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Strong convergence of a Euler-Maruyama scheme to a variable-order fractional stochastic differential equation driven by a multiplicative white noise

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ABSTRACT

We prove the existence and uniqueness of the solution to a variable-order fractional stochastic differential equation driven by a multiplicative white noise, which describes the random phenomena with nonlocal effect. We further develop a Euler-Maruyama scheme and prove the strong convergence of the scheme. Numerical experiments are presented to substantiate the mathematical analysis.

1. Introduction

The classical stochastic differential equation (SDE)

$$du = f(u(t))dt + b(u(t))dW \tag{1}$$

may be used to describe stochastic processes in many disciplined and applications [12,16,26,41]. For instance, in the modeling of the random movement of a Brownian particle in a surrounding viscous liquid, $u = u(t)$ denotes the velocity of the Brownian particle, $f(u)$ represents the mean resistance of the surrounding medium per unit mass of the Brownian particle, and $b(u(t))dW$ represents the random force that accounts for the effect of the noise that is modeled by a Brownian motion or Wiener process [12,18,26].

However, for the random movement of a Brownian particle in a viscoelastic medium, the resistance has memory effect that leads to fractional SDEs [4,6–11,14,15,17,20,22,24,25,28,39]. Moreover, in many scenarios the structure of the materials may evolve with time. For instance, in the random movement of a Brownian particle in a viscoelastic medium, the collisions of the particle with the molecules of the medium may change the structure of the medium. In nonconventional hydrocarbon or shale gas recovery hydrofracturing technique is used to increase the pore size of the medium to enhance the recovery of oil and shale gas. As the fractional order is determined by the fractal dimension of the media via the Hurst index [23], the structure change of porous materials leads to the change of the fractional order, yielding variable-order models [1,5,13,15,19,27,31,33,37,40].

Finally, classical fractional differential equations yield solutions with nonphysical initial weak singularity [32,36], which would affect the accuracy of their numerical approximations. The fundamental reason is that the classical fractional differential equations cannot accurately model the multiple time scales in the underlying physical processes. Recall that the classical time-fractional diffusion equation was derived as a stochastic limit when the number of particle jumps tends to infinity and hence holds only for large time [23,24]. A two time-scale variable-order time-fractional diffusion equation was analyzed and discretized in Wang and Zheng [38], Zheng and Wang [40], which catch both the Fickian diffusive transport behavior near the initial time and the power-law decaying behavior of the anomalously diffusive transport for large time. Similar phenomena happens in, e.g., the creep process of viscoelastic materials when an external loading is applied at the initial time instant, the strain of the material has a certain jump due to the elastic component in the material and then gradually increases due to the viscous component. Again, the conventional fractional

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differential equation model does not catch the elastic behavior at the initial time zone [2,15,17,21,30]. A two-time scale variable-order fractional SDE (fSDE) was proposed in Zheng et al. [42]

$$du = \left(-\lambda\, {}_0^R\mathcal{D}_t^{\alpha(t)} u + f(u)\right)dt + b(u)dW,\ t \in (0,T];\ u(0) = u_0, \quad (2)$$

which can describe the elastic behavior near the initial time zone while modeling the viscoelastic behavior for large time and holds on the entire time interval. Here $\lambda \geq 0$ and the variable-order fractional integral operator ${}_0\mathcal{I}_t^{1-\alpha(t)}$ and Riemann-Liouville fractional differential operator ${}_0^R\mathcal{D}_t^{\alpha(t)}$ are defined by Shi and Wang [35], Zheng et al. [42]

$$\begin{aligned}{}_0^R\mathcal{D}_t^{\alpha(t)} h &:= \frac{d}{dt}\left[{}_0\mathcal{I}_t^{1-\alpha(t)} h\right],\ {}_0\mathcal{I}_t^{1-\alpha(t)} h(t)\\ &:= \frac{1}{\Gamma(1-\alpha(t))}\int_0^t \frac{h(s)ds}{(t-s)^{\alpha(t)}}.\end{aligned}$$

Compared to conventional SDEs, the variable-order fSDE (2) possesses non-Lipschitz weakly singular kernel with variable order, and also loses the convolution structure. Extra work needs to be carried out in the discretization and the corresponding analysis of the model. In this paper we develop a generalized Euler-Maruyama scheme for the nonlinear variable-order fSDE (2) and prove its strong convergence. The rest of the paper is structured as follows: In Section 2 we prove the well-posedness of the variable-order fSDE (2). In Section 3 we develop a generalized Euler-Maruyama scheme for problem (2) and prove its strong convergence. In Section 4 we carry out numerical experiments to investigate the performance of the numerical scheme and to support the theoretical analysis. In Section 5 we draw concluding remarks. In Appendix A we present auxiliary results that are used in the analysis in the previous sections.

2. Wellposedness and estimates of the variable-order fSDE (2)

In this section we prove the well-posedness and moment estimates of the variable-order fSDE (2). We begin by reformulating the variable-order fSDE (2) as follows [12,26]: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $W(\cdot)$ be a Brownian motion and u_0 be a second-order random variable that is independent of $W(\cdot)$. Let $\mathcal{F}(t) := \mathcal{U}\big(W(s)\ \ (0 \leq s \leq t),\ u_0\big)$ denote the σ-algebra generated by u_0 and the history of the Brownian motion up to time t. Further, the data of Eq. (2) satisfy the following assumptions

(a) $\alpha \in C^1[0,T]$, the space of continuously differentiable functions on $[0,T]$, with $0 \leq \alpha(t) \leq \alpha^*$ for some $0 < \alpha^* < 1$.
(b) There exists a positive constant L such that

$$\begin{aligned}&|f(v)| \leq L(1+|v|), \quad |b(v)| \leq L(1+|v|), \quad \forall v \in \mathbb{R},\\ &|f(v_1) - f(v_2)| \leq L|v_1 - v_2|,\\ &|b(v_1) - b(v_2)| \leq L|v_1 - v_2|, \quad \forall v_1, v_2 \in \mathbb{R}. \quad (3)\end{aligned}$$

We integral the Eq. (2) from 0 to t and use the fact that

$$\begin{aligned}&\int_0^t -\lambda\, {}_0^R\mathcal{D}_s^{\alpha(s)} u(s)ds\\ &= \int_0^t -\lambda \frac{d}{ds}\left[\frac{1}{\Gamma(1-\alpha(s))}\int_0^s \frac{u(y)dy}{(s-y)^{\alpha(s)}}\right]ds\\ &= \int_0^t -\lambda\, d\left[\frac{1}{\Gamma(1-\alpha(s))}\int_0^s \frac{u(y)dy}{(s-y)^{\alpha(s)}}\right]\\ &= -\lambda\left[\frac{1}{\Gamma(1-\alpha(s))}\int_0^s \frac{u(y)dy}{(s-y)^{\alpha(s)}}\right]\Big|_{s=0}^{s=t}\\ &= \frac{-\lambda}{\Gamma(1-\alpha(t))}\int_0^t \frac{u(y)dy}{(t-y)^{\alpha(t)}}\\ &= \int_0^t \frac{-\lambda}{\Gamma(1-\alpha(t))(t-s)^{\alpha(t)}} u(s)ds =: \int_0^t k(t,s)u(s)ds \quad (4)\end{aligned}$$

to find a stochastic process $u(\cdot)$ on $[0,T]$ that is progressively measurable with respect to $\mathcal{F}(\cdot)$ such that for all times $0 \leq t \leq T$

$$u(t) = u_0 + \int_0^t k(t,s)u(s)ds + \int_0^t f(u(s))ds + \int_0^t b(u(s))dW \quad a.s. \quad (5)$$

where

$$k(t,s) := \frac{-\lambda}{\Gamma(1-\alpha(t))(t-s)^{\alpha(t)}}. \quad (6)$$

Throughout this paper we use Q,Q_i, and $\bar{Q}_i$ to denote generic positive constants, in which $\bar{Q}_i$ refer to the constant that occur from the lemmas cited in the appendix and Q may assume different values at different occurrences.

Theorem 2.1. *If Assumptions (a)-(b) hold, the variable-order fSDE (2) has a unique solution u such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|u(s)|^2\right] \leq Q_0 E_{1-\alpha^*}\left(Q_1\Gamma(1-\alpha^*)t^{1-\alpha^*}\right) < \infty, t \in [0,T]. \quad (7)$$

Here Q_0 and Q_1 are defined below, $E_p(z)$ is the Mittag-Leffler function [29]

$$\begin{aligned}Q_0 &= 4\mathbb{E}\left[u_0^2\right] + 8L^2T\left(T + \overline{Q}_1\right),\\ Q_1 &= \left(\frac{4\lambda^2 T^{1+\alpha^*}}{1-\alpha^*} + 8L^2\left(T + \overline{Q}_1\right)\right),\\ E_p(z) &:= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(pk+1)}, z \in \mathbb{R}, \quad p \in \mathbb{R}^+ \quad (8)\end{aligned}$$

with $\bar{Q}_1$ denoting the constant in Burkhölder-Davis-Gundy inequality (65) in Lemma A.1 which can be found in the Appendix.

