\phi|^2\,dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} c_i\,dx\Big)^{\frac{1}{2}}. \tag{4.5}$$

Plugging (4.3)-(4.5) into (4.2), and using (2.1) and (1.7), we obtain

$$\frac{d}{dt}m_1^i(t) \le \|c_i\|_{L^{\frac{4}{3}}(\mathbb{R}^n)} \|u\|_{L^4(\mathbb{R}^n)} + C\left(\|\nabla\sqrt{c_i}\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n} z_i^2 c_i|\nabla\phi|^2\,dx\Big)^{\frac{1}{2}}\right). \tag{4.6}$$

Since

$$\|c_i\|_{L^{\frac{4}{3}}(\mathbb{R}^n)} = \|\sqrt{c_i}\|^2_{L^{\frac{8}{3}}(\mathbb{R}^n)} \le C\|\sqrt{c_i}\|^{2(1-\theta)}_{L^2(\mathbb{R}^n)} \|\nabla\sqrt{c_i}\|^{2\theta}_{L^2(\mathbb{R}^n)} \le C\|\nabla\sqrt{c_i}\|^{2\theta}_{L^2(\mathbb{R}^n)}, \tag{4.7}$$

where $\theta=\frac{1}{4}$ for $n=2$, and $\theta=\frac{3}{8}$ for $n=3$. And using the Young inequality, we know that for any $\sigma>0$, it holds that

$$2\frac{d}{dt}m_1^i(t) \le C(\sigma) + \sigma\|\nabla\sqrt{c_i}\|^2_{L^2(\mathbb{R}^n)} + \sigma\int_{\mathbb{R}^n} z_i^2 c_i|\nabla\phi|^2\,dx + \frac{\sigma}{N}\|\nabla u\|^2_{L^2(\mathbb{R}^n)}, \tag{4.8}$$

$i=1,\dots,N$. Summing (4.8) from 1 to N, we get (4.1). ∎

Combining (2.5), (2.6) and (4.1) and taking $\sigma=\frac{1}{2}$, we deduce

$$\frac{d}{dt}(\mathcal{F}_1+\mathcal{F}_2+2m_1(t))+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2\,dx+2\int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)^2\,dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^n}(\sum_{i=1}^N z_i^2c_i)|\nabla\phi|^2\,dx+\frac{7}{2}\sum_{i=1}^N\int_{\mathbb{R}^n}|\nabla\sqrt{c_i}|^2\,dx\le C. \tag{4.9}$$

Based on Proposition 4.1 and the free energy-dissipation relation (1.8), we will deduce a series of a prior estimates, which are helpful for proving global existence of weak solutions to the model (1.1)-(1.4).

Lemma 4.1. *Assume that $u_0\in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_i^0\in L_+^1\cap L\log L(\mathbb{R}^n)$, $m_1(0)<\infty$ and $\mathcal{F}_1(0),\mathcal{F}_2(0)<\infty$. Then for any $T>0$, there is a constant C such that the following estimates hold*

$$\|u\|_{L^\infty(0,\infty;L^2(\mathbb{R}^n))}+\|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^n))}\le C, \tag{4.10}$$
$$\|\nabla\sqrt{c_i}\|_{L^2(0,T;L^2(\mathbb{R}^n))}\le C,\quad i=1,\dots,N, \tag{4.11}$$
$$\|c_i\|_{L^{\frac{8}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{R}^n))}\le C,\quad i=1,\dots,N, \tag{4.12}$$
$$\|\nabla\phi\|_{L^\infty(0,\infty;L^2(\mathbb{R}^n))}+\|\nabla\phi\|_{L^2(0,T;H^1(\mathbb{R}^n))}\le C. \tag{4.13}$$

Proof. By (1.7), we easily know that (4.10) holds, and thus

$$\|u\|_{L^4(0,\infty;L^4(\mathbb{R}^n))}\le C,\quad \text{for n=2}, \tag{4.14}$$
$$\|u\|_{L^{\frac{8}{3}}(0,\infty;L^4(\mathbb{R}^n))}\le C,\quad \text{for n=3}. \tag{4.15}$$