Proof. We employ the technique of Picard iteration to prove our existence and uniqueness theorem. We define a sequence $\{z_n\}_{n=0}^{\infty}$ by $z_0(t) := u_0$ and for $n \geq 1$

$$\begin{aligned}z_n(t) &:= u_0 + \int_0^t k(t,s)z_{n-1}(s)ds + \int_0^t f(z_{n-1})ds\\ &\quad + \int_0^t b(z_{n-1})dW(s). \quad (9)\end{aligned}$$

As $\Gamma(t)$ is decreasing on (0,1] and $0 < 1-\alpha^* \leq 1-\alpha(t) \leq 1$ for $t \in [0,T], \Gamma(1-\alpha(t)) \geq \Gamma(1) = 1$. We bound $k(t,s)$ in (6) by

$$\begin{aligned}|k(t,s)| &= \frac{\lambda(t-s)^{\alpha^*-\alpha(t)}}{\Gamma(1-\alpha(t))(t-s)^{\alpha^*}}\\ &\leq \frac{Q_2^{\alpha^*}\lambda}{(t-s)^{\alpha^*}}, \qquad Q_2 = \max\{1,T\}. \quad (10)\end{aligned}$$

For every $n \geq 1$ and $x \in [0,T]$, observe that

$$\begin{aligned}z_{n+1}(x) - z_n(x) &= \int_0^x k(x,s)(z_n(s) - z_{n-1}(s))ds\\ &\quad + \int_0^x f(z_n(s)) - f(z_{n-1}(s))ds\\ &\quad + \int_0^x b(z_n(s)) - b(z_{n-1}(s))dW(s). \quad (11)\end{aligned}$$

Use Jensen's inequality (66) in Lemma A.2 which can be found in the Appendix with $m = 3$ to bound (11) by

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le x\le t}|z_{n+1}(x)-z_n(x)|^2\Big]\\&\le 3\mathbb{E}\Big[\sup_{0\le x\le t}\Big|\int_0^x k(x,s)(z_n(s)-z_{n-1}(s))ds\Big|^2\\&\quad+\sup_{0\le x\le t}\Big|\int_0^x f(z_n(s))-f(z_{n-1}(s))ds\Big|^2\\&\quad+\sup_{0\le x\le t}\Big|\int_0^x b(z_n(s))-b(z_{n-1}(s))dW(s)\Big|^2\Big]=:\sum_{i=1}^3 H_i.\end{aligned}\tag{12}$$

Use Cauchy inequality, estimate (10) and (70) in Lemma A.5 in the Appendix to bound H_1 by

$$\begin{aligned}H_1&\le 3\mathbb{E}\Big[\sup_{0\le x\le t}\Big(\textstyle\int_0^x|k(x,s)|ds\int_0^x|k(x,s)||z_n(s)-z_{n-1}(s)|^2ds\Big)\Big]\\&\le 3\mathbb{E}\Big[\sup_{0\le x\le t}\Big(\tfrac{\lambda^2Q_2^{2\alpha^*}x^{1-\alpha^*}}{1-\alpha^*}\textstyle\int_0^x(x-s)^{-\alpha^*}|z_n(s)-z_{n-1}(s)|^2ds\Big)\Big]\\&\le\tfrac{3\lambda^2Q_2^{2\alpha^*}t^{1-\alpha^*}}{1-\alpha^*}\textstyle\int_0^t(t-s)^{-\alpha^*}\mathbb{E}\Big[\sup_{0\le r\le s}|z_n(r)-z_{n-1}(r)|^2\Big]ds.\end{aligned}\tag{13}$$

Apply Cauchy inequality along with assumption (3) to bound H_2 by

$$\begin{aligned}H_2&\le 3\mathbb{E}\Big[\sup_{0\le x\le t}\Big|\textstyle\int_0^x 1^2ds\int_0^x L^2|z_n(s)-z_{n-1}(s)|^2ds\Big|\Big]\\&\le 3tL^2\textstyle\int_0^t\mathbb{E}\Big[\sup_{0\le r\le s}|z_n(r)-z_{n-1}(r)|^2\Big]ds.\end{aligned}\tag{14}$$

Use Burkhölder-Davis-Gundy inequality (65) in Lemma A.1 in the Appendix and Itô isometry to bound H_3 by

$$\begin{aligned}H_3&=3\mathbb{E}\Big[\sup_{0\le x\le t}\Big|\textstyle\int_0^x b(z_n(s))-b(z_{n-1}(s))dW(s)\Big|^2\Big]\\&\le 3\overline{Q}_1\mathbb{E}\Big[\Big|\textstyle\int_0^t b(z_n(s))-b(z_{n-1}(s))dW(s)\Big|^2\Big]\\&=3\overline{Q}_1\mathbb{E}\Big[\textstyle\int_0^t|b(z_n(s))-b(z_{n-1}(s))|^2ds\Big]\\&\le 3\overline{Q}_1L^2\mathbb{E}\Big[\textstyle\int_0^t|z_n(s)-z_{n-1}(s)|^2ds\Big]\\&\le 3\overline{Q}_1L^2\textstyle\int_0^t\mathbb{E}\Big[\sup_{0\le r\le s}|z_n(r)-z_{n-1}(r)|^2\Big]ds.\end{aligned}\tag{15}$$

We substitute estimates (13), (14) and (15) into (12) to obtain

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le x\le t}|z_{n+1}(x)-z_n(x)|^2\Big]\\&\le Q_T\int_0^t\frac{\mathbb{E}\Big[\sup_{0\le r\le s}|z_n(r)-z_{n-1}(r)|^2\Big]}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds,\end{aligned}\tag{16}$$

where the positive constant Q_T is given by

$$Q_T=3\Gamma(1-\alpha^*)\Big(\frac{\lambda^2Q_2^{2\alpha^*}T^{1-\alpha^*}}{1-\alpha^*}+TL^2+\bar{Q}_1L^2\Big).\tag{17}$$

For $n\ge1$ we reformulate estimate (16) as follows

$$\begin{aligned}g_n(t)&:=\mathbb{E}\Big[\sup_{0\le x\le t}|z_{n+1}(x)-z_n(x)|^2\Big]\\&\le Q_T\int_0^t\frac{g_{n-1}(s)}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds,\end{aligned}\tag{18}$$

based on which we bound g_n by induction.

We bound g_0 similarly to (12) (except that using the growth condition in place of the Lipschitz condition in (3)) to get

$$\begin{aligned}g_0(t)&=\mathbb{E}\Big[\sup_{0\le x\le t}|z_1(x)-z_0(x)|^2\Big]\\&\le\mathbb{E}\Big[u_0^2\sup_{0\le x\le t}\Big(\textstyle\int_0^x|k(x,s)|ds\Big)^2+\sup_{0\le x\le t}\Big(\int_0^x|f(u_0)|ds\Big)^2\\&\quad+\sup_{0\le x\le t}\Big(\textstyle\int_0^x|b(u_0)|dW(s)\Big)^2\Big]\\&\le 3\mathbb{E}\Big[u_0^2\Big(\textstyle\int_0^t|k(x,s)|ds\Big)^2+\Big(\int_0^t|f(u_0)|ds\Big)^2\\&\quad+\overline{Q}_1\Big(\textstyle\int_0^t|b(u_0)|dW(s)\Big)^2\Big]\\&\le\frac{3\lambda^2Q_2^{2\alpha^*}T^{2(1-\alpha_*)}\mathbb{E}[u_0^2]}{(1-\alpha^*)^2}+6L^2\big(T^2+\overline{Q}_1T\big)\big(1+\mathbb{E}[u_0^2]\big)\\&=:Q_T'.\end{aligned}\tag{19}$$

We combine (18) and (19) to conclude that

$$g_1(t)\le Q_T\int_0^t\frac{Q_T'}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds=\frac{Q_T'Q_Tt^{(1-\alpha^*)}}{\Gamma((1-\alpha^*)+1)},\quad t\in[0,T].$$

Suppose that for any $1\le n\le m$, Then an induction argument using (18) shows that, for every $n\ge1$

$$g_n(t)\le\frac{Q_T'Q_T^nt^{n(1-\alpha^*)}}{\Gamma(n(1-\alpha^*)+1)},\quad t\in[0,T].\tag{20}$$

We combine (18) and (20) and use $s=t\theta$ to get

$$\begin{aligned}g_{m+1}(t)&\le Q_T\textstyle\int_0^t\frac{g_m(s)}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds\\&\le\frac{Q_T'Q_T^{m+1}}{\Gamma(m(1-\alpha^*)+1)}\int_0^t\frac{s^{m(1-\alpha^*)}}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds\\&\le\frac{Q_T'Q_T^{m+1}t^{(m+1)(1-\alpha^*)}B(m(1-\alpha^*)+1,1-\alpha^*)}{\Gamma(m(1-\alpha^*)+1)\Gamma(1-\alpha^*)}\\&\le\frac{Q_T'Q_T^{m+1}t^{(m+1)(1-\alpha^*)}}{\Gamma((m+1)(1-\alpha^*)+1)}.\end{aligned}$$

By mathematical induction, (20) holds for any $n\in\mathbb{N}$.