Using the Carleman-type inequality [4]

$$\int_{\{c_i\le 1\}}c_i|\log c_i|\,dx\le m_1^i(t)+\frac{8\pi}{e}, \tag{4.16}$$

we obtain that there is a constant C independent of t such that it holds that

$$\mathcal{F}_1+\mathcal{F}_2+2m_1(t)+C\ge\sum_{i=1}^N\int_{\mathbb{R}^n}c_i\log c_i\,dx+2m_1(t)+C\ge 0. \tag{4.17}$$

Combining (4.9) and (4.17), and using the initial assumptions, we have

$$\sum_{i=1}^N\int_0^T\int_{\mathbb{R}^n}|\nabla\sqrt{c_i}|^2\,dxdt+\int_0^T\int_{\mathbb{R}^n}(\sum_{i=1}^N z_ic_i)^2\,dxdt\le C. \tag{4.18}$$

Hence the Equation (1.3) and (4.18) imply that (4.11) and

$$\int_0^T\int_{\mathbb{R}^n}(\Delta\phi)^2\,dxdt\le C. \tag{4.19}$$

The fact

$$\int_{\mathbb{R}^n}(\Delta\phi)^2\,dx=\int_{\mathbb{R}^n}|\nabla^2\phi|^2\,dx,\quad\text{for any }0<t<T,$$

together with (4.19) imply that (4.13) holds. Moreover, combining (4.7) and (4.11), and noticing $2\theta \le \frac{3}{4}$, then by the Hölder inequality, we obtain

$$\int_0^T \|c_i\|_{\frac{4}{3}}^{\frac{8}{3}} dt \le C \int_0^T \|\nabla\sqrt{c_i}\|_2^{\frac{16\theta}{3}} dt = C\int_0^T \|\nabla\sqrt{c_i}\|_2^2 dt \le C.$$

That is (4.12). ∎

Now using (4.10)-(4.13), we derive some estimates on time derivative of $\nabla\phi$ and u.

LEMMA 4.2. *Assume that $u_0 \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_i^0 \in L^1_+ \cap L\log L(\mathbb{R}^n)$, $m_1(0) < \infty$, $\mathcal{F}_1(0) < \infty$ and $\mathcal{F}_2(0) < \infty$. Then for any $T > 0$, there is a constant C such that*

$$\|\partial_t(\nabla\phi)\|_{L^{\frac{4}{3}}(0,T;W^{-2,\frac{4}{3}}(\mathbb{R}^n))} \le C, \tag{4.20}$$

$$\|\partial_t u\|_{L^2(0,T;W^{-1,\frac{4}{3}}(\mathbb{R}^n))} \le C. \tag{4.21}$$

Proof. For any $v \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, v can be decomposed into

$$v = w + \nabla\psi, \quad \nabla\cdot w = 0.$$

Hence

$$\langle v, \partial_t(\nabla\phi)\rangle = \langle \nabla\psi, \partial_t(\nabla\phi)\rangle = \langle \psi, \partial_t\big(-\Delta\phi\big)\rangle = \langle \psi, \partial_t(\sum_{i=1}^N z_i c_i)\rangle. \tag{4.22}$$

Multiplying z_i to the Equation (1.2) and summing them from $i=1$ to N, we deduce that

$$\partial_t(\sum_{i=1}^N z_i c_i) + \nabla\cdot((\sum_{i=1}^N z_i c_i)u) = \Delta(\sum_{i=1}^N z_i c_i) + \nabla\cdot((\sum_{i=1}^N z_i^2 c_i)\nabla\phi).$$

So, we can compute

$$\langle \psi, \partial_t(\sum_{i=1}^N z_i c_i)\rangle = \langle \nabla\psi, (\sum_{i=1}^N z_i c_i)u\rangle + \langle \Delta\psi, (\sum_{i=1}^N z_i c_i)\rangle - \langle \nabla\psi, ((\sum_{i=1}^N z_i^2 c_i)\nabla\phi)\rangle. \tag{4.23}$$

Noticing that

$$|\langle \nabla\psi, ((\sum_{i=1}^N z_i^2 c_i)\nabla\phi)\rangle| \le \|\nabla\psi\|_{L^\infty}\left(\sum_{i=1}^N \big(\int_{\mathbb{R}^n} z_i^2 c_i\, dx\big)^{\frac{1}{2}}\big(\int_{\mathbb{R}^n} z_i^2 c_i |\nabla\phi|^2\, dx\big)^{\frac{1}{2}}\right), \tag{4.24}$$

and using the Hölder inequality for the right-hand side of (4.23), we have

$$\begin{aligned}\langle \psi, \partial_t(\sum_{i=1}^N z_i c_i)\rangle \le\ & \|\Delta\psi\|_{L^4}\|(\sum_{i=1}^N z_i c_i)\|_{L^{\frac{4}{3}}} + \|\nabla\psi\|_{L^\infty}\|(\sum_{i=1}^N z_i c_i)\|_{L^{\frac{4}{3}}}\|u\|_{L^4} \\ & + C\|\nabla\psi\|_{L^\infty}\big(\sum_{i=1}^N \int_{\mathbb{R}^n} z_i^2 c_i |\nabla\phi|^2\, dx\big)^{\frac{1}{2}}. \end{aligned} \tag{4.25}$$

By the free energy-dissipation relation (1.7), (4.12), (4.14) and (4.15), we deduce

$$\int_0^T \|\Delta\psi\|_{L^4}\|(\sum_{i=1}^N z_i c_i)\|_{L^{\frac{4}{3}}}\,dt \le C\|\Delta\psi\|_{L^{\frac{8}{5}}(0,T;L^4(\mathbb{R}^n))}, \tag{4.26}$$

$$\int_0^T \|\nabla\psi\|_{L^\infty}\|(\sum_{i=1}^N z_i c_i)\|_{L^{\frac{4}{3}}}\|u\|_{L^4}\,dt \le C\|\nabla\psi\|_{L^4(0,T;L^\infty(\mathbb{R}^n))}, \tag{4.27}$$

$$\int_0^T \|\nabla\psi\|_{L^\infty}(\sum_{i=1}^N \int_{\mathbb{R}^n} z_i^2 c_i|\nabla\phi|^2\,dx)^{\frac{1}{2}}\,dt \le C\|\nabla\psi\|_{L^2(0,T;L^\infty(\mathbb{R}^n))}. \tag{4.28}$$

Hence it holds that

$$\int_0^T \langle\psi,\partial_t(\sum_{i=1}^N z_i c_i)\rangle\,dt \le C\|\psi\|_{L^4(0,T;W^{2,4}(\mathbb{R}^n))}. \tag{4.29}$$

Therefore (4.22) and (4.29) imply that

$$\int_0^T \langle v,\partial_t(\nabla\phi)\rangle\,dt \le C\|v\|_{L^4(0,T;W^{2,4}(\mathbb{R}^n))},$$

which means that (4.20) holds. Using (4.10)-(4.13), a similar process can also give (4.21). This completes the proof of Lemma 4.2. ∎

Proof. (**Proof of Theorem 1.2.**) A regularized problem for (1.1)-(1.4) is given by the following equations

$$\partial_t u_\varepsilon + u_\varepsilon\cdot\nabla u_\varepsilon + \nabla p_\varepsilon = \Delta u_\varepsilon - (\sum_{i=1}^N z_i c_i^\varepsilon)(J_\varepsilon * \nabla\phi_\varepsilon), \tag{4.30}$$

$$\partial_t c_i^\varepsilon + \nabla\cdot(c_i^\varepsilon u_\varepsilon) = \Delta c_i^\varepsilon + \nabla\cdot(z_i c_i^\varepsilon J_\varepsilon * \nabla\phi_\varepsilon), \tag{4.31}$$

$$-\Delta\phi_\varepsilon = (\sum_{i=1}^N z_i c_i^\varepsilon) * J_\varepsilon, \tag{4.32}$$

$$\nabla\cdot u_\varepsilon = 0, \tag{4.33}$$

$$c_i^\varepsilon(x,0) = c_i^{\varepsilon 0}(x) := c_0^i(x) * J_\varepsilon, \tag{4.34}$$

$$u_\varepsilon(x,0) = u_{\varepsilon 0}(x) := u_0(x) * J_\varepsilon. \tag{4.35}$$