Consequently, the series defined by the right-hand side of (20) converges to the Mittag-Leffler function in (8)

$$\sum_{n=0}^\infty\frac{Q_T'Q_T^nt^{n(1-\alpha^*)}}{\Gamma(n(1-\alpha^*)+1)}=Q_T'E_{1-\alpha^*}\big(Q_Tt^{1-\alpha^*}\big)<\infty,\quad t\in[0,T],$$

which implies

$$\sum_{n=0}^\infty\mathbb{E}\Big[\sup_{0\le t\le T}\big|z_{n+1}(t)-z_n(t)\big|^2\Big]<\infty,\quad a.s.\tag{21}$$

We apply Chebyshev's inequality and use (20) and (21) to conclude that

$$\begin{aligned}P\Big(\sup_{t\in[0,T]}|z_{n+1}(t)-z_n(t)|\ge2^{-n}\Big)&\le4^n\mathbb{E}\Big[\sup_{t\in[0,T]}|z_{n+1}(t)-z_n(t)|^2\Big]\\&\le\frac{4^nQ_T'(Q_TT^{1-\alpha^*})^n}{\Gamma(n(1-\alpha^*)+1)}.\end{aligned}$$

By the Borel-Cantelli lemma, the sequence

$$z_n(t)=\sum_{m=1}^n\big(z_m(t)-z_{m-1}(t)\big)+u_0\tag{22}$$

converges uniformly on $[0,T]$ to a limit u that solves (5) a.s.. Together with the continuity of $\{z_n\}_{n=0}^\infty$,we conclude the continuity of u. That is, the variable-order fSDE (2) has a continuous solution u.

Suppose that there exists another solution $\tilde{u}$ to the variable-order fSDE (2), a similar derivation to (16) yields for all times $t \in [0,T]$

$$\mathbb{E}\Big[\sup_{0\le x\le t}\big|u(x)-\tilde{u}(x)\big|^2\Big] \le Q_T\int_0^t \frac{\mathbb{E}\Big[\sup\limits_{0\le r\le s}|u(r)-\tilde{u}(r)|^2\Big]}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds.$$

We apply Gronwall inequality (67) in Lemma A.3 in the Appendix to conclude that $u(t)=\tilde{u}(t)$ a.s., i.e., u and $\tilde{u}$ are indistinguishable. Therefore, the variable-order fSDE (2) has a unique solution.

We similarly bound the second moment of z_n for $n\ge 1$ and $t\in[0,T]$ by

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le x\le t}|z_n(x)|^2\Big]\\
&\le 4\mathbb{E}\big[u_0^2\big]+4\mathbb{E}\Big[\sup_{0\le x\le t}\Big(\textstyle\int_0^x k(x,s)z_{n-1}(s)ds\Big)^2\Big]\\
&\quad +4\mathbb{E}\Big[\sup_{0\le x\le t}\Big(\textstyle\int_0^x f(z_{n-1}(s))ds\Big)^2\Big]\\
&\quad +4\mathbb{E}\Big[\sup_{0\le x\le t}\Big(\textstyle\int_0^x b(z_{n-1}(s))dW(s)\Big)^2\Big]\\
&\le 4\mathbb{E}\big[u_0^2\big]+\frac{4\lambda^2Q_2^{2\alpha^*}T^{1-\alpha^*}}{1-\alpha^*}\int_0^t\frac{\mathbb{E}\Big[\sup\limits_{0\le r\le s}|z_{n-1}(r)|^2\Big]}{(t-s)^{\alpha^*}}ds\\
&\quad +8L^2\big(T+\overline{Q}_1\big)\int_0^t 1+\mathbb{E}\Big[\sup_{0\le r\le s}|z_{n-1}(r)|^2\Big]ds\\
&\le Q_0+Q_1\int_0^t\frac{\mathbb{E}\Big[\sup\limits_{0\le r\le s}|z_{n-1}(r)|^2\Big]}{(t-s)^{\alpha^*}}ds.
\end{aligned}$$

Here Q_0 and Q_1 are defined in (8). Pass to the limit as $n\to\infty$ and use Gronwall inequality (67) in Lemma A.3 which can be found in the Appendix with $\beta=1-\alpha^*$ to obtain (7). □

Remark 2.1. The above proof can be extended to estimate $\mathbb{E}\Big[\sup\limits_{0\le s\le t}|u(s)|^p\Big]$ for $2\le p<\infty$ provided $\mathbb{E}\big[|u_0|^p\big]<\infty$.

3. A Euler-Maruyama and its strong convergence

We derive a Euler-Maruyama scheme for the variable-order fSDE (2) and prove its strong convergence.

3.1. Derivation of the scheme

Define a uniform partition of $[0,T]$ by $t_n:=n\tau$ for $0\le n\le N$ with $\tau:=T/N$. At time step t_n for $1\le n\le N$, we use the Euler quadrature to discretize the integrals on the right-hand side of (5). We begin with the discretization of the second term by

$$\begin{aligned}
\int_0^{t_n} f(u(s))ds &= \textstyle\sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} f(u(s))ds\\
&\approx \textstyle\int_{t_l}^{t_{l+1}} f(u(t_l))ds=\tau\sum_{l=0}^{n-1}f(u(t_l)).
\end{aligned}\tag{23}$$

We similarly discretize the last term on the right-hand side of (5) by

$$\begin{aligned}
\int_0^{t_n} b(u(s))dW(s) &= \textstyle\sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} b(u(s))dW(s)\\
&\approx \textstyle\sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} b(u(t_l))dW(s)\\
&= \textstyle\sum_{l=0}^{n-1} b(u(t_l))\Delta W_l,
\end{aligned}\tag{24}$$

where $\Delta W_l:=W(t_{l+1})-W(t_l)\sim N(0,\tau)$ is a Gaussian random variable.

Note that the kernel in the first term on the right-hand side of (5) is nonlocal and has no convolution structure, which is not common in conventional SDEs or constant-order fSDEs. We extend the $L-1$ discretization to the current context

$$\begin{aligned}
&\textstyle\int_0^{t_n} k(t_n,s)u(s)ds\\
&= -\frac{\lambda}{\Gamma(1-\alpha(t_n))}\textstyle\sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}}\frac{u(s)}{(t_n-s)^{\alpha(t_n)}}ds\\
&\approx -\frac{\lambda}{\Gamma(1-\alpha(t_n))}\textstyle\sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}}\frac{u(t_l)}{(t_n-s)^{\alpha(t_n)}}ds\\
&= -\frac{\lambda u(t_l)}{\Gamma(2-\alpha(t_n))}\textstyle\sum_{l=0}^{n-1}\Big[(t_n-t_l)^{1-\alpha(t_n)}-(t_n-t_{l+1})^{1-\alpha(t_n)}\Big].
\end{aligned}\tag{25}$$

We incorporate the discretizations (23)–(25) into Eq. (5) to obtain a Euler-Maruyama scheme to the variable-order fSDE (2): Given the initial data u_0 in the variable-order fSDE (2), find v_n for $1\le n\le N$ such that

$$v_n=u_0+\sum_{l=0}^{n-1}b_{n,l}v_l+\tau\sum_{l=0}^{n-1}f(v_l)+\sum_{l=0}^{n-1}b(v_l)\Delta W_l \tag{26}$$

where the coefficients $b_{n,l}$ for $0\le l\le n-1$ are given by

$$b_{n,l}=-\lambda\frac{(t_n-t_l)^{1-\alpha(t_n)}-(t_n-t_{l+1})^{1-\alpha(t_n)}}{\Gamma\big(2-\alpha(t_n)\big)}. \tag{27}$$

Theorem 3.1. *For $1\le n\le N$, the solution v_n to the Euler-Maruyama scheme (26) satisfies the moment estimate*

$$\mathbb{E}\big[v_n^2\big]\le Q_3\big[1+E_{1-\alpha^*}(Q_{4,\lambda}\Gamma(1-\alpha^*))\big]=:M_{1,\lambda}. \tag{28}$$

Here the constants Q_3 and $Q_{4,\lambda}$ are given by

$$\begin{aligned}
Q_3 &= 4\mathbb{E}\big[u_0^2\big]+8L^2T(T+1),\\
Q_{4,\lambda} &= 8L^2T(T+1)+\frac{2^{2+\alpha^*}Q_2^{\alpha^*}T^{2(1-\alpha^*)}\lambda^2}{(1-\alpha^*)^2}.
\end{aligned}\tag{29}$$

Proof. Use Jensen's inequality (66) in the Appendix with $m=4$and Cauchy inequality to bound v_n in (26) by

$$\begin{aligned}
\mathbb{E}\big[v_n^2\big] &\le 4\mathbb{E}\big[u_0^2\big]+4\mathbb{E}\Bigg[\Bigg(\sum_{l=0}^{n-1}b_{n,l}v_l\Bigg)^2\Bigg]\\
&\quad +4\tau^2\mathbb{E}\Bigg[\Bigg(\sum_{l=0}^{n-1}f(v_l)\Bigg)^2\Bigg]\\
&\quad +4\mathbb{E}\Bigg[\Bigg(\sum_{l=0}^{n-1}b(v_l)\Delta W_l\Bigg)^2\Bigg].
\end{aligned}\tag{30}$$

We now bound the last three terms on the right-hand side. We use Cauchy inequality and assumption (3) to bound the third term on the right-hand side by

$$\begin{aligned}
\tau^2\mathbb{E}\Big[\Big(\textstyle\sum_{l=0}^{n-1}f(v_l)\Big)^2\Big] &\le 2L^2T\tau\textstyle\sum_{l=0}^{n-1}\mathbb{E}\big[\big(1+v_l^2\big)\big]\\
&\le 2L^2T^2+2L^2T\tau\textstyle\sum_{l=0}^{n-1}\mathbb{E}\big[v_l^2\big].
\end{aligned}\tag{31}$$

We use Ito's isometry and assumption (3) to bound the last term on the right-hand side of (30) by

$$\begin{aligned}
\mathbb{E}\Big[\Big(\textstyle\sum_{l=0}^{n-1}b(v_l)\Delta W_l\Big)^2\Big] &= \tau\textstyle\sum_{l=0}^{n-1}\mathbb{E}\big[b(v_l)^2\big]\\
&\le 2L^2\tau\textstyle\sum_{l=0}^{n-1}\mathbb{E}\big[1+v_l^2\big]\\
&\le 2L^2T+2L^2\tau\textstyle\sum_{l=0}^{n-1}\mathbb{E}\big[v_l^2\big].
\end{aligned}\tag{32}$$

We now turn to the estimate of the second term on the right-hand side of (30), which is not common in the analysis of numerical approximations to conventional SDEs. For $0\le l\le n-2$ we use the mean-value theorem to conclude

$$
\begin{aligned}
&(t_n-t_l)^{1-\alpha^*}-(t_n-t_{l+1})^{1-\alpha^*}\le(1-\alpha^*)\tau(t_n-t_{l+1})^{-\alpha^*}\\
&=\frac{(1-\alpha^*)T}{N}\left(\frac{[n-(l+1)]T}{N}\right)^{-\alpha^*}\\
&=\frac{(1-\alpha^*)T^{1-\alpha^*}}{N^{1-\alpha^*}(n-(l+1))^{\alpha^*}}=\frac{(1-\alpha^*)T^{1-\alpha^*}(n-l)^{\alpha^*}}{N^{1-\alpha^*}(n-l)^{\alpha^*}(n-l-1)^{\alpha^*}}\\
&=\frac{(1-\alpha^*)T^{1-\alpha^*}}{N^{1-\alpha^*}(n-l)^{\alpha^*}}\left(1+\frac{1}{n-l-1}\right)^{\alpha^*}\le\frac{(1-\alpha^*)2^{\alpha^*}}{(n-l)^{\alpha^*}}\left(\frac{T}{N}\right)^{1-\alpha^*}.
\end{aligned}
$$

We incorporate the estimate into (27) to bound $|b_{n,l}|$ for $0\le l\le n-2$ by

$$
\begin{aligned}
|b_{n,l}|&=\lambda\left|\frac{(t_n-t_l)^{1-\alpha(t_n)}-(t_n-t_{l+1})^{1-\alpha(t_n)}}{\Gamma(2-\alpha(t_n))}\right|\\
&\le\frac{2^{\alpha^*}\lambda}{\Gamma(1-\alpha(t_n))(n-l)^{\alpha^*}}\left(\frac{T}{N}\right)^{1-\alpha^*}\le\frac{2^{\alpha^*}\lambda}{(n-l)^{\alpha^*}}\left(\frac{T}{N}\right)^{1-\alpha^*}.
\end{aligned}
\tag{33}
$$

For $l=n-1$, we have

$$
\begin{aligned}
|b_{n,n-1}|&=\left|\frac{\lambda\tau^{1-\alpha(t_n)}}{\Gamma\left(2-\alpha(t_n)\right)}\right|=\frac{\lambda}{\left(1-\alpha(t_n)\right)\Gamma\left(1-\alpha(t_n)\right)}\left(\frac{T}{N}\right)^{1-\alpha^*}\\
&\le\frac{\lambda}{1-\alpha^*}\left(\frac{T}{N}\right)^{1-\alpha^*}.
\end{aligned}
\tag{34}
$$

We combine (33) and (34) to bound $|b_{n,l}|$ by

$$
|b_{n,l}|\le\frac{2^{\alpha^*}\lambda}{(1-\alpha^*)(n-l)^{\alpha^*}}\left(\frac{T}{N}\right)^{1-\alpha^*},\quad 0\le l\le n-1.
\tag{35}
$$

We use (10), (25) and (27) to bound

$$
\begin{aligned}
\sum_{l=0}^{n-1}|b_{n,l}|&=\frac{\lambda}{\Gamma(1-\alpha(t_n))}\sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}}\frac{ds}{(t_n-s)^{\alpha(t_n)}}\\
&=\frac{\lambda}{\Gamma(1-\alpha(t_n))}\int_0^{t_n}\frac{ds}{(t_n-s)^{\alpha(t_n)}}\\
&\le\int_0^{t_n}\frac{\lambda Q_2^{\alpha^*}ds}{(t_n-s)^{\alpha^*}}=\frac{Q_2^{\alpha^*}t_n^{1-\alpha^*}\lambda}{1-\alpha^*}.
\end{aligned}
\tag{36}
$$

Use (35) and (36) to bound the second term on the right side of (30) by

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{l=0}^{n-1}b_{n,l}v_l\right)^2\right]&\le\sum_{l=0}^{n-1}|b_{n,l}|\,\mathbb{E}[v_l^2]\sum_{l=0}^{n-1}|b_{n,l}|\\
&\le\frac{2^{\alpha^*}Q_2^{\alpha^*}T^{2(1-\alpha^*)}\lambda^2}{(1-\alpha^*)^2N^{1-\alpha^*}}\sum_{l=0}^{n-1}\frac{\mathbb{E}|v_l|^2}{(n-l)^{\alpha^*}}.
\end{aligned}
\tag{37}
$$

We incorporate estimates (31), (32) and (37) into (30) to obtain

$$
\begin{aligned}
\mathbb{E}\left[v_n^2\right]&\le4\mathbb{E}\left[u_0^2\right]+8L^2T(T+1)+8L^2(T+1)\tau\sum_{l=0}^{n-1}\mathbb{E}|v_l|^2\\
&\quad+\frac{2^{2+\alpha^*}Q_2^{\alpha^*}T^{2(1-\alpha^*)}\lambda^2}{(1-\alpha^*)^2N^{1-\alpha^*}}\sum_{l=0}^{n-1}\frac{\mathbb{E}|v_l|^2}{(n-l)^{\alpha^*}}\\
&\le Q_3+\frac{Q_{4,\lambda}}{N^{1-\alpha^*}}\sum_{l=0}^{n-1}\frac{\mathbb{E}|v_l|^2}{(n-l)^{\alpha^*}}
\end{aligned}
\tag{38}
$$

with Q_3 and $Q_{4,\lambda}$ being given in (29).