Since $u_0 \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, $c_+^0, c_-^0 \in L_+^1 \cap L\log L(\mathbb{R}^n)$, $m_1(0) < \infty$, $\mathcal{F}_1(0) < \infty$ and $\mathcal{F}_2(0) < \infty$, we have

$$\|u_{\varepsilon 0}\|_{L^2(\mathbb{R}^n)} \le \|u_0\|_{L^2(\mathbb{R}^n)}, \quad \|c_i^{\varepsilon 0}\|_{L^1(\mathbb{R}^n)} = \|c_0^i\|_{L^1(\mathbb{R}^n)} \tag{4.36}$$

$$m_1^\varepsilon(0) < m_1(0) + C, \quad \mathcal{F}_1^\varepsilon(0) \le \mathcal{F}_1(0), \qquad \mathcal{F}_2^\varepsilon(0) \le \mathcal{F}_2(0), \tag{4.37}$$

where C is independent of ε, $m_1^\varepsilon(t)$ and $\mathcal{F}_2^\varepsilon(t)$ are defined by

$$m_1^\varepsilon(t) := \sum_{i=1}^N \int_{\mathbb{R}^n} |x| c_i^\varepsilon\,dx, \quad \mathcal{F}_2^\varepsilon(t) = \sum_{i=1}^N \int_{\mathbb{R}^n} c_i^\varepsilon \log c_i^\varepsilon\,dx.$$

It is directly checked that the solutions $(u_\varepsilon, c_1^\varepsilon, \dots, c_N^\varepsilon)$ of the regularized problem (4.30)-(4.35) for $n=2,3$ satisfy the energy-dissipation relations (1.7) and (1.8), and the property of the first moment. Thus we have all the estimates in Lemma 4.1 and Lemma 4.2, i.e.,

$$\|u_\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^n))}+\|\nabla u_\varepsilon\|_{L^2(0,\infty;L^2(\mathbb{R}^n))}\le C, \tag{4.38}$$

$$\|\nabla\sqrt{c_i^\varepsilon}\|_{L^2(0,T;L^2(\mathbb{R}^n))}\le C,\quad i=1,\dots,N, \tag{4.39}$$

$$\|c_i^\varepsilon\|_{L^{\frac{8}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{R}^n))}\le C,\quad i=1,\dots,N, \tag{4.40}$$

$$\|\nabla\phi_\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^n))}+\|\nabla\phi_\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))}\le C, \tag{4.41}$$

$$\|\partial_t(\nabla\phi_\varepsilon)\|_{L^{\frac{4}{3}}(0,T;W^{-2,\frac{4}{3}}(\mathbb{R}^n))}\le C, \tag{4.42}$$

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;W^{-1,\frac{4}{3}}(\mathbb{R}^n))}\le C. \tag{4.43}$$

Hence using the Aubin-Lions-Simon lemma [12, Corollary 4] and the uniform estimates (4.38)-(4.43), we can obtain that there is a subsequence still denoted as u_ε, ϕ_ε, c_i^ε and limit functions u, ϕ and c_i satisfying the regularities (1.28)-(1.31) such that

$$u_\varepsilon\to u \quad \text{in } L^2(0,T;L^4_{\text{loc}}(\mathbb{R}^n)),$$
$$\nabla\phi_\varepsilon\to\nabla\phi \quad \text{in } L^2(0,T;L^4_{\text{loc}}(\mathbb{R}^n)).$$

Furthermore by the estimate (4.40), we deduce

$$c_i^\varepsilon\rightharpoonup c_i,\quad i=1,\dots,N,\qquad \text{in } L^2(0,T;L^{\frac{4}{3}}(\mathbb{R}^n)). \tag{4.44}$$

Hence the standard compactness argument implies that there is a global weak solution for the model (1.1)-(1.4). This completes the proof of Theorem 1.2. □

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