We apply the generalized discrete Gronwall's inequality (69) with $\beta=1-\alpha^*$ to arrive at (28). □

3.2. An auxiliary equation and its error estimates

To analyze the strong convergence of the Euler-Maruyama scheme, we define an auxiliary continuous time stochastic process $v(t)$ on $[0,T]$ using the step function $\hat{s}=\hat{s}(s)$ such that $\hat{s}:=t_n$ for $s\in[t_n,t_{n+1})$ and $0\le n\le N-1$

$$
v(t)=u_0+\int_0^t k(t,s)v(\hat{s})ds+\int_0^t f\big(v(\hat{s})\big)ds+\int_0^t b\big(v(\hat{s})\big)dW.
\tag{39}
$$

Lemma 3.2. *Let $\{v_n\}_{n=0}^N$ be the solution of the Euler-Maruyama scheme and v be the continuous time stochastic process defined by (39). Then $v(t_n)=v_n$ for $0\le n\le N$.*

Proof. It is clear that $v(0)=u_0=v_0$. Suppose that $v(t_m)=v_m$ for $0\le m\le n-1\le N-1$. We evaluate the integrals on the right-hand side of (39)similarly to (23)–(25) to obtain

$$
\begin{aligned}
v(t_n)&=u_0+\int_0^{t_n}k(t_n,s)v\big(\hat{s}\big)ds+\int_0^{t_n}f\big(v(\hat{s})\big)ds+\int_0^{t_n}b\big(v(\hat{s})\big)dW\\
&=u_0+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}k(t_n,s)v\big(\hat{s}\big)ds+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}f\big(v(\hat{s})\big)ds\\
&\quad+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}b\big(v(\hat{s})\big)dW\\
&=u_0-\frac{\lambda}{\Gamma(1-\alpha(t_n))}\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\frac{v(t_m)}{(t_n-s)^{\alpha(t_n)}}ds\\
&\quad+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}f(v(t_m))ds+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}b(v(t_m))dW\\
&=u_0+\sum_{m=0}^{n-1}b_{n,m}v(t_m)+\tau\sum_{m=0}^{n-1}f(v(t_m))+\sum_{m=0}^{n-1}b(v(t_m))\Delta W_m\\
&=u_0+\sum_{m=0}^{n-1}b_{n,m}v_m+\tau\sum_{m=0}^{n-1}f(v_m)+\sum_{m=0}^{n-1}b(v_m)\Delta W_m=v_n.
\end{aligned}
$$

By mathematical induction, we prove the lemma. □

Theorem 3.3. *The following estimate holds for the continuous time stochastic process v defined by (39) for any $t\in[t_n,t_{n+1})$ with $0\le n\le N-1$*

$$
\mathbb{E}\left[\left(v(t)-v(t_n)\right)^2\right]\le M_{2,\lambda}\tau^{2(1-\alpha^*)}+M_{3,\lambda}\tau,
\tag{40}
$$

where $M_{2,\lambda}$ and $M_{3,\lambda}$ are given by

$$
\begin{aligned}
M_{2,\lambda}&=6\lambda^2M_{1,\lambda}\left(4Q_2^{2\alpha^*}T^{1-\alpha^*}\|\alpha\|^2_{C^1[0,T]}+\frac{9+2Q_5Q_2^2}{(1-\alpha^*)^2}\right),\\
M_{3,\lambda}&=12L^2\left(1+M_{1,\lambda}\right),\quad Q_5=\left(\|\alpha\|_{C^1[0,T]}max_{t\in[1-\alpha^*,1]}|\Gamma'(t)|\right)^2.
\end{aligned}
\tag{41}
$$

In particular, if $\lambda=0$ (i.e. for conventional SDEs) estimate (40) reduces to

$$
\begin{aligned}
\mathbb{E}\left[\left(v(t)-v(t_n)\right)^2\right]&\le12L^2\Big\{\left[1+E_{1-\alpha^*}\left(8L^2T(T+1)\Gamma(1-\alpha^*)\right)\right]\\
&\quad\times\left[4\mathbb{E}\left[u_0^2\right]+8L^2T(T+1)\right]+1\Big\}\tau.
\end{aligned}
\tag{42}
$$

Proof. For $t\in[t_n,t_{n+1})$ with $0\le n\le N-1$,we subtract Eq. (39) at time t_n from Eq. (26) and apply Jensen's inequality with $m=3$ to obtain

$$
\begin{aligned}
&\mathbb{E}\left[(v(t)-v(t_n))^2\right]\\
&\le3\mathbb{E}\left[\left(\int_0^t k(t,s)v\big(\hat{s}\big)ds-\int_0^{t_n}k(t_n,s)v\big(\hat{s}\big)ds\right)^2\right]\\
&\quad+3\mathbb{E}\left[\left(\int_0^t f\big(v(\hat{s})\big)ds-\int_0^{t_n}f\big(v(\hat{s})\big)ds\right)^2\right]\\
&\quad+3\mathbb{E}\left[\left(\int_0^t b\big(v(\hat{s})\big)dW(s)-\int_0^{t_n}b\big(v(\hat{s})\big)dW(s)\right)^2\right]\\
&\le6\mathbb{E}\left[\left(\int_0^{t_n}(k(t,s)-k(t_n,s))v\big(\hat{s}\big)ds\right)^2\right]
\end{aligned}
$$

$$+6\mathbb{E}\left[\left(\int_{t_n}^{t} k(t,s)v(\hat{s})ds\right)^2\right]$$
$$+3\mathbb{E}\left[\left(\int_{t_n}^{t} f\big(v(\hat{s})\big)ds\right)^2\right] + 3\mathbb{E}\left[\left(\int_{t_n}^{t} b\big(v(\hat{s})\big)dW(s)\right)^2\right]. \quad (43)$$

We use assumption (3) and Theorem 3.1 to bound the third term by

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{t_n}^{t} f\big(v(\hat{s})\big)ds\right)^2\right] \\
&\le \tau \int_{t_n}^{t} \mathbb{E}\left[f\big(v(\hat{s})\big)^2\right]ds \\
&\le 2L^2\tau \int_{t_n}^{t} \big(1+\mathbb{E}[|v_n|]^2\big)ds \le 2L^2\big(1+M_{1,\lambda}\big)\tau^2. \quad (44)
\end{aligned}$$

We use Itô isometry and assumption (3) to bound the last term on the right-hand side of (43) by

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{t_n}^{t} b\big(v(\hat{s})\big)dW(s)\right)^2\right] &= \mathbb{E}\left[\int_{t_n}^{t} b\big(v(\hat{s})\big)^2 ds\right] \\
&\le 2L^2\big(1+\mathbb{E}\big[v_n^2\big]\big)\tau \quad (45)\\
&\le 2L^2\big(1+M_{1,\lambda}\big)\tau.
\end{aligned}$$

We are now in the position to the estimate of the first two terms on the right-hand side of (43), which are not common in the context of conventional SDEs and constant-order fSDEs. We use Cauchy inequality to bound the second term on the right side of (43) by

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{t_n}^{t} k(t,s)v(\hat{s})ds\right)^2\right] \\
&\le \int_{t_n}^{t} |k(t,s)|\,\mathbb{E}\big[v(\hat{s})\big]^2 ds \int_{t_n}^{t} |k(t,s)|ds \\
&\le M_{1,\lambda}\left(\int_{t_n}^{t} |k(t,s)|ds\right)^2 \\
&\le \frac{M_{1,\lambda}\lambda^2\tau^{2(1-\alpha^*)}}{(1-\alpha^*)^2}. \quad (46)
\end{aligned}$$

We now turn to the estimate of the first term on the right-hand side of (43)

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{0}^{t_n} (k(t,s)-k(t_n,s))v(\hat{s})ds\right)^2\right] \\
&\le \int_0^{t_n} |k(t,s)-k(t_n,s)|\,\mathbb{E}\big[v(\hat{s})\big]^2 ds \int_0^{t_n} |k(t,s)-k(t_n,s)|ds \\
&\le M_{1,\lambda}\lambda^2\left(\int_0^{t_n} \left|\frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} - \frac{(t_n-s)^{-\alpha(t_n)}}{\Gamma(1-\alpha(t_n))}\right|ds\right)^2 \\
&\le 2M_{1,\lambda}\lambda^2\left(\int_0^{t_n} \frac{|(t-s)^{-\alpha(t)}-(t_n-s)^{-\alpha(t_n)}|}{\Gamma(1-\alpha(t))}ds\right)^2 \\
&+2M_{1,\lambda}\lambda^2\left(\int_0^{t_n} \frac{1}{(t_n-s)^{\alpha(t_n)}}\left|\frac{1}{\Gamma(1-\alpha(t))} - \frac{1}{\Gamma(1-\alpha(t_n))}\right|ds\right)^2 \\
&= 2M_{1,\lambda}\lambda^2 J_1 + 2M_{1,\lambda}\lambda^2 J_2. \quad (47)
\end{aligned}$$

We bound the second term on the right-hand side of (47) by

$$\begin{aligned}
J_2 &\le \left(\int_0^{t_n} \frac{1}{(t_n-s)^{\alpha(t_n)}}\frac{|\Gamma(1-\alpha(t))-\Gamma(1-\alpha(t_n))|}{\Gamma(1-\alpha(t))\Gamma(1-\alpha(t_n))}ds\right)^2 \quad (48)\\
&\le Q_5\tau^2\left(\int_0^{t_n} \frac{ds}{(t_n-s)^{\alpha(t_n)}}\right)^2 \le \frac{Q_5Q_2^2\tau^2}{(1-\alpha^*)^2}
\end{aligned}$$

with Q_5being introduced in (41).

We use (10) and the facts that for $t \ge t_n, t^{1-\alpha} - t_n^{1-\alpha} \le (t-t_n)^{1-\alpha}$ and $(t-t_n)|\ln(t-s)|$ is bounded to obtain

$$\begin{aligned}
J_1 &\le 2\left(\int_0^{t_n} |(t-s)^{-\alpha(t_n)} - (t-s)^{-\alpha(t)}|ds\right)^2 \\
&\quad +2\left\{\int_0^{t_n}\left(\frac{1}{(t_n-s)^{\alpha(t_n)}} - \frac{1}{(t-s)^{\alpha(t_n)}}\right)ds\right\}^2 \\
&\le 2\left(\int_0^{t_n} \frac{|(t-s)^{\alpha(t)-\alpha(t_n)}-1|}{(t-s)^{\alpha(t)}}ds\right)^2 \\
&\quad + \frac{2\left(t_n^{1-\alpha(t_n)}+(t-t_n)^{1-\alpha(t_n)}-t^{1-\alpha(t_n)}\right)^2}{(1-\alpha(t_n))^2} \quad (49)\\
&\le 2Q_2^{2\alpha^*}\|\alpha\|^2_{C^1[0,T]}\left(\int_0^{t_n} \frac{(t-t_n)|\ln(t-s)|}{(t-s)^{\alpha^*}}ds\right)^2 + \frac{4(t-t_n)^{2(1-\alpha(t_n))}}{(1-\alpha(t_n))^2} \\
&\le 2Q_2^{2\alpha^*}\|\alpha\|^2_{C^1[0,T]}\tau^2\left(\int_0^{t_n} \frac{(t-s)^{\frac{1-\alpha^*}{2}}|\ln(t-s)|}{(t-s)^{(1+\alpha^*)/2}}ds\right)^2 + \frac{4\tau^{2(1-\alpha^*)}}{(1-\alpha^*)^2} \\
&\le \left(2Q_2^{2\alpha^*}T^{1-\alpha^*}\|\alpha\|^2_{C^1[0,T]} + \frac{4}{(1-\alpha^*)^2}\right)\tau^{2(1-\alpha^*)}.
\end{aligned}$$

We incorporate (48) and (49) into (47) to bound the first term on the right-hand side of (43) by

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^{t_n} (k(t,s)-k(t_n,s))v(\hat{s})ds\right)^2\right] \\
&\le 2M_{1,\lambda}\lambda^2\left(2Q_2^{2\alpha^*}T^{1-\alpha^*}\|\alpha\|^2_{C^1[0,T]} + \frac{4}{(1-\alpha^*)^2}\right)\tau^{2(1-\alpha^*)} \quad (50)\\
&\quad + \frac{2M_{1,\lambda}\lambda^2Q_5Q_2^2\tau^2}{(1-\alpha^*)^2}.
\end{aligned}$$

We combine (43), (44), (45), (46) and (50) to finish the proof of estimate (40). If $\lambda = 0, M_{2,\lambda}$ in (40) vanishes. $M_{1,\lambda}$ in (28) reduces to

$$M_{1,0} = Q_3\big[1 + E_{1-\alpha^*}(8L^2T(T+1)\Gamma(1-\alpha^*))\big] \quad (51)$$

with Q_3 given in (29). Incorporate (51) into (40) to finish the proof of estimate (42). □

3.3. Error estimate of the Euler-Maruyama scheme (26)

We now prove the main result of this paper, the strong convergence of Euler-Maruyama scheme (26). Now, we turn to estimate the error of $u(t) - v(t)$.

Theorem 3.4. *Let u and v be the solutions to the variable-order fSDE (2) and the auxiliarty Eq. (39), respectively. Then the following estimate holds*

$$\max_{t\in[0,T]} \mathbb{E}\big[|u(t)-v(t)|^2\big] \le M_{4,\lambda}\tau^{2(1-\alpha^*)} + M_{5,\lambda}\tau. \quad (52)$$

Here $M_{4,\lambda}$ and $M_{5,\lambda}$ are given by

$$\begin{aligned}
M_{4,\lambda} &= Q_6E_{1-\alpha^*}\big(Q_7\Gamma(1-\alpha^*)T^{1-\alpha^*}\big)M_{2,\lambda}, \\
M_{5,\lambda} &= Q_6E_{1-\alpha^*}\big(Q_7\Gamma(1-\alpha^*)T^{1-\alpha^*}\big)M_{3,\lambda}, \quad (53)\\
Q_6 &= \frac{6\lambda^2Q_2^{2\alpha^*}T^{2(1-\alpha^*)}}{(1-\alpha^*)^2} + 6L^2T(T+1), \\
Q_7 &= \frac{6\lambda^2Q_2^{2\alpha^*}T^{1-\alpha^*}}{1-\alpha^*} + 6L^2T^{\alpha^*}(T+1).
\end{aligned}$$

The following error estimate holds for the Euler-Maruyama scheme (26)

$$\max_{0\le n\le N} \mathbb{E}\big[|u(t_n)-v_n|^2\big] \le M_{4,\lambda}\tau^{2(1-\alpha^*)} + M_{5,\lambda}\tau. \quad (54)$$

In particular, if $\lambda = 0$ (when fSDE (2) reduces to a conventional SDE), the estimate (55) reduces to the following standard estimate for the Euler-Maruyama scheme

$$\max_{0\le n\le N} \mathbb{E}\Big[|u(t_n) - v_n|^2\Big] \le M_{5,0}\tau. \tag{55}$$

Proof. For any $t \in [0,T)$,let $t \in [t_n, t_{n+1})$for some $0 \le n \le N-1$. We subtract Eq. (39)from Eq. (5) to obtain

$$\begin{aligned} \mathbb{E}\Big[\big(u(t) - v(t)\big)^2\Big] \le\ & 3\mathbb{E}\Big[\Big(\textstyle\int_0^t k(t,s)\big(u(s) - v(\hat{s})\big)ds\Big)^2\Big] \\ & + 3\mathbb{E}\Big[\Big(\textstyle\int_0^t f(u(s)) - f\big(v(\hat{s})\big)ds\Big)^2\Big] \\ & + 3\mathbb{E}\Big[\Big(\textstyle\int_0^t b(u(s)) - b\big(v(\hat{s})\big)dW\Big)^2\Big] \\ =: & \textstyle\sum_{j=1}^3 I_j. \end{aligned} \tag{56}$$

We use Cauchy inequality, assumption (3) and Theorem 3.3 to bound $I_2$by

$$\begin{aligned} I_2 &\le 3TL^2 \int_0^t \mathbb{E}\Big[|u(s) - v\big(\hat{s}\big)|^2\Big]ds \\ &\le 6TL^2 \int_0^t \Big(\mathbb{E}\Big[|u(s) - v(s)|^2\Big] + \mathbb{E}\Big[|v(s) - v\big(\hat{s}\big)|^2\Big]\Big)ds \\ &\le 6TL^2 \int_0^t \mathbb{E}\Big[|u(s) - v(s)|^2\Big]ds + 6T^2L^2\Big(M_{2,\lambda}\tau^{2(1-\alpha^*)} + M_{3,\lambda}\tau\Big). \end{aligned} \tag{57}$$

We use Itô isometry, assumption (3), Theorem 3.3 and split $u(s) - v(\hat{s}) = \big(u(s) - v(s)\big) + \big(v(s) - v(\hat{s})\big)$ to bound I_3 by

$$\begin{aligned} I_3 &= 3\int_0^t \mathbb{E}\Big[\Big(b(u(s)) - b\big(v\big(\hat{s}\big)\big)\Big)^2\Big]dW \\ &\le 3L^2 \int_0^t \mathbb{E}\Big[|u(s) - v\big(\hat{s}\big)|^2\Big]ds \\ &\le 6L^2 \int_0^t \Big(\mathbb{E}\Big[|u(s) - v(s)|^2\Big] + \mathbb{E}\Big[|v(s) - v\big(\hat{s}\big)|^2\Big]\Big)ds \\ &\le 6L^2 \int_0^t \mathbb{E}\Big[|u(s) - v(s)|^2\Big]ds + 6TL^2\Big(M_{2,\lambda}\tau^{2(1-\alpha^*)} + M_{3,\lambda}\tau\Big). \end{aligned} \tag{58}$$

We similarly bound I_1 by

$$\begin{aligned} I_1 \quad & \le 6\mathbb{E}\Big[\Big(\textstyle\int_0^t k(t,s)(u(s) - v(s))ds\Big)^2\Big] \\ & + 6\mathbb{E}\Big[\Big(\textstyle\int_0^t k(t,s)\big(v(s) - v\big(\hat{s}\big)\big)ds\Big)^2\Big] =: I_{1,1} + I_{1,2}. \end{aligned} \tag{59}$$

We use Cauchy inequality and (10) to bound $I_{1,1}$ by

$$\begin{aligned} I_{1,1} &= 6\int_0^t \mathbb{E}\Big[|u(s) - v(s)|^2\Big]\,|k(t,s)|ds \int_0^t |k(t,s)|ds \\ &\le \frac{6Q_2^{2\alpha^*}T^{1-\alpha^*}\lambda^2}{1-\alpha^*}\int_0^t \frac{\mathbb{E}\Big[|u(s) - v(s)|^2\Big]}{(t-s)^{\alpha^*}}ds. \end{aligned} \tag{60}$$

We use Cauchy inequality and Theorem 3.3 to bound $I_{1,2}$ by

$$\begin{aligned} I_{1,2} &\le 6\textstyle\int_0^t \mathbb{E}\Big[|v(s) - v(\hat{s})|^2\Big]\,|k(t,s)|ds \int_0^t |k(t,s)|ds \\ &\le \tfrac{6\lambda^2 Q_2^{2\alpha^*}T^{2(1-\alpha^*)}}{(1-\alpha^*)^2}\Big(M_{2,\lambda}\tau^{2(1-\alpha^*)} + M_{3,\lambda}\tau\Big). \end{aligned} \tag{61}$$

Substitute estimates (57), (58), (59), (60) and (61) into (56) to obtain

$$\begin{aligned} &\mathbb{E}\Big[\big(u(t) - v(t)\big)^2\Big] \\ &\le Q_6\Big(M_{2,\lambda}\tau^{2(1-\alpha^*)} + M_{3,\lambda}\tau\Big) + Q_7\textstyle\int_0^t \frac{\mathbb{E}|u(s)-v(s)|^2}{(t-s)^{\alpha^*}}ds. \end{aligned}$$

Here Q_6 and Q_7 are given by (53).

Table 1
Convergence of scheme (26) for the linear fSDE (2) in Example 1.

$(\alpha(0), \alpha(1))$	$(0.2, 0.1)$		$(0.6, 0.3)$		$(0.8, 0.5)$	
N	e_τ	κ	e_τ	κ	e_τ	κ
8	3.32E−2		4.17E−2		4.73E−2	
16	2.23E−2	0.57	3.25E−2	0.36	4.16E−2	0.19
32	1.51E−2	0.55	2.52E−2	0.37	3.73E−2	0.16
64	1.00E−2	0.59	1.88E−2	0.42	3.30E−2	0.18
$\min\{1-\alpha^*, 0.5\}$		0.50		0.40		0.20

Apply the generalized Gronwall's inequality (67) in the Appendix to complete the proof of (52). If we choose $t = t_n$ in the estimate (52), $v(t_n) = v_n$ by Lemma 3.2. Then estimate (52) reduces to the estimate (54). In particular, if $\lambda = 0$,$M_{2,\lambda} = 0$ by (41). Hence, $M_{4,\lambda} = 0$ by (53). Thus, (54) reduces to (55). □

4. Numerical experiments

We carry out numerical experiments to investigate the performance of the Euler-Maruyama scheme (26). All the numerical experiments were implemented using MATLAB R2018b on a ThinkPad E431 Laptop with Inter Core i5 (2.60 GHz) CPU and 8.0G RAM.

4.1. Strong convergence of the Euler-Maruyama scheme (26)

We perform numerical experiments to test the strong convergence of the Euler-Maruyama scheme (26) for the variable-order fSDE (2). Let $u(t_n,\omega_j)$ be the jth independent sample path of the fSDE (2)evaluated at t_n with the numerical approximation $v_n(\omega_j)$ by the Euler-Maruyama scheme (26) for $n = 0,1,2,\dots,N$ and $j = 1,2,\dots,M$. Then we compute the sample mean of the error as follows

$$e_\tau := \max_{0\le n\le N}\left[\frac{1}{M}\sum_{j=1}^{M}\Big|u(t_n,\omega_j) - v_n(\omega_j)\Big|^2\right]^{\frac{1}{2}} \le QN^{-\kappa} \tag{62}$$

and fit the convergence rate $\kappa$by

$$\kappa := \log_2 \frac{e_\tau}{e_{\tau/2}}. \tag{63}$$

In the numerical experiments the time interval $[0,T] = [0,1]$, $\lambda = 1$, $M = 2^{10} = 1,024$ and the variable order α is chosen to be of the form

$$\alpha(t) = \alpha(1) + (\alpha(0) - \alpha(1))\left((1-t) - \frac{\sin(2\pi(1-t))}{2\pi}\right). \tag{64}$$

Since the true solution is not known a priori, we use a fine mesh size of $N_{ref} = 2^{10}$ to compute the reference solutions.

Example 1. A linear variable-order fSDE. We choose $f(u) = b(u) = -u$ and $u_0 = 0.1$ in the variable-order fSDE (2). We present the error e_τ for different mesh size N and difference choices of $\alpha(0)$ and $\alpha(1)$ in (64) in Table 1. We observe that the Euler-Maruyama scheme (26) demonstrates the convergence rates that are in agreement with the theoretical analysis in Theorem (3.4).

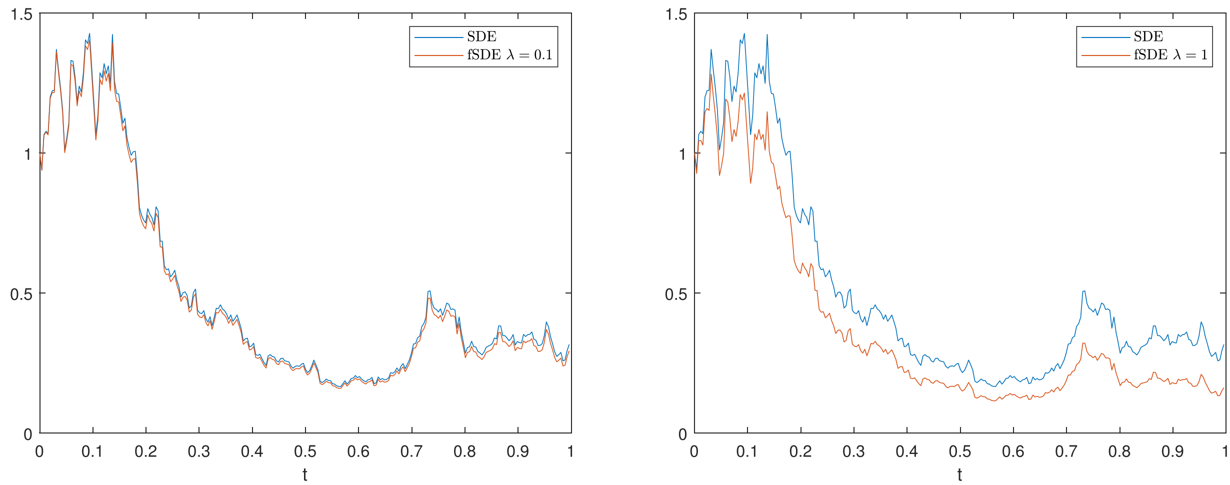

Fig. 1. Plots of SDE solutions with a same sample: integer-order SDE solutions ('blue color'), the variable-order fSDE solutions ('red color') with $(\alpha(0), \alpha(1)) = (0.2, 0.1)$ for cases (i) left $\lambda = 0.1$ and (ii) right $\lambda = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

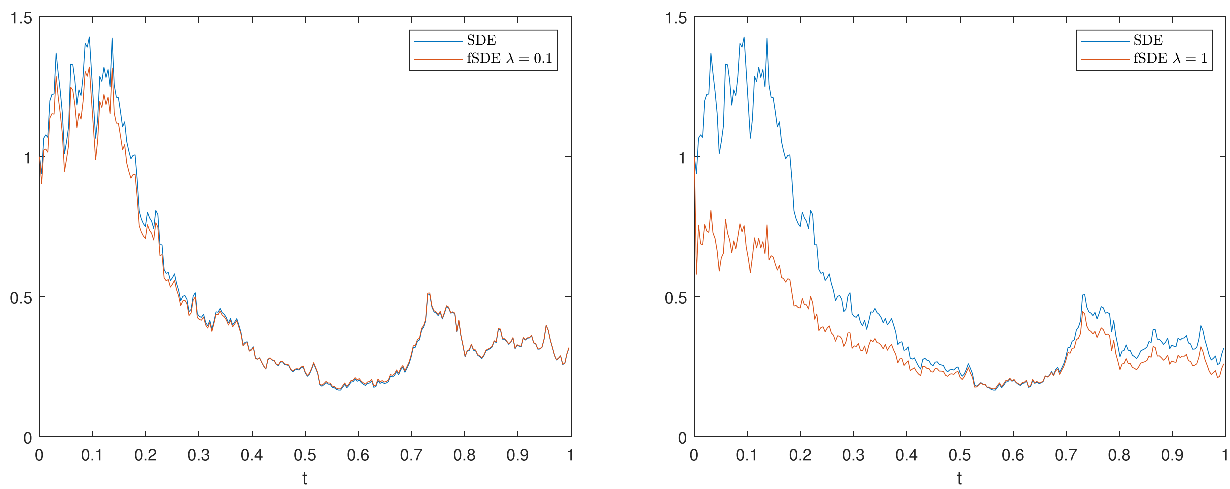

Fig. 2. Plots of SDE solutions with a same sample: integer-order SDE solutions ('blue color'), the variable-order fSDE solutions ('red color') with $(\alpha(0), \alpha(1)) = (0.8, 0.5)$ for cases (i) left $\lambda = 0.1$ and (ii) right $\lambda = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 2
Simulation of the linear fSDE (2) with $\lambda = 0$ in Example 1.

N	e_τ	κ
8	2.75E−2	
16	1.96E−2	0.49
32	1.38E−2	0.51
64	9.55E−3	0.53
Predict		0.50

Table 3
Convergence of scheme (26) for the nonlinear fSDE (2) in Example 2.

$(\alpha(0), \alpha(1))$	$(0.1, 0.2)$		$(0.6, 0.3)$		$(0.8, 0.5)$	
N	e_τ	κ	e_τ	κ	e_τ	κ
8	3.82E−2		5.01E−2		5.68E−2	
16	2.56E−2	0.57	3.90E−2	0.36	4.99E−2	0.19
32	1.74E−2	0.55	3.02E−2	0.37	4.47E−2	0.16
64	1.17E−2	0.57	2.26E−2	0.41	3.96E−2	0.18
$\min\{1-\alpha^*, 0.5\}$		0.50		0.40		0.20

In this example, we also test the case $\lambda = 0$ when the variable-order fSDE (2) naturally reduces to the conventional SDE. We present the numerical results in Table 2, which shows that the Euler-Maruyama scheme (26) naturally reduces to its analogue to conventional SDEs and has a standard convergence rate of first order. This is in consistent with the theoretical analysis in Theorem (3.4).

Example 2. A nonlinear variable-order fSDE We consider the numerical simulation to a nonlinear variable-order fSDE (2) with $f(u) = b(u) = -\sin(u)$. Other data are chosen to be the same as in Example 1. We present the numerical results in Table 3and have the same observations as in Example 1.

4.2. Performance of the variable-order fSDE (2)

We study the performance of the variable-order fSDE (2) in comparison with the conventional SDE, which corresponds to the variable-order fSDE (2) with $\lambda = 0$. In the study we choose $[0, T] = [0, 1], f(u) = b(u) = -u$ and $u_0 = 1$. The variable order α is still given by (64). We use the Euler-Maruyama scheme (26) with a mesh size of $N = 2^{10}$ to compute the solutions for different combinations of $(\alpha(0), \alpha(1))$. We present the results for $(\alpha(0), \alpha(1) = (0, 2, 0.1)$, (0.8,0,5) and in Figs. 1 and 2, respectively. Within each figure, we present four plots ranging from $\lambda = 0.1$ and 1, respectively. We observe that for $\lambda < 1$ (0.1 in this case), the solution of fSDE (2) is close to the solution to conventional SDE. In contrast, for $\lambda = 1$, the solution of fSDE (2) exhibits considerable difference from the solution to SDE.

5. Concluding remarks

In this paper, we study a variable-order fractional stochastic differential equation driven by a multiplicative noise, which contains a non-Lipschitz weakly singular kernel with a variable order, and loses the convolution structure due to the introduction of the variable-order fractional differential operator. We proved the wellposedness and moment estimates of the problem. We also developed a generalized Euler-Maruyama scheme for the problem and we proved the strong convergence of the scheme. The proved strong convergence results naturally reduce to the corresponding results for the classical SDE when the variable-order fSDE reduces to the classical one. We carried out numerical experiments to substantiate the theoretical results. In particular, the convergence behavior reduces to the well known convergence rates of the Euler-Maruyama scheme for conventional SDEs when the variable-order fSDE reduces to the conventional one.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Zhiwei Yang: Methodology, Formal analysis, Writing - original draft. **Xiangcheng Zheng:** Methodology, Formal analysis, Writing - original draft. **Zhongqiang Zhang:** Conceptualization, Supervision. **Hong Wang:** Methodology, Supervision, Funding acquisition.

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Appendix A

We list auxiliary results that were used earlier.

Lemma A.1. *(The Burkhölder-Davis-Gundy inequality [18, 41]) If Y is a continuous martingale on $[0,T]$, there exists a positive constant $\bar{Q}_1 = \bar{Q}_1(p)$ for $1 < p < \infty$ such that*

$$\mathbb{E}\Big[\sup_{x\in[0,t]} |Y(x)|^p\Big] \le \bar{Q}_1 \mathbb{E}\Big[|Y(t)|^p\Big], \quad 0 \le t \le T. \tag{65}$$

Lemma A.2. *(Jensen's inequality [18]) If $a_i, p \in \mathbb{R}$ with $p \ge 1$ and $m \in \mathbb{N}^+$, then*

$$|\sum_{i=1}^{m} a_i|^p \le m^{p-1} \sum_{i=1}^{m} |a_i|^p. \tag{66}$$

Lemma A.3. *(Generalized Gronwall inequality [34]) Let $\bar{Q}_2(t)$ be a non-negative and non-decreasing locally integrable function on $(a,b]$ and $\bar{Q}_3$ be a non-negative constant. Suppose $g(t)$ is a non-negative locally integrable function on $(a,b]$ with*

$$g(t) \le \bar{Q}_2(t) + \bar{Q}_3 \int_a^t \frac{g(s)}{(t-s)^{1-\beta}} ds, \quad \forall t \in (a,b], \quad 0 < \beta < 1,$$

then

$$g(t) \le \bar{Q}_2(t) E_\beta\big(\bar{Q}_3 \Gamma(\beta)(t-a)^\beta\big), \qquad \forall t \in (a,b]. \tag{67}$$

Lemma A.4. *(A generalized discrete Gronwall's inequality [3]) Suppose that a non-negative sequence $\{z_n\}_{n=1}^N$ and a non-negative and non-decreasing sequence $\{y_n\}_{n=1}^N$ satisfy the following relation*

$$z_n \le \frac{\bar{Q}_4}{N^\beta} \sum_{i=1}^{n-1} \frac{z_i}{(n-i)^{1-\beta}} + y_n, \quad 1 \le n \le N, \ 0 < \beta < 1, \ \bar{Q}_4 > 0. \tag{68}$$

Then the sequence $\{z_n\}_{n=1}^N$ can be bounded from above by

$$z_n \le y_n(1 + E_\beta(\bar{Q}_4 \Gamma(\beta))), \quad 1 \le n \le N. \tag{69}$$

Lemma A.5. *Let $\alpha \in \mathbb{R}^+$ with $0 < \alpha < 1$. The following estimate holds*

$$\sup_{0\le x\le t} \int_0^x (x-s)^{-\alpha} |\xi(s)| ds \le \int_0^t (t-s)^{-\alpha} \sup_{0\le r\le s} |\xi(r)| ds. \tag{70}$$

Proof. Let $\hat{\xi}(s) := \sup_{0\le r\le s} |\xi(r)|$. Use $\theta = s/x$ to bound the left side by

$$\begin{aligned} &sup_{0\le x\le t} \int_0^x (x-s)^{-\alpha} |\xi(s)| ds = sup_{0\le x\le t} \int_0^1 (x-\theta x)^{-\alpha} |\xi(\theta x)| x d\theta \\ &= sup_{0\le x\le t} x^{1-\alpha} \int_0^1 (1-\theta)^{-\alpha} |\xi(\theta x)| d\theta \\ &\le t^{1-\alpha} \int_0^1 (1-\theta)^{-\alpha} \hat{\xi}(\theta t) d\theta. \end{aligned} \tag{71}$$

The substitution $s = \theta t$ to the right side to rewrite this term as

$$\begin{aligned} t^{1-\alpha} \int_0^1 (1-\theta)^{-\alpha} \hat{\xi}(\theta t) d\theta &= t^{1-\alpha} \int_0^t \Big(1 - \frac{s}{t}\Big)^{-\alpha} \hat{\xi}(s) \frac{ds}{t} \\ &= \int_0^t (t-s)^{-\alpha} \hat{\xi}(s) ds = \int_0^t (t-s)^{-\alpha} \sup_{0\le r\le s} |\xi(r)| ds. \end{aligned} \tag{72}$$

We combine (71) and (72) to finish the proof of (70). □

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